B2: Symmetry and Relativity
Problem Set 1: Vectors and Tensors
MT 2020

Einstein summation convention is assumed, with indices taking values from 1 to N.

1. If we have two successive transformations from $u^i = u^i(x^1, x^2, \cdots x^N)$ to $v^i = v^i(y^1, y^2, \cdots y^N)$, and from v^i to $w^i = w^i(z^1, z^2, \cdots z^N)$, with $i = 1, 2, \cdots N$,

   $$v^i = \frac{\partial y^i}{\partial x^j} u^j,$$

   and

   $$w^i = \frac{\partial z^i}{\partial y^j} v^j,$$

   show that we can perform the transformation in one step via

   $$w^i = \frac{\partial z^i}{\partial x^j} u^j.$$

2. If ${A^{ij}}_k$ is a mixed tensor, ${B^{ij}}_k$ is another tensor of the same kind, and α and β are scalar invariants, show that $\alpha {A^{ij}}_k + \beta {B^{ij}}_k$ is yet another tensor of the same kind.

3. If ${A^i}_j$ are the components of a mixed tensor, show that ${A^i}_i$ transforms as a scalar invariant.

4. Assuming x and y transform as the components of a Euclidean vector, determine which of the following matrices are tensors:

   $$A^{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}, \quad C^{ij} = \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix}$$

   [based on E Butkov, *Mathematical Physics*]

5. Show that if the components of a contravariant vector vanish in one coordinate system, they vanish in all coordinate systems. What can be said of two contravariant vectors whose components are equal in one coordinate system?

6. Let A_{ij} be a skew-symmetric tensor with $A_{ij} = -A_{ji}$, and S_{ij} a symmetric tensor with $S_{ij} = S_{ji}$. Show that the symmetry properties are preserved in coordinate transformations. Also show that the quantities with raised indices, A^{ij} and S^{ij}, possess the same properties. From this, show that $A^{ij} S_{ij} = 0$ and $A_{ij} S^{ij} = 0$.

7. Let $C^{k\ell} = A^{ijk} {B^\ell}_{ij}$ be a rank-2 contravariant tensor given by contracting the N^3 functions A^{ijk} with the tensor ${B^\ell}_{mn}$, which is symmetric in the mn indices but otherwise arbitrary, *i.e.*, ${B^\ell}_{mn} = {B^\ell}_{nm}$. Show that $A^{ijk} + A^{jik}$ is a rank-3 contravariant tensor. Give reasons why the same is not true for A^{ijk} or A^{jik} separately.

8. In this problem, we will consider a transformation from Cartesian to polar coordinate systems in two Euclidean dimensions. Let $x^1 = x$ and $x^2 = y$ for the Cartesian system, and $\hat{x}^1 = r$ and $\hat{x}^2 = \theta$ for the polar, with the transformation

$$\begin{aligned} x^1 &= r\cos\theta = \hat{x}^1 \cos\hat{x}^2 \\ x^2 &= r\sin\theta = \hat{x}^1 \sin\hat{x}^2 \end{aligned}$$

The metric for the Cartesian system is $g_{ij} = \delta_{ij}$. Derive the metric tensor $\hat{g}_{ij}$ for the polar coordinate system, its reciprocal $\hat{g}^{ij}$, and the covariant polar components $\hat{x}_1$ and $\hat{x}_2$ in terms of r and θ. Why might it *not* be appropriate to calculate a length from the origin to a point specified by finite values of r and θ using these covariant components?

Show that the components of the metrics g_{ij} and $\hat{g}_{ij}$ do not change under rotations of the coordinate system through a fixed angle α around the origin.

B2: Symmetry and Relativity
Problem Set 2: Special Relativity
MT 2019

Conventions: Greek indices take values 0 through 3, while Latin indices take values 1 through 3. 3-vectors can also be indicated by boldface, *e.g.*, $\mathbf{a}$. Minkowski metric $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

1. Show, using algebra, a spacetime diagram, or otherwise,

   (i) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval;

   (ii) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval.

   (iii) for any time-like vector there exists a frame in which its spatial part is zero;

   (iv) any vector orthogonal to a time-like vector must be space-like;

   (v) with one exception, any vector orthogonal to a null vector is space-like, and describe the exception.

   (vi) the instantaneous 4-velocity of a particle is parallel to the worldline (*i.e.*, demonstrate that you understand the meaning of this claim—if you do then it is obvious);

   (vii) if the 4-displacement between any two events is orthogonal to an observer's worldline, then the events are simultaneous in the rest frame of that observer.

2. Define *proper time*. A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by $X^\mu = (ct, x, y, z)$ in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$.

   Two particles have 3-velocities $\mathbf{u}$ and $\mathbf{v}$ in some reference frame. The Lorentz factor for their relative 3-velocity $\mathbf{w}$ is given by

   $$\gamma_w = \gamma_u \gamma_v (1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

   Prove this twice, by using each of the following two methods:

   (i) In the given frame, the worldline of the first particle is $X^\mu = (ct, \mathbf{u}t)$. Transform to the rest frame of the other particle to obtain

   $$t' = \gamma_v t (1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

   Obtain dt'/dt and apply the result of the first part of this question.

   (ii) Use the invariant $U^\mu V_\mu$, first showing that it is equal to $-c^2\gamma_w$.

3. Derive a formula for the frequency ω of light waves from a moving source, in terms of the proper frequency ω_0 in the source frame and the angle in the observer's frame, θ, between the direction of observation and the velocity of the source.

   A galaxy with a negligible speed of recession from Earth has an active nucleus. It has emitted two jets of hot material with the same speed v in opposite directions, at an angle θ to the direction to the Earth. A spectral line in singly-ionised Mg (proper wavelength $\lambda_0 = 448.1$ nm) is emitted from both jets. Show that the wavelengths $\lambda_\pm$ observed on Earth from the two jets are given by

$$\lambda_\pm = \lambda_0\gamma(1 \pm (v/c)\cos\theta)$$

   (you may assume the angle subtended at Earth by the jets is negligible). If $\lambda_+ = 420.2$ nm and $\lambda_- = 700.1$ nm, find v and θ.

   In some cases, the receding source is difficult to observe. Suggest a reason for this.

4. The 4-angular momentum of a single particle about the origin is defined as

$$L^{\mu\nu} \equiv X^\mu P^\nu - X^\nu P^\mu.$$

   (i) Prove that, in the absence of forces, $dL^{\mu\nu}/d\tau = 0$.

   (ii) Exhibit the relationship between the space-space part L^{ij} and the 3-angular momentum vector $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$.

