Subatomic Physics: Prof. Guy Wilkinson

Problem Set I: Nuclear physics – the SEMF, properties of the nucleus and the strong nuclear force, general radioactive decay, and α decay

For questions involving the semi-empirical mass formula (SEMF) take the binding energy to have the following form:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C Z^2/A^{1/3} - a_A(A-2Z)^2/A + \delta(Z,A).$$

When required, assume the following values from a typical fit: $a_V = 15.56$ MeV, $a_S = 17.23$ MeV, $a_C = 0.697$ MeV, $a_A = 23.285$ MeV, and $\delta = -(+)12$ MeV$/A^{1/2}$ for odd-odd (even-even) nuclei and 0 for odd-A nuclei.

Also assume that the nuclear radius is given by $r = r_0 A^{\frac{1}{3}}$. with $r_0 = 1.2$ fm.

1. SEMF: Coulombic and gravitational terms

(a) Explain the physical significance and functional form of the various terms in the SEMF. (A detailed discussion of the Coulombic and asymmetry terms is not required, as these are explored further in part (b) and the next question.)

(b) Treating the nucleus as a sphere of uniform charge density, show that the constant

$$a_C = \frac{3e^2}{20\pi\epsilon_0 r_0} = 0.72\,\text{MeV}.$$

How would this calculation change for a model of the nucleus where all the protons were confined to the surface?

(c) A neutron star can be considered as a very large nucleus. Write down a modified form of the SEMF that you would consider appropriate to describe the binding energy for such a star. What is the minimum mass and radius this object can possess?

2. SEMF: origin of the asymmetry term

(a) Use the Fermi-gas model (assuming $N \approx Z$) to show that the energy ϵ_F of the Fermi level is given by

$$\epsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}.$$

(b) Estimate the total kinetic energy of the nucleons in an $^{16}_{8}$O nucleus.

(c) For a nucleus with neutron number N and proton number Z the asymmetry term in the semi-empirical mass formula is

$$\frac{a_A(N-Z)^2}{A}.$$

Assuming that $(N-Z) \ll A$ use the Fermi-gas model to justify this form and to estimate the value of a_A. Comment on the value obtained.

3. Using mirror nuclei to estimate nuclear radii

(a) Consider a pair of odd-A mirror nuclei (A,Z) and $(A,Z+1)$ for which $Z = (A \pm 1)/2$. Use the SEMF to show that the expected difference in binding energies between these nuclei is given by

$$\Delta B = \frac{3}{5}\frac{A\alpha\hbar c}{r},$$

where α is the fine-structure constant and r the radius (assumed to be the same for both members of the pair).

(b) Write down an expression for the released energy Q in a β^+ decay, in terms of ΔB, the change in nuclear masses, and the particles involved. (How would this expression differ if it were to be written in terms of the change in atomic masses?)

(c) Calculate r for the mirror pairs:

i. ($^{11}_{5}$B, $^{11}_{6}$C), where the maximum energy of the positron is $E_{\rm max} = 0.98$ MeV;

ii. ($^{23}_{11}$Na, $^{23}_{12}$Mg), where $E_{\rm max} = 2.95$ MeV;

iii. ($^{39}_{19}$K, $^{39}_{20}$Ca), where $E_{\rm max} = 5.49$ MeV.

Conpare your results to the expectation from $r = r_0 A^{1/3}$.

4. The deuteron

The deuteron is a weakly bound np system with no excited states. It has $J = 1$, and a symmetric wavefunction which is dominated by the orbital angular momentum state $l = 0$. There are no other two-nucleon bound states. What do these observations tell us about the nature of the nuclear interaction? (You may assume it to be charge independent).

(Note that further insights about the nature of the nuclear force come from consideration of the magnetic moment of the deuteron, but this takes us beyond the syllabus.)

5. Radioactive decay – general

(a) Write down the law of radioactive decay. Define the half-life and mean life of a radioactive nucleus and obtain the relation between them.

(b) The nucleus $^{87}_{37}$Rb decays into the ground state of $^{87}_{38}$Sr with a half-life of 4.7×10^{10} years. How can this half-life be measured in practice, and what are the experimental challenges? How would the approach change if the half life was, say, $\sim 10^1$ years?

(c) Five different samples of chondritic meteorites are found to have the following proportions of $^{87}_{37}$Rb, $^{87}_{38}$Sr and $^{86}_{38}$Sr.

Meteorite	$^{87}_{37}$Rb/$^{86}_{38}$Sr	$^{87}_{38}$Sr/$^{86}_{38}$Sr
Modoc	0.86	0.757
Homestead	0.80	0.731
Bruderheim	0.72	0.747
Kyushu	0.60	0.739
Bath Furnance	0.09	0.706

Given that the nucleus $^{86}_{38}$Sr is not the daughter product of any long-lived decay chain, show that most of these data are consistent with a common primordial ratio $^{87}_{38}$Sr/$^{86}_{38}$Sr and a common age for the meteorites, and find that age. Identify the anomalous set of data that does not conform to this picture.

6. Alpha decay and the SEMF

(a) Starting from the SEMF show that for large A and Z the energy released when a nucleus (Z, A) emits an α particle is approximately given by

$$Q = -4a_V + 8a_S/3A^{1/3} + 4a_C Z(1 - Z/3A)/A^{1/3} - 4a_A(A - 2Z)^2/A^2 + B(2, 4),$$

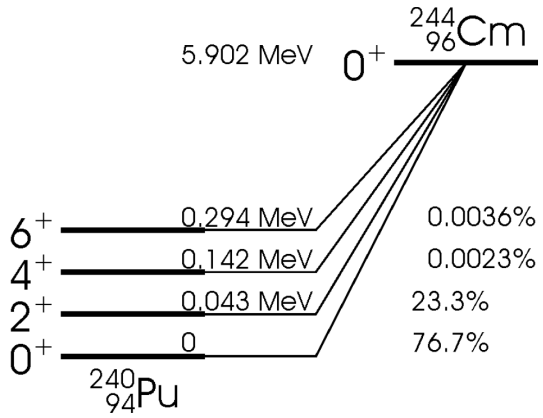
where $B(2, 4) = 28.30$ MeV is the binding energy of the α particle. Comment on why you have not been asked to expand $B(2, 4)$ into its constituent terms as given by the SEMF.

