Weak Interactions and Electroweak Theory

Notes 1: Weak interactions of leptons

Introduction

This part of the course is about weak interactions. You already know quite a lot about the weak force, β-decay of nuclei, the W and the Z, flavour changing interactions and the interactions of neutrinos. These lectures will consolidate this and introduce a few new topics: the idea of a 'current', universality, Cabibbo Theory, the GIM mechanism, left handed particles and electroweak unification. We will try to get most of the theoretical parts done in the lectures this term and then concentrate on examples and experimental measurements at the big accelerators: $p\overline{p}$, LHC and LEP in lectures in Trinity term.

The text book situation is a bit tricky because there is no book pitched at the level we need and you may find, particularly in this part of the course that you need to look at several of the books to find the description of the material which best suits. My main source will be Burcham and Jobes *(BJ)* but both Griffiths *(G)* and Halzen and Martin *(HM)* also have very clear descriptions on these subjects (with somewhat different approaches). The difficulty with both Griffiths and HM is that they are also used in post-graduate courses and use Feynman calculus and calculation of matrix elements (including spin considerations) in detail which is far beyond our syllabus, so you have to ignore some of the more complicated maths and just read for the discussion.

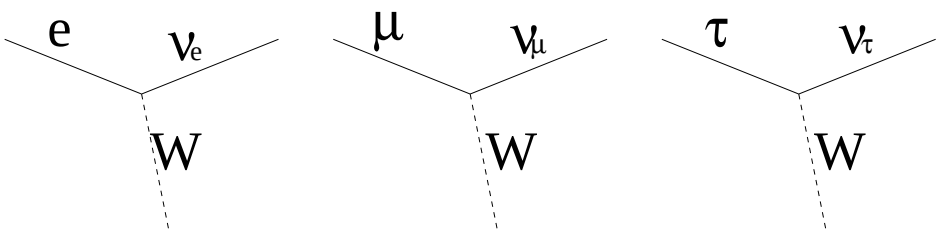
These notes are not detailed enough on the discussion of various important experiments for the M.Phys course — you can get more details from the lectures and the text books.

Lepton number conservation, the weak CC vertex

We are going to discuss the weak force in much more detail than previously, to try to explain the large range of different lifetimes of the particles.

(HM p251) The lifetimes of both the pion ($\pi^+ \to \mu^+\nu_\mu$, $\tau = 2.6\times10^{-8}$s) and the muon ($\mu^+ \to e^+\nu_e\overline{\nu}_\mu$, $\tau = 2.2\times10^{-6}$s) which involve the weak interaction are considerably longer than decays involving either the strong force or the electromagnetic force (e.g. the lifetime of the π^0 which decays electromagnetically to two photons is 10^{-16}s). The fact that the muon decays in the way given above is interesting. Energetically, it could decay by $\mu \to e\gamma$ and the fact that it does not leads us to the rule of conservation of lepton number L_e where we assign $L_e = +1$ to the e^- and ν_e, $L_e = +1$ to the e^+ and $\overline{\nu}_e$, and $L_e = 0$ to all other particles in order to describe the observed pattern of decays. There are exactly analogous conservation laws for each of the other two generations involving L_μ and L_τ.

The lepton number conservation laws along with charge conservation and baryon number conservation are reflected in the fact that there are a fixed number of 'permitted' vertices which one can draw in Feynman diagrams. For the weak charged current interaction (of leptons), these are as follows:



Fermi Theory

In the third year, you learned a slightly simplified version of Fermi's theory of β-decay involving the golden rule

$$w = \frac{2\pi}{\hbar}|M|^2\frac{dN}{dE} \tag{1}$$

where

$$M = \int \psi_f O \psi_i dV = \int \psi_p^* \psi_e^* \psi_\nu^* O \psi_n dV \tag{2}$$

involves the initial and final state wavefunctions ψ_i and ψ_f respectively and O is an operator describing the interaction which was assumed to be a constant G_F. The wavefunctions of the electron and the neutrino were assumed to be constant over the volume of the nucleus and were moved outside the integral; the volume over which they are normalised cancels with the volume used to calculate the density of states dN/dE. For a reminder of this, see Williams (Chapter 12.5). The simplification was that it was described in terms of Schrödinger wavefunctions ψ rather than Dirac spinors which are needed for spin-$\frac{1}{2}$ particles.

In these lectures, we are going to keep the calculation of dN/dE the same as you already know and look in more detail at the Matrix element part M.

Feynman rules

To do the calculation of an elementary particle interaction properly (i.e. to find the matrix element) was originally exceedingly complex, involving days or even weeks to complete a calculation. Richard Feynman invented a very elegant formalism which provided a considerable shortcut in calculations — doing full calculations with it is beyond the scope of the M.Phys major option course, but I'll give you a quick outline because it is used in both Griffiths and HM, and you'll recognize various aspects of it as we go along. It ends up being formulated as a set of "Feynman rules", in other words a recipe for performing the calculation. To show that Feynman rules actually work and give the right answer is even further beyond the scope of this course (and would take a good sized course in itself). We will go into some of the experimental observations about particle interactions, such as parity violation in weak interactions, which have gone into making the rules.

[This bit is beyond the syllabus] Feynman's recipe is as follows: Firstly, the process from incoming particles to outgoing particles is described in terms of "Feynman diagrams" where lines represent 'quanta' or particles. Lines which go to the outside represent 'on-mass-shell' particles, i.e. ones where $E^2 = p^2 + m^2$. Internal lines represent 'virtual particles'[1] where it is not necessarily the case that $E^2 = p^2 + m^2$. Diagrams can have as many complicated lines as you like (but all connected), although with weak and EM interactions, the ones with the fewest vertices (the lowest order diagrams) are the most important. To do the calculation, all diagrams up to a given order must be considered and the amplitudes added together to get the matrix element. In this way, it is possible to have interference (constructive or destructive) between two diagrams.

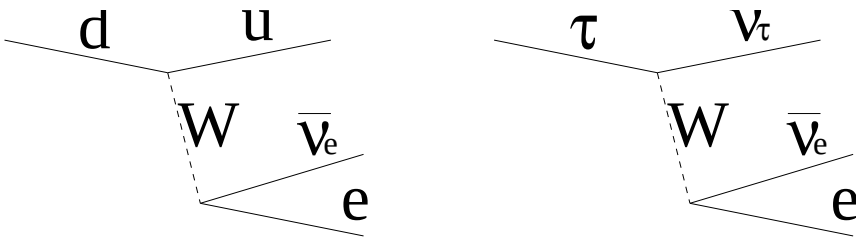
The second step in Feynman's recipe is to write down elements in an expression for the external lines, vertices and internal lines (propagators) according to the rules, and the third step is to evaluate the expression. If there are internal loops, the evaluation involves integrating over the unconstrained momenta on those particles. The recipe ensures energy and momentum conservation and it is also built into the rules that the resulting expression is Lorentz invariant.

[1] This virtualness is what we originally described as 'Energy and momentum does not have to be conserved in quantum mechanics - you are allowed to 'borrow' energy as long as it is within the time of the uncertainty principle $\Delta E \Delta t \leq \hbar/2$'

[Back on syllabus again] Obviously there are some things which you are familiar with here: the idea of the Feynman diagrams themselves gives us a very intuitive 'billiard ball'-like picture of something which is really a complicated interaction of waves and quanta; the idea of vertex factors (e.g. $\alpha = \frac{1}{137}$ for the electromagnetic interaction) comes from the Feynman rules for vertices; and the propagators $1/q^2$ for a massless photon and $1/(M_W^2 - q^2)$ for the W is part of the rules for internal lines.

