The wave equation

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1. Introduction

In these notes, we will derive the wave equation by considering the transverse motion of a stretched string, the compression and expansion of a solid bar, and the compression and expansion of gas in a pipe. All these systems can be described by the wave equation, a Partial Differential Equation (PDE) that we will study thoroughly for the rest of the term. We will learn why it is referred to as the wave equation in the next set of notes.

2. Transverse oscillations of a stretched string

Consider a string with linear density μ (kg/m) that is stretched from both sides with a force T. Here, the letter T stands for 'tension'. The string is sketched in figure 1. We want to understand how this string responds to small transverse displacements. We assume that the tension is sufficiently large that the string is horizontal to lowest order, that is, in the $x-y$ plane in figure 1, the string is very close to being the line $y=0$. This is, of course, only an approximation, and the string has in fact a vertical displacement $y(x,t)$ that depends on the position along the x-axis and the time t. For a string of length L, we assume that

$$\frac{y(x,t)}{L} \ll 1. \tag{2.1}$$

To describe the motion of the string, we use simple mechanics. We consider an infinitesimally small piece of string such as the one singled out in figure 1. Its length is given by

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2+\mathrm{d}y^2} = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,\mathrm{d}x \simeq \mathrm{d}x, \tag{2.2}$$

where we have used $\partial y/\partial x \sim y/L \ll 1$. Thus, the mass of this infinitesimal section is $\mu\,\mathrm{d}x$. This string piece is subjected to horizontal and vertical forces. Newton's second law gives

$$\mu a_x(x,t)\,\mathrm{d}x = F(x+\mathrm{d}x,t)\cos\theta(x+\mathrm{d}x,t) - F(x,t)\cos\theta(x,t) \tag{2.3}$$

and

$$\mu a_y(x,t)\,\mathrm{d}x = F(x+\mathrm{d}x,t)\sin\theta(x+\mathrm{d}x,t) - F(x,t)\sin\theta(x,t), \tag{2.4}$$

where $a_x(x,t)$ and $a_y(x,t)$ are the acceleration of the string in the x and y directions, and $F(x,t)$ and $F(x+\mathrm{d}x,t)$ are the forces that the rest of the string exerts on the infinitesimal piece that we are considering. Note that these forces must be of the order of T and that the string can only exert a force $F(x,t)$ parallel to itself. The angle $\theta(x,t)$ is determined by slope of $y(x,t)$,

$$\tan\theta(x,t) = \frac{\partial y}{\partial x} \sim \frac{y}{L} \ll 1. \tag{2.5}$$

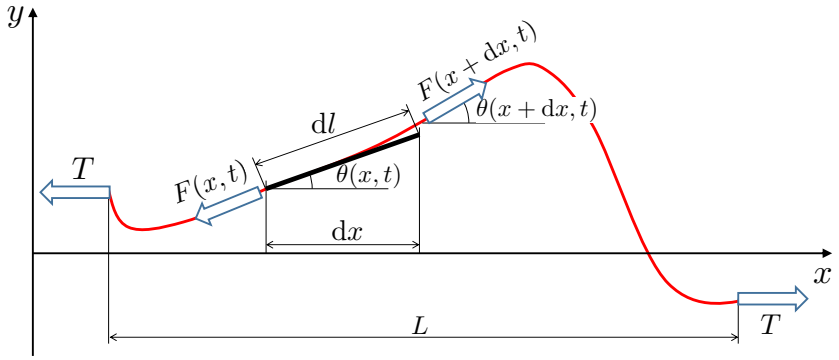


FIGURE 1. String (red line) stretched with tension T. Forces are represented as empty blue arrows. The thick black line represents an infinitesimally small piece of string.

Since $\tan\theta$ is small, we can use the small angle approximations to find

$$\theta(x,t) \simeq \frac{\partial y}{\partial x} \sim \frac{y}{L} \ll 1, \tag{2.6}$$

and hence the equations of motion simplify to

$$\mu a_x(x,t)\,\mathrm{d}x \simeq F(x+\mathrm{d}x,t) - F(x,t) \tag{2.7}$$

and

$$\mu a_y(x,t)\,\mathrm{d}x \simeq F(x+\mathrm{d}x,t)\theta(x+\mathrm{d}x) - F(x,t)\theta(x). \tag{2.8}$$

Using partial derivatives, we finally obtain

$$\mu a_x \simeq \frac{\partial F}{\partial x} \tag{2.9}$$

and

$$\mu a_y \simeq \frac{\partial}{\partial x}(F\theta). \tag{2.10}$$

We do not expect the displacement of the string in the x-direction to be much larger than the displacement in the y direction, giving $a_x \sim a_y$. The size of a_y can be obtained from equation (2.10): $a_y \sim T\theta/\mu L$. Using this result in equation (2.9), we find

$$\underbrace{\mu a_x}_{\sim T\theta/L \ll T/L} \simeq \underbrace{\frac{\partial F}{\partial x}}_{\sim T/L}. \tag{2.11}$$

Thus, the acceleration in the x direction can be neglected, giving that F is constant and equal to the tension exerted at the ends of the string,

$$F(x,t) = T. \tag{2.12}$$

Finally, using equation (2.12) and the fact that the acceleration in the y direction is

$$a_y(x,t) = \frac{\partial^2 y}{\partial t^2}, \tag{2.13}$$

equation (2.10) becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{2.14}$$

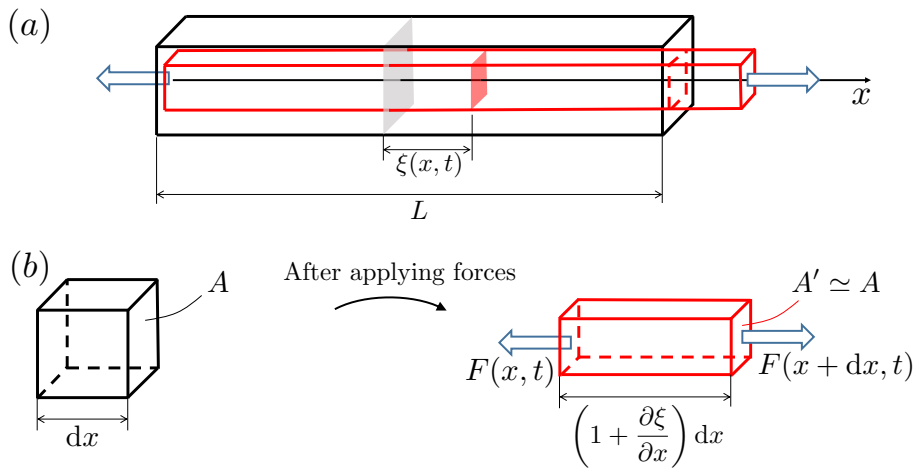


FIGURE 2. (a) Sketch of the deformation of a solid bar of constant cross section area A. The black lines represent the shape of the bar before deformation, and the red lines, the shape after deformation. Matter that was physically at the transparent grey square has moved to the position represented by the transparent red square. (b) Evolution of an infinitesimally small piece of the bar.

where

$$c = \sqrt{\frac{T}{\mu}} \tag{2.15}$$

is a constant with units of velocity. Equation (2.14) is the wave equation, and it will be our workhorse for most of this course.

3. Longitudinal oscillations of a solid bar

The wave equation also describes other systems of interest. For example, a solid bar can be stretched by pulling on it as shown in figure 2(a). We consider the displacements in the direction of the exerted force: a point initially located in the position x will move to the position $x + \xi(x,t)$ due to the forces on both sides of the bar (the Greek letter ξ is 'xi'). Note that, in addition to stretching (or compressing) in the direction of the exerted force, the bar will contract (or expand) in the direction perpendicular to the exerted force. These perpendicular deformations change the cross section area of the bar, but the changes in the cross section area turn out to be negligible in the limit that we consider in this course.

We consider the motion of an infinitesimally small piece of the bar of length $\mathrm{d}x$, shown in figure 2(b). If the volumetric density of the bar is ρ (kg/m^3), we find

$$\rho A a_x(x,t)\,\mathrm{d}x = F(x+\mathrm{d}x,t) - F(x,t), \tag{3.1}$$

where $a_x(x,t)$ is the acceleration of the infinitesimal piece of bar in the x-direction, A is the cross section area, and $F(x,t)$ and $F(x+\mathrm{d}x,t)$ are the forces exerted by the rest of the bar on the infinitesimal piece of interest. Using partial derivatives and the fact that the acceleration in the x direction is

$$a_x(x,t) = \frac{\partial^2 \xi}{\partial t^2}, \tag{3.2}$$

we find

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial F}{\partial x}. \tag{3.3}$$

The relationship between the displacement ξ and the force F depends on the geometry of the bar and constitutive properties of the material:

• A bar with a larger cross section area is more difficult to deform. The exerted force is spread over a wider area, and hence it will have a smaller effect. In other words, the relevant quantity is the stress F/A.

• The length of the bar is also crucial. It is not the same to extend 1 mm a bar of 1 m than a bar of 2 mm. It is the relative deformation that matters. An infinitesimally small piece of the bar that initially was of length $\mathrm{d}x$ ends up with a length

$$[x + \mathrm{d}x + \xi(x + \mathrm{d}x, t)] - [x + \xi(x, t)] = \mathrm{d}x + \frac{\partial \xi}{\partial x}\mathrm{d}x. \tag{3.4}$$

Thus, it has increased its length by $(\partial\xi/\partial x)\,\mathrm{d}x$ compared to its initial length of $\mathrm{d}x$: its relative deformation is $\partial\xi/\partial x$. This relative deformation depends on F/A. If the deformation is small, $\partial\xi/\partial x \sim \xi/L \ll 1$, the relationship is linear

$$\frac{\partial \xi}{\partial x} \propto \frac{F}{A}. \tag{3.5}$$

In this linear limit, the cross section area of the bar after the deformation, A', is approximately equal to the area before the deformation, A.

• The constant of proportionality that we are missing in equation (3.5) is a property of the material. It is known as Young modulus and we will denote it by E,

$$\frac{\partial \xi}{\partial x} = \frac{F}{EA}. \tag{3.6}$$

Typical values of the Young modulus range from 10^{11} N/m^2 for metals to 10^5 N/m^2 for rubber.

Solving for F in equation (3.6) and substituting the result into equation (3.3), we obtain the wave equation (2.14) again,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{3.7}$$

where the velocity c is now

$$c = \sqrt{\frac{E}{\rho}}. \tag{3.8}$$

4. Acoustic waves

By now, we can see a pattern in these calculations: the forces on both sides of an infinitesimal piece of material almost balance, leaving a residual force proportional to the spatial derivative of the force to accelerate the infinitesimal volume of material. The same technique can be applied to gas in a pipe (see figure 3). The only difference between the bar and the gas is the constitutive equation that relates the force and the deformation. Unlike in the bar, the forces on the infinitesimal volume of gas in figure 3 are always compression forces $F_C(x, t)$, i.e. they try to reduce the volume of the gas. Thus, Newton's equation gives

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial F_C}{\partial x}, \tag{4.1}$$

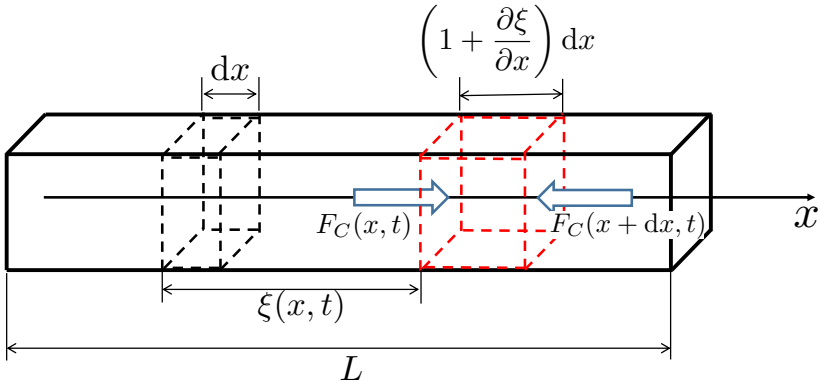


FIGURE 3. Pipe full of gas. The gas that at rest was within the infinitesimally small black dashed volume moves to the red dashed volume due to the compression forces $F_C(x,t)$ exerted by the rest of the gas.

where ρ is the density of the gas and A is the cross section area of the pipe.

The compression force F_C is proportional to the cross section of the pipe and the gas pressure $p(x,t)$,

$$F_C(x,t) = Ap(x,t). \tag{4.2}$$

The pressure depends on the properties of the gas, and in particular on its density and temperature. For rapid changes, the relationship between the pressure p and the density ρ is

$$p \propto \rho^\gamma, \tag{4.3}$$

that is, when the density decreases, the pressure decreases, and when it increases, the pressure increases. The constant γ is known as adiabatic constant and it depends on the gas and the temperature.

For simplicity, from here on, we consider a gas that is at a pressure p_0 and density ρ_0, and we assume that the changes in the pressure and the density are small, that is,

$$p(x,t) = p_0 + p_1(x,t), \quad \rho(x,t) = \rho_0 + \rho_1(x,t), \tag{4.4}$$

with $p_1/p_0 \ll 1$ and $\rho_1/\rho_0 \ll 1$. Then, relation (4.3) gives

$$p_0 + p_1 = \frac{p_0}{\rho_0^\gamma}(\rho_0+\rho_1)^\gamma. \tag{4.5}$$

Expanding in the smallness of $\rho_1/\rho_0 \ll 1$, we find

$$p_1 \simeq \frac{\gamma p_0}{\rho_0}\rho_1. \tag{4.6}$$

Now, we only need to relate the perturbation to the density, ρ_1, with the horizontal displacement ξ. Since the mass between $x+\xi(x,t)$ and $x+\mathrm{d}x+\xi(x+\mathrm{d}x,t)$ is the same that the mass between x and $x+\mathrm{d}x$, the relation between the density ρ_0 and the perturbed density $\rho_0+\rho_1$ is given by

$$(\rho_0+\rho_1)A\left[\mathrm{d}x + \xi(x+\mathrm{d}x,t) - \xi(x,t)\right] = \rho_0 A\,\mathrm{d}x. \tag{4.7}$$

Hence,

$$1 + \frac{\rho_1}{\rho_0} = \frac{1}{1+\partial\xi/\partial x}. \tag{4.8}$$

For ρ_1/ρ_0 to be small, we need $\partial\xi/\partial x \sim \xi/L \ll 1$ to be small. Then,

$$\frac{\rho_1}{\rho_0} \simeq -\frac{\partial \xi}{\partial x} \tag{4.9}$$

and equation (4.6) finally becomes

$$p_1 \simeq -\gamma p_0 \frac{\partial \xi}{\partial x}. \tag{4.10}$$

Thus, the compressional force in equation (4.2) is

$$F_C(x,t) \simeq A p_0 \left(1 - \gamma \frac{\partial \xi}{\partial x}\right). \tag{4.11}$$

Substituting equation (4.11) into equation (4.1) and using $\rho \simeq \rho_0$ give, yet again, the wave equation. In this case, the velocity c is

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}}. \tag{4.12}$$

The pressure and density perturbations that we have described are what sound is: once excited on air, these perturbation travel and reach our ears, where they move the eardrum and we interpret them as sound.

General solution to the wave equation

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1. Introduction

In these notes, we give the general solution to the wave equation. The wave equation is one of the rare PDEs that we can solve analytically with complete generality. We will use this analytical solution to explain that PDEs need two types of conditions: initial conditions (similar to those needed for Ordinary Differential Equations, ODEs) and boundary conditions (a new concept).

2. The d'Alembert solution

We start from the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{2.1}$$

where c is a constant with units of velocity. To solve this equation, we perform first a change of variables from x and t to

$$u = x - ct \tag{2.2}$$

and

$$v = x + ct. \tag{2.3}$$

Note that, since the relation between (u, v) and (x, t) is univocal, a solution in the new coordinates, $y(u, v)$, is necessarily the solution in x and t.