(b) Naturally occurring isotopes of silver and gold are $^{107}_{47}$Ag and $^{197}_{79}$Au. Discuss the stability of these

nuclei making use of your expression for Q.

7. Alpha-decay rates

(a) The figure below shows the possible α decays of $^{244}_{96}$Cm to $^{240}_{94}$Pu, giving the energies above the $^{240}_{94}$Pu ground state, and the relative rate of each transition.



Justify that we would expect the rates ω to satisfy an equation of the form

$$\ln \omega = C - \frac{D(Z-2)}{\sqrt{Q}},$$

where C and D are approximate constants for high A nuclides, and Q is the energy release of the decay.

(b) Taking $C = 132.8$ and $D = 3.97\,(\mathrm{MeV})^{1/2}$ calculate the expected lifetime of $^{244}_{96}$Cm.

(c) Calculate the expected branching fraction of the decay into the 6^+ state. Compare to the measured branching fraction listed in the figure and suggest an explanation for any discrepancy.

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Problem Set II: Nuclear physics – β decay, fission and fusion, cross sections and Rutherford scattering

For questions involving the semi-empirical mass formula (SEMF) take the binding energy to have the following form:

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When required, assume the following values from a typical fit: $a_V = 15.56$ MeV, $a_S = 17.23$ MeV, $a_C = 0.697$ MeV, $a_A = 23.285$ MeV, and $\delta = -(+)12$ MeV$/A^{1/2}$ for odd-odd (even-even) nuclei and 0 for odd-A nuclei.

Also assume that the nuclear radius is given by $r = r_0 A^{\frac{1}{3}}$. with $r_0 = 1.2$ fm.

1. Fermi theory of β decay

Fermi theory predicts that the rate of electrons emitted with momentm between p and $p + \mathrm{d}p$ is given by

$$\Gamma(p)\mathrm{d}p = \frac{G_F^2}{2\pi^3\hbar^7 c^3} p^2 (Q-E)^2 \mathrm{d}p$$

where G_F is the Fermi constant, E is the kinetic energy of the electron, and Q is the energy liberated in the reaction.

(a) Sketch this spectrum and justify the form of the result, stating the key assumptions that are made.

(b) Show that for $Q \gg m_e c^2$ the total rate is proportional to Q^5, *i.e.* Sargent's rule.

(c) What spin states are allowed for the combined system of the electron and the neutrino?

(d) Consider the following three decays, where the number in parentheses is the spin of the state and $t_{1/2}$ is the half life:

	Q (MeV)	$t_{1/2}$ (s)
${}^{6}_{2}\mathrm{He}(0) \rightarrow {}^{6}_{3}\mathrm{Li}(1)$	3.5	0.81
${}^{3}_{1}\mathrm{H}(\frac{1}{2}) \rightarrow {}^{3}_{2}\mathrm{He}(\frac{1}{2})$	0.018	4×10^8
${}^{10}_{4}\mathrm{Be}(0) \rightarrow {}^{10}_{5}\mathrm{B}(3)$	0.55	5×10^{13}

Show that the relative rates of these decays do not appear to follow Sargent's rule, and suggest reasons why this is so.

2. Stability against β decay

(a) Use the SEMF to predict the stable isobar with $A = 181$.

(b) A survey of all stable nuclei (where stable here means $t_{1/2} > 10^9$ years) shows that there are 177 even-even, 121 even-odd and 8 odd-odd nuclei. There are many cases where even-even nuclei have more than one stable isobar, but not for the even-odd and odd-odd cases.

i. Explain these observations graphically by considering how nuclear mass varies with Z for fixed A, and what transitions are allowed by $\beta^{\mp}$ decay and electron capture.

ii. These arguments whould suggest that ${}^{76}_{32}\mathrm{Ge}$ is *completely* stable, whereas it has a half-life of 2×10^{21} years. Explain why this is so.

3. Fission reactors

(a) Explain why $^{235}_{92}$U is a fissile material, and why $^{238}_{92}$U is, in general, not. Describe the role of a moderator in a reactor, and give the pros and cons of possible choices of moderator material. Why is the fissile fuel and moderator not mixed homogenously?

(b) The fission of $^{235}_{92}$U by thermal neutrons is asymmetric, the most probably mass numbers of the fragments being 93 and 140. Where are the daughter nuclei in relation to the valley of stability and what happens to them subsequently? Use the SEMF to find the most probably value of Z for these mass numbers. Making use of the values of binding energies given in nuclear tables, or from the SEMF, estimate the energy releasted in fission of $^{235}_{92}$U and hence the mass of $^{235}_{92}$U consumed each second in a typical commercial reactor with a power of 1 GW.

(c) Other possible fissile fuels include $^{233}_{92}$U and $^{239}_{94}$Pu, which have comparitively short half-lives and so are not found naturally. Explain how such fuel can be produced in reactors starting with either suitable isotopes of uranium or thorium.

4. Natural fission reactors

(a) Natural uranium found on Earth contains the isotopes $^{235}_{92}$U and $^{238}_{92}$U in the ratio 7.2×10^{-3} to 1. Assuming that they were produced equally, estimate the time since formation given the mean lives are 1.03×10^9 years and 6.49×10^9 years, respectively.

(b) Uranium deposits found in Oklo in Gabon, amounting to a mass of around 2×10^6 kg, possess a significantly lower ratio of around 5×10^{-3} to 1. It is hypothesised that this lower ratio can be explained by the uranium having undergone natural induced fission, with subterranean water acting as a modulator. Evidence suggests that this fission took place $\sim 2 \times 10^9$ years ago, and persisted for around a million years. Estimate the energy and mean power output of this reactor.