An example

Let's look at what the Feynman rules give for a particular example. In the section above, we used Fermi theory to do β-decay, i.e. $n \to pe^-\overline{\nu}$. For this example, I will take you through a very similar process[2], one of the decay modes of the tau-lepton $\tau^- \to \nu_\tau e^-\overline{\nu}$. See BJ p488. The Feynman diagrams of these two processes are



The Fermi theory expression for this process involving Schrodinger wavefunctions ψ is

$$M_{fi} = G_F \int \psi^*_{\nu_\tau} \psi^*_e \psi^*_{\overline{\nu}} \psi_\tau dV \tag{3}$$

The equivalent expression from Feynman rules, using Dirac spinors is

$$M_{fi} = \frac{1}{M_W^2 - q^2} \left[\frac{g}{\sqrt{2}} \overline{u}(\nu_\tau) \gamma_\mu \frac{1}{2}(1 - \gamma_5) u(\tau) \right] \left[\frac{g}{\sqrt{2}} \overline{u}(e) \gamma^\mu \frac{1}{2}(1 - \gamma^5) u(\nu_e) \right] \tag{4}$$

where $1/(M_W^2 - q^2)$ is the propagator and the two expressions in square brackets are the vertex factors. The vertex factors involve a particular way of combining the spinors — sandwiching a matrix, $\gamma_\mu \frac{1}{2}(1 - \gamma_5)$ in between the spinor and the adjoint spinor. This particular combination is called 'V−A' and there are more details in one of the next sections. The factor g is called the weak coupling constant and is analogous to the electric charge $e = \sqrt{4\pi\alpha}$ (in natural units) in electromagnetism.

q^2 in the propagator is the four-momentum squared of the W particle. It does not have to be equal to M_W^2 because the W is virtual. Indeed, in the case of τ-decay it must be below m_τ^2, and is usually lower because the ν_τ carries away some energy and momentum.

Since we are in the low energy limit $q^2 << M_W^2$, we can approximate $1/(M_W^2 - q^2) \mapsto 1/M_W^2$ in which case, if we set

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{5}$$

to relate the new constant g to the Fermi constant G_F, then the matrix element becomes

$$M_{fi} = \frac{4G_F}{\sqrt{2}} \left[\overline{u}(\nu_\tau) \gamma_\mu \frac{1}{2}(1 - \gamma_5) u(\tau) \right] \left[\overline{u}(e) \gamma^\mu \frac{1}{2}(1 - \gamma^5) u(\nu_e) \right] \tag{6}$$

and apart from the γ matrices this now looks very much like the expression from Fermi theory. The various factors of 4, and $\sqrt{2}$ are simply to keep the definition of G_F the same as originally defined by Fermi.

[2]The reason for the change is that fully leptonic processes are simpler with no hadronic strong force mess to deal with. I could have picked the decay of the muon as all the books do, however since the symbol μ turns up in things like γ^μ, I chose to avoid a particle called μ

Universality

Our model of weak interactions contains the postulate that

g is the same for *all* weak interactions[3]

There are many experiments to test universality, i.e. to measure g in various interactions and compare, and so far the postulate that it is the same is holding up very well. We will discuss some of these experiments in later lectures.

Left handed particles ...

You will recognize the $\frac{1}{2}(1-\gamma_5)$ in the operator is the left hand chirality projection operator P_L (which, in the high energy limit, is equivalent to a helicity projection). Defining the left handed part of the τ as $u_L(\tau)$,

$$u_L(\tau) = P_L u(\tau) = \frac{1}{2}(1-\gamma_5)u(\tau) \tag{7}$$

This means that the $\frac{1}{2}(1-\gamma_5)u(\tau)$ part of equation (6) can be replaced by $u_L(\tau)$. We say that the weak charged current interaction acts on only the left handed part of the particle. We will formulate this more rigorously when we come to electroweak unification.

... and right handed antiparticles

The $\frac{1}{2}(1-\gamma_5)$ operator, when acting on the spinor of an antiparticle (using v rather than u to denote a spinor of an antiparticle) projects out the right handed chirality of the antiparticle, i.e.

$$v_R = \frac{1}{2}(1-\gamma_5)v \tag{8}$$

The weak charged current acts only on the left handed part of particle wavefunctions and the right handed part of antiparticle wavefunctions.

V−A

We now return to the question of why the Feynman rules were constructed so that the matrix between the spinors was $\gamma_\mu \frac{1}{2}(1-\gamma_5)$. The first step is to work out how to construct a scalar quantity (i.e. a number which is Lorentz invariant) out of a spinor u. The answer is that $\overline{u}u$ is such a quantity. The buzzword to look for in the index of any text book to find this bit is "bilinear covariants" – the discussion in Griffiths p222-225 is particularly good as he gives a formula for how a Dirac spinor transforms under a Lorentz transform.

It is also interesting for this discussion to see what happens under a parity transformation. The combination $\overline{u}u$ remains the same under a parity transformation. We can also construct other numbers by inserting various combinations of gamma matrices in the middle. For example, $\overline{u}\gamma_5 u$ is also a Lorentz invariant scalar, however in this case, the parity operation produces $-\overline{u}\gamma_5 u$, i.e. a change in sign. This is therefore called a 'pseudoscalar'.

It turns out that there are 16 such combinations and these can be arranged in the following way to construct different types of Lorentz invariant quantities. It can be shown that this is the only way to do it and is the complete set of different combinations which can be made

[3] well, all weak charged current interactions involving leptons. To bring quarks into the picture, we need Cabibbo theory (next lecture) and to include the neutral current we need the Glashow theory of electroweak unification (next lecture but one).

	Name		#components	Parity transformation P
$\overline{u}u$	S	Scalar	1	+ under P
$\overline{u}\gamma_\mu u$	V	Vector	4	$(+,-,-,-)$
$\overline{u}\sigma_{\mu\nu}u$	T	Tensor	6	
$\overline{u}\gamma_\mu\gamma_5 u$	A	Axial vector	4	$(+,+,+,+)$
$\overline{u}\gamma_5 u$	P	Pseudo-scalar	1	− under P

where[4] $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$.

So the first step has given us choices of S,V,T,A or P (or some combination) for the interaction type for weak interactions. The second step is to turn to experiment to try to select the correct one. Burcham and Jobes p398-401 is good for this. It involves (i) looking at Fermi ($\Delta J = 0$ where J is the total spin of the nucleus) and Gamow-Teller ($\Delta J = 0, 1$) allowed β-decays and comparing various features of the decay electron spectra to see if the Fermi transitions are S or V or a combination of the two. The Gamow-Teller transitions tell us about the choice between A and T. After considerable experimental investigation, it was determined to be a combination of V and A with no contributions from S or T (P cannot produce a large contribution). Further measurements allow us to determine the relative contributions of V and A (equal) and the sign (negative). So:

The weak interaction is V−A, i.e. $\gamma_\mu(1-\gamma_5)$

Parity violation

We have just decided that the weak interaction has a V−A form, i.e. that the spinor combinations $\overline{u}\gamma_\mu u$ and $\overline{u}\gamma_\mu\gamma_5 u$ appear with equal amounts. Both combinations are Lorentz 4-vectors and so therefore the V−A combination

$$\overline{u}\gamma_\mu u - \overline{u}\gamma_\mu\gamma_5 u = \overline{u}\gamma_\mu(1-\gamma_5)u \tag{9}$$

is also a Lorentz 4-vector[5]

We now turn to the properties of these quantities under a parity operation. Both $\overline{u}\gamma_\mu u$ and $\overline{u}\gamma_\mu\gamma_5 u$ have definite parity transformation properties, however they are different. If the weak interaction were a single type, i.e. either V or A, then there would not be parity violation — the final state would have definite parity. The V−A combination $\overline{u}\gamma_\mu(1-\gamma_5)u$ however has a mixture of parity change or non-change and this results in parity violation. Since the V and A parts come with equal amounts, parity is said to be maximally violated - i.e. it doesn't just get flipped in a few percent of weak interactions, it's randomness is maximum.

