Outline of the course

The lecture course on *Superconductivity* will be given in 7 lectures. The topics covered include:

- Introduction to superconductivity
- The London equations
- Ginzburg-Landau theory
- The Josephson effect
- BCS theory and the energy gap
- Unconventional superconductors and superconducting technology

Reading

• '*Superconductivity, Superfluids and Condensates*', by J. F. Annett, OUP 2004: the best book for the course.

• '*Superconductivity of metals and cuprates* ', by J. R. Waldram (IOP), 1996: a good introduction to different concepts on superconductivity.

• '*Solid State Physics*' by N. W. Ashcroft and N. D. Mermin, chapters 34, is a good overview of some of the material in the course but written in CGS units. The Chapter 10 on Superconductivity in '*Introduction to Solid State Physics*' by Chales Kittel is also a good starting point for some of the introductory concepts.

• '*Introduction to Superconductivity*' by M. Tinkam, covers advanced aspects on superconductivity.

• '*The Physics of Superconductors Introduction to Fundamentals and Applications*' by V. V Schmidt, is a useful book for some of the topics.

• An advanced but informative description of the ideas concerning broken symmetry may be found in the second chapter of '*Basic Notions of Condensed Matter Physics*', by P. W. Anderson (Benjamin-Cummings 1984).

• A popular account of the history of superconductivity is presented in *Superconductivity: A Very Short Introduction*' by Stephen Blundell, OUP 2009.

Web-page for the course:

`https://canvas.ox.ac.uk/courses/124773`

Acknowledgements:

This is to acknowledge and thank the previous lecturers: Stephen Blundell, Paul Goddard and Peter Leek for sharing their notes.

This handout contains figures and material either complementary or additional to the content of the lectures.

NOTE: These notes will be updated until the end of the lectures in Hilary 2022.

Recap: Maxwell's equations:

In free space, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad (1)$$
$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (4)$$

and describe the relationships between the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, the charge density ρ and the current density $\mathbf{J}$.

Equation 1 shows that the flux of $\mathbf{E}$ through an close surface depends on the change inside; electric field diverges away from positive charges and converges into negative charges; charge density therefore acts as a source or a sink of electric field (*Gauss's law*). Equation 2 shows that the flux of $\mathbf{B}$ through an close surface is zero; thus the magnetic fields have no such divergence; thus, there are no magnetic charges (called monopoles) and the lines of $\mathbf{B}$ field must just exist in loops; they can never start or stop anywhere (*Gauss's law for magnetism*). Equation 3 shows that the line integral of $\mathbf{E}$ around a loop is directly related to the time variation of the flux of $\mathbf{B}$ through the loop; thus, one can only get loops of electric field around regions in space in which there is a changing magnetic field. This expression leads to *Faraday's law* of electromagnetic induction. Equation 4 shows that the integral of $\mathbf{B}$ around a loop is equal to the current though the loop and the time dependence of the flux of $\mathbf{E}$ through the loop; thus, in the absence of a changing electric field, loops of magnetic induction are found around electric currents leading to the *Ampere's law*.

In the presence of matter, the expressions become:

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad (5)$$
$$\nabla \cdot \mathbf{B} = 0 \quad (6)$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7)$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t}, \quad (8)$$

For materials without polarization and magnetization, the displacement field is $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$, where ϵ_0 is the permittivity of free space and μ_0 the permeability of free space.

Recap: Charges moving in fields

(i) *Magnetic Vector Potential*

The magnetic vector potential $\mathbf{A}$ is defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (9)$$

Since $\mathbf{A}$ is essentially derived from $\mathbf{B}$ via an integration, this definition is not complete. One can add another term to $\mathbf{A}$, which is the gradient of a scalar function χ, that keeps $\mathbf{B}$ unchanged.

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \quad (10)$$
$$\mathbf{B} \rightarrow \nabla \times (\mathbf{A} + \nabla\chi) = \nabla \times \mathbf{A} \quad (11)$$

A choice of χ is called a choice of *gauge*. In a given situation the gauge can be chosen so as to make the mathematics simple. In many cases the gauge is fixed so as to have $\nabla \cdot \mathbf{A} = 0$, this is known as the Coulomb gauge or, in the area of superconductivity, the London gauge.

Note that $\mathbf{A}$ also contributes to the electric field,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{12}$$

and although the electric scalar potential V is also altered by a change of gauge, the field remains unaffected

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \tag{13}$$

$$\phi \rightarrow V - \frac{\partial \chi}{\partial t} \tag{14}$$

$$\mathbf{E} \rightarrow -\nabla(V - \frac{\partial \chi}{\partial t}) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla\chi) = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{15}$$

(ii) *Canonical Momentum*[*]

In classical mechanics the Lorentz force $\mathbf{F}$ on a particle with charge q moving with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{16}$$

Using $\mathbf{F} = m\mathrm{d}\mathbf{v}/\mathrm{d}t$, $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla V - \partial\mathbf{A}/\partial t$, where m is the mass of the particle, this equation may be rewritten as

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -q\nabla V - q\frac{\partial \mathbf{A}}{\partial t} + q\mathbf{v} \times (\nabla \times \mathbf{A}). \tag{17}$$

Simplifying with the vector identity

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{18}$$

yields

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + q\left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}\right) = -q\nabla(V - \mathbf{v} \cdot \mathbf{A}). \tag{19}$$

Note that $m\,\mathrm{d}\mathbf{v}/\mathrm{d}t$ is the force on a charged particle measured in a coordinate system that moves with the particle. The partial derivative $\partial\mathbf{A}/\partial t$ measures the rate of change of $\mathbf{A}$ at a fixed point in space. We can rewrite Equation 19 as

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v} + q\mathbf{A}) = -q\nabla(V - \mathbf{v} \cdot \mathbf{A}) \tag{20}$$

where $\mathrm{d}\mathbf{A}/\mathrm{d}t$ is the *convective derivative* of $\mathbf{A}$, written as

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}, \tag{21}$$

which measures the rate of change of $\mathbf{A}$ at the location of the moving particle. Equation 20 takes the form of Newton's second law (i.e. it reads 'the rate of change of a quantity that looks like momentum

[*]This section is from Chapter 1 of the Magnetism in Condensed Matter by Stephen Blundell.

is equal to the gradient of a quantity that looks like potential energy'). We therefore define the *canonical momentum*

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \tag{22}$$

and an effective potential energy experienced by the charge particle, $q(V - \mathbf{v} \cdot \mathbf{A})$, which is velocity-dependent. The canonical momentum reverts to the familiar momentum $m\mathbf{v}$ in the case of no magnetic field, $\mathbf{A} = 0$. The kinetic energy remains the energy associated with actual motion throughout, i.e. equal to $\frac{1}{2}mv^2$, and can therefore be written in terms of the canonical momentum as $(\mathbf{p}-q\mathbf{A})^2/2m$.

In terms of quantum mechanics, when considering the wave functions of charged particles, one must make the replacement [†]

$$-i\hbar\nabla \rightarrow -i\hbar\nabla - q\mathbf{A}. \tag{23}$$

(iii) *Hamiltonian of a particle in an electromagnetic field*

The time-dependent Schrodinger equations,

$$\mathbf{H}\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad \text{and} \qquad \mathbf{H}\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t} \tag{24}$$

The total Hamiltonian for a charge particle can be written as

$$\mathbf{H} = \frac{[\mathbf{p} - q\mathbf{A}]^2}{2m} + qV = \frac{\mathbf{P}^2}{2m} + qV \tag{25}$$

where the first part if the kinetic energy, $\mathbf{P}$ is the *kinetic momentum operator* (which is guage invariant) and the second term is the potential energy. Application of the correspondence principle to the Hamiltonian function leads to:

$$\mathbf{H} = \frac{[-i\hbar\nabla - q\mathbf{A}]^2}{2m} + qV \tag{26}$$

which is the quantum mechanical operator associated with the kinetic energy of a charged particle in a magnetic field. The kinetic energy is a guage invariant term.

The current density satisfies the continuity equation, which expresses the (local) charge conservation. For a wave-function one can find a *current* of probability such that if there is any change in the probability density.

$$\frac{\partial\rho_v}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{27}$$

The probability density to find the particle at a position is given by the absolute square of the wave function. The charge density ρ_v is the probability density multiplied by the charge q.

$$\rho_v = q\psi^*\psi \qquad \text{and} \qquad \frac{\partial\rho_v}{\partial t} = q\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) \tag{28}$$

Using the Schrödinger equations and the continuity equation, the probability density current, which is real, becomes

$$\mathbf{J} = \frac{q}{2m}\left[\psi^*(-i\hbar\nabla - q\mathbf{A})\psi + \psi(i\hbar\nabla - q\mathbf{A})\psi^*\right] \tag{29}$$

[†] see, for example, Feynman Lectures on Physics, Vol. 3, Chapter 21; https : //www.feynmanlectures.caltech.edu/III$_2$1.html

Zero resistance

The conductivity of a normal metal is described by the Drude model applied to a Fermi gas:

$$\sigma = \frac{ne^2\tau}{m} \tag{30}$$

where n is the density of conduction electrons, m is the effective mass of the conduction electrons, e is the electron charge and τ is the average lifetime for free motion of the electrons between collisions with impurities or other electrons.

The conductivity is defined by the constitutive equation:

$$\mathbf{J} = \sigma\mathbf{E} \tag{31}$$

where $\mathbf{J}$ is the electrical current density which flows in response to the external electric field, $\mathbf{E}$.

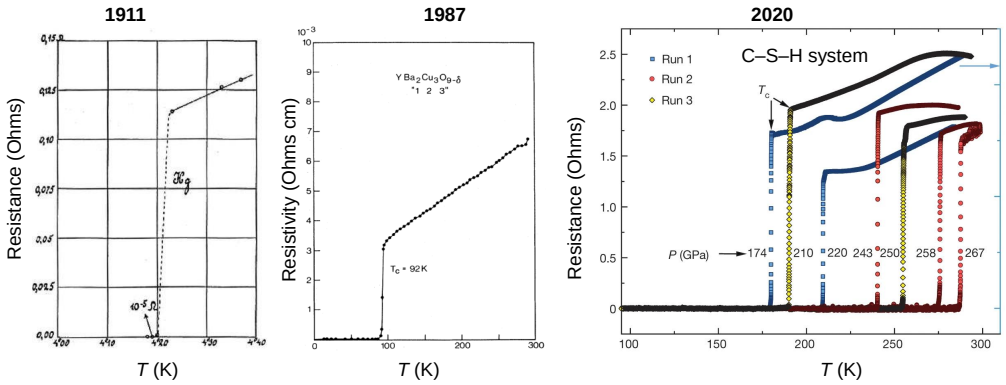
The resistivity of a metal follows:

$$\mathbf{E} = \rho\mathbf{J} \tag{32}$$

as ρ is simply the reciprocal of the conductivity, $\sigma = 1/\rho$. Thus, the resistivity is proportional to the scattering rate, τ^{-1} of the conduction electrons.

$$\rho = \frac{m}{ne^2}\tau^{-1} \tag{33}$$

In a normal metal the resistivity at low temperatures has a constant contribution from impurity scattering, ρ_0, a T^2 contribution from electron-electron scattering, and a T^5 contribution from phonon scattering.



A superconductor is characterized by its zero resistance below a critical temperature, T_c. As the resistance and resistivity is zero (R=0, ρ=0) implies, according to the Ohms's law $\mathbf{E} = \rho\mathbf{J}$, that the electric field inside of a superconductor is zero $\mathbf{E} = 0$. Thus, the vanishing of the resistivity at low temperatures in a superconductor is a clear indication of a new ground state.

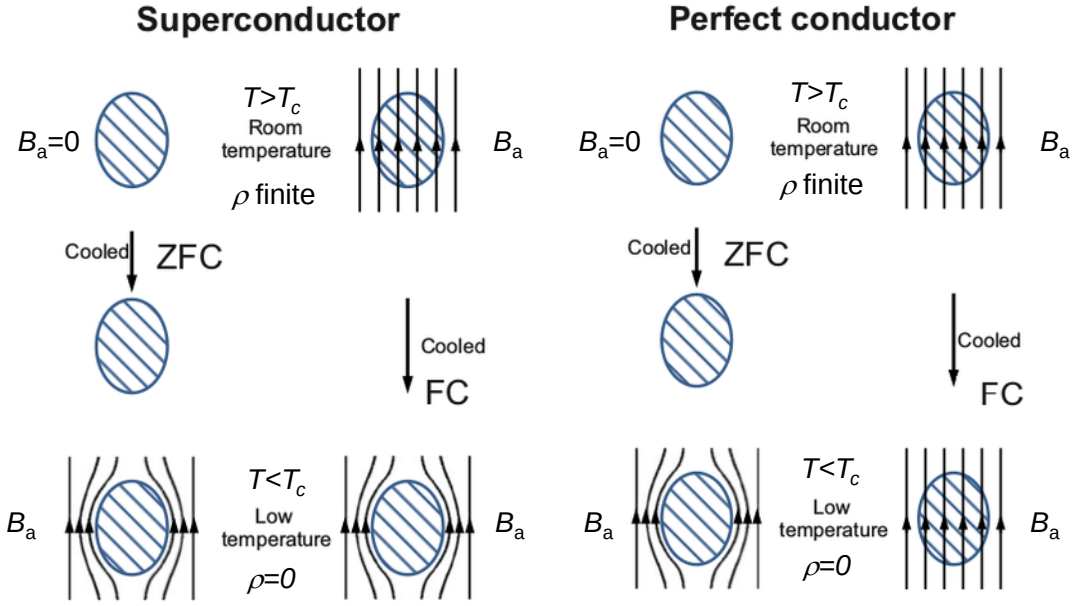
The superconductivity was first discovered by Kammerlingh Onnes in 1911 in pure samples of mercury with T_c ~4.2 K (left panel figure above). Materials that superconductor above liquid nitrogen temperature (77 K) were discovered in copper oxide superconductors (see $YBa_2Cu_3O_{7-\delta}$ with T_c ~92 K in the middle panel). Room temperature superconductivity has been reached in 2020 in a system containing light elements C-S-H (hydrides) under extreme high pressures above 267 GPa (right panel).

Superconducting transition temperatures

Substance	T_c (K)
Elemental superconductors -type I	
Al	1.196
Hg	4.15
In	3.40
Pb	7.19
Sn	3.72
Zn	0.875
Conventional superconductors - type II	
Nb	9.26
Nb_3Sn	18.1
V_3Si	17
Nb_3Ge	23.2
H_3S (hydride) (150 GPa)	203
LH_{10} (hydride) (170 GPa)	250
H_2S+CH_4 (hydride) (267 GPa)	287
Unconventional superconductors	
$BaPbO_3$	0.4
$Ba_xLa_{5-x}Cu_5O_y$	30–35
$YBa_2Cu_3O_{7-\delta}$	92
$Bi_2Sr_2Ca_2Cu_3O_{10}$	110
$Hg_{0.8}Pb_{0.2}Ba_2Ca_2Cu_3O_x$	133
$HgBa_2Ca_2Cu_3O_{8+\delta}$ at 30 GPa	164
URu_2Si_2	1.3
MgB_2	39
YNi_2B_2C	12.5
$(TMTSF)_2ClO_4$	1.4
K_3C_{60}	19
Cs_3C_{60} at 7 kbar	38
$(BEDT\text{-}TTF)_2Cu(NCS)_2$	10.4
$(BEDT\text{-}TTF)_2Cu[N(CN)_2]Br$	11.8
$Sm[O_{1-x}F_x]FeAs$	55
FeSe	9
1ML $FeSe/SrTiO_3$	65-75

Meissner effect

The Meissner state is characterized by the perfect diamagnetism and the complete expulsion of the magnetic flux inside of a superconductor ($\mathbf{B} = 0$ and $\mathbf{E} = 0$).



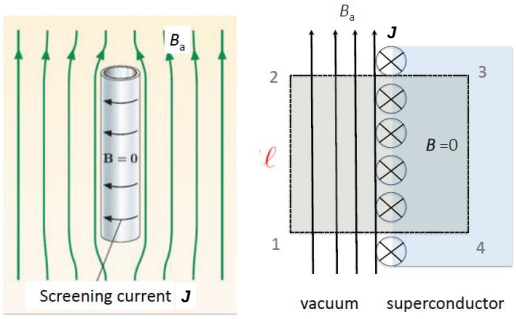
The figure illustrates the behaviour of a superconductor versus a perfect conductor in magnetic field and depending on the process of cooling down the material without (zero-field cooled, ZFC) or in field (field-cooled, FC). The low temperature ground state of a superconductor is not affected by the order of these processes.

Perfect conductor. Assuming that initially a perfect conductor is cooled down below the critical temperature in zero external magnetic field (ZFC in the right panel). After that, an external field is applied at low temperatures when it enter the zero resistance state. A magnetic field does not penetrate the interior the perfect conductor. According to Lenz's law, immediately the field penetrates the surface layer of the ideal conductor, an induced current is set up which generates a magnetic field in the direction opposite to that of the external field. Therefore, the total field in the interior of the specimen is zero, $\mathbf{B}$=0. However, the same situation (an ideal conductor at $T < T_c$ in an external magnetic field) can be reached through a different sequence of events. Firstly, by applying the external field at high temperatures where it has finite resistance and, therefore, the magnetic field penetrates into it. By cooling it down in magnetic field to reach the zero resistance state below $T < T_c$ (FC in the right panel), the magnetic field remains trapped inside. From Maxwell Equation 3, $d\mathbf{B}/dt$ is proportional to $\nabla \times \mathbf{E}$, so that zero resistivity implies $\mathbf{E} = 0$ and $d\mathbf{B}/dt$=0, but not $\mathbf{B}$=0. This suggests that the ground state of a perfect conductor is not unique and the flux through the metal cannot change on cooling through the transition.

Superconductor On the other hand, in a superconductor no matter how the system is cooled down (ZCF or FC in the left panel) it always expels the magnetic below $T < T_c$. Thus, a superconductor not only fulfills the zero-resistance condition with $\mathbf{E} = 0$ but also it expels the external magnetic field $\mathbf{B}$=0 having a unique ground state and one can use the equilibrium thermodynamics. The Meissner effect suggests that perfect diamagnetism is an essential property of the superconducting state.

Surface currents in superconductors

The Meissner effect has remarkable implications. For a superconducting cylinder in the presence of an external magnetic field, the internal magnetic field must disappear. This implies the presence of a surface current around the cylinder, such that the resulting solenoidal field exactly cancels the applied field throughout the volume of the rod (left panel). A current in a long solenoid (to eliminate the effect of the demagnetizing factors) produces a uniform field inside the solenoid parallel to the axis with a magnitude determined by the surface current density (current per unit length along the solenoid axis), and no field outside the solenoid. Magnetic field lines are tangential to the surface of the superconductor. Any change of the externally applied field will cause a change of the surface current that maintains zero field inside the cylinder. Surface currents can exist only when the superconductor is placed in an external magnetic field.



Using Maxwell's equation 4 for a superconductor with **E**=0 leads to the Ampere's law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{34}$$

This can be expressed as the line integral of **B** around a closed path is related to the current which passes through that enclosed path

$$\oint_{contour} \mathbf{B} \cdot d\ell = \mu_0 \int_{area} \mathbf{J} \cdot \mathbf{dS} = \mu_0 I \tag{35}$$

Considering an Amperean loop, the contour 1-2-3-4 -1, at the surface of a superconductor (dashed rectangle in right panel). Along the section 1-2 of length ℓ, which is parallel to the surface, we have

$$\oint_1^2 \mathbf{B} \cdot d\ell = \mu_0 I \tag{36}$$

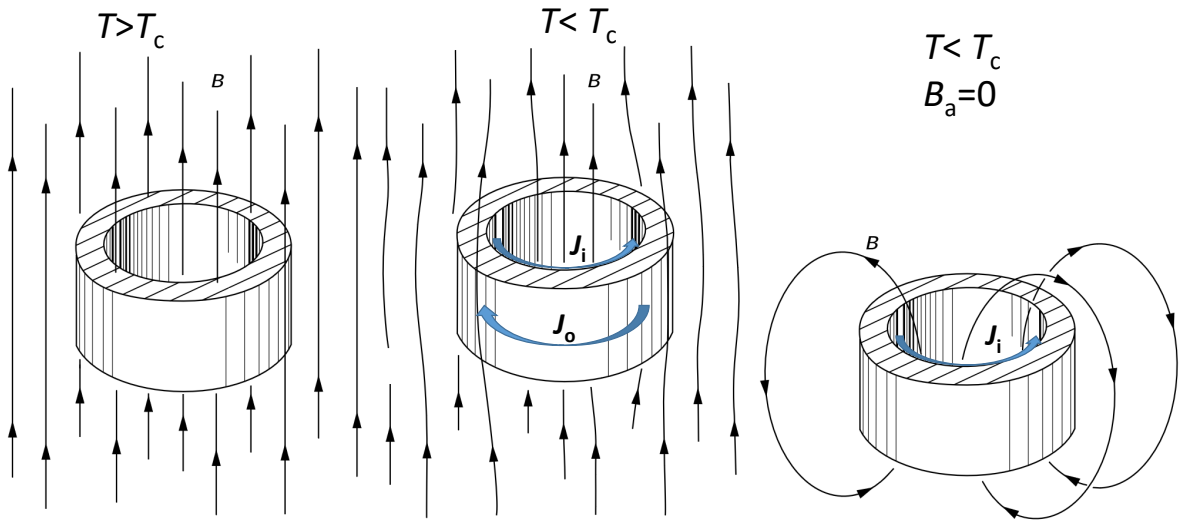
The contributions from the other sections 2-3 and 1-4 are both zero because, from symmetry arguments, the vector **B** in these two sections is orthogonal to the integration path. The contribution from the section 3-4 is also zero because $\mathbf{B} = 0$ in the interior of the superconductor.

$$\oint_2^3 \mathbf{B} \cdot d\ell = 0 \quad \oint_4^1 \mathbf{B} \cdot d\ell = 0 \quad \oint_3^4 \mathbf{B} \cdot d\ell = 0 \tag{37}$$

Thus, the surface current is given by

$$I = \frac{B\ell}{\mu_0} \tag{38}$$

which is similar to a solenoid with one turn.



Next, we consider the case of a hollow cylinder of material which is superconducting below T_c, as shown above (image after Feynman Lectures on Physics). Application of a field above T_c will result in full penetration of the flux lines both within the material of the tube and in the open volume within the inner surface (left panel). By reducing the temperature below T_c, the applied external field gives rise to a surface current on the outside of the cylinder, J_o such that there is in zero field inside the superconducting cylinder. By itself, this outside-surface current would also cancel the field in the open space inside the cylinder; however, the Faraday's law requires that the flux of the field inside the cylinder remain unchanged. Thus, a second current must appear on the inside surface of the cylinder J_i, in the opposite direction to the other surface current J_o, so that the flux in the open area is just that of the original applied field (middle panel). These two surface currents result in zero field inside the superconducting medium and an unchanged field in the central open region.

Flux trapping and persistent currents

If the applied magnetic field is turned off ($B_a = 0$ in the right panel), while the hollow cylinder (or a ring) is in the superconducting state, the flux inside the cylinder will remain unchanged. This implies that the inside-surface current remains J_i, whereas the outside-surface current disappears (J_o=0).

The flux inside the cylinder is defined by the surface integral

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} \tag{39}$$

where $d\mathbf{S}$ is the area of the hole.

Faraday's law gives

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{40}$$

and Stokes theorem gives

$$\int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{41}$$

Looking at the time-dependent flux variation:

$$\frac{d\Phi}{dt} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{42}$$

and considering a close loop ℓ_1 inside the superconductor where $\mathbf{E} = 0$, resulting in

$$\frac{d\Phi}{dt} = 0 \tag{43}$$

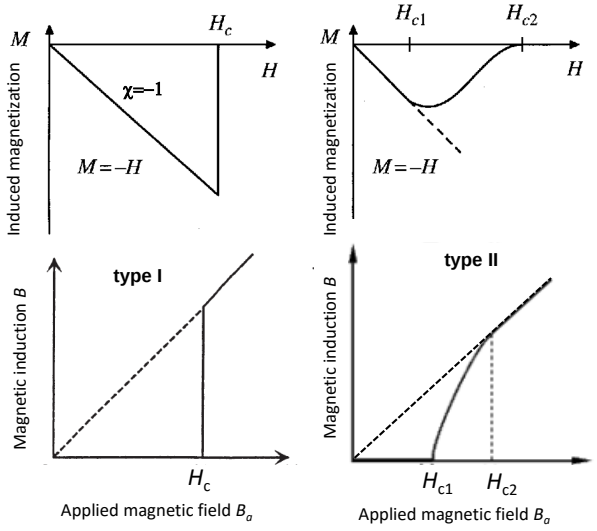
This means that the flux Φ=const. inside the cylinder (or ring) is constant; the flux is conserved or remains trapped. As long as the cylinder is superconducting the flux cannot change. The inside surface current is called persistent current and it will continue indefinitely as long as the medium is superconducting. Persistent currents have a time constant of $\sim 10^5$ years.

