Nuclear and particle physics

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Michaelmas Term 2021

Syllabus and order of course

From the course handbook:

B4 Nuclear and Particle Physics

Knowledge of the special relativity in the Prelims paper CP1 will be assumed

Scattering in quantum mechanics to first order, concept of a scattering cross section, form factors, propagators, virtual particle exchange.

Resonant scattering, decay widths, Breit-Wigner formula.

Nuclear mass, binding energy, the semi-empirical mass formula, stability, radioactivity, alpha and beta decay.

Basic fission and fusion reactions.

Quark model of hadrons: the light meson and baryon multiplets and quarkonium.

The Standard Model: quark and lepton families, fundamental interactions, Cabibbo mixing.

Strong interaction, a qualitative discussion of confinement, the concept of colour.

Weak interaction, parity violation, properties and decays of the W and Z bosons.

We will be covering these topics in the order: nuclear physics, scattering theory, and then particle physics, with approximate weights in time of 2:1:2.

As well as assuming knowledge of special relativity, we shall be making use of results from 2nd year quantum mechanics (*e.g.* perturbation theory, quantum spin *etc.*) and statistical mechanics (*e.g.* density of states).

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Text books

Again, from course handbook:

B4 Nuclear and Particle Physics

'Quantum Mechanics', Franz Mandl *

'Quantum Mechanics', Eugen Merzbacher *

'An Introduction to Nuclear Physics', W.N.Cottingham and D.A. Greenwood *

'Nuclear and Particle Physics', W.S.C Williams *

'Elements of Nuclear Physics', W.E.Burcham *

'Introductory Nuclear Physics', Kenneth S. Krane *

'Particle Physics', B.R.Martin and G.Shaw*

'Introduction to Elementary Particle Physics', Alessandro Bettini*

'Particle Physics in the LHC Era', G.Barr, R.Devenish, R.Walczak, T.Weidberg *

'Introduction to High Energy Physics', Donald H. Perkins, *

Most of the particle physics books straddle the 3rd and 4th year courses, so should be used selectively. The nuclear books generally contain more than we need, so should be dipped into (Krane is excellent). The best 'go-to' book with best (but not perfect) overlap with the syllabus remains Williams 'Nuclear and Particle Physics'.

Lectures and course material

Each week (or with variable frequency) I shall distribute handouts, which will be copies of my slides, but with gaps that you can fill in during the lectures.

Be alert for the inevitable errors. All spots will be suitably rewarded.

I will, however, post on Canvas complete sets of slides, again with ~weekly frequency, and a combined set of each main topic when we have finished that part of the course.

I will also post on Canvas the notes of the previous lecturer for this course, Georg Viehhauser. These approach things somewhat differently than me, but are comprehensive and excellent and may be preferred by the more mathematically inclined.

Finally, I have prepared five problem sets. These are already available on the Canvas page.

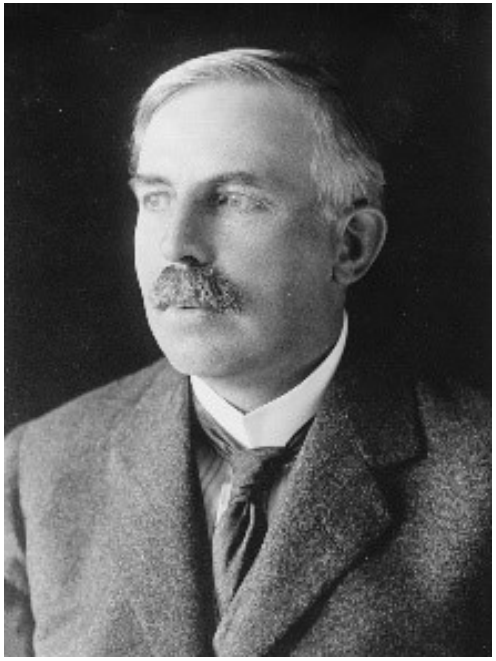
Nuclear physics

Dawn of nuclear physics

Nuclear physics began with Henri Bequerel's discovery of radiation in 1896, & the subsequent investigations of Marie Curie, Ernest Rutherford and others. Of particular importance was Rutherford's discovery of the nucleus in 1909, together with Geiger and Marsden.



Marie Curie, 1867-1934;
Polish-French citizenship;
Nobel Prize for Physics 1903
(with Bequerel and Pierre Curie);
Nobel Prize for Chemistry 1911



Ernest Rutherford, 1871-1937;
New Zealand – British citizenship;
Nobel Prize for Chemistry 1908

Nobel Prizes in Physics for subatomic discoveries: pre (& during) WWII

The Nobel Prize in Physics 1903

Antoine Henri Becquerel "in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity"

Pierre Curie and Marie Curie, née Sklodowska "in recognition of the extraordinary services they have rendered by their joint researches on the radiation phenomena discovered by Professor Henri Becquerel"

The Nobel Prize in Physics 1905

Philipp Eduard Anton von Lenard "for his work on cathode rays"

The Nobel Prize in Physics 1923

Robert Andrews Millikan "for his work on the elementary charge of electricity and on the photoelectric effect"

The Nobel Prize in Physics 1922

Niels Henrik David Bohr "for his services in the investigation of the structure of atoms and of the radiation emanating from them"

The Nobel Prize in Physics 1927

Arthur Holly Compton "for his discovery of the effect named after him"

Charles Thomson Rees Wilson"for his method of making the paths of electrically charged particles visible by condensation of vapour"

The Nobel Prize in Physics 1929

Prince Louis-Victor Pierre Raymond de Broglie "for his discovery of the wave nature of electrons"

The Nobel Prize in Physics 1933

Erwin Schrödinger and Paul Adrien Maurice Dirac "for the discovery of new productive forms of atomic theory"

The Nobel Prize in Physics 1932

Werner Karl Heisenberg "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen"

The Nobel Prize in Physics 1936

Victor Franz Hess "for his discovery of cosmic radiation"

Carl David Anderson "for his discovery of the positron"

The Nobel Prize in Physics 1935

James Chadwick"for the discovery of the neutron"

The Nobel Prize in Physics 1939

Ernest Orlando Lawrence"for the invention and development of the cyclotron and for results obtained with it, especially with regard to artificial radioactive elements"

The Nobel Prize in Physics 1938

Enrico Fermi"for his demonstrations of the existence of new radioactive elements produced by neutron irradiation, and for his related discovery of nuclear reactions brought about by slow neutrons"

The Nobel Prize in Physics 1945

Wolfgang Pauli "for the discovery of the Exclusion Principle, also called the Pauli Principle"

The Nobel Prize in Physics 1944

Isidor Isaac Rabi "for his resonance method for recording the magnetic properties of atomic nuclei"

The Nobel Prize in Physics 1943

Otto Stern "for his contribution to the development of the molecular ray method and his discovery of the magnetic moment of the proton"

Nobel Prizes in Physics for subatomic discoveries: 1951→1975

The Nobel Prize in Physics 1951

Sir John Douglas Cockcroft and Ernest Thomas Sinton Walton "for their pioneer work on the transmutation of atomic nuclei by artificially accelerated atomic particles"

The Nobel Prize in Physics 1950

Cecil Frank Powell "for his development of the photographic method of studying nuclear processes and his discoveries regarding mesons made with this method"

The Nobel Prize in Physics 1949

Hideki Yukawa "for his prediction of the existence of mesons on the basis of theoretical work on nuclear forces"

The Nobel Prize in Physics 1948

Patrick Maynard Stuart Blackett "for his development of the Wilson cloud chamber method, and his discoveries therewith in the fields of nuclear physics and cosmic radiation"

The Nobel Prize in Physics 1955

Willis Eugene Lamb "for his discoveries concerning the fine structure of the hydrogen spectrum"

Polykarp Kusch "for his precision determination of the magnetic moment of the electron"

The Nobel Prize in Physics 1954

Max Born "for his fundamental research in quantum mechanics, especially for his statistical interpretation of the wavefunction"

Walther Bothe "for the coincidence method and his discoveries made therewith"

The Nobel Prize in Physics 1969

Murray Gell-Mann "for his contributions and discoveries concerning the classification of elementary particles and their interactions"

The Nobel Prize in Physics 1968

Luis Walter Alvarez "for his decisive contributions to elementary particle physics, in particular the discovery of a large number of resonance states, made possible through his development of the technique of using hydrogen bubble chamber and data analysis"

The Nobel Prize in Physics 1967

Hans Albrecht Bethe "for his contributions to the theory of nuclear reactions, especially his discoveries concerning the energy production in stars"

The Nobel Prize in Physics 1965

Sin-Itiro Tomonaga, Julian Schwinger and Richard P. Feynman "for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles"

The Nobel Prize in Physics 1961

Robert Hofstadter "for his pioneering studies of electron scattering in atomic nuclei and for his thereby achieved discoveries concerning the structure of the nucleons"

Rudolf Ludwig Mössbauer "for his researches concerning the resonance absorption of gamma radiation and his discovery in this connection of the effect which bears his name"

The Nobel Prize in Physics 1960

Donald Arthur Glaser "for the invention of the bubble chamber"

The Nobel Prize in Physics 1959

Emilio Gino Segrè and Owen Chamberlain "for their discovery of the antiproton"

The Nobel Prize in Physics 1958

Pavel Alekseyevich Cherenkov, Il´ja Mikhailovich Frank and Igor Yevgenyevich Tamm "for the discovery and the interpretation of the Cherenkov effect"

The Nobel Prize in Physics 1957

Chen Ning Yang and Tsung-Dao (T.D.) Lee "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles"

The Nobel Prize in Physics 1963

Eugene Paul Wigner "for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles"

Maria Goeppert Mayer and J. Hans D. Jensen "for their discoveries concerning nuclear shell structure"

The Nobel Prize in Physics 1976

Burton Richter and Samuel Chao Chung Ting "for their pioneering work in the discovery of a heavy elementary particle of a new kind"

The Nobel Prize in Physics 1975

Aage Niels Bohr, Ben Roy Mottelson and Leo James Rainwater "for the discovery of the connection between collective motion and particle motion in atomic nuclei and the development of the theory of the structure of the atomic nucleus based on this connection"

Nobel Prizes in Physics for subatomic discoveries: 1979→

The Nobel Prize in Physics 1980

James Watson Cronin and Val Logsdon Fitch "for the discovery of violations of fundamental symmetry principles in the decay of neutral K-mesons"

The Nobel Prize in Physics 1979

Sheldon Lee Glashow, Abdus Salam and Steven Weinberg "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current"

The Nobel Prize in Physics 1984

Carlo Rubbia and Simon van der Meer "for their decisive contributions to the large project, which led to the discovery of the field particles W and Z, communicators of weak interaction"

The Nobel Prize in Physics 1988

Leon M. Lederman, Melvin Schwartz and Jack Steinberger "for the neutrino beam method and the demonstration of the doublet structure of the leptons through the discovery of the muon neutrino"

The Nobel Prize in Physics 1990

Jerome I. Friedman, Henry W. Kendall and Richard E. Taylor "for their pioneering investigations concerning deep inelastic scattering of electrons on protons and bound neutrons, which have been of essential importance for the development of the quark model in particle physics"

The Nobel Prize in Physics 1992

Georges Charpak "for his invention and development of particle detectors, in particular the multiwire proportional chamber"

The Nobel Prize in Physics 2015

Takaaki Kajita and Arthur B. McDonald "for the discovery of neutrino oscillations, which shows that neutrinos have mass"

The Nobel Prize in Physics 1995

"for pioneering experimental contributions to lepton physics"

Martin L. Perl "for the discovery of the tau lepton"

Frederick Reines "for the detection of the neutrino"

The Nobel Prize in Physics 1999

Gerardus 't Hooft and Martinus J.G. Veltman "for elucidating the quantum structure of electroweak interactions in physics"

The Nobel Prize in Physics 2002

Raymond Davis Jr. and Masatoshi Koshiba "for pioneering contributions to astrophysics, in particular for the detection of cosmic neutrinos"

Riccardo Giacconi "for pioneering contributions to astrophysics, which have led to the discovery of cosmic X-ray sources"

The Nobel Prize in Physics 2004

David J. Gross, H. David Politzer and Frank Wilczek "for the discovery of asymptotic freedom in the theory of the strong interaction"

The Nobel Prize in Physics 2008

Yoichiro Nambu "for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"

Makoto Kobayashi and Toshihide Maskawa "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"

The Nobel Prize in Physics 2013

François Englert and Peter W. Higgs "for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles, and which recently was confirmed through the discovery of the predicted fundamental particle, by the ATLAS and CMS experiments at CERN's Large Hadron Collider"

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Awarded (by my counting) on 46 / 115 occasions for discoveries in, or closely related to, subatomic physics.

Several times discoveries in subatomic physics have also been recognised by Nobel Prizes in Chemistry (*e.g.* Curie, Rutherford, Hahn…).

Occasionally particle physicists have been awarded the Physics Prize for other achievements (*e.g.* Parisi 2021, Barish 2017…).

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Nuclear physics: topic overview

- Definitions and initial observations
- The Semi-Empirical Mass Formula (SEMF)
- The true nature of the nuclear interaction

- Nuclear decay and radioactivity: an overview
- α decay
- β decay and Fermi theory

- Fission and fusion: an overview
- Induced fission
- Fusion reactions
- Solar fusion and nucleosynthesis

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Nuclear physics: initial definitions

A nucleus is constituted of Z protons and N neutrons, with mass number $A = N + Z$.

Protons and neutrons are collectively referred to as *nucleons.*

A *nuclide* refers to a nucleus with specific values of Z and N,

e.g. ^{4}He is the nuclide that has two protons and two neutrons.

Nuclides with equal Z but different N are called *isotopes*.

Nuclides with equal A but different Z are called *isobars*.

Nuclides with equal N but different Z are called *isotones*.

How many nuclides? How stable?

There are 286 *primordial nuclides*: these are nuclides that have existed in current form since before Earth was formed. Of these 252 are stable, and 34 have half-lives > 10^8 years.

The 34 unstable primordial nuclides

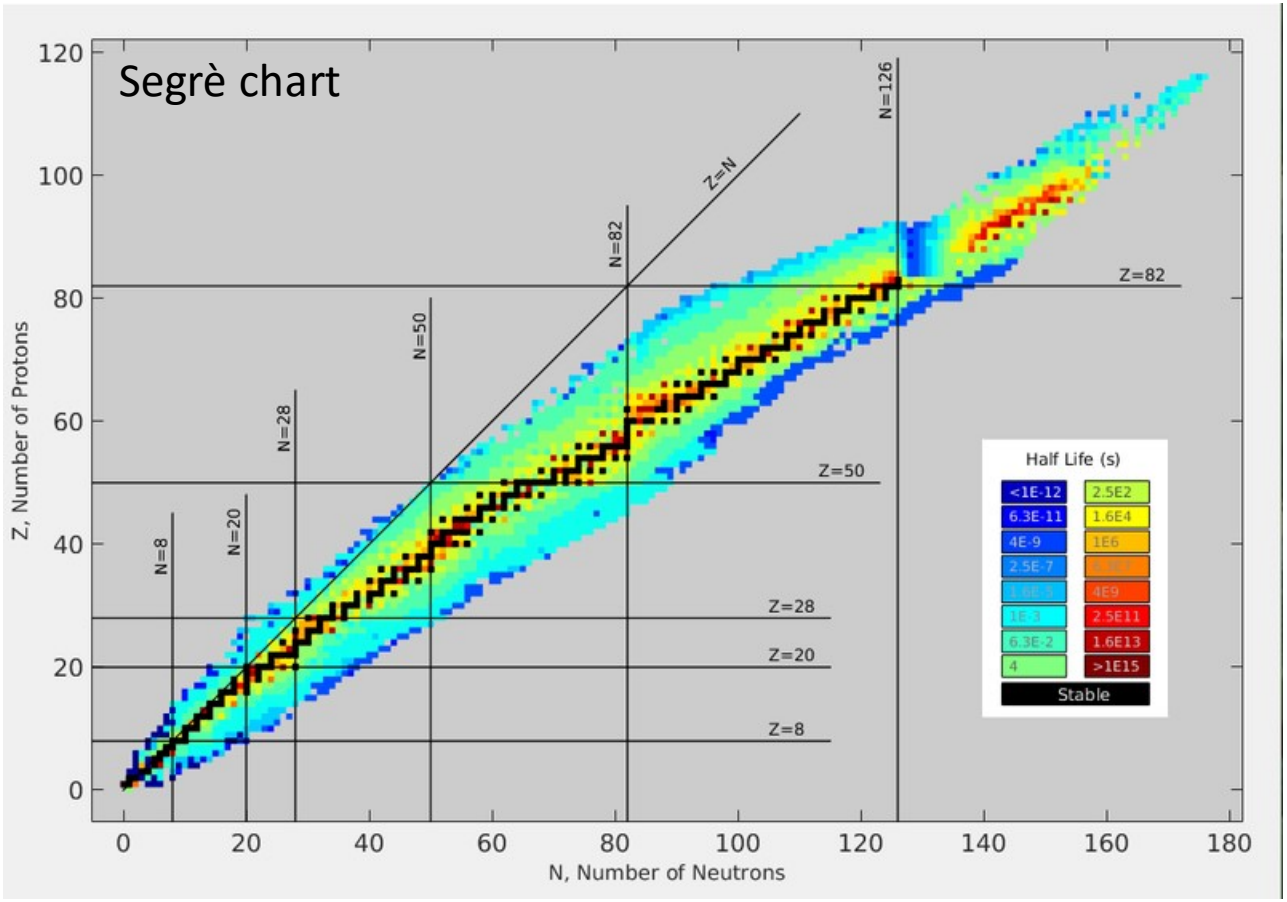
No.	Nuclide	Energy	Half-life (years)	Decay mode	Decay energy (MeV)	Approx. ratio half-life to age of universe
286	^{235}U	7.897198	7.038×10^8	α or SF	4.679	0.05
261	^{48}Ca	8.992452	2.301×10^{19}	2 β^-	4.274, .0058	2 billion
284	^{238}U	7.872551	4.468×10^9	α or SF or 2 β^-	4.270	0.3
283	^{232}Th	7.918533	1.405×10^{10}	α or SF	4.083	1
263	^{96}Zr	8.961359	2.0×10^{19}	2 β^-	3.4	1 billion
265	^{150}Nd	8.562594	7.905×10^{18}	2 β^-	3.367	600 million
277	^{190}Pt	8.267764	6.5×10^{11}	α	3.252	47
262	^{209}Bi	8.158689	2.01×10^{19}	α	3.137	1 billion
266	^{100}Mo	8.933167	7.804×10^{18}	2 β^-	3.035	600 million
259	^{82}Se	9.017596	1.1×10^{20}	2 β^-	2.995	8 billion
254	^{124}Xe	8.778264	1.8×10^{22}	KK	2.864	1 trillion
255	^{78}Kr	9.022349	9.2×10^{21}	KK	2.846	670 billion
273	^{186}Os	8.302508	2.002×10^{15}	α	2.823	100,000
260	^{116}Cd	8.836146	3.102×10^{19}	2 β^-	2.809	2 billion
258	^{130}Ba	8.742574	1.2×10^{21}	KK	2.620	90 billion
253	^{128}Te	8.743261	2.2×10^{24}	2 β^-	2.530	160 trillion
268	^{180}W	8.347127	1.801×10^{18}	α	2.509	100 million