Currents and fields

Now is a convenient point to define what is meant by a current, since it will come up a lot in the lecture on electroweak theory. See BJ, p488-490. The contents of the square brackets in equation 6 are each referred to as currents

$$J^+_\mu = \overline{u}(\nu_\tau)\gamma_\mu\frac{1}{2}(1-\gamma_5)u(\tau) \tag{10}$$

is a charge raising current (because the outgoing ν_τ is one positive increment in charge bigger than the incoming τ). Similarly,

$$J^{-\mu} = \overline{u}(e)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(\nu_e) \tag{11}$$

[4]the use of σ here has nothing to do with Pauli spin matrices

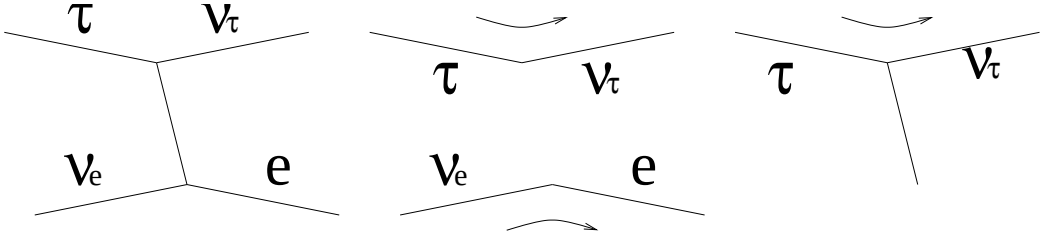
[5]Since a Lorentz invariant can be made from any pair of 4-vectors $a_\mu b^\mu = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$, we can make a Lorentz invariant quantity involving spinors e.g. $p_\mu\overline{u}\gamma^\mu(1-\gamma^5)u$

is a charge lowering current. These charge raising (lowering) currents are the reason why this sort of interaction is called a charged current — the particle in the middle is a W^+ or a W^- and charge needs to be conserved at each vertex. Of course, the other type of weak interaction is the neutral current and involves the exchange of the Z^0.

We can now write the matrix element in equation 4 as

$$M_{fi} = \frac{g}{\sqrt{2}} J_\mu^+ \cdot \frac{1}{M_W^2 - q^2} \cdot \frac{g}{\sqrt{2}} J^{-\mu} \tag{12}$$

This is a so called current-current formalism of an interaction, i.e. the interaction occurs between the two currents as shown in the middle diagram below.



Another variation on this theme is to treat the two halves of the Feynman diagram separately. We can 'plug-and-play' i.e. change the particles at one vertex (e.g. from (e, ν_e) to (d, u)) without changing the calculation at the other vertex. We do this by introducing the concept of a field W^μ.

$$W^\mu = \frac{1}{M_W^2 - q^2} \cdot \frac{g}{\sqrt{2}} J^{-\mu} \tag{13}$$

and the matrix element is

$$M_{fi} = \frac{g}{\sqrt{2}} J_\mu^+ \cdot W^\mu \tag{14}$$

This is shown on the right hand diagram above.

Since we are talking about currents and fields, the equivalent current for an electromagnetic interaction, e.g. on an electron is

$$J_\mu^{EM} = \overline{u}(e)\gamma_\mu u(e) \tag{15}$$

and the electromagnetic field is A^μ (p117 of HM for more details).

This discussion in terms of currents and fields is reminiscent of electric currents and magnetic fields, i.e. a quantity called a magnetic field is constructed at all points in space to describe the action of all the bits of current which are flowing in all circuits nearby. This can then be used to determine the force on a particular piece of current placed somewhere.

Note also that the continuity equation for the Dirac equation gives the expression for the Dirac spinor equivalent of a probability current as $\overline{u}\gamma_\mu u$ (one of the problem-set questions directs you to derive this by doing particular operations on the Dirac equation and the adjoint Dirac equation and combining).

Pion decay revisited

The V−A form of the weak interaction can now be used to calculate the decay $\pi^+ \to \mu^+\nu_\mu$ and compare it to the decay $\pi^+ \to e^+\nu_e$ which would be favoured if phase space arguments alone were used. This section of the notes is truncated because it is in a problem-set question. The result is

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 1.28 \times 10^{-4} \tag{16}$$

The predicted value agrees with the measured value of the branching ratio. This is at first sight a very surprising result because it correctly predicts that the muon mode is strongly preferred over the electron mode. A clue is suggested by equation (16), note that in the limit $m_e \to 0$ the electron mode would be disallowed completely. This can be explained as follows: The pion is spin zero and so, from conservation of angular momentum, in the pion rest frame, the electron and the neutrino must have opposite spins and therefore equal helicities. From V−A, the weak interaction only interacts with the chiral left handed part of the neutrino wavefunction and right handed part of the anti-electron. Assuming the neutrino is completely massless (i.e. chirality and helicity are the same), this means that the helicity of both particles is negative. Thus the anti-electron has the 'wrong' helicity and this is why the decay is so strongly suppressed.

Experiments

The following experiments are an important part of the course. Some will be covered in my lectures in Trinity term, some have been covered by other lecturers or in the class questions.

Discovery of the neutrino	Reins & Cowan using reactor antineutrinos; Griffiths p26
Neutrino helicity	See question from MT problem sheet 3, and BJ p371-375
Wu parity violation experiment	BJ p369
W decay angular distributions	BJ p557
$(1-y)^2$ in DIS formulae	Can be understood in terms of spin and left-hand particle weak interactions; BJ pp 459-464; HM pp273-5
Direct measurement of ν mass	β-spectrum end point on Kurie plot. BJ p167.
Neutrino interactions	See my lecture later on neutrinos
Discovery of the W	UA1 & UA2 in 1983. BJ p554 onwards
ν_τ 'discovery'	As recently as 2001 — first ν_τ interaction observed in DONUT at Fermilab
Universality	LEP (CERN) $W \to \nu e$, $W \to \nu\mu$, $W \to \nu\tau$ branching ratios compared. See my lecture later on LEP

Notes 2: Weak interactions including quarks

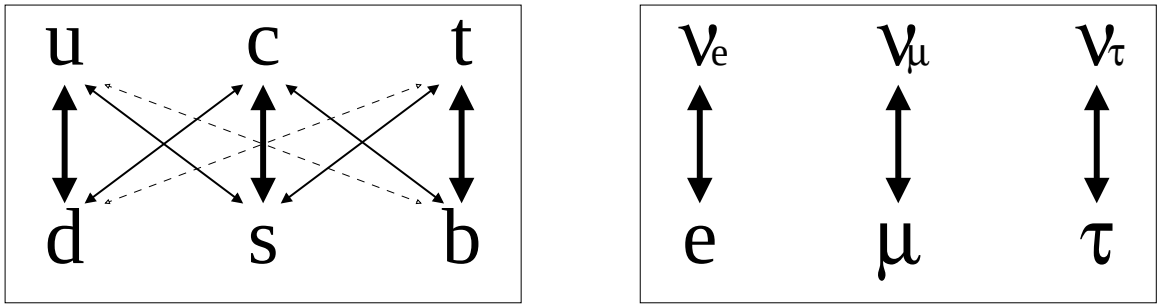
Introduction

Last time, we discussed the weak interaction and introduced the idea that the charged current has a V−A structure which maximally violates parity as indicated by experiment. We also introduced the idea that there is a single coupling constant g governing the strength of all charged current weak interactions of leptons. This is known as lepton-universality. It is saying that the *only* difference between an electron and a muon (and a τ-lepton) is that they have different masses. Other than this, they are identical.

We were fairly cagey last time about getting involved with weak interactions of quarks and that was waiting until this time when we introduce Cabibbo theory.

Cabibbo theory

The figure on the left below shows the quarks arranged in families with the possible charged current transitions marked (thicker lines denote the transitions with the highest probabilities). In contrast, the diagram on the right shows the lepton families. Extra transitions are possible to convert quarks of different generations which don't exist in the interaction of the leptons. If this were not the case, the K and Λ particles would be completely stable (as would some particles with b-quarks).



Cabibbo developed his theory when only u, d and s quarks were known. The theory states:

> The charged current interactions of quarks proceed with the same coupling constant g as for lepton interactions provided we associate a factor $\cos\theta_C$ to $u \leftrightarrow d$ transitions and a factor $\sin\theta_C$ to $u \leftrightarrow s$ transitions.

With this scheme, a wide number of weak interactions and decays are correctly predicted, the value of θ_C required is 13.1°.

However, one problem which still remained was that the decay $K_L^0 \mapsto \mu^+\mu^-$ (a second order weak interaction - it involves two W particles in the Feynman diagram) was predicted to go faster than experimentally measured. A solution to this paradox was suggested by Glashow, Iliopoulos and Maiani (GIM)...