We need to rewrite equation (2.1) in the new coordinates u and v. We will start rewriting the time derivatives. Using the chain rule, we find

$$\left.\frac{\partial y}{\partial t}\right|_x = \left.\frac{\partial u}{\partial t}\right|_x \left.\frac{\partial y}{\partial u}\right|_v + \left.\frac{\partial v}{\partial t}\right|_x \left.\frac{\partial y}{\partial v}\right|_u. \tag{2.4}$$

For clarity, we have made the coordinate being held constant explicit. Noting that $\partial u/\partial t|_x = -c$ and $\partial v/\partial t|_x = c$, we find

$$\left.\frac{\partial y}{\partial t}\right|_x = -c \left(\left.\frac{\partial y}{\partial u}\right|_v - \left.\frac{\partial y}{\partial v}\right|_u \right). \tag{2.5}$$

Using the chain rule again, we can write

$$\left.\frac{\partial^2 y}{\partial t^2}\right|_x = -c \left[\left.\frac{\partial}{\partial u}\right|_v \left(\left.\frac{\partial y}{\partial t}\right|_x \right) - \left.\frac{\partial}{\partial v}\right|_u \left(\left.\frac{\partial y}{\partial t}\right|_x \right) \right]. \tag{2.6}$$

Substituting equation (2.5) into equation (2.6), we finally obtain

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right), \tag{2.7}$$

where we are not indicating the coordinate being held constant to avoid cluttering notation. Similarly,

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \tag{2.8}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}. \tag{2.9}$$

Substituting equations (2.7) and (2.9) into equation (2.1), we obtain the extremely simple expression

$$\frac{\partial^2 y}{\partial u \partial v} = 0. \tag{2.10}$$

Equation (2.10) implies that $\partial y/\partial v$ is independent of u, that is,

$$\frac{\partial y}{\partial v} = h(v), \tag{2.11}$$

where $h(v)$ can be any function as long as it only depends on v. Integrating this equation, we find

$$y(u, v) = f(u) + \int h(v)\,\mathrm{d}v. \tag{2.12}$$

Here the function $f(u)$ is the 'constant' of integration. Renaming the indefinite integral $\int h(v)\,\mathrm{d}v$ as $g(v)$, we obtain the complete d'Alembert solution,

$$y(u, v) = f(u) + g(v). \tag{2.13}$$

In the coordinates x and t,

$$y(x, t) = f(x - ct) + g(x + ct). \tag{2.14}$$

Solution (2.14) is the reason why equation (2.1) is known as the wave equation. Any solution to the wave equation can always be split into the two functions $f(u)$ and $g(v)$ in equation (2.14), and these two functions move rigidly along x: the function f towards positive x and the function g towards negative x. Indeed, as time advances, the function that initially was $f(x)$ becomes $f(x - ct)$. Thus, the value f_0 of f located at $x = x_0$ moves to

$$x = x_0 + ct, \tag{2.15}$$

i.e. it moves towards positive x. All values of f move at the same speed c, giving the rigid displacement that we announced above. A similar argument can be applied to show that $g(v)$ moves rigidly towards negative x. These two rigidly moving functions are the waves of the wave equation. From here on, we refer to $f(x - ct)$ as the right-traveling wave, and to $g(x + ct)$ as the left-traveling wave.

We proceed to discuss what determines the functions $f(u)$ and $g(v)$: initial conditions and boundary conditions.

3. Initial conditions

The physical systems that we have studied (the stretched string, the solid bar and the gas in a pipe) are a good justification for the need of initial conditions: to determine the time evolution of a body in mechanics, we need the initial position and velocity. In the case of the stretched string, that means knowing the initial vertical displacement,

$$y(x, t = 0) = y_0(x), \tag{3.1}$$

and the initial vertical velocity,

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0(x). \tag{3.2}$$

This is a general property of any PDE: one needs as many initial conditions as the order of the highest order partial derivatives with respect to time in the PDE. In this case, there is a second order derivative with respect to time in the wave equation, and hence we need two initial conditions.

It is possible to find the solution given the initial conditions above (and ignoring boundary conditions, which we will discuss in the next section). Using the d'Alembert solution (2.14) and equation (2.5) for $\partial y/\partial t$, the two initial conditions become

$$f(x) + g(x) = y_0(x) \tag{3.3}$$

and

$$-c\,[f'(x) - g'(x)] = \dot{y}_0(x), \tag{3.4}$$

where f' and g' are the derivatives of f and g with respect to their arguments. We can integrate equation (3.4) in x to obtain

$$f(x) - g(x) = f(0) - g(0) - \frac{1}{c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s. \tag{3.5}$$

We will see shortly that we are free to choose the value of $f(x) - g(x)$ at $x = 0$. We can solve for $f(x)$ and $g(x)$ using equations (3.3) and (3.5), finding

$$f(x) = \frac{y_0(x)}{2} - \frac{1}{2c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s + \frac{f(0) - g(0)}{2} \tag{3.6}$$

and

$$g(x) = \frac{y_0(x)}{2} + \frac{1}{2c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s - \frac{f(0) - g(0)}{2}. \tag{3.7}$$

With these results, equation (2.14) gives

$$y(x, t) = \frac{y_0(x - ct) + y_0(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{y}_0(s)\,\mathrm{d}s. \tag{3.8}$$

Note that the difference $f(0) - g(0)$ does not contribute to the final answer, that is, the constant of integration in equation (3.5) is, in the end, irrelevant. The idea is that the functions $f(u)$ and $g(v)$ are only defined up to a constant: we can add a constant to $f(u)$ if we subtract the same constant from $g(v)$.

We proceed to solve a few examples of initial conditions.

3.1. *Stationary initial condition*

We start with a case in which $\dot{y}_0(x)$ is zero. Consider the infinite stretched string in figure 1(a). It is held in place such that the points at $x = -L$ and L are not displaced, $y(x = \pm L, t) = 0$, and such that the point at $x = 0$ is displaced a distance a, $y(x = 0, t) = a$. Since the string is being held in place, $\partial^2 y/\partial t^2 = 0$, and the wave equation (2.1) simply becomes $\partial^2 y/\partial x^2 = 0$. This equation is valid only for $x \neq -L, 0, L$ because at these points we are applying forces to hold the string in place. From $\partial^2 y/\partial x^2 = 0$, we find that the slope of y is constant, and hence we obtain the piecewise linear function in

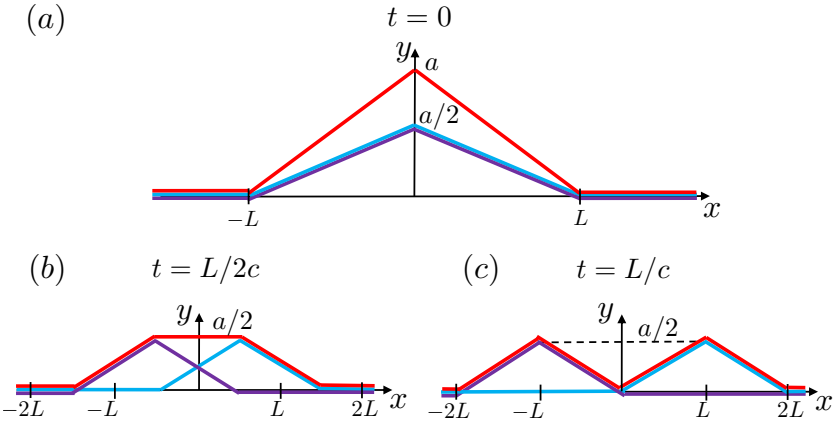


FIGURE 1. (a) Initial position of a stationary string (red) and the resulting right- and left-traveling waves (blue and purple, respectively). For times (b) $t = L/2c$ and (c) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string.

red in figure 1(a),

$$y(x, t = 0) = y_0(x) = \begin{cases} a(1 + x/L) & \text{for } -L \leqslant x < 0, \\ a(1 - x/L) & \text{for } 0 \leqslant x < L, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

In addition, we know that $\dot{y}_0(x) = 0$.

Using equations (3.6) and (3.7) and choosing $f(0) - g(0) = 0$, we find

$$f(u) = \frac{y_0(u)}{2} \tag{3.10}$$

and

$$g(v) = \frac{y_0(v)}{2}, \tag{3.11}$$

that is, the solution is composed of two equal waves moving in opposite directions. We represent these waves in blue and purple in figure 1(a). It is unsurprising that there are two waves moving in opposite directions because the initial conditions do not have a preferred direction.

By rigidly moving the two waves $f(u)$ and $g(v)$ and adding them, we can obtain the solution for any time t. We give examples of the solution at $t = L/2c$ and L/c in figures 1(b) and 1(c).

Note that the solution that we have obtained does not have continuous first order derivatives, and one might wonder how it could be a solution to a PDE that contains second order derivatives. We were able to bypass the problem with the second derivatives by using the d'Alembert solution (2.14). Thanks to the d'Alembert solution, we now know that the wave equation works even for discontinuous functions, and it only moves these discontinuities around, without smoothing them out. In a real physical systems, there is usually some dissipation present, and these discontinuities tend to disappear with time.

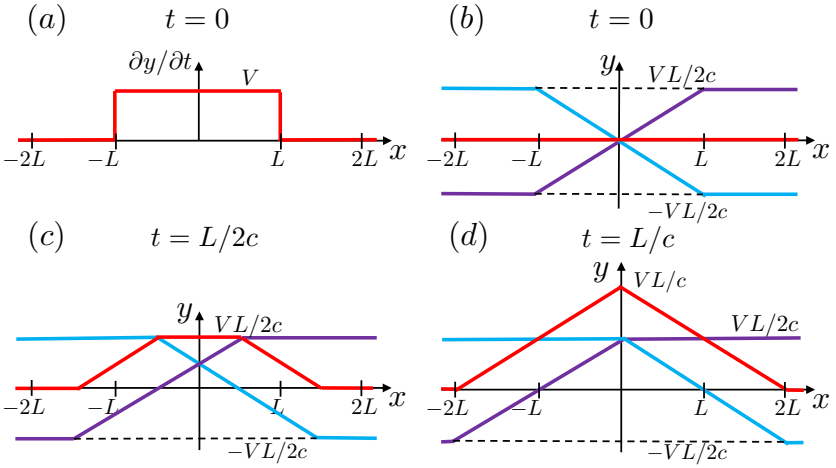


FIGURE 2. (a) Initial velocity of the wave (red). (b) Initial position of the string (red) and right- and left-traveling waves (in blue and purple, respectively). For times (c) $t = L/2c$ and (d) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string.

3.2. *Impulsive initial condition*

We now consider a stretched string that starts at $y_0(x) = 0$, but it is given an impulse such that its initial velocity is given by

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0 = \begin{cases} V & \text{for } -L \leqslant x \leqslant L, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

The function $\dot{y}_0$ is sketched in figure 2(a).

Using equations (3.6) and (3.7) with $f(0) - g(0) = 0$, we find

$$f(u) = -\frac{1}{2c}\int_0^u \dot{y}_0(s)\,\mathrm{d}s \tag{3.13}$$

and

$$g(v) = \frac{1}{2c}\int_0^v \dot{y}_0(s)\,\mathrm{d}s, \tag{3.14}$$

where

$$\frac{1}{2c}\int_0^x \dot{y}_0(s)\,\mathrm{d}s = \begin{cases} -VL/2c & \text{for } x < -L, \\ Vx/2c & \text{for } -L \leqslant x < L, \\ VL/2c & \text{for } x \geqslant L. \end{cases} \tag{3.15}$$

These waves are represented as blue and purple curves in figure 2(b). Using them, we can obtain the vertical position of the string at all times t (see figure 2 for a few time slices).

3.3. *Single wave initial condition*

In both cases considered so far, we have obtained two waves traveling in opposite directions. If we wanted a wave of the shape $y_0(x)$ traveling towards positive x, what would we need to do initially? We would simply need to ensure that it has the right initial velocity. Since we want to have

$$y(x, t) = f(x - ct) = y_0(x - ct), \tag{3.16}$$

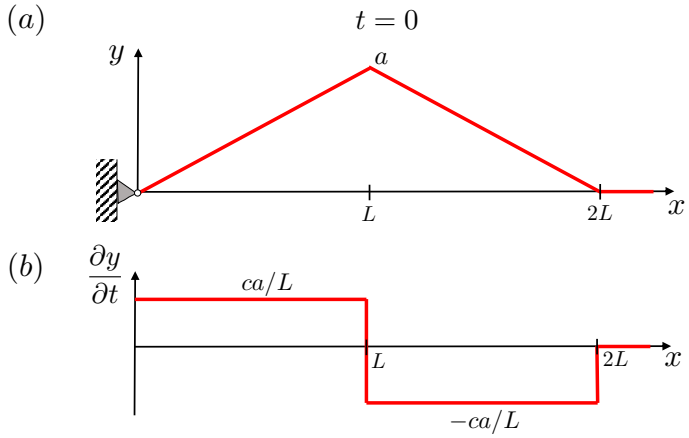


FIGURE 3. Initial (a) position and (b) velocity of a semi-infinite string with a pin at $x = 0$.

we need to impose the initial velocities

$$\dot{y}_0(x) = \frac{\partial y}{\partial t}(x, t = 0) = -c y_0'(x). \tag{3.17}$$

4. Boundary conditions

In all the cases in section 3, we have ignored the ends of the stretched string by assuming that it is infinite, but stretched strings do have ends where they are being pulled to ensure that they are tensioned. For example, we can consider a semi-infinite string that extends from $x = 0$ to $+\infty$, as sketched in figure 3. This string is kept in tension by being pinned to the wall at $x = 0$ and by the pull of something or someone a very long distance away from $x = 0$. The motion of such a string should be solvable given initial conditions.

To show the problems that arise for a semi-infinite string, we will try to solve the problem with initial conditions

$$y(x, t = 0) = y_0(x) = \begin{cases} ax/L & \text{for } 0 \leqslant x < L, \\ a(2 - x/L) & \text{for } L \leqslant x < 2L, \\ 0 & \text{for } x \geqslant 2L, \end{cases} \tag{4.1}$$

and

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0(x) = c y_0'(x) = \begin{cases} ca/L & \text{for } 0 \leqslant x < L, \\ -ca/L & \text{for } L \leqslant x < 2L, \\ 0 & \text{for } x \geqslant 2L, \end{cases} \tag{4.2}$$

sketched in figure 3. Using formulas (3.6) and (3.7) with $f(0) - g(0) = 0$, we find

$$f(u) = \begin{cases} ?? & \text{for } u < 0, \\ 0 & \text{for } u \geqslant 0, \end{cases} \tag{4.3}$$

and

$$g(v) = \begin{cases} ?? & \text{for } v < 0, \\ y_0(v) & \text{for } v \geqslant 0. \end{cases} \tag{4.4}$$

We cannot determine $f(u)$ or $g(v)$ for negative values of their arguments! This is not a problem for $g(v)$ since $v = x + ct$ is positive at all positive times for $x > 0$, but we need to do something for $f(u)$ since sufficiently small values of x will give negative values of

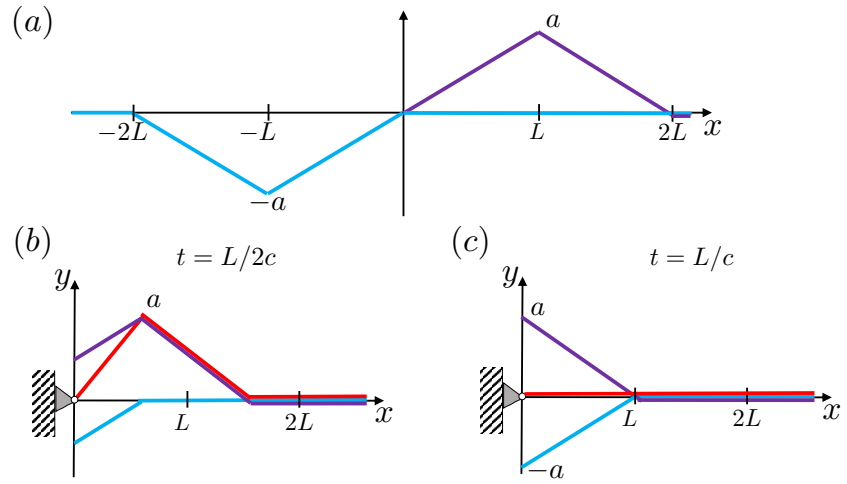


FIGURE 4. (a) Right- and left-traveling waves (in blue and purple, respectively) for the initial conditions and the boundary condition in figure 3. For times (b) $t = L/2c$ and (c) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string. Note that at $t = L/c$, the whole string satisfies $y(x, t = L/c) = 0$. Note as well that at all times, $y(x = 0, t) = 0$, as we wanted.

$u = x - ct$ at finite times t. The value of $f(u)$ for $u < 0$ is determined by the boundary condition at $x = 0$: we need to specify what the string is doing at $x = 0$.