No.	Nuclide	Energy	Half-life (years)	Decay mode	Decay energy (MeV)	Approx. ratio half-life to age of universe
274	^{174}Hf	8.392287	2.002×10^{15}	α	2.497	100,000
256	^{136}Xe	8.706805	2.165×10^{21}	2 β^-	2.462	150 billion
278	^{147}Sm	8.610593	1.061×10^{11}	α	2.310	7.7
269	^{50}V	9.055759	1.4×10^{17}	β^+ or β^-	2.205, 1.038	10 million
276	^{152}Gd	8.562868	1.1×10^{14}	α	2.203	8000
257	^{76}Ge	9.034656	1.8×10^{21}	2 β^-	2.039	130 billion
271	^{148}Sm	8.607423	7.005×10^{15}	α	1.986	500,000
267	^{151}Eu	8.565759	5.004×10^{18}	α	1.9644	300 million
272	^{144}Nd	8.652947	2.292×10^{15}	α	1.905	200,000
279	^{138}La	8.698320	1.021×10^{11}	K or β^-	1.737, 1.044	7.4
285	^{40}K	8.909707	1.251×10^9	β^- or K or β^+	1.311, 1.505, 1.505	0.09
282	^{176}Lu	8.374665	3.764×10^{10}	β^-	1.193	2.7
264	^{130}Te	8.766578	8.806×10^{18}	2 β^-	.868	600 million
275	^{115}In	8.849910	4.4×10^{14}	β^-	.499	30,000
270	^{113}Cd	8.859372	7.7×10^{15}	β^-	.321	600,000
280	^{87}Rb	9.043718	4.972×10^{10}	β^-	.283	3.6
281	^{187}Re	8.291732	4.122×10^{10}	β^-	.0026	3

In addition, there are 619 more nuclides with half-life > 1 hour, and 1000s more that are shorter lived. We shall seek to answer why some nuclides are stable, and others are not, and why there is such an enormous range of half-lives.

Segrè chart

Nuclides follow a characteristic distribution, which follows, and then diverges from, N=Z.



The most stable nuclides lie in the centre of this distribution – the *valley of stability*. Can this be understood, & how does position in plane relate to stability and mode of decay ?

Nuclear masses and binding energy

Unsurprisingly, mass scale of a nucleus is set by the number of neutrons and protons and their respective masses. We will use MeV/c^2 (and MeV for energy), *not* kg (and J).

Mass of neutron = 939.6 MeV/c^2

Mass of proton = 938.3 MeV/c^2

(both of these known with a relative precision of ~10^{-8})

Curious! These are very similar, but not quite identical; we'll come back to this later.

So a gold nucleus with A=197 has a mass of around 185,000 MeV/c^2 (185 GeV/c^2).

But a bound system, such as a nucleus, has also has *binding energy*, B:

$$m(Z,A)c^2 = Zm_p c^2 + Nm_n c^2 - B$$

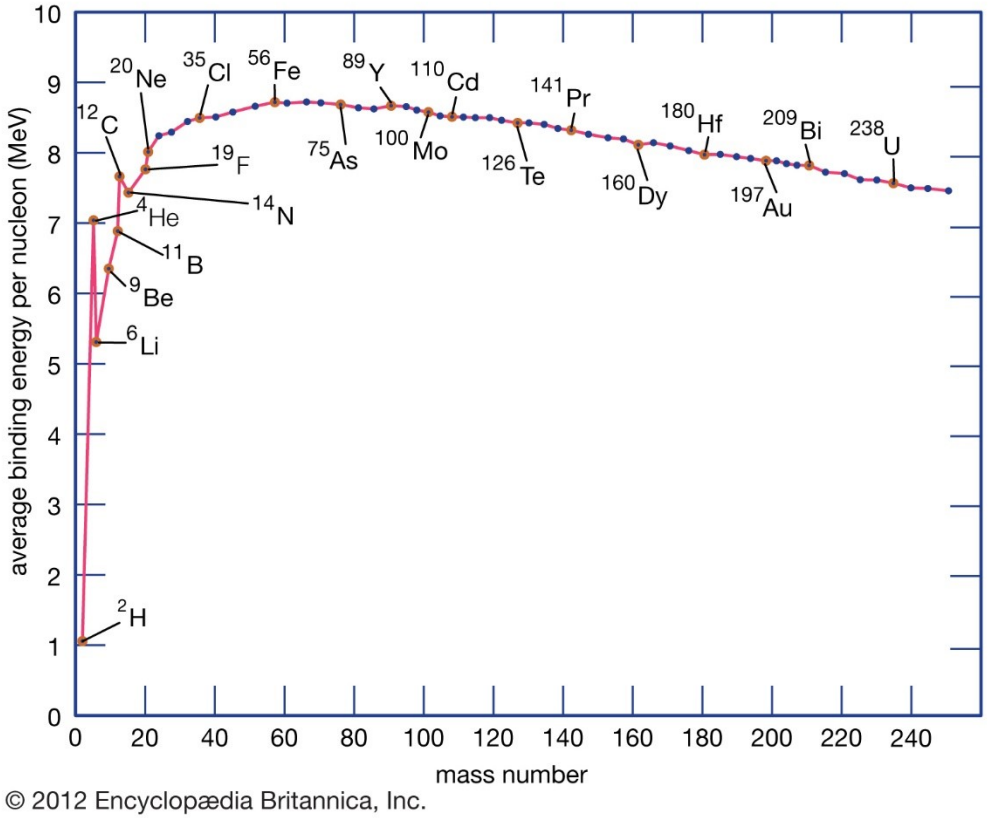
this being the energy required to pull the nucleus apart into its constituents.
As we shall shortly see, this is an order 10^{-2} correction to the naïve mass calculation.

(Be aware there is an alternative jargon of *mass defect*, which expresses B in terms of unified mass units, where everything is normalised to ^{12}C. We won't be using this.)

Nuclear binding energy

The binding energy, *B*, is the difference between the nuclear mass, and the mass contributed by its *Z* protons and *N* neutrons.

$$m(Z,A)c^2 = Zm_p c^2 + Nm_n c^2 - B$$



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The higher the binding energy per nucleon, the more tightly bound the nuclide.

Nickel-62 is the most tightly bound nucleus per nucleon, followed by iron-58 and iron-56.

The binding energy vs. mass number again has a characteristic shape – it is *approximately* constant for A>20. Can this be understood and described, and what are its implications?

Nuclear size

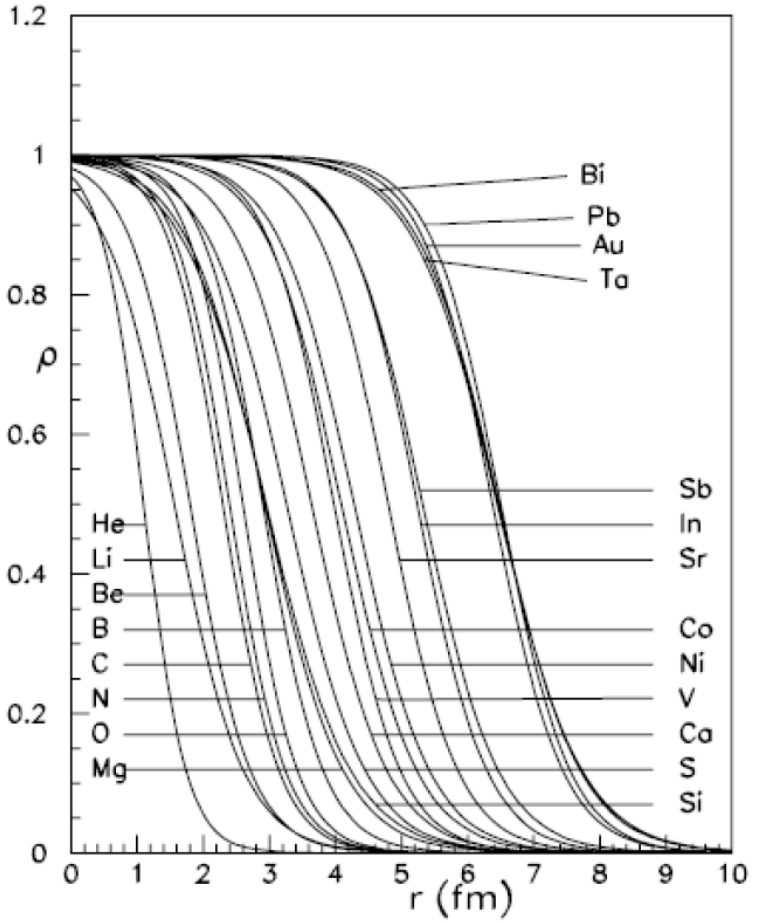
We know about the size of nuclei from scattering experiments, which we will discuss later.

From these we learn:

- Nuclei are approximately spherical with radii of a few femtometers (fm), *i.e.* ~10^{-15} m, to be compared with atomic scale of ~10^{-10} m.
- Distribution of neutrons and protons is similar.
- Size of the nucleus grows with A. Data can be fitted with:

$$r = r_0 A^{1/3}$$

with r_0=1.2 fm. This implies constant nuclear density across all nuclides in their interior, and suggests the nuclear matter is incompressible.



Spin, parity and excited states

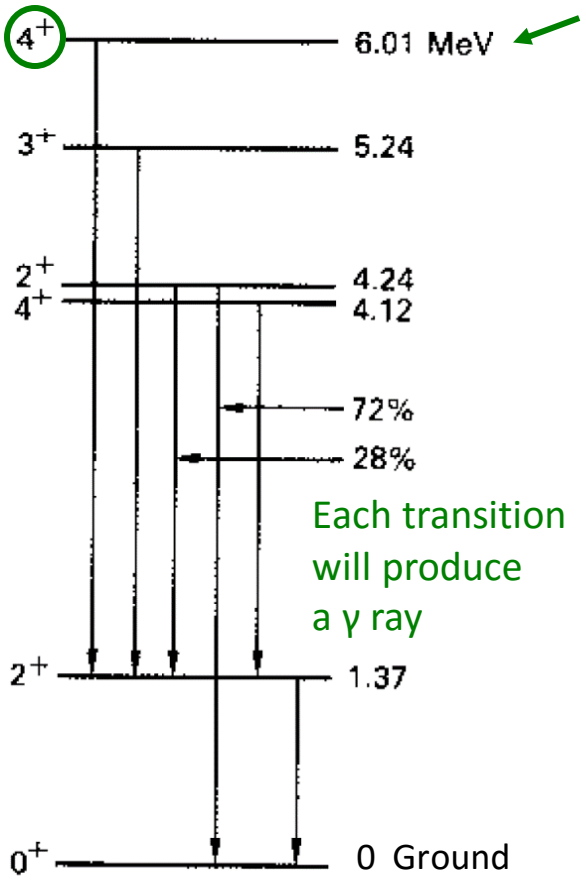
Nuclei, like atoms, can have excited states (*e.g.* shown here are the energy levels of $^{24}_{12}\mathrm{Mg}$). Excited nuclei will typically emit γ rays, eventually ending up at the ground state.

Nuclei also have spin *J* and parity *P* quantum numbers, labelled by J^P.

These quantum numbers will vary from state to state.

Spin you are familiar with. Parity we will discuss extensively in the particle physics lectures.

A quantitative knowledge of what is responsible for these properties is not required, but you should know they exist.



Energy levels of $^{24}_{12}\mathrm{Mg}$

These excited states are typically very short-lived, decaying 'promptly' on timescales of $\sim 10^{-12}$ s.

Some are noticeably longer lived, and are called *isomers*, *e.g.*

- $^{234\mathrm{m}}_{91}\mathrm{Pa}$ exists 80 keV above ground state & has a mean life of 101 s.
- $^{180\mathrm{m}}_{73}\mathrm{Ta}$ has a mean life of at least 10^{15} years !

The semi-empirical mass formula (SEMF)

We will seek to describe the main characteristics of the observed nuclides, in particular the binding energy, through the semi-empirical mass formula (SEMF).

Proposed by Bethe and Weizsäcker in 1935, taking as starting point the liquid-drop model developed by George Gamow in 1930.

The SEMF seeks to parametrise the binding energy vs. mass dependence in terms of the underlying physics that we know should be there.

The coefficients of the model are fitted from the data, but for many we can calculate expected values from first principles.

It is not a complete quantum-mechanical description, and there are many nuclear properties it cannot explain.



George Gamow (1904-1968); born in Russia, defected to the West in 1933



Hans Bethe (1906-2005); Nobel Prize 1967; German-American; worked on Manhattan project



Carl Friedrich von Weizsäcker (1912-2007); worked on 'German bomb' under Heisenberg

Towards the semi-empirical mass formula (SEMF): the liquid-drop model

Nuclei have roughly constant binding energy / nucleon;

Nuclei have roughly constant density (recall $r \propto A^{1/3}$)

analogy ⟷

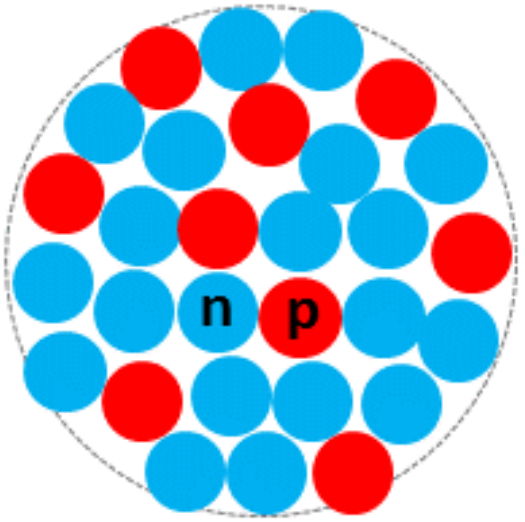
Water drops have constant binding energy / molecule (apart from surface tension effects);

Drop density is constant.

Constant binding energy / water molecule arises because inter-molecular forces are short range.

Implies strong-nuclear force is also attractive, short range and charge independent.

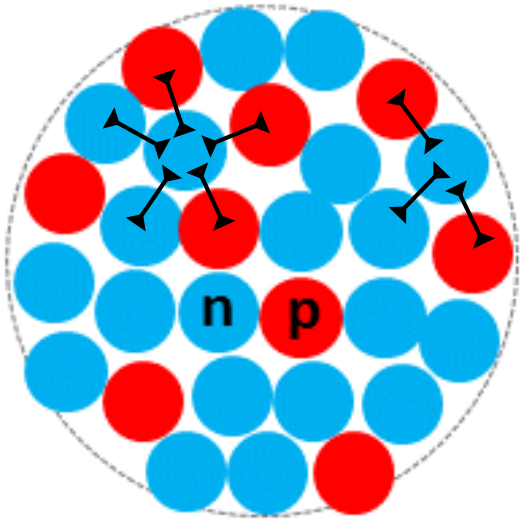
(But at even shorter distances there is a repulsive force preventing nuclear collapse, and hence keeping density constant).



