Electromagnetism (2nd year): Solutions to Problems

Problem sheet 0: (Summer vacation 2020)

At the start of the second year you will receive the second part of the Electromagnetism course. This vacation work contains a set of problems that will enable you to revise the material covered in the first year Electromagnetism course.

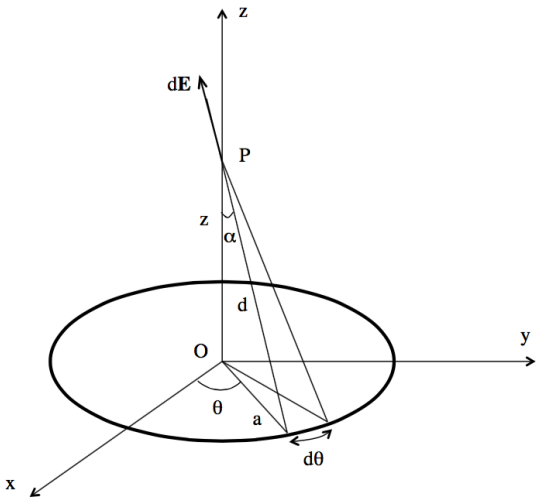
Some of the problems below are taken from:
Introduction to Electrodynamics, David J. Griffiths, 4th Edition
Electricity and Magnetism, Edward M. Purcell and David J. Morin, 3rd Edition.
(Problems first assembled by Prof. Caroline Terquem)

■ *Electrostatics*

0.1 Field and potential from charged ring

A thin ring of radius a carries a charge q uniformly distributed. Consider the ring to lie in the x–y plane with its centre at the origin.

a) *Find the electric field* $\mathbf{E}$ *at a point* P *on the z–axis.*



The arc along the ring that subtends the angle $d\theta$ contains a charge dq that creates the field $d\mathbf{E}$ at point P. Given the symmetry of the charge distribution, it can be seen that the total field at P is along the z–axis. Therefore, we are only interested in the z–component of $d\mathbf{E}$, which is $dE_z = dE\cos\alpha$. Coulomb's law gives:

$$dE = \frac{dq}{4\pi\epsilon_0 d^2}.$$

The charge per unit length along the ring is $\lambda = q/(2\pi a)$. With $\cos\alpha = z/d$ and $dq = \lambda a d\theta$, we obtain:

$$dE_z = \frac{\lambda a z d\theta}{4\pi\epsilon_0 d^3}.$$

Using $d = \sqrt{a^2 + z^2}$ and the above expression for λ, this yields:

$$dE_z = \frac{qzd\theta}{8\pi^2\epsilon_0\left(a^2+z^2\right)^{3/2}}$$

The field $\mathbf{E}$ due to the ring is obtained by summing over θ:

$$\mathbf{E} = E_z\hat{\mathbf{z}} = \hat{\mathbf{z}}\int_0^{2\pi} dE_z = \boxed{\frac{qz}{4\pi\epsilon_0\left(a^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}},}$$

where $\hat{\mathbf{z}}$ is the unit vector along the z–axis.

In the limit $z \gg a$, we recover the field due to a point charge: $\mathbf{E} = q\hat{\mathbf{z}}/(4\pi\epsilon_0 z^2)$ (the ring looks like a point charge from far away).

b) *Find the electric potential V at P.*

The arc along the ring that subtends the angle $d\theta$ creates at P the potential:

$$dV = \frac{dq}{4\pi\epsilon_0 d} = \frac{\lambda a d\theta}{4\pi\epsilon_0 d} = \frac{q d\theta}{8\pi^2\epsilon_0\sqrt{a^2+z^2}},$$

where we have chosen the origin of the potential at infinity. The potential V due to the ring is therefore:

$$V = \int_0^{2\pi} dV = \boxed{\frac{q}{4\pi\epsilon_0\sqrt{a^2+z^2}}.}$$

Here again, we recover the potential of a point charge, $V = q/(4\pi\epsilon_0 z)$, in the limit $z \gg a$.

c) *A charge $-q$ with mass m is released from rest far away along the axis. Calculate its speed when it passes through the centre of the ring. (Assume that the ring is fixed in place).*

The charge has no initial velocity, so its initial kinetic energy is zero. At it is initially far away from the ring, its initial potential energy, $-qV$, is also zero ($V = 0$ at infinity). The total energy of the charge is therefore zero initially. As energy is conserved, the energy of the particle when is passes through the origin has to be zero, which means $mv^2/2 - qV(0) = 0$, where v is the speed of the particle. This yields:

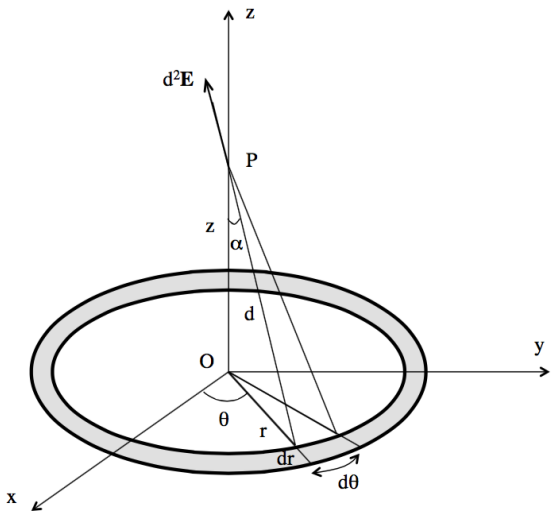
$$\frac{1}{2}mv^2 = \frac{q^2}{4\pi\epsilon_0 a}, \text{ so that } \boxed{v = \frac{q}{\sqrt{2\pi\epsilon_0 m a}}.}$$

If $a \to \infty$, $v \to 0$: the finite charge of the ring is at infinity and does not affect the charge going through the origin. If $a \to 0$, $v \to \infty$: the ring is equivalent to a charge q at the origin, and the charge $-q$ is accelerated by a potential which becomes infinite close to the ring.