In general, a PDE requires as many boundary conditions as the order of its highest order derivative with respect the spatial variable. The wave equation has a second order spatial derivatives and hence it requires two boundary conditions, one on each end of the string. One then might wonder how we solved the problems in section 3 without imposing boundary conditions on the 'ends' of the infinite string. We did in fact impose hidden boundary conditions: we did not let any waves travel into our domain from infinity, that is, we assumed that no other person was twiddling with the string at a large distance from us (here large means large compared to the extent of the initial condition). These boundary conditions are known as outgoing wave boundary conditions because we allow waves to leave our domain but we do not allow any waves to come into it.

We proceed to discuss three examples of boundary conditions for the semi-infinite string and one example with two simultaneous boundary conditions.

4.1. *Dirichlet boundary condition*

The Dirichlet boundary conditions are the ones in which the value of the function $y(x, t)$ is set at one point. One example of a Dirichlet boundary condition is the case in figure 3 that shows a string pinned at $x = 0$,

$$y(x = 0, t) = 0. \tag{4.5}$$

Imposing boundary condition (4.5) on the d'Alembert solution (2.14), we find

$$f(-ct) + g(ct) = 0. \tag{4.6}$$

Using the change of variables $u = -ct$, we obtain

$$f(u) = -g(-u), \tag{4.7}$$

that is, $f(u)$ for negative values of u is a reflection of $g(v)$ with respect to the point

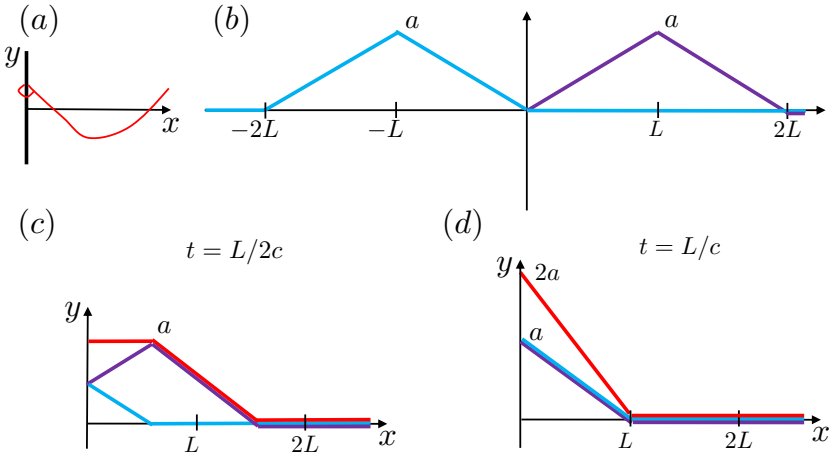


FIGURE 5. (a) Neumann boundary condition for a stretched string. (b) Right- and left-traveling waves (in blue and purple, respectively) for the initial conditions in figure 3. For times (c) $t = L/2c$ and (d) $t = L/c$, we sketch, in blue and purple, the right- and left-traveling waves, and in red the position of the string. Note that $\partial y/\partial x(x = 0, t)$ vanishes for almost every time. Time $t = L/c$ is one of the times at which the boundary condition is not satisfied, but note that it is satisfied for times infinitesimally close to $t = L/c$. This problem is due to the fact that the initial condition does not have continuous first derivatives.

$x = 0, y = 0$. Thus, the function $f(u)$ is

$$f(u) = \begin{cases} -y_0(-u) & \text{for } u < 0, \\ 0 & \text{for } u \geqslant 0. \end{cases} \tag{4.8}$$

The functions $f(u)$ and $g(v)$ are sketched in figure 4(a). By rigidly moving these functions and summing them, we find $y(x, t)$, as shown in figure 4.

4.2. *Neumann boundary condition*

The Neumann boundary conditions are the ones in which the value of the spatial derivative $\partial y(x, t)/\partial x$ is set at one point. One example of Neumann boundary condition is the set-up shown in figure 5(a). The string is kept stretched in this case by tying it in a knot around a frictionless column (represented as a thick vertical line in figure 5(a)). Since the column is frictionless, the only vertical force on the knot is the one exerted by the string,

$$F_y(x = 0, t) \simeq T\theta(x = 0, t) \simeq T\frac{\partial y}{\partial x}(x = 0, t). \tag{4.9}$$

We assume that the knot has negligibly small mass, giving $F_y(x = 0, t) = 0$, that is,

$$\frac{\partial y}{\partial x}(x = 0, t) = 0. \tag{4.10}$$

To show the effect of the Neumann boundary condition on the string, we solve the problem with the initial conditions (4.1) and (4.2) that lead to the partial solutions for $f(u)$ and $g(v)$ in equations (4.3) and (4.4). Imposing boundary condition (4.10) on the d'Alembert solution (2.14), we find

$$f'(-ct) + g'(ct) = 0. \tag{4.11}$$

Using the change of variable $u = -ct$, this expression becomes

$$f'(u) = -g'(-u). \tag{4.12}$$

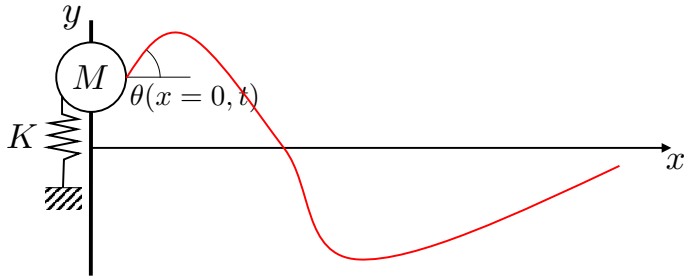


FIGURE 6. Example of general boundary condition for a stretched string.

Integrating this equation once and recalling that we had chosen $f(0) = g(0)$ to obtain equations (4.3) and (4.4), we find

$$f(u) = g(-u), \tag{4.13}$$

that is, $f(u)$ for negative u is a reflection of $g(v)$ with respect to the axis $x = 0$. With this result, we obtain the value of $f(u)$ for negative u,

$$f(u) = \begin{cases} y_0(-u) & \text{for } u < 0, \\ 0 & \text{for } u \geqslant 0. \end{cases} \tag{4.14}$$

The functions $f(u)$ and $g(v)$ are sketched in figure 5(b), and we can construct the solution for $t > 0$ by displacing them rigidly and summing them, as shown in figure 5 for a few times.

4.3. *General boundary condition*

In general, boundary conditions give $\partial y/\partial x$ at a given point x as a function of y and its time derivatives at that point. An example of a general boundary condition is given in figure 6. In this case, the string is knotted around a frictionless column, but the knot has a mass M and it is tied to a vertical spring with spring constant K. Thus, in addition to the vertical force due to the string, $T(\partial y/\partial x|_{x=0})$, the knot is pulled down by the force of the spring, $-Ky(x = 0, t)$, giving the equation of motion

$$M\frac{\partial^2 y}{\partial t^2}(x = 0, t) = T\frac{\partial y}{\partial x}(x = 0, t) - Ky(x = 0, t). \tag{4.15}$$

The application of this boundary condition to the d'Alembert solution is left as an exercise for the reader. We will see in a few lectures that these complex boundary conditions are usually easier to treat by using a technique different from the one that we have used so far.

4.4. *Problems with two boundary conditions*

We will now consider a finite stretched string with two boundary conditions. The string is of length $2L$ and it is pinned at both ends, located at $x = 0$ and $2L$. Thus, it has the Dirichlet boundary conditions

$$y(x = 0, t) = 0 \tag{4.16}$$

and

$$y(x = 2L, t) = 0. \tag{4.17}$$

We consider the motion of this string for the initial conditions

$$y(x, t = 0) = y_0(x) = \begin{cases} ax/L & \text{for } 0 \leqslant x < L, \\ a(2 - x/L) & \text{for } L \leqslant x < 2L, \end{cases} \tag{4.18}$$

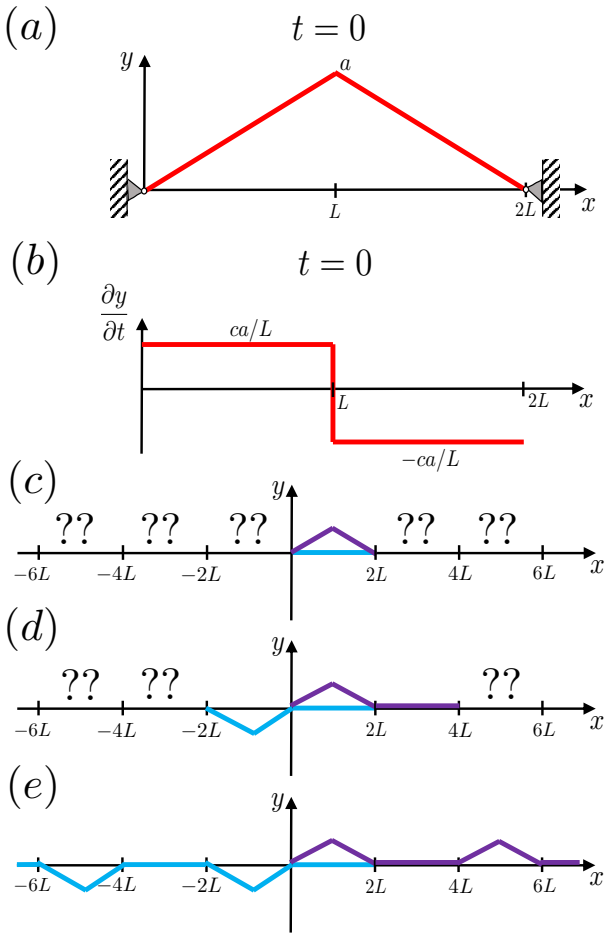


FIGURE 7. (a) Initial position of a string pinned at $x = 0$ and $x = 2L$. (b) Initial velocity of the same string. (c) Right- and left-traveling waves (blue and purple, respectively) given by the initial condition. (d) Right- and left-traveling waves after applying the boundary conditions to the functions in (c). (e) Final form of the right- and left-traveling waves after repeated application of the boundary conditions.

and

$$\frac{\partial y}{\partial t}(x, t = 0) = \dot{y}_0(x) = c y_0'(x) = \begin{cases} ca/L & \text{for } 0 \leqslant x < L, \\ -ca/L & \text{for } L \leqslant x < 2L. \end{cases} \tag{4.19}$$

These initial conditions are sketched in figures 7(a) and 7(b). Using formulas (3.6) and (3.7) with $f(0) - g(0) = 0$, we find

$$f(u) = \begin{cases} ?? & \text{for } u < 0, \\ 0 & \text{for } 0 \leqslant u \leqslant 2L, \end{cases} \tag{4.20}$$

and

$$g(v) = \begin{cases} y_0(v) & \text{for } 0 \leqslant v < 2L, \\ ?? & \text{for } v \geqslant 2L, \end{cases} \tag{4.21}$$

These pieces of the right- and left-traveling waves are sketched in figure 7(c).

Applying the same technique that we used to solve the wave equation with one Dirichlet boundary condition, we find that $f(u) = -g(-u)$ due to the boundary condition at $x = 0$,

and $g(v) = -f(4L - v)$ due to the boundary condition at $x = 2L$. Given the solutions in equations (4.20) and (4.21), we can use $f(u) = -g(-u)$ and $g(v) = -f(4L - v)$ to determine $f(u)$ for $-2L \leqslant u < 0$ and $g(v)$ for $2L \leqslant v < 4L$,

$$f(u) = \begin{cases} ?? & \text{for } u < -2L, \\ -y_0(-u) & \text{for } -2L \leqslant u < 0, \\ 0 & \text{for } 0 \leqslant u \leqslant 2L, \end{cases} \tag{4.22}$$

and

$$g(v) = \begin{cases} y_0(v) & \text{for } 0 \leqslant v < 2L, \\ 0 & \text{for } 2L \leqslant v < 4L, \\ ?? & \text{for } v \geqslant 4L, \end{cases} \tag{4.23}$$

These extended right- and left-traveling waves are sketched in figure 7(d). With the left-traveling wave $g(v)$ known for $2L \leqslant v < 4L$ and $f(u) = -g(-u)$, we can obtain the right-traveling wave $f(u)$ from $-4L \leqslant u < -2L$, and with $f(u)$ known for $-2L \leqslant u < 0$ and $g(v) = -f(4L - v)$, we can calculate $g(v)$ for $4L \leqslant v < 6L$. Thus, we can continue constructing $f(u)$ and $g(v)$ to obtain the solution to the motion of the string for all positive times, as shown in figure 7(e).

The most important lesson to learn from this calculation is that the wave equation needs two boundary conditions, and each of these two boundary conditions must be applied at the two opposite extremes of the string. Over-constraining the solution by imposing two boundary conditions at $x = 0$ and no boundary conditions at $x = 2L$ would not work. At both $x = 0$ and $x = 2L$, we need sufficient information to determine the right- and left-traveling waves. At $x = 2L$, the initial condition determines the right-traveling wave, but without a boundary condition we would not be able to find the left-traveling wave. Conversely, at $x = 0$, the extra boundary condition would be impossible to impose because the left-traveling wave is determined by the initial condition, leaving only the right-traveling wave as an unknown.

Understanding how many boundary conditions are needed and where they should be imposed is one of the fundamental problems in Partial Differential Equations (PDEs). The wave equation demonstrates that to solve this problem, we need to understand how the information is transported through the system.

It is clear how to use the d'Alembert solution to construct solutions in finite strings with two boundary conditions, but it is also clear that such construction can be tedious, and it can hide fundamental properties of the solutions, such as the fact that it is periodic, as we will prove shortly. We will present a different technique to solve these problems with two boundary conditions in a few lectures. Before doing so, we discuss the energy balance for the stretched string to demonstrate how energy is transported via waves in such a system.

Energy conservation in the stretched string

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(This version is of 7 January 2021)

1. Introduction

In these notes, we present the energy conservation equation for the stretched string. The equation for energy conservation will reveal interesting physics, such as the fact that waves can transport energy from one place to another even though matter is not being transported in the process.

2. Energy density

We start by calculating the energy in an infinitesimally small piece $\mathrm{d}x$ of a stretched string with linear density μ tensioned by a force T. The string and the infinitesimal piece of interest are represented in figure 1. The infinitesimal piece of string contains infinitesimal amounts of kinetic and potential energy.

The kinetic energy of the infinitesimal piece of string is

$$\mathrm{d}E_k = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x, \tag{2.1}$$

where we have used the fact that the mass of the piece of string is $\mu\,\mathrm{d}x$. Note that we have ignored the velocity of the infinitesimal piece of string in the x direction. We can do this because the motion in the x and y direction are decoupled from each other.

The potential energy is due to the work done by the tension of the string. To move the string in the y direction, we have increased the original length of the string from $\mathrm{d}x$ to

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\,\mathrm{d}x \simeq \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x, \tag{2.2}$$

where we have used $\partial y/\partial x \sim y/L \ll 1$. Due this increase in length, the rest of the string is doing work on the infinitesimal length of string of interest, and this work is

$$\mathrm{d}E_p = T(\mathrm{d}l - \mathrm{d}x) \simeq \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. \tag{2.3}$$

The total energy of the string can be calculated by summing the kinetic and potential energies of all the infinitesimal pieces of string,

$$E = \int \mathrm{d}E_k + \int \mathrm{d}E_p = \int \left[\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x. \tag{2.4}$$

The quantity

$$\epsilon = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \tag{2.5}$$

is the energy density and gives the energy per unit length.

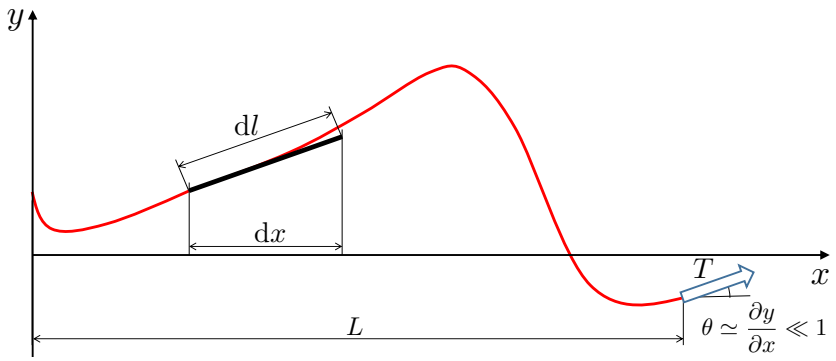


FIGURE 1. Stretched string in red and an infinitesimal piece of it in black.