Towards the semi-empirical mass formula (SEMF): the liquid-drop model

So, our first attempt to write down the binding energy consists of two terms:

$$B(Z,A) = a_V A \; - a_S A^{2/3}$$



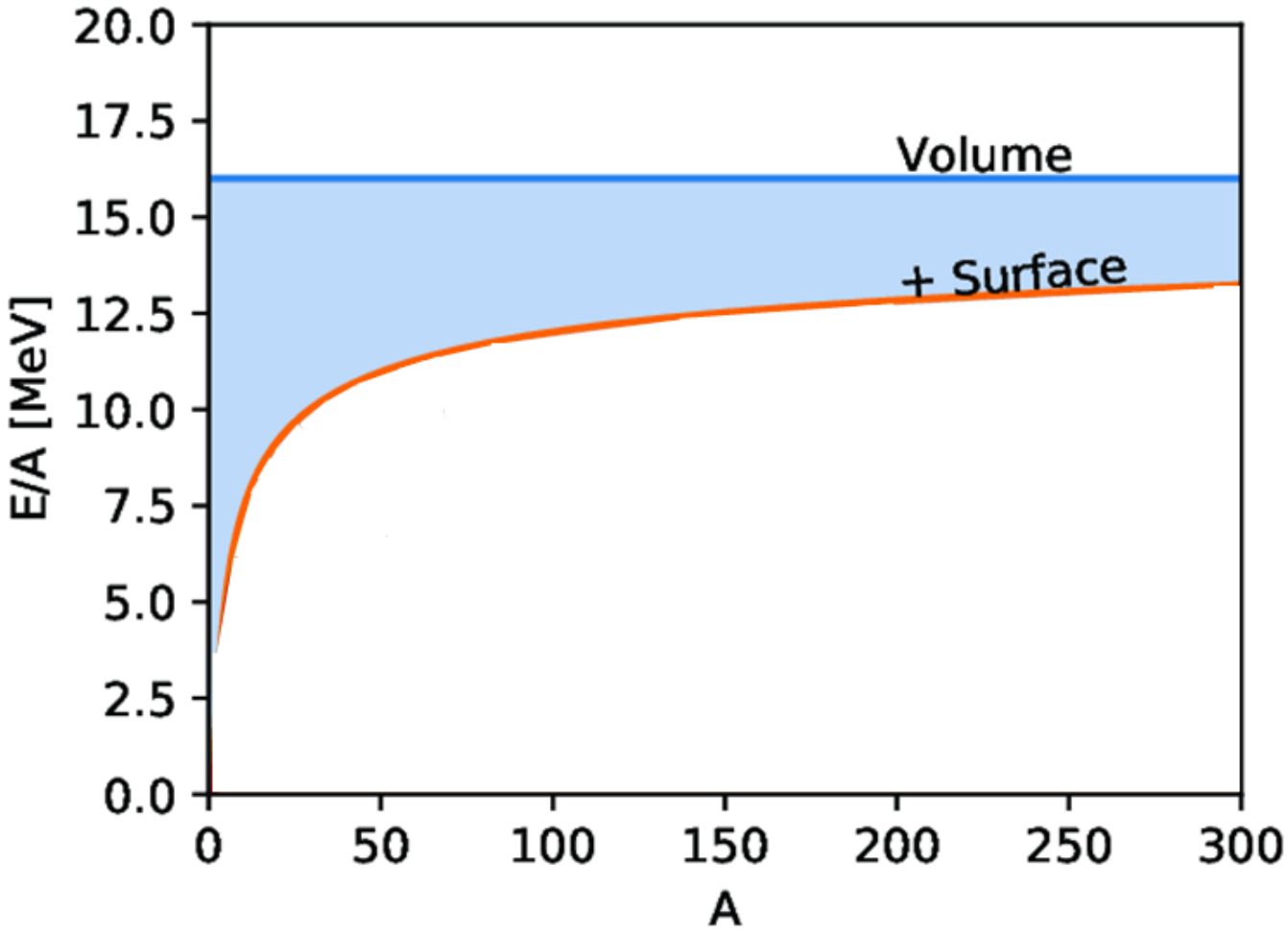
$a_V A$: volume term

Because the force is very short range each nucleon will only interact with its neighbour and the term is $\propto A$ (if the force was long range the dependence would be $\propto A(A-1) \sim A^2$). a_V is set by the strength of the nuclear force, but is fitted from data and is $\approx$ 16 MeV.

$-a_S A^{2/3}$: surface term

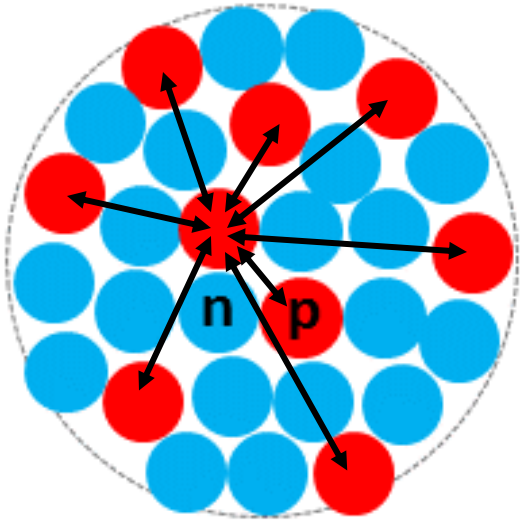
The volume term needs a correction, because nucleons at surface have fewer neighbours. It is proportional to surface area, *i.e.* $\sim r^{\,2} \sim A^{2/3}$. $a_S \approx 17$ MeV.

Towards the semi-empirical mass formula (SEMF): the liquid-drop model



Towards the SEMF: the Coulomb term

Although the nuclear force is dominant, the Coulombic repulsion between protons cannot be neglected.



Know from year-1 electromagnetism that potential energy of a sphere of uniformly distributed charge is given by:

$$U = \frac{3}{5}\left(\frac{Q^2}{4\pi\epsilon_0 r}\right)$$

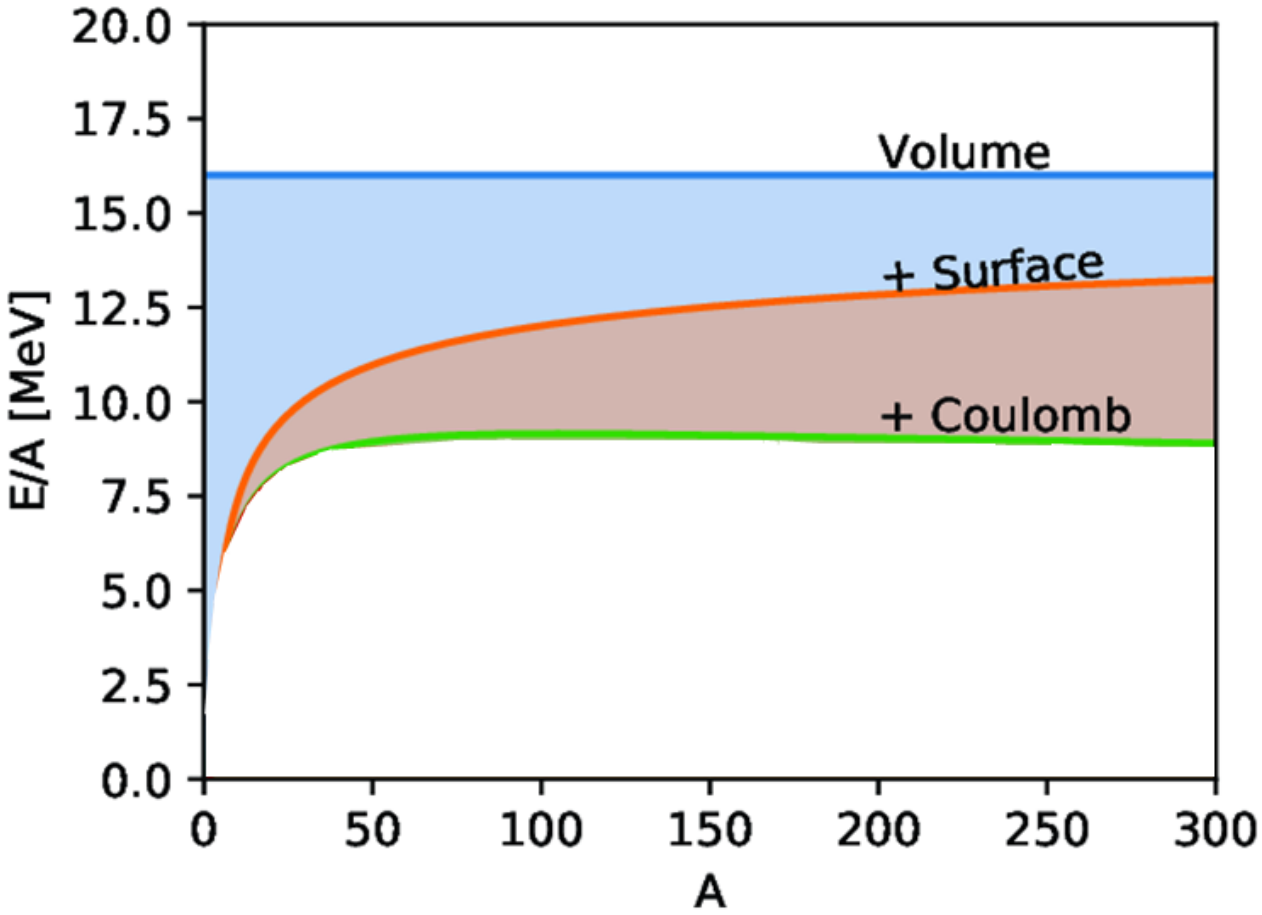
Thus for discrete, but uniformly charge distribution the correction to the binding energy will be negative (repulsive force) and of a size

$$a_c \frac{Z(Z-1)}{A^{1/3}} \qquad \text{with} \qquad a_c = \frac{3}{5}\left(\frac{e^2}{4\pi\epsilon_0 r_0}\right)$$

Where the coefficient can be calculated to be $a_c = 0.72$ MeV with $r_0 = 1.2$ fm. Because $Z(Z-1) \sim Z^2$ for most nuclei, & because the SEMF has other terms proportional to Z (see next slides) the Coulombic term is usually simplified to Z^2. Hence we now have:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_c Z^2/A^{1/3}$$

Towards the SEMF: the Coulomb term



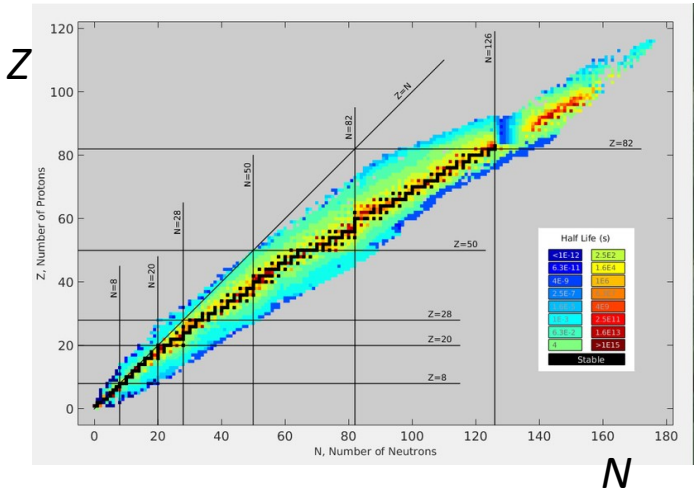
Things are improving, but the formula $B(Z,A) = a_V A - a_S A^{2/3} - a_c Z^2/A^{1/3}$ is clearly inadequate, as it predicts $Z = 0$, which is not realised in the physical world. To complete SEMF we need to add two other terms, quantum-mechanical in nature.

Towards the SEMF: the asymmetry term

At first order nuclides have $N \approx Z$.

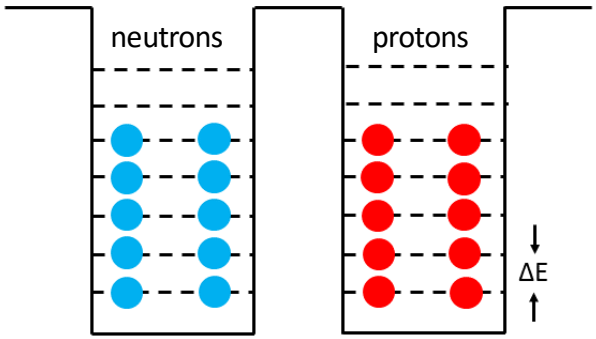
There is clearly a correction coming to this, causing $N > Z$, which becomes more important at large *N*,*Z*. This, of course, is driven by the Coulomb repulsion.

But why the first order effect? Neutrons and protons are fermions, populating separate potential wells, obeying Pauli exclusion principle.



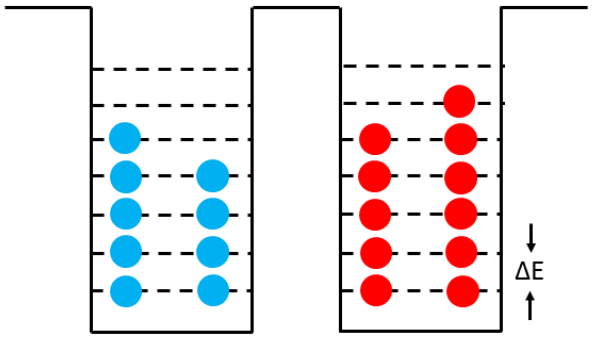
Assume equal energy spacings in both wells (although in reality spacing decreases with *A*). Two fermions allowed in each level. Take as example *A=20*, & consider three possibilities:

(N,Z)=(10,10)



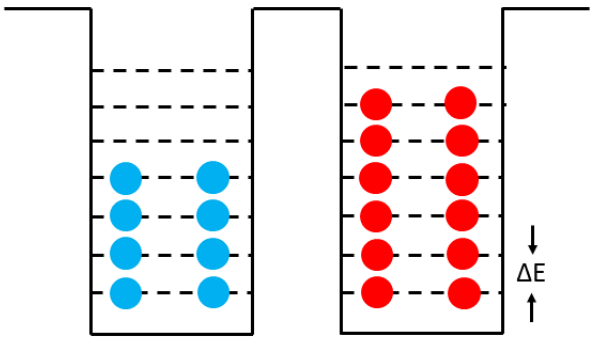
Let's say it requires E_0 to populate this configuration

(N,Z)=(9,11)



Then this requires $E_0 + \Delta E$ (here neglecting Coulomb)

(N,Z)=(8,12)



And this requires $E_0 + 2\Delta E$

→clear penalty for having |N-Z|>0 !

Towards the SEMF: the asymmetry term

Let us estimate the strength of this asymmetry term, by using the Fermi-gas model of the nucleus. As usual, we take the nucleus to be a sphere of volume $V = \frac{4}{3}\pi\left(r_0 A^{1/3}\right)^3$.

Consider first the neutrons. The number of filled states Is the neutron number *N*:

$$N = \frac{4\pi p_F^3}{3}\frac{V}{(2\pi\hbar)^3} \times 2$$

Here p_F is the maximum momentum available, sometimes called the *Fermi momentum*, & the factor 2 accounts for the possibility of spin-up and spin-down states. This becomes:

$$N = \frac{4p_F^3 r_0^3 A}{9\pi\hbar^3} \qquad p_F = \left(\frac{9\pi N}{4A}\right)^{1/3}\frac{\hbar}{r_0}$$

Now let's make the approximation $N \approx A/2$ and calculate the *Fermi energy* $E_F = p_F^2/2m$:

$$E_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{2/3} = 33.3\ \text{MeV}$$

Here m = 940 MeV/c^2 is the neutron mass, which we will take to be the same as the proton mass in this discussion.

Towards the SEMF: the asymmetry term

Let us pause to calculate the mean kinetic energy of a neutron in this nucleus. This will be given by the energy of each neutron, weighted by a density-of-states factor, divided by the normalised density-of-states factor:

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{1}{2m} \frac{\int_0^{p_F} p^2 \times 4\pi p^2 \frac{V}{(2\pi\hbar)^3} \mathrm{d}p}{\int_0^{p_F} 4\pi p^2 \frac{V}{(2\pi\hbar)^3} \mathrm{d}p} = \frac{3}{5} E_F$$

The total energy of the nucleus will receive contributions from this and the proton KE:

$$E_{\mathrm{tot}} = \frac{3N}{5} E_F + \frac{3Z}{5} E_F$$

which for, say, a $^{24}_{12}\mathrm{Mg}$ nucleus will give 480 MeV, or 20 MeV per nucleon.

Towards the SEMF: the asymmetry term

Not let us evaluate how the total energy changes if $N \neq Z$.

$$E_{\text{tot}} = \frac{3}{5}\frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi N}{4A}\right)^{2/3} N + \frac{3}{5}\frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi Z}{4A}\right)^{2/3} Z$$

$$= \frac{3}{5}E_F\left(\frac{2}{A}\right)^{2/3}\left(N^{5/3} + Z^{5/3}\right)$$

If we write $\delta = (Z - N)/A$ this becomes

$$E_{\text{tot}} = \frac{3}{5}E_F\frac{A}{2}\left((1-\delta)^{5/3} + (1+\delta)^{5/3}\right)$$

which when expanded for small δ gives

$$\boxed{E_{\text{tot}} = \frac{3}{5}E_F A\left(1 + \frac{5}{9}\frac{(A-2Z)^2}{A^2}\right)}$$

Towards the SEMF: the asymmetry term

We have:

$$E_{\text{tot}} = \frac{3}{5} E_F A \left(1 + \frac{5}{9} \frac{(A-2Z)^2}{A^2} \right)$$

So indeed the total energy rises in the $N \neq Z$ case, thus reducing the binding energy.

