**Waves**
**Problem Set I**

1.1 **Stretched string under gravity.** Consider the transverse oscillations of a string of linear density $\mu$ tensioned with force $T$ under the influence of the gravitational acceleration $g$.

(a) Sketch the forces on an infinitesimal piece of string of length $\mathrm{d}x$ including gravity.

(b) Using the diagram in part (a), show that the equation for the transverse displacements of the spring becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - g.$$

(c) Solve the equation in part (b) for a stationary string $y_s(x)$ ($\partial y_s/\partial t = 0$) held in place at $x = 0$ and $x = L$, that is, $y_s(0) = 0$ and $y_s(L) = 0$. What is the equation for the oscillations $\tilde{y}(x,t)$ around this solution?

1.2 **Longitudinal and transverse oscillations of a stretched string.** (a) Using the equations for the longitudinal oscillations of a solid bar, find the longitudinal deformation $\xi_s(x)$ of a stationary stretched string ($\partial \xi_s/\partial t = 0$) of Young modulus $E$, volumetric density $\rho$, cross section area $A$ and length $L$. Assume that the string is held in place at $x = 0$, that is, $\xi_s(0) = 0$, and that there are no transverse oscillations. Thus, show that under tension $T$, the string has stretched a distance $\xi_s(L) = T/K$, and determine the constant $K$ as a function of quantities that you know.

(b) Consider the longitudinal oscillations $\tilde{\xi}(x,t)$ and the transverse oscillations $y(x,t)$ of the string around the steady state, that is, the point of the string originally at $(x+\xi_s(x), 0)$ moves to $(x+\xi_s(x)+\tilde{\xi}(x,t), y(x,t))$. For $\xi/L \ll 1$ and $y/L \ll 1$, show that an infinitesimal piece of string of length $(1 + \mathrm{d}\xi_s/\mathrm{d}x)\,\mathrm{d}x$ elongates to be of length

$$\mathrm{d}l \simeq \left[1 + \frac{\mathrm{d}\xi_s}{\mathrm{d}x} + \frac{\partial \tilde{\xi}}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x.$$

(c) Using the result in part (b), show that the equation for $\tilde{\xi}(x,t)$ is

$$\frac{\partial^2 \tilde{\xi}}{\partial t^2} = \frac{E}{\rho}\frac{\partial}{\partial x}\left[\frac{\partial \tilde{\xi}}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right].$$

(d) In deriving the equation for the transverse oscillations of the stretched string $y(x,t)$, we neglected the variation of the longitudinal force $F(x,t)$ with $x$, $\partial F/\partial x = 0$. Keeping this derivative of $F$ in the derivation, show that the longitudinal and transversal oscillations of the string are coupled by the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[\left(\frac{T}{\rho A} + \frac{E}{\rho}\left(\frac{\partial \tilde{\xi}}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right)\right)\frac{\partial y}{\partial x}\right].$$

(e) Using parts (c) and (d), argue that the longitudinal and transversal oscillations decouple when $(y/L)^2 \ll \tilde{\xi}/L \ll T/EA$.

1.3 **Initial conditions for the d'Alembert solution.** (a) Sketch as a function of time the displacement at $x = 0$ and at $x = L$, $y(x = 0, t)$ and $y(x = L, t)$, of an infinite stretched string with wave speed $c$ for the initial conditions

$$y(x, t = 0) = \begin{cases} a(1 + x/L) & \text{for } -L \le x < 0, \\ a(1 - x/L) & \text{for } 0 \le x < L, \\ 0 & \text{otherwise,} \end{cases}$$

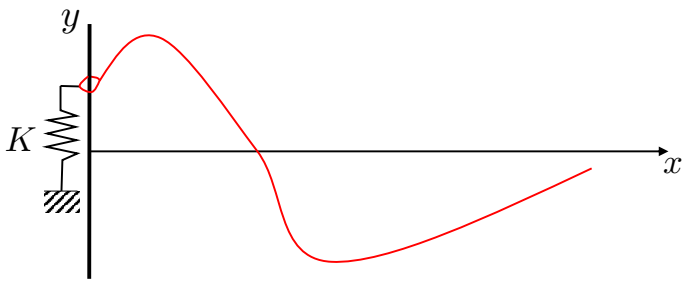
and $\partial y/\partial t(x, t = 0) = 0$.