Magnetic susceptibility and perfect diamagnetism

In free space, where there is no magnetization, the magnetic induction or the magnetic flux density is described by $\mathbf{B} = \mu_0\ \mathbf{H}$, where $\mu_0 = 4\pi \times 10^7$ H/m is the permeability of the free space. The magnetic susceptibility is given by $\chi = \frac{\mathbf{M}}{\mathbf{H}}$ where $\mathbf{M}$ is the magnetization and $\mathbf{H}$ is the magnetic field strength. In a magnetic system or a superconductor the magnetic flux density or magnetic induction is expressed as $\mathbf{B} = \mu_0\ (\mathbf{H} + \mathbf{M}) = \mu_0 \mathbf{H}\ (1+\chi)$ (in SI units) (or $\mathbf{B} = \mathbf{H}\ (1+4\pi\chi_{CGS})$ in CGS units).

Magnetization of superconductors

A superconductor is a perfect diamagnet, $\chi = -1$. As the $\mathbf{B} = 0$ inside of a superconductor, the magnetization $\mathbf{M}$ is completely opposite to the applied field $\mathbf{H}$ (type I), $\mathbf{M}$=-$\mathbf{H}$. There is another type of superconductor, type II, that allows partial flux penetration as flux lines.



The behaviour of the induced magnetization $\mathbf{M}$ and of the magnetic induction $\mathbf{B}$ for different types of superconductors as a function of an applied external magnetic field is shown in figure above. This refers to the magnetization curve of a long cylinder superconductor (to eliminate the effect of the demagnetizing factors) in a longitudinal external magnetic field. For a type I superconductor, the perfect Meissner diamagnetism is maintained until magnetic field reaches its critical value, $H_{\rm c}$, beyond which superconductivity is destroyed. The induction inside the superconductor $\mathbf{B}$ remains zero until suddenly the superconductivity is destroyed, where is varies linearly, assuming that the normal state is paramagnetic.

For type II superconductor, the perfect diamagnetism occurs only below $H_{\rm c1}$. The flux starts to penetrate the specimen at a field $H_{\rm c1}$. The specimen is in a vortex state between $H_{\rm c1}$ and $H_{\rm c2}$, and it

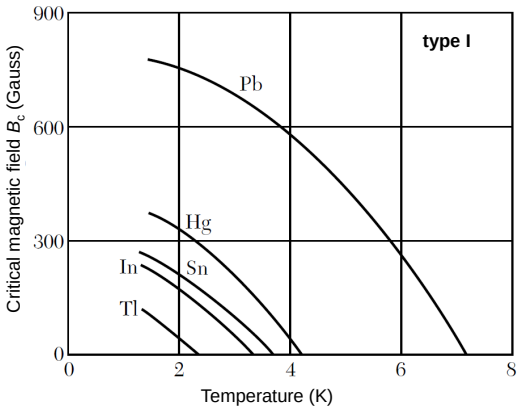
has superconducting electrical properties up to H_{c2}. Above H_{c2}, the specimen is a normal conductor in every respect, except for possible surface effects.

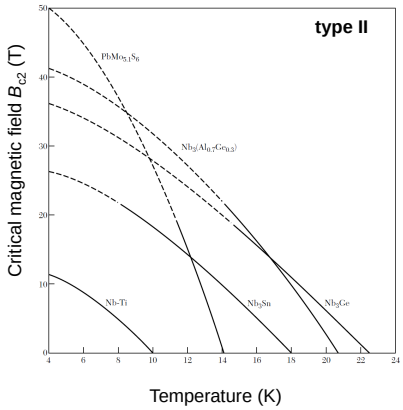
Critical magnetic fields

A superconducting state is eventually destroyed (becomes normal metal) by a large applied external magnetic field that can fully penetrate the material for values larger than its critical field. For a type I superconductor, the value of the critical applied magnetic field to destroy the superconducting state is given by: $B_c = \mu_0 H_c$. This value was first observed experimentally by Onnes in 1914. The experimental values of B_c are typically very low for type I superconductors (below 1000 Gauss=0.1 T)

Type II superconductors the upper critical field, $B_{c2} = \mu_0 H_{c2}$, defines the boundary between the mixed or the vortex phase and the normal phase. Inside the vortex phase, the magnetic field penetrates the material in discrete vortex lines, as discussed later. The resistance remains zero up to B_{c2} if the vortices are not allowed to move by getting them pinned to pieces of disorder in a material. This phenomenon of vortex pinning being necessary in order to have a true zero resistance state is very similar to a ferromagnet, where domain walls must be pinned by disorder in order to have a true ferromagnet that retains magnetization even in zero field.

The experimental values of B_{c2} for type II superconductor are typically very high (can exceed 100 T) and thus these materials are suitable for producing high magnetic fields using superconducting solenoids. Currently it is technologically possible to construct 32 T superconducting magnets.





Type I	B_c(T)
Al	0.01
Sn	0.03
Hg	0.04
Pb	0.08
Type II	B_{c2}(T)
Nb	0.8
V	1
Nb_3Sn	25
$YBa_2Cu_3O_{7-\delta}$	120

The graph above shows the temperature dependence of the critical fields for a type I (left panel) and type II superconductors (middle panel) (after Kittel). The temperature dependence of the upper critical field can be approximated by the empirical relationship: $B_c(T)$=B_0(1-$(T/T_c)^2$). A table of critical fields for a selection of materials is shown in the right panel.

Critical current and Silsbee's rule

A superconducting state state is eventually destroyed if the current densities exceeds the critical current values J_c. For a wire of radius a, the critical current, I_c is related to the critical field that destroys superconductivity by $I_c = 2\pi a B_c/\mu_0$, which is the Silsbee's rule. Experiments that measured critical currents in thin cylinders confirmed the Silsbee's rule and that the supercurrents only travel in a narrow sheath at the surface. This gives the possibility of very high current densities $J_c = I_c/(\pi a^2)$. For example, $J_c \sim 2.6 \times 10^9$ A m^{-2} in 12 T and 4.2 K for superconducting NbTi wire versus $J_c \sim 5 \times 10^6$ A m^{-2} for Copper wires. This means that the superconducting wires are not only energy efficient (as they show no dissipation due to zero resistance) but they are also space

efficient (superconducting wires with 100 times less cross-sectional area is needed).

Thermodynamics

Superconductors have a definite ground state and we can make use of equilibrium thermodynamics. Considering the Gibbs free energy (at constant pressure), and assume changing the applied magnetic field gives $\mathrm{d}G = -S\,\mathrm{d}T - m\,\mathrm{d}B$. Therefore, for a superconductor at constant T ($T < T_\mathrm{c}$)

$$G_\mathrm{s}(T,B) - G_\mathrm{s}(T,0) = -\int_0^B m\,\mathrm{d}B. \tag{44}$$

Here $m = MV$ is magnetic moment, V is the sample volume and magnetization is $M = -H = -B/\mu_0$. This implies that

$$G_\mathrm{s}(T,B) - G_\mathrm{s}(T,0) = -\int_0^B m\,\mathrm{d}B = \frac{VB^2}{2\mu_0}. \tag{45}$$

The integral is simply the area under the curve of M as a function of H shown earlier. When the magnetic field reaches the critical field, $B = B_\mathrm{c} = \mu_0 H_\mathrm{c}$, the system reaches its normal state. As the superconducting and normal states are in equilibrium at (phase coexistence) in the normal state at B_c and assuming that the normal state is non-magnetic and has no field dependence (as shown in the right panel below) then

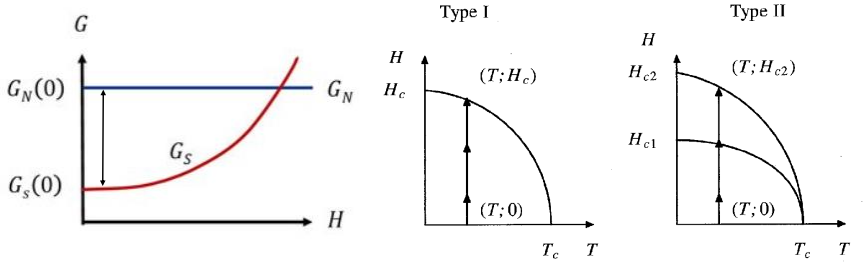
$$G_\mathrm{s}(T,B_\mathrm{c}) = G_\mathrm{n}(T,B_\mathrm{c}) \simeq G_\mathrm{n}(T,0) \tag{46}$$

Hence, the difference in Gibbs free energies of superconducting and normal states at zero field is:

$$G_\mathrm{n}(T,0) - G_\mathrm{s}(T,0) = \frac{VB_\mathrm{c}^2}{2\mu_0} > 0, \tag{47}$$

Thus, the condensation energy per unit of volume which is the gain in energy for becoming a superconductor is

$$\text{Condensation energy} = \frac{B_\mathrm{c}^2}{2\mu_0} \tag{48}$$



In an applied magnetic field $B < B_\mathrm{c}$, the difference in Gibbs free energies of superconducting and normal states is:

$$G_\mathrm{n}(T,B) - G_\mathrm{s}(T,B) = \frac{V(B_\mathrm{c}^2 - B^2)}{2\mu_0}, \tag{49}$$

The entropy can be obtained by differentiating the Gibbs's energy. In a type I superconductor there is a finite change in entropy between the normal and superconducting states,

$$S_\mathrm{n}(T,B) - S_\mathrm{s}(T,B) = -\left(\frac{d(G_\mathrm{n} - G_\mathrm{s})}{dT}\right)_B = -\frac{V}{\mu_0} B_\mathrm{c}(T)\frac{\mathrm{d}B_\mathrm{c}(T)}{\mathrm{d}T} \tag{50}$$

The difference $S_{\rm n}(T,B) - S_{\rm s}(T,B) > 0$ is positive, as the temperature dependence of the upper critical field $B_{\rm c}$ of different real superconductors that separates the superconducting from the normal state has an approximate mean-field behaviour given by the empirical relationship (see schematic panels above):

$$B_{\rm c}(T) = B_0 \left(1 - \left(\frac{T}{T_{\rm c}}\right)^2\right) \tag{51}$$

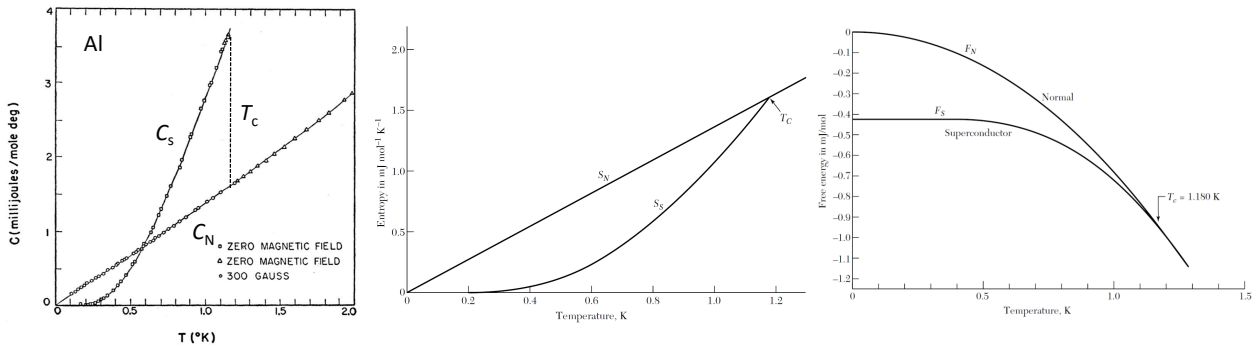
$$\frac{{\rm d}B_{\rm c}(T)}{{\rm d}T} < 0. \tag{52}$$

Therefore the *entropy of the superconducting state is lower than the normal state* being a more ordered state. The expression 50 shows that the phase transition at constant temperature the superconducting to normal boundary at $B_{\rm c}$, is generally first-order as it has a finite latent heat. Latent energy is given by $L = T(S_{\rm n}(T,B) - S_{\rm s}(T,B))$. In zero external field, $B = 0$, at $T_{\rm c}$, this entropy difference goes to zero, and the phase transition is second-order.

Further differentiation of the entropy relation from equation 50 yields the heat capacity.

$$C_{\rm n}(T,B) - C_{\rm s}(T,B) = \left(T\frac{d(S_{\rm n} - S_{\rm s})}{dT}\right)_B = -\frac{TV}{\mu_0}\left[B_{\rm c}(T)\frac{{\rm d}^2 B_{\rm c}(T)}{{\rm d}T^2} + \left(\frac{{\rm d}B_{\rm c}(T)}{{\rm d}T}\right)^2\right]. \tag{53}$$

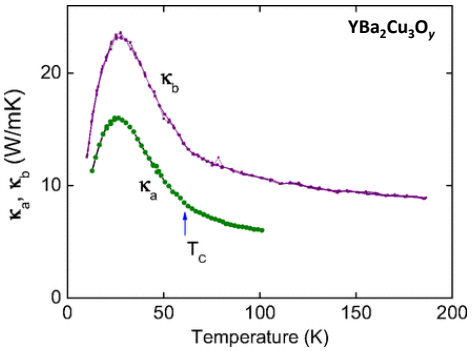
This result gives a direct measure of the discontinuity of the jump in the heat capacity of a superconductor at $T_{\rm c}$ in zero magnetic field.



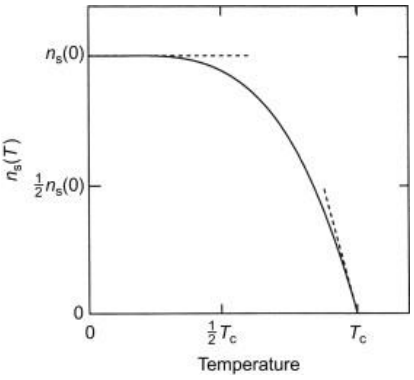
The plot above shows the heat capacity of Aluminium with a critical temperature of $T_{\rm c} \sim 1.18$ K (after Norman E. Phillips, Phys. Rev. 114, 676, 1959). The heat capacity in the superconducting state has a distinct jump at $T_{\rm c}$. A magnetic field of 300 Gauss is used to suppress the superconducting state and to measure the heat capacity of Al in the normal state. The entropy and the free energy of the superconducting state of Aluminiun measured in zero field is lower than that of the normal state measured in 300 Gauss (after Kittel, Chapter 10). A lower entropy in the superconducting state suggests that the electrons are more ordered than in the normal state. No jump in entropy (or no latent heat) at $T_{\rm c}$ implies that the superconducting to normal phase transition in zero magnetic field is a second order phase transition, which is a continuous phase transition.

Poor thermal conductivity

The thermal conductivity of a metal is affected when it enters the superconducting state. Most of the heat flow along a normal metal in a temperature gradient is carried by the conduction electrons. In the superconducting phase, the superelectrons do not interact with the lattice in the same way so that they cannot exchange energy and they cannot pick up the heat from one part of the specimen to deliver it to the other part. Thus, the thermal conductivity in the superconducting state is reduced. The image on the right shows the thermal conductivity of the $YBa_2Cu_3O_y$ (after PNAS November 8, 2016 113 (45) 12667-12672).



London equation



London theory explained the Meissner-Ochsenfeld effect by adopting a two-fluid model. The basic assumption of the model is that there are two types of electrons in a metal in the superconducting state: n_s(T) (density of superconducting charge carriers) and n_n (density of total number of normal conducting electrons that behave normally and can carry an ohmic current), and the total number of carriers is kept constant (n=n_s+n_n). As the system is cooled down at low temperatures $T < T_c$, the number of superconducting charge carriers increases, such that at $T \to 0$ all the charge carriers are superconducting carriers $n = n_s \to n_s(0)$, as shown in figure on the right. The number of superconducting carriers decreases while warming up and at $T \to T_c$ $n_s(T) \to 0$ as they disappear at Tc.

London theory was looking for a Ohm's law for a superconductor in terms of the Drude theory for a fluid with no scattering. If a material is a perfect conductor, application of a an electric field freely accelerates the electric charge. Define a superconducting current density $\mathbf{J}$ with q the charge for the superconducting carriers and n_s is the density of superconducting charge carriers:

$$\mathbf{J} = n_s q \mathbf{v} \tag{54}$$

From electromagnetism the canonical momentum (a particle in magnetic field in quantum mechanics) is given by equation 22 and the magnetic vector potential $\mathbf{A}$ is is defined in equation 9 and repeated below:

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{55}$$

Thus,

$$\mathbf{J} = \frac{n_s q}{m}(\mathbf{p} - q\mathbf{A}) \tag{56}$$

Another assumption made by the London theory is that all superconducting carrier occupy the same energy state in which they have zero canonical momentum $\mathbf{p} = \mathbf{0}$ suggesting that they condensed in the momentum space. This assumption was based on the empirical justification that there was no hysteresis in the superconducting state of a type I superconductor and that the supercurrents do not dissipate. This implied that there are no nearby energy states for superconducting carriers to scatter into nor can they save energy by rearranging their momentum distribution. Thus, there is no continuum of states available but a single $\mathbf{p = 0}$ state.

Using this gives the London equation:

$$\boxed{\mathbf{J} = -\frac{n_s q^2}{m}\mathbf{A}} \tag{57}$$

An important consequences of the London equation is that the superconducting current $\mathbf{J}$ is due entirely to a magnetic field without the need for an electric field.

J can be understood as the screening current required to keep the magnetic field out of the bulk of the superconductor. Regarding the single momentum state, the London theory could have chosen $\mathbf{p} \neq 0$ but some other scalar. That $\mathbf{p} = 0$ was chosen was mostly for convenience. [‡]

London equations

There are in fact two London equations that can be summarized in the unique expression 57 above using the Maxwell equations in a case of a superconductor.

Applying the Drude theory to a superconductor with no scattering gives:

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{F}}{m} = \frac{q\mathbf{E}}{m} \tag{58}$$

and using

$$\mathbf{J} = n_s q \mathbf{v} \tag{59}$$

$$\frac{\partial \mathbf{J}}{\partial t} = \frac{n_s q^2}{m}\mathbf{E} \tag{60}$$

which is often called *the first London equation.*

Then applying the curl to this equation

$$\nabla \times \frac{\partial \mathbf{J}}{\partial t} = \frac{n_s q^2}{m}(\nabla \times \mathbf{E}) \tag{61}$$

and considering Faraday's law

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \tag{62}$$

$$\nabla \times \frac{\partial \mathbf{J}}{\partial t} = \frac{n_s q^2}{m}\left(-\frac{\partial \mathbf{B}}{\partial t}\right) \tag{63}$$

Integrating on both sides at some initial time and assuming that any constant C from the integral is independent of time and zero everywhere one obtains *the second London equation.*

$$\nabla \times \mathbf{J} = -\frac{n_s q^2}{m}\mathbf{B} \tag{64}$$

Gauge theory and the London guage

The London equation $\mathbf{J} = -\frac{nq^2}{m}\mathbf{A}$ only works in one choice of gauge, known as the London gauge.

The continuity equation, which expresses the (local) charge conservation, states that the divergence of the current density J (in amperes per square metre) is equal to the negative rate of change of the charge density ρ_v (in coulombs per cubic metre): $\frac{\partial \rho_v}{\partial t} + \nabla \cdot \mathbf{J} = 0$. This means that the current is the movement of charge. The continuity equation implies that the charge is moving out of a differential volume (i.e., divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge. In the DC case is just $\nabla \cdot \mathbf{J} = 0$ which is what is needed for a steady state of supercurrent flow. The London gauge amounts to choosing $\nabla \cdot \mathbf{A} = 0$.

[‡]The BCS theory of superconductivity finds that the charge corresponds to a pair of electrons $|q| = 2e$. In the free-electron approximation, the mass would be $m = 2m_e$ and the carrier density $n = n_s/2$, where n_s is the number density of single electrons in the condensate. Thus, the London penetration depth remains unchanged by the pairing.

This is sometimes called the rigid gauge, since the phase of never changes and therefore $\nabla\theta = 0$ when we switch on a magnetic field or introduce a finite supercurrent.

Notice that the canonical momentum $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$ is therefore not gauge invariant either:

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \nabla\chi \\ \mathbf{p} &\rightarrow \mathbf{p} + q\nabla\chi \end{aligned} \tag{65}$$

Assuming a wave function $\psi(\mathbf{r}) = \psi e^{i\theta(\mathbf{r})}$ with a phase θ that depends on position in space, then $\mathbf{p} = -i\hbar\nabla = \hbar\nabla\theta$, and

$$\hbar\nabla\theta \rightarrow \hbar\nabla\theta + q\nabla\chi \tag{66}$$

or

$$\theta \rightarrow \theta + \frac{q\chi}{\hbar}, \tag{67}$$

and so the phase (and hence the wave function) is also not gauge invariant.

Note, however, that the operator associated with kinetic energy, $(\hbar\nabla\theta - q\mathbf{A})^2/2m$, **is gauge invariant**, with the effect of the gauge transformations on both θ and $\mathbf{A}$ cancelling each other out.

Quantities such as $\mathbf{A}$, or θ describe physical variables but are not themselves observable. There exists an infinite number of possible vector potentials that can describe the correct magnetic flux density. This suggests that we can obtain a well defined value for the measurable quantity superconducting current density $\mathbf{J}$ only if we can measure both θ and $\mathbf{A}$, which have been introduced only for mathematical convenience and are not physical observables. The way out of this dilemma is to recognize the fact that the relation between phase and vector potential is not arbitrary but fixed. In this way we can measure the supercurrent but still are not able to determine θ and A. That is, we demand that expression $\mathbf{J}$ is independent of the special choice of A. The specific choice of $\mathbf{A}$ is usually referred to as the gauge and, hence, we have to make the expression for the supercurrent gauge invariant. The same scalar function χ is changing both the phase and the vector potential. In this way the supercurrent always has the same value and can be measured regardless the specific gauge chosen. The important conclusion that can be drawn is that the expression for the supercurrent is gauge invariant and therefore we do no longer be concerned about this issue.

Penetration depth

London equation can determine the distance over which a magnetic field penetrates a superconductor at the surface and explains theoretically the existence of the Meissner effect.

By applying the curl of J to the London equation 57:

$$\nabla \times \mathbf{J} = -\frac{n_s q^2}{m}(\nabla \times \mathbf{A}) \tag{68}$$

$$\nabla \times \mathbf{J} = -\frac{n_s q^2}{m}\mathbf{B} \tag{69}$$

$\mathbf{J}$ and $\mathbf{B}$ related by the static Maxwell equations.

The Ampere's law with $\mathbf{E} = 0$ for a superconductor is given by:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{70}$$

By combining these two equations and using the vector calculus identity:

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \tag{71}$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\frac{n_s q^2 \mu_0}{m} \mathbf{B} \tag{72}$$

and as $\nabla(\nabla \cdot \mathbf{B}) = 0$ as there are no monopoles gives the expression:

$$\nabla^2 \mathbf{B} = \frac{n_s q^2 \mu_0}{m} \mathbf{B} \tag{73}$$

This leads to a differential equation (a "diffusion" equation for the magnetic field $\mathbf{B}$) of the form

$$\boxed{\nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda^2}}, \tag{74}$$

where λ is the London penetration depth. This equation establishes that the value for $\mathbf{B}$ inside the superconductor must be zero unless the penetration depth is infinite (i.e., not a superconductor). The penetration depth is the distance over which the magnetic field lines penetrate inside of a superconductor and it is directly linked to the density of superconducting charge carriers n_s:

$$\lambda = \sqrt{\frac{m}{n_s q^2 \mu_0}} \tag{75}$$

Example: penetration of the magnetic field inside the surface of a superconductor

The differential equation 74 can be solved in various geometries.