The change in binding energy is given by:

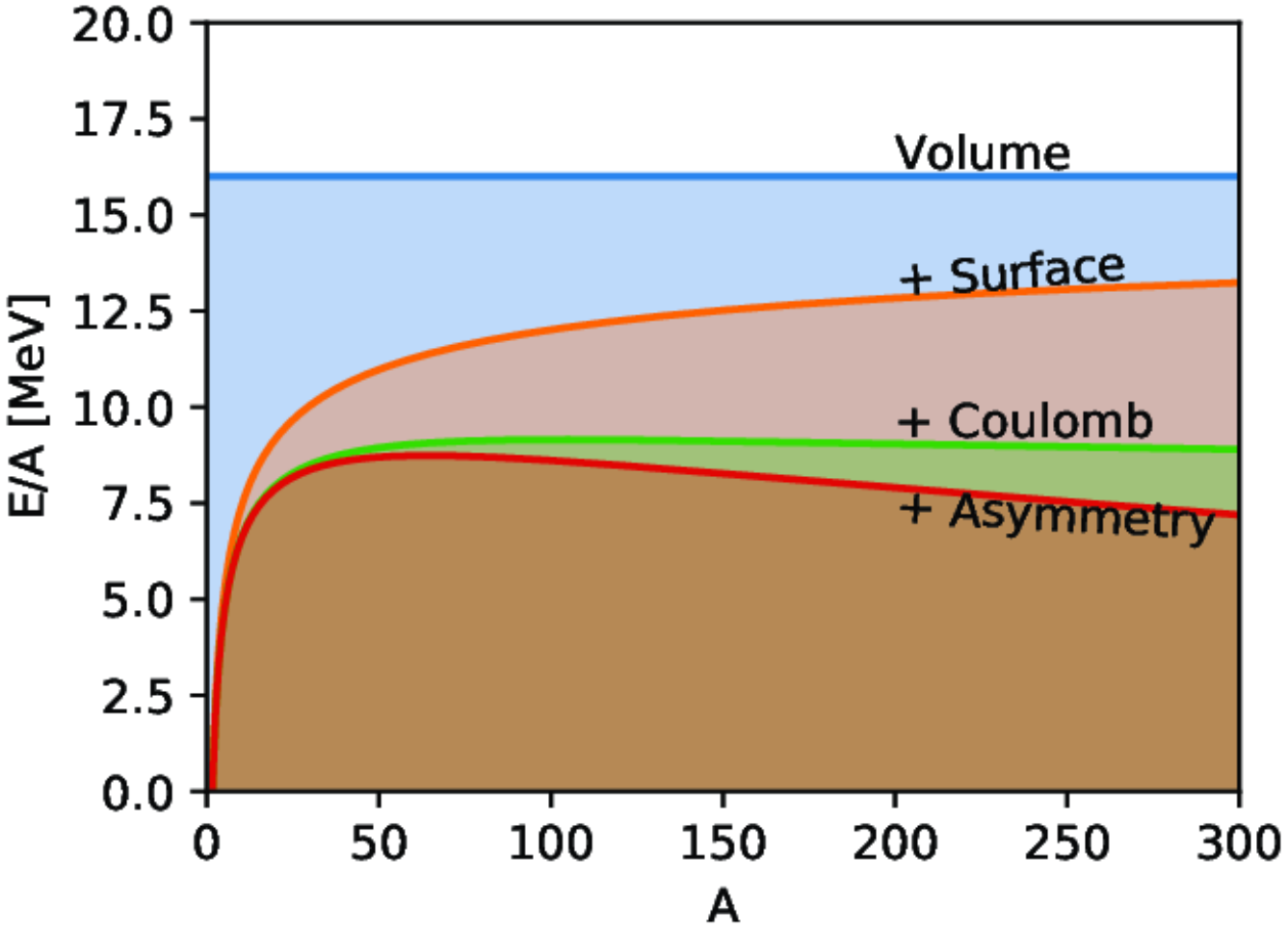
$$-a_A \frac{(A-2Z)^2}{A}$$

with

$$a_A = \frac{1}{3} E_F$$

The numerical value of a_A from this derivation is around 10 MeV, which is about a factor of two smaller than the empirically fitted value. We can attribute this difference to the fact that we have only considered the difference in kinetic energy from this asymmetry term. In fact the total potential energy is also changed by $N \neq Z$.

Towards the SEMF: the asymmetry term



So we now have $B(Z,A) = a_V A - a_S A^{2/3} - a_c Z^2/A^{1/3} - a_A (A-2Z)^2/A$

Completing the SEMF: the pairing term

Two like nucleons in the same spatial state but different spin states will couple pairwise to give spin-zero configurations. These pairs are more tightly bound than unpaired nucleons.

Hence, we expect:

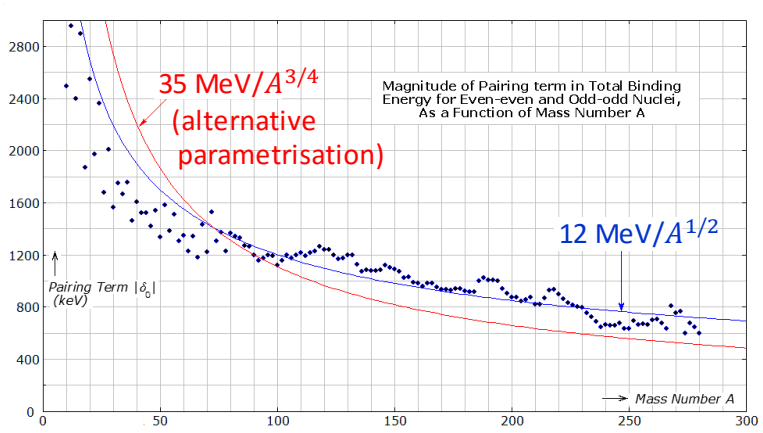
even N, even Z — to be more tightly bound than — even N, odd Z or odd N, even Z — to be more tightly bound than — odd N, odd Z

Indeed there are only four `odd-odd' stable nuclei ($^2$H, $^6$Li, $^{10}$B, $^{14}$N), and >150 `even-even'.

Parametrise in SEMF by adding a *pairing term* that is 0 for `even-odd (eo)/odd-even (oe)' case & positive for tightly bound 'even-even (ee)' & negative for loosely bound 'odd-odd (oo)'.

Empirical fit to data shows a slow A dependence, with ~12 MeV $/ A^{1/2}$ doing a decent modelling of inputs.

$$+\delta(Z,A) = \begin{cases} -12\ \text{MeV}/A^{1/2} & \text{oo} \\ 0 & \text{oe/eo} \\ +12\ \text{MeV}/A^{1/2} & \text{ee} \end{cases}$$



And so finally

$$B(Z,A) = a_V A - a_S A^{2/3} - a_c Z^2/A^{1/3} - a_A(A-2Z)^2/A + \delta(Z,A)$$

Cross checks: mirror nuclei

The SEMF assumes that strong force does not distinguish between protons and neutrons, and so apart from the Coulomb term, we should see essentially perfect symmetry in behaviour of *mirror nuclei*: pairs of nuclides with *(Z,N)=(x,y)* and *(Z,N)=(y,x)*.

Some odd-A mirror nuclei examples: $^{7}_{3}\mathrm{Li}$ and $^{7}_{4}\mathrm{Be}$, $^{11}_{5}\mathrm{B}$ and $^{11}_{6}\mathrm{C}$, $^{19}_{9}\mathrm{F}$ and $^{19}_{10}\mathrm{Ne}$ …

If we evaluate difference in binding energy between two such mirror nuclei, everything apart from the difference between the Coulomb terms should cancel:

$$\Delta B(Z,A) = a_c\left((Z+1)^2 - Z^2\right)/A^{1/3}$$

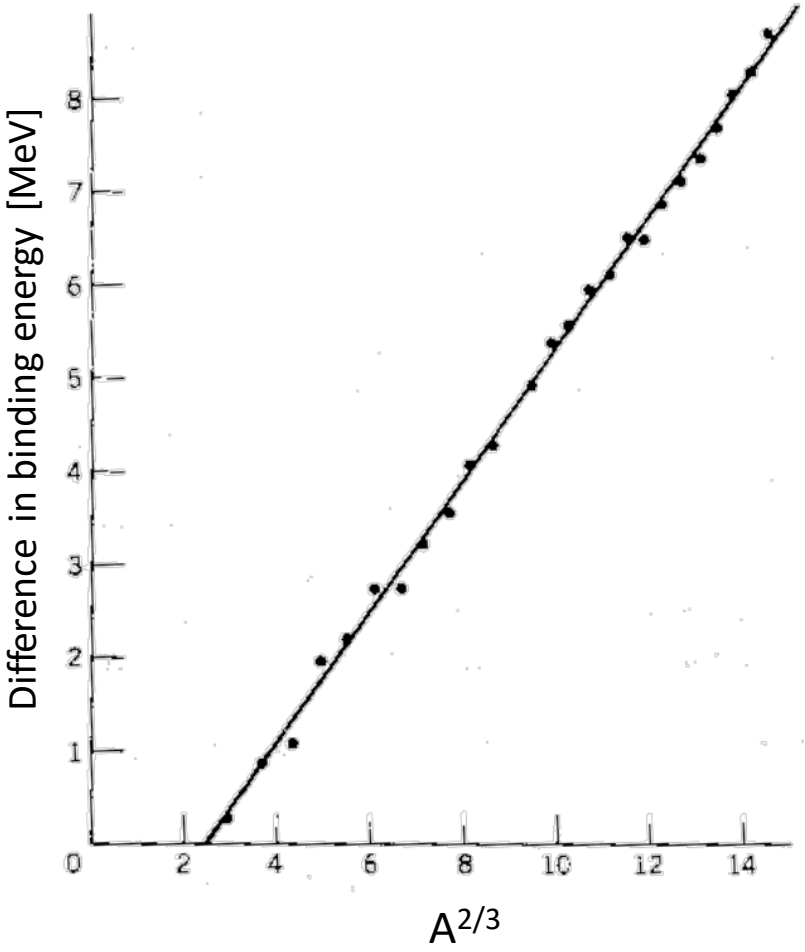
$$\approx a_c A^{2/3}$$ where we have made the rough approximation $Z \sim A/2$.

Also recall that $a_c = \frac{3}{5}\left(\frac{e^2}{4\pi\epsilon_0 r_0}\right)$.

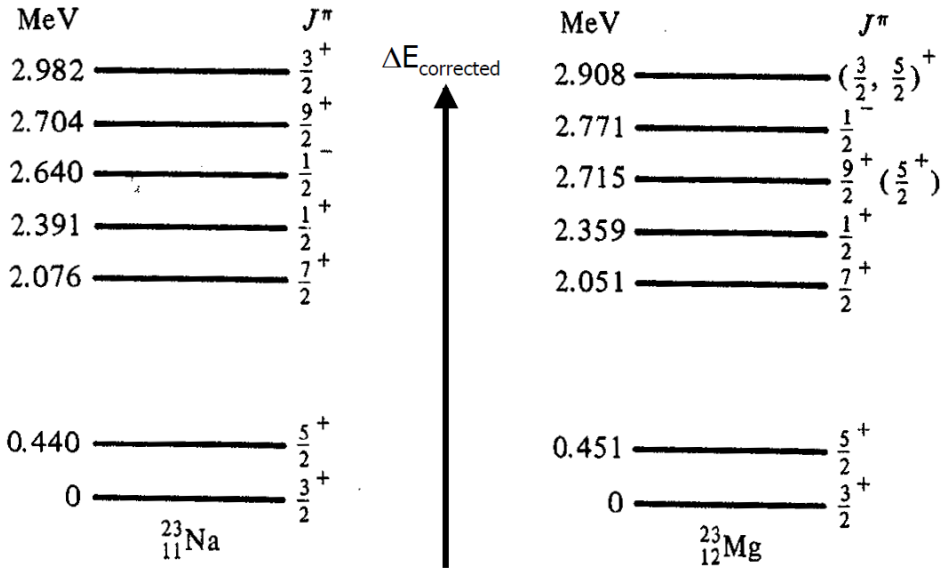
Hence we can check our SEMF assumptions by looking for $\Delta B \propto A^{2/3}$ dependence, and if we find this, use the fit parameters to make an independent determination of r_0.

Cross checks: mirror nuclei

Yes, the differences in binding energies follow $A^{2/3}$, and the slope gives r_0=1.2 fm !



Other properties of mirror nuclei, *e.g.* energy levels of excited states, look very similar once corrected for Coulombic and small m_p, m_n effects.



This validates *charge symmetry* – that *nn* & *pp* forces are identical. Studies of triplets of nuclei *e.g.* $^{6}_{2}$He, $^{6}_{3}$Li and $^{6}_{4}$Be can validate that *np* force is also the same → complete *charge independence*.

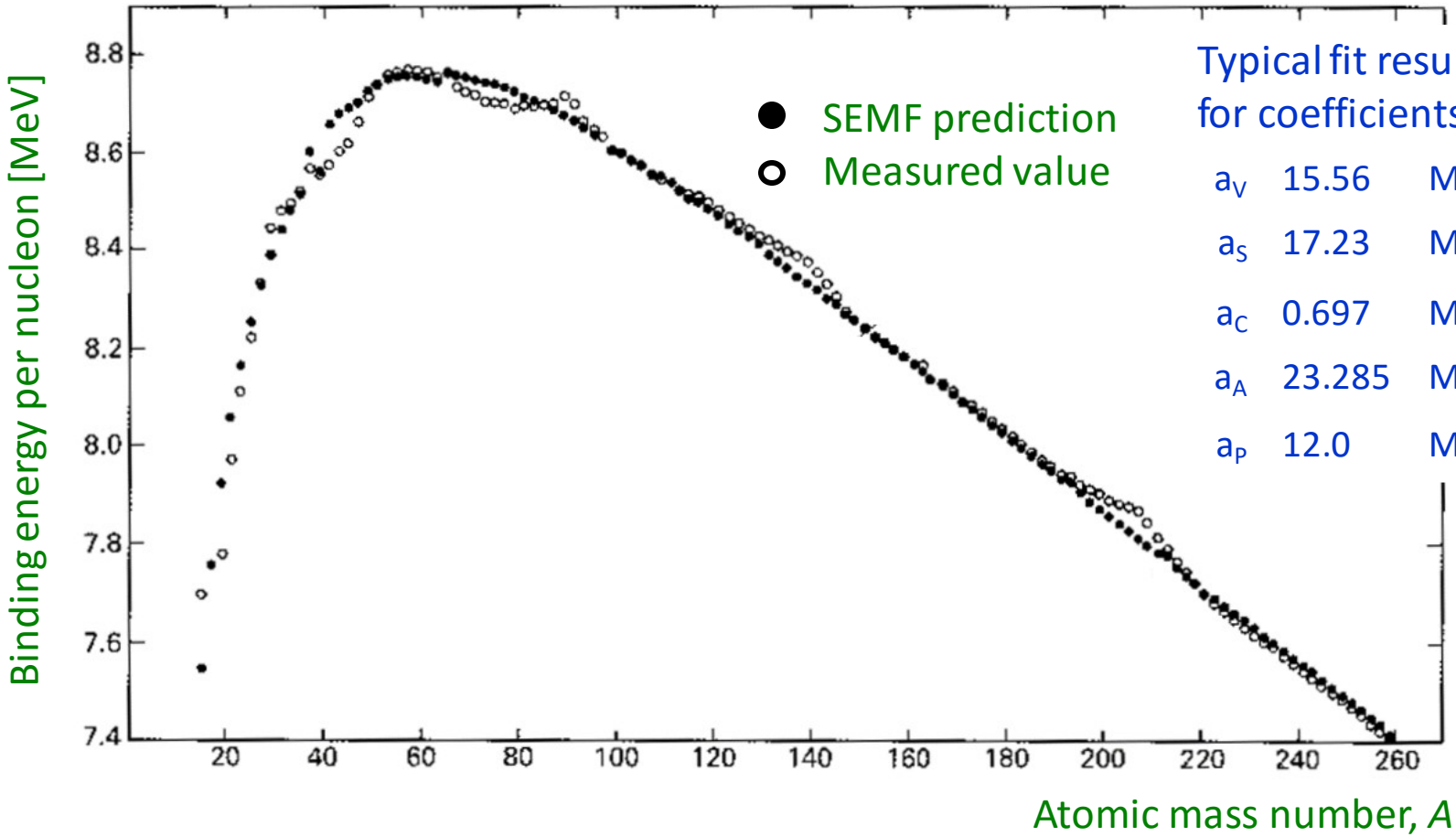
SEMF: a strange beast

The SEMF is certainly not an elegant model, but it in general does a good job.



SEMF: so how well does it do?

Fit SEMF to data, but typically restrict range to $A>20$, as description poor for light nuclides.
e.g. results here shown for odd-A nuclei from A=15-259.

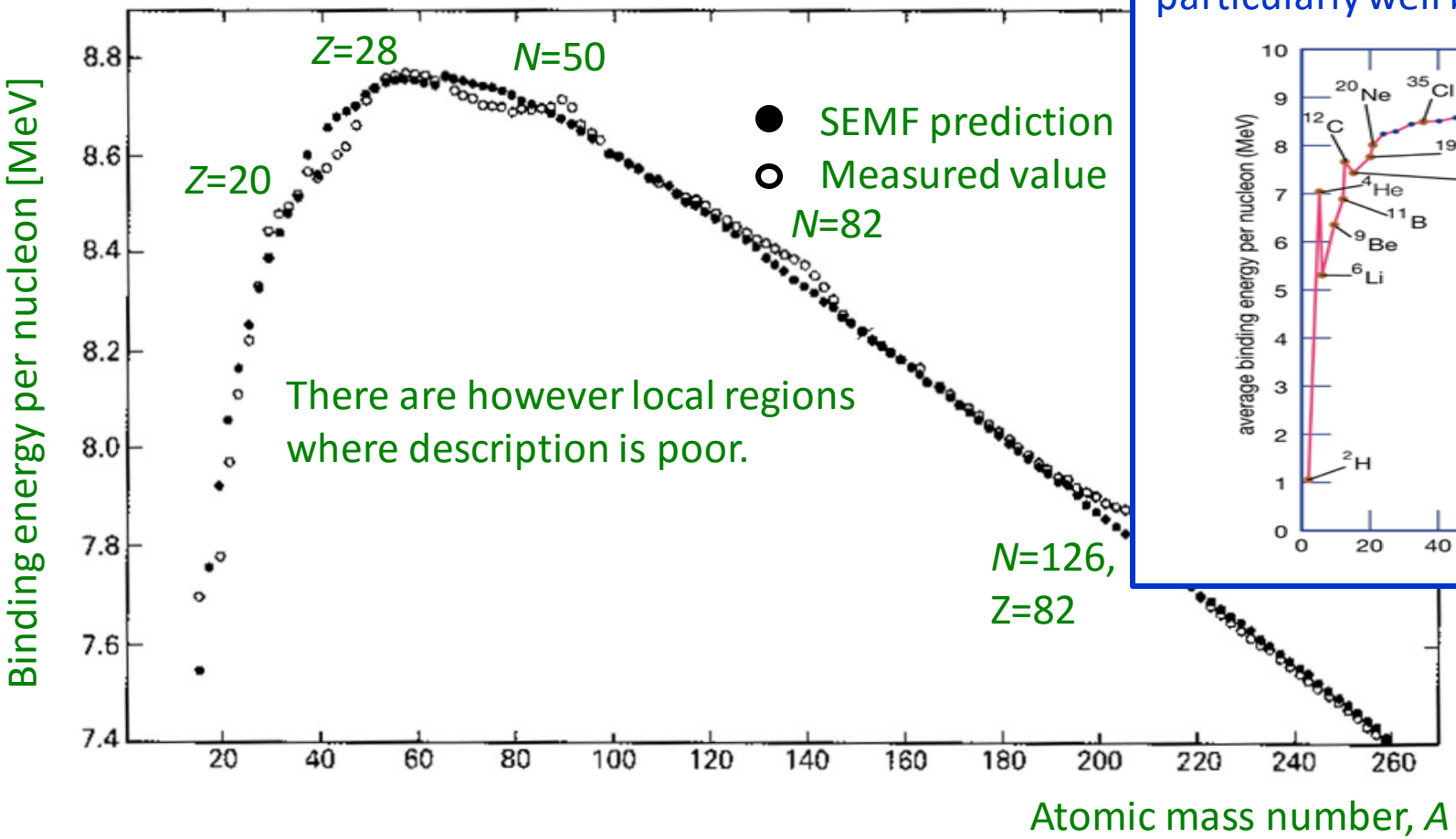


In general SEMF does an excellent job, describing data to within 0.1 MeV.