3. Energy conservation equation

We proceed to study the time evolution of the energy density. By simple differentiation, we find

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right] = \mu\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t \partial x}. \tag{3.1}$$

Employing the wave equation to write $\partial^2 y/\partial t^2 = c^2(\partial^2 y/\partial x^2)$, we rewrite this equation as

$$\frac{\partial \epsilon}{\partial t} = \mu c^2 \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t \partial x}. \tag{3.2}$$

Finally, recalling that $c^2 = T/\mu$, we find the nice expression

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right). \tag{3.3}$$

To understand what the term in the right side of equation (3.3) means, we consider the energy of a finite piece of string that extends from $x = 0$ to $x = L$. Integrating equation (3.3) from $x = 0$ to $x = L$, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\bigg|_{x=L} - T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\bigg|_{x=0}. \tag{3.4}$$

The fact that the term in the right side of equation (3.3) is a spatial derivative implies that the time evolution of the energy of a finite length of string only depends on what happens at its boundaries $x = 0$ and $x = L$. Physically, this is not surprising because the boundary terms are the work done by the rest of the string on the piece of string that we are considering. Indeed, in figure 1 we show that the vertical force F_y exerted by the rest of the string on $x = L$ is

$$F_y = T\frac{\partial y}{\partial x}(x = L, 0). \tag{3.5}$$

Thus, the first term in the right side of equation (3.4) is the power due to this vertical force, $F_y v_y$, where $v_y = \partial y/\partial t$ is the vertical velocity of the string. A similar result can be found for the term at $x = 0$.

The boundary terms in equation (3.4) can also be interpreted as energy fluxes in and out of the length of string of interest. Using the d'Alembert solution $y(x,t) = f(x-ct)+$

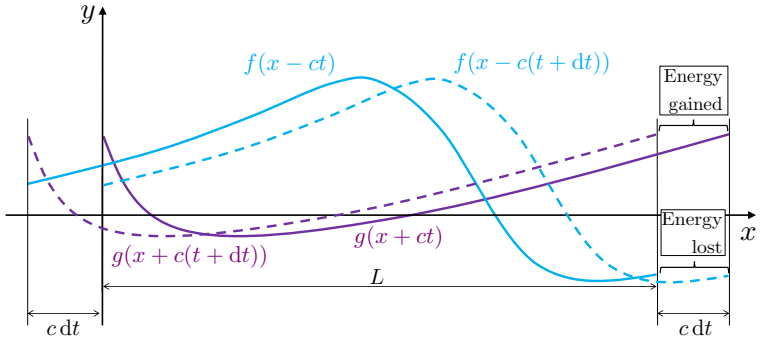


FIGURE 2. Right- and left-traveling waves (in blue and purple, respectively) at times t (solid lines) and $t + \mathrm{d}t$ (dashed lines).

$g(x + ct)$, we can rewrite the energy density and the boundary terms as

$$\epsilon = \frac{1}{2}\mu(-cf' + cg')^2 + \frac{1}{2}T(f' + g')^2 \tag{3.6}$$

and

$$T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = T(f' + g')(-cf' + cg'), \tag{3.7}$$

where f' and g' are the derivatives of $f(u)$ and $g(v)$ with respect to their arguments. Using $c^2 = T/\mu$, these expressions can be converted into

$$\epsilon = T(f')^2 + T(g')^2 \tag{3.8}$$

and

$$T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -T(f')^2 c + T(g')^2 c. \tag{3.9}$$

Both the energy density and the boundary terms can be split into two additive terms (even though they are nonlinear quantities!). In particular, the energy density can be split into the energy density of the right-traveling wave, $\epsilon_f = T(f')^2$, and the energy density of the left-traveling wave, $\epsilon_g = T(g')^2$. As a result, the boundary term can be written as

$$T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -\epsilon_f c + \epsilon_g c, \tag{3.10}$$

and equation (3.4) becomes

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -c\,\epsilon_f|_{x=L} + c\,\epsilon_g|_{x=L} + c\,\epsilon_f|_{x=0} - c\,\epsilon_g|_{x=0}\,. \tag{3.11}$$

To understand these terms, we consider what happens at $x = L$ with the right-traveling wave $f(x - ct)$. In a time $\mathrm{d}t$, the wave moves a distance $c\,\mathrm{d}t$ to the right – see figure 2. The energy of the piece of $f(x - ct)$ that leaves the region between $x = 0$ and $x = L$, $\epsilon_f c\,\mathrm{d}t$, is lost, giving a rate of energy loss of $-\epsilon_f c$, as seen in equation (3.11). Similarly, the left-traveling wave $g(x + ct)$ at $x = L$ moves into the region between $x = 0$ and $x = L$, feeding energy into the piece of string of interest at a rate $\epsilon_g c$. For this reason, the boundary terms in equation (3.4) can be interpreted as energy fluxes.

We finish by noting that the energy conservation equation is satisfied for the left- and right-traveling waves independently. By simply noting that f depends on $x - ct$ and g depends on $x + ct$, it can be shown that

$$\frac{\partial \epsilon_f}{\partial t} + c\frac{\partial \epsilon_f}{\partial x} = 0 \tag{3.12}$$

and

$$\frac{\partial \epsilon_g}{\partial t} - c\frac{\partial \epsilon_g}{\partial x} = 0. \tag{3.13}$$

Sinusoidal waves, boundary conditions and transmission problems

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1. Introduction

In these notes, we introduce sinusoidal waves, a very useful tool because it simplifies significantly some of the calculations. In particular, we will show how to use them to obtain solutions with complex boundary conditions and with inhomogeneous wave speed c. In future lectures, we will see that sinusoidal waves can also be used for equations that are not solved by the d'Alembert solution.

2. Sinusoidal waves

We have seen that the wave equation is solved by the d'Alembert solution $y(x,t) = f(x-ct) + g(x+ct)$. A particularly interesting option for $f(u)$ and $g(v)$ are sines and cosines. For example, we can choose

$$f(x-ct) = C\cos(k(x-ct)+\varphi). \tag{2.1}$$

The sinusoidal wave is charaterised by

- **Wavenumber** $= k$,
- **Wavelength** $= 2\pi/k$,
- **Angular frequency** $\omega = kc$,
- **Frequency** $f = \omega/2\pi$,
- **Period** $\tau = 2\pi/\omega$,
- **Amplitude** $= C$, and
- **Phase** $= \varphi$.

It is usually convenient to rewrite the sinusoidal wave as a complex exponential,

$$C\cos(kx-\omega t+\varphi) = \frac{1}{2}Ce^{\mathrm{i}\varphi}e^{\mathrm{i}kx-\mathrm{i}\omega t} + \text{complex conjugate}, \tag{2.2}$$

where the complex number $Ce^{\mathrm{i}\varphi}$ is the complex amplitude. Both the sinusoidal wave and the complex exponential are solutions to the wave equation, and can be used interchangeably.

Why would we focus on sinusoidal solutions to the wave equation? The most important reason is that one can construct any function using them. Indeed, under certain assumptions, one can write any function $f(u)$ as a linear combination of sinusoidal waves,

$$f(u) = \int_{-\infty}^{\infty} F(k)e^{\mathrm{i}ku}\,\mathrm{d}k. \tag{2.3}$$

You will see a proof of this next year, in Mathematical Methods. For this course, it is sufficient to know that this is possible to do, and hence one should be interested in sinusoidal waves.

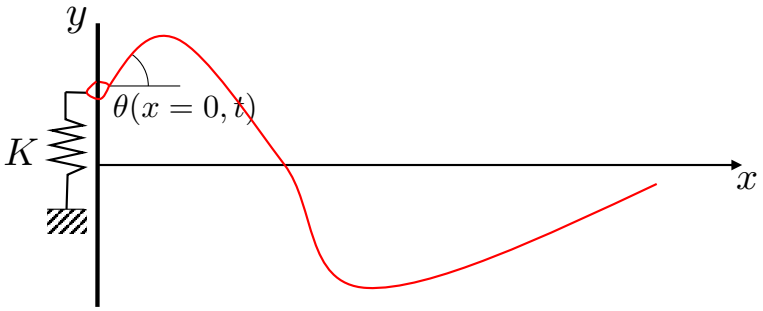


FIGURE 1. Boundary condition at $x = 0$.

3. Boundary conditions for sinusoidal waves

In this section of the notes, we will consider two problems to illustrate the advantages of using sinusoidal waves: a stretched string with a general boundary condition, and transmission lines.

3.1. *Stretched string with general boundary condition*

We consider a semi-infinite string that extends from $x = 0$ to $x = +\infty$. At $x = 0$, the string is knotted around a frictionless column, and it is attached to a vertical spring with spring constant K. The knot is massless. The configuration is sketched in figure 1. Force balance at the knot gives

$$T\frac{\partial y}{\partial x}(x = 0, t) - Ky(x = 0, t) = 0. \tag{3.1}$$

Applying boundary condition (3.1) to the d'Alembert solution is possible but tedious. The problem becomes more tractable if we use sinusoidal waves. We consider a sinusoidal wave traveling from $x = +\infty$ to $x = 0$,

$$g(x + ct) = \frac{1}{2}Ce^{-\mathrm{i}k(x+ct)} + \text{complex conjugate}. \tag{3.2}$$

Here, C can be a complex number. Using boundary condition (3.1), we can obtain the wave traveling to the right (reflected wave), $f(x - ct)$, from the incoming wave $g(x + ct)$. With a sinusoidal incoming wave $g(x + ct)$, obtaining the reflected wave turns out to be trivial. Since all the coefficients in boundary condition (3.1) are constant, the reflected wave is also sinusoidal,

$$f(x - ct) = \frac{1}{2}Re^{\mathrm{i}k(x-ct)} + \text{complex conjugate}. \tag{3.3}$$

Note that only the complex constant R is not known here. Note as well that we have written the complex exponential such that the time dependence is the same for both the right- and the left-traveling waves, i.e. they are both proportional to $e^{-\mathrm{i}\omega t}$. This is a choice that simplifies the equations below. It is always possible to make this choice because the solution is a complex exponential plus its complex conjugate.

We can check that the reflected wave in equation (3.3) is the solution by checking that it satisfies boundary condition (3.1). Since $y(x, t) = f(x - ct) + g(x + ct)$, the different functions that enter in the boundary conditions are

$$y(x = 0, t) = \frac{1}{2}(C + R)e^{-\mathrm{i}kct} + \text{complex conjugate} \tag{3.4}$$

and

$$\frac{\partial y}{\partial x}(x=0,t) = \frac{1}{2}\mathrm{i}k(-C+R)e^{-\mathrm{i}kct} + \text{complex conjugate}. \tag{3.5}$$

Replacing all these results into equation (3.1), we find

$$\mathrm{i}kT(-C+R) - K(C+R) = 0. \tag{3.6}$$

This equation gives R as a function of C,

$$\frac{R}{C} = -\frac{K+\mathrm{i}kT}{K-\mathrm{i}kT}. \tag{3.7}$$

Note that the modulus of R/A is one, and hence

$$\frac{R}{C} = e^{\mathrm{i}\varphi}, \tag{3.8}$$

where

$$\varphi = -\pi + 2\arctan\left(\frac{kT}{K}\right) \tag{3.9}$$

is the phase between the right- and the left-traveling wave. In the limit $kT/K \to 0$, the spring is too stiff and the point at $x = 0$ barely moves, giving $y(x = 0, t) = 0$ and $\varphi = -\pi$, i.e. when the extreme of the string is pinned, the right- and left-traveling wave are in anti-phase (cf with the result that we obtained for the d'Alembert solution). In the limit $kT/K \to \infty$, the spring barely exerts any force, and the spring behaves as if there was no vertical force at $x = 0$, leading to $\partial y(x = 0, t)/\partial x = 0$ and $\varphi = 0$. In this case, the two waves are in phase, as one would expect from the calculation that we performed in the notes about the d'Alembert solution.

3.2. *Transmission lines and wave impedance*

To transmit electric power at high frequencies, one cannot use simple wires as these would radiate the power away. Instead, we use transmission lines such as co-axial cables. These transmission lines can be conceptualized as two conducting wires next to each other, as sketched in figure 2. One of the conductors absorbs the waves emitted by the other, preventing losses due to radiation. The potential difference $V(x, t)$ between the two conductors and the current $I(x, t)$ flowing through them depend on the position x along the line and the time t. Note that the current on one of the conductors is opposite to the current on the other to maintain zero charge. Focusing on an infinitesimal piece of the line of length $\mathrm{d}x$, we note two important facts:

- Consider the piece of the top conductor of length $\mathrm{d}x$ shown in figure 2 as a dashed box. Charge $\mathrm{d}Q$ accumulates within this piece of conductor due to the different currents at its two extremes,

$$\frac{\partial}{\partial t}\mathrm{d}Q = I(x) - I(x+\mathrm{d}x) = -\frac{\partial I}{\partial x}\,\mathrm{d}x. \tag{3.10}$$

The same charge but with opposite sign accumulates in the other conductor. Thus, the two pieces of conductor of length $\mathrm{d}x$ work as a capacitor. Since the capacitance is proportional to the length of the conductor, we can write the capacitance as $C'\,\mathrm{d}x$, where C' is a capacitance per unit length. Thus, $\mathrm{d}Q = C'V\,\mathrm{d}x$. Using this result and assuming that C' is constant in time, equation (3.10) gives

$$C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}. \tag{3.11}$$

- The currents through the conductor produce time varying magnetic fields that in

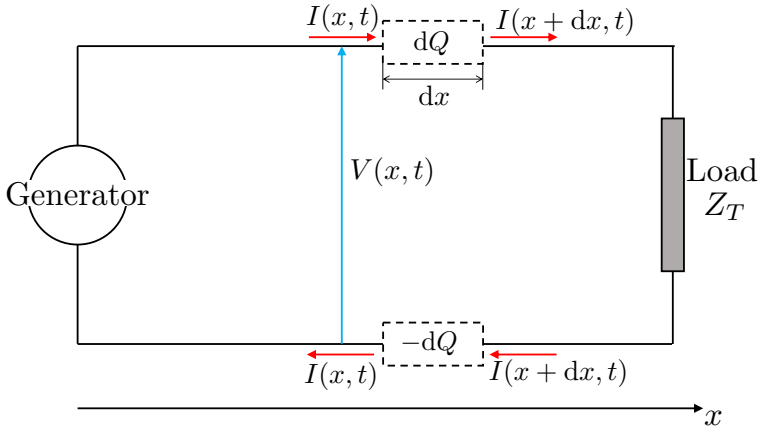


FIGURE 2. Transmission line with currents in red and potential difference in blue.

turn drive electromotive forces in the conductors. The resulting inductance is proportional to the length, and hence, for a piece of the transmission line of length $\mathrm{d}x$, it can be written as $L'\,\mathrm{d}x$, where L' is an inductance per unit length. The potential drop due to the inductance $L'\,\mathrm{d}x$ is

$$V(x) - V(x+\mathrm{d}x) = L'\frac{\partial I}{\partial t}\,\mathrm{d}x. \tag{3.12}$$

This equation then gives

$$-\frac{\partial V}{\partial x} = L'\frac{\partial I}{\partial t}. \tag{3.13}$$

Combining equations (3.11) and (3.13), we find a wave equation. For example, differentiating equation (3.11) with respect to time, we find

$$\frac{\partial^2 V}{\partial t^2} = -\frac{1}{C'}\frac{\partial^2 I}{\partial t \partial x}. \tag{3.14}$$

To calculate $\partial^2 I/\partial t\partial x$, we differentiate equation (3.13) with respect to x (assuming that L' is constant along the transmission line), finally obtaining

$$\frac{\partial^2 V}{\partial t^2} = c^2\frac{\partial^2 V}{\partial x^2}, \tag{3.15}$$

where the wave speed in this case is given by

$$c = \frac{1}{\sqrt{L'C'}}. \tag{3.16}$$

Thus, we can use the d'Alembert solution for the voltage,

$$V(x,t) = f_V(x-ct) + g_V(x+ct). \tag{3.17}$$

A similar set of manipulations can be used to show that the current $I(x,t)$ also satisfies the wave equation and can hence be written as

$$I(x,t) = f_I(x-ct) + g_I(x+ct). \tag{3.18}$$

Typically, transmission lines end at electric loads that are characterized by their complex impedance Z_T that relates a sinusoidal voltage of frequency ω with the sinusoidal current that this voltage induces in the load. We consider such a load at the end of a

semi-infinite transmission line that extends from $x = -\infty$ to $x = 0$. For a voltage

$$V(x = 0, t) = \frac{1}{2} V_0 e^{-\mathrm{i}\omega t} + \text{complex conjugate} \tag{3.19}$$

and a current

$$I(x = 0, t) = \frac{1}{2} I_0 e^{-\mathrm{i}\omega t} + \text{complex conjugate} \tag{3.20}$$

at $x = 0$, the load imposes

$$V_0 = Z_T I_0. \tag{3.21}$$

Let us assume that we have launched a right-traveling wave towards the load with a voltage of the form

$$f_V(x - ct) = \frac{1}{2} A e^{\mathrm{i}k(x-ct)} + \text{complex conjugate}. \tag{3.22}$$

The presence of the boundary condition (3.21) will induce a reflected wave

$$g_V(x + ct) = \frac{1}{2} R e^{-\mathrm{i}k(x+ct)} + \text{complex conjugate}. \tag{3.23}$$

To get the ratio R/A, we can use the useful concept of transmission impedance. This is the ratio of the voltage and the current of the right-traveling wave,

$$Z_L = \frac{f_V(x - ct)}{f_I(x - ct)}. \tag{3.24}$$

Using equation (3.11) and $c = 1/\sqrt{L'C'}$, we find

$$Z_L = \frac{1}{cC'} = \sqrt{\frac{L'}{C'}}. \tag{3.25}$$

The same ratio for the left-traveling wave gives an impedance with the opposite sign,

$$\frac{g_V(x + ct)}{g_I(x + ct)} = -Z_L. \tag{3.26}$$

Using these results in equation (3.21) gives

$$A + R = Z_T \left(\frac{A}{Z_L} - \frac{R}{Z_L} \right), \tag{3.27}$$

which can be solved to obtain

$$\frac{R}{A} = \frac{Z_T - Z_L}{Z_T + Z_L}. \tag{3.28}$$

Thus, we have found that in general loads reflect waves. These reflected waves are undesirable because they mean that not all the energy transmitted has been used by the load, and because they can damage the electric generator that produced the incoming wave. Thus, one needs to design the load such that $Z_T = Z_L$. This is known as impedance matching, and it is achieved by adding capacitors, inductors and resistors to the final load.