0.2 Field from charged disc

The ring in the previous problem is replaced by a thin disc of radius a carrying a charge q uniformly distributed. Consider the disc to lie in the x–y plane with its centre at the origin.

a) *Find the electric field $\mathbf{E}$ at a point P on the z–axis.*



In Problem 1.a, we have shown that the field at P due to a thin ring of radius r and charge q is $qz\hat{\mathbf{z}}/\left[4\pi\epsilon_0\left(r^2+z^2\right)^{3/2}\right]$. Therefore, the field $d\mathbf{E}$ at P due to the circular ring of radius r and width dr (shaded area on the figure) is:

$$d\mathbf{E} = \frac{z\,dq}{4\pi\epsilon_0\left(r^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}},$$

where dq is the charge of the ring.

The charge per unit surface area on the disc is $\sigma = q/(\pi a^2)$, and $dq = 2\pi r dr\sigma$ (as the surface of the shaded area is $2\pi r dr$). This yields:

$$d\mathbf{E} = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{r dr}{\left(r^2+z^2\right)^{3/2}}.$$

To obtain the total field due to the disc, we integrate over radius:

$$\mathbf{E} = \int_o^a d\mathbf{E} = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\int_o^a \frac{r dr}{\left(r^2+z^2\right)^{3/2}} = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[-\frac{1}{\sqrt{r^2+z^2}}\right]_0^a = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{1}{|z|}-\frac{1}{\sqrt{a^2+z^2}}\right).$$

Using the above expression for σ, we obtain:

$$\boxed{\mathbf{E} = \frac{q\hat{\mathbf{z}}}{2\pi\epsilon_0 a^2}\left(\frac{z}{|z|}-\frac{z}{\sqrt{a^2+z^2}}\right).}$$

b) *Check that the values of* $\mathbf{E}$ *at* $z=0$ *and in the limit* $z \gg a$ *are consistent with expectations.*

When $z \gg a$, we have:

$$\frac{1}{\sqrt{a^2+z^2}} = \frac{1}{|z|}\left(1+\frac{a^2}{z^2}\right)^{-1/2} \simeq \frac{1}{|z|}\left(1-\frac{a^2}{2z^2}\right).$$

Substituting into the above expression for $\mathbf{E}$, we obtain:

$$\mathbf{E} \simeq \frac{q\hat{\mathbf{z}}}{2\pi\epsilon_0}\frac{1}{2z|z|} = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2}\frac{z}{|z|},$$

where the term $z/|z|$ only indicates that the field reverses direction as z changes sign. As expected, this is the field due to a point charge q at the origin (from far away, the disc looks like a point charge).

At $z=0$, we have $\mathbf{E} = \pm\sigma\hat{\mathbf{z}}/(2\epsilon_0)$, where the $+$ and $-$ signs apply for $z>0$ and

$z < 0$, respectively. As expected, we recover the expression of the field due to an infinite charged plane.

0.3 Hydrogen atom

According to quantum mechanics, the hydrogen atom in its ground state can be described by a point charge $+q$ (charge of the proton) surrounded by an electron cloud with a charge density $\rho(r) = -Ce^{-2r/a_0}$. Here a_0 is the Bohr radius, 0.53×10^{-10} m, and C is a constant.

a) *Given that the total charge of the atom is zero, calculate C.*
The total negative charge is $\int_0^\infty \rho(r) 4\pi r^2 dr$ and it has to be equal to $-q$. Therefore:

$$-q = -4\pi C \int_0^\infty e^{-2r/a_0} r^2 dr.$$

The integral can be calculated using integration by parts. It is equal to $a_0^3/4$. Therefore:

$$\boxed{C = \frac{q}{\pi a_0^3}}.$$

b) *Calculate the electric field at a distance r from the nucleus.*
Using Gauss's law:

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{q(r)}{\epsilon_0},$$

where $q(r)$ is the total charge contained in the sphere or radius r and the integral is over the surface Σ of the sphere. As the charge distribution has a spherical symmetry, the electric field is radial and depends only on r, so that $\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r)$. We also have:

$$q(r) = q - 4\pi C \int_0^r e^{-2r'/a_0} r'^2 dr'.$$

The integral can be calculated using integration by parts. It is equal to:

$$\frac{a_0^3}{4} - \frac{a_0}{2}\left(\frac{a_0^2}{2} + a_0 r + r^2\right) e^{-2r/a_0}.$$

Therefore:

$$\boxed{E(r) = \frac{q}{4\pi\epsilon_0 r^2}\left(1 + \frac{2r}{a_0} + \frac{2r^2}{a_0^2}\right) e^{-2r/a_0}.}$$

c) *Calculate the electric potential, $V(r)$, at a distance r from the nucleus. We give:*

$$\int \left(\frac{1}{\alpha r'} + 1\right) \frac{e^{-\alpha r'}}{r'} dr' = -\frac{e^{-\alpha r}}{\alpha r}.$$

We have $\mathbf{E} = -\boldsymbol{\nabla} V$. As $\mathbf{E}$ is radial, this reduces to $E = -dV/dr$, or:

$$V(\infty) - V(r) = -\int_r^\infty E(r')dr'.$$

This yields:

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0 r}\left(1 + \frac{r}{a_0}\right)\mathrm{e}^{-2r/a_0},}$$

where we have chosen $V(\infty) = 0$.

0.4 Energy of a charged sphere

We consider a solid sphere of radius a and charge Q uniformly distributed.

a) *Calculate the electric field $E(r)$ and the electric potential $V(r)$ at a distance r from the centre of the sphere.*

The electric field at a distance r from the centre of the sphere can be calculated using Gauss's law:

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{q(r)}{\epsilon_0},$$

where $q(r)$ is the total charge contained in the sphere or radius r and the integral is over the surface Σ of the sphere. As the charge distribution has a spherical symmetry, the electric field is radial and depends only on r, so that $\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r)$. If $r > a$, $q(r) = Q$. If $r < a$, $q(r) = (4/3)\pi r^3 \rho$ with $\rho = Q/[(4/3)\pi a^3]$, which implies $q(r) = (r/a)^3 Q$. This yields:

$$\boxed{E(r) = \frac{Qr}{4\pi\epsilon_0 a^3} \text{ for } r < a} \text{ and } \boxed{E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \text{ for } r > a.}$$

We have $\mathbf{E} = -\boldsymbol{\nabla} V$. As $\mathbf{E}$ is radial, this reduces to $E = -dV/dr$. For $r \geq a$, we obtain:

$$V(\infty) - V(r) = -\int_r^\infty E(r')dr' = -\int_r^\infty \frac{Q}{4\pi\epsilon_0 r'^2}dr'.$$