SEMF: so how well does it do?

Fit SEMF to data, but typically restrict range to $A>20$, as descripti

e.g. results here shown for odd-A nuclei from A=15-259.



Note spiky behaviour at low masses, with ^{4}He particularly well bound.

In general SEMF does an excellent job, describing data to within 0.1 MeV.

Deficiencies of the SEMF

SEMF is a liquid-drop / quantum-mechanical hybrid, which cannot explain:

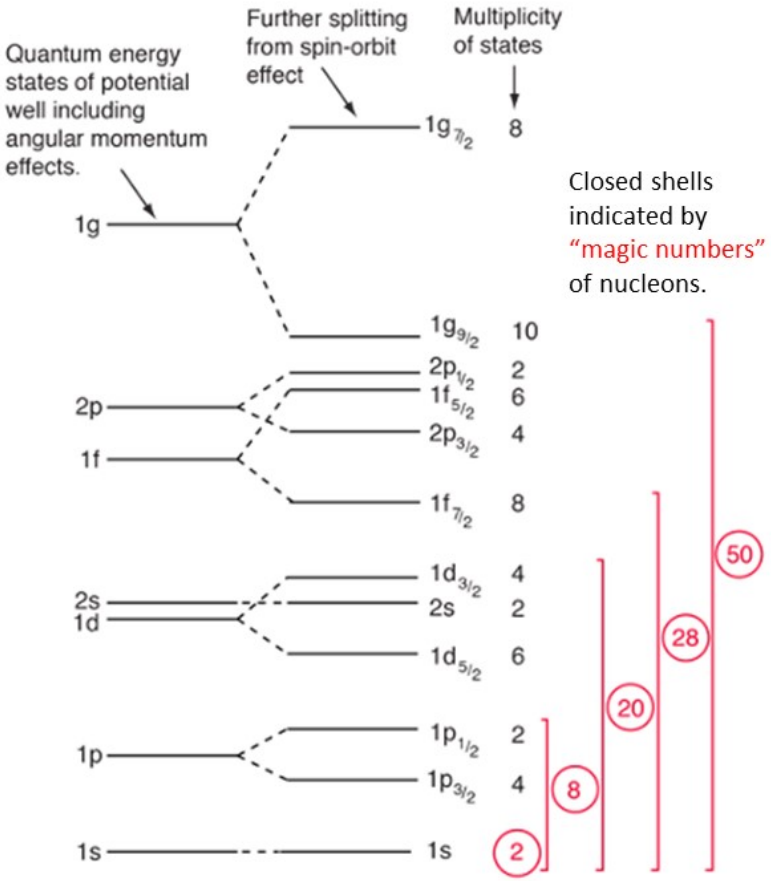
- The very tightly bound nuclides;
- Spin and parity of nuclides;
- Electromagnetic moments;
- *etc…*
- Existence and structure of excited states (although, the liquid-drop model allows for vibrational & rotational degrees of motion);

Some of these properties can be explained by alternative descriptions, *e.g.* the *shell model.*

Not dissimilar to atomic model, that explains periodic table, in particular the inert nobel gases.

- But rather than a well defined Coulomb potential coming from nucleus, here we have to guess a potential coming from other nucleo

- Then accounting for spin-orbit effects we can explain *e.g.* the tightly-bound nuclides. These have closed shells of protons &/or nucleons: *magic numbers* of 2, 8, 20, 28, 50, 82, 126. Nuclides can be *doubly magic*, *e.g.* ^{4}He.



Deficiencies of the SEMF

SEMF is a liquid-drop / quantum-mechanical hybrid, which cannot explain:

- The very tightly bound nuclides;
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- *etc*…
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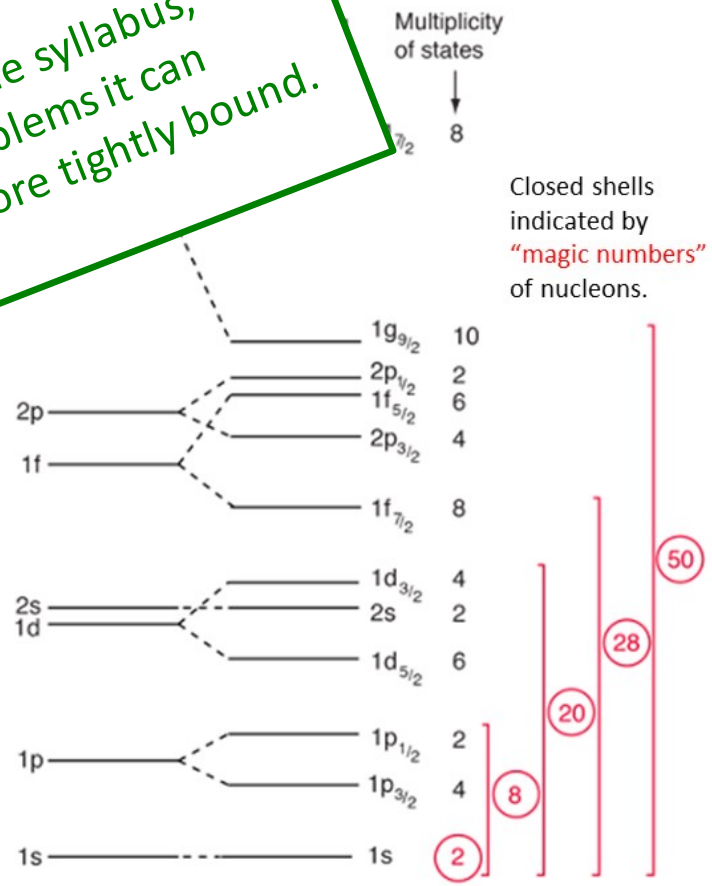
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Note the shell model is no longer on the syllabus, but you should be aware of what problems it can solve, and why some nuclides are more tightly bound.

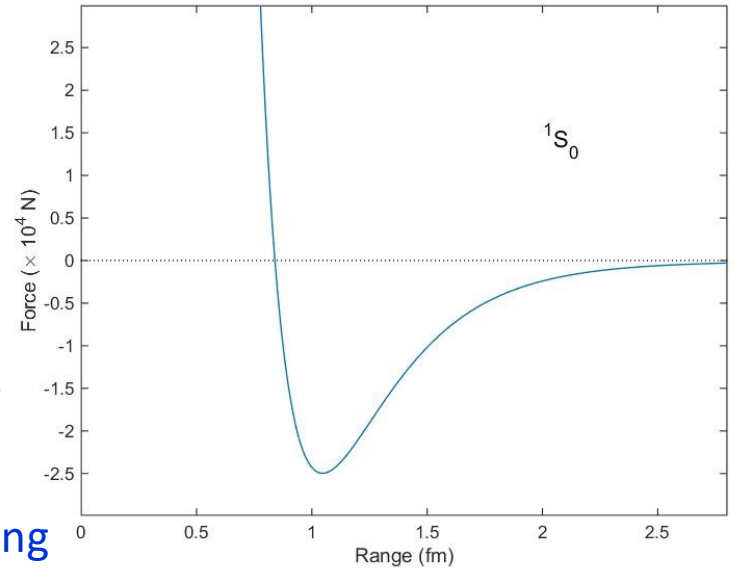


What's really going on? The true nature of the nuclear interaction

So far, we have dodged the question of what the nuclear interaction, *i.e.* the force binding together the nucleus, actually is. Let's briefly address this question, which again goes beyond the syllabus, by first summarising its key characteristics:

- At short distances, ~1 fm, it is attractive and stronger than Coulomb repulsion;
- At longer distances, ~2.5 fm, it is feeble;
- At very short distances, ~0.7 fm, it is repulsive;
- It is very nearly the same for protons and neutrons ('charge independence');
- It does not affect the atomic electrons;
- It has significant spin dependence, *i.e.* sensitive to whether nucleons are parallel / antiparallel;
- It has a non-central or tensor component, leading to a partial non-conservation of angular momentum.

Force between spin-aligned *n* and *p* according to one model (Reid potential)



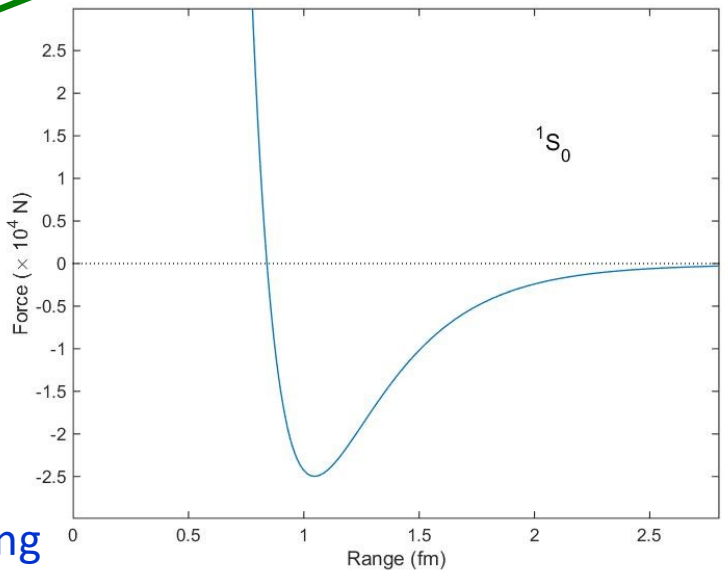
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- At ver[illegible] pulsive;
- [illegible] protons and n[illegible] dependence');
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- It has a non-central or tensor component, leading to a partial non-conservation of angular momentum.

How do we know all this? Mainly from scattering experiments (see later). Much can also be learned from considering deuteron and other possible two-nucleon systems (see problem sheets).

[illegible] spin-aligned [illegible] *p* according to one model (Reid potential)

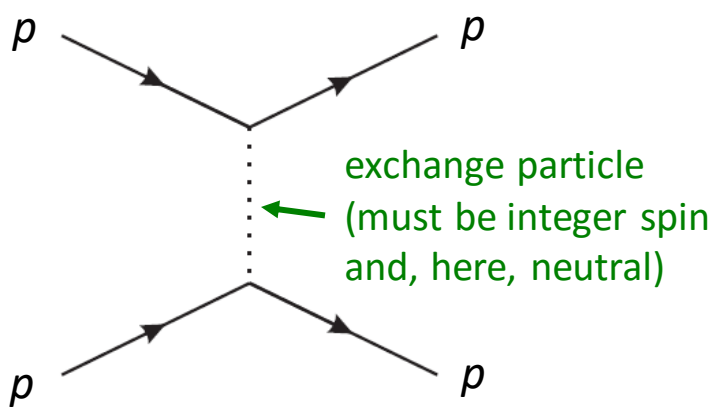


The nuclear interaction as an exchange force

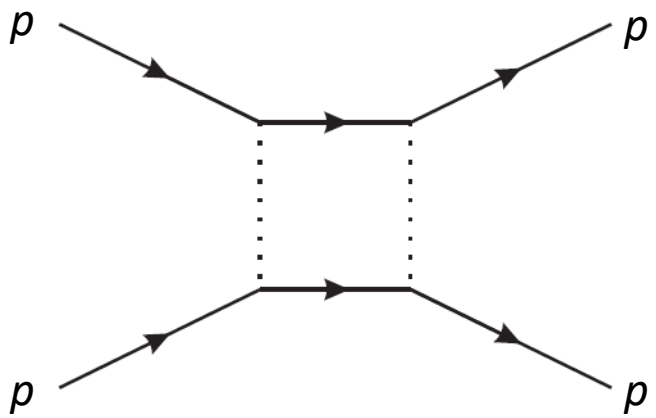
Quantum-field theory describes forces as being mediated by virtual exchange particles. The nuclear interaction can be pictured in such a manner, where the exchange particle, when on-shell, must be massive in order to explain short-range nature of interaction.

Our first exposure to Feynman diagrams. Time runs from left to right. Formally, each diagram represents an order in an expansion. Let's focus on first order contribution.

First order: one exchange particle



Second order: two exchange particles *etc.*

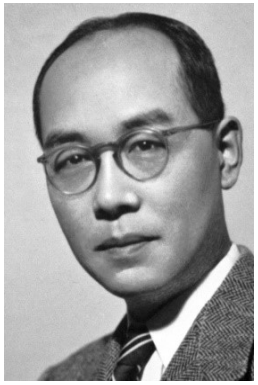


The exchange particles are the *mesons* of particle physics (*e.g.* pions). The nuclear interaction is a *residual force*, or *emergent force*, which is generated by the strong force that binds the quarks in the nucleons together, itself mediated by gluons. Many-body problem of nucleons within nucleus make calculations very difficult, hence important role of models.

The nuclear interaction as an exchange force

Realisation that nuclear interaction can be described as an exchange force first came in 1935 from Hideki Yukawa.

Japanese physicist Hideki Yukawa (1907-1981), Nobel Prize 1949



Describe attractive part of interaction with a Yukawa potential (recall it is repulsive at very short distances):

$$V(r) = \frac{-g^2}{4\pi}\frac{e^{-r/a}}{r}$$

Similar to a Coulomb potential, with two key differences:

- Strength given by $g^2/4\pi$ rather then $e^2/4\pi\varepsilon_0$ with g being the coupling of the exchange particle to nucleon;
- The exponential factor $e^{-r/a}$ limits the range of the force to distance scales of a.

This makes sense. Formally this is a static potential which is a time-independent radial solution $V(r) = \psi(\boldsymbol{r}, t)$ to the Klein-Gordon equation $\left(\nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$.

If we use the uncertainty relation to set an allowed propagation time t for the virtual particle of mass m, we have $t = \frac{\hbar}{mc^2}$. Now exchange particle, assumed relativistic, can travel a distance $x = ct = \frac{\hbar c}{mc^2} = a$, the range of the nuclear interaction ~ 1 fm.

Discovery of the primary exchange particle: the pion

So the distance scale *a* for an exchange particle of mass *m* is given by $\frac{\hbar c}{mc^2}$ = 1 fm.

Time to put in some numbers… a *very* useful numerical trick is to note that $\hbar c = 197$ MeV fm (or $\approx$ 200 MeV fm, if we're working to one significant figure).

$$1 \text{ fm} = 200 \text{ MeV fm } /mc^2$$

So the mass of the exchange particle is m ≈ 200 MeV/c^2. This is in the ball park of the *pion,* which we now know to have a mass of 140 MeV/c^2. Yukawa's hypothesis stimulated efforts to discover the mediating particle as an on-shell, decaying meson.

Success ! The pion discovered by C.F. Powell and others in cosmic rays in 1947.

[Lattes, Muirhead, Occhialini and Powell, Nature 159 (1947) 694]

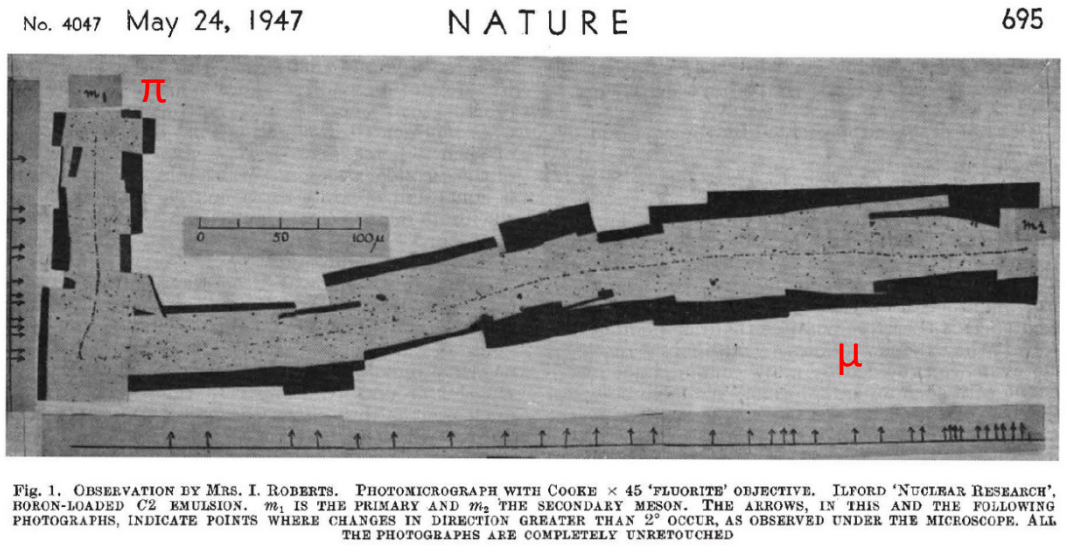
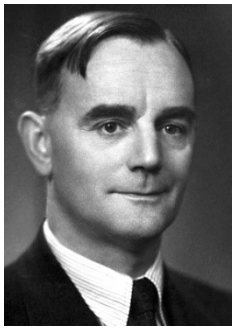


Fig. 1. Observation by Mrs. I. Roberts. Photomicrograph with Cooke × 45 'fluorite' objective. Ilford 'Nuclear Research', boron-loaded C2 emulsion. m_1 is the primary and m_2 the secondary meson. The arrows, in this and the following photographs, indicate points where changes in direction greater than 2° occur, as observed under the microscope. All the photographs are completely unretouched

Cecil Powell (1903-1969), Univ. of Bristol, Nobel Prize 1950



Nuclear physics: topic overview

- Definitions and initial observations
- The Semi-Empirical Mass Formula (SEMF)
- The true nature of the nuclear interaction

- Nuclear decay and radioactivity: an overview
- α decay
- β decay and Fermi theory

- Fission and fusion: an overview
- Induced fission
- Fusion reactions
- Solar fusion and nucleosynthesis

Nuclear decay and radioactivity

Nuclides have positive binding energy and are therefore bound states. These states are however not necessarily stable (only 252 are genuinely so), and many of them can undergo radioactive decay, through the following processes:

- **Alpha decay**

  Occurs when the nucleus emits an alpha particle (= helium nucleus).