4. Transmission problems

Up to this point, we have considered wave propagation in homogeneous media. We will now start considering simple inhomogeneous systems. As an example, we proceed to

study a stretched string composed of two strings a and b of different linear density, μ_a and μ_b. These two strings are tied together at $x = 0$.

We know that the vertical motion of both strings, $y_a(x,t)$ and $y_b(x,t)$, is described by the wave equation, but the wave speed is different in each of them. Due to horizontal force balance at the knot, the tension has to be the same in both strings, giving the speeds $c_a = \sqrt{T/\mu_a}$ and $c_b = \sqrt{T/\mu_b}$. But in addition to knowing how each string moves, we need to state how they interact. Since they are tied together at $x = 0$, we must have continuity of vertical displacement,

$$y_a(x=0,t) = y_b(x=0,t). \tag{4.1}$$

Assuming that the knot is massless, vertical force balance at the knot gives

$$T\frac{\partial y_a}{\partial x}(x=0,t) = T\frac{\partial y_b}{\partial x}(x=0,t). \tag{4.2}$$

As an example of the motion in this two-string system, we consider a right-traveling wave

$$f_a(x-c_at) = \frac{1}{2}Ae^{\mathrm{i}k(x-c_at)} + \text{complex conjugate} \tag{4.3}$$

moving towards the knot at $x = 0$. The knot will reflect part of this wave, giving

$$y_a(x,t) = f_a(x-c_at) + g_a(x+c_at), \tag{4.4}$$

with

$$g_a(x+c_at) = \frac{1}{2}Re^{-\mathrm{i}k(x+c_at)} + \text{complex conjugate}. \tag{4.5}$$

The complex constant R still needs to be determined. The displacement of the knot at $x = 0$ will also launch a right-traveling wave in the second string,

$$y_b(x,t) = f_b(x-c_bt), \tag{4.6}$$

where

$$f_b(x-c_bt) = \frac{1}{2}Be^{\mathrm{i}k_b(x-c_bt)} + \text{complex conjugate}. \tag{4.7}$$

The wavenumber k_b and the complex constant B still need to be determined.

Imposing conditions (4.1) and (4.2), we will be able to find k_b, R and B. Indeed, condition (4.1) gives

$$Ae^{-\mathrm{i}kc_at} + Re^{-\mathrm{i}kc_at} = Be^{-\mathrm{i}k_bc_bt}. \tag{4.8}$$

Thus, the frequency of all the waves must coincide,

$$\omega = kc_a = k_bc_b, \tag{4.9}$$

giving

$$k_b = k\frac{c_a}{c_b} = k\sqrt{\frac{\mu_b}{\mu_a}}. \tag{4.10}$$

Then, condition (4.1) simplifies to

$$A + R = B \tag{4.11}$$

Condition (4.2) gives

$$kA - kR = k_bB \tag{4.12}$$

Using all these equations, we finally obtain

$$\frac{B}{A} = \frac{2c_b}{c_a+c_b} = \frac{2\sqrt{\mu_a}}{\sqrt{\mu_a}+\sqrt{\mu_b}} \tag{4.13}$$

and

$$\frac{R}{A} = \frac{c_b - c_a}{c_a + c_b} = \frac{\sqrt{\mu_a} - \sqrt{\mu_b}}{\sqrt{\mu_a} + \sqrt{\mu_b}}. \tag{4.14}$$

Thus, there is always a reflected wave and not all the energy travels through the knot.

It is instructive to consider energy balance in this problem. We have already seen that the work done by a stretched string is

$$-T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = \epsilon_f c - \epsilon_g c. \tag{4.15}$$

where $\epsilon_f := T(f')^2$ and $\epsilon_g := T(g')^2$. For sinusoidal waves

$$f(x - ct) = \frac{1}{2}A_f e^{\mathrm{i}k(x-ct)} + \text{complex conjugate} \tag{4.16}$$

and

$$g(x + ct) = \frac{1}{2}A_g e^{-\mathrm{i}k(x+ct)} + \text{complex conjugate}, \tag{4.17}$$

both ϵ_f and ϵ_g are periodic in time with period $\tau = 2\pi/\omega = 2\pi/kc$. Thus, it is convenient to use the time average

$$\langle F \rangle_\tau = \frac{1}{\tau}\int_t^{t+\tau} F(t')\,\mathrm{d}t'. \tag{4.18}$$

With this time average, the work done by the string becomes

$$-T\left\langle \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \right\rangle_\tau = \langle \epsilon_f \rangle_\tau c - \langle \epsilon_g \rangle_\tau c, \tag{4.19}$$

where

$$\langle \epsilon_f \rangle_\tau = \frac{1}{2}k^2 T|A_f|^2, \quad \langle \epsilon_g \rangle_\tau = \frac{1}{2}k^2 T|A_g|^2. \tag{4.20}$$

Applying this result to $x = 0$, where the work done by the string a should be equal to the work received by the string b, we find that

$$\frac{1}{2}k^2|A|^2 c_a - \frac{1}{2}k^2|R|^2 c_a = \frac{1}{2}k_b^2|B|^2 c_b, \tag{4.21}$$

that is, the energy of the incoming wave in string a must be equal to the energy of the reflected wave in string a and of the outgoing wave in string b. This equation can be checked to be satisfied using equations (4.10), (4.13) and (4.14).

Separation of variables and stationary waves

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1. Introduction

In these notes, we show how to obtain solutions for the wave equation with two boundary conditions without resorting to D'Alembert's solution. The new technique is known as separation of variables, and can simplify the problem significantly in some cases.

2. Separation of variables

The idea behind separation of variables is to reproduce the calculation that we have already performed for systems of ODEs. The motion of a system of oscillators is described by a system of ODEs of the form

$$\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}t^2} = \mathbf{D}\cdot\mathbf{q}, \tag{2.1}$$

where $\mathbf{q}(t)$ is the vector that contains the position of the different oscillators, and $\mathbf{D}$ is a constant square matrix. To solve system (2.1), we propose solutions of the form

$$\mathbf{q}(t) = \Theta(t)\mathbf{Q}, \tag{2.2}$$

where the vector $\mathbf{Q}$ does not depend on time. With this assumption, $\Theta(t)$ and $\mathbf{Q}$ must satisfy

$$\mathbf{D}\cdot\mathbf{Q} = \frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2}\mathbf{Q}. \tag{2.3}$$

Thus, $\mathbf{Q}$ must be an eigenvector of matrix $\mathbf{D}$, and $\Theta^{-1}(\mathrm{d}^2\Theta/\mathrm{d}t^2)$ the corresponding eigenvalue. For matrices $\mathbf{D}$ that have a complete basis of eigenvectors $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ with their corresponding eigenvalues $\{\Lambda_1,\Lambda_2,\ldots,\Lambda_n\}$, the most general solution is

$$\mathbf{q}(t) = \sum_{i=1}^{n}\left[A_i\cos(\sqrt{-\Lambda_i}t) + B_i\sin(\sqrt{-\Lambda_i}t)\right]\mathbf{e}_i, \tag{2.4}$$

where the constants A_i and B_i are determined by the initial conditions. Note that we have written the solution in a convenient form for negative eigenvalues Λ_i, but the expression is still valid for complex eigenvalues.

The wave equation can be written analogously to equation (2.1),

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}. \tag{2.5}$$

Here, the operator $c^2(\partial^2/\partial x^2)$ plays the role of matrix $\mathbf{D}$. We solved system (2.1) by assuming that all oscillators moved in unison, but with different amplitudes, as shown in equation (2.2). For the wave equation, we can use the continuum limit of this idea, that is, each position x has its corresponding oscillator that moves with amplitude $Q(x)$, giving

$$y(x,t) = \Theta(t)Q(x). \tag{2.6}$$

With this assumption, we can write equation (2.5) as

$$\frac{1}{c^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} = \frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}x^2}. \tag{2.7}$$

Note that here we are setting a function of t equal to a function of x. Since these are functions of different variables, they can only be equal if they are constant, that is,

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} = \Lambda c^2\Theta \tag{2.8}$$

and

$$\frac{\mathrm{d}^2Q}{\mathrm{d}x^2} = \Lambda Q, \tag{2.9}$$

where the constant Λ can only take a limited set of values – in complete analogy to the eigenvalues of matrix $\mathbf{D}$. To determine these special values of Λ that are eigenvalues of the operator $\partial^2/\partial x^2$, we need to consider the boundary conditions. We proceed to find the eigenvalues for a series of simple examples. You can also find an example of separation of variables with a more general boundary condition in Appendix A.

2.1. *Two Dirichlet boundary conditions*

We consider a stretched string that is held in place at $x = 0$ and $x = L$, that is, we use the boundary conditions

$$y(x = 0, t) = 0, \quad y(x = L, t) = 0. \tag{2.10}$$

For a solution of the form (2.6) to satisfy these boundary conditions, we need to impose

$$Q(0) = 0, \quad Q(L) = 0. \tag{2.11}$$

Solving for $Q(x)$ using equation (2.9), we find

$$Q(x) = C_+ e^{\sqrt{\Lambda}x} + C_- e^{-\sqrt{\Lambda}x}. \tag{2.12}$$

Imposing conditions (2.11), we obtain a linear system of equations for the unknown constants C_+ and C_-,

$$\begin{pmatrix} 1 & 1 \\ e^{\sqrt{\Lambda}L} & e^{-\sqrt{\Lambda}L} \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.13}$$

This linear system of equations for C_+ and C_- gives a nontrivial solution only if the matrix is singular. By setting the determinant of the matrix equal to zero, we find the equation

$$e^{2\sqrt{\Lambda}L} = 1. \tag{2.14}$$

Using $1 = e^{2\pi k\mathrm{i}}$, with k an integer number, this equation leads to

$$\Lambda = -\frac{k^2\pi^2}{L^2}. \tag{2.15}$$

Since k appears squared, we only need to consider $k \geqslant 0$ from here on.

In principle, k could be equal to zero, giving $\Lambda = 0$. For this value of Λ, the solution for $Q(x)$ is not equation (2.12), but $Q(x) = C + \widetilde{C}x$, where C and $\widetilde{C}$ are constants that we need to determine using boundary conditions (2.11). It is not possible to find $C \neq 0$ and $\widetilde{C} \neq 0$ that satisfy these boundary conditions, so $\Lambda = 0$ is not of interest. Thus, the final set of eigenvalues is

$$\Lambda_k = -\frac{k^2\pi^2}{L^2}, \tag{2.16}$$

with $k = 1, 2, 3, \ldots$ The corresponding eigenfunctions $Q_k(x)$ satisfy equation (2.9) and boundary conditions (2.11),

$$Q_k(x) = \sin\left(\frac{k\pi x}{L}\right). \tag{2.17}$$

The functions $Q_k(x)$ are defined up to a constant that can be absorbed into $\Theta(t)$. The functions $Q_k(x)$ are the **modes** of the system.

For eigenvalue $\Lambda_k = -k^2\pi^2/L^2$, we can solve equation (2.8) for $\Theta(t)$, finding

$$\Theta_k(t) = A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right). \tag{2.18}$$

The constants A_k and B_k must be determined by the initial conditions, as we will see shortly.

Combining all these results, we find that the most general solution to this problem is

$$y(x,t) = \sum_{k=1}^{\infty}\left[A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right)\right]\sin\left(\frac{k\pi x}{L}\right). \tag{2.19}$$

Note that this solution is periodic in time with period $\tau = 2L/c$.

We finish by showing how one imposes initial conditions. The initial conditions for the string are its position and velocity,

$$y(x, t=0) = y_0(x), \quad \frac{\partial y}{\partial t}(x, t=0) = \dot{y}_0(x). \tag{2.20}$$

It turns out that any function of x between 0 and L can be written as an infinite series of sines,

$$f(x) = \sum_{k=1}^{\infty} F_k \sin\left(\frac{k\pi x}{L}\right), \tag{2.21}$$

and the coefficients $\{F_1, F_2, F_3, \ldots\}$ of this decomposition are unique, that is, there is no other set of coefficients $\{F_1', F_2', F_3', \ldots\}$ that gives the same function $f(x)$. Next year, in Mathematical Methods, you will learn how to calculate these coefficients from the function $f(x)$. For now, it is sufficient to know that they exist and that they are unique. Thus, we can write $y_0(x)$ and $\dot{y}_0(x)$ as

$$y_0(x) = \sum_{k=1}^{\infty} Y_k \sin\left(\frac{k\pi x}{L}\right), \quad \dot{y}_0(x) = \sum_{k=1}^{\infty} \dot{Y}_k \sin\left(\frac{k\pi x}{L}\right). \tag{2.22}$$

With these decompositions of $y_0(x)$ and $\dot{y}_0(x)$, we can impose initial conditions (2.20) on the solution (2.19) to find

$$A_k = Y_k, \quad \frac{k\pi c}{L} B_k = \dot{Y}_k. \tag{2.23}$$

For this course, you will usually be given initial conditions that are finite sums of sines so that it is easy for you to identify the coefficients Y_k and $\dot{Y}_k$. For example, for

$$y(x, t=0) = h \sin\left(\frac{\pi x}{L}\right), \quad \frac{\partial y}{\partial t}(x, t=0) = 0, \tag{2.24}$$

the coefficients $\dot{Y}_k$ are zero for all values of k, and the coefficients Y_k are zero for $k \neq 1$. For $k = 1$, $Y_1 = h$, giving the final solution

$$y(x,t) = h \cos\left(\frac{\pi ct}{L}\right) \sin\left(\frac{\pi x}{L}\right). \tag{2.25}$$

2.2. *Two Neumann boundary conditions*

In this section, we consider a stretched string that slides along frictionless columns at $x = 0$ and $x = L$,

$$\frac{\partial y}{\partial x}(x = 0, t) = 0, \quad \frac{\partial y}{\partial x}(x = L, t) = 0. \tag{2.26}$$

Following the procedure above for this case gives the general solution

$$y(x, t) = A_0 + \widetilde{A}_0 t + \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right) \right] \cos\left(\frac{k\pi x}{L}\right). \tag{2.27}$$

In this case, the string can slide freely along the columns, as demonstrated by the term linear in time. Apart from this linear term, the solution is again periodic in time with period $\tau = 2L/c$.

2.3. *One Dirichlet boundary condition and one Neumann boundary condition*

As an example of a wave equation with two different boundary conditions, we consider a stretched string that is held in place at $x = 0$ and slides along a frictionless column at $x = L$,

$$y(x = 0, t) = 0, \quad \frac{\partial y}{\partial x}(x = L, t) = 0. \tag{2.28}$$

In this case, separation of variables gives the general solution

$$y(x, t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{(k - 1/2)\pi ct}{L}\right) + B_k \sin\left(\frac{(k - 1/2)\pi ct}{L}\right) \right] \sin\left(\frac{(k - 1/2)\pi x}{L}\right). \tag{2.29}$$

This solution is periodic in time with period $\tau = 4L/c$.