Choosing $V(\infty) = 0$, this yields:

$$\boxed{V(r) = \frac{Q}{4\pi\epsilon_0 r} \text{ for } r \geq a.}$$

For $r \leq a$, we obtain:

$$V(a) - V(r) = -\int_r^a E(r')dr' = -\int_r^a \frac{Qr'}{4\pi\epsilon_0 a^3}dr' = -\frac{Q}{8\pi\epsilon_0 a^3}\left(a^2 - r^2\right).$$

The potential has to be continuous at $r = a$ (otherwise the field would diverge), so

that $V(a) = Q/(4\pi\epsilon_0 a)$. Therefore, we obtain:

$$\boxed{V(r) = \frac{Q}{8\pi\epsilon_0 a}\left(3 - \frac{r^2}{a^2}\right) \quad \text{for } r \leq a.}$$

Alternatively, E can be calculated using the local form of Gauss's law: $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$. As $\mathbf{E}$ is radial,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E\right),$$

in spherical coordinates. For $r < a$, this yields:

$$\frac{d}{dr}\left(r^2 E\right) = \frac{\rho r^2}{\epsilon_0}, \text{ so that } E(r) = \frac{\rho r}{3\epsilon_0} + \frac{C}{r^2},$$

where C is a constant. To avoid a singularity at $r = 0$, we have to take $C = 0$. Therefore $E(r) = Qr/(4\pi\epsilon_0 a^3)$, as above. For $r > a$, $\rho = 0$ so that

$$\frac{d}{dr}\left(r^2 E\right) = 0, \text{ so that } E(r) = \frac{C'}{r^2},$$

where C' is a constant. The potential is obtained as above, and C' can be calculated by requiring the potential to be continuous at $r = a$.

Find the energy U stored in the sphere three different ways:

b) *Use the potential energy of the charge distribution due to the potential $V(r)$:*

$$U = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau,$$

where ρ is the charge density and the integral is over the volume $\mathcal{V}$ of the sphere. Using the expression for V found above for $r \leq a$, and with $d\tau = 4\pi r^2 dr$, we have:

$$U = \frac{Q\rho}{4\epsilon_0 a}\int_0^a \left(3r^2 - \frac{r^4}{a^2}\right) dr = \frac{Q\rho a^2}{5\epsilon_0}.$$

With $\rho = Q/[(4/3)\pi a^3]$, we obtain:

$$\boxed{U = \frac{3Q^2}{20\pi\epsilon_0 a}.}$$

c) *Use the energy stored in the field produced by the charge distribution:*

$$U = \int_{\text{space}} \frac{\epsilon_0 E^2}{2} d\tau,$$

where the integral is over all space.

Using the expressions for E found above for both $r < a$ and $r > a$, and with $d\tau = 4\pi r^2 dr$, we have:

$$U = \frac{Q^2}{8\pi\epsilon_0 a^6}\int_0^a r^4 dr + \frac{Q^2}{8\pi\epsilon_0}\int_a^\infty \frac{dr}{r^2} = \frac{3Q^2}{20\pi\epsilon_0 a},$$

as above.

d) *Calculate the work necessary to assemble the sphere by bringing successively thin charged layers at the surface.*

Assume we have assembled the sphere up to a radius r. We have seen in question (a) above that the potential at the surface of a uniformly charged sphere of radius a is $Q/(4\pi\epsilon_0 a)$. Therefore, the potential at the surface of the sphere of radius r is $V_r = q(r)/(4\pi\epsilon_0 r) = \rho r^2/(3\epsilon_0)$, where $q(r) = (4/3)\pi r^3\rho$ is the charge of the sphere. The work necessary to add a thin layer with a uniform charge density ρ between the radii r and $r + dr$ is $dW = V_r dq$, where dq is the charge of the layer. We have $dq = 4\pi r^2 dr\rho$, so that $dW = 4\pi\rho^2 r^4 dr/(3\epsilon_0)$. The energy of the sphere is equal to the work necessary to assemble the whole sphere, so that:

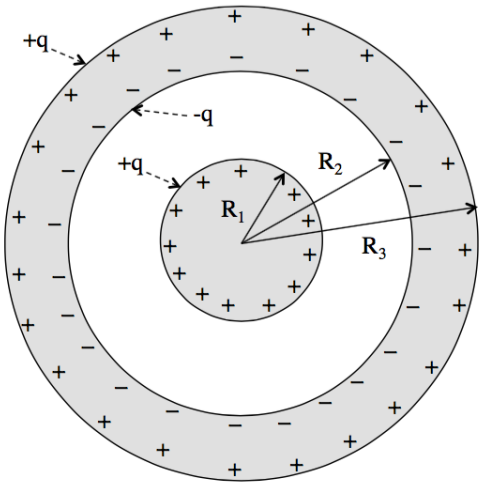
$$U = \int_0^a dW = \frac{4\pi\rho^2 a^5}{15\epsilon_0} = \frac{3Q^2}{20\pi\epsilon_0 a},$$

where we have used $\rho = Q/[(4/3)\pi a^3]$. We recover the same expression for U as above.

0.5 Conductors

A metal sphere of radius R_1, carrying charge q, is surrounded by a thick concentric metal shell of inner and outer radii R_2 and R_3. The shell carries no net charge.

a) *Find the surface charge densities at R_1, R_2 and R_3.*



Any net charge on a conductor has to reside on the surface, otherwise there would be an electric field in the conductor. Therefore, the charge q of the metal sphere is distributed at its surface. Because of the spherical symmetry, the charge is uniformly distributed. As a consequence, the surface charge density at R_1 is:

$$\boxed{\sigma_1 = \frac{q}{4\pi R_1^2}.}$$

For the electric field to be zero in the metal outer shell, by Gauss's law, a charge $-q$ has to be present at R_2. (If the charge at R_2 were, say, smaller than $-q$, there

would be an outward electric field in the conductor that would bring electrons toward the surface at R_2. They would pile up there until the field in the conductor were zero, that is to say until the total charge at R_2 were $-q$.) Since the charge at the surface of the inner sphere is uniformly distributed, the charge at R_2 is also uniformly distributed. The surface charge density at R_2 is therefore:

$$\boxed{\sigma_2 = \frac{-q}{4\pi R_2^2}.}$$

As the charge $-q$ has piled up at R_2, and the outer shell is neutral, there is a charge $+q$ left in the rest of the shell. Here again, the charge has to reside at a surface, so it has to be at R_3. The charge is uniformly distributed and the surface charge density at R_3 is therefore:

$$\boxed{\sigma_3 = \frac{q}{4\pi R_3^2}.}$$

b) *Find the potential at the centre, choosing $V = 0$ at infinity.*

We first calculate the electric field. At a distance r from the centre of the sphere, it can be calculated using Gauss's law:

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{Q(r)}{\epsilon_0},$$

where $Q(r)$ is the total charge contained in the sphere or radius r and the integral is over the surface Σ of the sphere. As the charge distribution has a spherical symmetry, the electric field is radial and depends only on r, so that $\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r)$. If $r < R_1$ or $R_2 < r < R_3$, $Q(r) = 0$ so that $E = 0$ (as it should be in a conductor!). If $R_1 < r < R_2$ or $r > R_3$, $Q(r) = q$ so that $E = q/(4\pi\epsilon_0 r^2)$.

We have $\mathbf{E} = -\boldsymbol{\nabla} V$. As $\mathbf{E}$ is radial, this reduces to $E = -dV/dr$. Therefore:

$$V(\infty) - V(0) = -\int_0^\infty E dr = -\int_{R_1}^{R_2} \frac{q}{4\pi\epsilon_0 r^2} dr - \int_{R_3}^{\infty} \frac{q}{4\pi\epsilon_0 r^2} dr.$$

We take the reference point at infinity, that is to say $V(\infty) = 0$, so that we obtain:

$$\boxed{V(0) = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{R_2} + \frac{1}{R_1} + \frac{1}{R_3}\right).}$$

c) *Now the outer surface is grounded. Explain how that modifies the charge distribution. How do the answers to questions (a) and (b) change?*

When we connect the surface at R_3 to the ground, electrons flow from the ground to that surface because they are attracted by the positive charges (electrons would flow from the surface to the ground if the surface were negatively charged), until the surface becomes neutral ($\sigma_3 = 0$). So now $E = 0$, and therefore V is constant, for $r > R_2$. As $V = 0$ at infinity, $V = 0$ everywhere for $r \geq R_2$.

To calculate $V(0)$, we now have to take into account the electric field only between

R_1 and R_2, so that:

$$V(\infty) - V(0) = -\int_0^\infty E dr = -\int_{R_1}^{R_2} \frac{q}{4\pi\epsilon_0 r^2} dr,$$

which yields:

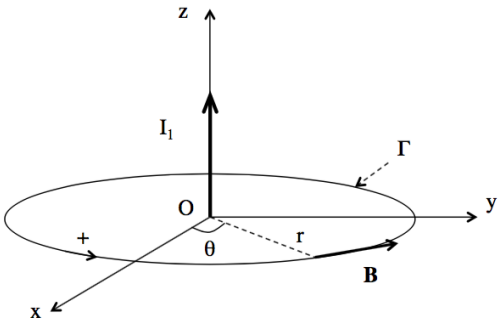
$$\boxed{V(0) = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{R_2} + \frac{1}{R_1}\right).}$$

■ *Magnetostatics*

0.6 **Force on a loop**

A long thin wire carries a current I_1 in the positive z–direction along the axis of a cylindrical co-ordinate system. A thin, rectangular loop of wire lies in a plane containing the axis, as represented on the figure. The loop carries a current I_2.

a) *Find the magnetic field due to the long thin wire as a function of distance r from the axis.*



We consider a contour Γ as shown on the figure. Given the direction in which I_1 flows, we define a "+" direction along the contour using the "right–hand rule". The Biot–Savart's law implies that the magnetic field $\mathbf{B}$ is orthoradial and the right–hand rule gives its orientation.

According to Ampère's law:

$$\int_\Gamma \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1,$$

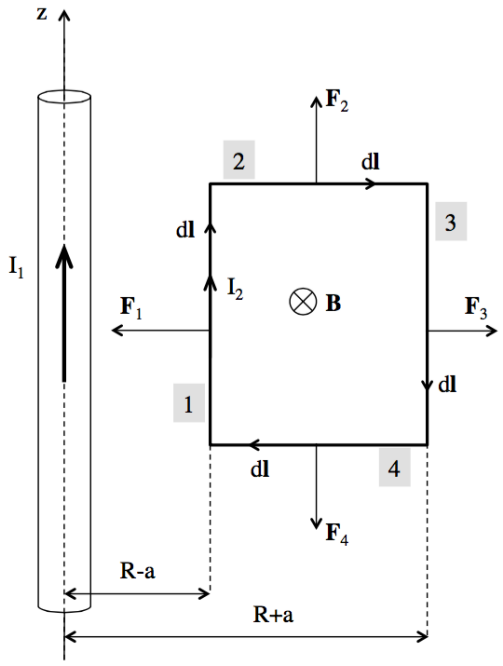
where $d\mathbf{l}$ is a small line element along the contour in the "+" direction. Then $\mathbf{B} \cdot d\mathbf{l} = B dl$. Because of the symmetry of the problem, B depends only on the distance r to the axis of the wire, so that it is constant along the contour. Therefore,

$$\int_\Gamma \mathbf{B} \cdot d\mathbf{l} = B(r) \int_\Gamma dl = 2\pi r B(r),$$

and Ampère's law yields:

$$\boxed{B(r) = \frac{\mu_0 I_1}{2\pi r}.}$$

b) *Find the vector force on each side of the loop which results from this magnetic field.*



We label the different sides $1 \ldots 4$ as shown on the figure. The force exerted on one of the sides is given by:

$$\int_{\text{side}} I_2 d\mathbf{l} \times \mathbf{B},$$

where $d\mathbf{l}$ has the same orientation as the current I_2 in the side we consider, and $\mathbf{B}$ is the magnetic field at the location of the side.
Side 1: $dl = dz$ and B is evaluated at $r = R - a$, so that:

$$F_1 = \int_{z=0}^{b} \frac{\mu_0 I_1 I_2}{2\pi(R-a)} dz = \boxed{\frac{\mu_0 I_1 I_2 b}{2\pi(R-a)}.}$$