The GIM mechanism, flavour changing neutral currents

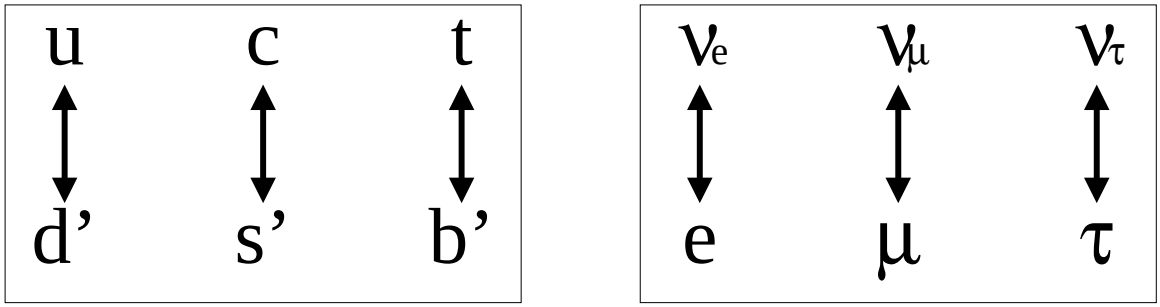
A fourth quark c was introduced into the theory[6] with couplings $\cos\theta_C$ for $c \leftrightarrow s$ transitions and $-\sin\theta_C$ for $c \leftrightarrow d$. This produces a second Feynman diagram in the process $K_L^0 \mapsto \mu^+\mu^-$ which interferes destructively with the one involving only u, d and s quarks and suppresses the decay to a level compatible with experiment [see e.g. BJ p411, Griffiths p320 or HM p282].

[6]This was in 1970, four years before the J/ψ was discovered

With four quarks, the GIM mechanism gives a nice conceptual picture of what is going on. If we define a linear supposition of quarks (of flavour eigenstates d and s) to form d' and s' as follows

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_c \\ -\sin\theta_C & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \tag{17}$$

then it is possible to view weak charged current interactions of quarks as occurring within generations with the same coupling g as for leptons provided we consider the rotated states d' and s' rather than d and s as follows



The original motivation of the GIM mechanism was to suppress weak interactions which change flavour but not the quark charge, such as $s \mapsto d$ which are not observed in nature. The idea is as follows: The Cabbibo rotation in equation (17) shows that we have to consider weak charged-current interaction as being between quarks $u \leftrightarrow d'$ and $c \leftrightarrow s'$ and that these form two families which don't mix at all just like the leptons (without neutrino oscillations). Therefore, for the weak neutral-current interaction, we can hypothesise that the same thing happens, i.e. that the quark families don't mix at all in the rotated basis; the only interactions allowed are $u \leftrightarrow u$, $d' \leftrightarrow d'$, $s' \leftrightarrow s'$ and $c \leftrightarrow c$.

The question is, now we restrict ourselves to the above hypothesis, is it still possible to see flavour changing interactions of the neutral current, i.e. $d \leftrightarrow s$? Well, let's see. If it were possible, then the overlap between d and s should not be zero. Take the rotation defined in equation (17), invert it and use it to write

$$\begin{aligned} ds &= (d'\cos\theta_C - s'\sin\theta_C)(d'\sin\theta_C + s'\cos\theta_C) && (18)\\ &= (d'd' - s's')\cos\theta_C\sin\theta_C + d's'\cos^2\theta_C - s'd'\sin^2\theta_C && (19)\\ &= 0 && (20) \end{aligned}$$

where we reach the last line by using the hypothesis that in the new basis, there are no interactions between quark families, i.e. $d'd' = s's' = 1$ and $d's' = s'd' = 0$. So flavour changing neutral currents are not possible under the GIM mechanism. This then agrees with experiment, since no flavour changing neutral currents are observed.

CKM matrix

As hinted in the diagram above, the scheme can be extended to three quark generations with a 3×3 matrix, which was proposed by Kobayashi and Maskawa (the resulting mixing matrix is known as the CKM matrix where C=Cabibbo).

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{21}$$

The matrix has values which come from experiment. They are given in the PDG (2010) as follows:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00389 \pm 0.00044 \\ 0.230 \pm 0.011 & 1.023 \pm 0.036 & 0.0406 \pm 0.0013 \\ \text{Fit} = 0.0086 & \text{Fit} = 0.040 & \text{Fit} = 0.99915 \end{pmatrix} \tag{22}$$

Note that the bottom row and right column are rather close to being all zero or one, i.e. the third generation does not mix much with the other two. This is the reason why the decay of particles involving b-quarks are very slow. The elements involving the top quark have not been well measured yet; the values given in the table above are obtained by assuming the matrix is unitary.

Kobayashi and Maskawa were originally searching for an excuse to allow a nontrivial complex number into the 2×2 mixing matrix - they realized[7] it is a way of inserting the (at that time recently discovered) CP violation into the theory. Because it is possible to add a phase to each quark without altering the theory (measurable quantities are proportional to $|M|^2$), this could not be done with a 2×2 matrix. If the number of quark generations was extended to 3, the matrix is allowed to possess one non-trivial CP-violation generating phase. [If you want a more precise description of this, see HM p285]. There is currently an extensive programme of research, including groups in this laboratory exploring the consistency of the experimental situation to this picture of the CKM matrix.

It is conventional to treat the charge $-\frac{1}{3}$ quarks (d, s, b) as the ones which mix to (d', s', b'), while the others (u, c, t) do not change. This could have been done the other way round (or even considering a combination of rotations of both the charge $-\frac{1}{3}$ and charge $+\frac{2}{3}$ quarks), but it can be shown to simplify to the same situation as used here.

The measurement of the elements of the CKM elements proceeds by measuring different processes, for example, V_{ud} is measured by comparing beta decays to the decay of the muon, V_{us} is measured by comparing the decay $K \to \pi e \nu$ with a non-strange decay. V_{ub} comes from the study of b-meson decays where there is no K-meson in the final state. V_{cd} can be measured by observing the rate of charm production in charged current neutrino interactions. It is worth a visit to the particle data group website (www.pdg.org) to see the current state of measurements of the CKM matrix and the various ways in which each element has been measured.

Assuming that the CKM matrix is unitary, it can be parametersised in terms of three independent mixing angles θ_1, θ_2, θ_3 and the one complex phase δ as discussed above. One popular way to parameterize the matrix (from the PDG) is as follows:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{23}$$

Here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ where i and j are generation labels from 1 to 3. In the limit $\theta_{23} = \theta_{13} = 0$ the third generation decouples from the first two and $\theta_{12} = \theta_C$ (the Cabibbo angle). The phase δ can be used to include CP-violation in the standard model. CP violation will be covered further in the lectures by Dr. Cobb.

MNS matrix

More recently, it has been discovered that neutrinos oscillate. This will be dealt with in detail in Dr. Cobb's lectures. A matrix analogous to the CKM matrix called the MNS mixing matrix has been proposed for the lepton sector. In the case of the quarks, the

[7] which got them the 2008 Nobel prize

matrix translates between the flavour eigenstates (distinguished by being conserved in both strong and electromagnetic interactions) and the weak eigenstates with the primes which participate in the weak interaction. For leptons, there is no matrix associated with the interactions of particles because the neutrinos can only interact by the weak interaction — there is no other basis with which to refer them to.

With the discovery of neutrino oscillations, there *is* now a second way of distinguishing neutrinos — by their mass. The PMNS matrix translates between the mass eigenstates of the neutrinos and the eigenstates with which they interact weakly. Somewhat surprisingly, the two mixing matrices are entirely different from each other. The mixing angles in the PMNS matrix are large.

Decays of hadrons containing heavy quarks

We are now in a position to try to make predictions of how weak decays of mesons and baryons containing heavy quarks might proceed. A useful concept is the 'spectator quark' concept. Normally during the existence of a meson, gluon exchange occurs continuously and the strong force is important in understanding what is happening. The idea is that when the hadron decays, one of it's quarks changes flavour, emitting a virtual W particle — while this happens the other quark(s) are 'spectators' i.e. they play no role in the weak interaction itself.