(c) What additional pieces of evidence could be searched for to test this natural-reactor hypothesis? Why is it very unlikely that any such natural reactors are in operation currently?

5. Fusion

(a) The cross section for the fusion of two nuclides is approximately given by

$$\sigma = \frac{S}{E} e^{-C/\sqrt{E}},$$

where S is a constant. Justify this form and, through analogy with α decay, write down an expression for C in terms of Z and A of the fusing nuclides. Write down an expression for the rate of fusion of two nuclides. Show that this rate has a peak (the 'Gamow peak'), evaluate the energy at which this peak is located, and show that its height is given by:

$$R \propto \exp\left[-1.89(C^2/k_B T)^{\frac{1}{3}}\right].$$

(b) Consider a star in which fusion proceeds dominantly in the proton-proton cycle. For a temperature of 10^7K in the core of the star the energy output is 1 mW/kg. Assuming that the time scale of the process is determined by the fusion of two protons into a deuteron, estimate the power per unit mass output if the temperature i) falls to 10^6K, or ii) rises to 10^8K.

(c) Estimate the ratio of the ^{2}H + ^{3}H fusion rate to the ^{2}H + ^{3}He rate at a temperatue of 2×10^8K.

(d) In a gas of atoms undergoing fusion estimate the temperature required to overcome the Coulomb barrier and the energy release for the fusion of two nuclei of i) $^{10}_5$B, ii) $^{12}_6$C, and iiii) $^{16}_8$O.

6. Making use of cross sections

A beam of neutrons with an intensity of 10^5 particles per second transverses a thin foil of $^{235}_{92}$U for which the product of the mass density and thickness is $10^{-1}\,\mathrm{kg\,m^{-2}}$. Three types of interaction may

occur: elastic scattering with a cross section of 10^{-2} b; absorption followed by fission with a cross section of 200 b; absorption followed by γ-ray emission with a cross section of 70 b.

(a) Calculate what fraction of the beam emerges unscattered from the foil.

(b) Calculate the rate of fission reactions that are directly induced by the incident beam.

(c) Calculate the flux of scattered neutrons at a point 10 m from the foil, assuming an isotropic distribution.

7. Rutherford scattering

In 1909 Geiger and Marsden studied scattering off thin metal foils, using 5.6 MeV α particles. They observed a small fraction of incident particles was scattered backwards, being about 1 in 8000 in the case of a platinum target with density $21.5\,\mathrm{g/cm^3}$ and atomic mass 195.1 u. These results led Rutherford to propose his model of the atom and the accompanying formula for the differential cross section.

(a) Integrate the differential cross section over the backwards hemisphere to determine the cross section for backscattering with $\theta > 90^\circ$ and hence predict the ratio $N_\mathrm{b}/N_\mathrm{in}$, where N_in is the number of incoming and N_b is the number of backscatter α particles. Compare your calculation with the result of Geiger and Marsden assuming a target of thickness $1\,\mu\mathrm{m}$.

(b) One scattered α particle per second was detected on a screen of area $1\,\mathrm{mm^2}$ at a distrance of 1 cm from the foil at a scattering angle of 120°. From this estimate the rate of α particles hitting the foil.

(c) Later measurements with improved angular resolution showed no deviation from the Rutherford scattering formula up to a scattering angle of 140°. From this observation estimate an upper limit for the radius of the platinum nucleus.

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Problem Set III: Nuclear and particle physics – scattering and resonances

1. Calculating a cross section

The differential cross section for elastic scattering of a state of momentum $\hbar\mathbf{k}$ is given by the square of the scattering amplitude

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\phi)|^2 \tag{1}$$

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})\mathrm{d}^3\mathbf{r} \tag{2}$$

where $V(\mathbf{r})$ is the perturbing potential and $\hbar\mathbf{q}$ is the momentum transfer.

(a) Discuss the assumptions implicit in arriving at these expressions.

(b) The Yukawa potential is given by

$$V(r) = V_0\frac{e^{-r/a}}{r}$$

where V_0 characterises the strength of the interaction and a is a distance scale. Show that from expressions 1 and 2 it follows that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{2mV_0}{\hbar^2}\right)^2\left(\frac{1}{4k^2\sin^2\theta/2 + (1/a)^2}\right)^2 \tag{3}$$

where $\hbar k$ is the momentum of the incident particle and θ is the scattering angle.

(c) By making a suitable substitution for V_0 and allowing the range of the interaction to become infinite show that Eq. 3 becomes identical to the differential cross section for Rutherford scattering.

(d) Integrate Eq. 3 to show that the total cross section is given by

$$\sigma = \left(\frac{4maV_0}{\hbar^2}\right)^2\frac{\pi}{4k^2 + (1/a)^2}. \tag{4}$$

Calculate this cross section for scattering of a neutron with $p = 200$ MeV/c in a potential with $V_0 = 35$ MeV fm and $a = 1$ fm.

2. More cross-section calculations

(a) Show that for a Gaussian potential

$$U(r) = U_0e^{-r^2/2a^2}$$

the scattering amplitude is given by

$$f(\theta,\phi) = Ae^{-a^2k^2(1-\cos\theta)}$$

where θ is the scattering angle relative to the direction of the incoming wave and determine the factor A. (Here, the standard integral $\int_{-\infty}^{\infty} e^{-\frac{1}{2}cx^2+dx}\mathrm{d}x = (2\pi/c)^{1/2}e^{d^2/2c}$ may be of use).

(b) Show that the total cross section has the form

$$\sigma(k) = B\left(1 - e^{-4a^2k^2}\right)$$

and determine the factor B.

(c) Estimate the mean free path of a 5 MeV neutron in water for elastic scatters on hydrogen nuclei. Take a to be of the order of the nucleon radius ~ 1 fm and $U_0 = 35$ MeV.