   (iii) The total angular momentum of a collection of particles about the pivot R^λ is defined as

$$L^{\mu\nu}_{\text{tot}}(R^\lambda) = \sum_i (X_i^\mu - R^\mu)P_i^\nu - (X_i^\nu - R^\nu)P_i^\mu$$

   where the sum runs over the particles (that is, X^μ and P^μ are 4-vectors, not 2nd-rank tensors; i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

5. The 4-vector field F^μ is given by $F^\mu = 2x^\mu + k^\mu(x^\nu x_\nu)$ where k^μ is a constant 4-vector and $x^\mu = (ct, x, y, z)$ is the 4-vector displacement in spacetime. Evaluate the following:

   (i) $\partial_\lambda x^\lambda$

   (ii) $\partial^\mu(x_\lambda x^\lambda)$

   (iii) $\partial^\mu\partial_\mu x^\nu x_\nu$

   (iv) $\partial_\lambda F^\lambda$

   (v) $\partial^\mu(\partial_\lambda F^\lambda)$

   (vi) $\partial^\mu\partial_\mu \sin(k_\lambda x^\lambda)$

   (vii) $\partial^\mu x^\nu$

6. A particle of rest mass m and kinetic energy $3mc^2$ strikes a stationary particle of rest mass $2m$ and combines with it while still conserving energy and momentum. Find the rest mass and speed of the composite particle.

7. Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction, in terms of E_0 and the electron rest mass m.

   High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4} eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

8. A particle Y decays into three other particles, with labels indicated by $Y \to 1+2+3$. Working throughout in the CM frame:

   (i) Show that the 3-momenta of the decay products are coplanar.

   (ii) Show that the energy of particle 3 is given by
   $$E_3 = \frac{(m_Y^2 + m_3^2 - m_1^2 - m_2^2)c^4 - 2E_1E_2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2}{2m_Y c^2}.$$

   (iii) Show that the maximum value of E_3 is
   $$E_{3,\max} = \frac{m_Y^2 + m_3^2 - (m_1+m_2)^2}{2m_Y}c^2$$
   and explain under what circumstances this maximum is attained.

   (iv) Show that, when particle 3 has its maximum possible energy, particle 1 has the energy
   $$E_1 = \frac{m_1(m_Y c^2 - E_{3,\max})}{m_1 + m_2}.$$
   [Hint: first argue that 1 and 2 have the same speed in this situation.]

   (v) Now let's return to the more general circumstance, with E_3 not necessarily maximal. Let X be the system composed of particles 1 and 2. Show that its rest mass is given by
   $$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2.$$

   (vi) Write down an expression for the energy E^* of particle 2 in the rest frame of X, in terms of m_1, m_2, and m_X.

   (vii) Show that, when particle 3 has an energy of intermediate size, $m_3c^2 < E_3 < E_{3,\max}$, the energy of particle 2 in the original frame (the rest frame of Y) is in the range
   $$\gamma(E^* - \beta p^* c) \le E_2 \le \gamma(E^* + \beta p^* c)$$
   where E^* and p^* are the energy and momentum of particle 2 in the X frame, and γ and β refer to the speed of that frame relative to the rest frame of Y.

9. Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $P^\mu + P_e^\mu = (P')^\mu + (P_e')^\mu$. [Hint: we would like to eliminate the final electron 4-momentum $(P_e')^\mu$, so make this the subject of the equation and square.] A collimated beam of X-rays of energy 17.52 keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of 90°, including a quantitative indication of the scale.

Additional questions

10. Derive the equations describing the transformation of velocity:

$$\begin{aligned} \mathbf{u}'_{\parallel} &= \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2} \\ \mathbf{u}'_{\perp} &= \frac{\mathbf{u}_{\perp}}{\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)} \end{aligned}$$

11. For any two future-pointing time-like vectors U^μ and V^μ, prove that $U^\mu V_\mu = -uv\cosh\rho$, where ρ is the relative rapidity of frames in which U^μ and V^μ are purely temporal and with $U^\mu U_\mu = -u^2$ and $V^\mu V_\mu = -v^2$.

12. In a given inertial frame S, two particles are shot out from a point in orthogonal directions with equal speeds v. At what rate does the distance between the particles increase in S? What is the speed of each particle relative to the other?

13. In a frame S a guillotine blade in the (x, y) plane falls in the negative y direction towards a block level with the x axis and centred at the origin. The angle of the edge of the blade is such that the point of intersection of blade and block moves at a speed in excess of c in the positive x direction. Show that in some frames S' in standard configuration with S, this point moves in the *opposite* direction along the block. [For simplicity, assume that the blade drops with constant speed u.]

    Now suppose that when the centre of the blade arrives at the block, the whole blade instantaneously evaporates in frame S (for example, it could be vapourized by a very powerful laser beam incident from the z direction). A piece of paper placed on the block is therefore cut on the negative x-axis only. Explain this in S'.

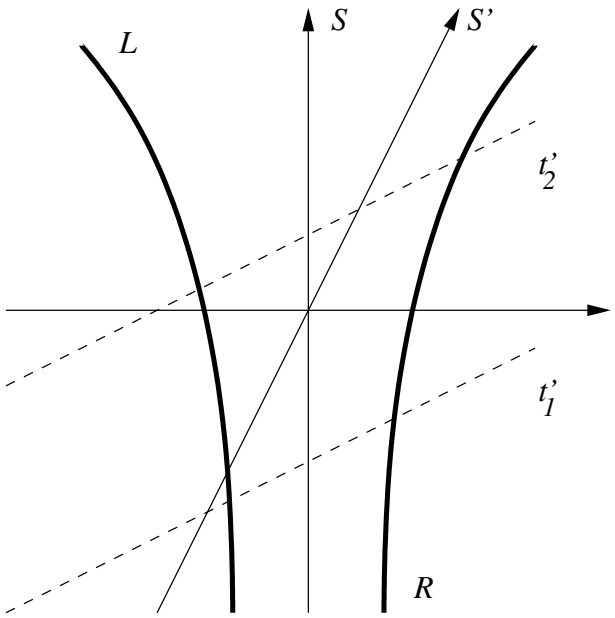
Energy-momentum conservation

14. The upper atmosphere of the Earth receives electromagnetic energy from the sun at the rate 1400 Wm^{-2}. Find the rate of loss of mass of the sun due to all its emitted radiation. [The Earth-sun distance is 499 light seconds.]

15. Calculate the mass increase of a block of copper heated from 0°C to 1000°C, assuming the specific heat capacity is constant at 420 $\mathrm{J\,kg^{-1}K^{-1}}$ and the initial mass is 10 kg.