Side 3: Same as for side 1 but with B evaluated at $r = R + a$, so that:

$$\boxed{F_3 = \frac{\mu_0 I_1 I_2 b}{2\pi(R+a)}.}$$

Side 2: $dl = dr$ and B is evaluated between $r = R - a$ and $r = R + a$:

$$F_2 = \int_{R-a}^{R+a} \frac{\mu_0 I_1 I_2}{2\pi r} dr = \boxed{\frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{R+a}{R-a}.}$$

Side 4: By symmetry, $\boxed{F_4 = F_2}$, but the forces are oriented in opposite directions.

c) *Find the resultant force on the loop.*
As $\mathbf{F}_2 + \mathbf{F}_4 = \mathbf{0}$, the resultant force is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_3$. Given the orientation of $\mathbf{F}_1$ and $\mathbf{F}_3$, we have:

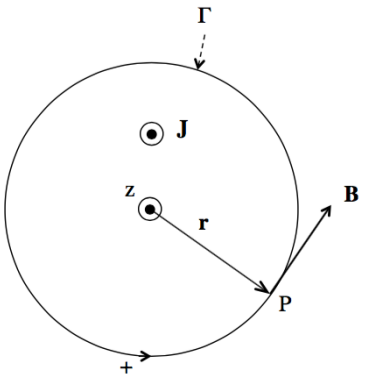
$$\boxed{F = F_1 - F_3 = \frac{\mu_0 I_1 I_2 ab}{\pi(R^2 - a^2)},}$$

and $\mathbf{F}$ is oriented toward the z–axis.

0.7 Magnetic field in off–centre hole

A cylindrical rod carries a uniform current density J. A cylindrical cavity with an arbitrary radius is hollowed out from the rod at an arbitrary location. The axes of the rod and cavity are parallel. A cross section is shown on the figure. The points O and O' are on the axes of the rod and cavity, respectively, and we note $\mathbf{a} = \mathbf{OO'}$.

a) *Show that the field inside a solid cylinder can be written as $\mathbf{B} = (\mu_0 J/2)\hat{\mathbf{z}} \times \mathbf{r}$, where $\hat{\mathbf{z}}$ is the unit vector along the axis and $\mathbf{r}$ is the position vector measured perpendicularly to the axis.*



We calculate the magnetic field in the same way as in question (a) of problem 6:

$$\int_\Gamma \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_\Sigma \mathbf{J}\cdot d\mathbf{S},$$

where Σ is the surface enclosed by the contour Γ. If the contour is inside the cylinder, then $\int_\Sigma \mathbf{J}\cdot d\mathbf{S} = \pi r^2 J$. Therefore $B = \mu_0 \pi r^2 J/(2\pi r)$, that is to say $B = \mu_0 J r/2$.

Because of the symmetry of the problem, the field is orthoradial, pointing in the direction given by the unit vector $\hat{\mathbf{z}}\times\mathbf{r}/r$ (if we assume that $\mathbf{J}$ is in the direction of positive z). Using the above expression for B, we then obtain:

$$\boxed{\mathbf{B} = \frac{\mu_0 J}{2}\hat{\mathbf{z}}\times\mathbf{r}.}$$

b) *Show that the magnetic field inside the cylindrical cavity is uniform (in both magnitude and direction).*

We can view the system as the superposition of a cylinder (the rod) with current density $\mathbf{J}$ and a second cylinder (the cavity) with current density $-\mathbf{J}$, embedded in the first one. We consider a point P which is inside the second cylinder (and therefore also inside the first cylinder). The first cylinder produces at point P the magnetic field $\mathbf{B}_1 = (\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{OP}$. The second cylinder produces at point P the magnetic field $\mathbf{B}_2 = (-\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{O'P}$. Therefore, the total field at P is $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = (\mu_0 J/2)\hat{\mathbf{z}}\times(\mathbf{OP} - \mathbf{O'P})$, that is to say:

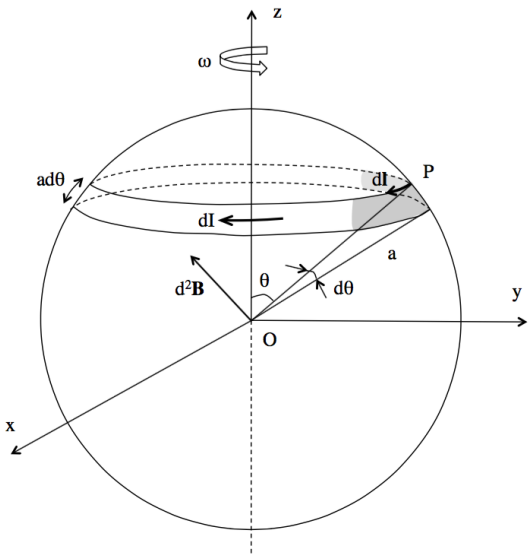
$$\boxed{\mathbf{B} = \frac{\mu_0 J}{2}\hat{\mathbf{z}}\times\mathbf{a}.}$$

It is uniform in both magnitude and direction.

0.8 Magnetic field at the centre of a sphere

A spherical shell with radius a and uniform surface charge density σ spins with angular frequency ω around a diameter. Find the magnetic field at the centre.

We choose the z–axis to be the axis of rotation. We consider the ring shown on the figure, and located between the angles θ and $\theta + d\theta$, measured from the z–axis. The radius of the ring is $R = a\sin\theta$. Since the sphere is rotating, the charges are carried around and a current dI is produced at the surface of the ring.



The shaded area along the ring around the point P (shown on the figure) produces the field $d^2\mathbf{B}$ at the centre. From the Biot–Savart's law:

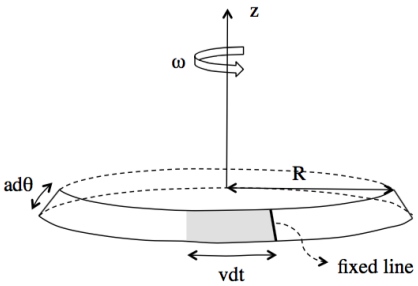
$$d^2\mathbf{B} = \frac{\mu_0 dI}{4\pi}\frac{d\mathbf{l}\times\mathbf{PO}}{PO^3}.$$