3. Form factors and determining the nuclear radius through scattering

(a) Consider the scattering of an electron from a nucleus with extended charge distribution with density given by $\rho(\mathbf{r})$ such that the potential is

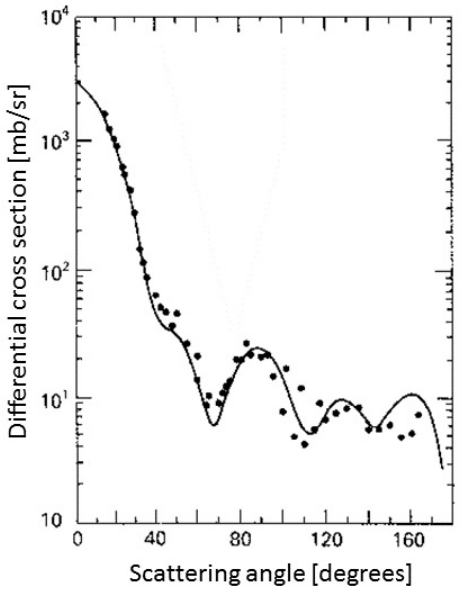
$$V(\mathbf{r}) = \int \frac{Ze^2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}') \mathrm{d}^3\mathbf{r}'$$

Show that making use of this in Eq. 1 and 2 of question 1 allows the differential cross section to be written as the product of the Rutherford equation and the square of the form factor $F(q)$ where

$$F(q) = \int e^{-i\mathbf{q}\cdot\mathbf{r}} \rho(\mathbf{r}) \mathrm{d}^3\mathbf{r}.$$

(b) The relations you have derived apply equally well for other interactions, such as the nuclear force. Consider two possible models of the distribution of nucleons within the nucleus: (i) a uniform distribution in the region $0 < r \leq a$; (ii) a distribution of the form $e^{-r/a}/r$. Calculate the form factor for both. Determine the factor by which the differential cross section that is expected for point-like scattering will be suppressed in both models for the case $qa = 1$.

(c) The below figure shows the scattering of neutrons of kinetic energy 14 MeV by nickel (Bauer *et al.*, 1963).



By equating the first minimum in this distribution to the value of the first minimum as predicted by model (i) make an estimate of the radius of the nickel nucleus. Compare this with your expectation. Comment on the fact that the observed distribution in the figure does not agree well with either of the two proposed models.

4. The Breit-Wigner and neutron scattering

The Breit-Wigner for a reaction cross section is given by

$$\sigma(E) = \frac{\pi g}{k^2} \frac{\Gamma_i \Gamma_f}{(E_0 - E)^2 + \Gamma^2/4}.$$

(a) Explain the meaning of the symbols in this equation.

(b) Neutrons of energy 2.2 eV are incident on a target of $^{123}_{52}$Te, which has $J = 1/2$, giving rise to either radiative capture or elastic scattering. The peak total and elastic cross sections are measured to be 82.4 kb and 7.4 kb, respectively, with an uncertainty of around 5%. Determine the branching fraction of the intermediate state into neutrons. Use the data provided to give your best estimate of the spin of this state.

5. Resonances and compound nuclei

When neutrons collide with a nucleus they can get captured to form a short-lived compound nucleus, which in non-fissile cases has a decay that is dominated by photon emission. The width of the state Γ is independent of the incident neutron energy.

(a) The neutron partial width Γ_n is proportional to the transition rate for the compound nucleus to decay by emitting a neutron. Show that the density of states available to the neutron is proportional to its momentum. Hence show that the cross section for the process $n + A \rightarrow B + \gamma$ has a $1/v$ dependence at low energies, where v is the velocity of the neutron.

(b) $^{113}_{48}$Cd has a cross section for capture by the $n + A \rightarrow B + \gamma$ reaction that has the following energy dependence:

E [eV]	0.01	0.02	0.03	0.05	0.08	0.10	0.15	0.20	0.25	0.30	0.40	0.50
σ [b]	3250	2500	2300	2400	2800	3400	6800	6000	2400	1100	320	140

The cross sections are accurate to $\pm 5\%$. The maximum cross section is 7750 ± 150 b. The spin of $^{113}_{48}$Cd is $1/2$ and that of the compound nucleus is 1.

By plotting $\sigma\sqrt{E}$ vs E determine the pole energy of the resonance E_R, the total width Γ and the neutron partial width Γ_n at the resonance energy E_R. You may assume $\Gamma_\gamma \approx \Gamma$. Determine the maximum cross section for elastic neutron scattering off $^{113}_{48}$Cd.

(c) What does the fact that the compound nucleus displays a narrow resonance and dominantly decays by photon emission tell us about the way the energy of the incoming neutron is absorbed by the nucleus?

6. Resonances in pion-proton scattering

The cross section for the reaction $\pi^- p \rightarrow \pi^0 n$ shows a promient peak when measured as a function of the π^- energy. The peak corresponds to a Δ resonance which has a mass of 1232 MeV with Γ=120 MeV. The partial widths for the incoming and outgoing states are Γ_i=40 MeV and Γ_f=80 MeV.

(a) At what pion beam energy will the cross section be maximal for a stationary proton?

(b) Describe and explain the similarities and difference you would expect between the cross sections for $\pi^- p \rightarrow \pi^0 n$ and the one for $\pi^- p \rightarrow \pi^- p$ for centre-of-mass energies not far from 1.2 GeV. Use quark-flow diagrams to explain what is happening.

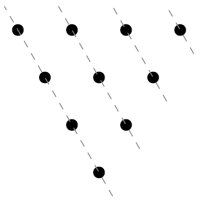
(c) By considering the quark content of the intermediate states, discuss whether you would expect similar peaks in the cross sections for the reactions (i) $K^- p \rightarrow$ products and (ii) $K^+ p \rightarrow$ products.