16. Show that if a 4-vector has a component which is zero in *all* frames, then the entire vector is zero. What insight does this offer into energy and momentum?

17. The following spacetime diagram shows the worldlines of two accelerating particles L and R, and two inertial observers S and S'. The dashed lines are lines of simultaneity in frame S' at two times t'_1 and t'_2.



The two particles have the same mass.

First consider the observations in frame S. At any given time in S, the particles have equal and opposite momenta, therefore their total 3-momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_L + \mathbf{p}_R = 0$. In other words, $\mathbf{p}_{\text{tot}}$ is constant (and zero).

Now consider the observations in frame S'. Initially the two particles have almost the same velocity relative to S'. Then, between times t'_1 and t'_2, R comes to rest relative to S', while L changes its velocity relative to S' by only a small amount. Therefore the total particle momentum in frame S' roughly halves between times t'_1 and t'_2. The total momentum of the two particles is certainly not constant in S'.

What does the situation tell us about total momentum? Is total momentum a meaningful physical concept? If so, then is it always conserved? Under what conditions does it transform as part of a 4-vector?

18. A system consists of two photons, each of energy E, propagating at right angles in the laboratory frame. Find the rest mass of the system and the velocity of its CM frame relative to the laboratory frame.

Particle formation

19. A particle of mass m and energy E (in the laboratory frame) hits a free stationary target of mass M. If E is greater than a threshold energy $E_{\rm th}$, the collision produces a number of collision products with masses m_i. Show that $E_{\rm th}$ is given by

$$E_{\rm th} = \frac{(\sum_i m_i)^2 - m^2 - M^2}{2M} c^2.$$

20. A particle formation experiment creates reactions of the form $b + t \to b + t + n$ where b is an incident particle of mass m, t is a target of mass M at rest in the laboratory frame, and n is a new particle. Define the "efficiency" of the experiment as the ratio of the rest energy of the new particle to the supplied kinetic energy of the incident particle. Show that, at threshold, the efficiency thus defined is equal to

$$\frac{M}{m + M + m_n/2}.$$

tem Pion formation can be achieved by the process $p + p \to p + p + \pi^0$. A proton beam strikes a target containing stationary protons. Calculate the minimum kinetic energy which must be supplied to an incident proton to allow pions to be formed, and compare this to the rest energy of a pion.

21. A photon is incident on a stationary proton. Find, in terms of the rest masses, the threshold energy of the photon if a neutron and a pion are to emerge.

Particle decay

22. Particle tracks are recorded in a bubble chamber subject to a uniform magnetic field of 2 tesla. A vertex consisting of no incoming and two outgoing tracks is observed. The tracks lie in the plane perpendicular to the magnetic field, with radii of curvature 1.67 m and 0.417 m, and separation angle 21°. It is believed that they belong to a proton (mass 939.3 MeV/c^2) and pion (mass 139.6 MeV/c^2), respectively. Assuming this, and that the process at the vertex at the vertex is decay of a neutral particle into two products, find the rest mass of the neutral particle.

23. A particle with known rest mass M and energy E decays into two products with known rest masses m_1 and m_2. Find the energies E_1 and E_2 (in the lab frame) of the products, by the following steps:

    (i) Find the energies E_1' and E_2' of the products in the CM frame.

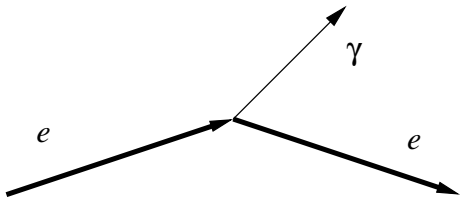
    (ii) Show that the momentum of either decay product in the CM frame is

    $$p = \frac{c}{2M}[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2 m_2^2]^{1/2}.$$

    (iii) Find the Lorentz factor and the speed v of the CM frame relative to the lab.

    (iv) Write down, in terms of v, γ, p, E_1' and E_2', expressions for E_1 and E_2 when the products are emitted (1) along the line of flight, and (2) at right angles to the line of flight in the CM frame.

24. This diagram illustrates a process in which an electron emits a photon:



    Prove that the process is impossible. Prove also that a photon cannot transform into an electron-positron pair in free space, and that a photon in free space cannot decay into a pair of photons with differing directions of propagation.

25. A decay mode of the neutral Kaon is $K^0 \to \pi^+ + \pi^-$. The Kaon has momentum 300 MeV/c in the laboratory, and one of the pions is emitted, in the laboratory, in a direction perpendicular to the velocity of the Kaon. Find the momenta of both pions. [Use 497.611 MeV/c^2 as the mass of the kaon, and 139.57061 MeV/c^2 as the mass of the pion.]

Scattering

26. Consider a head-on elastic collision between a moving "bullet" of rest mass m and a stationary target of rest mass M such that the bullet recoils in the opposite of its original direction. Show that the post-collision Lorentz factor γ of the bullet cannot exceed $(m^2 + M^2)/(2mM)$. (This means that for large energies almost all the energy of the bullet is transferred to the target, very different from the classical result.) [Hint: consider $P_t^\mu - Q_b^\mu$, where P_t^μ is the initial 4-momentum of the target and Q_b^μ is the final 4-momentum of the bullet.]

27. Particles of mass m and kinetic energy T are incident on similar particles at rest in the laboratory. Show that, if elastic scattering takes place, then the minimum angle between the final momenta in the laboratory is given by

    $$\cos\theta_{\min} = \frac{T}{T + 4mc^2}.$$

B2: Symmetry and Relativity
Problem Set 3: Forces and fields
MT 2019

1. Obtain the transformation equations for 3-force, by starting from the Lorentz transformation of energy-momentum, and then differentiating with respect to t'. [Hint: argue that the relative velocity $\mathbf{v}$ of the reference frame is constant, and use or derive an expression for dt/dt'.]

2. Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows:

   (i) Write down the solution for $\mathbf{p}$ as a function of time, taking as initial condition $\mathbf{p}(0) = \mathbf{p}_0$.

   (ii) Show that the Lorentz factor as a function of time is given by $\gamma^2 = 1 + \alpha^2$ where $\boldsymbol{\alpha} = (\mathbf{p}_0 + \mathbf{f}t)/mc$.

   (iii) You can now write down the solution for $\mathbf{v}$ as a function of time. Do so.