3. Stationary waves

The solutions that we have found using separation of variables are stationary waves: they are oscillating patterns that do not move along x.

It is instructive to see how to construct stationary waves from the D'Alembert solution. The stationary wave solutions that we have found are of the form $\sin(kx+\theta)\sin(kct+\varphi)$, and they can be constructed from left- and right-traveling sinusoidal waves of the same amplitude. Indeed, for $f(x - ct) = A\cos(k(x - ct) + \theta - \varphi)$ and $g(x + ct) = -A\cos(k(x + ct) + \theta + \varphi)$, we find

$$y(x, t) = f(x - ct) + g(x + ct) = 2A\sin(kx + \theta)\sin(kct + \varphi). \tag{3.1}$$

The time-averaged energy fluxes due to the left- and right-traveling waves cancel each other because the two waves have the same amplitude. Indeed, using the time average over one period,

$$\langle \ldots \rangle_\tau = \frac{kc}{2\pi} \int_t^{t+2\pi/kc} (\ldots)\, \mathrm{d}t', \tag{3.2}$$

the time averaged energy fluxes are $\langle \epsilon_f \rangle_\tau c = Tk^2A^2c/2$ and $-\langle \epsilon_g \rangle_\tau c = -Tk^2A^2c/2$. Thus, these fluxes cancel at every point x, ensuring that the energy does not grow or decay at any given point on average, although there are instantaneous changes in the energy as it sloshes from one x location to another.

4. Energy and modes

The modes that we have calculated in the examples in sections 2.1, 2.2 and 2.3 have one surprising property: the total energy contained in any wave can be split into the sum of the energies in each mode despite the fact that the energy is a quadratic quantity. This is a consequence of the simple boundary conditions that we have imposed.

The solutions that we have found in sections 2.1, 2.2 and 2.3 are of the form

$$y(x,t) = A_0 + \widetilde{A}_0 t + \sum_{k=1}^{\infty} C_k \cos\left(\sqrt{-\Lambda_k}ct - \varphi_k\right) Q_k(x), \tag{4.1}$$

where the functions $Q_k(x)$ satisfy equation

$$\frac{\mathrm{d}^2 Q_k}{\mathrm{d}x^2} = \Lambda_k Q_k \tag{4.2}$$

and boundary conditions of the type $Q_k = 0$ or $\mathrm{d}Q_k/\mathrm{d}x = 0$ at $x = 0$ and $x = L$. For example, solution (2.19) can be recovered with $A_0 = 0$, $\widetilde{A}_0 = 0$, $C_k = \sqrt{A_k^2 + B_k^2}$, $\tan\varphi_k = B_k/A_k$, $\Lambda_k = -k^2\pi^2/L^2$ and $Q_k(x) = \sin(k\pi x/L)$.

If $y(x,t)$ is the transverse displacement of a stretched string of linear density μ and tension T, the energy contained in the string motion is

$$E = \frac{\mu}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x + \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. \tag{4.3}$$

Integrating the second term by parts, we find

$$E = \int_0^L \left[\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{Ty}{2}\frac{\partial^2 y}{\partial x^2}\right] \mathrm{d}x + \frac{T}{2}\left[y\frac{\partial y}{\partial x}\right]_{x=0}^{x=L}. \tag{4.4}$$

The combination $y(\partial y/\partial x)$ vanishes due to the boundary conditions imposed on $Q_k(x)$ at $x = 0$ and $x = L$. Using this fact and equation (4.2), we rewrite the energy as

$$\begin{aligned} E = \int_0^L \Bigg[\frac{\mu}{2}\Bigg(\widetilde{A}_0 - c\sum_{k=1}^{\infty}\sqrt{-\Lambda_k}C_k \sin\left(\sqrt{-\Lambda_k}ct - \varphi_k\right) Q_k(x)\Bigg)^2 \\ - \frac{T}{2}\left(\sum_{k=1}^{\infty} C_k \cos\left(\sqrt{-\Lambda_k}ct - \varphi_k\right) Q_k(x)\right)\left(\sum_{l=1}^{\infty} \Lambda_l C_l \cos\left(\sqrt{-\Lambda_l}ct - \varphi_l\right) Q_l(x)\right)\Bigg] \mathrm{d}x. \end{aligned} \tag{4.5}$$

If equation (4.5) is expanded, we find integrals of the form $\widetilde{A}_0 C_k \int_0^L Q_k(x)\,\mathrm{d}x$ and $C_k C_l \int_0^L Q_k(x)Q_l(x)\,\mathrm{d}x$, giving a complicated expression for the energy E. This is hardly surprising given the quadratic nature of the energy. What is surprising is that the modes satisfy $\widetilde{A}_0 \int_0^L Q_k(x)\,\mathrm{d}x = 0$ and $\int_0^L Q_k(x)Q_l(x)\,\mathrm{d}x = 0$. This result can be checked by direct integration, or using equation (4.2), as we proceed to show. Multiplying equation (4.2) by $Q_l(x)$ and integrating over x, we find

$$\int_0^L Q_l \frac{\mathrm{d}^2 Q_k}{\mathrm{d}x^2}\,\mathrm{d}x = \Lambda_k \int_0^L Q_l Q_k\,\mathrm{d}x. \tag{4.6}$$

Integrating the term on the left side of the equation by parts, we obtain

$$\left[Q_l\frac{\mathrm{d}Q_k}{\mathrm{d}x}\right]_{x=0}^{x=L} - \int_0^L \frac{\mathrm{d}Q_l}{\mathrm{d}x}\frac{\mathrm{d}Q_k}{\mathrm{d}x}\,\mathrm{d}x = \Lambda_k \int_0^L Q_l Q_k\,\mathrm{d}x. \tag{4.7}$$

The combination $Q_l(\mathrm{d}Q_k/\mathrm{d}x)$ vanishes at $x=0$ and $x=L$ due to the boundary conditions, leaving

$$-\int_0^L \frac{\mathrm{d}Q_l}{\mathrm{d}x}\frac{\mathrm{d}Q_k}{\mathrm{d}x}\,\mathrm{d}x = \Lambda_k \int_0^L Q_l Q_k\,\mathrm{d}x. \tag{4.8}$$

We obtained this expression by multiplying equation (4.2) for $Q_k(x)$ by $Q_l(x)$ and integrating over x. One can multiply the equation for $Q_l(x)$ by $Q_k(x)$ and integrate over x to find the alternative equation

$$-\int_0^L \frac{\mathrm{d}Q_l}{\mathrm{d}x}\frac{\mathrm{d}Q_k}{\mathrm{d}x}\,\mathrm{d}x = \Lambda_l \int_0^L Q_l Q_k\,\mathrm{d}x. \tag{4.9}$$

Note that the left sides of equations (4.8) and (4.9) are the same, and thus, if we subtract one equation from the other, we find

$$(\Lambda_k-\Lambda_l)\int_0^L Q_l Q_k\,\mathrm{d}x = 0. \tag{4.10}$$

Thus,

$$\int_0^L Q_l Q_k\,\mathrm{d}x = 0 \quad \text{for } k\neq l. \tag{4.11}$$

This equation also proves that $\widetilde{A}_0\int_0^L Q_k(x)\,\mathrm{d}x$ vanishes. The coefficient $\widetilde{A}_0$ is different from zero only when $\Lambda_0=0$ is an eigenvalue and hence $Q_0(x)=1$ is a mode. As a result, either $\widetilde{A}_0=0$ or $\int_0^L Q_0Q_k\,\mathrm{d}x=\int_0^L Q_k\,\mathrm{d}x=0$ for $k\geqslant 1$, giving

$$\widetilde{A}_0\int_0^L Q_k\,\mathrm{d}x = 0 \quad \text{for } k\geqslant 1. \tag{4.12}$$

Using equations (4.11) and (4.12), the cross-terms in equation (4.5) for the energy vanish, giving the much simpler result

$$\begin{aligned} E =& \frac{\mu}{2}\left(\widetilde{A}_0^2 L - c^2\sum_{k=1}^{\infty}\Lambda_k C_k^2 \sin^2\left(\sqrt{-\Lambda_k}ct-\varphi_k\right)\int_0^L Q_k^2(x)\,\mathrm{d}x\right) \\ &- \frac{T}{2}\sum_{k=1}^{\infty}\Lambda_k C_k^2\cos^2\left(\sqrt{-\Lambda_k}ct-\varphi_k\right)\int_0^L Q_k^2(x)\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Using $c=\sqrt{T/\mu}$ finally gives the explicitly time-independent expression

$$E = \frac{\mu L\widetilde{A}_0^2}{2} - \frac{T}{2}\sum_{k=1}^{\infty}\Lambda_k C_k^2\int_0^L Q_k^2(x)\,\mathrm{d}x. \tag{4.14}$$

Thus, the total energy is simply the sum of the energy of each mode. Expression (4.14) can be further simplified by realizing that the functions $Q_k(x)$ in sections 2.1, 2.2 and 2.3 satisfy $\int_0^L Q_k^2(x)\,\mathrm{d}x = L/2$, finally leading to

$$E = \frac{\mu L\widetilde{A}_0^2}{2} - \frac{TL}{4}\sum_{k=1}^{\infty}\Lambda_k C_k^2. \tag{4.15}$$

For the example in equation (2.19), with $\widetilde{A}_0=0$, $\Lambda_k=-k^2\pi^2/L^2$ and $C_k=\sqrt{A_k^2+B_k^2}$, the energy is then

$$E = \frac{\pi^2 T}{4L}\sum_{k=1}^{\infty}k^2(A_k^2+B_k^2). \tag{4.16}$$

As we pointed out at the start of this section, the split of the energy into a sum of the energies of the individual modes was possible due to the simple boundary conditions considered in sections 2.1, 2.2 and 2.3. For more complex boundary conditions, one needs to carefully choose what to include or exclude in the definition of the energy of the system. See the end of Appendix A for an example of these more general boundary conditions.

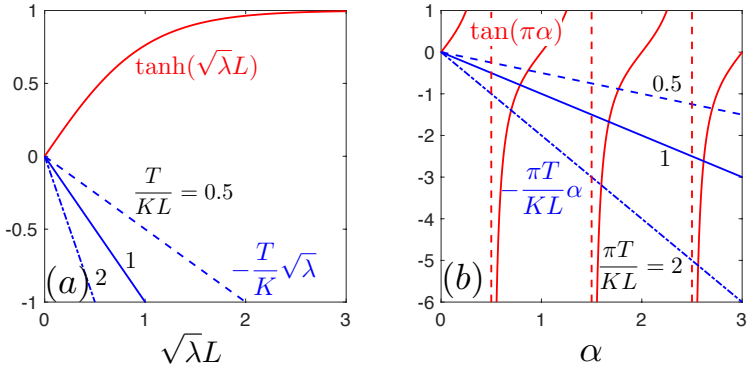


FIGURE 1. Solutions to equation (A 4). (a) For $\Lambda > 0$, the solutions are intersections between the curves $\tanh(\sqrt{\Lambda}L)$ (in red) and $-(T/K)\sqrt{\Lambda}$ (in blue for different values of T/KL: dashed line for $T/KL = 0.5$, solid for $T/KL = 1$ and dash-dot for $T/KL = 2$). (b) For $\Lambda < 0$, we use $\alpha = \sqrt{-\Lambda}L/\pi$. The solutions are intersections between the curves $\tan(\pi\alpha)$ (in red) and $-(\pi T/KL)\alpha$ (in blue for different values of $\pi T/KL$: dashed line for $\pi T/KL = 0.5$, solid for $\pi T/KL = 1$ and dash-dot for $\pi T/KL = 2$).

Appendix A. Example of separation of variables for a general boundary condition

We finish these notes by considering a situation in which the eigenvalues and modes are not as simple as the ones described so far: a stretched string that is held in place at $x = 0$ and that, at $x = L$, is knotted to a frictionless column and to a spring of constant K,

$$y(x = 0, t) = 0, \quad \frac{\partial y}{\partial x}(x = L, t) + \frac{K}{T} y(x = L, t) = 0. \tag{A 1}$$

For assumption (2.6) to work, we need to impose

$$Q(0) = 0, \quad \frac{\mathrm{d}Q}{\mathrm{d}x}(L) + \frac{K}{T} Q(L) = 0. \tag{A 2}$$

Using solution (2.12) for $Q(x)$, we find that these boundary conditions lead to the linear system of equations

$$\begin{pmatrix} 1 & 1 \\ (\sqrt{\Lambda} + K/T)e^{\sqrt{\Lambda}L} & (-\sqrt{\Lambda} + K/T)e^{-\sqrt{\Lambda}L} \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A 3}$$

for the constants C_+ and C_-. This system of equations gives nontrivial solutions only when the matrix is singular. Requiring that the determinant of the matrix vanishes leads to

$$\tanh(\sqrt{\Lambda}L) = -\frac{T}{K}\sqrt{\Lambda}. \tag{A 4}$$

One can show that Λ is real. Thus, we need to distinguish two cases to solve this equation:

- **Positive Λ.** For positive values of Λ, the solutions to equation (A 4) are the intersections between a hyperbolic tangent and a straight line, as shown in figure 1(a). Only $\Lambda = 0$ is a solution, and as we have seen, the solution for this value of Λ is $Q(x) = C + \widetilde{C}x$. Since it is not possible to find values $C \neq 0$ and $\widetilde{C} \neq 0$ that satisfy the boundary conditions, Λ is in fact not an eigenvalue.
- **Negative Λ.** For negative values of Λ, we can rewrite equation (A 4) using the new

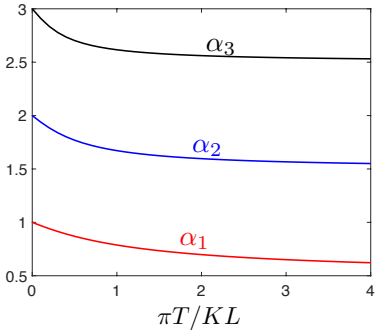


FIGURE 2. Coefficients α_1, α_2 and α_3 for different values of $\pi T/KL$.

variable $\alpha = \sqrt{-\Lambda}L/\pi$,

$$\tan(\pi\alpha) = -\frac{\pi T}{KL}\alpha. \tag{A 5}$$

The solutions to this equation are then the intersections between $\tan(\pi\alpha)$ and a straight line, as shown in figure 1(b). There are infinitely many solutions $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. For example, for $\pi T/KL = 1$, we find

$$\alpha_1 = 0.788, \quad \alpha_2 = 1.672, \quad \alpha_3 = 2.616, \quad \alpha_4 = 3.587, \ldots \tag{A 6}$$

In figure 2 we plot how α_1, α_2 and α_3 depend on $\pi T/KL$. For $\pi T/KL = 0$, we find $\alpha_k = k$, whereas for $\pi T/KL \to \infty$, $\alpha_k \to k - 1/2$.

As a result of the discussion above, we find that the eigenvalues are

$$\Lambda_k = -\frac{\alpha_k^2 \pi^2}{L^2}, \tag{A 7}$$

with $k = 1, 2, 3, \ldots$ The corresponding general solution is then

$$y(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{\alpha_k \pi c t}{L}\right) + B_k \sin\left(\frac{\alpha_k \pi c t}{L}\right) \right] \sin\left(\frac{\alpha_k \pi x}{L}\right). \tag{A 8}$$

In this system, the energy that splits into a nice sum of the energies of the modes must include the potential energy of the spring at $x = L$,

$$E = \frac{\mu}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x + \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}Ky^2(x = L, t) \tag{A 9}$$

$$= \frac{\pi^2 T}{2L^2} \sum_{k=1}^{\infty} \alpha_k^2 (A_k^2 + B_k^2) \int_0^L \sin^2\left(\frac{\alpha_k \pi x}{L}\right) \mathrm{d}x. \tag{A 10}$$

The energy of the stretched string alone, $(\mu/2)\int_0^L (\partial y/\partial t)^2 \,\mathrm{d}x + (T/2)\int_0^L (\partial y/\partial x)^2 \,\mathrm{d}x$, does not satisfy this property. It is easy to see that this is the case: the energy of the spring

$$\frac{1}{2}Ky^2(x = L, t) = \frac{K}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left[A_k \cos\left(\frac{\alpha_k \pi c t}{L}\right) + B_k \sin\left(\frac{\alpha_k \pi c t}{L}\right) \right]$$
$$\times \left[A_l \cos\left(\frac{\alpha_l \pi c t}{L}\right) + B_l \sin\left(\frac{\alpha_l \pi c t}{L}\right) \right] \sin(\alpha_k \pi) \sin(\alpha_l \pi) \tag{A 11}$$

cannot be split into a sum of the energies of the modes and hence the energy of the stretched string, given by $E - Ky^2(x = L, t)/2$, cannot either.