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Problem Set IV: Particle physics – the light quark model, quarkonia and e^+e^- collisions

1. The light-quark model

The $J^P = \frac{3}{2}^+$ decuplet of hadrons is represented below. The dashes indicate lines of equal charge.



Label	Mass (MeV/c^2)	Strangeness
Δ	1238	0
Σ^*	1385	-1
Ξ^*	1532	-2
Ω^-	?	-3

(a) Explain how the stated characteristics of these hadrons can be accommodated within the quark model.

(b) The Ω^- was discovered through the process

$$K^- + p \to \Omega^- + K^0 + X$$

Assuming the process to have proceeded through the strong interaction, state the quark content of the unspecified X hadron and draw a quark-flow diagram for this process.

(c) Give, with reasoning, a prediction for the mass of the Ω^- baryon. Calculate the minimum momentum of the kaon beam required for this reaction to occur.

(d) What does the existence of the Ω^- tell us about the colour charge of the strong interaction?

(e) Write down the quark content of the lightest five-quark baryon (a 'pentaquark') that would be both neutral and also stable under the strong interaction.

2. Valid strong reactions and quark content

Consider the following reactions:

(a) $K^- + p \to \bar{K^0} + n$,

(b) $\pi^+ + p \to K^+ + \Sigma^+$,

(c) $\Xi^- + p \to \Lambda^0 + \Lambda^0$,

(d) $\pi^- + p \to K^- + \Sigma^+$,

(e) $\bar{K^0} + p \to K^- + p + \pi^+$,

(f) $\bar{p} + p \to \pi^+ + \pi^+ + \pi^- + \pi^- + \pi^+$,

(g) $\pi^+ + p \to K^0 + \Sigma^0 + \pi^+ + K^+ + \bar{K^0}$,

(h) $K^- + p \to \Sigma^+ + n + \pi^-$,

(i) $\pi^- + p \to \Sigma^+ + \Sigma^- + K^0 + \bar{p} + \bar{\Sigma}^+ + n$,

(j) $\pi^- + p \to \bar{\Sigma}^- + \Sigma^0 + p$.

Here $\bar{\Sigma}^{\pm}$ indicates the antiparticle of the $\Sigma^{\mp}$.

Determine which of these processes are allowed, and whether they can all proceed by the strong interaction.

3. Measuring the parity of the pion

When π^- mesons are stopped in deuterium they form 'pionic atoms' ($\pi^- d$), which usually undergo transitions to an atomic s-state, whereupon the capture reaction $\pi^- d \to nn$ occurs. Determine the parity of the pion from these observations, given that it has spin 0, that it is reasonable to assume that protons and neutrons have the same parity, and that the deuteron has spin one and is predominantly s state. Compare this result with the prediction of the quark model.

4. Masses and decays of the vector mesons

The ρ^0 and the K^{*0} are the $J^P = 1^-$ counterparts of the π^0 and K^0 mesons, respectively.

(a) Comparison of their masses with those of the ground-state mesons shows a significant difference, which can be attributed to a contribution from the spin-coupling energy, analogous to hyperfine splitting in atomic physics. This contribution is given by

$$E = a\frac{\overrightarrow{S_q} \cdot \overrightarrow{S_{\bar{q}'}}}{m_q m_{\bar{q}'}}$$

where a is a constant and q and $\bar{q}'$ are the constitutent quarks. Taking $m_u = m_d = 310$ MeV, $m_s = 480$ MeV and $a = 6.2 \times 10^7$ MeV^3, calculate the masses of the π^0, K^0, ρ^0 and K^{*0}, and compare to observation.

(b) Draw a quark-flow or Feynman diagram for the predominant decay mode of the K^{*0}.

(b) Consider the following decay modes of the ρ^0:

i. $\rho^0 \to \pi^0\gamma$;

ii. $\rho^0 \to \pi^+\pi^-$;

iii. $\rho^0 \to \pi^0\pi^0$;

iv. $\rho^0 \to e^+e^-$

Draw Feynman diagrams for each of these processes. One of the decays does not occur; explain why this is so. Rank the others in order of probability.

5. e^+e^- collisions and quarkonia

Various resonances are observed in e^+e^- collisions. A closely grouped set of three occurs has one state with a mass of 3096 MeV/c^2 and a width of 0.093 MeV (the J/ψ), a second at a mass of 3686 MeV/c^2 with width 0.294 MeV (the ψ' or $\psi(2S)$) and a third at a mass of around 3774 MeV/c^2 with a width of 27 MeV (the ψ'' or $\psi(3770)$). The resonances are determined to have $J^P = 1^-$.

(a) Calculate the lifetimes of these particles. Explain how their quantum numbers, masses, and variations in lifetimes indicate the existence of a fourth quark: the charm (c) quark.

(b) The model that explains these observations predicts analogous states involving charm but with different quantum numbers. An example is the group of 0^+, 1^+ and 2^+ states χ_{co}, χ_{c1} and χ_{c2}, with masses of 3415 MeV/c^2, 3511 MeV/c^2 and 3556 MeV/c^2, respectively. How are these states explained in the quark model and why are these not directly seen as resonances in e^+e^- collisions? Explain, however, how they could still be studied at an experiment running at collision energies of 3686 MeV/c^2.

c) The ϕ meson, with mass 1019 MeV, decays into two kaons 83% of the time, and into three pions 15% of the time. Explain why considerations of phase space make these facts puzzling. Suggest an

explanation, connected to your discussion on the variation in lifetimes of the J/ψ, $\psi(2S)$ and ψ'' resonances. Illustrate your answer with quark flow or Feynman diagrams of the decay.

6. Quarkonia potential

The potential for a $q\bar{q}$ system can be approximated by

$$V(r) = -\frac{4}{3}\frac{\hbar c \alpha_s}{r} + kr.$$

(a) Sketch $V(r)$ against r. Show that for $r \ll r_0 = \sqrt{\hbar c \alpha_s / k}$ that the $1/r$ term dominates.