(b) Sketch $y(x = 0, t)$ and $y(x = L, t)$ for the initial conditions $y(x, t = 0) = 0$ and

$$\frac{\partial y}{\partial t}(x, t = 0) = \begin{cases} V & \text{for } -L \le x \le L, \\ 0 & \text{otherwise.} \end{cases}$$

(c) What happens in the case that the string starts with the displacement of part (a) and the velocity of part (b)? For your sketches of $y(x = 0, t)$ and $y(x = L, t)$, assume $VL/c < a$.

1.4 **General boundary condition for the d'Alembert solution.** Consider a semi-infinite stretched string with wave speed $c$ that extends from $x = 0$ to $+\infty$. At $x = 0$, the string is tied with a massless knot to a frictionless column and a spring of constant $K$, as shown below.




(a) Show that, given the left-traveling wave $g(x + ct)$, the equation for the right-traveling wave $f(x - ct)$ is

$$Tf'(u) - Kf(u) = -Tg'(-u) + Kg(-u),$$

where $f'$ and $g'$ are the derivatives of $f$ and $g$ with respect to their argument.

From here on, consider the initial conditions

$$y(x, t = 0) = \begin{cases} ax/L & \text{for } 0 \le x < L., \\ a & \text{for } x \ge L, \end{cases}$$

and $\partial y/\partial t(x, t = 0) = 0$

(b) Use the initial conditions to find $f(u)$ for positive $u$, and $g(v)$.

(c) Employing the differential equation in part (a), solve for $f(u)$ for negative $u$. Sketch $f(u)$.

(d) What is the solution for $y(x, t)$ for $t \to +\infty$?

1.5 **Two boundary conditions for the d'Alembert solution.** A stretched string with wave speed $c$ is pinned at $x = 0$ and is tied to a frictionless column at $x = L$, that is,

$y(x=0,t)=0$ and $\partial y/\partial x(x=L,t)=0$. The string is plucked at $x=L/4$ at a distance $h$ until it comes to rest. It is then released at $t=0$.

(a) Show that the initial displacement is

$$y(x,t=0)=\begin{cases} 4hx/L & \text{for } 0\le x<L/4,\\ h & \text{for } L/4\le x\le L.\end{cases}$$

(b) Using the initial conditions, determine the right- and left-traveling waves $f(x-ct)$ and $g(x+ct)$ for $0<x-ct<L$ and $0<x+ct<L$.

(c) Show that the boundary condition at $x=0$ imposes $f(u)=-g(-u)$, and that the boundary condition at $x=L$ requires $g(v)=g(L)-f(L)+f(2L-v)$.

(d) With the results in parts (b) and (c), construct $f(x-ct)$ for $x-ct<L$ and $g(x+ct)$ for $x+ct>0$.

(e) Sketch $y(x,t)$ for $t=L/4c, L/c, 5L/4c, 2L/c$. Recall that $y(x,t)$ is only defined for $0<x<L$!

1.6 **Energy equation.** (a) For the longitudinal oscillations of a bar of volumetric density $\rho$, Young modulus $E$ and cross section area $A$, show that the energy equation

$$\frac{\partial}{\partial t}\left[\frac{\rho A}{2}\left(\frac{\partial \xi}{\partial t}\right)^2+\frac{EA}{2}\left(\frac{\partial \xi}{\partial x}\right)^2\right]=\frac{\partial}{\partial x}\left(EA\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t}\right)$$

is satisfied. Explain physically what each of the terms represents.

(b) For the longitudinal oscillations of a gas of pressure $p_0$, density $\rho_0$ and adiabatic constant $\gamma$ in a pipe of cross section $A$, show that the energy equation

$$\frac{\partial}{\partial t}\left[\frac{\rho_0 A}{2}\left(\frac{\partial \xi}{\partial t}\right)^2+\frac{\gamma p_0 A}{2}\left(\frac{\partial \xi}{\partial x}\right)^2\right]=\frac{\partial}{\partial x}\left(\gamma p_0 A\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t}\right)$$

is satisfied. Explain physically what each of the terms represents.