Consider a magnetic field $\mathbf{B}$ (B_0, 0, 0) along the x direction parallel with the surface of a superconductor (along z direction) (as shown in the left panel of the figure below):

$$\frac{\partial^2 B_x}{\partial z^2} = \frac{1}{\lambda^2} B_x \tag{76}$$

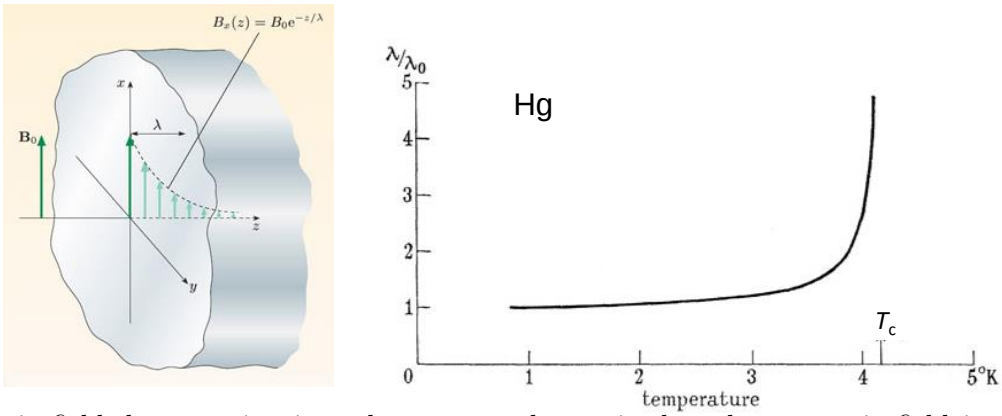
A solution is

$$B_x(z) = B_1 e^{-z/\lambda} + B_2 e^{z/\lambda} \tag{77}$$

Defining boundary conditions: $B_x(z = 0) = B_0$ and $B_x(\infty) = 0$, as the second term becomes infinity for $z \to \infty$.

$$B_x(z) = B_0 e^{-z/\lambda} \tag{78}$$

The magnetic field penetrates a superconductor and it decays exponentially with the characteristic length scale, λ, so that $\mathbf{B} = 0$ inside the bulk of the superconductor.



The reason the magnetic field decays going into the superconductor is that the magnetic field is screened by persistent superconducting electric currents. This is similar to Lenz's law: starting with a superconductor in zero magnetic field, then we turn on magnetic field and currents flow in the superconductor to prevent the change of the magnetic field and these currents persist indefinitely since there is no resistance to current flow. However, the screening of magnetic field in a superconductor is fundamentally more than just Lenz's law.

Experiments on superconducting microparticles of Hg (Shoenberg, Proc. Royal Society A 175, 960, 1940) verified the existence of penetration depth and its temperature dependence (right panel in the graph above). This work also confirms the London's proposal that superconducting carrier density is zero $n_s \to 0$ at the critical temperature $T \to T_c$ while $\lambda \to \infty$ diverges ($\lambda \sim 1/\sqrt{n_s}$). Later experiments have shown that typical values of $\lambda \sim 30 - 110$ nm for elemental SCs.

A Macroscopic Quantum State

Fritz London, working with Heinz London, realised that superconductivity was due to a **macroscopic quantum phenomenon** in which there was long range order of the momentum vector. This implies **condensation in momentum space**. He suggested that all the superconducting carriers exist as a single entity which can be described with a single-value quantum mechanical wavefunction.

$$\psi = \psi_0 e^{i\theta} \tag{79}$$

This quantum state is similar to the state of an electron orbiting an atom, but here the physical extend of the wavelength is not of the order of a few Bohr radii ($a_0 = 0.53$ Å) but it is extended throughout the superconducting sample. After many theoretical developments, it was shown that ψ is the wavefunction of the superconducting carriers and equivalent to the superconducting order parameter.

Fritz London also realised that it is the rigidity of the superconducting wave function ψ which is responsible for perfect diamagnetism.

Flux quantization

This is the final consequence of the London equation. Fritz London in the late 1940 considered the consequence of flux trapping in multiply connected superconductors on the postulated wave-function from equation 79.

Superconducting current density, **J** is given by the product:

$$\mathbf{J} = n_s q \mathbf{v} \tag{80}$$

and using the canonical momentum from equation 22

$$q\mathbf{v} = \frac{q}{m}(\mathbf{p} - q\mathbf{A}) \tag{81}$$

it becomes

$$\mathbf{J} = n_s \frac{q}{m}(\mathbf{p} - q\mathbf{A}) \tag{82}$$

The quantum mechanical operator associated with momentum from equation 84:

$$\mathbf{p} = -i\hbar\nabla \tag{83}$$

and for a charged particle one must make the replacement with the kinetic momentum operator

$$\mathbf{P} = -i\hbar\nabla - q\mathbf{A}. \tag{84}$$

which is guage invariant and associated with the kinetic energy of a charged particle.

In Quantum Mechanics terminology, the probability density function, $|\psi|^2$, is linked to the superconducting carrier density, n_s:

$$n_s = \psi^*\psi = |\psi|^2 \tag{85}$$

The probability current of the wave function ψ of a charge particle of mass m in an external electromagnetic field is given by [§]:

$$\mathbf{J} = \frac{q}{2m}\left[\psi^*\left(-i\hbar\nabla - q\mathbf{A}\right)\psi + \psi\left(-i\hbar\nabla - q\mathbf{A}\right)^*\psi^*\right] \tag{86}$$

which can be expressed as a function of the canonical momentum $\mathbf{p}$ as

$$\mathbf{J} = \frac{q}{2m}\left[\psi^*\left(\mathbf{p} - q\mathbf{A}\right)\psi + \psi\left(\mathbf{p} - q\mathbf{A}\right)^*\psi^*\right] \tag{87}$$

which can be re-written as

$$\mathbf{J} = \frac{q}{2m}\left[(\psi^*\mathbf{p}\psi - \psi\mathbf{p}\psi^*) - 2q\mathbf{A}|\psi|^2\right] \tag{88}$$

The first term, the momentum current, results from the interference between the real and imaginary parts. It is zero in a wave function that is pure real or pure imaginary. This is usually the most important part in most problems. So if a wave function does not have an "i" in it somewhere, the momentum current is zero.

The first term related to the canonical momentum $\mathbf{p}$ in expression 88 can be expressed in quantum mechanical way as:

$$\psi^*\mathbf{p}\psi = \psi_0 e^{-i\theta}(-i\hbar)(i\nabla\theta)\psi_0 e^{i\theta} = n_s\hbar\nabla\theta \tag{89}$$

and the term with the magnetic vector potential, $\mathbf{A}$ becomes:

$$\psi^*\mathbf{A}\psi = q\mathbf{A}|\psi|^2 = n_s\mathbf{A} \tag{90}$$

The term caused by the vector potential just depends on the square of the wave function at the point of interest.

[§]See Feymann lectures https://www.feynmanlectures.caltech.edu/III_{21}.html

Thus, the current density becomes:

$$\mathbf{J} = \frac{n_s q}{m}(\hbar \nabla\theta - q\mathbf{A}) \tag{91}$$

Take the integral around a loop:

$$\oint \mathbf{J} \cdot \mathrm{d}\mathbf{l} = \frac{n_s q}{m}(\hbar \oint \nabla\theta \cdot \mathrm{d}\mathbf{l} - q \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l}) \tag{92}$$

The flux is given by

$$\Phi = \int_{dS} \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} \tag{93}$$

Due to the Stoke's theorem:

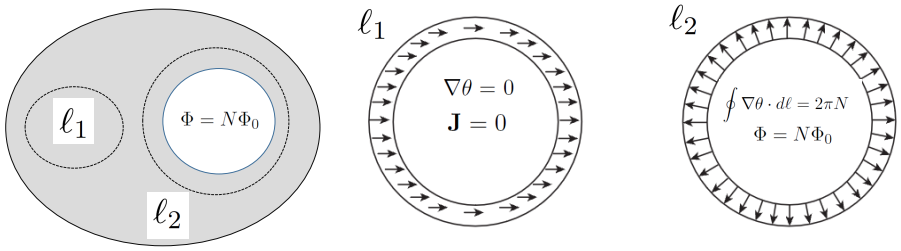
$$\int_{dS} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} \tag{94}$$

Thus, the current density becomes:

$$\oint \mathbf{J} \cdot \mathrm{d}\mathbf{l} = \frac{n_s q}{m}(\hbar \oint \nabla\theta \cdot \mathrm{d}\mathbf{l} - q\Phi) \tag{95}$$

For a closed loop ℓ_1 inside the superconductor (left panel) that contains no flux, both $\mathbf{J}$ and $\Phi = 0$ ($\mathbf{B} = 0$ inside the bulk implies $\nabla\theta = 0$ and $\mathbf{p} = 0$, as assumed by London. We can pick any loop we like and this applies throughout the bulk of the SC. The arrows in the middle panel indicate the local phase angle at each point in space for a ℓ_1 loop; having arrows parallel to each other indicates that the phase is constant ($\theta(r)$=const.) and the ring is in a low energy, zero current state.

For $\nabla\theta = 0$ the equation 95 links $\mathbf{J}$ to $\mathbf{A}$, as established by London equation 57 and it is called rigidity of the wavefunction. A superconductor with a uniform order parameter with a constant phase is described by the London theory.



For a closed loop ℓ_2 which contains flux but it is still inside the superconductor $\mathbf{J} = 0$ but $\Phi \neq 0$.

$$\Phi = \frac{\hbar}{q} \oint \nabla\theta \cdot d\ell \tag{96}$$

For ψ to be single-valued (one value for the macroscopic wave function), the phase needs to vary in units of 2π. This is because the phase factor $e^{i\theta}$ must take the same values at either end of the loop, of course, otherwise the order parameter would have a jump somewhere. The arrows in the right panel indicate the local phase angle varies ($\theta(r) \neq$ constant) for a ℓ_2 loop. This indicates that there is a current flow when the flux in constant (persistent current).

$$\oint \nabla\theta \cdot d\ell = 2\pi N \tag{97}$$

where N is integer

$$\Phi = \frac{\hbar}{q} 2\pi N = N\frac{h}{q} \tag{98}$$

Flux quantization leads to the equation $\Phi = N\Phi_0$ where N is an integer and Φ_0 is the **flux quantum**:

$$\Phi_0 = \frac{h}{2e} = 2.067 \times^{15} \text{ Wb.} \tag{99}$$

The $|q| = 2e$ in this equation represents the charge of the superconducting carriers, and the experiment implies that the superconducting carrier consists of a **pair** of electrons. The N integer is an invariant under the smooth deformation of the orientation of the phase and thus is a topological invariant (also known as the winding number).

The Fluxoid

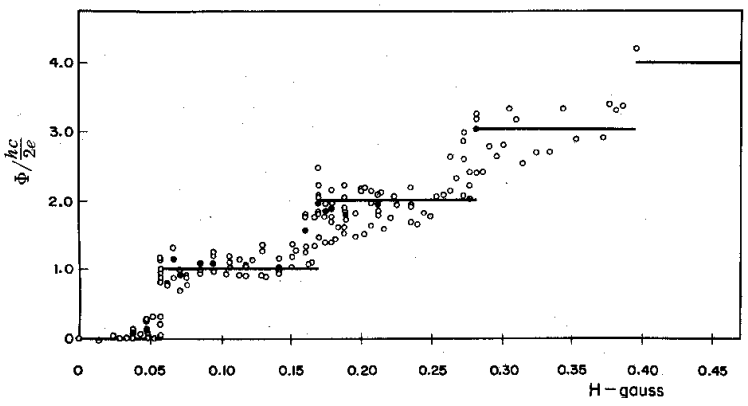
Fritz London noted that it is not strictly speaking flux that is conserved and quantized in a superconductor. If one considers the loop ℓ_2 close to the edge of the superconductor then it is a combination of flux and surface currents that has these properties. This combination is called a fluxoid and is given by:

$$\Phi' = \Phi + \frac{m}{n_s q^2} \oint \mathbf{J} \cdot \mathrm{d}\mathbf{l} \tag{100}$$

Deep inside the superconductor, the second term on the right will be zero and flux and fluxoids are the same, but within a few penetration depths of the surface the **J**-term is still present. Here, we would have $\mathrm{d}\Phi'/\mathrm{d}t = 0$ and $\Phi' = Nh/2e$.

Experimental evidence of flux quantization

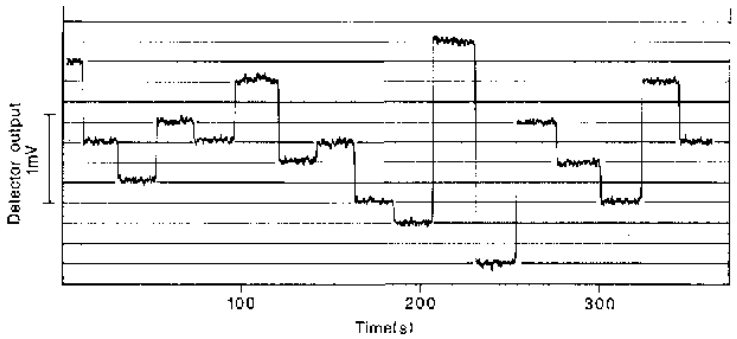
The first evidence for this pairing came from the data shown below [B. S. Deaver and W. M. Fairbank, Phys. Rev. Lett. **7**, 43 (1961)]; this is from an experiment on a cylinder made of tin. Note the CGS units used here in which c value is multiplied with the total flux. Another second experiment performed in 1961 to prove the flux quantization in superconducting cylinders was performed by R. Doll and M. Näbauer at the Walther-Meissner-Institute in Munich. [R. Doll, M. Näbauer, Phys. Rev. Lett. 7, 51 (1961)].



In the experiment by Deaver and Fairbank a tiny Sn tube with a length of about 0.9 mm, an inner diameter of 13 μm and a wall thickness of 1.5μm was used. The cylinder was vibrated in the axial direction at a frequency of about 100 Hz and the resulting rf signal was detected via a pair of pick-up coils.

The aim of the experiments was to show that the flux enclosed by a superconducting cylinder of wall thickness $d \gg \lambda$ can have only discrete values $\Phi = N\Phi_0$. In order to do so different amounts of magnetic flux have to be frozen in during cooling down the cylinder in an applied magnetic field and then the amount of trapped flux has to be measured with a precision much better than a single flux quantum. In order to obtain a large relative change of the magnetic flux from measurement to measurement only small values of magnetic field have to be used resulting in a small number of trapped flux quanta. Since $\Phi = \mathbf{B}S$, both small applied fields and a small cross-sectional area S of the cylinder have to be used. For a cylinder area of S= 1mm^2 a flux density B of only about 10^{-9} T is required to generate one flux quantum. This flux density is much smaller than that due to the earth magnetic field $B_{\text{Earth}} \sim 2 \cdot 10^{-5}$T. Both experimental groups were using small hollow cylinders with an outer diameter of the order of 10 μ. For such diameter, a flux density of $B \sim 2 \cdot 10^{-5}$T is required to generate one flux quantum Φ_0=$2 \cdot 10^{-15}$T/m^2. Since this flux density is of the same order of magnitude as the one due to the earth magnetic field, a careful shielding of the earth magnetic field and other perturbing fields was necessary.

The experiment shown below [C. E. Gough et al, Nature **326**, 855 (1987)] tried a similar experiment, this time using a ring made out of a high-T_c superconductor ($Y_{1.2}Ba_{0.8}CuO_4$). The ring had an outer-diameter 10.0 mm, inner-diameter 4.5 mm and thickness 4.0 mm. and the figure shows the output of an rf-SQUID magnetometer. The ring was exposed to a source of electromagnetic noise so that the flux varied through the ring. Once the output of the rf-SQUID was calibrated, one could show that the flux jumps were 0.97±0.04 ($h/2e$), confirming that the charge carriers in the high-T_c materials were pairs of electrons.



Phase rigidity or stiffness

A bulk superconductor has a ground state with a constant order parameter $\psi = \psi_0 e^{i\theta}$ and therefore it must have the same phase everywhere, as $\nabla\theta = 0$ and θ=const. There must be a phase-stiffness, or an energy cost associated with changing the phase from one part of the solid to another. The phase $\theta(r)$ strongly resists deformation. Thus, the existence of the rigidity or stiffness of the phase explains the existence of the persistent currents. The surface currents protect the superconducting state and thus the phase needs to change ($\theta(r)$ has to vary) for these currents to exist.

Pippard coherence length

The London equation is a local equation, with the value of the magnetic potential at a point determining the current density at that point. It was found that experimentally that the derived values of London penetration depths were frequently greater than those estimated from the London theory. To remedy this, Brian Pippard (1953) suggested a non-local modification of the theory in which a disturbance in magnetic potential would be felt by all superconducting carriers within a certain distance of the perturbation. Deep in the superconducting state of a material free from impurities this distance is known as the Pippard coherence length, ξ_0.

The modified form of the London equation proposed by Pippard generalizes the London equation by relating the current at a point **r** in the solid, **J** (**r**), to the vector potential $\mathbf{A}(\mathbf{r'})$ at nearby points **r'**. The expression proposed was:

$$\mathbf{J}(\mathbf{r}) = -\frac{n_s q^2}{m}\frac{3}{4\pi\xi_0}\int \frac{\mathbf{R}(\mathbf{R}\cdot\mathbf{A}(\mathbf{r'}))}{R^4} e^{-R/r_0} d^3r' \tag{101}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r'}$. The points which contribute to the integral are separated by distances of order r_0 defined by

$$\frac{1}{r_0} = \frac{1}{\xi_0} + \frac{1}{l} \tag{102}$$

where l is the mean free path of the electrons at the Fermi surface of the metal, $l = v_F \cdot \tau$, with τ the scattering time from the Drude conductivity formula, and v_F, the electron band velocity at the Fermi surface.

The Pippard coherence length, ξ_0, has the physical size of the Cooper pair bound state in BCS theory, confirmed by the microscopic theory of Bardeen, Cooper and Schrieffer (discussed later).

$$\xi_0 \sim \frac{\hbar v_F}{\pi\Delta} \tag{103}$$

where v_F is the Fermi velocity and Δ the superconducting gap energy (from the BCS theory). For a type I superconductor, ξ_0, can have a large size up to $\xi_0 \sim 1000$ Å.

Ginzburg-Landau theory

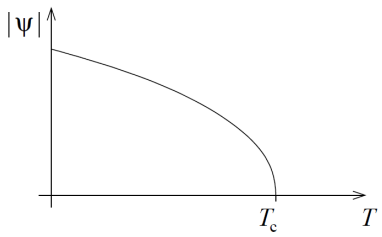
Landau theory was developed in 1936 to describe a second order phase transition in terms of an order parameter, which is zero above the critical temperature T_c but takes finite value below, tending smoothly towards zero as $T \to T_c$. Landau noticed that typically second-order phase transitions, such as the Curie temperature in a ferromagnet, involve some change in symmetry of the system. For example, a magnet above the Curie temperature, T_c , has no magnetic moment. But below T_c a spontaneous magnetic moment develops. In principle this could point in any one of a number of different directions, each with an equal energy, but the system spontaneously chooses one particular direction.

The order parameter may be real scalar, complex scalar, real vector (e.g., ferromagnet), but its crucial feature is that it transforms under some symmetry group G which commutes with the Hamiltonian. (e.g., the hamiltonian of a ferromagnet remains invariant under the rotation of spins in the absence of external magnetic field). This has the consequence that the form of free energy, as a function of the order parameter, must be so constructed as to be invariant under the symmetry G.

The Ginzburg-Landau theory of superconductivity is built upon a general approach to the theory of second-order phase transitions developed by Landau. Ginzburg-Landau postulated that an order parameter ψ should be a complex number, and the macroscopic wave function for a superconductor. Later in the microscopic BCS theory of superconductivity there appears a energy gap parameter, $\Delta_{\mathbf{k}}$, which is also complex. Gor'kov was able to derive the Ginzburg-Landau theory from BCS theory, and show that ψ is essentially the same as $\Delta_{\mathbf{k}}$, except for some constant numerical factors.

Gingzburg-Landau theory is applied to superconductors considering specifically the case of a complex order parameter which has both an amplitude and a complex phase that can also depend on the position $\mathbf{r}$,

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r})\mathrm{e}^{i\theta(\mathbf{r})} \tag{104}$$



Basic idea is that the complex order parameter ψ is zero in the normal state above T_c, but becomes finite below T_c:

$$\begin{aligned} \psi &\neq 0, \qquad T < T_c \\ \psi &= 0, \qquad T > T_c \end{aligned}$$

The free energy of the system must be real and invariant under rotation resulting that for the spatially uniform case, such that only even powers of $|\psi|^2$ can occur to obtain real solutions. Thus, one can write down the free energy for a small variation of the order parameter in the vicinity of T_c, then minimize this free energy with respect to ψ. Since ψ goes to zero at the critical temperature, T_c, we can Taylor expand the free energy in powers of $|\psi|$. For temperatures close to T_c, only the first two terms in the expansion should be necessary (as for $|\psi|^2 \ll 1$), and so the free energy density ($f = F/V$ which is the free energy per unit volume) must be of the form

$$f_s(T) = f_n(T) + a(T)\,|\psi|^2 + \frac{1}{2}b(T)\,|\psi|^4 + ... \tag{105}$$

where $f_s(T)$ and $f_n(T)$ are the superconducting state and normal state free energy densities, respectively and the coefficients $a(T)$ and $b(T)$ are temperature dependent phenomenological smooth functions to be determined.

For the system to be thermodynamically stable, the coefficient of the highest even power of the order parameter must be positive. In this case, we find that $b(T)>0$, such that the free energy is bounded, otherwise the free energy density would have no minimum, which would be unphysical (or we would have to extend the expansion to include higher powers such as $|\psi|^6$).

a) Bulk Phase Transition ($\psi \neq \psi(\mathbf{r})$)

Next we consider the case of a spatially uniform superconductor in which the order parameter is constant throughout the bulk of the material and the order parameter does not depend on position ψ= constant ($\psi \neq \psi(\mathbf{r})$).

The equilibrium value is obtained by minimizing $f(T) = f_s(T) - f_n(T)$ with respect to ψ and considering that $b(T) > 0$.

$$\frac{df}{d|\psi|} = 0 = 2a(T)|\psi| + \frac{4}{2}b(T)|\psi|^3 = 2|\psi| \cdot (a(T) + b(T)|\psi|^2) \tag{106}$$

At fixed T there are two possible solutions that give a minimum in the free energy $f = f_s - f_n$:

$$|\psi| = 0 \qquad \text{and} \qquad |\psi| = \sqrt{-\frac{a(T)}{b(T)}} \tag{107}$$

They correspond to the normal state which has:

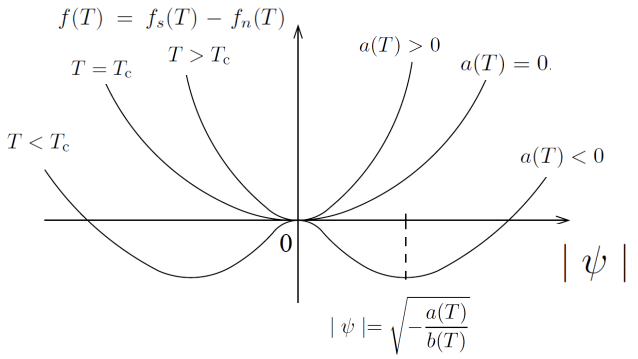
$$|\psi| = 0 \qquad \text{and} \qquad f_s(T) = f_n(T) \tag{108}$$

and the superconducting state

$$|\psi| = \sqrt{-\frac{a(T)}{b(T)}} \qquad \text{for} \qquad f_s(T) = f_n(T) - \frac{a(T)^2}{b(T)} + \frac{a(T)^2}{2b(T)} = f_n(T) - \frac{a(T)^2}{2b(T)} \tag{109}$$

Thus, for the superconducting state with finite order parameter $|\psi| = \sqrt{-\frac{a(T)}{b(T)}}$, this solution is physical only if $a(T) < 0$ as $b(T) > 0$.

The normal state has the global free energy minimum ($f(T)$=0) for $|\psi| = 0$ and $a(T) > 0$ (as shown on the right). The superconducting state has the global minimum at $|\psi| \neq 0$ for $a(T) < 0$ and the minimum free energy solution changes. This means that $a(T)$ has to change sign and vanish at T_c. Therefore, we can identify the temperature where $a(T) = 0$ as the critical temperature $T = T_c$.



Thus, near to this critical temperature, T_c , assuming that the coefficients $a(T)$ and $b(T)$ change smoothly with temperature, one can use the Taylor expansion (there may be higher terms but they are not considered here),

$$a(T) \simeq a_0(T - T_c), a_0 > 0 \tag{110}$$

and

$$b(T) \simeq b > 0 \tag{111}$$

Replacing these expressions, one can find the order parameter $|\psi|$ obtained from the free energy f minimization as follows:

$$|\psi| = \left(\frac{a_0}{b}\right)^{1/2} \sqrt{T_\mathrm{c} - T}, \qquad T < T_\mathrm{c} \tag{112}$$
$$|\psi| = 0, \qquad T > T_\mathrm{c}$$

The corresponding curve of $|\psi|$ as a function of temperature is shown in the figure on the previous page. There is an abrupt change from zero to nonzero values at the critical temperature T_c. This curve is qualitatively similar to those obtained for other types of second-order phase transitions; for example, the curve looks like the magnetization M, which is the order parameter in a ferromagnet near its Curie point in the Stoner theory of magnetism (Blundell 2001).