(b) Ignoring the linear term, and through analogy to the energy levels of the atom, calculate an effective value of α_s from the mass difference between the 2^3S_1 and 1^3S_1 states of the $c\bar{c}$ system. Take m_c to be 1870 MeV, the mass of the D^+ meson.

(c) Use your result to calculate the expected mass of the $\Upsilon(2S)$, given that the $\Upsilon(1S)$ has a mass of 9460 MeV. Take m_b to be 5280 MeV, the mass of the B^- meson. Compare with the measured value of 10023 MeV and comment on any difference.

(d) Still ignoring the linear term, compute the Bohr radius of the J/ψ and $\Upsilon(1S)$. Taking $\sqrt{k} \approx 400\,\mathrm{MeV}/\sqrt{\hbar c}$ verify that it was reasonable to consider only the $1/r$ term.

7. e^+e^- collisions and the R ratio

At collision energies E in the continuum regime $m_\mu c^2 \ll E \ll m_Z c^2$ the cross section for the process $e^+e^- \to \mu^+\mu^-$ is given by

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\frac{(\alpha\hbar c)^2}{E^2}$$

(a) Sketch the Feynman diagram for this process and justify the α and E dependence.

(b) Describe the expected behaviour of the cross-section ratio

$$R = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

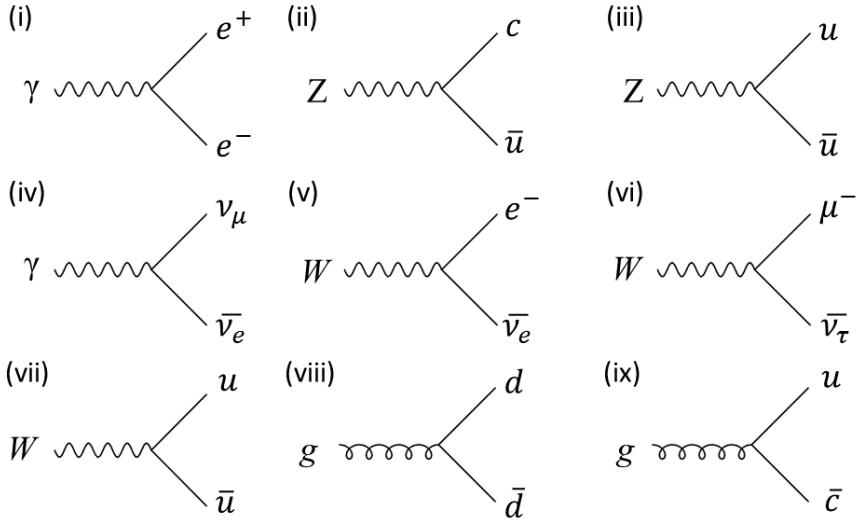
between $E = 2$ GeV and 15 GeV. Would this ratio look the same if constructed with $\sigma(e^+e^- \to e^+e^-)$ or $\sigma(e^+e^- \to \tau^+\tau^-)$ in the denominator?

Subatomic Physics: Prof. Guy Wilkinson

Problem Set V: Particle physics – (mostly) the weak interaction

1. Vertices and decay diagrams

(a) Below are shown various vertices coupling gauge bosons (*i.e.* force carying particles) into pairs of fermions. Identify which vertices are forbidden and give a reason why. Where possible, for these cases, show the allowed diagram by changing the identify of the bottom fermion.



(b) Draw Feynman diagrams for the following weak decays or processes. Do this at quark level for those involving hadrons.

i. $K^0 \to \pi^- + e^+ + \nu_e$,

ii. $D^+ \to \bar{K^0} + \mu^+ + \nu_\mu$,

iii. $\tau^+ \to \pi^+ + \bar{\nu_\tau}$,

iv. $\Lambda^0 \to p + e^- + \bar{\nu_e}$,

v. $\Xi^- \to \Lambda^0 + \pi^-$;

vi. $K^+ \to \pi^+ + \pi^- + \pi^+$,

vii. $B_s^0 \to \bar{B_s^0}$.

2. Neutrino scattering

Consider a high energy ($E_\nu > 10$ GeV) muon neutrino beam interacting with matter in a detector through the process

$$\nu_\mu + N \to \mu^- + \text{hadrons},$$

whose signature is a long penetrating track and localised activity associated with low-energy hadronic debris.

(a) From consideration of the de Broglie wavelength show that the experiment is sensitive to the substructure of the nucleons.

(b) The cross-section for this process is given by

$$\sigma = \frac{G_F^2 E_\nu m_p c^2}{4\pi(\hbar c)^4}.$$

Justify the linear dependence on E_ν and evaluate σ at $E_\nu = 100$ GeV.

(c) Experimentally it is indeed observed that σ/E_ν is constant over all energies so far probed. What conclusion can we draw from this fact about the nature of quarks?

(d) Independent of your answer to part (c), do you expect this σ/E_ν relation to remain true to arbitrarily high energies? If not, what changes the dependence, and at what neutrino energy do you expect this effect to become significant?

(e) Other interactions are observed in which no penetrating long track is observed. What is causing these events? Draw Feynman diagrams for the underlying physics process for both categories of events.

3. Cabibbo suppression

(a) Draw the Feynman diagrams for the decays $\tau^- \to \pi^-\nu_\tau$ and $\tau^- \to K^-\nu_\tau$. The former has a branching fraction of 10.8% and the latter one of 7.0×10^{-3}. Use this information to determine a value for the Cabibbo angle.

(b) Draw the Feynman diagrams for the decays $D^0 \to K^-\pi^+$ and $D^0 \to K^+\pi^-$. Calculate the expected ratio of the branching fraction of the latter mode to that of the former, and compare to the measured value of 0.0038. Make also an approximate estimate of the ratio of branching fractions of $D^0 \to K^+K^-$ to $D^0 \to K^-\pi^+$.

(c) Consider the decays $B^- \to D^0\mu^-\bar{\nu_\mu}$ and $B^- \to \pi^0\mu^-\bar{\nu_\mu}$. Which of these do you expect to have the higher branching fraction, and why?