   (iv) Now restrict attention to the case where $\mathbf{p}_0$ is perpendicular to $\mathbf{f}$. Taking the x-direction along $\mathbf{f}$ and the y-direction along $\mathbf{p}_0$, show that the trajectory is given by

$$\begin{aligned} x &= \frac{c}{f}(m^2c^2 + p_0^2 + f^2t^2)^{1/2} + \text{const} \\ y &= \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const} \end{aligned}$$

   where you may quote that $\int (a^2 + t^2)^{-1/2} dt = \log(t + \sqrt{a^2 + t^2})$.

   (v) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation. [N.B. the calculation as a function of proper time is best done another way, see later problems.]

3. For motion under a pure (rest mass preserving) inverse square law force $\mathbf{f} = -\alpha\mathbf{r}/r^3$, where α is a constant, derive the energy equation $\gamma mc^2 - \alpha/r = \text{constant}$.

4. Prove that the time rate of change of the angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ of a particle about an origin O is equal to the couple $\mathbf{r} \wedge \mathbf{f}$ of the applied force about O.

   If $L^{\mu\nu}$ is a particle's 4-angular momentum, and we define the 4-couple $G^{\mu\nu} \equiv X^\mu F^\nu - X^\nu F^\mu$, prove that $(d/d\tau)L^{\mu\nu} = G^{\mu\nu}$, and that the space-space part of this equation corresponds to the previous 3-vector result.

5. Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$ and ϕ such that

$$\begin{aligned} \mathbf{B} &= \nabla \wedge \mathbf{A} \\ \mathbf{E} &= -\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \nabla\phi. \end{aligned}$$

(i) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ and ϕ.

(ii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators, and Lorentz scalars (a *manifestly covariant* form).

6. How does a 2nd rank tensor change under a Lorentz transformation? By transforming the field tensor, and interpreting the result, prove that the electromagnetic field transforms as

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}) \\ \mathbf{B}'_{\perp} &= \gamma(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2) \end{aligned}$$

[Hint: you may find the algebra easier if you treat $\mathbf{E}$ and $\mathbf{B}$ separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]

Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.

7. The electromagnetic field tensor $F^{\mu\nu}$ (sometimes called the Faraday tensor) is defined such that the 4-force on a charged particle is given by

$$f^{\mu} = qF^{\mu\nu}U_{\nu}.$$

By comparing this to the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

which defines the electromagnetic and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $dP^{\mu}/d\tau$), show that the components of the field tensor are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

8. Show that the field equation

$$\partial_{\lambda}F^{\lambda\nu} = -\mu_0\rho_0U^{\nu}$$

is equivalent to

$$\partial^{\lambda}\partial_{\lambda}A^{\nu} - \partial^{\nu}(\partial_{\lambda}A^{\lambda}) = -\mu_0 J^{\nu}$$

where $J^{\nu} \equiv \rho_0 U^{\nu}$ (here ρ_0 is the proper charge density, and J^{ν} is the 4-current density). Comment.

9. Show that the following two scalar quantities are Lorentz invariant:

$$\begin{aligned} D &= B^2 - E^2/c^2 \\ \alpha &= \mathbf{B}\cdot\mathbf{E}/c. \end{aligned}$$

[Hint: for the second, introduce the dual field tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$.]

Show that if $\alpha = 0$ but $D \neq 0$ then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example such frame (by specifying its velocity relative to one in which the fields are $\mathbf{E}$, $\mathbf{B}$). Suggest a type of field for which both $\alpha = 0$ and $D = 0$.

Additional questions

10. Twin paradox:

    (i) Evaluate the acceleration due to gravity at the Earth's surface (9.8 m/s^2) in units of years and light-years.

    (ii) In the twin paradox, the travelling twin leaves Earth on board a spaceship undergoing motion at constant proper acceleration of 9.8 m/s^2. After 5 years of proper time for the spaceship, the direction of the rockets are reversed so that the spaceship accelerates towards Earth for 10 proper years. The rockets are then reversed again to allow the spaceship to slow and come to rest on Earth after a further 5 years of spaceship proper time. How much does the travelling twin age? How much does the stay-at-home twin age?

Motion under a given force

11. Consider a particle moving in a straight line with speed v, rapidity ρ, and proper acceleration a_0. Prove that $d\rho/d\tau = a_0/c$. [Hint: use the fact that colinear rapidities are additive.]

12. The axis of a cylinder lies along the x' axis. The cylinder has no translational motion in S', but it rotates about its axis with angular speed ω'. When observed in S the cylinder travels and rotates.

    (i) Prove that in S at any instant the cylinder is twisted, with a twist per unit length $\gamma\omega' v/c^2$. [Hint: consider the rotation of the flat surfaces at the two ends of the cylinder; a line painted on either surface rotates like the hand of a clock.]

    (ii) Is the cylinder in mechanical equilibrium? Comment on whether or not you expect there to be internal shear forces in the cylinder in frame S.

13. A "photon rocket" propels itself by emitting photons in the rearwards direction. The rocket is initially at rest with mass m. Show that when the rest mass has fallen to αm the speed (as observed in the original rest frame) is given by

$$\frac{v}{c} = \frac{1-\alpha^2}{1+\alpha^2}.$$

[Hint: don't bother with equations of motion, use conservation of momentum.]

It is desired to reach a speed giving a Lorentz factor of 10. What value of α is required? Supposing the rocket cannot pick up fuel en route, what proportion of its initial mass must be devoted to fuel if it is to make a journey in which it first accelerates to $\gamma = 10$, then decelerates to rest at the destination (the destination being a star with negligible relative speed to the sun)?

14. A rocket propels itself by giving portions of its mass m a constant velocity $\mathbf{u}$ relative to its instantaneous rest frame. Let S' be the frame in which the rocket is at rest at time t. Show that, if v' is the speed of the rocket in S', then to first order in dv',

$$(-dm)u = mdv'.$$

Hence prove that, when the rocket attains a speed v relative to its initial rest frame, the ratio of final to initial rest mass of the rocket is

$$\frac{m_f}{m_i} = \left(\frac{1 - v/c}{1 + v/c}\right)^{c/2u}.$$

Note that the least expenditure of mass occurs when $u = c$, *i.e.*, the "photon rocket".

Prove that if the rocket moves with constant proper acceleration a_0 for a proper time τ, then $m_f/m_i = \exp(-a_0\tau/u)$.

15. Show that the motion of a particle in a uniform magnetic field is in general helical, with the period for a cycle independent of the initial direction of the velocity. [Hint: what can you learn from $\mathbf{f} \cdot \mathbf{v}$?]