Recalling that the probability density function, $|\psi|^2$, is linked to the superconducting carrier density, n_s:

$$n_s = \psi^* \psi = |\psi|^2 \tag{113}$$

gives

$$\lambda = \sqrt{\frac{m}{n_s q^2 \mu_0}} = \sqrt{\frac{mb}{4\mu_0 e^2 |a(T)|}} \tag{114}$$

which gives penetration depth diverging at T_c, in agreement with experiments.

$$\lambda = \lambda_0 \left| \frac{T_\mathrm{c}}{T_\mathrm{c} - T} \right|^{1/2} \tag{115}$$

Condensation energy and thermodynamic critical field

Substitute back the expression order parameter inside the free energy density gives:

$$f_s(T) - f_n(T) = -\frac{a_0^2}{2b}(T - T_\mathrm{c})^2 \tag{116}$$

This is the condensation energy just below T_c.

Recall the condensation energy from the previous thermodynamics section for a type I superconductor is which can be expressed as the Gibbs energy density:

$$g_n(0) - g_s(0) = (G_n(0) - G_s(0))/V = \frac{B_c^2}{2\mu_0} \tag{117}$$

This energy $\frac{B_c^2}{2\mu_0}$ is the condensation energy per unit volume which is the gain in free energy for a normal conducting material on becoming superconducting.

By combining the two expressions for the condensation energy one can get:

$$-\frac{B_\mathrm{c}^2}{2\mu_0} = -\frac{a_0^2}{2b}(T - T_\mathrm{c})^2 \tag{118}$$

the thermodynamic critical field near T_c,

$$B_c = a_0\sqrt{\frac{\mu_0}{b}}(T_c - T) \tag{119}$$

Entropy and heat capacity

Knowing the free energy, one can extract other relevant physical quantities, such as the entropy and heat capacity $f = u - Ts$. Differentiating f with respect to temperature T gives the difference in entropy between the superconducting and normal states as

$$s_s(T) - s_n(T) = -\frac{\partial}{\partial T}(f_s(T) - f_n(T)) \tag{120}$$

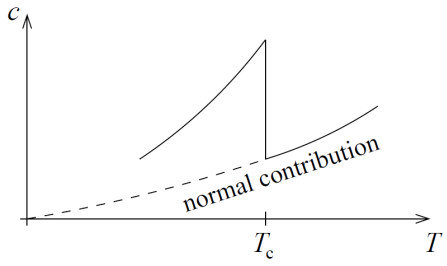
$$\begin{aligned} s_s(T) - s_n(T) &= -\frac{a_0^2}{b}(T_c - T), \qquad T < T_c \\ s_s(T) - s_n(T) &= 0, \qquad T > T_c \end{aligned} \tag{121}$$

The entropy is lower in the superconducting state but there is no discontinuity at T_c or latent heat implying that the normal to superconducting phase transition confirming that the Ginzburg-Landau model corresponds to a second-order thermodynamic phase transition.

Next, we can assess the change in the heat capacity per unit volume given by $c = C/V = T\frac{ds}{dT}$.

$$\begin{aligned} c_s(T) - c_n(T) &= \frac{a_0^2 T}{b}, T < T_c \\ c_s(T) - c_n(T) &= 0, T > T_c \end{aligned} \tag{122}$$

which shows that there is a sudden change in specific heat at T_c. The metallic normal state heat capacity per unit volume is linear in T, ${}_cn = \gamma T$, with γ the Sommerfeld constant.



The Ginzburg-Landau theory can correctly predict that there is a jump in the heat capacity at the transition, but the functional form is not quite in agreement with experiment. The Ginzburg-Landau theory can only be reliably used at temperatures close to T_c, in particular in low-T_c superconductors. Therefore, the calculated specific heat is only correct near to T_c. This theory, which relies on a mean-field approximation, has neglected important thermal fluctuation effects that can lead to amplitude and phase fluctuations in particular in high-temperature superconductors. In a mean-field theory one can consider many individual components that interact with each other but the effect of all the other individuals on any given individual is approximated by a single averaged effect, thus reducing a many-body problem to a one-body problem.

b) Inhomogeneous systems ($\psi = \psi(\mathbf{r})$)

To be able to describe also spatially non-uniform situations, Ginzburg and Landau had to go beyond the Landau description for a constant order parameter. To do so, they included gradients and define a macroscopic condensate wave function $\psi(\mathbf{r})$, which is essentially given by a wave-function averaged over length scales large compared to atomic distances. We expect any spatial changes of $\psi(\mathbf{r})$ to cost energy; this is analogous to the energy of domain walls in magnetic systems.

Consider the total energy (not just the energy density) and include a kinetic energy term to account for kinetic energy of charge particles and assume $q = -2e$ using the kinetic momentum operator, $\mathbf{P}$, which is guage invariant;

$$\frac{\mathbf{P}^2}{2m} \rightarrow \frac{(-i\hbar\nabla - q\mathbf{A})^2}{2m} \tag{123}$$

where the parameter m determines the energy cost associated with gradients in $\psi(\mathbf{r})$ and it has dimensions of mass, and it plays the role of an effective mass for the quantum system with macroscopic wave function $\psi(\mathbf{r})$. Note that the notation m^* and q^* is often used for the GL theory to distinguish the superelectrons from normal electrons.

One can also include a magnetic field energy term, where B_0 is applied field

$$\frac{(B(\mathbf{r}) - B_0)^2}{2\mu_0} \tag{124}$$

Putting all these together the (gauge-invariant) Ginzburg-Landau expression for the free-energy of a superconductor is given by:

$$F_{\mathrm{s}}(T) = F_{\mathrm{n}}(T) + \int \mathrm{d}^3 r \left[a(T)|\psi(\mathbf{r})|^2 + \frac{b}{2}|\psi(\mathbf{r})|^4 + \frac{1}{2m}| - i\hbar\nabla\psi(\mathbf{r}) - (-2e)\mathbf{A}\psi(\mathbf{r})|^2 + \frac{(B(\mathbf{r}) - B_0)^2}{2\mu_0} \right], \tag{125}$$

where $\psi(\mathbf{r}) = \psi_0 \mathrm{e}^{i\theta(\mathbf{r})}$ is the complex order parameter and B_0 is the applied field.

We are going to want to minimize this now and things from here on get theoretically complex.

Inhomogeneous systems $\psi = \psi(\mathbf{r})$ and A=0

Essentially setting the magnetic potential to zero means that we are not going to consider variations in phase. That this is the case can be seen from the fact that this equation is entire real. Thus what we are looking at here is how the amplitude of the order parameter affects the free energy.

When the magnetic field can be ignored by setting $\mathbf{A} = 0$, we have

$$F_{\mathrm{s}}(T) = F_{\mathrm{n}}(T) + \int \mathrm{d}^3 r \, f, \tag{126}$$

where

$$f = a(T)|\psi(\mathbf{r})|^2 + \frac{b}{2}|\psi(\mathbf{r})|^4 + \frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r})|^2. \tag{127}$$

The extensive derivation of this part can be found in Annett (pages 74-76). A few steps are given below:

Considering an infinitesimal variation of the order parameter in order to find a free energy minimum (writing without some modulus sign to connect later):

$$\psi \rightarrow \psi + \mathrm{d}\psi \tag{128}$$

If ψ is varied, then

$$\mathrm{d}f = 2a\psi \, \mathrm{d}\psi + 2b\psi^3 \, \mathrm{d}\psi + \frac{\hbar^2}{2m}\mathrm{d}|\nabla\psi|^2, \tag{129}$$

and to the first order

$$\mathrm{d}|\nabla\psi|^2 = |\nabla(\psi + \mathrm{d}\psi)|^2 - |\nabla\psi|^2 \simeq 2\nabla\psi \cdot \nabla(\mathrm{d}\psi). \tag{130}$$

We can re-write this term using the following math identity:

$$\nabla \cdot [\nabla\psi\, \mathrm{d}\psi] = (\nabla^2\psi)\mathrm{d}\psi + \nabla\psi \cdot \nabla(\mathrm{d}\psi) \tag{131}$$

As we are integrating over a volume of a superconductor and using the Gauss theorem

$$\int d^3r\ \nabla \cdot [\nabla\psi \mathrm{d}\psi] = \int \mathbf{dS}\ \nabla\psi \mathrm{d}\psi = 0 \tag{132}$$

since $\mathrm{d}\psi \to 0$ surface contribution can be taken to be zero for a superconducting to normal interface.

Then in expression 131 becomes

$$-(\nabla^2\psi)\mathrm{d}\psi = \nabla\psi \cdot \nabla(\mathrm{d}\psi) \tag{133}$$

Thus the variation of the free energy

$$df = \left[-\frac{\hbar^2}{2m}\nabla^2\psi + a\psi + b\psi \mid \psi \mid^2\right] 2(\delta\psi) \tag{134}$$

Hence

$$\mathrm{d}f = 2\mathrm{d}\psi[(a + b\psi^2)\psi - \frac{\hbar^2}{2m}\nabla^2\psi] \tag{135}$$

and minimizing for any $\mathrm{d}\psi$ (ψ taken to be real)

$$\frac{\mathrm{d}f}{\mathrm{d}\psi} = 2[(a + b\psi^2)\psi - \frac{\hbar^2}{2m}\nabla^2\psi] = 0 \tag{136}$$

for any ψ.

From energy minimization one finds:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + (a(T) + b \mid \psi(\mathbf{r}) \mid^2)\psi(\mathbf{r}) = 0 \tag{137}$$

This looks like a non-linear Schrödinger equation. The non-linearity has consequences for the interpretation of the order parameter; however, this equation is difficult to solve. Because of this nonlinearity the quantum mechanical principle of superposition does not apply, and the normalization of ψ is different from the usual one in quantum mechanics.

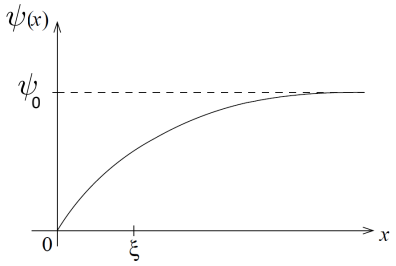
Surface of superconductors and coherence lenght

The effective nonlinear Schrödinger equation has several useful applications. In particular, it can be used to study the response of the superconducting order parameter to external perturbations. Important examples include the properties of the surfaces and interfaces of superconductors.

Consider a simple model for the interface between a normal metal and a superconductor. Suppose that the interface lies in the yz plane separating the normal metal in the $x < 0$ region from the superconductor in the $x > 0$ region (see picture of $\psi(x)$ vs x). On the normal metal side of the interface the superconducting order parameter, must be zero, $\psi(\mathbf{r}) = 0$. Assuming that $\psi(\mathbf{r})$ must be continuous, one must solve the nonlinear Schrödinger equation 141.

Since all coefficients are real, the solution can be chosen real. For $x \to \infty$ we should obtain the uniform solution recalling that in a uniform system below T_c we have

$$\psi_0 = \sqrt{-\frac{a(T)}{b}} \tag{138}$$



Here is useful to define a functional

$$\psi(x) = \psi_0 \cdot g(x) \tag{139}$$

so that for a uniform system we have $g = 1$.

The nonlinear Schrödinger equation can be written

$$-\frac{\hbar^2}{2m}\nabla^2 g(x) + a(T)g(x) + b|\psi_0|^2|g(x)|^2 g(x) = 0 \tag{140}$$

by dividing everything by $a(T)$ and using $|\psi_0|^2 = -a(T)/b$ the expression becomes

$$-\xi^2\nabla^2 g(x) + g(x) - g(x)^3 = 0 \tag{141}$$

where

$$\xi = \sqrt{\frac{\hbar^2}{2m|a(T)|}}. \tag{142}$$

This quantity ξ, which has dimensions of length, is called the Ginzburg-Landau coherence length. It is an important physical parameter characterizing the superconductor.

A solution of the resulting order parameter using the boundary conditions
at the surface ($\psi \to 0$ for $x \to 0$) and inside the superconductor $\psi(x) \to \psi_0$ for $x \to \infty$ is

$$\psi(x) = \psi_0 \tanh\left(\frac{x}{\sqrt{2}\xi(T)}\right) \tag{143}$$

and is represented in the figure above.

From the figure above one can see that ξ is a measure of the distance from the surface over which the order parameter $\psi(x)$ has recovered back to nearly its bulk value, ψ_0. The Ginzburg-Landau coherence length, ξ, is the natural length scale associated with the nonlinear Schrödinger equation. It is effectively a stiffness length for twisting the phase or magnitude of the order parameter.

The GL coherence length arises in almost all problems of inhomogenous superconductors, including surfaces, interfaces, defects, and vortices.

Using $a(T) = a_0(T - T_c)$, the coherence length $\xi(T)$ can be rewritten as:

$$\xi = \sqrt{\frac{\hbar^2}{2ma_0|(T - T_c)|}}. \tag{144}$$

which can be written

$$\xi = \xi_0 \left|\frac{T_c}{T - T_c}\right|^{1/2} \tag{145}$$

This expression makes it clear that the coherence length, $\xi(T)$, has a strong temperature dependence and diverges at the critical temperature, T_c. Its divergence is characterized by a critical exponent of 1/2 which is typical for mean-field theories such as the GL model. The zero temperature value of ξ_0, is apart from some numerical factors of order unity, essentially the same as the Pippard coherence length for superconductors, introduced earlier for a clean superconductor. In the BCS theory, the coherence length relates to the physical extend of a single Cooper pair.

Ginzburg-Landau equations in magnetic field

Minimizing this equation with respect to variations in ψ and $\mathbf{A}$ yields the Ginzburg-Landau equations:

$$-\frac{\hbar^2}{2m}\left(-i\nabla+\frac{2e}{\hbar}\mathbf{A}\right)^2\psi+a(T)\psi+b\psi^3=0, \tag{146}$$

and

$$\frac{\nabla\times\mathbf{B}}{\mu_0}=\mathbf{J}=-i\hbar\frac{(2e)}{(2m)}[\psi^*\nabla\psi-\psi\nabla\psi^*]-\frac{(2e)^2}{m}|\psi|^2\mathbf{A}. \tag{147}$$

These equations imply that, in the absence of a magnetic field, there is an energy cost associated with having a variation in the order parameter $\psi(\mathbf{r})$ from one part of the system to another. In particular, the superconducting ground state is one in which the phase of the order parameter takes a constant, but arbitrary, value throughout the sample. This **macroscopic phase coherence** is the essence of the long-range order exhibited by a superconductor.

Note that the equations are complex and that the first term on the right hand side of the last equation 147 looks like the *flow of probability current* in Quantum Mechanics but here ψ represents the order parameter.

These equations should be solved simultaneously but as a simple starting point we can consider $\nabla\psi=\nabla\psi^*$=0 in equation 147 (equivalent to constant θ or $\mathbf{p}=0$ in the bulk of the superconductor)

$$\mathbf{J}=-\frac{(2e)^2}{m}|\psi|^2\mathbf{A} \tag{148}$$

which is just the London equation with $q=2e$ as in experiments and $|\psi|^2=n_s$, as suggested by London. Note that since each Cooper pair has a net charge of $-2e$, then this is the correct effective charge q. Actually the sign can equally well be taken as $q=+2e$, since we can think of Cooper pairs of holes as readily as pairs of electrons. In fact no observable effects in the GL theory differ when we take a different convention for the sign.

The GL equations also predict the Meissner effect, flux trapping and quantization, as well as yielding expressions for the penetration depth λ and the Ginzburg-Landau coherence length ξ.

Consequences of GL theory: Type I and II superconductors

GL theory can be used to solve different problems in superconductivity, such as the superconductor to normal boundary. From GL theory the coherence length is:

$$\xi(T) = \sqrt{\frac{\hbar^2}{2m|a(T)|}}. \tag{149}$$

and the penetration depth

$$\lambda(T) = \sqrt{\frac{m \cdot b}{4\mu_0 e^2 |a(T)|}} \tag{150}$$

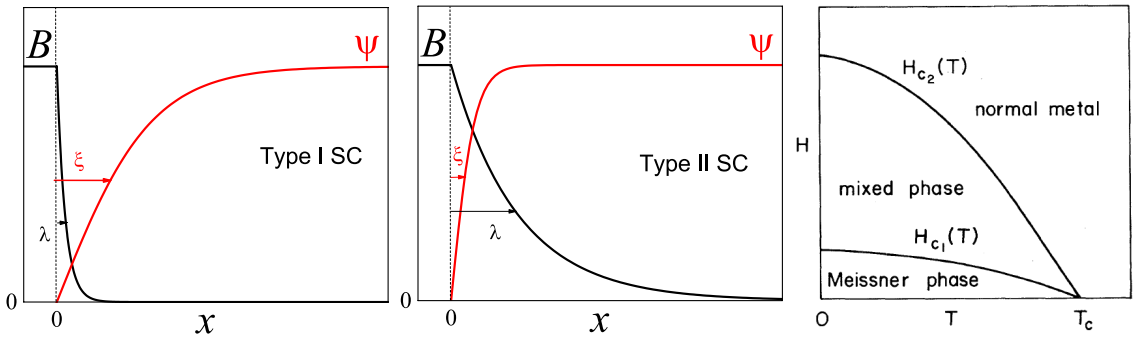
which has the same temperature dependence as the coherence length. $\lambda(T)$ in the distance over the magnetic field penetrates a superconductor and $\xi(T)$ in the distance over which ψ is diminished from its full bulk value ψ_0.

One can define the temperature independent Ginzburg-Landau parameter, κ:

$$\kappa = \frac{\lambda(T)}{\xi(T)} \tag{151}$$

Using the Ginzburg-Landau parameter one can define the type I and type II superconductor as: A type type I superconductor is defined if $\kappa < 1/\sqrt{2}$ and $\xi > \lambda(T)\sqrt{2}$, i.e. the coherence length is larger than the penetration length;
A type type II superconductor is defined if $\kappa > 1/\sqrt{2}$ and $\xi < \lambda(T)\sqrt{2}$, i.e. the coherence length is much smaller than the penetration length;

The left panel in the figure below shows the behaviour of magnetic field (B) and order parameter (ψ) close to the surface of a type I superconductor that exists for $x > 0$. Here the surface costs energy because there is an extended region for which field is being expelled, but where the order parameter has not reached its bulk value. The middle figure is the same plot for a type II superconductor showing that the surface saves energy because there are regions that have the full bulk value of the order parameter, but are not completely expelling the magnetic field. This qualitatively accounts for the formation of the mixed or vortex phase. The upper critical versus temperature phase diagram is obtained by solving the Ginzburg-Landau equations, as shown above on the right panel.



Now we can consider qualitatively what happens at the surface when we have two competing terms.

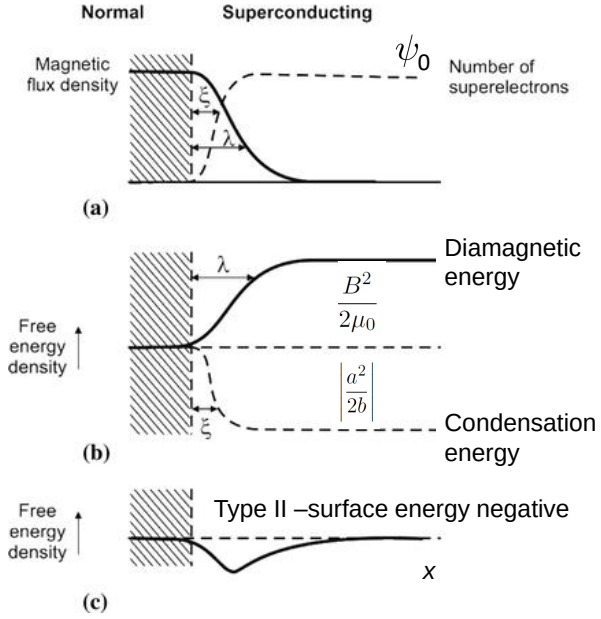
Condensation energy saved on becoming superconducting is:

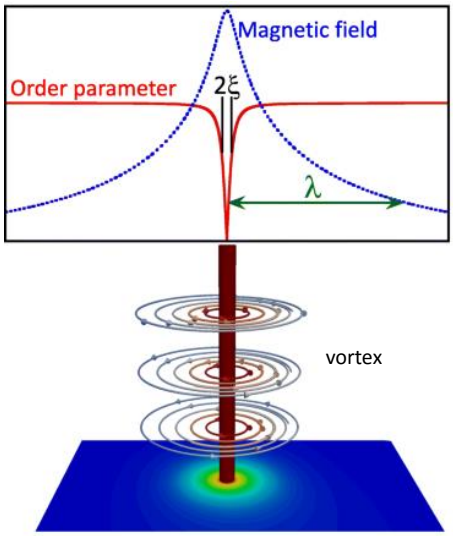
$$E_{condensation} = \left| \frac{a^2}{2b} \right| \tag{152}$$

and the diamagnetic energy cost on expelling the magnetic field is

$$E_{diamagnetic} = \frac{B^2}{2\mu_0} \tag{153}$$

In a type II superconductor, there is a region where the energy is being doubly saved by making full use of the condensation energy but also not fully expelling the field (see figure below). Thus, here, *the surface saves energy*. In type I superconductor, the surface area is minimized and we only have a superconductor-normal interface where absolutely necessary i.e. at the actual surface of the sample. The bulk of the sample is in the Meissner state. In type II superconductor the surface area should be maximized and this leads to fragmentation of the bulk of the superconductor into separate normal and superconducting regions (vortex or mixed state or Abrikosov phase) with flux threading the sample (as shown on the right). The supercurrent circulates around the normal (i.e. non-superconducting) core of the vortex. The core has a size of the coherence length ξ and the supercurrents decay on the distance about the penetration length λ from the core. The circulating supercurrents induce magnetic fields with the total flux equal to a single flux quantum Φ_0 and an (Abrikosov) vortex is often called a fluxon. The average vortex density (flux density) approximately equal to the externally applied magnetic field and the superconducting state is destroyed when all the normal vortex cores overlap.





Solving the GL equations quantitatively yields different characteristics critical fields leading to the phase diagram of different superconductors. In a type I superconductor there is a single critical field (thermodynamic critical field), B_c, which is the magnetic field associated with the condensation energy.

For a type II superconductor one can defined different critical fields:

$$B_{c1} = \mu_0 H_{c1} \approx \frac{\Phi_0}{4\pi\lambda^2} \ln \kappa \tag{154}$$

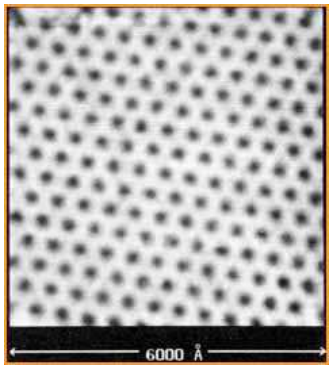
is the magnetic field at which one flux quantum can enter the sample;

$$B_{c2} = \mu_0 H_{c2} = \frac{\Phi_0}{2\pi\xi^2} \tag{155}$$

is the magnetic field at which superconductivity is destroyed as the vortices overlap. The formation of the mixed state allows the superconductivity to survive at fields much higher than would be otherwise possible. Vortices, each containing one flux quantum, will form into a lattice for fields between B_{c1} and B_{c2} (as shown schematic above and experimentally vortex lattice was observed). Experimentally, B_{c2} in a type II superconductor is typically much higher that thermodynamic B_c in a type I superconductor (it was also shown that $B_{c2} = \sqrt{2}\kappa B_c$).

There is also a B_{c3} associated with the surface of real superconductors; a superconducting surface sheet of thickness ξ can survive up to $B_{c3} = 1.695 B_{c2}$, despite the fact that the bulk is already normal and $\psi \to 0$.

Abrikosov found a solution to the GL equations in the case of a bulk superconductor in a magnetic field. He predicted that just below B_{c2} the order parameter forms into a periodic structure of vortices. Each vortex carries a magnetic flux, hence explaining how magnetic flux enters superconductors in the mixed state between B_{c1} and B_{c2}. Abrikosov's prediction of the flux lattice was confirmed experimentally (see below), showing not just that vortices occur, but they also tend to align in a regular triangular lattice, as predicted by the theory. Each normal core contains a single flux quantum and the supercurrent of each vortex flowing around the core, screens the SC regions in between. As the magnetic field goes up, the number of vortices increases and the upper critical field is reached, when they begin to overlap the system becomes normal. This periodic lattice of vortices was perhaps the first example in physics of *emergent phenomena in complex systems.*



This figure shows the vortex lattice imaged in $NbSe_2$ (a type II superconductor with a transition temperature of 7.2 K and a critical field of 3.2 T) by tunnelling into the superconducting gap edge with a low-temperature scanning-tunnelling microscope. The magnetic field used is 1 T. H. F. Hess *et al.*, Phys. Rev. Lett. **62**, 214 (1989).