1.7 **Energy conservation.** A stretched string with wave speed $c$ has an initial displacement and an initial velocity such that the right- and the left-traveling waves are

$$f(u)=\begin{cases} A\sin(ku) & \text{for } -2\pi/k\le u<-\pi/k,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$g(v)=\begin{cases} A\sin(kv) & \text{for } \pi/k\le v<2\pi/k,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Sketch $y(x,t=0)$ and $y(x,t=3\pi/2kc)$.

(b) Calculate the energy at $t=0$ and at $t=3\pi/2kc$.

Comment on the results.

**Waves**
**Problem Set II**

2.1 **General boundary condition for sinusoidal waves.** A semi-infinite string of linear density $\mu$ extends from $x = 0$ to $+\infty$ and is under tension $T$. At $x = 0$ the string is tied to a rod that lies perpendicular to the string. The friction force between the string and the rod is $\gamma v_y$, where $\gamma$ is a constant and $v_y$ is the velocity of the knot in the direction perpendicular to the string. Determine the relation between the reflected wave and an incident sinusoidal wave of amplitude $A$. Calculate the average power lost due to friction and discuss the limits $T \ll \gamma c$ and $T \gg \gamma c$, where $c = \sqrt{T/\mu}$.

2.2 **More general boundary conditions for sinusoidal waves.** A bar of uniform cross section $A$, density $\rho$ and Young's modulus $E$ transmits longitudinal elastic waves. Waves of frequency $\omega$ traveling in the bar are reflected at an end which has a mass $M$ rigidly attached to it. Find the phase change on reflection and discuss the cases $M = 0$ and $M \to \infty$.

2.3 **Transmission lines.** A semi-infinite transmission line, of capacitance $C'$ and inductance $L'$ per unit length, extends from $-\infty$ to $x = 0$ and is terminated by an impedance $Z_T$ at $x = 0$. Find the ratio of the amplitude and the phase difference for the reflected and incident waves if

(a) $Z_T = \sqrt{L'/C'}$,

(b) $Z_T = 2\sqrt{L'/C'}$,

(c) $Z_T$ is a capacitor of capacitance $C_0$.

In (a) and (b), what type of impedance is required?

2.4 **Transmission problem in a stretched string.** Two long strings lie along the $x$-axis under tension $T$. They are joined at $x = 0$ so that for $x < 0$ the line density $\mu = \mu_a$, and for $x > 0$, $\mu = \mu_b$. A mass $M$ is attached to the join and it is connected to a fixed support by a light spring of stiffness $K$. This spring exerts a transverse force on the mass when the latter is displaced from $y = 0$. Show that at the join

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x=0^+} - \left.\frac{\partial y}{\partial x}\right|_{x=0^-}\right) = M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0} + Ky(x=0).$$

Small transverse sinusoidal oscillations propagate along these strings from $x = -\infty$. Show that the phase of the transmitted wave lags behind that of the incident wave by an angle

$$\arctan\left(\frac{c_a c_b(M\omega^2 - K)}{\omega T(c_a + c_b)}\right),$$

where $c_a$ and $c_b$ are the speeds of the waves for $x < 0$ and $x > 0$, respectively.

Check that the reflected and transmitted waves satisfy energy conservation.

2.5 **Transmission problem in pipe full of gas.** A pipe of constant cross section $A$ contains two gases with densities $\rho_a$ (at $x < 0$) and $\rho_b$ (at $x > 0$) separated by an elastic membrane at $x = 0$. The elastic membrane moves to the left or the right following the law

$$A[p(x = 0^-) - p(x = 0^+)] = K\xi(x = 0),$$

where $K$ is a constant. We consider perturbations to a background pressure $p_0$ that is the same for both gases.

(a) Find the relationships between $\xi(x = 0^-)$, $\xi(x = 0^+)$, $\partial\xi/\partial x|_{x=0^-}$ and $\partial\xi/\partial x|_{x=0^-}$.

(b) Use these relationships to determine the reflected and transmitted waves that result from an incident harmonic wave with frequency $\omega$ traveling from $-\infty$ to $x = 0$.

(c) Discuss the limits of small and large $\omega$.