16. A particle moves hyperbolically with proper acceleration a_0, starting from rest at $t = 0$. At $t = 0$ a photon is emitted towards the particle from a distance c^2/a_0 behind it. Prove that in the instantaneous rest frames of the particle, the distance to the photon is always c^2/a_0.

Electromagnetic field tensor

17. Assuming the relation of fields $\mathbf{E}$ and $\mathbf{B}$ to potentials ϕ and $\mathbf{A}$, show that the field tensor can be written

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

(Note, the right hand side here is the 4-vector equivalent of a curl operation.) [Hint: use cyclic permutation to avoid unnecessary repetition.] Now write down $\partial^\lambda F^{\mu\nu}$ in terms of ∂ operators and A^μ. By keeping track of the sequence of indices, show that

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0.$$

(In an axiomatic approach, one could argue in the opposite direction, asserting the above as an axiom and then deriving the relation of fields to potentials.)

18. If $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ are two different electromagnetic fields, show that $\mathbf{E}\cdot\mathbf{E}' - c^2\mathbf{B}\cdot\mathbf{B}'$ and $\mathbf{E}\cdot\mathbf{B}' + \mathbf{B}\cdot\mathbf{E}'$ are invariants.

B2: Symmetry and Relativity
Problem Set 4: Radiation
MT 2019

1. Obtain the electric field of a uniformly moving charge, as follows: place the charge at the origin of the primed frame S' and write down the field in that frame, then transform to S using the equations for the transformation of the fields (not the force transformation method) and the coordinates. Be sure to write your result in terms of coordinates in the appropriate frame. Sketch the field lines. Prove (from the transformation equations, or otherwise) that the magnetic field of a uniformly moving charge is related to its electric field by $\mathbf{B} = \mathbf{v} \wedge \mathbf{E}/c^2$.

2. A sphere of radius a in its rest frame is uniformly charged with charge density $\rho = 3q/4\pi a^3$ where q is the total charge. Find the fields due to a moving charged sphere by two methods, as follows.

   [N.B. it will be useful to let the rest frame of the sphere be S' (not S) and to let the frame in which we want the fields be S. This will help to avoid a proliferation of primes in the equations you will be writing down. Let S and S' be in the standard configuration.]

   (i) Field method: write down the electric field as a function of position in the rest frame of the sphere, for the two regions $r' < a$ and $r' \geq a$ where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. Use the field transformation equations to find the electric and magnetic fields in frame S (re-using results from previous questions where possible), making clear in what regions of space your formulae apply.

   (ii) Potential method: in the rest frame of the sphere the 3-vector potential is zero, and the scalar potential is

   $$\phi' = \frac{q}{8\pi\varepsilon_0 a}(3 - r'^2/a^2)$$

   for $r' < a$, and

   $$\phi' = \frac{q}{4\pi\varepsilon_0 r'}$$

   for $r' \geq a$.

   Form the 4-vector potential, transform it, and thus show that both ϕ and $\mathbf{A}$ are time-dependent in frame S. Hence derive the fields for a moving sphere. [Beware when taking gradients that you do not muddle $\partial/\partial x$ and $\partial/\partial x'$, etc.]

3. In a frame S a point charge first moves uniformly along the negative x-axis in the positive x direction, reaching the point $(-d, 0, 0)$ at $t = -\Delta t$, and then it is slowed down until it comes to rest at the origin at $t = 0$. Sketch the lines of electric field in S at $t = 0$, in the region $(x + d)^2 + y^2 + z^2 > (c\Delta t)^2$.

4. Give a 4-vector argument to show that the 4-vector potential of a point charge q in an arbitrary state of motion is given by

$$A^\mu = \frac{q}{4\pi\varepsilon_0}\frac{U^\mu/c}{-R_\nu U^\nu}$$

where U^μ and R^μ are suitably chosen 4-vectors which you should define in your answer.

5. The electromagnetic field of a charge in an arbitrary state of motion is given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0\kappa^3}\left(\frac{\hat{\mathbf{n}}-\mathbf{v}/c}{\gamma^2 r^2} + \frac{\hat{\mathbf{n}}\wedge[(\hat{\mathbf{n}}-\mathbf{v}/c)\wedge\mathbf{a}]}{c^2 r}\right)$$

where $\hat{\mathbf{n}} = \mathbf{r}/r$ and $\kappa = 1 - v_r/c = 1 - \hat{\mathbf{n}}\cdot\mathbf{v}/c$, and

$$\mathbf{B} = \hat{\mathbf{n}}\wedge\mathbf{E}/c$$

where $\mathbf{r}$ is the vector from the source point to the field point, and $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration of the charge at the source event. Without detailed derivation, outline briefly how this result may be obtained. How is the source event identified?

A charged particle moves along the x axis with constant proper acceleration ("hyperbolic motion"), its worldline being given by

$$x^2 - t^2 = \alpha^2$$

in units where $c = 1$. Find the electric field at $t = 0$ at points in the plane $x = \alpha$, as follows:

(i) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$. Show that the source event is at

$$x_s = \alpha + \frac{y^2}{2\alpha}.$$

(ii) Show that the velocity and acceleration at the source event are

$$\begin{aligned} v_s &= -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s} \\ a_s &= \frac{\alpha^2}{x_s^3}. \end{aligned}$$

(iii) Consider the case $\alpha = 1$, and the field point $y = 2$. Write down the values of x_s, v_s, and a_s. Draw on a diagram the field point, the source point, and the location of the charge at $t = 0$. Mark at the field point on the diagram the directions of the vectors $\hat{\mathbf{n}}$, $\mathbf{v}$, $\mathbf{a}$, and $\hat{\mathbf{n}}\wedge(\hat{\mathbf{n}}\wedge\mathbf{a})$. Hence, by applying the formula above, establish the direction of the electric field at $(t, x, y, z) = (0, 1, 2, 0)$.

(iv) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x axis such that their separation is constant, comment on the forces they exert on one another.

6. The far field due to an elementary wire segment dz carrying oscillating current I is given by

$$dE = \frac{I \sin\theta}{2\varepsilon_0 cr} \frac{dz}{\lambda} \cos(kr - \omega t).$$

Compare and contrast the case of a short antenna and the *half-wave dipole antenna*. Roughly estimate E in the far field for each case by proposing a suitable model for the distribution of current $I(z)$ in the antenna. What happens (qualitatively) for still longer antennae?

7. Show that the space-space part of the energy-momentum tensor

$$T^{\mu\nu} = \varepsilon_0 c^2 \left(-F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right)$$

is

$$\sigma^{ij} = \frac{1}{2} \varepsilon_0 (E^k E_k + c^2 B^k B_k) \delta^{ij} - \varepsilon_0 (E^i E^j + c^2 B^i B^j)$$

(Greek indices run over space and time, and Latin indices over space only.)