4. Weak decays and lifetimes

The below table lists the mass and lifetime of mesons and leptons of different flavours, and the branching fraction for their semileptonic decay.

Particle	Mass [MeV/c^2]	Lifetime [s]	Branching fraction
B^+	5279	1.6×10^{-12}	BF($B^+ \to X_c\, e^+\nu_e$)=10.8%
D^+	1870	1.0×10^{-12}	BF($D^+ \to K^0 e^+\nu_e$)=8.7%
K^+	494	1.2×10^{-8}	BF($K^+ \to \pi^0 e^+\nu_e$)=5.1%
τ^+	1777	2.9×10^{-13}	BF($\tau^+ \to e^+\bar{\nu_\tau}\nu_e$)=17.8%
μ^+	106	2.2×10^{-6}	BF($\mu^+ \to e^+\bar{\nu_\mu}\nu_\mu$)=100%

Here X_c designates an unspecified charm hadron. Other potentially useful data are that the mass of the π^0 is 135 MeV/c^2 and the Cabibbo-Kobayashi-Maskawa matrix elements V_{us}, V_{cs} and V_{cb} have magnitudes of 0.225, 0.987 and 0.041, respectively.

(a) Draw the Feynman diagrams for the τ^+ and D^+ decays.

(b) Show that, despite spanning seven orders of magnitude, the variation in lifetimes can be explained by a common picture that, with the data given, gives predictions which are good to well within a factor of ten. Comment on whether you get better internal consistency for the decays of the leptons or hadrons, and suggest a reason for this.

(c) Demonstrate that the π^+, with mass 140 MeV, lifetime 2.6×10^{-8} s, and dominant decay $\pi^+ \to \mu^+\nu_\mu$ does not fit into this picture at all, and explain why this is so.

(c) At the LHC B^+ hadrons are produced with a wide spectrum in momentum, with a mean value of around 50 GeV/c. What is the approximate mean distance these hadrons travel before decaying? Explain how you would measure the lifetime of a given decay, commenting on whether the listed decay mode in the table would be suitable for this purpose.

5. Parity violation

(a) Given your knowledge of the parity of the pion, and the spins of the pion and kaon, show that the two decays

$$\begin{aligned} K^- &\rightarrow \pi^-\pi^0 \\ K^- &\rightarrow \pi^-\pi^-\pi^+ \end{aligned}$$

have final states with different parity, thus suggesting parity is not conserved in weak decays.

(b) When cooled to low temperature and placed in a magentic field ^{60}Co nuclei ($J^P = 5^+$) polarise along the field direction. These nuclei undergo β^- decay to ^{60}Ni ($J^P = 4^+$). The electrons are observed to be emitted predominantly in an opposite direction to the polarisation direction of the nuclei. Explain how this observation indicates parity violation.

(c) The intensity of the electrons has a dependence not just on direction θ (defined to be the polar angle with respect to the B field), but also on speed v, with an intensity distribution:

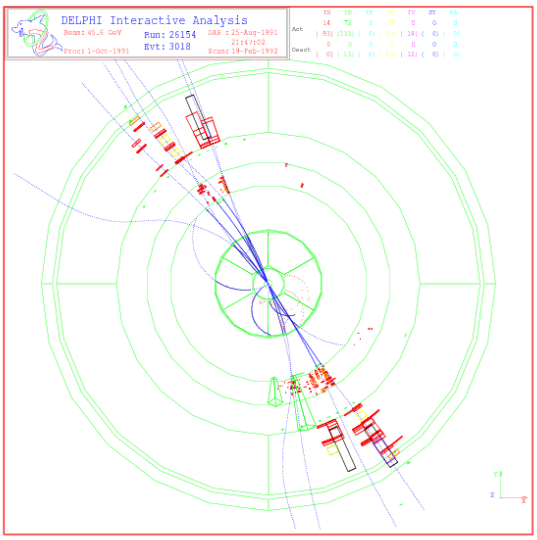
$$I(v, \theta) \propto 1 - \frac{v}{c}\cos\theta.$$

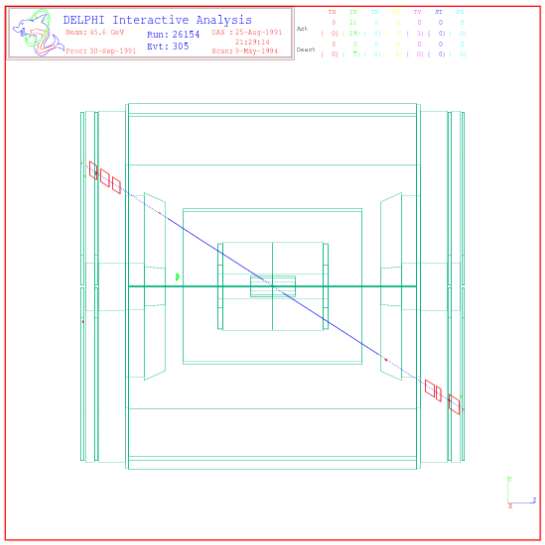
What does this tell us about the helicity of the electrons emitted in the decay.

(d) The decay $\pi^- \rightarrow e^-\bar{\nu}_e$ is suppressed by around a factor 10^4 compared to the decay $\pi^- \rightarrow \mu^-\bar{\nu}_\mu$. Why is this observation a priori surprising, and how can it be explained?

6. e^+e^- collisions at the Z^0 resonance

(a) In the early 1990s the DELPHI experiment at the LEP accelerator recorded the decay products in e^+e^- collisions at centre-of-mass energies between 88 and 94 GeV/c^2. The 'events' resulting from the collisions could be categorised into various 'topologies' or 'final states'. Example events of two such topologies are illustrated below.