As $d\mathbf{l}$ and $\mathbf{PO}$ are perpendicular (they point in the directions of two unit vectors of the spherical coordinate system), we obtain $d^2B = (\mu_0 dI/4\pi)dl/a^2$. It can be seen that, because of the symmetry of the problem, the field produced by the entire ring is vertical. Therefore we are only interested in the vertical component of $d^2\mathbf{B}$, which is $d^2B_z = d^2B\sin\theta$. The magnetic field due to the ring is therefore:

$$d\mathbf{B} = \int_{\text{ring}} d^2B_z\,\hat{\mathbf{z}} = \int_{\text{ring}} \frac{\mu_0 dI}{4\pi a^2} dl\sin\theta\,\hat{\mathbf{z}} = \frac{\mu_0 dI}{4\pi a^2}\sin\theta\,\hat{\mathbf{z}}\int_{\text{ring}} dl = \frac{\mu_0 dI}{2a^2}R\sin\theta\,\hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the unit vector in the vertical direction. With $R = a\sin\theta$, we obtain:

$$d\mathbf{B} = \frac{\mu_0 dI}{2a}\sin^2\theta\,\hat{\mathbf{z}}.$$



To calculate the current dI, we consider a fixed (non rotating) line on the ring. The amount of charges dq that passes through that line during a time interval dt is the amount of charges contained in the surface area of length vdt and width $ad\theta$ (shaded area on the figure), where v is the velocity of the charges. Therefore, $dq = \sigma a v d\theta dt$. With $v = R\omega = a\omega\sin\theta$, we obtain: $dI = dq/dt = \sigma\omega a^2\sin\theta d\theta$.

Therefore:

$$d\mathbf{B} = \frac{\mu_0 \sigma \omega a}{2} \sin^3\theta d\theta \, \hat{\mathbf{z}},$$

and the magnetic field produced by the sphere at the centre is:

$$\mathbf{B} = \int_{\theta=0}^{\pi} d\mathbf{B} = \frac{\mu_0 \sigma \omega a}{2} \, \hat{\mathbf{z}} \int_0^{\pi} \sin^3\theta d\theta.$$

We can calculate the integral by using $\sin^3\theta = \sin\theta(1 - \cos^2\theta)$:

$$\int_0^{\pi} \sin^3\theta d\theta = \int_0^{\pi} \left(\sin\theta - \sin\theta\cos^2\theta\right) d\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{4}{3},$$

so that:

$$\boxed{\mathbf{B} = \frac{2}{3}\mu_0 \sigma \omega a \, \hat{\mathbf{z}}.}$$

0.9 Motion of a charged particle in a magnetic field

A long thin wire carries a current I in the positive z–direction along the axis of a cylindrical co-ordinate system. A particle of charge q and mass m moves in the magnetic field produced by this wire. We will neglect the gravitational force acting on the particle as it is very small compared to the magnetic force.

a) *Is the kinetic energy of the particle a constant of motion?*

Yes, because the magnetic force does not work. It can be shown in the following way. From Newton's law: $m d\mathbf{v}/dt = \mathbf{F}$, where $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$ is the magnetic force. Therefore:

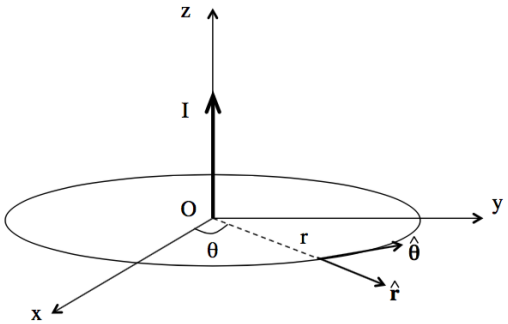
$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = q(\mathbf{v}\times\mathbf{B}) \cdot \mathbf{v}.$$

Since $\mathbf{v}\times\mathbf{B}$ is perpendicular to $\mathbf{v}$, the right–hand side of this equation is zero. We then obtain:

$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{1}{2} m \frac{dv^2}{dt} = 0,$$

which means that the kinetic energy $mv^2/2$ is constant.

b) *Find the force $\mathbf{F}$ on the particle, in cylindrical coordinates.*



We use the cylindrical coordinates (r, θ, z) and note $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ the unit vectors. The magnetic field has been calculated in question (a) of problem 6, and it is equal to:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}.$$

The position of the particle is $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}$, and therefore its velocity is $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\boldsymbol{\theta}}} + \dot{z}\hat{\mathbf{z}}$, where the dot denotes a time derivative. Using the above expression for $\mathbf{B}$, we obtain:

$$\boxed{\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) = \frac{\mu_0 I q}{2\pi r}\left(-\dot{z}\hat{\mathbf{r}} + \dot{r}\hat{\mathbf{z}}\right).}$$

c) *Obtain the equation of motion, $\mathbf{F} = m d\mathbf{v}/dt$, in cylindrical coordinates for the particle.*

The acceleration in cylindrical coordinates is $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}$. Therefore the equation of motion $m\mathbf{a} = \mathbf{F}$ yields the three equations:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu_0 I q}{2\pi m r}\dot{z}, \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0, \\ \ddot{z} &= \frac{\mu_0 I q}{2\pi m r}\dot{r}. \end{aligned}$$

d) *Suppose the velocity in the z–direction is constant. Describe the motion.*

We assume that $\dot{z} = v_0$ is a constant. Then the third equation above yields $\dot{r} = 0$, that is to say r is a constant which we note r_0. The first equation then becomes:

$$\dot{\theta}^2 = \frac{\mu_0 I q}{2\pi m r_0^2} v_0.$$

This implies that $v_0 > 0$ if $q > 0$ and $v_0 < 0$ if $q < 0$. The particle is then rotating around the z–axis with an angular velocity

$$\omega \equiv \dot{\theta} = \sqrt{\frac{\mu_0 I q}{2\pi m r_0^2} v_0}.$$

The particle is moving along a helix which axis is the z–axis. If $q > 0$ the particle is moving toward increasing z, whereas it is moving toward decreasing z if $q < 0$.