By inspecting the CKM matrix above, we can see that since V_{cs} is large, a c-quark within a meson is going to decay preferentially to an s-quark, i.e. that decays such as $D^+ \rightarrow K +$ (anything) will have a high branching ratio (in fact K^- + (anything) 28%, $K^0/\overline{K}^0$ + (anything) 61% and K^- + (anything) 5.5%). An indication that the spectator model is correct would be if the lifetime of a charged D meson is the same as the lifetime of a neutral D. This is not quite the case; $\tau(D^+) = 1.040$ ps and $\tau(D^0) = 0.410$ ps; there are effects such as that the D^0 has more annihilation diagrams than the D^+ and that non-perturbative effects from the strong interaction are still significant.

The B mesons are expected to decay mostly to particles involving charm (which will subsequently decay to things with an s-quark) because V_{cb} is about a factor 10 bigger than V_{ub}, and this is indeed the case. Since neither V_{cb} or V_{ub} is very big, the b-quark will decay relatively slowly. Comparing the lifetimes, $\tau(B^+) = 1.67$ ps and $\tau(B^0) = 1.54$ ps, so the spectator quark test works better here - we are at higher energies, $\alpha_s(m_b)$ is smaller and perturbative effects are less important.

Looking again at the CKM matrix, we see that V_{tb} is nearly one. This means that the top quark will decay nearly always to the b-quark and not directly to s- or d-quarks. Since it is very heavy, the phase space will be large and it will decay rapidly — indeed too rapidly for any meson to be formed.

A quick note about naming of heavy mesons. The letter B or D with no subscript means the meson contains a heavy quark and either a u or a d in order to make up the charge of the meson. More exotic mesons are denoted with a subscript which indicates the less massive quark, e.g. B_s, D_s or even B_c.

Notes 3: Introduction to Electroweak Unification

In the previous two sections we have characterized the charged current weak interaction on both quarks and leptons.

- The interaction proceeds with a coupling constant g for all weak charged current processes. $G_F/\sqrt{2} = g^2/8M_W^2$.
- The quark flavour eigenstates are rotated through the Cabibbo angle $\theta_C \sim 13.1°$ (or for six quarks by the 3×3 CKM matrix).
- The 'vertex factor' is found to be (for example for $u \to d$) $gV_{ud}J_\mu^-$ where the weak charged current $J_\mu^- = \overline{u}(u)\gamma_\mu \frac{1}{2}(1-\gamma_5)u(d)$ and V_{ud} is the appropriate element of the CKM matrix.

By comparing the properties of the EM and weak forces, it becomes apparent that these have some very similar properties. Perhaps it might be possible to unify them, i.e. to provide a theory which covers both forces as two aspects of a more complete theory. This was indeed what was done by Glashow, Salam and Weinberg and we will go through the arguments here. This theory was constructed before neutral currents had been discovered, and indeed, the theory predicts the properties of neutral currents which have since been experimentally verified. Therefore, in our discussion, we will start with just the EM and weak charged current interactions and see how the neutral current emerges from the theory.

First, let's list the properties of the forces more carefully.

EM interaction	**Weak CC interaction**
Maxwell's equations	e.g. $\nu_e n \to p e^-$, $\mu^- \to e^- \overline{\nu}_e \nu_\mu$ $K^0 \to e^+ \pi^0 \nu_e$
Long range	Short range
γ = intermediate spin 1 boson	Massive $W^\pm$ spin 1 bosons
$F = (1/4\pi\epsilon_0)Q_1Q_2/r^2$	Propagator $1/(M_W^2 - q^2)$
Acts on particles depending on their charge. Therefore neutral particles not affected by EM	Acts on left handed particles and right handed antiparticles only
Conserves parity	Not parity conserving
Comes from Dirac equation when insisting on local gauge invariance	
$J_\mu^{EM} = \overline{\psi}\gamma_\mu Q\psi$ where Q is the electric charge in units where the electron has charge -1.	$J_\mu = \overline{\psi}\gamma_\mu \frac{1}{2}(1-\gamma_5)\psi$
Coupling constant $e = \sqrt{4\pi\alpha}$	Coupling constant g for all weak interactions (universality); use CKM matrix elements with quarks.

Well, they don't look too similar to start with. However here is what Glashow did to unify the forces. I have divided the process up into 4 steps as it is complicated. We will use the $\tau^- \to \nu_\tau e^- \overline{\nu}$ decay from notes 1 as an example.

Prelude: Starting point

As we just said, we start from: the known weak interactions c.1964 ($d \to u e^- \overline{\nu}_e$ (+conj.), $s \to u e^- \overline{\nu}_e$, no neutral currents) and the success of the Feynman diagram view of perturbation theory. In exactly the same way that local gauge invariance in the Dirac equation

leads to an additional spin 1 field (=photon) and the correct photon-electron interaction, we use the same mechanism to insert the weak charged current interaction on the fermions by a spin 1 boson (the $W^{\pm}$)[8]. We assume the V−A structure and spin =1 as guided by experiment, so

$$J^{+}_{\mu} = \overline{u}(\nu)\gamma_{\mu}\tfrac{1}{2}(1-\gamma_5)u(\tau) \tag{24}$$

Step 1: Left handed particles

See BJ p494. Since $P_L = \frac{1}{2}(1-\gamma_5)$ is a projection operator, we can operate on a spinor with it twice in succession without changing the effect, so $\frac{1}{2}(1-\gamma_5)u(\tau) = \frac{1}{2}(1-\gamma_5)\frac{1}{2}(1-\gamma_5)u(\tau)$, and

$$\begin{aligned}
J^{+}_{\mu} &= \overline{u}(\nu)\gamma_{\mu}\frac{1}{2}(1-\gamma_5)\frac{1}{2}(1-\gamma_5)u(\tau) && (25)\\
&= \overline{u}(\nu)\frac{1}{2}(1+\gamma_5)\gamma_{\mu}\frac{1}{2}(1-\gamma_5)u(\tau) && (26)\\
&= \left[\overline{u}(\nu)\frac{1}{2}(1+\gamma_5)\right]\gamma_{\mu}\left[\frac{1}{2}(1-\gamma_5)u(\tau)\right] && (27)\\
&= \overline{u}(\nu_L)\gamma_{\mu}u(\tau_L) && (28)\\
&= \overline{u}(\nu_L)\gamma_{\mu}(2I_3)u(\tau_L) && (29)
\end{aligned}$$

where in line (26), we have used the identity $\gamma_{\mu}\gamma_5 + \gamma_5\gamma_{\mu} = 0$.

Line (27) is the main part of step 1 towards unification — we associate the bits with the $(1 \pm \gamma_5)$ with the spinors rather than the operator. By doing so, we are left with an operator γ_{μ} which looks like the EM operator. This looks more obvious in line (28) when we write the left handed parts of the spinors directly. The table below (from BJ p494 or G p332) shows the correspondence between left and right handed spinors and adjoint spinors for both particles and antiparticles.

Particles	**Antiparticles**
$u_L = \frac{1}{2}(1-\gamma_5)u$	$v_L = \frac{1}{2}(1+\gamma_5)v$
$u_R = \frac{1}{2}(1+\gamma_5)u$	$v_R = \frac{1}{2}(1-\gamma_5)v$
$\overline{u}_L = \overline{u}\frac{1}{2}(1+\gamma_5)$	$\overline{v}_L = \overline{v}\frac{1}{2}(1-\gamma_5)$
$\overline{u}_R = \overline{u}\frac{1}{2}(1-\gamma_5)$	$\overline{v}_R = \overline{v}\frac{1}{2}(1+\gamma_5)$

R,L correspond to helicity +1, −1 if m=0 and approximately if $m \sim 0$.

$W^{\pm}$ vertex factors are the same as EM if we act *only* on the LH part of the spinor.

Interlude: Weak Isospin

We now add some numerology to formalize step 1. We categorize the particles — indeed the separate left and right handed bits[9] — according to how they interact weakly. This is perhaps a bit of an overkill, but is how everyone (including the textbooks) do it. We define a quantity called *weak isospin* in an analogous way to isospin (and ordinary spin), and arrange the left hand parts of the particles in $I = \frac{1}{2}$ doublets e.g. $(\nu_{\mu L}, \mu_L)$ or (u_L, d'_L) containing the particles which can change into each other at a weak CC vertex. The member of each multiplet with the more positive charge is assigned $I_3 = +\frac{1}{2}$ and the more negative charge with $I_3 = -\frac{1}{2}$.