2.6 **Separation of variables.** (a) For a stretched string with wave velocity $c$ that satisfies the boundary conditions $\partial y/\partial x(x = 0) = 0 = \partial y/\partial x(x = L)$, show that the general solution is of the form

$$y(x,t) = A_0 + \widetilde{A}_0 t + \sum_{k=1}^{\infty} \left[ A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right) \right] \cos\left(\frac{k\pi x}{L}\right).$$

(b) For a stretched string with wave velocity $c$ that satisfies the boundary conditions $y(x = 0) = 0 = \partial y/\partial x(x = L)$, show that the general solution is of the form

$$y(x,t) = \sum_{k=1}^{\infty} \left[ A_k \cos\left(\frac{(k-1/2)\pi ct}{L}\right) + B_k \sin\left(\frac{(k-1/2)\pi ct}{L}\right) \right] \sin\left(\frac{(k-1/2)\pi x}{L}\right).$$

2.7 **Initial conditions for separation of variables.** (a) Determine the motion of a stretched string with wave velocity $c$ held in place at $x = 0$ and $x = L$ with the initial conditions $y(x, t = 0) = h\sin(\pi x/L) + 2h\sin(2\pi x/L)$ and $\partial y/\partial t(x, t = 0) = 0$.

(b) Determine the motion of a stretched string with wave velocity $c$ that can slide along frictionless columns at $x = 0$ and $x = L$ with the initial conditions $y(x, t = 0) = 0$ and $\partial y/\partial t(x, t = 0) = V\sin^2(\pi x/L)$.

(c) Determine the motion of a stretched string with wave velocity $c$ that is held in place at $x = 0$ and can slide along a frictionless column at $x = L$ with the initial conditions $y(x, t = 0) = h\sin(\pi x/2L)$ and $\partial y/\partial t(x, t = 0) = V\sin(3\pi x/2L)\cos(\pi x/L)$.

2.8 **Dispersion.** (a) Show that an alternative expression for group velocity $v_g$ is

$$v_g = v_p + k\frac{\mathrm{d}v_p}{\mathrm{d}k},$$

where $v_p$ is the phase velocity.

(b) Evaluate $v_p$ and $v_g$ as a functions of $k$ for the following cases:

i. Long wavelength surface waves on water $\omega = \sqrt{gk}$ (where $g$ is the acceleration due to gravity).

ii. Short wavelength ripples on water $\omega = \sqrt{\sigma k^3/\rho}$ (where $\sigma$ is the surface tension and $\rho$ the density).

iii. In the crossover region where both effects are important $\omega^2 = gk + \sigma k^3/\rho$.

iv. Guided electromagnetic waves in a waveguide (with a non-zero longitudinal component of either $\mathbf{E}$ or $\mathbf{B}$) $\omega^2 = \omega_0^2 + k^2c^2$ (where $c$ is the speed of light).

(c) In the first two cases but not the other two you should have found $v_g = \alpha v_p$, where the constant $\alpha$ is different in the two cases. What type of dispersion relation leads to this result?

(d) In the fourth case you should have found $v_p v_g = c^2$, so that either $v_p$ or $v_g$ is greater than $c$. Which is it, and why does this *not* allow signalling faster than the speed of light?

2.9 **Stationary phase or group velocity.** In the long wavelength limit of question 2.8(b)i., $v_p$ and $v_g$ are decreasing functions of $k$, while in the short-wavelength limit of 2.8(b)ii., they increase with $k$. Thus in the cross-over region of question 2.8(b)iii., both pass through minima.

(a) At the minimum of $v_p$, we have, using the result of question 2.8(a), $v_p = v_g$. Find the values of $k$ and $\omega$ at which this occurs. Verify this using the dispersive wavepacket plotter on the course web page, and describe the propagation of a wavepacket centred around this wavenumber (for example, $k_{\min} = k - 25\,\mathrm{m}^{-1}$, $k_{\max} = k + 25\,\mathrm{m}^{-1}$). Don't forget that the length unit used in this section of the DWP is 10 cm.

(b) Calculate the wavenumber $k$ and frequency $\omega$ at which $v_g$ has its minimum. Verify this using the DWP. (Note that the displayed value of $v_g$ is evaluated for the centre frequency of the wavepacket, so that it can be read out at steps of $5\,\mathrm{m}^{-1}$ in the cross-over region by using appropriate combinations of $k_{\min}$ and $k_{\max}$.) Around this frequency, $v_g$ is essentially constant, making the envelope approximation particularly accurate.