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\mathrm{He} $$

  Nuclear fission is closely related to alpha decay, and will be discussed separately.

- **Beta decay**

  Mediated by the weak interaction and has three manifestations:

  β⁻ decay

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + e^{-} + \bar{\nu}_e $$

  ← anti-neutrino

  β⁺ decay

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}Y + e^{+} + \nu_e $$

  ← neutrino

  electron capture (EC)

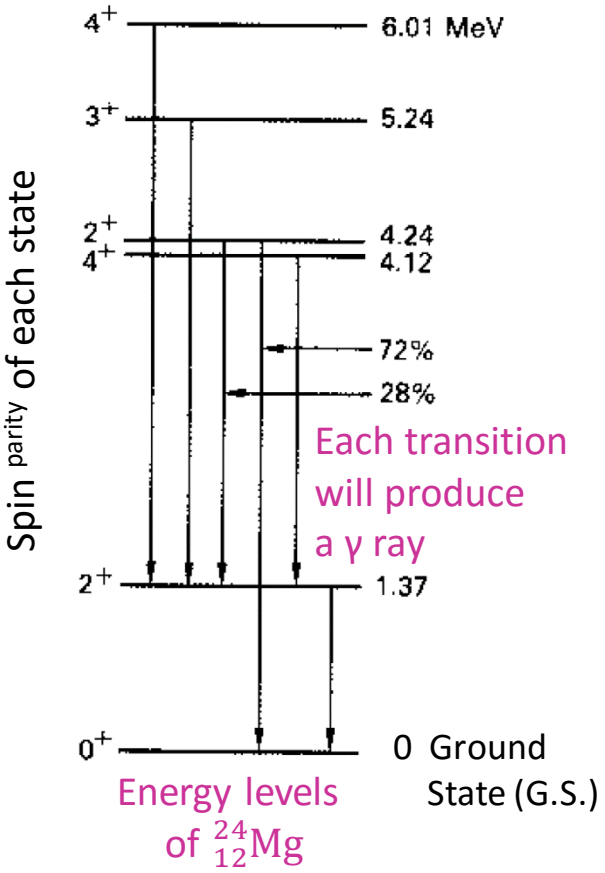
  ← atomic electron (usually captured from K-shell but can come from higher shell)

$$ {}^{A}_{Z}X + e^{-} \rightarrow {}^{A}_{Z-1}Y + \nu_e $$

Nuclear decay and radioactivity

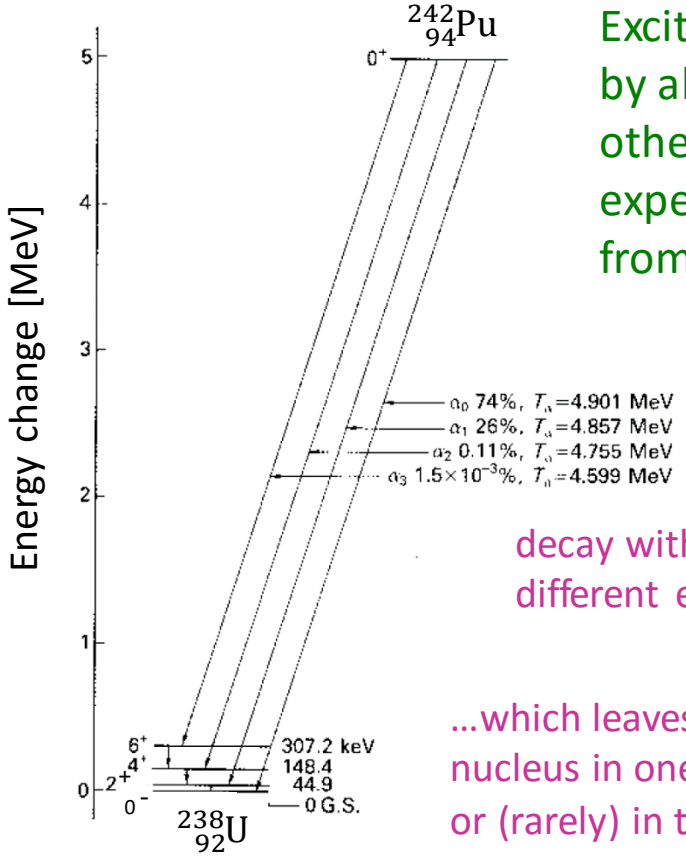
- **Gamma decay** (not on syllabus)

Occurs when an excited nuclear state decays to a lower energy state with emission of a gamma ray, which in this case is a photon of a few keV up to a few MeV.



Each transition will produce a γ ray

Energy levels of $^{24}_{12}\mathrm{Mg}$



Excited states are populated by alpha or beta decay of other nuclides (or, in experiments or reactors, from nucleon capture).

Here we see $^{242}_{94}\mathrm{Pu}$ can undergo alpha decay with transitions of four different energies possible…

…which leaves the daughter $^{238}_{92}\mathrm{U}$ nucleus in one of three excited states or (rarely) in the ground state.

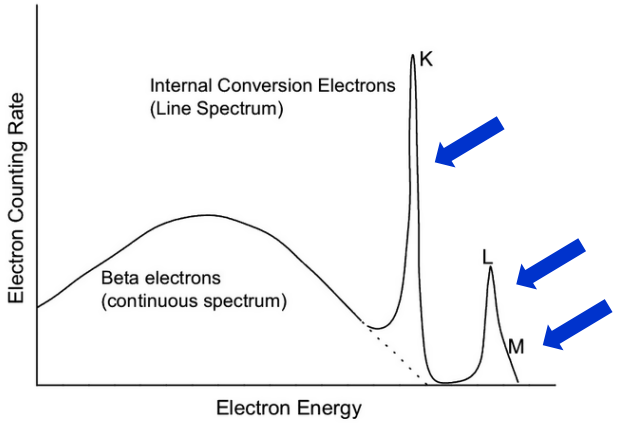
Requires shell model (or similar) for a proper understanding.

Nuclear decay and radioactivity

- **Internal conversion** (not on syllabus)

  An electromagnetic process that competes with γ emission for excited states. In internal conversion an atomic electron is ejected, but there is no transmutation of the nucleus. Not to be confused with β decay !

electron spectrum of ^{203}Hg



- **Spontaneous neutron or proton emission** (not on syllabus)

  Nuclei with an excess of neutrons have a low binding energy (asymmetry term) and can decay through *spontaneous emission* of a neutron. Such nuclei are said to lie beyond the *neutron drip line*.

  e.g. $^{13}\text{Be} \rightarrow {}^{12}\text{Be} + n$, $^{5}\text{He} \rightarrow {}^{4}\text{He} + n$

  Spontaneous proton emission is also possible for proton rich nuclei.

  e.g. $^{53}\text{Co} \rightarrow {}^{52}\text{Fe} + p$

Radioactive decay law

If we have $N(t)$ nuclei at time t, and we define ω as the transition *rate*, which is the probability that a nucleus decays to another state in one second. Then we have:

$$\mathrm{d}N(t) = -\omega N(t)\mathrm{d}t \qquad \boxed{\frac{\mathrm{d}N}{\mathrm{d}t} = -\omega N(t)}$$

$$\Longrightarrow \quad \boxed{N(t) = N(0)e^{-\omega t}}$$

So the mean life $\boxed{\tau = \omega^{-1}}$ and the half-life $\boxed{\tau_{1/2} = \tau \ln 2}$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\omega_1 N(t) - \omega_2 N(t)$$

$$\Longrightarrow \quad N(t) = N(0)e^{-(\omega_1+\omega_2)t}$$

So now the mean life $\tau = (\omega_1 + \omega_2)^{-1}$. For this bimodal decay, the probability of decaying through transition 1 (2) is called the *branching fraction* and is given by.

$$\mathrm{BF}_{1(2)} = \frac{\omega_{1(2)}}{\omega_1 + \omega_2}$$

or generalising to states with many decay paths

$$\boxed{\mathrm{BF}_i = \frac{\omega_i}{\sum_j \omega_j}}$$

Studying radioactive decay

Two units of commonly used units of radioactivity:

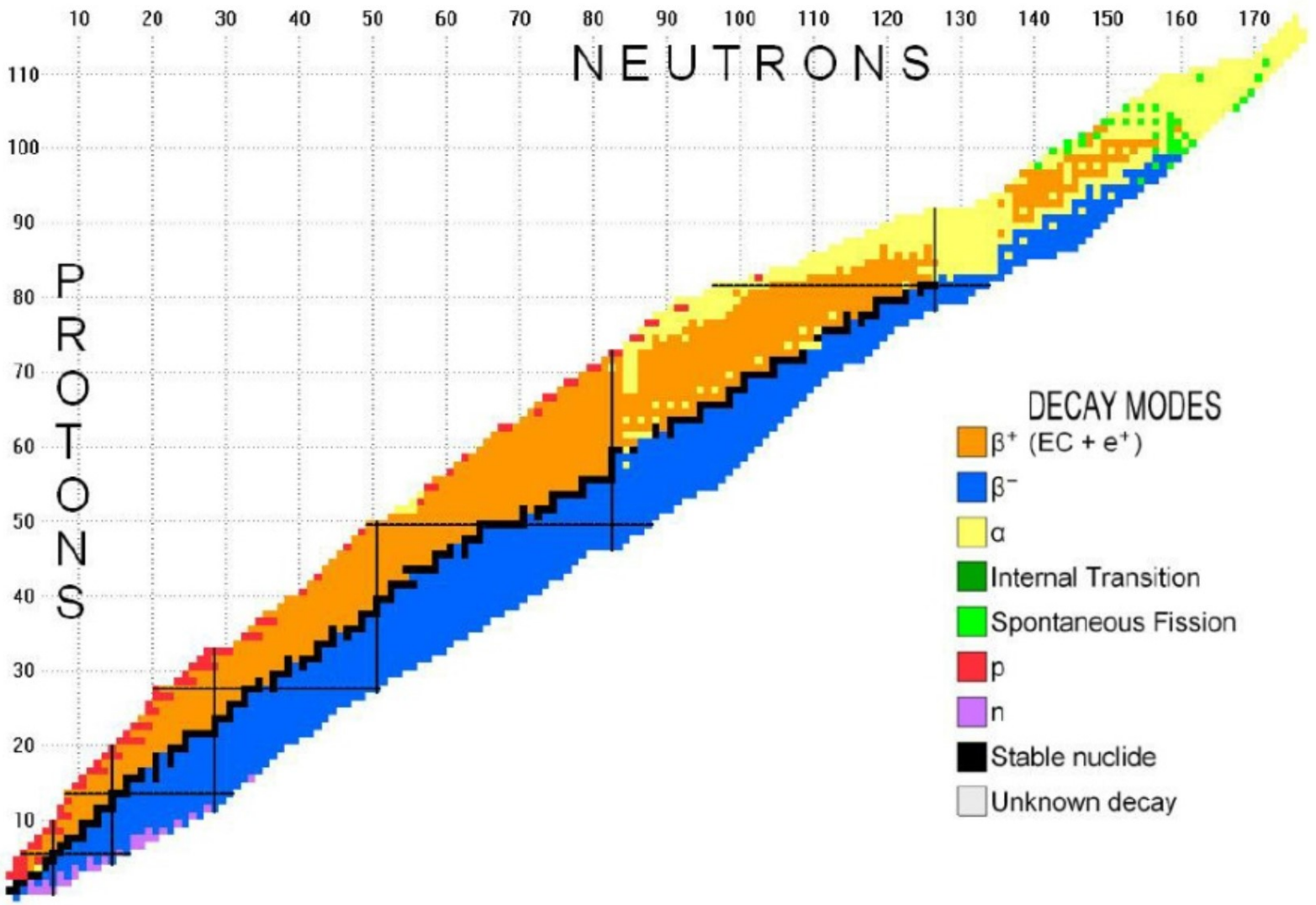
Curie

One curie (Ci) is the amount of radioactive material in which the disintegrations in one second is the same as that of one gram of pure radium which is $3.7 \times 10^{10}\ s^{-1}$.

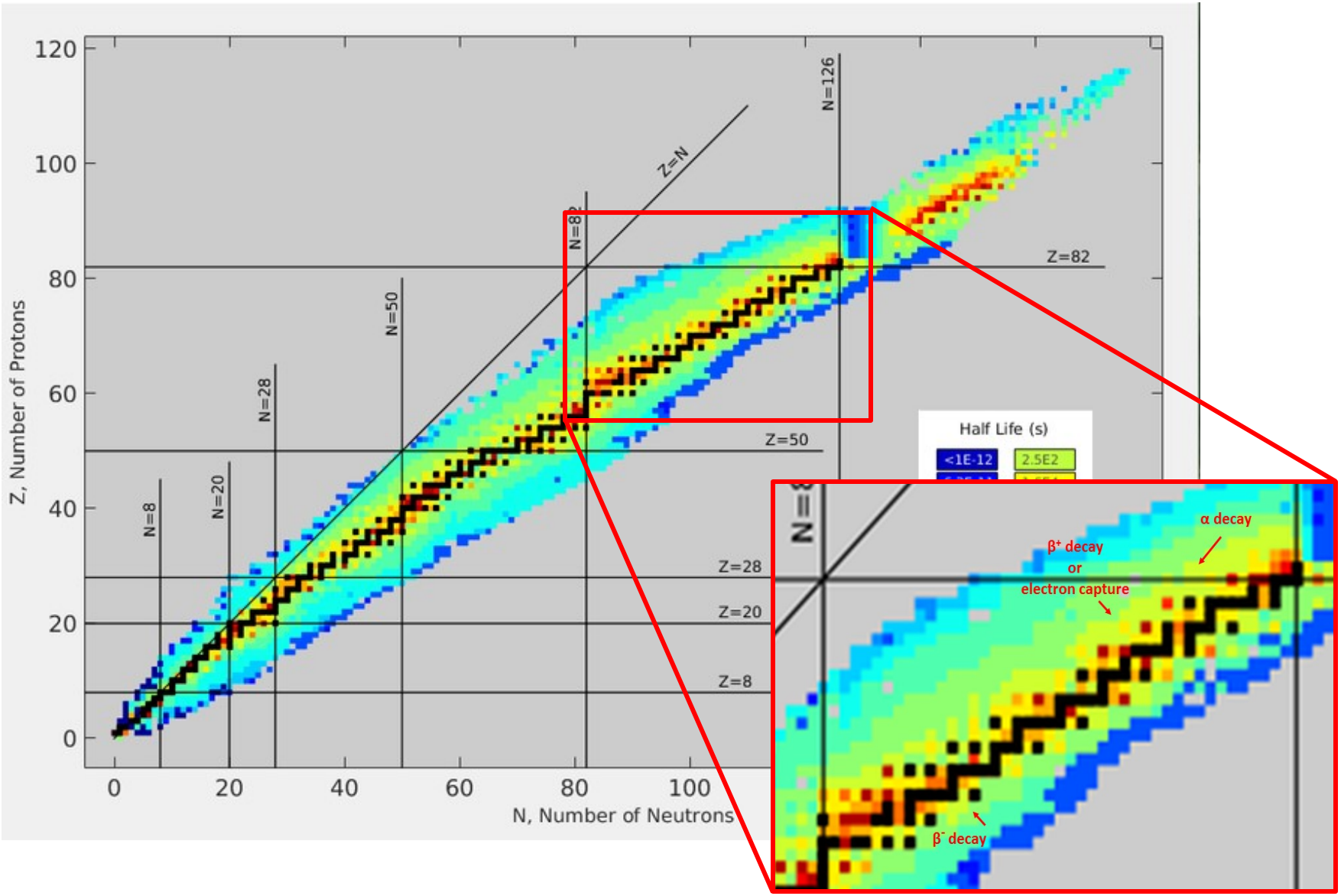
Bequerel

One Becquerel (Bq) is the amount of radioactive material in which the average number of disintegrations in one second is one.