[8]We sweep under the carpet for now the fact that the local gauge invariance mechanism only works with massless particles like photons, we will return to this later

[9]Note, these left and right handed parts are not paricles in their own right. An electron spinor $u(e)$ describes the real electron, we split it up $u(e) = u(e_L) + u(e_R)$ simply to make the similarity between EM and weak apparent in the maths

I	I_3						
$\frac{1}{2}$	$+\frac{1}{2}$	ν_{eL}	$\nu_{\mu L}$	$\nu_{\tau L}$	u_L	c_L	t_L
$\frac{1}{2}$	$-\frac{1}{2}$	e_L	μ_L	τ_L	d'_L	s'_L	b'_L

The right hand parts of all the particles are assigned to weak isospin singlets with $I = 0$ and $I_3 = 0$.

I	I_3									
0	0	e_R	μ_R	τ_R	u_R	c_R	t_R	d'_R	s'_R	b'_R

I have left the right handed neutrinos out of the table — their nature is being studied. They also have $I_3 = 0$. It is possible that neutrinos are 'Majorana' particles, i.e. their own antiparticles (in which case, flipping the helicity of a neutrino ν_L produces the antineutrino ν_R; neutrinoless double β-decay becomes possible), or are Dirac particles in which neutrino and antineutrino are distinct, and therefore the ν_R and $\overline{\nu}_L$ don't interact with any known force. If neutrinos had been completely massless, these two situations would be experimentally indistinguishable.

The I_3 label which is either $+\frac{1}{2}$, $-\frac{1}{2}$ or 0 is used for the weak charged current interaction in a similar way to the charge in an electromagnetic interaction, i.e. when it is 0, there is no interaction.

Step 2: W^0

We next appeal to symmetry; postulate a W^0 neutral partner to the $W^{\pm}$. This interacts producing no change in charge within each doublet.

- Same coupling constant g.
- Interacts only with the left handed states.

This is NOT the Z^0 which also interacts with right handed states of quarks and leptons.

Step 3: B^0

Instead of the electromagnetic interaction, we introduce another field, the B^0. The particular things we do in this step are contrived to make the following step work (i.e. produce the correct electromagnetic interaction). The B^0 interacts with a strength proportional to a quantity called weak hypercharge Y where Y is defined by

$$Q = \quad I_3 + \tfrac{1}{2}Y \tag{30}$$

The B^0 interacts with a current J^Y_μ;

$$J^Y_\mu = 2J^{EM}_\mu - 2J^0_\mu \tag{31}$$

We give the B^0 interaction it's own coupling constant $g'/2$ (the factor of 2 is just convention). Examples of B^0 currents involving left handed up-quarks and right handed electrons are

$$J^Y_\mu = \overline{u}(u_L)\gamma_\mu Y(u_L)u(u_L); \qquad \overline{u}(e_R)\gamma_\mu Y(e_R)u(e_R) \tag{32}$$

where $Y(u_L) = +\frac{1}{3}$ and $Y(e_R) = -2$ using equation (30). The table at the end of these notes gives Y for all the particles.

Step 4: The Weinberg angle

We are going to use the notation introduced in the first lecture with currents and fields. The table below gives a summary of the symbols used (the same as the ones above with the addition of μ indices on the currents and fields and dropping the ^{0} where we can for brevity). Since all the fields are vector or axial vector fields (see lecture 1), they include an index $\mu = 0, 1, 2, 3$. When a current and a field are combined to form a matrix element, there is an implied summation over μ.

	Current	Field	Coupling const.	'Charge'
Charge raising weak	J^+_μ	$W^{\mu,+}$	g	I_3
Charge lowering	J^-_μ	$W^{\mu,-}$	g	I_3
Symmetric W^0 field (step 2)	J^0_μ	$W^{\mu,0}$	g	I_3
B^0 field (step 3)	J^Y_μ	B^μ	$g'/2$	Y
Electromagnetic	J^{EM}_μ	A^μ	e	Q
Neutral current	J^{NC}_μ	Z^μ		

BJ p498, The electromagnetic interaction A^μ is a linear combination of $W^{\mu,0}$ and B^μ and the orthogonal combination produces a [new] interaction which is the weak neutral current Z^0. A convenient way to form the linear combination is with a rotation angle. We introduce a new rotation angle θ_W, the 'weak mixing angle' or 'Weinberg angle'.

$$A^\mu = +B^\mu \cos\theta_W + W^{\mu,0} \sin\theta_W \quad (33)$$
$$Z^\mu = -B^\mu \sin\theta_W + W^{\mu,0} \cos\theta_W \quad (34)$$

We now write down the total Glashow-Salam-Weinberg electroweak interaction as a $(\text{current})_\mu(\text{field})^\mu$.

$$g\left[J^+_\mu \frac{W^{\mu,+}}{\sqrt{2}} + J^-_\mu \frac{W^{\mu,-}}{\sqrt{2}} + J^0_\mu W^{\mu,0}\right] + \frac{g'}{2} J^Y_\mu B^\mu \quad (35)$$

The factors of $\sqrt{2}$ come from group theory (a more complicated treatment which you will see in the textbooks constructs the W^+ and W^- in a way which makes the symmetry with W^0 more apparent).

Now, invert equation (34)

$$B^\mu = +A^\mu \cos\theta_W - Z^{\mu,0} \sin\theta_W \quad (36)$$
$$W^\mu = +A^\mu \sin\theta_W + Z^{\mu,0} \cos\theta_W \quad (37)$$

and write the neutral part of equation (35) in terms of A^μ and Z^μ by substituting in from equation (37).

$$gJ^0_\mu W^{\mu,0} + \frac{g'}{2} J^Y_\mu B^\mu = \left(g \sin\theta_W J^0_\mu + g' \cos\theta_W \frac{J^Y_\mu}{2}\right) A^\mu + \left(g \cos\theta_W J^0_\mu - g' \sin\theta_W \frac{J^Y_\mu}{2}\right) Z^\mu \quad (38)$$

The EM interaction

Let's look at the two parts of the right hand side of equation (38) separately. The first part must be set equal to $eJ^{EM}_\mu A^\mu$, otherwise the GSW theory will not reproduce the

electromagnetic physics which is described by QED. See BJ p 499. From equation (31), $J_\mu^{EM} = J_\mu^0 + \frac{1}{2}J_\mu^Y$, and so

$$\left(g \sin\theta_W J_\mu^0 + g' \cos\theta_W \frac{J_\mu^Y}{2}\right) A^\mu = e\left(J_\mu^0 + \frac{1}{2}J_\mu^Y\right) A^\mu \tag{39}$$

For this to be satisfied, we must set

$$\boxed{e = g \sin\theta_W = g' \cos\theta_W} \tag{40}$$

These two equalities are called the 'Unification condition'.

The neutral current interaction

The second part of equation (38) is the weak neutral current $g_z J_\mu^Z$. This is completely specified (i.e. no free parameters) by the charged weak current and EM.

$$\begin{aligned} g_z J_\mu^Z &= g \cos\theta_W J_\mu^0 - g' \sin\theta_W \frac{J_\mu^Y}{2} && (41)\\ &= \frac{g}{\cos\theta_W}\left[\cos^2\theta_W J_\mu^0 - \sin^2\theta_W (J_\mu^{EM} - J_\mu^0)\right] && (42)\\ &= \frac{g}{\cos\theta_W}\left[J_\mu^0 - \sin^2\theta_W J_\mu^{EM}\right] && (43) \end{aligned}$$

We can now put explicit forms of $J_\mu^0 = \overline{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)u$ and $J_\mu^{EM} = \overline{u}\gamma_\mu u$ into this expression

$$\begin{aligned} g_z J_\mu^Z &= \frac{g}{\cos\theta_W}\overline{u}\gamma_\mu\left[\frac{1}{2}(1-\gamma_5)I_3 - \sin^2\theta_W Q\right] u && (44)\\ &= \frac{g}{\cos\theta_W}\overline{u}\gamma_\mu\left[\frac{1}{2}(1-\gamma_5)I_3 - \sin^2\theta_W Q\left(\frac{1}{2}(1-\gamma_5) + \frac{1}{2}(1+\gamma_5)\right)\right] u && (45)\\ &= \frac{g}{\cos\theta_W}\overline{u}\gamma_\mu\left[\frac{1}{2}g_L(1-\gamma_5) + g_R\frac{1}{2}(1+\gamma_5)\right] u && (46) \end{aligned}$$

where $g_L = I_3 - Q\sin^2\theta_W$ and $g_R = -Q\sin^2\theta_W$ are the couplings of the left and right handed particles respectively. Note that apart from neutrinos for which $Q = 0$ and so $g_R = 0$, the neutral current interacts with both the left and right handed parts of the particle, with different amounts. This is different from the charged current interaction which only interacts with the left part of all particles.