Dispersive waves, phase velocity and group velocity

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1. Introduction

In these notes, we use sinusoidal waves to solve linear PDEs. The resulting waves do not behave as the solution to the wave equation, and wavepackets are one interesting instance in which the difference between general sinusoidal solutions and the solutions to the wave equation is very clear. By studying the behavior of wavepackets, we will learn about phase and group velocities.

2. Dispersive waves and dispersion relations

We have used sinusoidal solutions of the form $e^{\mathrm{i}kx-\mathrm{i}\omega t}$ to solve the wave equation. We found that, to be solutions of the wave equation, the frequency of these sinusoidal waves had to satisfy $\omega(k) = \pm kc$. Importantly, solutions of the form $e^{\mathrm{i}kx-\mathrm{i}\omega t}$ are not only solutions to the wave equation, but they can be solutions to a variety of systems of linear PDEs as long as the frequency has the right value, $\omega(k)$, determined by the PDEs. The relation between the frequency ω and the wavenumber k is known as **dispersion relation**. To show that we can use sinusoidal waves for a variety of linear PDEs, we consider two examples:

- to demonstrate that sinusoidal waves are solutions to linear PDEs with constant coefficients, we solve for the corrected motion of a stretched string, and
- to show that dispersion relations arise even in systems of linear PDEs without constant coefficients, we discuss the motion of the interface surface between a liquid and the atmosphere.

2.1. *Bending correction to the motion of a stretched string*

For the motion of a stretched string of linear density μ and tension T, one of the approximations that led to the wave equation was that the force exerted by the rest of the string on an infinitesimal piece of string can only point along the string. In reality, there is always a small component of the force perpendicular to the string, $S(x,t)$, as shown in figure 1. This force resists bending, that is, it opposes changes of the angle $\theta(x,t) \simeq \partial y/\partial x$ with x. By introducing the force $S(x,t)$, we prevent the appearance of sharp corners in $y(x,t)$.

The force $S(x,t)$ is related to the derivatives of $y(x,t)$ by

$$S(x,t) = -TL_b^2 \frac{\partial^3 y}{\partial x^3}, \tag{2.1}$$

where L_b is a length that depends on the tension T, properties of the material and the shape of the cross section of the string. See Appendix A for a very brief discussion of how to obtain equation (2.1).

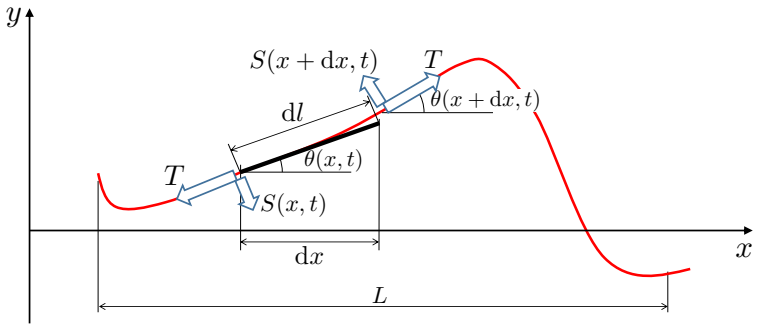


FIGURE 1. Transverse oscillations of a stretched string of linear density μ tensioned by a force T including the small forces $S(x,t)$ perpendicular to the string.

When we keep $S(x,t)$ in the equations for the motion of the string, vertical force balance of the infinitesimal piece of string in figure 1 gives

$$\mu\,\mathrm{d}x\frac{\partial^2 y}{\partial t^2} = T\sin\theta(x+\mathrm{d}x,t) + S(x+\mathrm{d}x,t)\cos\theta(x+\mathrm{d}x,t) - T\sin\theta(x,t) - S(x,t)\cos\theta(x,t). \tag{2.2}$$

Using $\theta(x,t) \ll 1$, we find

$$\mu\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\theta + S\right). \tag{2.3}$$

Employing $\theta \simeq \partial y/\partial x$ and equation (2.1) for the force $S(x,t)$, we find the modified wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} - L_b^2\frac{\partial^4 y}{\partial x^4}\right), \tag{2.4}$$

where $c = \sqrt{T/\mu}$.

Equation (2.4) does not have a d'Alembert solution, but it can be solved by using the assumption $y(x,t) \propto e^{\mathrm{i}kx-\mathrm{i}\omega t}$. With this assumption for $y(x,t)$, differentiation becomes an algebraic operation,

$$\frac{\partial y}{\partial t} = -\mathrm{i}\omega y, \quad \frac{\partial y}{\partial x} = \mathrm{i}k y, \tag{2.5}$$

and since the coefficients c and L_b are constants, the PDE in equation (2.4) gives

$$\left[k^2c^2(1+k^2L_b^2) - \omega^2\right] y(x,t) = 0. \tag{2.6}$$

Thus, the sinusoidal wave is a solution if ω satisfies

$$\omega(k) = \pm kc\sqrt{1+k^2L_b^2}. \tag{2.7}$$

The sinusoidal waves that we have found move rigidly with a velocity known as **phase velocity** v_p, $y(x,t) = f(x-v_pt)$. Indeed, for a general dispersion relation $\omega(k)$, we find

$$y(x,t) = Ce^{\mathrm{i}kx-\mathrm{i}\omega(k)t} = Ce^{\mathrm{i}k(x-v_p(k)t)}, \tag{2.8}$$

where the phase velocity is

$$v_p(k) = \frac{\omega(k)}{k}. \tag{2.9}$$

In the case of the wave equation, we find that $v_p(k) = \pm c$ does not depend on k, and all sinusoidal waves, regardless of their wavenumber, move at the same velocity. For waves

such as the ones represented by the dispersion relation (2.7), however, the phase velocity depends on k, and waves with different wavenumber k move at different speeds. Thus, a shape constructed by adding several sinusoidal waves will not keep its shape while moving, but will disperse in a collection of waves that move at different speeds. For this reason, waves that do not have a phase velocity independent of the wavenumber are known as **dispersive waves**.

Note that equation (2.7) contains two possible branches for $\omega(k)$, given by

$$\omega_+(k) = kc\sqrt{1+k^2L_b^2}, \quad \omega_-(k) = -kc\sqrt{1+k^2L_b^2}. \tag{2.10}$$

These two branches represent two waves traveling in opposite direction. We have constructed the two branches of the solution in (2.10) to satisfy the property

$$\omega(-k) = -\omega^*(k), \tag{2.11}$$

where * indicates complex conjugation. This condition is satisfied by any dispersion relation that solves a linear PDE with real coefficients. The solutions to the equation have to be real, and to be able to construct real solutions using the wave $e^{\mathrm{i}kx-\mathrm{i}\omega(k)t}$, its complex conjugate $e^{-\mathrm{i}kx+\mathrm{i}\omega^*(k)t}$ has to be a solution to the PDE. This is only possible if condition (2.11) is satisfied.

The two different waves in the dispersion relation (2.10) are required when imposing the two necessary initial conditions: displacement and velocity of the wave. Indeed, consider the two initial conditions

$$y(x,t=0) = \frac{1}{2}Ye^{\mathrm{i}kx} + \text{complex conjugate} \tag{2.12}$$

and

$$\frac{\partial y}{\partial t}(x,t=0) = \frac{1}{2}\dot{Y}e^{\mathrm{i}kx} + \text{complex conjugate}, \tag{2.13}$$

where Y and $\dot{Y}$ are two complex constants. The solution for $t>0$ will only have the two waves

$$y(x,t) = \frac{1}{2}\left(C_+e^{\mathrm{i}kx-\mathrm{i}\omega_+(k)t} + C_-e^{\mathrm{i}kx-\mathrm{i}\omega_-(k)t}\right) + \text{complex conjugate}. \tag{2.14}$$

The constants C_+ and C_- are determined by the initial conditions,

$$\begin{pmatrix} 1 & 1 \\ -\mathrm{i}\omega_+(k) & -\mathrm{i}\omega_-(k) \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix}. \tag{2.15}$$

Solving this system of equations, we find the solution

$$y(x,t) = \frac{1}{2}\left(\frac{\mathrm{i}\omega_-(k)Y+\dot{Y}}{\mathrm{i}\omega_-(k)-\mathrm{i}\omega_+(k)}e^{\mathrm{i}kx-\mathrm{i}\omega_+(k)t} - \frac{\dot{Y}+\mathrm{i}\omega_+(k)Y}{\mathrm{i}\omega_-(k)-\mathrm{i}\omega_+(k)}e^{\mathrm{i}kx-\mathrm{i}\omega_-(k)t}\right) + \text{complex conjugate}. \tag{2.16}$$

In complete analogy to this initial condition problem, boundary conditions can be imposed and transmission problems can be solved using sinusoidal waves with the appropriate dispersion relation $\omega(k)$. In the particular case of PDE (2.4), we need four boundary conditions (as many conditions as the order of the highest order derivative with respect to x in the equation). By symmetry, two of these boundary conditions would have to be imposed on one end of the stretched string, and the other two on the other end. The treatment of boundary conditions and transmission problems for dispersive waves is outside of the scope of this course, so we will stop here and consider a different type of problem.

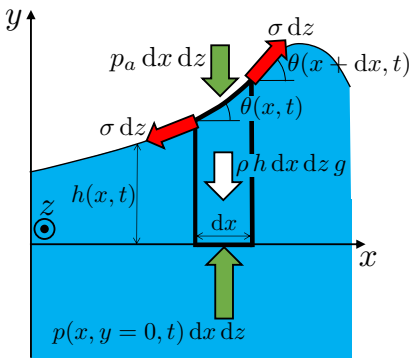


FIGURE 2. Interface $h(x,t)$ between a liquid and the atmosphere. The z axis is pointing towards the reader. The thick black line is a volume of liquid of height $h(x,t)$ and base $\mathrm{d}x\,\mathrm{d}z$ (the extent in the z direction is not represented). The forces with vertical components applied on this volume are gravity (in white), pressure (in green) and surface tension (in red).

2.2. *Gravity waves with surface tension*

Dispersion relations can also be found for more complex problems. As an example, we consider a liquid of density ρ that has a free surface in contact with the atmosphere at or near $y=0$, as shown in figure 2. The atmosphere is at a pressure p_a.

We proceed to find the equations for small displacements $h(x,t)$ of the free surface around $y=0$. At $y=0$, pressure not only opposes the atmospheric pressure p_a, but also sustains the weight of the liquid above it and balances the surface tension. The surface tension σ is a force per unit length that appears at the interface between different fluids and points along the interface – see, for example, the surface tension on the infinitesimal piece of interface highlighted in figure 2. For a water-air interface, surface tension is $\sigma = 7.4\times10^{-2}\,\mathrm{N/m}$. To calculate the pressure $p(x,y,t)$ at $y=0$, we apply vertical force balance to the volume of height $h(x,t)$ and base $\mathrm{d}x\,\mathrm{d}z$ shown in figure 2,

$$p(x,y=0,t)\,\mathrm{d}x\,\mathrm{d}z - p_a\,\mathrm{d}x\,\mathrm{d}z - \rho g h(x,t)\,\mathrm{d}x\,\mathrm{d}z + \sigma\sin\theta(x+\mathrm{d}x,t)\,\mathrm{d}z \\ -\sigma\sin\theta(x,t)\,\mathrm{d}z = 0. \tag{2.17}$$

Note that we have neglected inertia because it is quadratic in the small displacement $h(x,t)$ – it scales as $h(\partial h/\partial t)$. For small $h(x,t)$, $\theta(x,t)\simeq\partial h/\partial x \ll 1$, giving the result

$$p(x,y=0,t) = p_a + \rho g h(x,t) - \sigma\frac{\partial^2 h}{\partial x^2}(x,t). \tag{2.18}$$

The non-uniform pressure at $y=0$ pushes the liquid to move horizontally. Indeed, considering an infinitesimal volume of liquid of size $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ at $y=0$, shown in figure 3, we find that the horizontal forces give

$$\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,a_x(x,y=0,t) = p(x,y=0,t)\,\mathrm{d}y\,\mathrm{d}z - p(x+\mathrm{d}x,y=0,t)\,\mathrm{d}y\,\mathrm{d}z, \tag{2.19}$$

where $a_x(x,y=0,t)$ is the acceleration in the x direction of the infinitesimal volume. Replacing equation (2.18) into equation (2.19), we obtain the equation for the horizontal acceleration

$$a_x(x,y=0,t) = -\frac{\partial}{\partial x}\left(g h(x,t) - \frac{\sigma}{\rho}\frac{\partial^2 h}{\partial x^2}(x,t)\right). \tag{2.20}$$

Due to this acceleration, the infinitesimal volume of liquid that we are considering moves in the x direction, displacing other parts of the liquid in its path and ultimately affecting the shape of the interface $h(x,t)$. To find the equations for the evolution of the interface

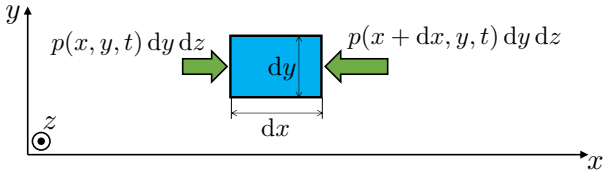


FIGURE 3. Horizontal forces on an infinitesimally small volume of liquid (the extent in the z direction is not represented; the z axis is pointing towards the reader).

shape, we need to relate the acceleration $a_x(x, y = 0, t)$ to the interface shape $h(x, t)$, and to obtain this relationship, we need to solve the 2D fluid motion below the surface, described by a system of PDEs given in Appendix B. The resulting oscillations of the interface are known as **gravity waves**.

The problem might seem daunting, but it can be simplified significantly. The coefficients of the problem depend on y (they are discontinuous at the interface), but they do not depend on x or t. Hence, by using the assumption $h(x, t) \propto e^{\mathrm{i}kx - \mathrm{i}\omega t}$, we eliminate all the derivatives with respect to x and t, and equation (2.20) gives

$$a_x(x, y = 0, t) = -\mathrm{i}k \left(g + \frac{\sigma k^2}{\rho} \right) h(x, t). \tag{2.21}$$

Solving the 2D fluid motion below the surface (see Appendix C) gives the unsurprising result that the horizontal acceleration of the fluid at $y = 0$ is related to the vertical acceleration of the interface $\partial^2 h/\partial t^2 = -\omega^2 h$,

$$a_x(x, y = 0, t) = -\frac{\mathrm{i}k}{|k|} \omega^2 h(x, t). \tag{2.22}$$

With this result, equation (2.21) gives the dispersion relation

$$\omega(k) = \pm \sqrt{g|k| + \frac{\sigma |k|^3}{\rho}}. \tag{2.23}$$

Note that we got a dispersion relation despite the fact that the problem was not uniform in the y direction.

There are many other examples of linear PDEs that lead to dispersion relations (Schrödinger's equation for the motion of a quantum particle, electromagnetic waves in a wave guide, electromagnetic waves in an ionized gas...). We will now focus on wavepackets.

3. Wavepackets: phase velocity and group velocity

We have already established that, unlike the solutions to the wave equation, dispersive waves do not maintain its initial shape while they move. There is a particular case in which analytical progress can be made: wavepackets. Wavepackets are composed of a carrier wave with wavelength k_c and an envelope that can have any shape (see the example in figure 4).