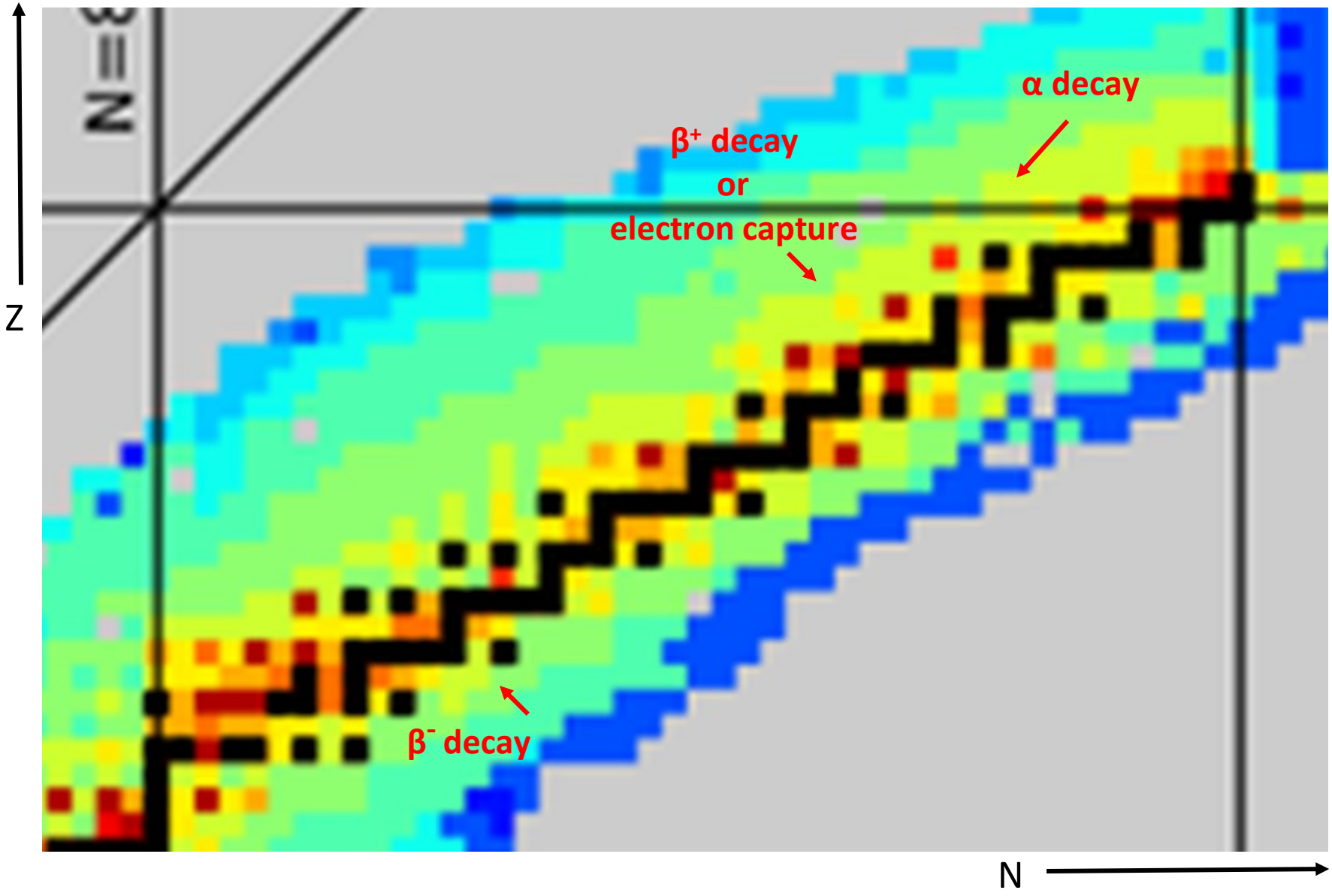
Stability and decay in the Segrè chart



α and β decays as transmutations in the Segrè chart



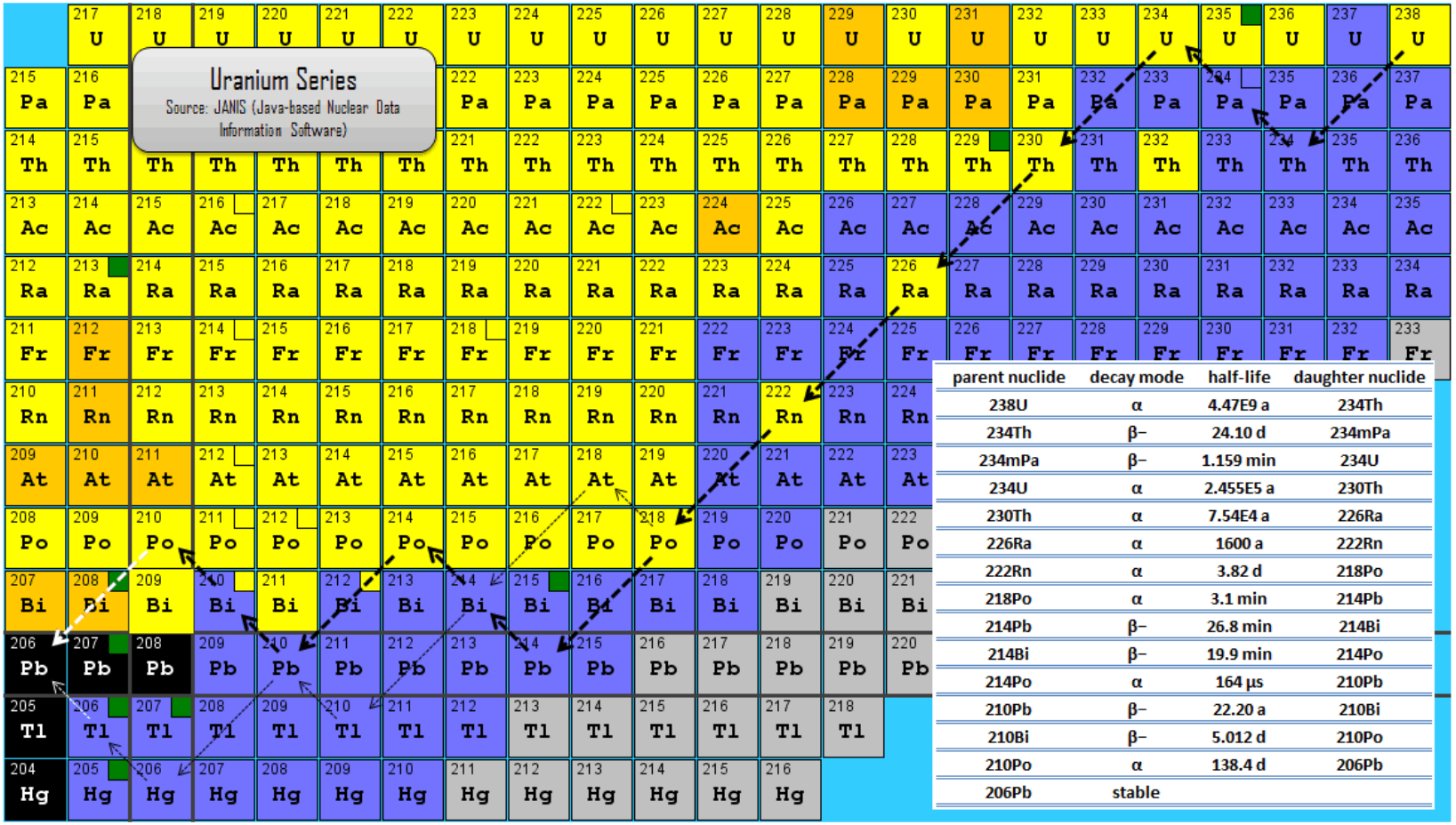
α and β decays as transmutations in the Segrè chart



Radioactive decay chains

Decay chain involving alpha and beta processes allows unstable nucleus to reach stability.

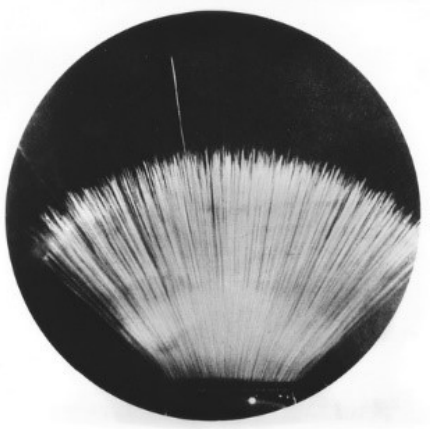
There are three 'classical series' of decay chains (thorium, uranium and actinium) beginning with primordial unstable nuclei. *e.g.* the `uranium series' beginning with ^{238}U.



parent nuclide	decay mode	half-life	daughter nuclide
238U	α	4.47E9 a	234Th
234Th	β−	24.10 d	234mPa
234mPa	β−	1.159 min	234U
234U	α	2.455E5 a	230Th
230Th	α	7.54E4 a	226Ra
226Ra	α	1600 a	222Rn
222Rn	α	3.82 d	218Po
218Po	α	3.1 min	214Pb
214Pb	β−	26.8 min	214Bi
214Bi	β−	19.9 min	214Po
214Po	α	164 µs	210Pb
210Pb	β−	22.20 a	210Bi
210Bi	β−	5.012 d	210Po
210Po	α	138.4 d	206Pb
206Pb	stable		

[from https://www.nuclear-power.com/]

Alpha decay: initial observations



Energy of emitted alpha particles is constant for a given decay (here manifested by tracks of ~ constant length in a cloud chamber) → confirms it is a two-body decay.

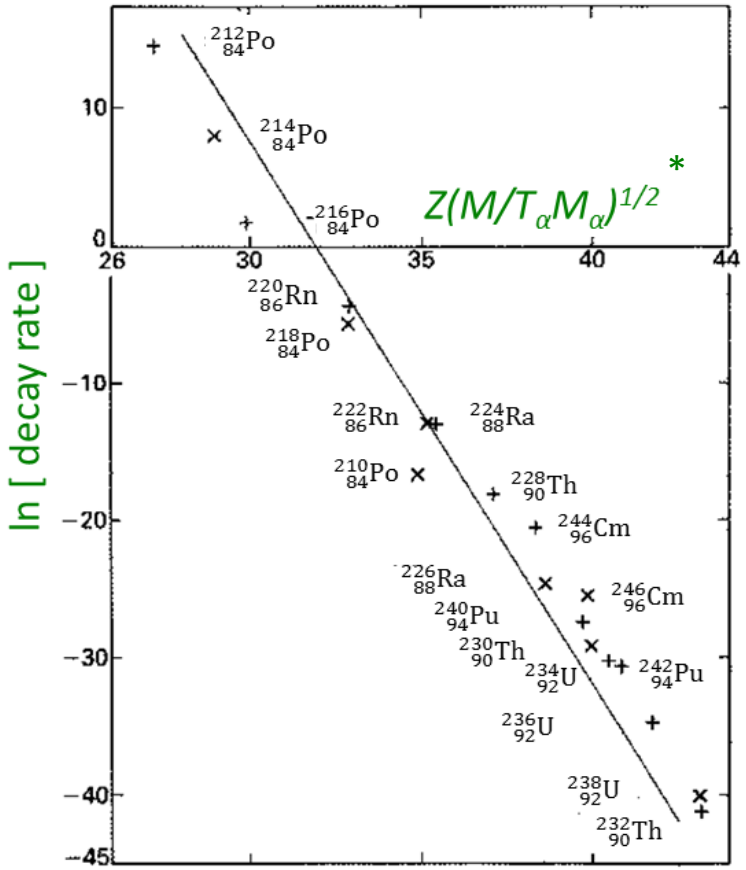
There is an enormous range in lifetime / decay rate, *e.g.* mean lifetime of $^{212}_{84}$Po is 10^{-7} s, and that of $^{232}_{90}$Th is 10^{10} years !

Lifetime has a dependence on energy of alpha particle T_α (= `*Q value*' of decay).

Empirically described by Geiger-Nuttal rule:

$$\log_{10} 1/\tau = -a\, Z/\sqrt{T_\alpha} + b$$

where τ is mean lifetime, *a* and *b* are constants (formulated in 2011 in terms of α path length).



[modified from Williams]

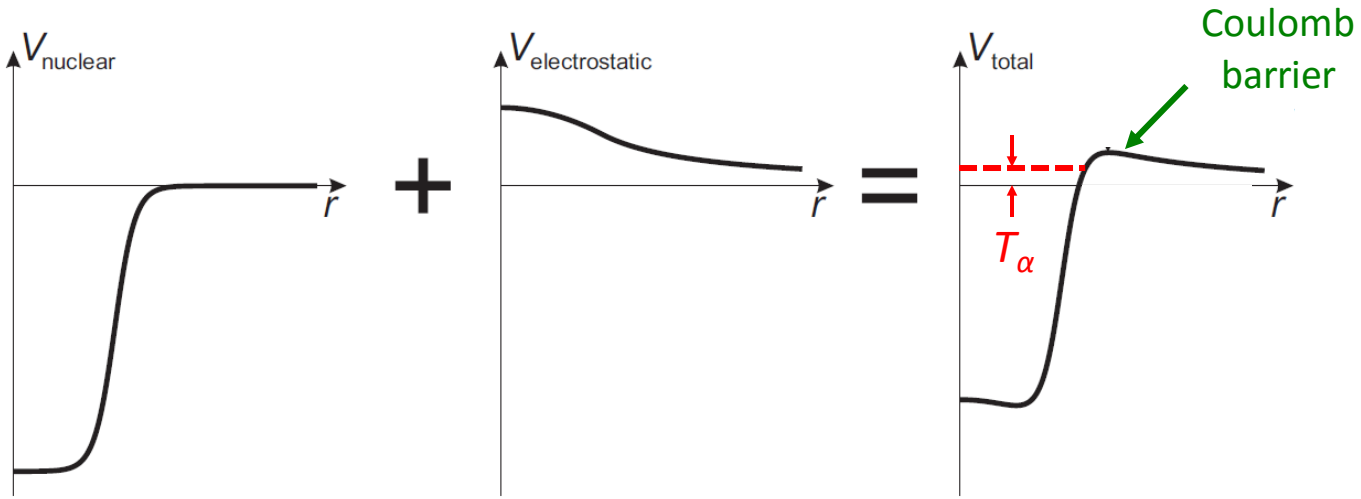
* in this plot M is reduced mass of α – daughter nucleus system ≈ M_α, so horizontal axis is ≈ $Z/(T_\alpha)^{1/2}$

Alpha decay: a quantum-mechanical model

A model to describe alpha decay and explain the Geiger-Nuttal rule was developed in 1928 in Gamow and, independently, by Gurney and Condon.

Suppose protons and neutrons agglomerate within the parent nucleus and form an alpha particle. The high binding energy that is liberated may be sufficient to allow particle to escape. To see if this is feasible we need to consider shape of potential.

Contributions from both nuclear and Coulombic forces:



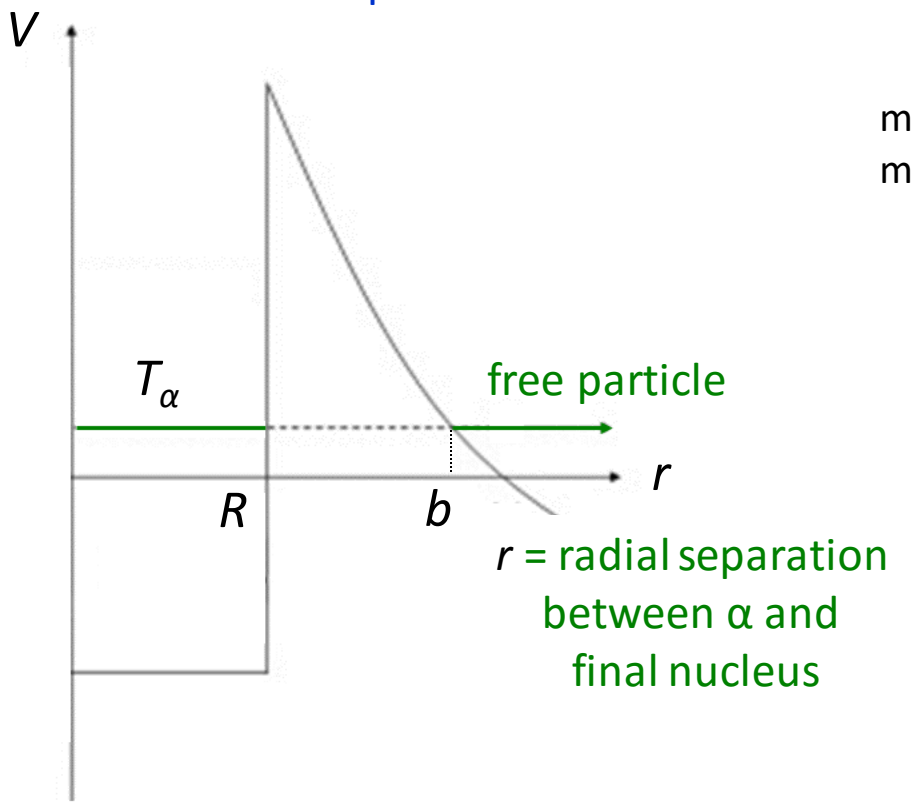
Most alpha emitters have A~200, and so $r = r_0 A^{1/3} \approx 7$ fm. Coulombic potential for such a nucleus with Z~100 is $\sim 2Ze^2/4\pi\varepsilon_0 r \approx 17$ MeV, which is more than the typical Q-value / T_α for most emitters which is <10 MeV. Alpha particle is trapped !

Alpha decay: a quantum-mechanical model

It is possible, however, for the alpha particle to quantum tunnel through the barrier !

To facilitate the calculation, we shall simplify the picture to the superposition of a box potential and a 1/r Coulombic repulsion. We start by recalling 2nd year QM result for transition probability for tunnelling through a 1-D potential wall.