It is also possible to rearrange equation (46) to define coupling constants c_V and c_A in terms of vector and axial couplings.

$$\begin{aligned} g_z J_\mu^Z &= \frac{g}{\cos\theta_W}\overline{u}\gamma_\mu\left[\frac{1}{2}(g_L + g_R) - \frac{1}{2}(g_L - g_R)\gamma_5\right] u && (47)\\ &= \frac{g}{\cos\theta_W}\overline{u}\gamma_\mu\frac{1}{2}\left[c_V - c_A\gamma_5\right] u && (48) \end{aligned}$$

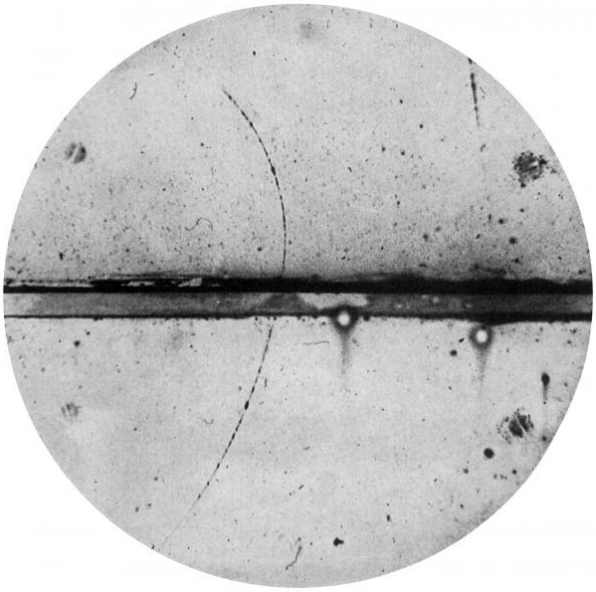
where $c_V = I_3 - 2Q\sin^2\theta_W$ and $c_A = I_3$. The values of c_V^f and c_A^f for each fermion f along with other properties of particles discussed in this lecture are shown in the table below.

Fermion (f)	I	I_3	Q	Y	c_V^f	c_A^f
$\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$	$\frac{1}{2}$	$+\frac{1}{2}$	0	-1	$+\frac{1}{2}$	$+\frac{1}{2}$
e_L, μ_L, τ_L	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$
e_R, μ_R, τ_R	0	0	-1	-2	$2\sin^2\theta_W$	0
u_L, c_L, t_L	$\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{2}{3}$	$+\frac{1}{3}$	$+\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$+\frac{1}{2}$
u_R, c_R, t_R	0	0	$+\frac{2}{3}$	$+\frac{4}{3}$	$-\frac{4}{3}\sin^2\theta_W$	0
d_L, s_L, b_L	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$
d_R, s_R, b_R	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{2}{3}\sin^2\theta_W$	0

Experiments

Again, the following experiments are an important part of the course. Most will be covered in my lectures in Trinity term, some have been covered by other lecturers or in the class questions.

Appearance of neutrino interactions	Problem set on detectors: Super Kamiokande; Iron/scintillator detectors; bubble chambers
Discovery of neutral currents	Gargamelle at CERN — problem set question
Discovery of W^0 and Z^0	At UA1/UA2
Measurement of W and Z mass at $\overline{p}p$ colliders	Missing momentum; transverse mass
Accurate m_Z measurement	LEP; My lecture next term
Accurate Z couplings	Forward-backward asymetry at LEP
Number of neutrinos	LEP
Z-width Γ_Z	
Triple gauge-boson coupling	
m_W and Γ_W at LEP	



The Oxford MPhys Course

C4: Particle Physics

Problem Set Three

1. The spin wave function for a left-handed electron, with momentum p and energy E, can be written in terms of helicity two-component spinor eigenstates as

$$\Omega = a_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{where} \quad a_\pm = \left[\frac{E \mp p}{m}\right]^{\frac{1}{2}}.$$

(a) Show that $\Omega^\dagger\Omega$ and $-\Omega^\dagger\sigma_z\Omega$ form two components of a four-vector (σ_z is the usual Pauli matrix). Show that the expectation value of the helicity of the electron is $-p/E$.

(b) Explain how neutrino scattering off quarks via W boson exchange is related to the decays $\pi^- \to \mu^- + \bar{\nu}_\mu$ and $\pi^- \to e^- + \bar{\nu}_e$. Show that the phase space factor $\rho \propto p^2 \mathrm{d}p/\mathrm{d}E$ for the decay $\pi^- \to \mu^- + \bar{\nu}_\mu$ is

$$\rho \propto \frac{(m_\pi^2 - m_\mu^2)^2}{m_\pi^3} E_\mu,$$

where E is the total energy.

(c) Explain why the rate for the decay $\pi^- \to \mu^- + \bar{\nu}_\mu$ depends on the probability that the muon has helicity $+1$. Show that the ratio R of the rate for the decay

$\pi^- \to \mu^- + \bar{\nu}_\mu$ to the rate for the decay $\pi^- \to e^- + \bar{\nu}_e$ is given by

$$R = \frac{m_\mu^2(m_\pi^2 - m_\mu^2)^2}{m_e^2(m_\pi^2 - m_e^2)^2}.$$

Evaluate the numerical value of this ratio and check from the PDG tables how well this agrees with the measured value.

2. Describe briefly (but stress the essential physical arguments) the classic experiments that give information about...

(a) ... the helicity of the neutrino. [Phys. Rev. **109**, 1015 (1958)]

(b) ... the parity of the π^-. [Phys. Rev. **81**, 565 (1951)]

(c) What can be deduced about the spin of the π^0 from the existence of the decay $\pi^0 \to \gamma\gamma$? If the π^0 has $J^P = 0^-$, how is the decay $\pi^0 \to \gamma\gamma$ compatible with parity conservation? The branching ratio for the decay $\pi^0 \to 3\gamma$ is $< 3.1 \times 10^{-8}$ (at 90% confidence level). Why is it so small?

3. The τ lepton is the heaviest charged lepton in the sequence: electron, muon, tau. Apart from mass, the three leptons have close similarities.

(a) What are the possible decay modes of the τ, and why?

(b) Given that the lifetime of the muon is 2×10^{-6} seconds, estimate the expected lifetime. How might the lifetime of the τ be measured?

(c) Neglecting density of states factors, what is the expected ratio of branching ratios for

$$\frac{\tau^+ \to K^+\overline{\nu}_\tau}{\tau^+ \to \pi^+\overline{\nu}_\tau}?$$

(d) Starting with an intense 800 GeV proton beam, how could a high-energy neutrino beam, enriched in ν_τ, be produced? Explain the principles of a suitable detector for identifying the interactions of ν_τ and distinguishing it from other neutrino flavours. [Details – DONUT experiment Phys. Rev. **D78**, 052002 (2008).]

4. In the Cabibbo model, the quarks couple to the weak currents through doublets

$$\begin{pmatrix} u \\ d_{\rm c} \end{pmatrix} = \begin{pmatrix} u \\ d\cos\theta_{\rm c} + s\sin\theta_{\rm c} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ s_{\rm c} \end{pmatrix} = \begin{pmatrix} c \\ s\cos\theta_{\rm c} - d\sin\theta_{\rm c} \end{pmatrix}$$

where $\theta_{\rm c} = 0{\cdot}23\,\text{rad}$.