Use the stress-energy tensor $T^{\mu\nu}$ to find the forces exerted by the magnetic field inside a long cylindrical solenoid of radius 3 cm and field 1 tesla. *Mu-metal* is an alloy of high magnetic permeability that can be used to provide shielding against magnetic fields. If a piece of mu-metal is placed against the end of a solenoid, it "confines" the magnetic field to the interior of the solenoid. By interpreting the stress-energy tensor for the field on each side of the mu-metal sheet, discover whether the latter is attracted or repelled by the solenoid, and find the net force.

8. Write down the stress-energy tensor and the 4-wave vector for an electromagnetic plane wave propagating in the x direction.

Such a wave is observed in two frames in standard configuration. Show that the values of radiation pressure P, momentum density g, energy density u, and frequency ν in the two frames satisfy

$$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2}$$

(Optional: can you prove this for any relative motion of the frame? [Hint: write $T^{\mu\nu}$ in terms of K^μ]).

A confused student proposes that these quantities should transform like ν'/ν rather than ν'^2/ν^2, on the grounds that energy-momentum $N^\mu = (uc, \mathbf{N})$ is a 4-vector and so should transform in the same way as the wave vector. What is wrong with this argument?

Additional questions

9. An isolated parallel plate capacitor has charge $\pm Q$ on the plates. It is initially at rest in the laboratory frame. Assuming the capacitor's proper dimensions are fixed, what uniform motion should be given to the capacitor in order to increase the electric field between the plates? Does this result in an increased force of attraction between the plates?

10. A current-carrying wire is electrically neutral in its rest frame S. The wire has cross-sectional area A and a current I flows uniformly through this cross-section. Write down the 4-vector current density in the rest frame of the wire. Obtain the 4-vector current density in a rest frame S' moving with non-zero velocity v parallel to the wire. Hence show that in this frame there is a non-zero charge density in the wire. Does this imply that charge is not Lorentz invariant? Explain. Find the electric field in S' produced by the charge density of the wire, in the region outside the wire, and show that it is the same as the field obtained by transformation of the magnetic field in frame S.

Retarded potentials and radiative emission

11. Retarded potential:

    (i) Write down the solution to Poisson's equation for the case of a point charge q.

    (ii) In electrostatics, how is the electric potential at a point in space obtained if the charge distribution is known?

    (iii) Now consider the wave equation

    $$\partial_\mu \partial^\mu \phi = -\frac{\rho}{\varepsilon_0}.$$

    Show that the spherical wave form $\phi = \kappa g(t - r/c)/r$ (where $\kappa \equiv 1/4\pi\varepsilon_0$) is a solution of the wave equation for $r \neq 0$ if ρ is zero everywhere except at the origin.

    (iv) We would like to show that this is a solution also as $r \to 0$, if the charge density ρ is concentrated at a point at the origin. Using your knowledge of Poisson's equation, or otherwise, show that this is true as long as $g(t) = \int \rho(t) dV$.

    (v) Hence write down the solution to the wave equation for a given arbitrary time-dependent distribution of charge.

    (vi) Why is this called a *retarded* solution?

Field energy and momentum

12. In this problem we will compare power loss in electron accelerators.

    (i) The electric field in a linear accelerator is 10^6 V/m. Find the power emitted by an electron travelling down the accelerator. Express your results in eV per metre assuming the electrons travel at close to the speed of light. (You may quote Larmor's formula for emitted power.)

    (ii) A magnetic field of 1 tesla is used to maintain electrons in their orbits around a synchrotron of radius 10 m. Show that the electron energy is approximately 3 GeV. Find the radiative energy loss per revolution.

    (iii) What is the main reason why the loss rate is so much higher in the synchrotron case, compared with the linear accelerator?

13. Various identities—mostly needs good organisation to attack the algebra:

(i) Show from Maxwell's equations, or otherwise, that

$$\rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial \mathbf{g}}{\partial t} + \varepsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\nabla \wedge \mathbf{E}) \wedge \mathbf{E} + c^2(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}] \qquad (1)$$

where $\mathbf{g} = \mathbf{N}/c^2$ and $\mathbf{N}$ is the Poynting vector.
We wish to show that this can be written

$$\rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial \mathbf{g}}{\partial t} - \mathbf{s} \qquad (2)$$

such that $s^i = \partial_j \sigma^{ij}$ with

$$\sigma^{ij} = \frac{1}{2}\varepsilon_0(E^k E_k + c^2 B^k B_k)\delta^{ij} - \varepsilon_0(E^i E^j + c^2 B^i B^j).$$

To this end, take the following steps (or use another method if you prefer):

(ii) Show that the x component of $(\nabla \wedge \mathbf{E}) \wedge \mathbf{E}$ is

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) E_z - \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) E_y.$$

(iii) Evaluate just the electric field contribution to s^1, the x component of $\mathbf{s}$.

(iv) Confirm that you have matched the x component of the electric field part of the square bracket in equation 1. Explain why the magnetic part also follows.

(v) Is it necessary to calculate explicitly the other components?

(vi) Multiplying equation 2 by a small volume δV, we have

$$\delta V \rho\mathbf{E} + \delta V \mathbf{j} \wedge \mathbf{B} = -\delta V \frac{\partial \mathbf{g}}{\partial t} - \delta V \mathbf{s}. \qquad (3)$$

Explain the physical significance of the terms in this equation (and thus justify all this hard work!).

14. A pair of parallel particle beams separated by a distance d have the same uniform charge per unit length λ. In the laboratory frame, a magnetic field is applied with a direction and strength just sufficient to overcome the repulsion between the beams, so that they both propagate in a straight line at constant speed v. Find the size B of this magnetic field, by both of the following methods:

(i) Do the whole calculation in the laboratory frame.

(ii) Start with a calculation of the force exerted by either beam on a particle in the other, in the rest frame of the beams. Transform this force to the laboratory frame and hence deduce the required B field in that frame. What form does the externally applied field take in the rest frame of the beams?

15. Consider the general problem of motion of a particle of charge q in a uniform constant electromagnetic field. The equation of motion is

$$qF^{\mu}{}_{\nu}U^{\nu} = m\frac{dU^{\mu}}{d\tau}$$

For a uniform constant field, $F^{\mu\nu}$ is independent of space and time, and therefore this matrix equation is precisely the same as the one obtained in a classical normal modes problem, and can be solved by the same methods.