For a square vortex lattice of spacing d, the $\Phi_0 = Bd^2$ and so $d = (\Phi_0/B)^{1/2}$. For the much more common triangular vortex lattice $d = (2\Phi_0/\sqrt{3}B)^{1/2}$.

Phase coherence from Ginzburg-Landau theory

In 1960, shortly after BCS theory was published, Lev Gor'kov was able to show that GL order parameter was equivalent to the centre of mass of the wave-function of individual SC electron pairs in the BCS theory. Because all the pairs condense into the *same state*, the complex wave function $\psi(\mathbf{r}) = \psi_0 e^{i\theta(\mathbf{r})}$ is the macroscopic wave-function of the superconducting state as a whole. The ordering that develops at the transition temperature T_c is an ordering of this wave-function.

In the ground state of a bulk superconductor, the superconducting parameter is constant and such the amplitude ψ_0

$$\psi_0 = \sqrt{n_s} = \text{const.} \qquad \text{and} \qquad \theta = \text{const.} \tag{156}$$

Therefore, it must have the same phase θ everywhere. There must be a *phase-stiffness*, or an energy cost associated with changing θ from one part of the solid to another.

The gauge transformation is used for the magnetic vector potential $\mathbf{A(r)}$, the cannonical momentum $\mathbf{p}$, phase of the order parameter, $\theta(\mathbf{r})$, that depend on the choice of a gauge. However, the GL theory satisfies *the local gauge invariance* as all physical observable like free energy, F, magnetic field $\mathbf{B}$ are gauge invariant.

Considering a superconductor with an order parameter with constant magnitude, $|\psi| = \psi_0$, and a phase $\theta(\mathbf{r})$, which varies only slowly with position $\mathbf{r}$, then the total GL free energy of the superconducting state from equation 125 becomes

$$F_s(T) = F_s^0(T) + \frac{\hbar^2 n_s}{2m} \int d^3r \left(\nabla\theta + \frac{2e}{\hbar}\mathbf{A} \right)^2 \tag{157}$$

where $F_s^0(T)$ is the total free energy ground state in the ground state ($\theta(\mathbf{r})$= constant, $\mathbf{A(r)} = 0$). A superfluid stiffness can be defined by

$$\rho_s = \frac{\hbar^2 n_s}{2m} \tag{158}$$

According to the free energy having a variation in phase $\nabla\theta$ costs energy.

For a particular gauge for $\mathbf{A(r)}$, such as the London gauge, $\nabla\mathbf{A}(\mathbf{r})$= 0, there is now a free energy cost associated with further gradients in $\theta(\mathbf{r})$. To minimize the gradient energy, one must minimize the gradients, by making $\theta(\mathbf{r})$= constant as possible throughout the system. In the case of zero applied magnetic field, we can choose $\mathbf{A(r)} = 0$, and clearly then $\theta(\mathbf{r})$ will be constant everywhere in the system. So in the ground state of all the SC carriers have the same phase. In the same way a ferromagnet picks an arbitrary direction for the spins to spontaneously align (i.e. the direction of M, the FM order parameter), so SC picks up an arbitrary phase of the entire sample. This is the nature of the symmetry that is broken at the SC phase transition, known as *phase coherence*. The energy cost associated with variations in phase $\nabla\theta$ represents *the rigidity of the system* which is the reluctance to change the way the symmetry is broken.

Because the long-range order is in the phase variable (which is not normally a physical observable in quantum mechanics) the system has spontaneously *broken global gauge symmetry*. The global gauge symmetry refers to changing $\theta(\mathbf{r})$ by a constant amount everywhere in the whole solid (which does not require any change in $\mathbf{A}(\mathbf{r})$. This is in contrast to local gauge symmetry in which $\theta(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are changed simultaneously.

Recall the second GL equation for the current density $\mathbf{J}$:

$$\psi = \sqrt{n_s} e^{i\theta} \tag{159}$$

$$\mathbf{J} = -i\frac{(2e)\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] - \frac{(2e)^2}{m}|\psi|^2\mathbf{A} \tag{160}$$

Allowing both the amplitude and phase to vary

$$n_s = n_s(\mathbf{r}) \tag{161}$$

$$\theta = \theta(\mathbf{r}) \tag{162}$$

$$\mathbf{J} = \frac{(2e)\hbar}{m} n_s \nabla\theta - \frac{(2e)^2}{m} n_s \mathbf{A} = \frac{(2e) n_s \hbar}{m}(\nabla\theta - \frac{(2e)}{\hbar}\mathbf{A}) \tag{163}$$

Note that in the absence of a magnetic potential ($\mathbf{A}$=0), superconducting current can still be caused by gradients in phase.

$$\mathbf{J} = \frac{(2e) n_s \hbar}{m}(\nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A}) \tag{164}$$

The supercurrent density $\mathbf{J}$ in a bulk superconductor depends on the gauge invariant phase gradient γ.

$$\gamma = (\nabla\theta - \frac{2\pi}{\Phi_0}\mathbf{A}) \tag{165}$$

The phase invariant phase gradient can be replaced by the gauge-invariant phase difference:

$$\phi = \oint_1^2 \gamma(\mathbf{r}) = \oint_1^2 \nabla\theta d\mathbf{l} - \frac{2\pi}{\Phi_0}\oint_1^2 \mathbf{A} d\mathbf{l} \tag{166}$$

$$\phi = \oint_1^2 \gamma(\mathbf{r}) = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0}\oint_1^2 \mathbf{A} d\mathbf{l} \tag{167}$$

Taking an integral around a loop ℓ_2 inside the bulk of a superconductor (where $\mathbf{J} = 0$) (see London's theory section)

$$\oint \mathbf{A}\cdot d\mathbf{l} = \Phi = \frac{\hbar}{2e}\oint \nabla\theta \cdot d\mathbf{l} \tag{168}$$

gives flux quantization.

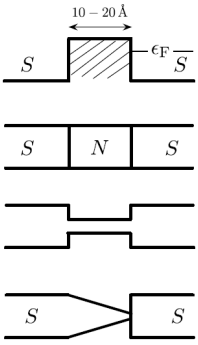
$$\Phi = \frac{Nh}{2e} \tag{169}$$

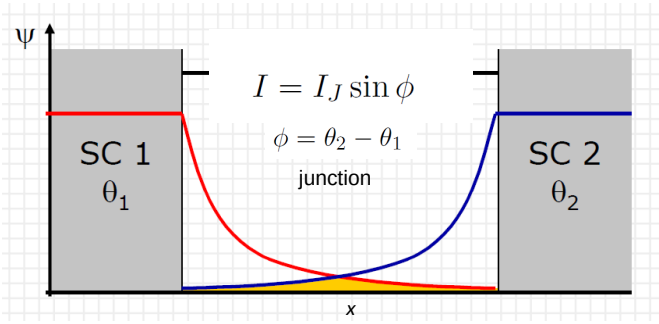
So phase coherence is the defining quantity of a superconductor. Although it costs energy for a superconductor to vary the phase, it can be induced to do exactly that in order to protect the superconducting state.

Typically in Quantum Mechanics the absolute phase of a wave function is not that important. It is the phase differences that causes potentially observable phenomena, such as superposition.

The Josephson effect [¶]

Josephson equation 1





Considering the tunneling through a junction between two SCs where a weak link can be insulator (SIS junction with $d \sim 2$nm), normal metal (SNS junction with $d \sim 500$ nm), a constriction or point contact (see left panel).

Phase is fixed in SC1 ($\psi_1 \sim e^{i\theta_1}$) and SC2 ($\psi_2 \sim e^{i\theta_2}$) but there will be a phase difference between them (se right panel). Josephson asserted that if the SC are weakly coupled, a supercurrent will flow across the sample due to the phase difference. Josephson worked out the details using BCS theory, but it can be understood from simple arguments.

The supercurrent density **J** from the second GL theory in the absence of a magnetic field and a vector potential (**A**=0):

$$\mathbf{J} = \frac{2e\hbar}{m} n_s \nabla\theta \tag{170}$$

and the currents from phase gradient (and various prefactors) are

$$I = I(\phi) \tag{171}$$

that depend on the phase difference between the two junctions $\phi = \theta_2 - \theta_1$. If the current density through the junction is zero then the phase difference is zero.

Since the variation of the phase θ through one of the electrodes does not change anything physically, it is evident that the current is periodic function with period 2π such that $I(\phi) = I(2\pi+\phi)$. Changing the sign of the current must change the sign of the phase difference $I(\phi) = -I(-\phi)$.

[¶]For learning about Josephson junctions for quantum computing read the review *Superconducting quantum circuits, qubits and computing* by Wendin and Shumeiko (http://arxiv.org/abs/cond-mat/0508729).

The tunneling probability can be obtained by solving GL equations. SC carriers tunnel through barrier due to phase difference ϕ.

$$I(L \to R) \sim e^{i(\theta_2 - \theta_1)} = e^{i\phi} \tag{172}$$

$$I(R \to L) \sim e^{i(\theta_1 - \theta_2)} = e^{-i\phi} \tag{173}$$

The net current is the total sum

$$I = I(L \to R) + I(R \to L) \sim (e^{i\phi} + e^{-i\phi}) \sim 2\sin(\phi) \tag{174}$$

$$I = I_J \sin\phi \tag{175}$$

This expression is the Josephson equation 1 and I_J is a characteristic current for each junction that depends on temperature, magnetic field.

Josephson equation 2

Next we consider the effect of an applied voltage across the device. This will change the potential energy of the electron pairs on the left and right side of the junction.

In Quantum Mechanics we can relate the energy to the time evolution from time-dependent Schrodinger equation.

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = E\psi(\mathbf{r}, t) \tag{176}$$

The time evolution of the microscopic wave-function in equilibrium is similar to that of a single electron in a stationary state.

$$\psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-i\varepsilon t/\hbar} \tag{177}$$

where ε is the energy associated with SC pair.

Considering the SC wave-functions for the different sides of the junction with their amplitudes independent of time:

$$\psi_1 = |\psi_1| e^{i\theta_1(t)} \tag{178}$$

and

$$\psi_2 = |\psi_2| e^{i\theta_1(t)} \tag{179}$$

gives

$$-\hbar \frac{\partial \theta_1}{\partial t} = E_1 \tag{180}$$

$$-\hbar \frac{\partial \theta_2}{\partial t} = E_2 \tag{181}$$

so the energy difference is

$$\Delta E = E_2 - E_1 = -\hbar \frac{\partial \phi}{\partial t} \tag{182}$$

but $\Delta E = qV$, where $q = -2e$ for SC charge carriers.

The voltage difference V is

$$V = \frac{\hbar}{2e}\frac{\partial \phi}{\partial t} \tag{183}$$

This expression is the Josephson equation 2.

DC Josephson effect (no applied voltage)

Let's look at the physical consequences of these two equations.

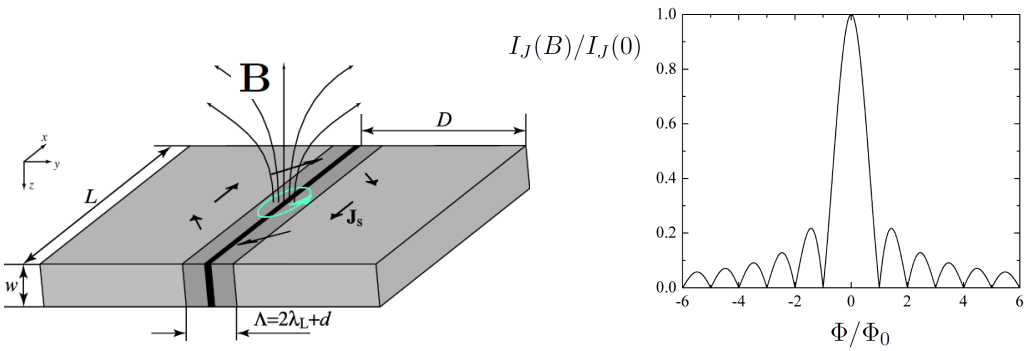
The first equation

$$I = I_J \sin \phi \tag{184}$$

means spontaneous supercurrent. In the absence of an applied voltage, a constant, spontaneous supercurrent flows across the junction.

Maximum current (or critical) $I_{max} = \pm I_J$ (typical values vary from $\sim 1\mu$A to 10 mA). I_J not the same as the critical current in a superconductor I_c but it is rather dependent on the junction details.

DC Josephson effect in magnetic field



Considering a Josephson junction placed in an external magnetic field parallel to the plane of the junction. A screening supercurrent will be generated at the outer surfaces of the slabs. This current circulates within a surface layer and, in doing so, has to cross the weak link, where the critical current density is very small. Therefore, in order to maintain its dissipation-free flow, the current must spread over a rather wide area protruding into the junction.

Taking the finite size of the Josephson contacts into account, one can find a correlation between I_J and the magnetic flux Φ through the junction. Solving London (or GL equation in magnetic field for the supercurrent), one can find that the JJ current I_J varies with the magnetic field due to the flux treading through the junction.

$$I_J(B) = I_J(0)\left|\frac{\sin(\pi\Phi/\Phi_0)}{\pi\Phi/\Phi_0}\right| \tag{185}$$

where the flux though the junction is $\Phi = (2\lambda_L + d) \cdot L \cdot B$ and d is the thickness of the non-superconducting layer, L is the size of the contact perpendicular to the magnetic field and λ_L is the London penetration depth.

The effect of the applied magnetic field is to modulate the Josephson phase difference across the width of the junction. Thus, from first Josephson equation the supercurrent density for a rectangular junction must vary across the width of the junction.

The overall effect of the applied field on $I_J(B)/I_J(0)$ is shown in the right panel. This is analogous to the interference pattern created by photons in a laser beam, observed in optics with Young's experiment.

Just as with type-II superconductors, when the external field exceeds a certain critical value B_{c1}, which is a characteristic of the junction, the field starts to penetrate into the junction in the form of superconducting vortices, each carrying one magnetic flux quantum Φ_0. In the case of a Josephson junction, they are called Josephson vortices. Having penetrated into the junction, the vortices form a linear chain and the junction goes into the mixed state. Unlike an Abrikosov vortex of a type II superconductor, a Josephson vortex does not have a normal core.

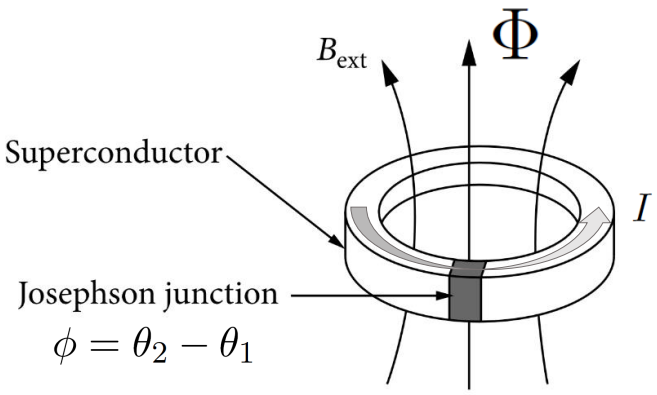
Superconducting loop with a Josephson junction in magnetic field

Consider a loop of superconductor and introduce a Josephson junction into it. This loop is exposed to a magnetic field and flux lines Φ treading though the loop. This is the extension of the flux quantization through a loop in the presence of a Josephson junction.

From London equation:

$$\oint \mathbf{J} \cdot d\mathbf{l} = \frac{n_s q}{m}(\hbar \oint \nabla\theta \cdot d\mathbf{l} - q \oint \mathbf{A} \cdot d\mathbf{l}) \tag{186}$$

Inside the SC $\mathbf{J} = 0$ and $q = 2e$ (the charge can be positive assuming pairing of holes);



Considering a single Josephson junction having the phase difference across the junction $\phi = \theta_2 - \theta_1$,

$$\oint \nabla\theta \cdot d\mathbf{l} = \frac{2e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} + \phi \tag{187}$$

As the wave-function must be single-valued, the change in phase across a closed loop needs to occur in units of 2π

$$\oint \nabla\theta \cdot d\mathbf{l} = 2\pi n \tag{188}$$

$$2\pi n = 2\pi \frac{\Phi}{\Phi_0} + \phi, \tag{189}$$

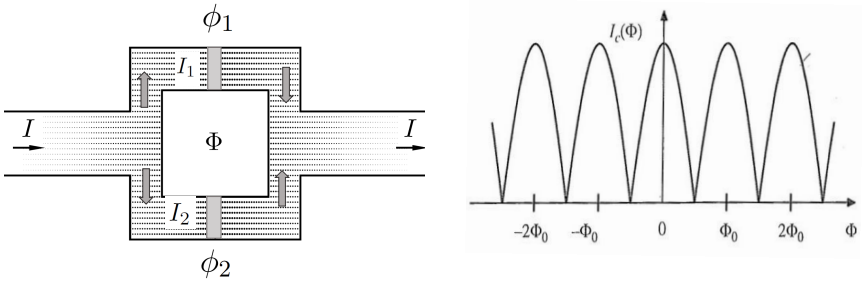
where $\Phi_0 = \frac{h}{2e}$ is flux quantum.

So when one changes the flux transverses the ring, the phase across the junction must change to keep the phase an integer multiple of 2π. Thus, the supercurrent across the across the Josephson junction changes as the flux changes.

SQUID -Superconducting interference device

A practical solution for a sensitive measurement of the magnetic flux through the loop is considering the use of two Josephson junctions in a loop.

Having two Josephson junctions and integrating around a complete loop the flux quantization through the loop is maintained in the presence of two phase difference around the loop which have opposite signs:



$$\oint \nabla\theta \cdot d\mathbf{l} = \frac{2e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} + \phi_1 - \phi_2 \tag{190}$$

$$2\pi n = 2\pi \frac{\Phi}{\Phi_0} + \phi_1 - \phi_2 \tag{191}$$

where $2\pi n$ for a single valued wave-function and n is an integer.

Assuming zero voltage across the junctions ($V = 0$ and using a contour such $J = 0$) and that the two junctions are perfectly balanced having the same I_J, the total current is given by

$$I = I_1 + I_2 = I_J \sin\phi_1 + I_J \sin\phi_2 \tag{192}$$

$$I = 2I_J \sin(\frac{\phi_1 + \phi_2}{2}) \cos(\frac{\phi_1 - \phi_2}{2}) \tag{193}$$

$$\phi_0 = \frac{\phi_1 + \phi_2}{2} \tag{194}$$

$$I = I_c(\Phi) \sin\phi_0 \tag{195}$$

and the total critical current

$$I_c(\Phi) = 2I_J \cos(\frac{\phi_1 - \phi_2}{2}) \tag{196}$$

When a magnetic flux threads the area

$$\cos\frac{(\phi_1 - \phi_2)}{2} = \cos\left(\pi n - \pi\frac{\Phi}{\Phi_0}\right) = (-1)^n \ \cos\left(\pi\frac{\Phi}{\Phi_0}\right) \tag{197}$$

and the total critical current becomes

$$I_c(\Phi) = 2I_J \mid \cos\left(\pi\frac{\Phi}{\Phi_0}\right) \mid \tag{198}$$

This looks like the DC Josephson equation with the critical current that depends strongly on flux. The figure above shows in the left panel is the modulation of the supercurrent by an applied magnetic field passing through the loop of the SQUID ring.

Gauge-invariant phase difference across a junction

Alternatively, one can derive the phase difference by invoking the gauge-invariant phase across the junction. The magnetic vector potential can have the gradient of an arbitrary scalar function added to it without changing anything physically. This occurs as part of a gauge transformation. However, this would seem to imply that the current density could take on any value. This would be unphysical. To ensure that the current density is the same irrespective of gauge, the phase $\nabla\theta$ must change too. Defining a new gauge χ the vector potential becomes

$$\mathbf{A}' \quad \rightarrow \quad \mathbf{A} + \nabla\chi \tag{199}$$

$$\mathbf{J} = \frac{q\hbar}{m} n_s \nabla\theta - \frac{(q)^2}{m} n_s \mathbf{A} \tag{200}$$

$$\nabla\theta' = \frac{m}{q\hbar} n_s \mathbf{J} - \frac{2\pi}{\Phi_0}\mathbf{A}' = \frac{m}{q\hbar} n_s \mathbf{J} - \frac{2\pi}{\Phi_0}\mathbf{A} - \frac{2\pi}{\Phi_0}\nabla\chi \tag{201}$$

Thus

$$\nabla\theta' = \nabla\theta - \frac{2\pi}{\Phi_0}\nabla\chi \tag{202}$$

$$\theta' = \theta - \frac{2\pi}{\Phi_0}\chi + C \tag{203}$$

So for a guage in $\mathbf{A}' \rightarrow \mathbf{A} + \nabla\chi$ the phase must change $\theta' \rightarrow \theta - \frac{2\pi}{\Phi_0}\chi + C$ up to a constant, C ($q = -2e$).

Let ϕ_1 be the gauge invariant phase at point 1 and ϕ_2 be the gauge invariant phase at point 2. The gauge-invariant phase difference follows easily as

$$\phi_1 = \theta_1 - \frac{2\pi}{\Phi_0}\chi_1 \tag{204}$$

$$\phi_2 = \theta_2 - \frac{2\pi}{\Phi_0}\chi_2 \tag{205}$$

$$\phi_2 - \phi_1 = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0}(\chi_2 - \chi_1) = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0}\int_1^2 \nabla\chi \cdot d\mathbf{l} \tag{206}$$

$$\phi_2 - \phi_1 = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A} \cdot d\mathbf{l} \tag{207}$$

So the gauge-invariant phase difference is the same regardless of the chosen gauge function χ.

$$\phi = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A} \cdot d\mathbf{l} \tag{208}$$

AC Josephson effect

Here we consider the case in which a voltage is applied to a Josephson junction. By solving Josephson equation 2, the phase difference is given by:

$$\phi = \frac{2eV}{\hbar}t + \phi_0 \tag{209}$$

substitute in Equation 1

$$I = I_J \sin(\omega_J t + \phi_0) \tag{210}$$

where Josephson frequency

$$\omega_J = \frac{2eV}{\hbar} \tag{211}$$

So an applied DC voltage yields a purely sinusoidal AC supercurrent and a Josephson junction is a natural tunable oscillator. Note that for $V \simeq 10\mu V$, $\omega_J/2\pi$ is ~ 4.8 GHz. This is known as the ac Josephson effect.

Inverse AC Josephson effect

Given that a AC current is generated when a DC voltage is applied to a Josephson junction, now consider the case when an AC voltage is applied.

$$V = V_0 + V_{rf}\cos\omega t \tag{212}$$

Using Equation 2 for JJ

$$\frac{\partial\phi}{\partial t} = \frac{2e}{\hbar}V_0 + \frac{2e}{\hbar}V_{rf}\cos\omega t \tag{213}$$

results in

$$\phi = \phi_0 + \frac{2e}{\hbar}V_0 t + \frac{2e}{\hbar\omega}V_{rf}\sin\omega t \tag{214}$$

substituting back in Equation 1 and the Josephson junction frequency is given by

$$\omega_J = \frac{2eV_0}{\hbar} \tag{215}$$

$$I = I_J \sin(\phi_0 + \frac{2e}{\hbar}V_0 t + \frac{2e}{\hbar\omega}V_{rf}\sin\omega t) \tag{216}$$

$$I = I_J \sin(\phi_0 + \omega_J t + \frac{2e}{\hbar\omega}V_{rf}\sin\omega t) \tag{217}$$

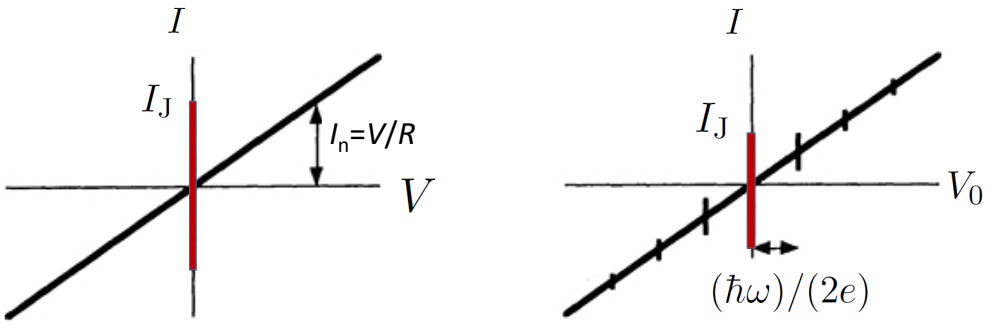
which is the inverse AC Josephson effect. In this expression the phase of the supercurrent is modulated and the frequency of the supercurrent is not the same as that of the driving ac voltage source.