(a) Draw diagrams for the following decays and discuss the dependence of the decay rates on $\theta_{\rm c}$:

(i) $\mu^+ \to {\rm e}^+ + \bar{\nu}_\mu + \nu_{\rm e}$

(ii) $\rm K^+ \to \mu^+ + \nu_\mu$

(iii) $\pi^+ \to \mu^+ + \nu_\mu$

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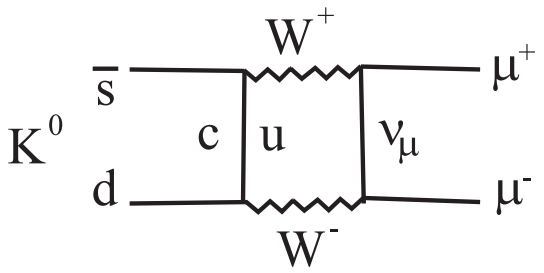
(iv) $\mathrm{D}^+ \to \mathrm{K}^- + \pi^+ + \pi^+$

(v) $\mathrm{D}^+ \to \mathrm{K}^+ + \pi^+ + \pi^-$

(b) Explain how this model accounts for the absence of flavour-changing weak neutral currents.

(c) The Λ has a mean lifetime of $2{\cdot}6 \times 10^{-10}$s and decays into $\mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}$ with a branching fraction of $8{\cdot}3 \times 10^{-4}$. The Λ_c^+ (udc) has a mean lifetime of $2{\cdot}1 \times 10^{-13}$s. Estimate the branching fraction of the Λ_c^+ into $\Lambda + \mathrm{e}^+ + \nu_\mathrm{e}$, comment on how your result compares with the measured value.
[$m(\Lambda_\mathrm{c}^+) = 2{\cdot}285\,\mathrm{GeV}/c^2$, $BR(\Lambda_c^+ \to \Lambda e^+ \nu_e) = (2{\cdot}1 \pm 0{\cdot}6)\%$]

(d) By considering second-order box diagrams, show qualitatively how the introduction of the charm quark and of weak transitions $c \to s_\mathrm{c}$ as well as $u \to d_\mathrm{c}$ can account for the highly suppressed decay $\mathrm{K}^0 \to \mu^+ + \mu^-$.



5. Universality describes the fact that the strength of the coupling of the $W \to l\nu_l$ is the same for the three generations $l = e,\ \mu$ and τ. This independence has been tested precisely by comparing the rates of $\tau^-(\mu^-) \to e^- \bar{\nu}_e \nu_{\tau(\mu)}$ and $\tau^- \to e^-(\mu^-) \bar{\nu}_{e(\mu)} \nu_\tau$. (See Perkins 4th edition p. 195 *et seq.*). In addition, the strength of the couplings of the $Z^0 \to l^+ l^-$ and $Z^0 \to \nu_l \bar{\nu}_l$ are independent of the generations.

(a) Give a definition of the Cabibbo–Kobayashi–Maskawa (CKM) matrix and explain how this is used to generalize the universality of these interactions from the lepton sector to the quark sector.

(b) Explain how each of the following measurements could be used to determine a single element of the CKM matrix or a combination of elements; in each case also explain why knowledge of the strong interaction is important for interpreting the results. First, the ratio of μ pair production to single-μ production in ν_μ interactions on nuclei.

(c) Then the decay rate of $D_s^+ \to \tau^+ \nu_\tau$?

(d) Then the decay rate of $\bar{B}^0 \to D^{*+} \mu^- \nu_\mu$.

(e) And last, the region of the μ momentum spectrum near the kinematic endpoint from $\bar{B} \to X \mu \nu_\mu$.

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6. The decay rate for a particle of mass M is given by $d\Gamma = \dfrac{(2\pi)^4}{2M}\overline{|\mathcal{M}|^2}\,\mathrm{dLips}$, where $\overline{|\mathcal{M}|^2}$ is the spin summed and averaged matrix element squared.

(a) Using the result of Problem Set One for the integrated two-body phase space $R_2(E) = \dfrac{\pi}{(2\pi)^6}\dfrac{p}{E}$, where p is the centre-of-mass momentum, show that the total rate for the two-body decay $M \to m_1 + m_2$ (symbols label both particles and masses) is given by $\Gamma = \dfrac{p}{8\pi M^2}\overline{|\mathcal{M}|^2}$. Why may the integrated phase-space term be used in this case?

(b) Draw Feynman diagrams for the decays $\mathrm{K}^+ \to \pi^+\pi^0$ (BR 20.7%) and $\mathrm{K}^+ \to \mu^+\nu_\mu$ (BR 63.5%). What CKM matrix elements should be associated with the W exchange in each case? Could the difference in rates of these two decays be explained by phase space and CKM elements alone?

(c) For the decay $\mathrm{K}^+ \to \mu^+\nu_\mu$ $\overline{|\mathcal{M}|^2}$ is $4G_F^2(V_{ab})^2 f_K^2\, m_\mu^2(m_K^2 - m_\mu^2)$, where V_{ab} is the relevant CKM matrix element and f_K is the K decay constant which takes account of the K wave-function in the matrix element. Show that

$$\Gamma(\mathrm{K}^+ \to \mu^+\nu_\mu) = \frac{G_F^2}{4\pi}(V_{ab})^2 f_K^2\,\frac{m_\mu^2(m_\mathrm{K}^2 - m_\mu^2)^2}{m_\mathrm{K}^3}.$$

(d) A similar equation holds for $\Gamma(\pi^+ \to \mu^+\nu_\mu)$. Decay constants for the light-quark hadrons cannot be calculated accurately at present, but well measured decay rates provide a way of estimating them. Using expressions derived in this question show that $f_K/f_\pi \approx 1{\cdot}3$. What does this result say about the hadronic wave functions for the π^+ and K^+ and an approximate SU(3) flavour symmetry for the u, d, s quarks?

[$\tau_\pi = 2{\cdot}603\times10^{-8}$ s; $\tau_\mathrm{K} = 1{\cdot}238\times10^{-8}$ s. Masses (MeV): $m_\mathrm{K} = 493{\cdot}7$; $m_\mu = 105{\cdot}7$; $m_{\pi^\pm} = 139{\cdot}6$; $m_{\pi^0} = 135{\cdot}0$. The BR($\pi \to \mu\nu$) may be taken to be 100% and CKM matrix element values may be found in the PDG tables or the C4 data sheet.]

7. The Standard Model unifies the Weak Interactions with the Electromagnetic Interactions and predicts the Z^0 boson. The interaction is of the form

$$-i\left(g\mathbf{j}_\mu \cdot \mathbf{W}^\mu + \frac{g'}{2}j_\mu^Y B^\mu\right)$$

where

$$\mathbf{j}_\mu \cdot \mathbf{W}^\mu = \frac{1}{\sqrt{2}}j_\mu^+ W^{\mu+} + \frac{1}{\sqrt{2}}j_\mu^- W^{\mu-} + j_\mu^3 W^{\mu 3}$$

[Note that we have skipped a bit of the maths related to group theory to get to this step. The standard model starts with three fields which are members of an SU(2) group, W^1, W^2 and W^3 acting on three currents j_1, j_2 and j_3 all with the same coupling constant g. Linear combinations of W^1 and W^2, are used to identify combinations of j_1 and j_2 with the charged currents (CC) and this results in the $1/\sqrt{2}$ factors in the above equations. See e.g. p498 of Burcham and Jobes]

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(a) Introducing the weak mixing angle θ_W, use the interactions associated with the W^3 and B to find the neutral current (NC) interactions in terms of the photon (A) and the Z^0 boson. Give the relationship between the charge of an electron e and the couplings g and g'. In addition, show that the current associated with the Z^0 can be written as:

$$j_Z^\mu = -i\frac{g}{\cos\theta_W}\overline{\psi}\gamma^\mu\left[\frac{1}{2}(1-\gamma_5)I_3 - \sin^2\theta_W Q\right]\psi$$

where I_3 is the third component of weak isospin and Q is the charge of the fermion in units of the electron charge.

(b) Show that the fermion couplings to the Z^0 can be written in terms of left- and right-handed couplings (g_L, g_R). Furthermore, show that the couplings can be written in terms of vector and axial couplings (c_V, c_A). Give each coupling in terms of I_3, Q and θ_W.

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