The second equation in question (c) means that the vertical component of the angular momentum $\mathbf{J}$ of the particle is constant. Indeed, $\mathbf{J} = m\mathbf{r}\times\mathbf{v} = m(-r\dot{z}\hat{\boldsymbol{\theta}} + r^2\dot{\theta}\hat{\mathbf{z}})$. Therefore $\dot{J}_z = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$ and this is zero from question (c). Note that we also have $\dot{J}_\theta = 0$, as $\dot{r} = \ddot{z} = 0$. The fact that $\dot{\mathbf{J}} = \mathbf{0}$ is a consequence of the force $\mathbf{F}$ being radial (see question b with $\dot{r} = 0$). Indeed, $m\dot{\mathbf{v}} = \mathbf{F}$ implies $m\mathbf{r}\times\dot{\mathbf{v}} = \mathbf{r}\times\mathbf{F}$. As $\mathbf{F}$ and $\mathbf{r}$ are parallel, the right–hand side of this equation is zero. Then $\dot{\mathbf{J}} = m\, d(\mathbf{r}\times\mathbf{v})/dt = m\mathbf{r}\times\dot{\mathbf{v}} = \mathbf{0}$.

■ *Electromagnetic induction*

0.10 Growing current in a solenoid

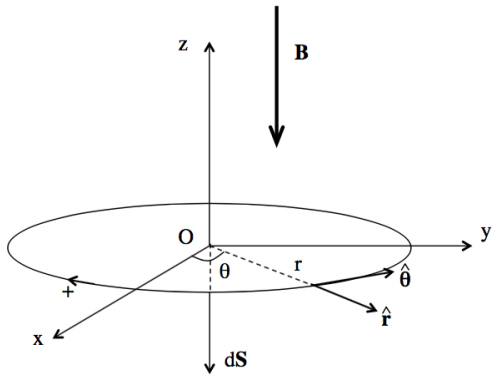
An infinite solenoid has radius a and n turns per unit length. The current grows linearly with time, according to $I(t) = kt$, $k > 0$. The solenoid is looped by a circular wire of

radius r, coaxial with it. We recall that the magnetic field due to the current in the solenoid is $B = \mu_0 nI$ inside the solenoid and zero outside.

a) *Without doing any calculation, explain which way the current induced in the loop flows.*

The magnetic field in the solenoid is $B = \mu_0 nI$. Therefore, an increase of I leads to an increase of the magnetic flux through the surface delimited by the loop. By Lenz's law, the current induced in the loop will produce a field that will oppose this increase of the flux. Therefore, within the loop, it will point in the opposite direction to the field produced by the solenoid. This implies that the current induced in the loop flows in opposite way to the current in the solenoid.

b) *Use the integral form of Faraday's law, which is $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt$, to find the electric field in the loop for both $r < a$ and $r > a$. Check that the orientation of $\mathbf{E}$ agrees with the answer to question (a).*



We use the cylindrical coordinates (r, θ, z) and note $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ the unit vectors, with the z–axis pointing upward. The magnetic field due to the current in the solenoid is $\mathbf{B} = -\mu_0 nI\hat{\mathbf{z}}$ inside the solenoid, and zero outside. We *choose* a positive orientation along the loop as indicated on the figure. Then the orientation of $d\mathbf{S}$, which is given by the right–hand rule, is such that the flux of $\mathbf{B}$ through the surface Σ delimited by the loop is positive.

Faraday's law:

$$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{S},$$

where $d\mathbf{l}$ is oriented in the positive direction chosen along the loop (we could have chosen the positive direction in the opposite way, but then *both* $d\mathbf{S}$ and $d\mathbf{l}$ would have had to be reversed). The electric field induced in the loop is orthoradial: $\mathbf{E} = E_\theta \hat{\boldsymbol{\theta}}$, where E_θ can be either positive or negative. In addition, because of the symmetry of the problem, E_θ depends only on r. With $d\mathbf{l} = -dl\hat{\boldsymbol{\theta}}$, we obtain $\mathbf{E} \cdot d\mathbf{l} = -E_\theta \, dl$, and $\oint \mathbf{E} \cdot d\mathbf{l} = -2\pi r E_\theta$. If $r > a$, the flux of $\mathbf{B}$ through the loop is $\pi a^2 \mu_0 nI$, whereas it is $\pi r^2 \mu_0 nI$ if $r < a$. Therefore, Faraday's law yields: $2\pi r E_\theta = \pi a^2 \mu_0 nk$ is $r > a$ and $2\pi r E_\theta = \pi r^2 \mu_0 nk$ is $r < a$. The electric field is then given by:

$$\boxed{\mathbf{E} = \frac{\mu_0 nka^2}{2r}\hat{\boldsymbol{\theta}} \text{ for } r > a, \text{ and } \mathbf{E} = \frac{\mu_0 nkr}{2}\hat{\boldsymbol{\theta}} \text{ for } r < a.}$$

Because the electric field is in the direction of $\hat{\boldsymbol{\theta}}$, the electric force on the electrons in the loop points in the $-\hat{\boldsymbol{\theta}}$ direction, and the current in the loop points in the $+\hat{\boldsymbol{\theta}}$ direction. As the current in the solenoid is opposite, we recover the result of

question (a).

c) *Verify that your result satisfies the local form of the law,* $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$.

In cylindrical coordinates, and given that $\mathbf{E}$ is orthoradial and depends only on r:

$$\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{r}\frac{d(rE_\theta)}{dr}\hat{\mathbf{z}}.$$

By using the above expressions for $\mathbf{E}$, it is straightforward to verify that $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$.

■ *Maxwell's equations*

0.11 **Energy flow into a capacitor**

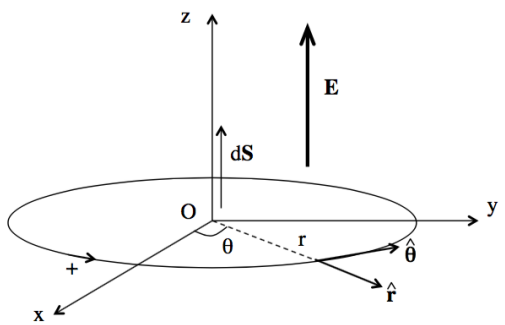
A capacitor has circular plates with radius a and is being charged by a constant current I. The separation of the plates is $w \ll a$. Assume that the current flows out over the plates through thin wires that connect to the centre of the plates, and in such a way that the surface charge density σ is uniform, at any given time, and is zero at $t = 0$.