(i) Propose a solution $U^{\mu} = U_0^{\mu}\exp(\Gamma\tau)$ and thus convert the equation into an eigenvalue equation with eigenvalues $\lambda = m\Gamma/q$. (Note that $F^{\mu}{}_{\nu}$ is not symmetric, so the right-eigenvectors are not the same as the left-eigenvectors. We only need the right-eigenvectors here.)

(ii) Without loss of generality, we can take the z-axis along $\mathbf{B}$, and $\mathbf{E}$ in the xz plane. Show that the eigenvalues are

$$\lambda^2 = -\frac{D}{2} \pm \sqrt{D^2/4 + \alpha^2}$$

where $D = |\mathbf{B}|^2 - |\mathbf{E}|^2/c^2$ and $\alpha = \mathbf{B}\cdot\mathbf{E}/c$.

(iii) Consider the case $\alpha = 0$. What does this tell us about the fields? Interpret the solution corresponding to a zero eigenvalue [Hint: Lorentz force].

(iv) Find $U^{\mu}(\tau)$ for a particle initially at rest in a uniform purely electric field.

(v) Show that the right eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

may be written

$$\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Hence find $U^{\mu}(\tau)$ for a particle moving in a plane perpendicular to a uniform purely magnetic field.

16. In a frame S there is a uniform electric field E along the y direction and a uniform magnetic field $B = 5E/3c$ along the z direction. A particle of mass m and charge q is released from rest on the x axis. What time elapses before it returns to the x axis?

17. In synchrotron radiation, in which direction is most of the energy emitted in the rest frame of the accelerating charge? Describe qualitatively the pattern of the radiation field in the rest frame of the centre of the synchrotron apparatus.

18. Why is it not feasible for mobile phones to use radio waves?

Energy-momentum tensor

19. In a certain frame S_0 having 4-velocity U^μ, a 2nd-rank tensor $T^{\mu\nu}$ has but one non-zero component, $T^{00} = c^2$. Find the components of T in the general frame S, in which $U^\mu = \gamma_u(c, \mathbf{u})$.

20. Prove that for the electromagnetic stress-energy tensor, $T^\lambda{}_\lambda = 0$.

21. Prove that for the electromagnetic stress-energy tensor, $T^\mu{}_\lambda T^\lambda{}_\nu = (I\varepsilon_0/2)^2\delta^\mu_\nu$ where $I^2 = (|\mathbf{E}|^2 - c^2|\mathbf{B}|^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2c^2$. [Hint: start by establishing the identity in a particular frame, such as one in which $\mathbf{E}$ is parallel to $\mathbf{B}$.]

B2: Symmetry and Relativity
Problem Set 5: Groups and Revision
MT 2019

1. Show that the Lorentz transformations in a single spatial direction form a group.

2. Show that $e^{in\theta}$, with θ a constant, is a representation of the group of integers n under the addition operator. If $\theta = \pi/N$, how many elements does the representation have, and in what sense is it still a representation of the infinite group of integers?

3. Write down a set of 3×3 matrices to represent the permutation group on three objects, such that the action of swapping the second and third objects is the matrix

$$(D_{132})^i{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   Show that this matrix representation is reducible by the following steps.

   (i) Find a common eigenvector for all the $D^i{}_j$ matrices.

   (ii) Write down a suitable similarity transformation matrix $S^i{}_j$, such that $(D')^i{}_j = S^i{}_m D^m{}_n (S^{-1})^n{}_j$ with the common eigenvector becoming $(1, 0, 0)$ in the new basis.

   (iii) Show that the transformation matrices in the new basis take on block-diagonal form.

4. Show that the following matrix generates a rotation around the x axis

$$(J_1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

   using the exponential $R(\theta) = e^{-i\theta J_1}$.

   Show that the matrix

$$(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   generates a boost in the x direction with $\Lambda(\eta) = e^{-i\eta K_1}$.

   Write the matrix form of the generator K_2 for infinitesimal boosts along the y axis. Multiply an infinitesimal boost along x by another along y. What does the form of the matrix indicate about whether non-aligned Lorentz transformations can form a group?

5. The canonical $j = 1$ representation of the generators of 3D rotations can be derived from following rules:

$$\begin{aligned} J_3|m\rangle &= m|m\rangle \\ J_\pm|m\rangle &= [j(j+1) - m(m\pm 1)]^{1/2}|m\pm 1\rangle \\ J_\pm &= J_1 \pm iJ_2 \end{aligned}$$

Write down matrices representing the J_i generators using the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$.

Verify that the generators satisfy the same Lie algebra as that of the $SO(3)$ group, *i.e.*,

$$[J_j, J_k] = i\sum_m \epsilon_{jkm} J_m.$$

Using the appropriate spherical harmonics as basis of the $j = 1$ representation space, show that J_3 generates rotations around $\hat{\mathbf{z}}$. Similarly, show that the spherical harmonics representing a direction in the yz plane are rotated around $\hat{\mathbf{x}}$ by J_1.

6. Consider the Dirac equation of a fermion field in the presence of an electromagnetic field A^μ:

$$(i\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu - m)\psi = 0$$

where the γ^μ are 4×4 matrices with the anti-commutation relationship

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$$

(The important thing to keep in mind here is merely that there are 4 distinct matrices; you shouldn't need the anti-commutation relationship itself.) Apply the local gauge transformation to the fermion field

$$\psi \to \psi' = e^{iq\alpha}\psi$$

where α is also a function of spacetime. Show that local gauge invariance can be restored by applying a simultaneous gauge transformation to the electromagnetic field,

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi.$$

What must the relationship be between χ and α?

Past exam problems

7. (Based on B2 2014 Q3.) Write down the relationship between electric and magnetic fields and the scalar and vector potentials. The Faraday tensor is related to the four-vector potential by the expression

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Use this to find the expression for $F^{\mu\nu}$ in terms of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

The electromagnetic field satisfies $\partial_\lambda F^{\lambda\kappa} = -\mu_0 J^\kappa$ where J^κ is the four-current. Use this equation to obtain two of the Maxwell equations. Also, obtain the other two Maxwell equations given that $F^{\lambda\kappa}$ can be obtained from a 4-potential.

The stress-energy tensor of the electromagentic field may be written

$$T^{\mu\nu} = \varepsilon_0 c^2(-F^{\mu\lambda}F_\lambda{}^\nu - \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}).$$

Find the top row of this tensor (*i.e.*, $T^{0\nu}$) in terms of $\mathbf{E}$ and $\mathbf{B}$ for a general field, and identify the physical quantities obtained. Find the $\nu = 0$ component of $\partial_\lambda T^{\lambda\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, and state how it is related to the current density $\mathbf{j}$.