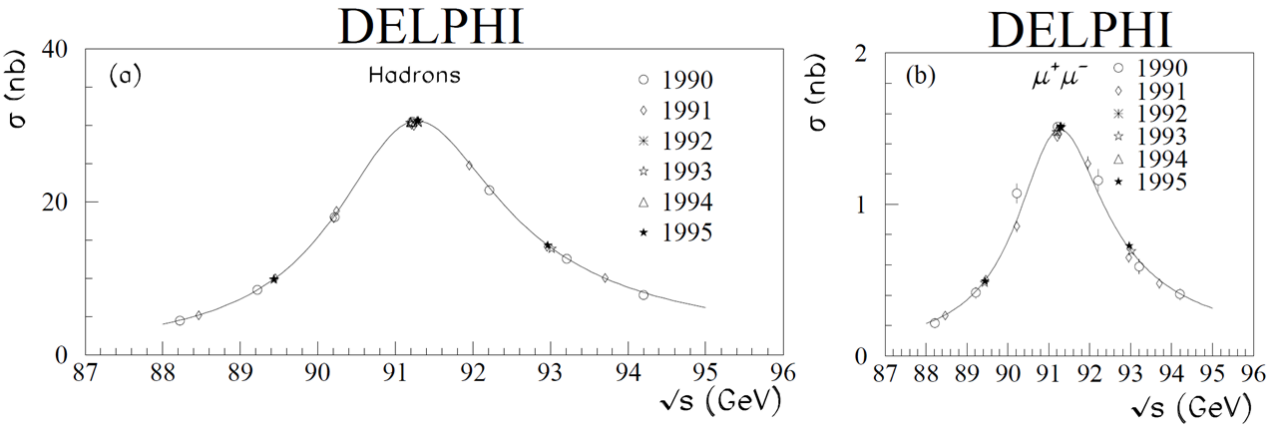




The left event, shown in cross-section through the detector, has two back-to-back jets, each containing many tracks, several of which have left significant energy deposits in the outermost layers of the calorimetry system, but nothing beyond. The right event, shown side-on, has two back-to-back tracks which have left a much fainter, but continuous trail of energy in the calorimeter accompanied by hits in the outermost tracking stations.

Explain what is happening in the two cases. Draw the dominant Feynman diagrams for both. Do you understand why there are so many particles in the left hand plot?

(b) The cross-sections (designated by σ) of both topologies were measured as a function of centre-of-mass energy (designated as E_{CM} or $\sqrt{s}$). Distinctive resonances were seen, shown below, where the ordering is the same as in the event pictures.



The resonances for final state X can be fitted to a Breit-Wigner of the form:

$$\sigma(e^+e^- \to X) = \frac{12\pi M^2}{E_{\mathrm{CM}}^2}\left[\frac{\Gamma_{e^+e^-}\Gamma_X}{(E_{\mathrm{CM}}^2 - M^2)^2 + M^2\Gamma^2}\right]. \tag{1}$$

Note this expression is written in natural units, for which $0.389\ \mathrm{mb} = 1\,\mathrm{GeV}^{-2}$.

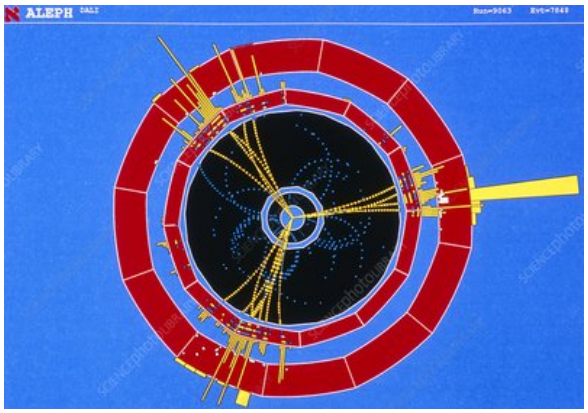
- Why do you think that $\Gamma_{e^+e^-}$ is present in the expression, even for cases where the final state is *not* e^+e^- ?
- Given that the resonance is reasonanly narrow, show that this expression is equivalent in form to that we encountered for the Breit-Wigner when discussing nuclear physics, *i.e.* question 4 of sheet III.

(c) In fact Eq. 1 is not a perfect description of the resonances as measured, as initial-state radiation, where a photon is radidated by the electron or positron prior to collision, distorts the lineshape. In consequence the peak cross section corresponding to this expression is about 1.4 times larger than what is measured, and the width 0.83 times smaller.

- This effect also introduces a small but noticeable asymmetry in the lineshape. Explain why this is so.
- Determine the mass of the resonance and from measurements of the cross sections and width show that the *invisible width*, *i.e.* partial width into undetected states, is about 510 MeV. State any assumptions you have made in this excercise.

(c) The Standard Model predicts with confidence that $\Gamma_{\nu\bar{\nu}}/\Gamma_{e^+e^-} = 2$. Use this information and the result of your measurement above to estimate how many lepton generations there are. What assumptions have you made?

(d) Closer inspection of the 'jet' events shows a subset ($\sim 15\%$) of this category which have *more* than two jets, such as the three-jet 'Mercedes' event show below, as recorded in ALEPH, another experiment on LEP.



What is happening here, as understood in the context of the strong interaction?

7. Studies of the W boson

(a) The W boson has a mass of 80.4 GeV was first observed at the SPS $p\bar{p}$ collider at CERN (as was the Z^0) . Draw the Feynman diagram for its production in this environment and explain what is the least challenging decay mode in which to infer its presence.

(b) The W boson may also be studied in e^+e^- collisions. Draw the Feynman diagrams for W production in this environment and give the minimum collision energy required.

(c) In the e^+e^- environment three distinct event topologies are observed.

i. Four hadronic jets;

ii. Two hadronic jets together with a single track , or low number of other tracks;

iii. A single track (or low number of tracks) together with a single track (or low number of tracks).

Explain what underlies these topologies, and make an estimate of the relative rate of each.

(d) Suggest a method by which the mass of the W boson could be measured from these events.