The current can be expressed as a Fourier series

$$I = I_{\rm J}\sum_{n=-\infty}^{\infty}(-1)^n J_n\left(\frac{2eV_{\rm rf}}{\hbar\omega}\right)\sin[(\omega_{\rm J} - n\omega)t + \phi_0] \tag{218}$$

using the identity $\sin(a + z\sin\theta) = \sum_{n=-\infty}^{\infty}(-1)^n J_n(z)\sin(a - n\theta)$, and J_n are Bessel functions.

This implies that whenever $\omega_J = n\omega$ (multiple of the applied frequency) which is equivalent to $2eV_0 = n\hbar\omega$, the voltage becomes $V_0 = n(\hbar\omega)/(2e)$ and the supercurrent is DC again (Josephson equation 1) .



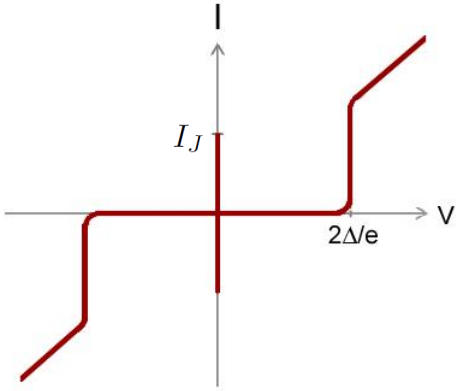
For a IV curve the supercurrent is DC again and we get spikes called Shapiro spikes. Their spike amplitude follows Bessel function $2I_J \cdot |J_n\left(\frac{2eV_{\text{rf}}}{\hbar\omega}\right)|$. These Shapiro spikes correspond to a condition the energy $2eV_0 = n\hbar\omega$. Because the Bessel functions oscillate as ac frequency is increased, these Shapiro spikes shrink to zero and then grow again several times as the microwave power is increased. Finally, at large powers, the Bessel functions all tend to zero and all super current spikes (including the zero voltage spike) are suppressed and only the normal DC current remains visible in the DC I–V characteristic. The appearance of the Shapiro spikes is the Josephson inverse AC effect.

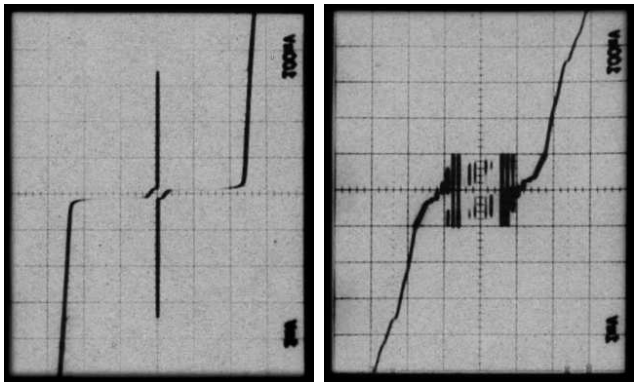
IV curves in the voltage-source model

For a current-voltage curve for a Josephson junction depends whether the current or the voltage is varied. For the DC voltage biased Josephson junction one would typically see:

- At $V = 0$, a DC SC flows due to superconducting pair tunneling;
- At $V > 0$, an AC SC flows (the mean voltage $< V >= 0$); normal tunneling can also occurs in real devices at finite temperatures;
- At V greater than a threshold value $|2\Delta/e|$ the SC breaks down it behaves as a normal barrier (Δ is the superconducting gap from BCS theory. Normal quasiparticle tunneling current is seen above $|2\Delta/e|$.

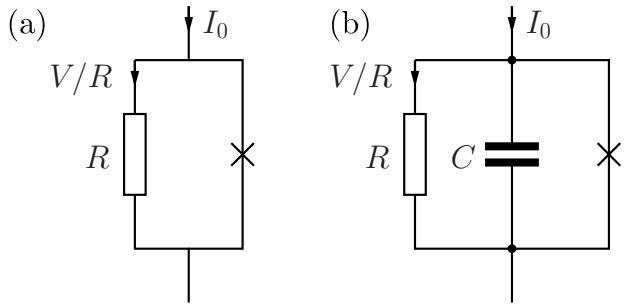
Figure below shows an ideal IV curve for the voltage-source model. (middle and right) The current-voltage characteristic (I horizontal, V vertical) for a superconductor-insulator-superconductor (SIS) Nb-Al_2O_3-Nb junction. (middle) The spike in the centre is caused by the DC and AC Josephson effects, while currents at larger values of V are caused by normal electron transport. (right) Applied microwave radiation of 70 GHz yields the Shapiro spikes of the inverse AC Josephson effect.





Real Josephson junctions: IV characteristic in the current-source model

A perfect Josephson junction (signified by the cross) can be inserted in various electrical circuits. (A real Josephson junction may well have some real resistance or capacitance so this circuit can be thought of as an attempt to model real junctions.) Single junctions typically have low impedances compared to the transmission lines and so can be thought of as being driven by a constant current source. This means that the normal and supercurrents can no longer be added and one must solve a differential equations.



(a) Josephson junction in parallel with a resistor ($V \neq 0$, $C = 0$)

While in the voltage-source model, the supercurrent and the normal current flowing in the shunt impedance could be calculated separately and simply added together, in the current-source model supercurrents generated in the Josephson element have to be compensated by equivalent negative currents in the shunt R. These compensating currents generate voltages which in turn affect the supercurrent.

Consider the case of a resistively shunted Josephson model (the Josephson junction in parallel with the R which acts as a shunt in the panel a) with the total current I_0 obtained by applying the Kirchoff's law in this circuit (V the total voltage):

$$I_0 = I_J \sin\phi + \frac{V}{R} \tag{219}$$

and using the second Josephson equation:

$$V = \frac{\hbar}{2e}\frac{d\phi}{dt} \tag{220}$$

$$I_0 - I_J \sin\phi = \frac{\hbar}{2eR}\frac{d\phi}{dt} \tag{221}$$

The phase variation of this circuit becomes

$$\frac{d\phi}{dt} \sim I_0 - I_J \sin\phi \tag{222}$$

which is a first order non-linear differential equation. This can be solved assuming a constant current I_0.

a) if $I_0 < I_J$ the phase does not change with time and is therefore stationary (the steady state solution) $\phi = \arcsin(I_0/I_J)$ and the voltage is zero. The phase reaches $\phi = \pi/2$ for $I_0 = I_J$.

b) if $I_0 > I_J$, the phase grows with time and we get a non-vanishing voltage. This is the dynamic solution as $\frac{d\phi}{dt} \neq 0$ is never zero and the phase ϕ changes continuously.

The IV curve for such a junction has the mean voltage (over time t) across the junction is given by:

a) $\langle V\rangle$=0 for $I_0 < I_J$ the current flows without resistance. At I_J , non-zero DC and AC voltages set in gradually.

b) For $I_0 > I_J$

To determine the average voltage $\langle V\rangle$, we calculate the time t_0 is takes for the phase to increase with 2π by integrating for such a period of time the following relationship:

$$\frac{\hbar}{2eR}\frac{d\phi}{I_0 - I_J \sin\phi} = dt \tag{223}$$

$$t_0 = \frac{\hbar}{2eR}\frac{2\pi}{\sqrt{I_0^2 - I_J^2}} \tag{224}$$

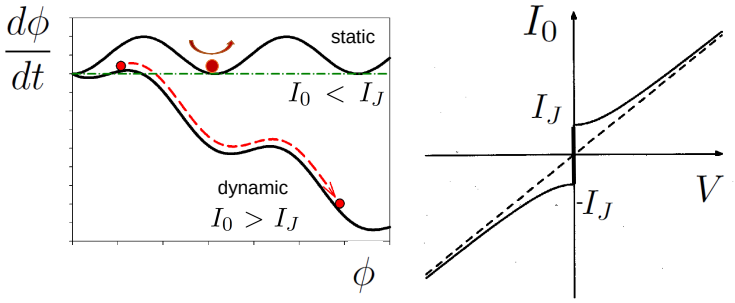
The voltage varies over this period but we can find its average

$$\langle V\rangle = \frac{1}{t_0}\int_0^{t_0} V dt = \frac{\hbar}{et_0}\int_0^{t_0}\frac{d\phi}{dt}dt = \frac{2\pi\hbar}{2et_0} \tag{225}$$

resulting in

$$\langle V\rangle = R\sqrt{I_0^2 - I_J^2} \tag{226}$$

This shows that the mean voltage drop across the junction has a step a $V = 0$ rather than a spike (as compared with the voltage driven case). Since the normal current is still V/R it is evident that the d.c. supercurrent is no longer zero at finite voltage in the current-source model. For $I_0 \gg I_J$, the DC voltage approaches the ohmic result for a normal contact.

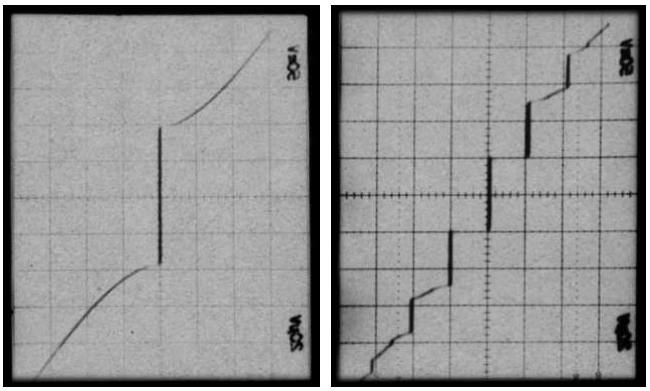


The figure above is the tilted washboard analogue for the change of phase in a current-driven Josephson junction; a particle slides down the washboard through viscous treacle under the influence of gravity. The phase ϕ corresponds to distance down the washboard, and the applied d.c. current I_0 corresponds to the tilt of the washboard. If we have $I_0 > I_J$, we get an unsteady motion: ϕ will increase continuously, but will do so most slowly when $\sin\phi = 1$ and most rapidly when $\sin\phi = -1$. For a gentle tilt ($I_0 < I_J$) the particle comes to rest (no voltage) in one of the hollows of the board. For a larger tilt ($I_0 > I_J$), the particle slides continuously but unsteadily (finite but varying voltage).

If we apply a radio-frequency excitation to this real junction, one gets Shapiro steps rather than Shapiro spikes and these steps will occur at

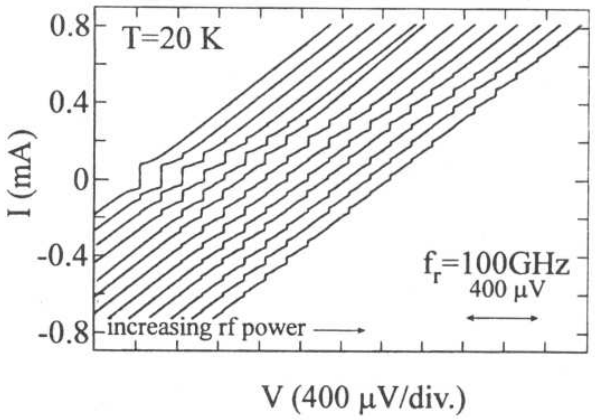
$$V_0 = n\frac{\hbar\omega}{2e} \tag{227}$$

where ω is the applied frequency.



The current-voltage characteristic (I vertical, V horizontal) for a superconductor-normal-superconductor (SNS) Nb–PdAu–Nb junction This corresponds well with the current-source model, in which the spikes are replaced by steps. [Left] The DC and AC effects. [Right] Microwave radiation of 10 GHz produces Shapiro steps.

The current-voltage characteristic (I vertical, V horizontal) for a high-T_c Josephson junction under microwave illumination. Data are shown for increasing microwave power.



(b) Josephson junction in parallel with a resistor and capacitance ($V \neq 0$, $C \neq 0$)

Considering now the case of a real circuit containing a Josephson junction in parallel with a resistor and capacitance such that the total current is given by:

$$I_0 = I_J \sin\phi + \frac{\hbar}{2eR}\frac{d\phi}{dt} + \frac{\hbar C}{2e}\frac{d^2\phi}{dt^2} \tag{228}$$

Re-arranging the above expression as:

$$\frac{\hbar C}{2e}\frac{d^2\phi}{dt^2} + \frac{\hbar}{2eR}\frac{d\phi}{dt} = I_0 - I_J \sin\phi \tag{229}$$

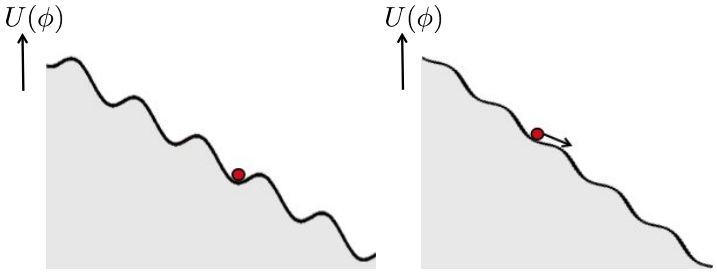
we can re-write the general differential equation like:

$$m\frac{d^2\phi}{dt^2} + \gamma\frac{d\phi}{dt} = -\frac{\partial U}{d\phi} \tag{230}$$

This looks like an equation of motion for a ball rolling through a viscous liquid down a bumpy slope and the different terms are:
- the inertial term is the acceleration of the phase, $m\frac{d^2\phi}{dt^2}$, where $m = \frac{\hbar C}{2e}$;
- the damping term is the dissipation, $\gamma\frac{d\phi}{dt}$, where $\gamma = \frac{\hbar}{2eR}$;
- the bumpy slope is the washboard potential is the force from a potential U, $\frac{\partial U}{d\phi}$ where the potential is given by

$$U(\phi) = -I_0\phi - I_J \cos\phi \tag{231}$$

as the *tilted washboard potential* $I_0\phi$ -tilt and $I_J\cos\phi$ -bumps.



Two regimes for the tilted washboard potential: $I_0 < I_J$, which leads to minima in $U(\phi)$ [left]; or $I_0 > I_J$ for which there are no minima [right]. In the latter case, ϕ and hence $I_J \sin\phi$ continue to vary with time. When minima are present it is possible for ϕ to reach a steady solution.

When

$$\frac{\partial U(\phi)}{\partial\phi} = 0,\ I_0 = I_J \sin\phi \tag{232}$$

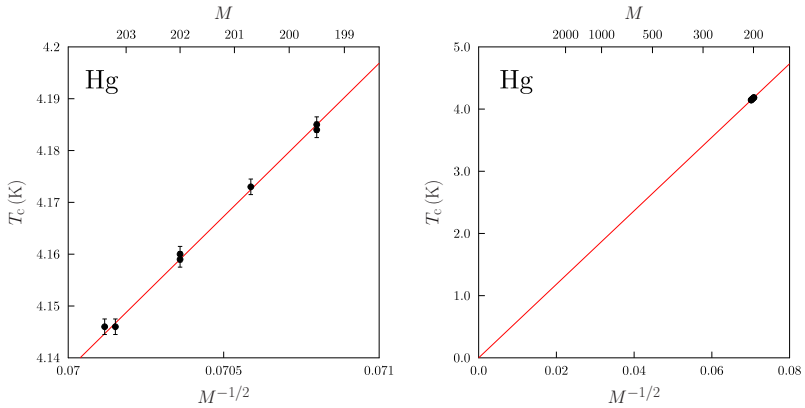
When $I_0 << I_J$ the particle is trapped in a minimum with constant phase and there are steady solutions when the ball gets stuck in a minima $U(\phi)$ and depending on the inertial term. There could be two solutions ($I_0 = 0$ and $I_0 = 0.5I_J$), one in which the ball in trapped with zero voltage as for the R+JJ, with a DC Josephson current ($< V >= 0$). Another solution is when the ball can slowly roll, such that the dissipation cancels the acceleration. This double valued solution give rise to an hysteresis effect in real IV characteristics, i.e. when the current is increased the voltage takes the value given by forward arrow, whereas when the current is decreased it takes the other value.

When $I_0 >> I_J$, the particle rolls down rapidly with almost linearly increasing phase and $I = V/R$ again. If the tilt is large, compared to the depth of the bumps, then there are no minima in $U(\phi)$, and there is no steady solution and the supercurrent is constantly varying with time.

BCS theory: the microscopic theory of superconductivity

The isotope effect

For conventional superconductors there is experimental evidence that the coupling to the lattice via phonos plays an important role in superconducting pairing of electrons. This evidence comes from the *isotope effect*, where different isotopes of the same material are obtained by adding and subtracting a few neutrons from the nucleus of atoms. and this affects the frequency of vibration of the atoms by changing their masses. Thus, it is often found that the critical temperature varies as $T_c \propto M^{-1/2}$, where M is the mass of the isotope.



This is very good evidence for the role of *phonons* in superconductivity.

Attractive interaction between electrons

The Coloulomb repulsion of two independent electrons in free space is given by:

$$V(\mathbf{r}-\mathbf{r}') = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{233}$$

To be able to have attractive interaction between the electrons the Coulomb energy needs to be reduced.

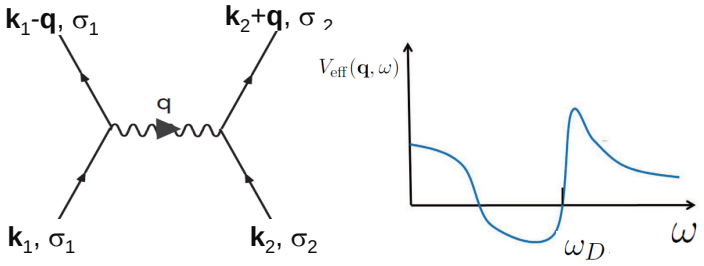
In a metal one can refer to the charge carriers as quasiparticles, not bare electrons. A quasiparticle is an excitation of a solid consisting of a moving electron together with its surrounding. When the electron moves other electrons must move out of the way. They must do this both because the exclusion principle prevents two electrons of the same spin being at the same point (this is called the exchange interaction) and because they must also try to minimize the repulsive Coulomb energy.

The effective direct Coulomb interaction between electrons is shielded by the other electrons and the electrons also shield the ions involved in the vibrational motion. The Coulomb interaction is screened by the conduction electrons in a solid is given by:

$$V(\mathbf{r}-\mathbf{r}') = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}e^{-|\mathbf{r}-\mathbf{r}'|/r_{\mathrm{TF}}} \tag{234}$$

where r_{TF} is the Thomas Fermi screening length. Screening reduces the Coulomb repulsion and the effective repulsive force is now short ranged in space, vanishing for $\mathbf{r} \to r_{\mathrm{TF}}$, which is much weaker than the original $1/\mathbf{r}$ potential. The Fermi sea screens quasiparticles from each other and reduces repulsion between electrons and their Coulomb repulsion, but it is never attractive.

Consider now the effect of interaction via phonons which was first formulated by Frölich in 1950.



The electrons can interact with each other via their interaction with the phonons of the crystal lattice. In the language of Feynman diagrams an electron with momentum $\hbar\mathbf{k_1}$ can be scattered by a phonon of crystal momentum $\hbar\mathbf{q}$, changing the electron in a state with momentum $\hbar(\mathbf{k_1}-\mathbf{q})$. Later a second electron can absorb the phonon $\hbar\mathbf{k_2}$ and pick up the momentum $\hbar\mathbf{q}$ and changes its momentum to $\hbar(\mathbf{k_2}+\mathbf{q})$. This gives rise to Feynman diagrams (left panel in the figure), which correspond to an effective interaction between the electrons.

The effective interactions between electrons due to exchange of phonon is of the form and one needs to sum over different phonon nodes and wavevectors $\mathbf{q}$:

$$V_{\mathrm{eff}}(\mathbf{q},\omega) = |g_{\mathbf{q},\lambda}|^2\frac{1}{\omega^2-\omega^2_{\mathbf{q},\lambda}} \tag{235}$$

for a virtual phonon with wave vector q, frequency, $\omega_{\mathbf{q},\lambda}$ and $|g_{\mathbf{q},\lambda}|^2$ is related to the matrix element for electron scattering $\mathbf{k} \to \mathbf{k}+\mathbf{q}$.

Full treatment is very complex, but the BCS theory simplified this interaction. They neglected dependence of the interaction on the wave vector $\mathbf{q}$ and phonon branch λ, replacing the interaction by an approximate to an effective interaction, $V_{eff}(\mathbf{q},\omega)$, which effectively averages over all values of $\mathbf{q}$. The frequency $\omega_{\mathbf{q},\lambda}$ is replaced by, ω_{D}, which is a typical phonon frequency taken to be the Debye frequency of the phonons. Thus the

$$V_{\mathrm{eff}}(\mathbf{q},\omega) \sim |g_{\mathrm{eff}}|^2 \frac{1}{\omega^2 - \omega_D^2} \tag{236}$$

ω_D was Debye frequency is the maximum frequency that electron-phonon coupling can exist in a solid. This expression suggests that the electrons will feel an attractive interaction ($V_{\mathrm{eff}} < 0$) when $\omega < \omega_D$ and a repulsive interaction ($V_{\mathrm{eff}} > 0$) when $\omega > \omega_D$, as shown in the right panel of the figure above. In the actual many-body system, only the electrons near the Fermi level will be affected by the attractive interaction. The matrix element for scattering of a pair of electrons near the Fermi surface from exchange of virtual phonons is negative (attractive), if the energy difference between the electron states involved is less than the phonon energy.

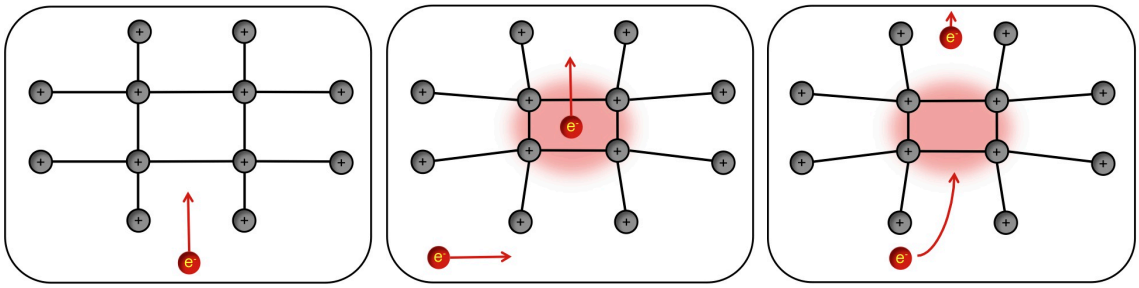
As superconductivity in many systems happens at low temperatures one can consider the electrons that can scatter within $\pm k_B T$ of the Fermi surface at E_{F} having the condition that $k_B T << \hbar\omega_D$ that correspond to the electrons within a thin shell of Bloch states around the Fermi energy $\hbar\omega_D$ around E_F or $\epsilon \sim |\epsilon_{\mathbf{k}} - \mu| < \hbar\omega_{\mathrm{D}}$ (as shown in the figure below).

$$V_{\mathrm{eff}}(\mathbf{q},\omega) \approx -|g_{\mathrm{eff}}|^2 < 0 \tag{237}$$

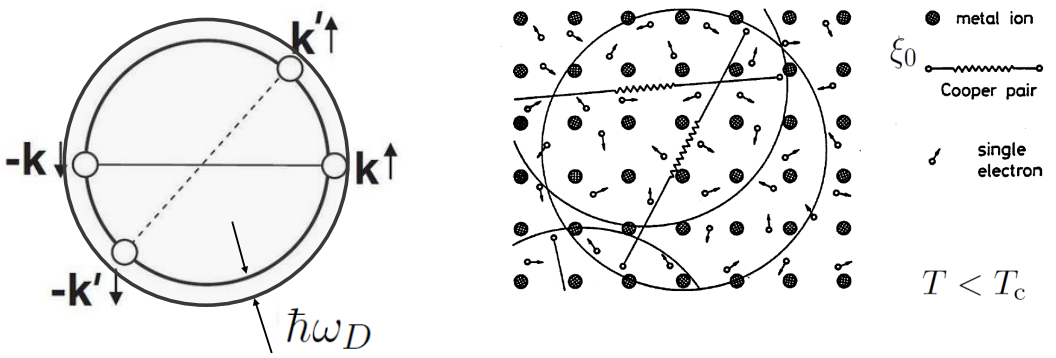
Thus, the electrons close to the Fermi surface feel an attractive interaction $V_{\mathrm{eff}} < 0$ no matter how small this $|g_{\mathrm{eff}}|^2$ parameter is. In the BCS model the isotope effect arises because the thickness of the energy shell around the Fermi surface is $\hbar\omega_D$, and not from changes in the coupling constant, $|g_{\mathrm{eff}}|$.