At $t = 0$, a wavepacket has the form

$$y(x, t = 0) = E(x) \cos(k_c x + \varphi), \tag{3.1}$$

where $E(x)$ is the envelope. We stated (without proof) that most functions can be ex-

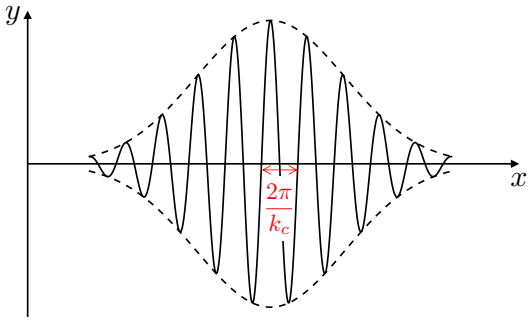


FIGURE 4. Example of wavepacket (solid black line) with a carrier wave of wavelength k_c. The envelope is the dashed black line.

pressed as sums or integrals of sines and cosines. Thus,

$$E(x) = \int C(K) e^{\mathrm{i}Kx} \,\mathrm{d}K, \tag{3.2}$$

where the coefficients $C(K)$ have to satisfy the property $C(-K) = C^*(K)$ to ensure that $E(x)$ is real. With this form for $E(x)$, the wavepacket can be written as

$$y(x, t = 0) = \frac{1}{2} \int C(K) e^{\mathrm{i}(k_c+K)x+\mathrm{i}\varphi} + \text{complex conjugate}, \tag{3.3}$$

and hence the solution for $t > 0$ is

$$y(x, t) = \frac{1}{2} \int C(K) e^{\mathrm{i}(k_c+K)x-\mathrm{i}\omega(k_c+K)t+\mathrm{i}\varphi} + \text{complex conjugate}. \tag{3.4}$$

This solution can be rewritten as

$$y(x, t) = \frac{1}{2} e^{\mathrm{i}k_c x-\mathrm{i}\omega(k_c)t+\mathrm{i}\varphi} \int C(K) e^{\mathrm{i}Kx-\mathrm{i}\Omega(K)t} \,\mathrm{d}K + \text{complex conjugate}, \tag{3.5}$$

where we have used the shorthand $\Omega(K) = \omega(k_c + K) - \omega(k_c)$. In general, the integral $\int C(K) e^{\mathrm{i}Kx-\mathrm{i}\Omega(K)t+\mathrm{i}\varphi} \,\mathrm{d}K$ gives a complex function $F(x, t)e^{\mathrm{i}\psi(x,t)}$, and the solution for $t > 0$ is $y(x, t) = F(x, t) \cos(k_c x - \omega(k_c)t + \varphi + \psi(x, t))$. This solution will rapidly become unrecognizable as a wavepacket if the function $\psi(x, t)$ becomes large. Fortunately, there is a limit in which this does not happen.

For an envelope with a characteristic length much longer than the wavelength of the carrier wave k_c,

$$\frac{1}{k_c} \left| \frac{1}{E} \frac{\mathrm{d}E}{\mathrm{d}x} \right| \sim \frac{K}{k_c} \ll 1, \tag{3.6}$$

the variable K is much smaller than k_c, and we can expand the dispersion relation,

$$\omega(k_c + K) \simeq \omega(k_c) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_c)K. \tag{3.7}$$

Then, $\Omega(K)$ in equation (3.5) simplifies to $\Omega(K) \simeq (\mathrm{d}\omega/\mathrm{d}k)K$, giving

$$\int C(K) e^{\mathrm{i}Kx-\mathrm{i}\Omega(K)t} \,\mathrm{d}K \simeq \int C(K) e^{\mathrm{i}K[x-\mathrm{i}(\mathrm{d}\omega/\mathrm{d}k)t]} \,\mathrm{d}K = E\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right). \tag{3.8}$$

Substituting this result into equation (3.5), we obtain the nice formula

$$y(x, t) = E\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right) \cos(k_c x - \omega(k_c)t + \varphi). \tag{3.9}$$

The envelope and the carrier wave move at different speeds. The carrier wave moves at the phase velocity $v_p(k_c) = \omega(k_c)/k_c$, whereas the envelope moves at the **group velocity**

$$v_g(k_c) = \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_c). \tag{3.10}$$

For non-dispersive waves, $\omega(k) = kc$, both velocities coincide, but for dispersive waves they do not. In the Dispersive Wavepacket Plotter (DWP), created by Christopher Palmer and available in the wavepage of the course, one can try different initial wavepackets for different dispersion relation – all of them based on dispersion relation (2.23) for gravity waves with surface tension.

The coherence of the wavepacket has a finite life time. The next order corrections to $\Omega(K)$ become important if we wait for a sufficiently long time. The next order term in the Taylor expansion of $\Omega(K)$ gives a phase

$$\frac{K^2}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}t, \tag{3.11}$$

and this phase cannot be neglected when the time t is of order

$$t \sim \frac{1}{K^2}\frac{1}{\mathrm{d}^2\omega/\mathrm{d}k^2} \gg \frac{1}{\omega}. \tag{3.12}$$

Larger envelopes (= smaller K) take a longer time to disperse. The carrier wave also matters. If we choose a k_c for which $\mathrm{d}^2\omega/\mathrm{d}k^2$ is small, the envelope will keep its shape for longer.

The group velocity is important for more than wavepackets. It also matters for energy and information transfer, as you will see in future courses.

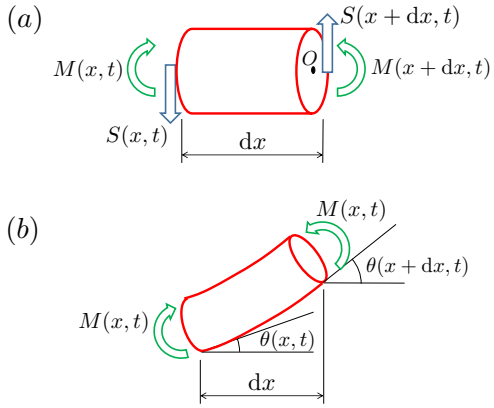


FIGURE 5. (a) Torque balance with respect to the point O for an infinitesimal length of string. (b) Bending of an infinitesimal piece of string due to torques on its ends.

Appendix A. Relation between the force $S(x,t)$ and the derivatives of $y(x,t)$

The main effect of the small force $S(x,t)$ in figure 1 is to introduce torques $M(x,t)$. Indeed, to keep the infinitesimal piece of string shown in figure 5(a) in equilibrium, we must balance torques. We proceed to calculate torques with respect to point O in the middle of the cross section at $x+\mathrm{d}x$. The torque due to the force $S(x,t)$ at x, of size $S(x,t)\,\mathrm{d}x$, must be balanced by other torques. Thus, the rest of the string not only needs to apply the tension T along the direction of the string and the forces $S(x,t)$ and $S(x+\mathrm{d}x,t)$ in the direction perpendicular to it, but also torques $M(x,t)$ and $M(x+\mathrm{d}x,t)$, represented in figure 5(a) as green curved arrows. These torques can be applied due to the finite thickness of string: one can exert a stronger force on the top of the cross section than on the bottom. To balance all the torques with respect to the point O, we need $M(x+\mathrm{d}x,t) = M(x,t) - S(x+\mathrm{d}x,t)\,\mathrm{d}x$, giving

$$S(x,t) = -\frac{\partial M}{\partial x}. \tag{A 1}$$

Note that we have neglected the angular momentum of the infinitesimal piece of string because it scales as $(\mathrm{d}x)^2$.

The torques $M(x,t)$ in turn lead to bending, as shown in figure 5(b). The change in the slope of the string shown in figure 5(b), $\theta(x+\mathrm{d}x) - \theta(x)$, is proportional to the torque $M(x,t)$, and the length of the piece of string being bent, $\mathrm{d}x$, as longer strings are easier to bend. Hence, $\theta(x+\mathrm{d}x,t) - \theta(x,t) \propto M(x,t)\,\mathrm{d}x$. The constant of proportionality depends on the cross section of the string, of area A, and on its Young modulus E. In fact, the constant of proportionality is EAr_b^2,

$$M(x,t)\,\mathrm{d}x = EAr_b^2\left[\theta(x+\mathrm{d}x,t) - \theta(x,t)\right], \tag{A 2}$$

where r_b is a length of the order of the thickness of the string that only depends on the shape of the cross section. For our purposes, it is convenient to write the constant of proportionality as TL_b^2, where $L_b = r_b\sqrt{EA/T} \gg r_b$. Then,

$$M(x,t)\,\mathrm{d}x = TL_b^2\left[\theta(x+\mathrm{d}x,t) - \theta(x,t)\right]. \tag{A 3}$$

Using partial derivatives and $\theta(x,t) \simeq \partial y/\partial x$, we obtain

$$M(x,t) = TL_b^2\frac{\partial^2 y}{\partial x^2}. \tag{A 4}$$

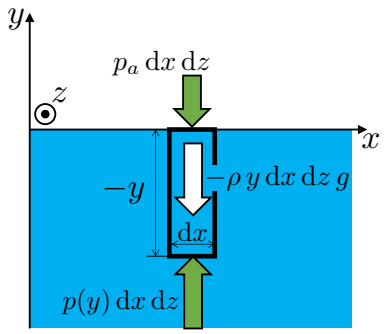


FIGURE 6. Horizontal interface between a liquid and the atmosphere. The z axis is pointing towards the reader. The thick black line is a volume of height $-y$ and base $\mathrm{d}x\,\mathrm{d}z$ (the extent in the z direction is not represented). The forces with vertical components applied on the liquid inside this volume are gravity (in white) and pressure (in green). The surface tension is not included in the vertical force balance because it is pointing in the horizontal direction.

With this result and equation (A 1), we find formula (2.1) for the force $S(x,t)$ perpendicular to the string.

Appendix B. Equations for gravity waves with surface tension

We start by calculating the steady state pressure of the liquid. We consider the case in which the interface is horizontal and is located at $y = 0$, as sketched in figure 6. To calculate the pressure within the liquid, we use infinitesimal volumes of height $-y$ and base $\mathrm{d}x\,\mathrm{d}z$, such as the one represented in figure 6. The vertical forces on these volumes are the atmospheric pressure force $p_a\,\mathrm{d}x\,\mathrm{d}z$ on the top, the pressure force $p(y)\,\mathrm{d}x\,\mathrm{d}z$ on the bottom, and the weight of the volume of liquid $-\rho\,y\,\mathrm{d}x\,\mathrm{d}z\,g$. Balancing all these forces, we find $p(y) = p_a - \rho g y$. Thus, the pressure increases as we go deeper because the pressure must sustain all the weight of the fluid above it.

We now proceed to perturb the shape of the interface. To describe the motion of the liquid below the interface, we use the displacements $\xi(x,y,t)$ and $\eta(x,y,t)$: a point originally at (x,y,z) moves to $(x+\xi(x,y,t), y+\eta(x,y,t), z)$. The vertical displacement η of the line $y = 0$ is the interface shape that we are trying to calculate,

$$h(x,t) = \eta(x, y = 0, t). \tag{B 1}$$

Due to the displacement $h(x,t)$, the pressure is perturbed, that is,

$$p(x,y,t) = p_a - \rho g y + p_1(x,y,t). \tag{B 2}$$

The displacements ξ and η are not independent of each other in a liquid because the density ρ is constant. Thus, infinitesimal volumes of liquid cannot change their volume as they move. Consider the infinitesimal volume of sides $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ sketched in figure 7. The result of its deformation, also sketched in figure 7, has a volume

$$\left[\left(1+\frac{\partial\xi}{\partial x}\right)\left(1+\frac{\partial\eta}{\partial y}\right) - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{B 3}$$

For small ξ and η, this volume simplifies to $(1+\partial\xi/\partial x+\partial\eta/\partial y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. For this volume to be the same as the original volume, $\xi(x,y,t)$ and $\eta(x,y,t)$ must satisfy the condition

$$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} = 0. \tag{B 4}$$

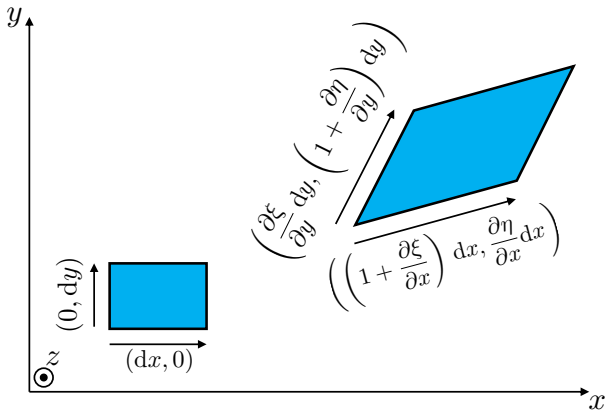


FIGURE 7. Volume of sides dx, dy and dz, and its deformation due to the motion of the fluid (the extent of the volumes in the z direction are not represented; the z axis is pointing towards the reader).

We proceed to calculate the equations of motion for ξ and η. The horizontal force balance that we used to obtain equation (2.20) is valid for any y, and not only for $y = 0$. Thus, $\rho a_x(x, y, t) = -\partial p/\partial x$. Using expression (B 2) and $a_x = \partial^2\xi/\partial t^2$, we can write the equation of motion for ξ in terms of the perturbed pressure $p_1(x, y, t)$,

$$\rho\frac{\partial^2\xi}{\partial t^2} = -\frac{\partial p_1}{\partial x}. \tag{B 5}$$

To obtain the equation of motion for the vertical displacement, we consider the vertical forces on an infinitesimal volume such as the one in figure 8, where can see that the volume's weight contributes a force $-\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,g$. We find that

$$\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\partial^2\eta}{\partial t^2}(x, y, t) = p(x, y, t)\,\mathrm{d}x\,\mathrm{d}z - p(x, y + \mathrm{d}y, t)\,\mathrm{d}x\,\mathrm{d}z - \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,g, \tag{B 6}$$

leading to

$$\rho\frac{\partial^2\eta}{\partial t^2} = -\frac{\partial p}{\partial y} - \rho g. \tag{B 7}$$

Using equation (B 2), we finally obtain

$$\rho\frac{\partial^2\eta}{\partial t^2} = -\frac{\partial p_1}{\partial y}. \tag{B 8}$$

Equations (B 4), (B 5) and (B 8) are the PDEs that we need to solve to obtain the motion of the liquid below the interface. These equations have to be solved with the boundary condition given by equation (2.18). Using equation (B 2), equation (2.18) implies that

$$p_1(x, y = 0, t) = gh(x, t) - \frac{\sigma}{\rho}\frac{\partial^2 h}{\partial x^2}(x, t), \tag{B 9}$$

where $h(x, t)$ is related to $\eta(x, y, t)$ by equation (B 1). In addition to this boundary condition, we impose that the displacements $\xi(x, y, t)$ and $\eta(x, y, t)$ and the pressure perturbation $p_1(x, y, t)$ vanish far away from the the liquid interface, that is, they go to zero for $y \to -\infty$.

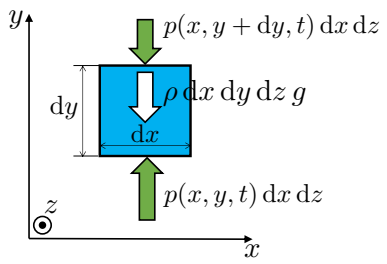


FIGURE 8. Vertical forces on an infinitesimally small volume of liquid (the extent in the z direction is not represented; the z axis is pointing towards the reader).

Appendix C. Solution to equations (B 4), (B 5) and (B 8) for a sinusoidal interface $h(x,t)$

The pressure perturbation $p_1(x,y,t)$ has to ensure that equation (B 4) is satisfied at all times. Differentiating equation (B 5) with respect to x and equation (B 8) with respect to y, and summing these equations, we find

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}\right)=-\frac{\partial^2 p_1}{\partial x^2}-\frac{\partial^2 p_1}{\partial y^2}. \tag{C 1}$$

Thus, to satisfy equation (B 4), $p_1(x,y,t)$ must satisfy the equation

$$\frac{\partial^2 p_1}{\partial x^2}+\frac{\partial^2 p_1}{\partial y^2}=0. \tag{C 2}$$

For $h(x,t)\propto e^{\mathrm{i}kx-\mathrm{i}\omega t}$, equation (B 9) gives $p_1(x,y=0,t)\propto e^{\mathrm{i}kx-\mathrm{i}\omega t}$. This suggests looking for solutions of the form $p_1(x,y,t)=P_1(y)e^{\mathrm{i}kx-\mathrm{i}\omega t}$. With this assumption, equation (C 2) becomes

$$\frac{\mathrm{d}^2 P_1}{\mathrm{d}y^2}-k^2P_1=0. \tag{C 3}$$

Recalling that we expect $P_1\to 0$ for $y\to-\infty$, we can solve this ODE to find

$$p_1(x,y,t)=p_1(x,y=0,t)e^{|k|y}. \tag{C 4}$$

This solution implies that

$$\frac{\partial p_1}{\partial x}=\mathrm{i}kp_1,\quad \frac{\partial p_1}{\partial y}=|k|p_1, \tag{C 5}$$

and using these results in equations (B 5) and (B 8) for the accelerations in the x and y directions, we obtain relation (2.22).