A parallel-plate capacitor has its plates parallel to the xz plane and moves relative to the laboratory in the x direction at speed v. Let E be the electric field between the plates in the rest frame of the capacitor. Write down the Faraday tensor for this field in the rest frame, and hence obtain the stress-energy tensor, first in the rest frame and then in the laboratory. Hence find the Poynting vector of the field observed in the laboratory.

Repeat the calculation for a capacitor moving in the same way whose plates are parallel to the yz plane. In both cases describe the physical processes whereby the capacitor's stored energy is transported.

8. (Based on B2 2014 Q4.) Write down the scalar and vector potential for the field of a charged particle at rest. Hence, carefully explaining your reasoning, show that the 4-vector potential of an arbitrarily moving charged particle is given by

$$A^\mu = \frac{q}{4\pi\varepsilon_0}\frac{U^\mu/c}{(-R^\nu U_\nu)}$$

and define the quantities U^μ and R^ν involved in this expression. Make sure you justify the claim that your argument leads to a genuinely covariant result.

A particle of charge q is moving at constant, non-relativistic speed around a circle in the xy plane, such that its position is given by $(x, y, z) = (a\cos\omega t_s, a\sin\omega t_s, 0)$ at any given time t_s. It is desired to obtain the electric field $\mathbf{E}$ at points on the z axis. Let $\mathbf{v}_s$ be the velocity of the particle at the source time. Find an expression for A^μ in terms of q, a, z, $\mathbf{v}_s$, and fundamental constants. Hence obtain E_z.

To find the other components of $\mathbf{E}$, the gradient of the scalar potential ϕ is required. Consider a field event at $(\Delta x, 0, z)$ at time t, and obtain the dependence of ϕ on Δx to first order. Hence show that the x component of the electric field at $(0, 0, z)$ is given by

$$E_x = \frac{qa}{4\pi\varepsilon_0 c^2\sqrt{a^2+z^2}}\left(\left(\omega^2 - \frac{c^2}{a^2+z^2}\right)\cos\omega t_s + \frac{\omega c}{\sqrt{a^2+z^2}}\sin\omega t_s\right).$$

9. (Based on B2 2017 Q1.) For a particle of mass m moving along a worldline $X^\mu = X^\mu(\tau)$ in the inertial reference frame S, define the 4-velocity U^μ and the 4-force F^μ. Show that if the scalar product of two non-zero 4-vectors D^μ and G^μ is zero, and D^μ is time-like, then G^μ must be space-like. Prove that if the particle has a 4-momentum P^μ

such that $P^\mu P_\mu = 0$, then the particle has a rest mass equal to zero. Give an example of such a particle.

Define proper time and pure force. Show that if F^μ is a pure force, then $F^\mu U_\mu = 0$, where U^μ is the 4-velocity. Show that if 3-velocity $\mathbf{v}$ and 3-acceleration $\mathbf{a}$ are orthogonal to each other, *i.e.*, $\mathbf{v} \perp \mathbf{a}$, then the 4-acceleration invariant $A^\mu A_\mu$ is given by $A^\mu A_\mu = \gamma^4 a^2$.

A free particle, having rest mass M_0, is in vacuum and initially at rest in the lab frame S. It undergoes an acceleration under the action of a constant pure force with $\mathbf{f} = (f, 0, 0)$. Find its 4-velocity U^μ as a function of time and force. Sketch the graphs of the dependence of normalised 3-velocity $\boldsymbol{\beta} = \mathbf{v}/c$ and the Lorentz factor $\gamma(t)$ of the particle as a function of time t.

An electron and a positron are annihilated during a head-on collision. Before the collision they had 3-velocities $\mathbf{v}_e = (v_x, 0, 0)$ and $\mathbf{v}_p = (-v_x, 0, 0)$, respectively. After the collision, some number of photons are detected. What is the minimum number of photons that can be registered in this experiment? Explain your answer. Find the energy of each of the minimal number of photons taking into account that the rest mass of the electron and the positron is each equal to 0.511 MeV and their total kinetic energy before the collisions was 1 GeV.

An electron is accelerated from rest through a gap of $L = 3$ m by an electric field of strength 10 MV m^{-1} which is constant throughout the gap. Find γ and β at the other end of the gap.

10. (Based on part of B1 2004 Q3.) For an isolated system of particles, let

$$s^2 = \left(\sum E_i\right)^2 - \left(\sum \mathbf{p}_i c\right)^2$$

where the sums are taken over the particles in the system at some given time. What is s for a single particle of mass m?

In the laboratory frame a particle of mass m and momentum p_m is incident on a particle of mass M, at rest. Find an expression for the total available energy in the centre-of-mass frame.

Show that the momentum of the particle of mass m in the centre-of-mass frame is given by $p'_m = Mc^2 p_m / s$.

11. (Based on B1 2006 Q8.) It is proposed to generate a pure beam of either electron neutrinos or electron antineutrinos by accelerating ions of unstable nuclei to relativistic speeds and then allowing them to decay in a long straight section of the accelerator.

An unstable ion of rest mass M decays after it has been accelerated to total energy E and Lorentz factor $\gamma = E/Mc^2$ and emits a neutrino of energy E_ν at an angle of θ to the beam direction. (i) Derive an expression for the neutrino's energy E^*_ν in the rest frame of the ion in terms of E_ν, θ, and the velocity of the ion βc. (ii) Show that in the rest frame of the ion, the neutrino's path is inclined to the beam direction by the angle θ^* that satisfies

$$\cos\theta^* = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}.$$

Ions are accelerated to $\gamma = 100$ and decay in the straight section of the accelerator. A cylindrical detector that is coaxial with the beam and has radius $r = 30$ m, is placed $D = 300$ km downstream. Show that the angle between the beam direction and that of a neutrino which will hit the outer edge of the detector, measured in the rest frame of the ion, is approximately given by

$$\cos\theta^* = \frac{1 - \gamma^2\theta^2}{1 + \gamma^2\theta^2 - \theta^2/2}$$

where $\theta \simeq r/D$. Given that the emission of neutrinos is isotropic in the ion rest frame, find the fraction of the neutrinos that pass through the detector.

Show that in the ion rest frame the detector subtends an angle $2\theta_r^*$ at the ion at the emission event, where $\theta_r^* = \tan^{-1}(\gamma r/D)$. Why does θ_r^* differ from θ^*?