Cooper pairs

The superconducting carriers in BCS theory are Cooper pairs, which consist of two electrons with equal and opposite crystal momentum and spin, bound together by a phonon-mediated interaction. Below is a hand-wavy cartoon of how the interaction works: an electron perturbs the lattice as it moves past and a nearby electron is attracted to the perturbation.



Not all attractive interactions necessary result in bond pairs of electrons but in 1956 Leon Cooper showed that this was indeed the case for two electrons lying within the Debye energy of the Fermi surface, even for very weak interaction. So this is why superconducting pairs are called Cooper pairs and the pairing mechanism is mediated by the electron-phonon interaction.



Cooper considered a Fermi liquid at $T = 0$ with two electrons excited outside the Fermi surface, and considered their interaction via phonons. The wave function of the Cooper pair (CP) is given by:

$$\Psi_{\rm CP}(\mathbf{r_1}, \sigma_1, \mathbf{r_2}, \sigma_2) = e^{i\mathbf{K}_{\rm CM}\mathbf{R}_{\rm CM}}\ \Psi(\mathbf{r_1} - \mathbf{r_2})\ \Phi_{\sigma_1,\sigma_2} \tag{238}$$

where $e^{i\mathbf{K}_{\rm CM}\mathbf{R}_{\rm CM}}$ in the centre of mass of the Cooper pair, $\mathbf{R}_{\rm CM}$=$(\mathbf{r_1} + \mathbf{r_2})/2$
$\Psi(\mathbf{r_1} - \mathbf{r_2})$ gives the orbital part of the wave-function and
Φ_{σ_1,σ_2} gives the spin part of the wave-function.

The minimum energy for this state occurs when there is no centre of mass motion (CM is at rest so that the two electrons can chase each other around the lattice). The means with deal with electron pairs with equal and opposite momentum $\mathbf{K}_{\rm CM} = 0$ and the quasiparticles pair has $(+\mathbf{k},-\mathbf{k})$ ($S = 0$ and L=0) which is a Cooper pair (as shown in the left panel below).

Because we are dealing with two fermions, the total wave function needs to be *antisymmetric*:

$$\Psi_{\rm CP}(\mathbf{r_1}, \sigma_1, \mathbf{r_2}, \sigma_2) = -\Psi_{\rm CP}(\mathbf{r_2}, \sigma_2, \mathbf{r_1}, \sigma_1) \tag{239}$$

Cooper chose to look at the lowest angular momentum $L = 0$, which has the *s-wave* orbital state and is symmetric, and it needs to have spin singlet (S=0), meaning antisymmetric spin wave-function.

$$\Psi(\mathbf{r_1} - \mathbf{r_2}) = +\Psi(\mathbf{r_2} - \mathbf{r_1}) \tag{240}$$

$$\Phi_{\sigma_1,\sigma_2} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{241}$$

This turns out to make sense for the BCS effective interaction, because is independent of phonon wave-vector, i.e. a constat function in momentum space. This means that is a δ function in real space.

In real space

$$V_{\text{eff}}(\mathbf{r_1} - \mathbf{r_2}) = -|g_{\text{eff}}|^2 \delta(\mathbf{r_1} - \mathbf{r_2}) \tag{242}$$

which is a point contact interaction. This vanishes unless we have $\Psi(0) \neq 0$ and only works for s-wave.

Next Cooper solve the Schrodinger for this state and found out a binding energy for the electron pair of:

$$\boxed{E_{\text{CP}} = -2\hbar\omega_D e^{-1/\lambda}} \tag{243}$$

where $\lambda = |g_{\text{eff}}|^2 g(E_{\text{F}}) << 1$ is the electron-phonon coupling parameter (not to be confused with the penetration depth of a magnetic field in a superconductor) and $g(E_F)$ is density of states at the Fermi level. In any dimension, in the presence of a Fermi sea an arbitrarily weak attractive potential will form a bound state and the Pauli exclusion principle facilitates electron binding.

Creation and annihilation operators

We define a creation operator $\hat{a}^\dagger$ and an annihilation operator $\hat{a}$ for the harmonic oscillator problem:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2. \tag{244}$$

Since $[\hat{x}, \hat{p}] = i\hbar$ we have that $[\hat{a}, \hat{a}^\dagger] = 1$. Furthermore, we can write

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{245}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{246}$$

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle, \tag{247}$$

and hence $\hat{a}^\dagger\hat{a}$ is the number operator. The Hamiltonian becomes

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \tag{248}$$

and the eigenvalues are $E = (n + \frac{1}{2})\hbar\omega$. Note that

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle. \tag{249}$$

Coherent states

A coherent state $|\alpha\rangle$ is defined by

$$|\alpha\rangle = C\left[|0\rangle + \frac{\alpha}{\sqrt{1!}}|1\rangle + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + \frac{\alpha^3}{\sqrt{3!}}|3\rangle + \cdots\right], \tag{250}$$

where $\alpha = |\alpha|e^{i\theta}$ is a complex number. Hence

$$|\alpha\rangle = C\left[1 + \frac{\alpha\hat{a}^\dagger}{\sqrt{1!}} + \frac{(\alpha\hat{a}^\dagger)^2}{\sqrt{2!}} + \frac{(\alpha\hat{a}^\dagger)^3}{\sqrt{3!}} + \cdots\right]|0\rangle, \tag{251}$$

This coherent state can be written

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2}\exp(\alpha\hat{a}^\dagger)|0\rangle}. \tag{252}$$

A coherent state $|\alpha\rangle$ is defined to be the (unique) eigenstate of the annihilation operator $\hat{a}$ associated to the eigenvalue α, so that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{253}$$

and hence

$$\langle\alpha|\hat{a}|\alpha\rangle = \alpha \neq 0 \tag{254}$$

The mean number of particles $\langle\hat{n}\rangle$ in a coherent state $|\alpha\rangle$ is given by

$$\langle\hat{n}\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \langle\alpha|n|\alpha\rangle = |\alpha|^2 \tag{255}$$

as

and hence

$$\langle n|\hat{a}|n\rangle = 0 \tag{256}$$

Coherent states do not have a definite value of the quantum number n simply because they are not eigenvalues of the number operator $\hat{n}$.

The uncertainty in the number of particles in the coherent state is given by the Poisson number distribution of n.

$$\Delta\hat{n} = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2} = |\alpha| \tag{257}$$

It can be shown that the number-phase uncertainty

$$\Delta n\Delta\theta \geq 1/2 \tag{258}$$

where n and θ of the ground state are canonically conjugate. So if n is well-defined θ is uncertain and vice-versa.

To describe a superconducting state that has a well-defined phase one can use the coherent state. A coherent state can be defined for any complex number, α, as $\alpha = |\alpha|e^{i\theta}$. Importantly, a coherent state $|\alpha\rangle$ has a definite phase, thought it does not have a definite quantum number n. Coherent states were employed to describe the many-body wave-function of a superconductor with phase coherence at its heart.

BCS wave function

Schriffer proposed that the superconducting phase was a coherent phase of Cooper pairs of uncertain numbers but with a well-defined phase. BCS theory is different from most of the theoretical investigations in condensed matter physics. It does not solve the wave function from a many-body Hamiltonian, which is an extremely difficult task.

- $\hat{c}^\dagger_{\mathbf{k}\sigma}$ is a creation operator for a single electron with momentum $\mathbf{k}$ and spin σ.
- $\hat{c}_{\mathbf{k}\sigma}$ is a annihilation operator for a single electron with momentum $\mathbf{k}$ and spin σ.
- $\hat{P}^\dagger_k = \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow}$ is the Cooper pair creation operator.
  This operator creates a pair with zero net momentum ($L = 0$) and spin ($S = 0$).

The Fermi sea can be written as

$$|\text{Fermi sea}\rangle = \prod_{\sigma,k<k_\text{F}} \hat{c}^\dagger_{\mathbf{k}\sigma}|0\rangle \tag{259}$$

where $|0\rangle$ is the is the vacuum state is the state with no electrons at all in the band of Bloch states at the Fermi surface. This zero temperature Fermi sea, in which all states with $k < k_\text{F}$ and different spin σ are occupied and the rest are unoccupied. One can equally well view the BCS state as a condensate of electron pairs above a filled electron Fermi sea or a condensate of hole pairs below an empty *hole sea*.

Coherent state for fermion pairs:

$$|\Psi_\text{BCS}\rangle = C\left(e^{\sum_\mathbf{k}\alpha_\mathbf{k}\hat{P}^\dagger_\mathbf{k}}\right)|0\rangle \tag{260}$$

where $\alpha_\mathbf{k}$ are complex numbers to be determined.

Even though these pair operators do not obey Bose commutation laws, it can be shown that the pair operator commutes with itself

$$[\hat{P}^\dagger_{\mathbf{k1}}, \hat{P}^\dagger_{\mathbf{k2}}] = 0 \tag{261}$$

and

$$[\hat{P}^\dagger_\mathbf{k}, \hat{P}^\dagger_\mathbf{k}] = 0 \tag{262}$$

due to Pauli exclusion principle.

The BCS wave function will be written as a product of coherent states of pairs using the fact that these operators commute one for each $\mathbf{k}$ point,

$$|\Psi_\text{BCS}\rangle = C \times \prod_\mathbf{k} \exp(\alpha_\mathbf{k}\hat{P}^\dagger_\mathbf{k})|0\rangle. \tag{263}$$

where C is a constant and this coherent state wave-function has an indefinite number of particles.

Expanding each of the operator exponentials

$$|\Psi_\text{BCS}\rangle = C\prod_\mathbf{k}(1 + \alpha_\mathbf{k}\hat{P}^\dagger_\mathbf{k})|0\rangle \tag{264}$$

where the quadratic and higher order quadratic terms can be lost as

$$[\hat{P}^\dagger_\mathbf{k}, \hat{P}^\dagger_\mathbf{k}] = (\hat{P}^\dagger_\mathbf{k})^2 = 0 \tag{265}$$

due to Pauli exclusion principle.

Note. While the individual particles that make the pairs are fermions (spin 1/2), the pair themselves are not strictly bosons with spin 0.

Finally, normalizing to get the constant , C,

$$\langle \Psi_{\rm BCS} | \Psi_{\rm BCS} \rangle = 1 \tag{266}$$

$$\langle 0 | (1 + \alpha^*_{\mathbf{k}} \hat{P}_{\mathbf{k}})(1 + \alpha_{\mathbf{k}} \hat{P}^\dagger_{\mathbf{k}}) | 0 \rangle = 1 = (1 + |\alpha_{\mathbf{k}}|^2) \tag{267}$$

Thus, the BCS wave-function is given by:

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} (u^*_{\mathbf{k}} + v^*_{\mathbf{k}} \hat{P}^\dagger_{\mathbf{k}}) |0\rangle, \tag{268}$$

where the variational parameters

$$u^*_{\mathbf{k}} = \frac{1}{\sqrt{1 + |\alpha_{\mathbf{k}}|^2}} \tag{269}$$

and

$$v^*_{\mathbf{k}} = \frac{\alpha_{\mathbf{k}}}{\sqrt{1 + |\alpha_{\mathbf{k}}|^2}} \tag{270}$$

are complex by convention and

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \tag{271}$$

where $|u_{\mathbf{k}}|^2$, is the probability of the pair state to be occupied and $|v_{\mathbf{k}}|^2$ is the probability of the pair state to be unoccupied.

Finally, the way the BCS state was originally written, as described above, treats electrons and holes in a relatively unsymmetrical manner. We start with a vacuum $|0\rangle$ and add pairs of electrons. But what about pairs of holes? In fact these are also included in the theory.

BCS theory

The derivation of BCS theory is rather involved and only an outline is given here. For more details, consult the books by Annett and by Schrieffer. This material provided in this section is *for interest only* and the less-interested reader need only focus on the boxed results.

The BCS trial wave function can be written as a product of coherent states of pairs:

$$|\Psi_{\rm BCS}\rangle = \text{constant} \times \prod_{\mathbf{k}} \exp(\alpha_{\mathbf{k}} \hat{P}^\dagger_{\mathbf{k}}) |0\rangle, \tag{272}$$

where $\hat{P}^\dagger_{\mathbf{k}}$ is a pair creation operator.

The BCS trial wave function can be re-written as detailed above as:

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} (u^*_{\mathbf{k}} + v^*_{\mathbf{k}} \hat{P}^\dagger_{\mathbf{k}}) |0\rangle, \tag{273}$$

where $u^*_{\mathbf{k}}$ and $v^*_{\mathbf{k}}$ are variational parameters which can be adjusted to minimise the energy.

The BCS Hamiltonian is

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - |g_{\text{eff}}|^2 \sum_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}, \tag{274}$$

where the first term represents the kinetic energy and the second term accounts for the electron phonon interaction.

This Hamiltonian is applied to the BCS trail wave function and the energy is minimised with respect to the parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$. An exact solution of this problem is hard, but one can adopt a variational approach to find an approximate solution.

The energy is given by:

$$E = \langle \Psi_{\text{BCS}} | H_{\text{BCS}} | \Psi_{\text{BCS}} \rangle \tag{275}$$

The constrains are that the mean total number of particles is conserved:

$$\langle N \rangle = const. \tag{276}$$

(the particle number is not a good quantum number for the coherent BCS state) and the variational parameters follow the relationship:

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \tag{277}$$

The average occupation of a Block state is

$$\langle n_{\mathbf{k}\uparrow} \rangle = \langle \Psi_{\text{BCS}} | c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} | \Psi_{\text{BCS}} \rangle \tag{278}$$

and it can be shown (see Annet's book) that the mean total number of electrons is

$$\langle N \rangle = \langle n_{\mathbf{k}\uparrow} + n_{\mathbf{k}\downarrow} \rangle = 2 \sum_{\mathbf{k}} |v_{\mathbf{k}}|^2 \tag{279}$$

where the factor of 2 is just for the two spin states.

The kinetic energy contribution to the total energy is:

$$\langle \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \rangle = 2 \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} |v_{\mathbf{k}}|^2 \tag{280}$$

The expectation value of the interaction part of the BCS Hamiltonian is :

$$\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle = v_{\mathbf{k}} v^{*}_{\mathbf{k}'} u_{\mathbf{k}'} u^{*}_{\mathbf{k}} \tag{281}$$

Thus, the total energy expressed as a function of the variational parameters becomes:

$$E = 2 \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} |v_{\mathbf{k}}|^2 - |g_{\text{eff}}|^2 \sum_{\mathbf{k},\mathbf{k}'} v_{\mathbf{k}} v^{*}_{\mathbf{k}'} u_{\mathbf{k}'} u^{*}_{\mathbf{k}} \tag{282}$$

To find the minimum energy solution, the method of Langrange multipliers is used.

Beforehand, this constrain:

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \tag{283}$$

is used to re-write the kinetic energy of the total energy, E, and the total mean number of particles, $< N >$ such that

$$E = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}(|v_{\mathbf{k}}|^2 - |u_{\mathbf{k}}|^2) + 1) - |g_{\text{eff}}|^2 \sum_{\mathbf{k},\mathbf{k}'} v_{\mathbf{k}} v^*_{\mathbf{k}'} u_{\mathbf{k}'} u^*_{\mathbf{k}} \tag{284}$$

and

$$\langle N \rangle = \sum_{\mathbf{k}} (|v_{\mathbf{k}}|^2 - |u_{\mathbf{k}}|^2 + 1) \tag{285}$$

Using the Langrange multipliers and differentiating with respect to $u^*_{\mathbf{k}}$ and $v^*_{\mathbf{k}}$ variational parameters (independent variables from $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$) to get the minimum values of energy:

$$0 = \frac{\partial E}{\partial u^*_{\mathbf{k}}} - \mu \frac{\partial < N >}{\partial u^*_{\mathbf{k}}} + E_{\mathbf{k}} u_{\mathbf{k}} \tag{286}$$

$$0 = \frac{\partial E}{\partial v^*_{\mathbf{k}}} - \mu \frac{\partial < N >}{\partial v^*_{\mathbf{k}}} + E_{\mathbf{k}} v_{\mathbf{k}} \tag{287}$$

where the chemical potential μ enters as a Langrange multiplier which enforces the constant mean number of particles and E_k is the Langrange multiplier associated with the constrain $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$.

This gives the pair of linear equations:

$$(\epsilon_{\mathbf{k}} - \mu) u_{\mathbf{k}} + \Delta_{\mathbf{k}} v_{\mathbf{k}} = E_{\mathbf{k}} u_{\mathbf{k}} \tag{288}$$

$$\Delta^*_{\mathbf{k}} u_{\mathbf{k}} - (\epsilon_{\mathbf{k}} - \mu) v_{\mathbf{k}} = E_{\mathbf{k}} v_{\mathbf{k}} \tag{289}$$

where $\Delta_{\mathbf{k}}$ is introduced as:

$$\Delta_{\mathbf{k}} = |g_{\text{eff}}|^2 \sum_{\mathbf{k}} u_{\mathbf{k}} v^*_{\mathbf{k}} \tag{290}$$

In order to determine the parameter $\Delta_{\mathbf{k}}$ one can solve the above coupled equations for $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ that can be solved. From here one can see that ($u_{\mathbf{k}}$, $v_{\mathbf{k}}$) is the eigenvector of the two-by-two matrix and the $E_{\mathbf{k}}$ is its eigenvalue.

After the energy minimization, the $E_{\mathbf{k}}$ values are

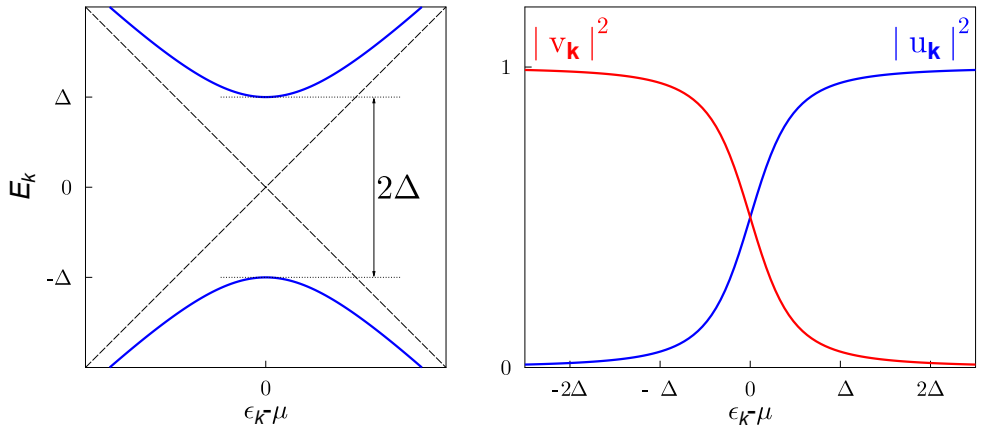
$$E_{\mathbf{k}} = \pm \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta_{\mathbf{k}}|^2}. \tag{291}$$

which are the excitation energy for an extra electron or hole to be added to the BCS ground state that is formed of condensed pairs. This is also called Bogoliubon energy as the quasiparticles are often called Bogoliubov quasiparticles.

By evaluating the $(u_{\mathbf{k}}, v_{\mathbf{k}})$ eigenvector, it can be shown that

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right) \tag{292}$$

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right) \tag{293}$$



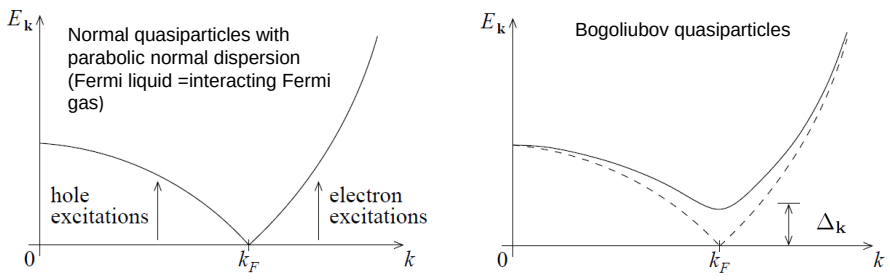
These results of the BCS theory are plotted in the graphs above.

$|v_{\mathbf{k}}|^2$ and $|u_{\mathbf{k}}|^2$ can be interpreted as the probabilities of a pair state being respectively occupied and unoccupied. $|v_{\mathbf{k}}|^2$ is the probability that the excitation is an electron if one measures its charge (for energies outside the gap) and $|u_{\mathbf{k}}|^2$ is the probability that is a hole.

$E_{\mathbf{k}}$ is the energy of elementary excitations of the superconducting state. The dashed lines are the excitations in the normal state $+\epsilon_{\mathbf{k}}$ for adding an electron to an empty state and $-\epsilon_{\mathbf{k}}$ for removing an electron (or adding a hole).

The superconducting excitations $E_{\mathbf{k}}$ are fermions and are similar to the electron and hole excitations of the Fermi sea, but they have a minimum electronic excitation energy, the energy gap Δ. The gap is the result of *hybridizing* electron-like and hole-like quasiparticles leading to new quasiparticles as a superposition of electron- and hole-like character called Bogoliubov quasiparticles.

Note that both electron and hole solutions emerge meaning that both pairs of electrons and pair of holes are permitted.



Landau developed the idea of quasiparticle excitations in the context of interacting Fermi systems, known as the Fermi liquid theory. Figure above shows the excitation energies of a Fermi liquid $E_{\mathbf{k}} = \epsilon_{\mathbf{k}}$ (left) - it costs energy to create a hole in the Fermi sea and also to insert an electron into an empty state outside of the Fermi sea. Superconductivity opens an energy gap of magnitude $\Delta_{\mathbf{k}}$ in the excitation spectrum $E_{\mathbf{k}}$ and the new dispersions correspond to Bogoliubov quasiparticles (right).

The superconducting energy gap

The BCS gap parameter, $\Delta_{\mathbf{k}}$, is introduced as:

$$\Delta_{\mathbf{k}} = |g_{\text{eff}}|^2 \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}^* \tag{294}$$

can be re-written as

$$\Delta_{\mathbf{k}} = |g_{\text{eff}}|^2 \sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \tag{295}$$

is the expectation value that of the Cooper pair annihilation operator $\langle P_{\mathbf{k}} \rangle$.

Thus, $\Delta_{\mathbf{k}}$ is the BCS energy gap (at $T = 0$) which is the energy required to destroy a Cooper pair.

The parameter $\Delta_{\mathbf{k}}$ is also found from

$$u_{\mathbf{k}} v_{\mathbf{k}}^* = \frac{\Delta_{\mathbf{k}}}{(2E_{\mathbf{k}})} \tag{296}$$

These two equations lead to the **BCS gap equation** at $T = 0$:

$$\Delta_{\mathbf{k}} = |g_{\text{eff}}|^2 \sum_{\mathbf{k}} \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}. \tag{297}$$

Cancelling $\Delta_{\mathbf{k}}$ on both sides and introducing the expression of E_k of the elementary excitation the BCS gap equation becomes

$$1 = \frac{|g_{\text{eff}}|^2}{2} \sum_{\mathbf{k}} \frac{1}{\sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta_{\mathbf{k}}|^2}} \tag{298}$$

This equation is central to the BCS theory since predicts both the magnitude of the gap parameter $|\Delta|$ at zero temperature $|\Delta|(0)$ and the critical temperature T_{c}.

Next we will assume that $\Delta_{\mathbf{k}}$=Δ for an isotropic superconducting gap around the Fermi surface. This is the *BCS approximation* in which is that the gap function has no **k**-dependence. In order to evaluate this expression is important not to forget that we are working within a thin shell of Bloch states around the Fermi energy with $\epsilon \sim |\epsilon_{\mathbf{k}} - \mu| < \hbar\omega_{\mathrm{D}}$, where BCS electron-phonon coupling is attractive.