a) *Find the electric field between the plates as a function of t.*

Since $w \ll a$, we can ignore edge effects, and the electric field produced by the lower plate is the same as that produced by an infinite plane with surface density σ, that is to say $\hat{\mathbf{z}}\sigma/(2\epsilon_0)$, where $\hat{\mathbf{z}}$ is the unit vector in the vertical direction, pointing upward. The electric field produced by the upper plate is the same as that produced by the lower plate. Therefore, the total electric field is $\mathbf{E} = \hat{\mathbf{z}}\sigma/\epsilon_0$. The total charge on the lower plate is $Q = \pi a^2\sigma$, and we have $I = dQ/dt$. As I is constant, this yields $Q = It + Q_0$, where Q_0 is the charge on the plate at $t = 0$. As $Q_0 = 0$, we obtain $Q = It$. Therefore:

$$\boxed{\mathbf{E} = \frac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$$

b) *Consider the circle of radius $r < a$ shown on the figure (and centered on the axis of the capacitor). Using the integral form of Maxwell's equation $\boldsymbol{\nabla}\times\mathbf{B} = \epsilon_0\mu_0\partial\mathbf{E}/\partial t$ over the surface delimited by the circle, find the magnetic field at a distance r from the axis of the capacitor.*



The integral form of Maxwell's equation is:

$$\iint_\Sigma (\boldsymbol{\nabla}\times\mathbf{B})\cdot d\mathbf{S} = \iint_\Sigma \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\cdot d\mathbf{S},$$

where the integral is over the surface Σ delimited by the circle.

We choose a positive orientation on the circle, as indicated on the figure, and the orientation of $d\mathbf{S}$ is then given by the right–hand rule. We use Stokes's theorem:

$$\iint_{\Sigma} (\boldsymbol{\nabla}\times\mathbf{B})\cdot d\mathbf{S} = \int_{\text{circle}} \mathbf{B}\cdot d\mathbf{l},$$

where $d\mathbf{l} = dl\hat{\boldsymbol{\theta}}$ is oriented in the positive direction along the circle. The magnetic field is orthoradial, that it to say $\mathbf{B} = B_\theta\hat{\boldsymbol{\theta}}$ and, because of the symmetry of the problem, depends only on r. Therefore, in the integral above, $\mathbf{B}\cdot d\mathbf{l} = B_\theta(r)dl$. Maxwell's equation then yields:

$$2\pi r B_\theta(r) = \epsilon_0\mu_0\pi r^2\frac{\partial E_z}{\partial t},$$

where we have written $\mathbf{E} = E_z\hat{\mathbf{z}}$. With the expression of E_z found in question (a), we obtain:

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 I r}{2\pi a^2}\,\hat{\boldsymbol{\theta}}.}$$

c) *Find the energy density u and the Poynting vector $\mathbf{S}$ in the gap. Check that the relation:*

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{S},$$

is satisfied.

The energy density is given by:

$$u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}.$$

With the expressions of E and B found above, we obtain:

$$\boxed{u = \frac{I^2}{2\pi^2 a^4}\left(\frac{\mu_0 r^2}{4} + \frac{t^2}{\epsilon_0}\right).}$$

The Poynting vector is given by:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}E_z\hat{\mathbf{z}}\times B_\theta\hat{\boldsymbol{\theta}} = -\frac{1}{\mu_0}E_z B_\theta\hat{\mathbf{r}} = \boxed{-\frac{I^2 r t}{2\pi^2\epsilon_0 a^4}\hat{\mathbf{r}}.}$$

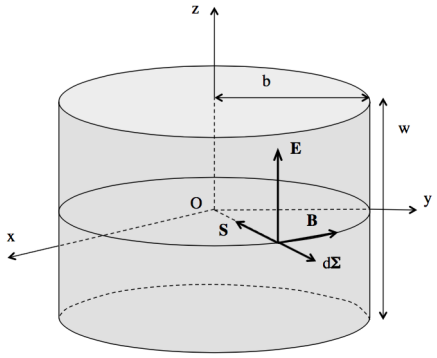
The Poynting vector points radially inward, which means that energy *enters* the capacitor. This makes sense as E, and therefore u, is increasing with time.

The above expression for u yields $\partial u/\partial t = I^2 t/(\pi^2\epsilon_0 a^4)$. Given that $\mathbf{S}$ is radial and depends only on the coordinate r, its divergence in cylindrical coordinates is:

$$\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{1}{r}\frac{d(rS)}{dr},$$

and it is straightforward to check that $\partial u/\partial t = -\boldsymbol{\nabla}\cdot\mathbf{S}$.

d) *Consider a cylinder of radius $b < a$ and length w inside the gap. Determine the total energy in the cylinder, as a function of time. Calculate the total power flowing into the cylinder, by integrating the Poynting vector* $\mathbf{S}$ *over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the cylinder.*



The total energy U in the cylinder is:

$$U = \iiint_{\text{cylinder}} urd\theta drdz = 2\pi w \int_0^b urdr,$$

$$\boxed{U = \frac{I^2 b^2 w}{2\pi a^4}\left(\frac{\mu_0 b^2}{8} + \frac{t^2}{\epsilon_0}\right).}$$

The Poynting vector is an energy flux density, which means an energy per unit time per unit surface area. Therefore, the total power (energy per unit time) flowing into the cylinder is:

$$P = \left| \iint_\Sigma \mathbf{S} \cdot d\mathbf{\Sigma} \right|,$$

where the integral is over the surface Σ of the cylinder and $d\mathbf{\Sigma}$ is perpendicular to a surface element of the cylinder and pointing outward. As $\mathbf{S}$ is radial and pointing inward, $\mathbf{S} \cdot d\mathbf{\Sigma} = -S\, d\Sigma$ on the lateral surface of the cylinder and zero on the horizontal surfaces. Since S is uniform over the lateral surface of the cylinder, $P = S(b) \times 2\pi bw$. With the above expression for S, we obtain:

$$\boxed{P = \frac{I^2 b^2 wt}{\pi \epsilon_0 a^4}.}$$

As expected from energy conservation, we check that $\partial U/\partial t = P$.

e) *When $b = a$, and assuming that we can still neglect edge effects in that case, check that the total power flowing into the capacitor is:*

$$\frac{d}{dt}\left(\frac{1}{2}QV\right),$$

where V is the voltage across the capacitor (since $QV/2$ is the energy stored in the electric field in the capacitor).

When $b = a$, the above expression for P becomes $P = I^2 wt/(\pi\epsilon_0 a^2)$. With $Q = It$, $V = Ew$ and E given in question (a), we obtain $QV/2 = wI^2 t^2/(2\pi\epsilon_0 a^2)$. Therefore $d(QV/2)/dt = P$, as expected.