Within this shell ϵ the sum over **k** is replaced by an integral over energy, also assuming that the density of states (per spin) at the Fermi energy is constant.

$$\sum_{\mathbf{k}} \rightarrow g(\epsilon_{\mathrm{F}}) \int d\epsilon \tag{299}$$

Writing the *electron-phonon coupling parameter* as

$$\lambda = |g_{\mathrm{eff}}|^2 g(\epsilon_{\mathrm{F}}) \tag{300}$$

where $g(E_{\mathrm{F}})$ is the density of states at the Fermi level,

The gap equation at $T = 0$ becomes

$$\Delta = \lambda \int_0^{\hbar\omega_{\mathrm{D}}} \frac{\Delta \, \mathrm{d}\epsilon}{\sqrt{|\Delta|^2 + \epsilon^2}}, \tag{301}$$

and so

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_{\mathrm{D}}} \frac{\mathrm{d}\epsilon}{\sqrt{\Delta^2 + \epsilon^2}} = \sinh^{-1}\left(\frac{\hbar\omega_{\mathrm{D}}}{|\Delta|}\right). \tag{302}$$

In most cases, Δ is of the order of a few meV, much smaller than ω_{D}, which is of the order of a few hundreds of meV. Since $\Delta \ll \hbar\omega_{\mathrm{D}}$, we can expand the $\arcsin(x)$ for large x and have $e^{1/\lambda}/2 \approx \hbar\omega_{\mathrm{D}}/|\Delta|$ to obtain

$$\frac{1}{\lambda} = \sinh^{-1}\left(\frac{\hbar\omega_{\mathrm{D}}}{|\Delta|}\right) \approx \ln\left(\frac{2\hbar\omega_{\mathrm{D}}}{|\Delta|}\right) \tag{303}$$

This gives the famous BCS value of the gap parameter at zero temperature

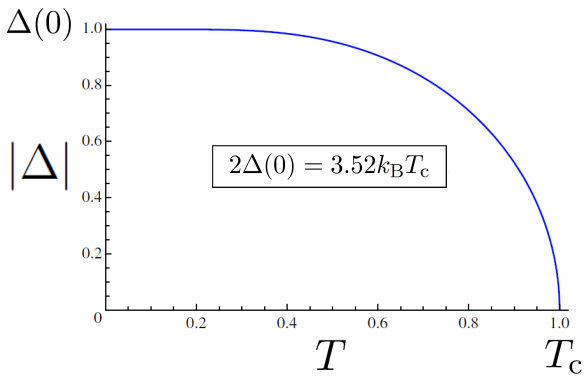
$$\boxed{|\Delta(0)| \approx 2\hbar\omega_{\rm D} e^{-1/\lambda}}. \tag{304}$$

At finite temperature T, the occupation of the quasiparticle state with energy $E_{\mathbf{k}}$ is given by Fermi-Dirac distribution, $f(E_{\mathbf{k}}) = 1/(e^{E_{\mathbf{k}}/k_{\rm B}T} + 1)$ and the superconducting gap becomes:

$$\Delta_{\mathbf{k}} = |g_{\rm eff}|^2 \sum_{\mathbf{k}} v_{\mathbf{k}} u^*_{\mathbf{k}} (1 - 2f(E_{\mathbf{k}})) \tag{305}$$

resulting in a gap equation which is temperature dependent

$$\Delta_{\mathbf{k}} = |g_{\rm eff}|^2 \sum_{\mathbf{k}} \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2k_{\rm B}T}\right) \tag{306}$$



Converting to an integral the BCS gap for a s-wave isotropic gap $\Delta_{\mathbf{k}} = \Delta$ equation becomes:

$$1 = \lambda \int_0^{\hbar\omega_{\rm D}} d\epsilon \frac{1}{E} \tanh\left(\frac{E}{2k_{\rm B}T}\right) \tag{307}$$

where $E = \sqrt{\epsilon^2 + |\Delta|^2}$ and ϵ is relative to the chemical potential (μ) within a shell $\hbar\omega_{\rm D}$, λ is the electron-phonon coupling parameter and $g(E_{\rm F})$ is the density of states at the Fermi level.

Hence, for $T = T_{\rm c}$ we have that $\Delta = 0$ and so

$$\frac{1}{\lambda} = \int_0^{x_{\rm D}} \frac{\tanh x}{x} {\rm d}x \tag{308}$$

where $x_{\rm D} = \hbar\omega_{\rm D}/2k_{\rm B}T_{\rm c}$ and hence $\hbar\omega_{\rm D} \gg k_{\rm B}T_{\rm c}$.

The integral can be approximated by:

$$\int_0^{x_0} \frac{\tanh x}{x} {\rm d}x \sim \ln(x_0 B) \tag{309}$$

where the constant $B = 4\gamma/\pi \approx 2.26$, $\gamma = e^C \approx 1.78$ and $C = 0.577$ is the Euler constant.

The BCS equation for the critical temperature becomes:

$$\boxed{k_B T_c = 1.13 \hbar \omega_D e^{-1/\lambda}}. \tag{310}$$

which has almost exactly the same form as the formula for the binding energy in the Cooper problem.

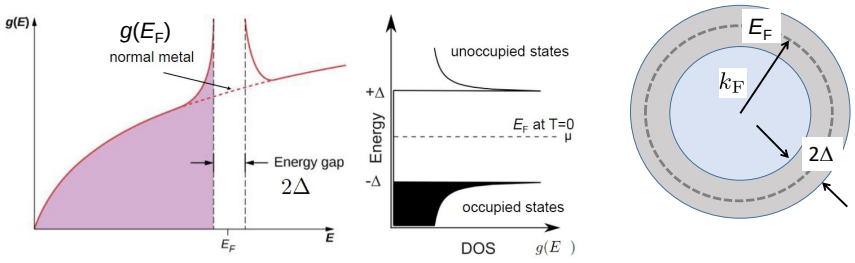
For a small λ typical for low-T_c conventional superconductors, the BCS theory predicts an upper limit on the critical temperature of about 30 – 40 K. However, in 2020 T_c reached room temperature in hydrates under very large applied pressure.

Equations 304 and 310 can be combined to yield

$$\boxed{2\Delta(0) = 3.52 k_B T_c}. \tag{311}$$

The BCS theory makes a quantitative prediction for the function $\Delta(T)$. Near T_c, the variation of the gap with temperature obeys $\Delta(T) \sim (T_c - T)^{1/2}$.

Density of states in a superconductor



The attractive pairing interaction $V_{\text{eff}} = -|g_{\text{eff}}|^2$ via effective electron-phono interaction (for $|\omega| < \omega_D$) leads to an instability of the Fermi surface and the opening of an energy gap Δ at the Fermi level (on both sides of the Fermi energy). Without a Fermi sea, there is no bound state between electrons to lower the energy and no gap.

In a superconductor there are no quasiparticle states inside the gap with energies $E < |\Delta|$ and the density of states is zero.

Near the Fermi surface, the normal electron density of states $g_n(E)$ can be approximated as a constant density of states, $g(E_F)$ over a small energy interval $d\epsilon$ to get the density of states of superconducting states:

$$g_s(E)dE = g_n(E)d\epsilon \approx g_n(E_F)d\epsilon \tag{312}$$

$$g_s(E) = g_n(E_F)\frac{d\epsilon}{dE} \tag{313}$$

Using a small energy interval (in relation to the chemical potential) $\epsilon \sim \epsilon_{\mathbf{k}} - \mu$ and the superconducting excitation energy for electron-like excitations $E = \sqrt{\epsilon^2 + |\Delta|^2}$

$$\epsilon = \sqrt{E^2 - |\Delta|^2}. \tag{314}$$

The density of states (DOS) in the superconductor can be written as:

$$g_s(E) = 0, \qquad E \leq |\Delta| \tag{315}$$

$$g_s(E) = g(E_{\rm F}) \frac{E}{\sqrt{E^2 - |\Delta|^2}}, \qquad E > |\Delta| \tag{316}$$

$$\tag{317}$$

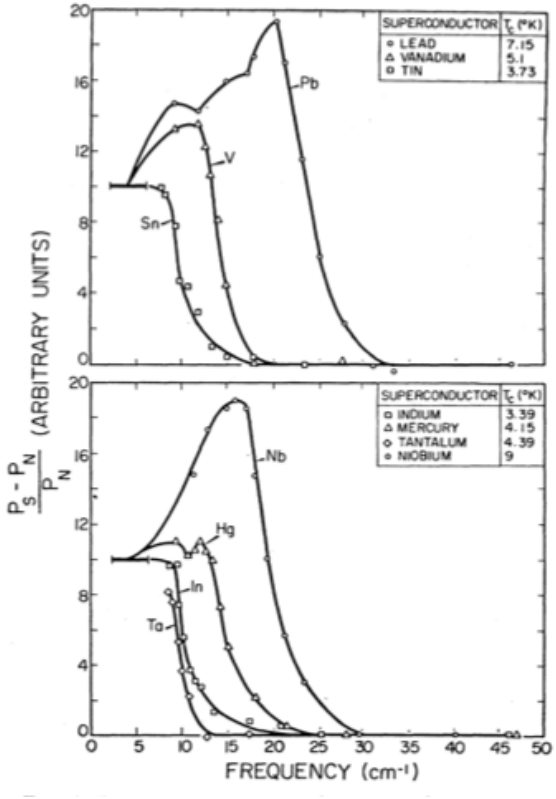
Figure above shows the density of states of a superconductor as a function of energy. There are no available states for $-\Delta < E < \Delta$. The DOS has a square-root singularity at $E \to |\Delta|$. It approaches the normal-state value $g_n(E_{\rm F})$ for $E \gg |\Delta|$. The dashed line is the expected density of states for a normal metal. This density of states has an infinite cusp at the gap edge as the states that had energy below $|\Delta|$ in the normal state are pushed up above $|\Delta|$ to form a peak in the superconducting state. This density of states is easily measured in tunneling experiments where one tunnels unpaired electrons into the superconductor.

Evidence for the energy gap

The existence of the energy gap is a very important property of the superconductor and explains many aspects of its behavior and it has a number of experimental implications.

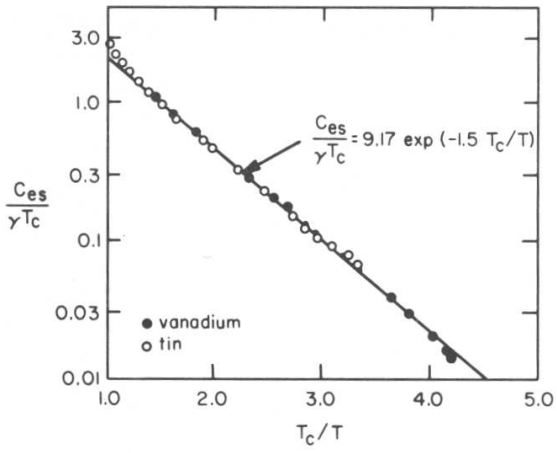
Absorption of infra-red radiation

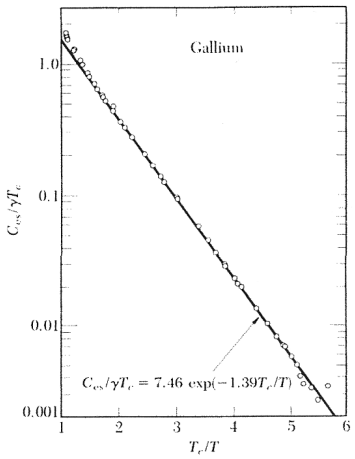
P. L. Richards and M. Tinkham, Physical Review **119**, 575 (1960).



Heat capacity data

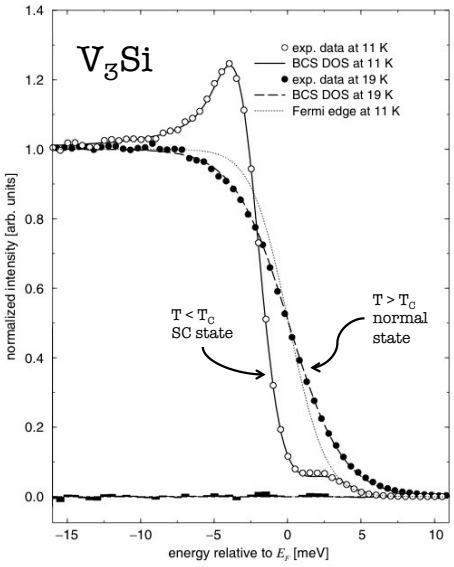
Norman E. Phillips, Physical Review **134**, A386 (1963).





Photoemission spectroscopy

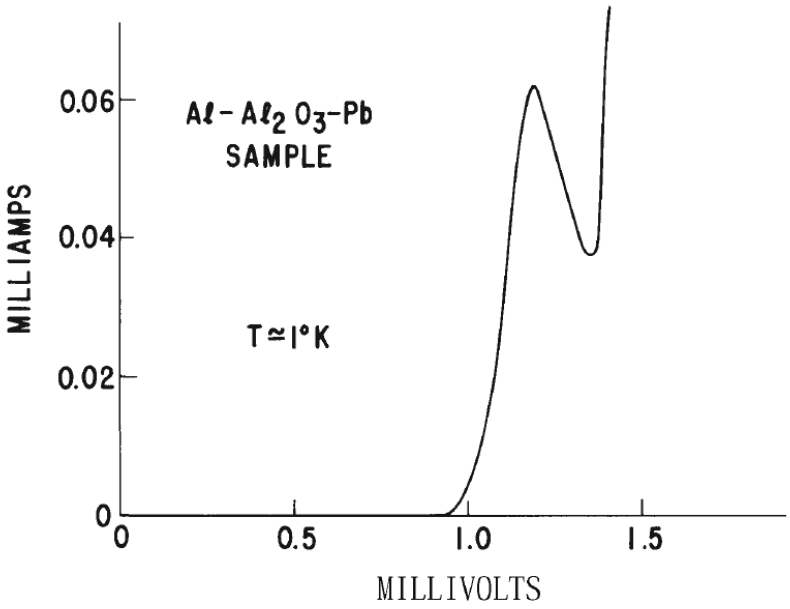
F. Reinert *et al.*,
Physical Review Letters **85**, 3930 (2000).



The tunnelling spectra between two different superconductors

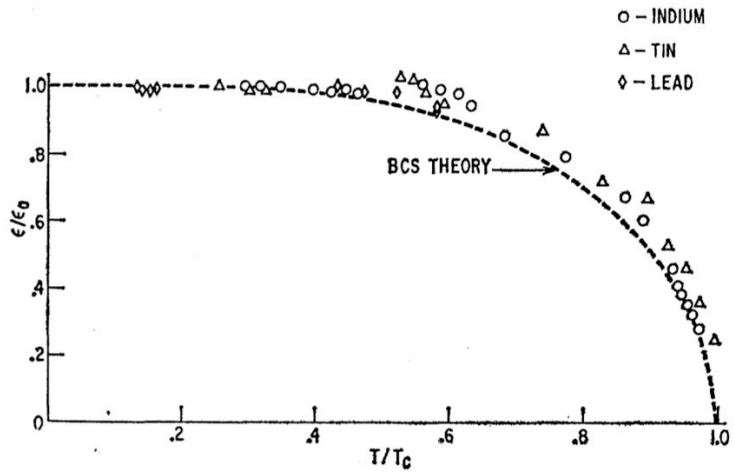
The current-voltage characteristic (I vertical, V horizontal) for the tunneling of superconductor-normal-superconductor (SNS) made of Al - Al_2O_3 - Pb. The presence of a gap results in no tunneling current. The graph here shows the tunneling effect arising in the presence of two different superconducting gaps at finite temperature.

Ivar Giaever,
Phys. Rev. Lett. **5**, 464 (1960).



Temperature dependence of the gap

I. Giaever and K. Mergerle,
Phys. Rev. **122**, 1101 (1961).



Experimental values of T_c, $2\Delta(0)/k_B T_c$ and $(C_s - C_n)/C_n$, the jump in the heat capacity at T_c, are given in the following table (BCS predicted values for the last two are 3.52 and 1.43, respectively):

Material	T_c(K)	$2\Delta(0)/k_B T_c$	$(C_s - C_n)/C_n$
Zn	0.9	3.2	1.3
Al	1.2	3.4	1.6
In	3.4	3.6	
Sn	3.7	3.5	1.6
Ta	4.5	3.7	1.6
V	5.4	3.4	1.6
Hg	4.2	4.6	2.2
Pb	7.2	4.3	2.7
Nb	9.3	3.7	2.1
K_3C_{60}	19	3.6	
$YBa_2Cu_3O_{7-\delta}$	93	4.0	2.9

Noteable aspects of BCS superconductivity

- The superconducting condensate is made up of a large number of Cooper pairs *all of which have the same ground state wave function.*

- Cooper pairs are constructed from electrons that have equal and opposite crystal momenta and opposite spin. The pairs are formed via an attractive electron-phonon interaction.

- The part of the Cooper-pair wave function that deals with centre-of-mass motion is equivalent to the order parameter in Ginzburg-Landau theory.

- The full Cooper-pair wave function contains the characteristic real-space length scale of the Cooper pair, which is of the order of the Pippard coherence length, ξ_0. This can be much bigger than the average distance between electrons in the normal state.

- BCS theory deals with the condensate as a whole and defines a *coherent many-body wave function*, $|\Psi_{\mathrm{BCS}}\rangle$.

- The theory predicts the existence of a *complex gap function*, Δ. The magnitude of this function within BCS theory is independent of $\mathbf{k}$ and represents the size of the energy gap that occurs at the Fermi energy, separating the superconducting ground state from the quasiparticle excitations.

- The formation of the condensate is a cooperative effect, with the size of the energy gap dependent on the number of quasiparticles that condense. The quantitative predictions of the gap that emerge from BCS have been experimentally verified in a large number of superconducting materials.

- *The gap function is proportional to the Ginzburg-Landau order parameter.* The GL equations (and hence the London equations, flux quantization, and the Meissner & Josephson effects) can be derived from BCS theory.

Unconventional superconductors beyond BCS

The deviations in $2\Delta(0)/k_{\rm B}T_{\rm c}$ for some elemental materials (including Pb and Hg) from the BCS predictions can be explained through the extending the theory to include **strong coupling** of the electrons and phonons.

It is now apparent that some materials have properties that may not be explained entirely within either weakly or strongly-coupled BCS theories. These include some heavy-fermion superconductors, organic superconductors, the high-temperature or cuprate superconductors, and the recently discovered iron-based superconductors. In several of these materials it is likely that the *symmetry of the gap function* is different to the isotropic gap predicted by BCS.

In the normal state above $T_{\rm c}$, a given superconducting material has a symmetry group that incorporates the symmetry of the underlying crystal structure, spin rotation symmetries, and so-called $U(1)$ symmetry. Below $T_{\rm c}$ the $U(1)$ symmetry is spontaneously broken, giving rise to phase coherence. If this is the only symmetry that is broken on going through the transition, then the material is called a **conventional superconductor**, and will have an isotropic or *s-wave* gap function (the latter in reference to the atomic $l = 0$, isotropic s-orbital). If, in addition to $U(1)$, the system breaks one or more of the other symmetries it possessed above the transition, then it is known as an **unconventional superconductor**. ‖

The superconducting gap, $\Delta_{\mathbf{k}}$, for a $d_{x^2-y^2}$-wave high temperature superconductor can vary like $\Delta_{\mathbf{k}} = \Delta_0(\cos(k_x) - \cos(k_y))$ which leads to nodes in magnitude and discontinuities in phase along certain directions. The nodes are points on the Fermi surface where the gap is zero and the superconducting pairs can be broken without any cost in energy.

These unconventional materials still contain superconducting pairs of electrons and exhibit much of the superconducting phenomena we discuss in the course. However, the interaction mechanism responsible for pairing the electrons in these materials is not yet established and, in some cases, the critical temperatures can be in excess of 100 K, indicating the likelihood of something more exotic than the "simple" BCS electron-phonon interaction.

‖For an introduction to the symmetry aspects of superconductivity see James Annett's book, or his article in *Contemporary Physics* vol. 36, p423 (1995).

1. (a) Describe briefly the Meissner effect in a superconductor. Explain why a superconductor cannot be considered as a merely a metal with perfect electrical conduction.

   (b) The charge carriers in superconductor (mass m^*, charge q, number n per unit volume) carry a current density $\mathbf{J}$. By equating the canonical momentum $\mathbf{p} = m^*\mathbf{v} + q\mathbf{A}$ to zero, derive the London equation

   $$\nabla \times \mathbf{J} = -\frac{\mathbf{B}}{\mu_0 \lambda^2},$$

   and find λ in terms of q, n and m^*. By considering a magnetic field applied tangentially to the $x = 0$ plane, show that this result leads to the Meissner effect in a superconductor located in the half space $x > 0$.

   (c) Estimate λ for a typical superconductor using reasonable values of n and m^*.

2. Lead has a critical field B_c of about 0.07 T at 4.2 K. A magnetic field is applied along the axis of a lead cylinder which has an axial hole of 1 cm inner diameter. The applied field is reduced to zero from an initial value of 1 T.

   (a) Draw sketches to show the lines of flux at the various stages of this process.

   (b) Estimate the magnetic dipole moment per unit length of the cylinder with its trapped flux. Is flux quantization in this problem something about which to be concerned?

   (c) Calculate the magnitude of the surface currents per unit length circulating within the surface of the central hole. Estimate the current density within a London penetration depth of the surface.

   (d) A sample of tin containing a hole of diameter 10^{-5} m is cooled through its transition temperature in a magnetic field of 10^{-4} T applied parallel to the axis of the hole. Calculate the number of trapped flux quanta.

3. Empirically it is found that the temperature dependence of the critical field B_c of a type-I superconductor is given by

   $$B_\mathrm{c}(T) = B_0 \left[1 - \left(\frac{T}{T_\mathrm{c}}\right)^2\right].$$

   Use this expression to show that the entropy S_s per unit volume of the superconducting state is lower than that of the normal state S_n at constant temperature T below T_c at the superconducting to normal boundary at B_c. Furthermore, show that $S_\mathrm{n} - S_\mathrm{s} = aT - bT^3$ where a and b are constants, and that $S_\mathrm{n} = S_\mathrm{s}$ when $T = T_\mathrm{c}$.

   Sketch the heat capacity of such a superconductor with $B = 0$ as a function of T. Show that the drop in heat capacity at T_c has magnitude $\Delta C = 4B_0^2/(\mu_0 T_\mathrm{c})$ per unit volume. Calculate ΔC for a conventional superconductor with $B_0 = 0.08$ T and $T_\mathrm{c} = 7$ K.

4. (a) What is the evidence for the *energy gap* in a superconductor?

   (b) Why does the energy gap lead to superconductivity in certain metals but to large resistivity in intrinsic semiconductors?

   (c) Explain why an energy gap might be expected in superconductors on the basis of the BCS theory?

5. Why do superconductors conduct electricity with zero resistance? Why do they have poor thermal conductivity?

6. Assuming a coherent ground state, show that any magnetic flux trapped within a multiply connected superconductor must be quantized. Derive an expression for the flux quantum.

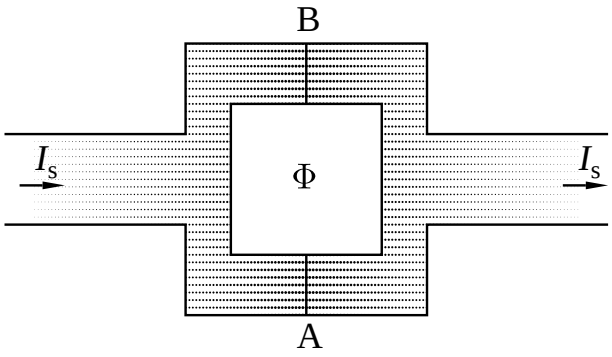
7. The wave function of the coherent pair state in a superconductor may be written as $\psi = \sqrt{n}\,\mathrm{e}^{\mathrm{i}\theta}$, where n is the density of electron pairs. A simplified form of the Schrödinger equation for the wave functions ψ_1 and ψ_2 on each side of the Josephson junction with potential V across the junction are

$$\begin{aligned} \mathrm{i}\hbar\dot{\psi}_1 &= \hbar T\psi_2 - eV\psi_1 \\ \mathrm{i}\hbar\dot{\psi}_2 &= \hbar T\psi_1 + eV\psi_2, \end{aligned}$$

where T is a characteristic of the junction. Show that the current across the junction is given by

$$I = -2e\dot{n}_1 = 4eT\sqrt{n_1 n_2}\,\sin\left(\delta(0) + \frac{2eVt}{\hbar}\right)$$

where $\delta(0)$ is the value of $\theta_1 - \theta_2$ at time zero and one can assume $|n_1 - n_2| \ll (n_1 + n_2)$.



Consider two such junctions A and B with a magnetic flux Φ threading the loop (see above). No voltage is applied. Use the condition $\delta_\mathrm{B} - \delta_\mathrm{A} = 2e\Phi/\hbar + 2m\pi$, where m is an integer and δ_i is the phase difference across junction i, to show that a superconducting current I_s flows where

$$I_\mathrm{s} = (-1)^m 8eT\sqrt{n_1 n_2}\,\sin\left(\frac{\delta_\mathrm{A} + \delta_\mathrm{B}}{2}\right)\cos\left(\frac{e\Phi}{\hbar}\right).$$

Explain briefly how this type of device may be used to measure magnetic fields.