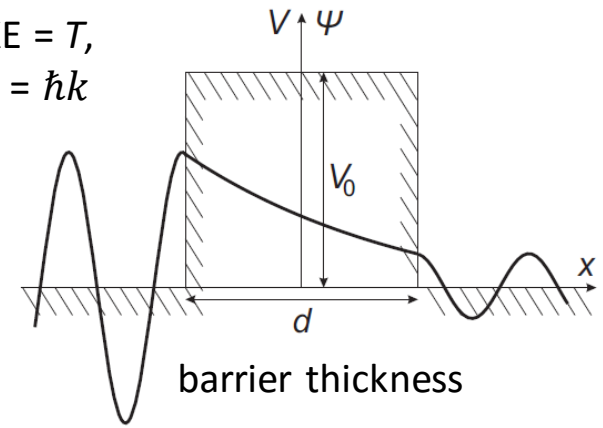
Idealised potential



r = radial separation between α and final nucleus

Reminder of QM barrier-transition probability

mass = *M*, KE = *T*, momentum = $\hbar k$



Transition probability ≈ e^{-2Kd}

where $\hbar K = \sqrt{2M(V_0 - T)}$

in the case $\hbar k \ll \hbar K$.

Alpha decay: a quantum-mechanical model

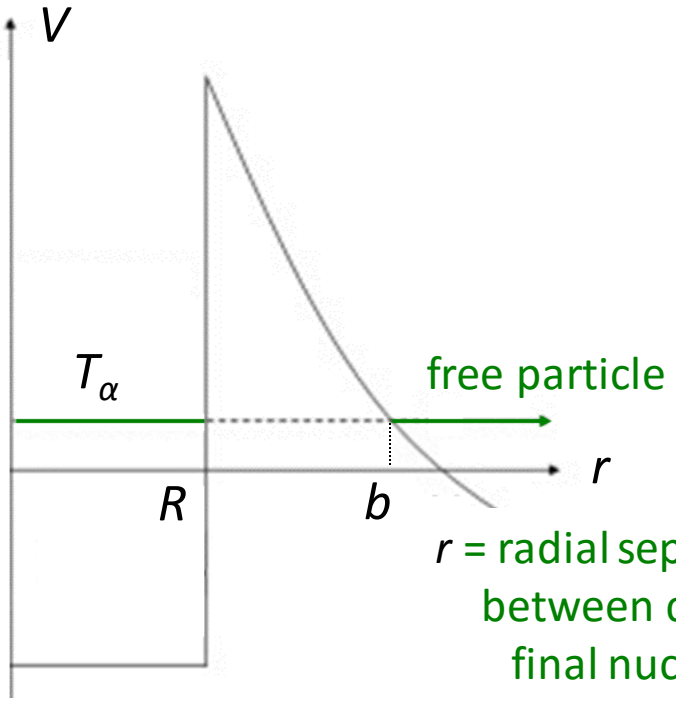
Step 1 - modify transition-probability result to account for Coulombic potential:

$$e^{-2Kd} \longrightarrow e^{-2G}$$

where *G* is the *Gamow factor*

$$G = \int_R^b K \, \mathrm{d}r$$

$$= \int_R^b \frac{\sqrt{2M_\alpha}}{\hbar} \sqrt{[V(r) - T_\alpha)]} \, \mathrm{d}r$$



with $V(r) = 2(Z-2)e^2/4\pi\varepsilon_0 r$

$R = r_0 A^{1/3}$

$b = 2(Z-2)e^2/4\pi\varepsilon_0 T_\alpha$ so $V(r)/T_\alpha = b/r$ & using sub^n. $r = b\,(\cos\theta)^2$

gives

$$G = \frac{\sqrt{2M_\alpha T_\alpha}}{\hbar} \frac{2(Z-2)e^2}{4\pi\varepsilon_0 T_\alpha} \left(\cos^{-1}\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b} - \left(\frac{R}{b}\right)^2} \right)$$

Alpha decay: a quantum-mechanical model

Step 2 - approximate to case $b \gg R$ (*i.e.* T_α small compared to height of barrier):

$$G \approx \frac{\sqrt{2M_\alpha T_\alpha}}{\hbar}\frac{2(Z-2)e^2}{4\pi\varepsilon_0 T_\alpha}\frac{\pi}{2} \approx 2\ \mathrm{MeV}^{1/2}\frac{Z-2}{\sqrt{T_\alpha}}$$

Step 3 - calculate the decay rate ω (or inverse lifetime $1/\tau$):

Frequency at which α particle approaches barrier = f (approximately $v_\alpha/2R$)

Thus:

$$\omega = f e^{-2G}$$

$$\log_{10}\omega = \log_{10} f - 2G\log_{10} e$$

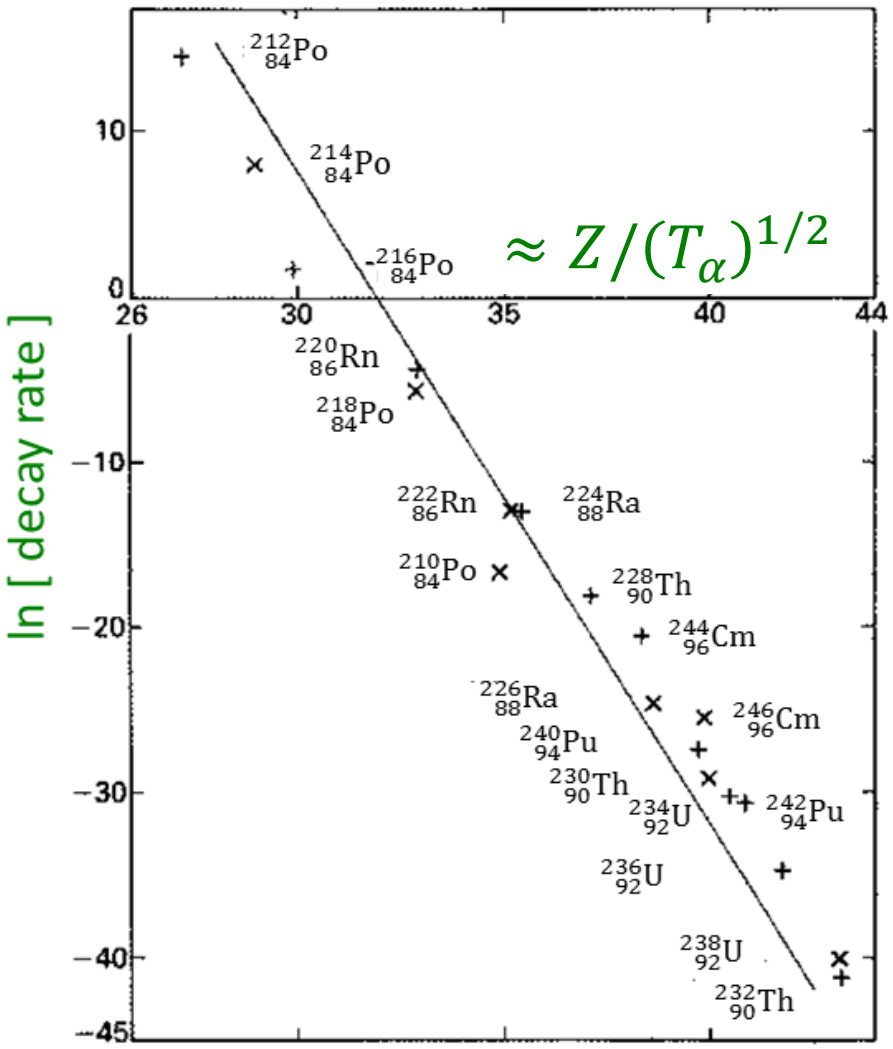
$$\log_{10} 1/\tau = \log_{10} f - 1.7\ \mathrm{MeV}^{1/2}\frac{Z-2}{\sqrt{T_\alpha}}$$

Since $\log_{10} f$ is approximately constant for high *A* α-emitters, & $Z \sim (Z-2)$ we have

$$\log_{10} 1/\tau \approx \text{constant} - 1.7\ \mathrm{MeV}^{1/2}\frac{Z}{\sqrt{T_\alpha}}$$

has form of Geiger-Nuttal !

Alpha decay: theory confronting data



Gradient of plot around - 4 MeV $^{1/2}$.

Our result, expressed in natural logs:

$$\ln 1/\tau \approx \text{constant} - 4\ \text{MeV}^{1/2}\ \frac{Z}{\sqrt{T_\alpha}}$$

Matches data well !

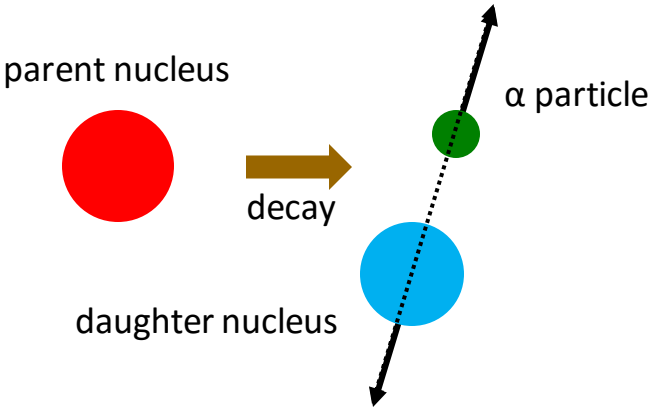
So we have successfully explained the enormous variation of alpha-emitter lifetimes, and their dependence on T_α (= Q value). It is due to the rapid variation of the barrier-penetration factor with this quantity.

Angular-momentum barrier

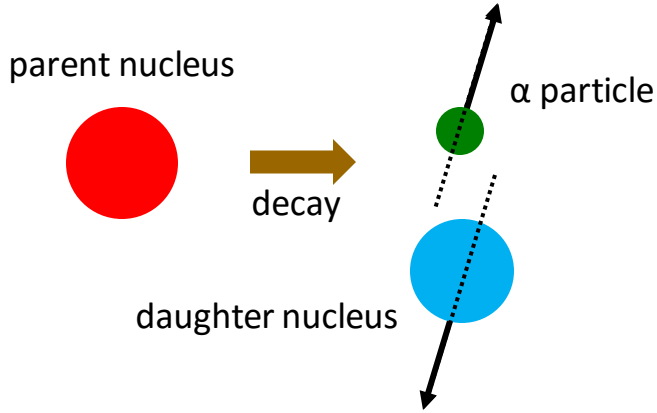
Our derivation of Geiger-Nuttal rule contained many simplifications, *e.g.*:

- Idealised shape of potential;
- Probability that α particle exists inside nucleus is independent of nature of nucleus (in reality, its much higher for an ee nucleus);
- Neglects shell effects;
- *etc.*

One neglected issue to be highlighted, as it has applications for nuclear & particle decays & reactions, is the effect of angular momentum. What happens if l of final system $\neq 0$?



α is a spin-0 particle



If spin of parent and daughter is the same then no orbital angular momentum required in final state.

If spin of parent and daughter is different then α and daughter must be in state of $l > 0$.

Angular-momentum barrier

As we wish to stress that the angular momentum is between the α particle and the daughter nucleus we will work in the centre-of-mass frame (as we should have strictly done anyway).

Total energy of system: $E = KE + V(r)$

We split the kinetic energy into its radial and azimuthal parts:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r)$$

where μ is the reduced mass of the system

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$$

with L the angular momentum, which is both conserved and quantised $L^2 = l(l+1)\hbar^2$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)$$

So we can replace the potential by an effective potential $\frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)$.

The Coulomb barrier $V(r)$ has been enhanced by an angular-momentum, or centrifugal barrier, $\frac{l(l+1)\hbar^2}{2\mu r^2}$ that further inhibits the decay.

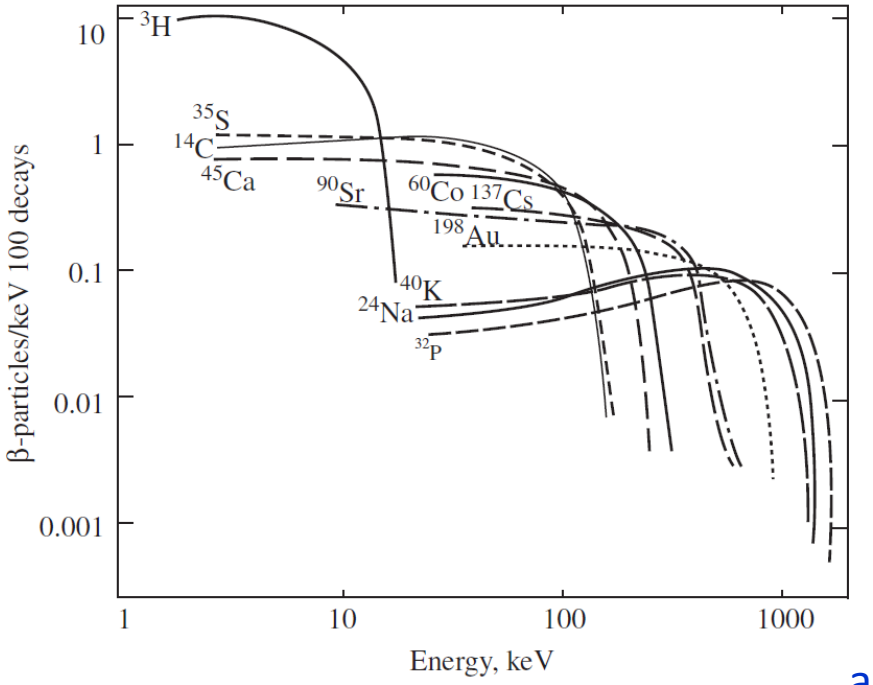
Example: *Z=86*, T_α*=4.88* MeV, *R=9.87* MeV [Blatt and Weisskopf, 1952]

l	0	1	2	3	4	5	6
Suppression factor	1	0.7	0.37	0.137	0.037	0.0071	0.0011

Beta decay: initial observations

In β^- decay we observe the emission of an electron & the transformation of one isobar into another, through a *n* changing into a *p*. This *cannot* be explained in terms of electromagnetism or the nuclear interaction & requires a new interaction – the *weak force*.

Energy of emitted electrons is continuous, with a mean value for each emitter that spans a wide range of energies from a few keV (tritium) up to several MeV.



There is also a very wide range in lifetimes among beta emitters, which appears correlated to the energy available.

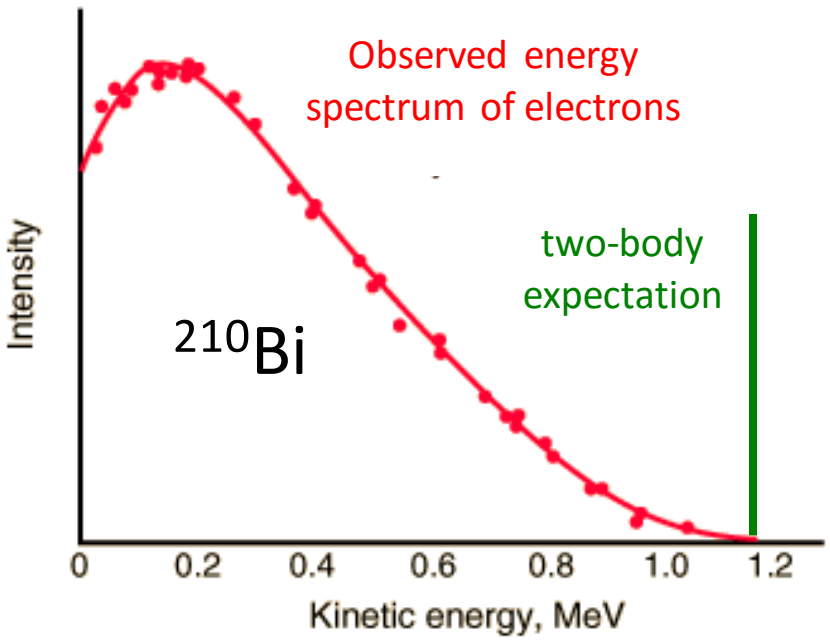
		Energy of process	Half life
^{23}Ne	β^-	4.38 MeV	38 s
^{99}Tc	β^-	0.29 MeV	2.1×10^5 y
^{25}Al	β^+	3.26 MeV	7.2 s
^{124}I	β^+	2.14 MeV	4.2 d
^{15}O	EC	2.75 MeV	1.2 s
^{41}Ca	EC	0.43 MeV	1.0×10^5 y

(There are also striking characteristics in the angular distribution of the $e^{-/+}$ in $\beta^{-/+}$ decay which we will consider in the particle physics discussion.)

Beta decay: historical perspective

In the first studies of β decay, in the early 1930s, only the nucleus & electron were thought to participate in the process (proton discovered in 1909, the neutron in 1932).

Striking observation: e^- decay spectrum is *continuous*. In a two-body decay it would have been a delta-function.



Another problem – some decays did not seem to conserve angular momentum:

$$e.g. \quad {}^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} + e^-$$

J=0 J=1 S= ½

Or consider decay of free neutron:

$$n \rightarrow p + e^-$$

S= ½ S= ½ S= ½

Again, this doesn't work.

So these simple observations indicated that existing picture of physics was incomplete.

Beta decay: historical perspective

An audacious solution to these problems was proposed in 1930 by Wolfgang Pauli in a letter to colleagues attending a conference in Tübingen, Germany.

("Unfortunately, I cannot personally appear at Tübingen since I am indispensable here in Zurich because of a ball.")

Pauli's "desperate remedy" was to suggest a third particle is present in the final state, which is spin-1/2, neutral and very light – the neutrino *.

The neutrino was first directly detected in 1956 – it resisted detection for so long as it only interacts through the weak interaction. On receiving the news Pauli telegraphed: "Thanks for message. Everything comes to him who knows how to wait. Pauli."



Wolfgang Pauli (1900-1958);
Nobel Prize 1945;
Austrian, Swiss, USA citizenship.

* in fact Pauli originally suggested calling it the neutron, but he didn't mean what we now call the neutron

Stability against beta decay

For beta decay to occur, the mass of the final state particles must be less than that of the initial state particles. The difference between the two, expressed as an energy, is called the *Q value*, and is shared between the kinetic energies of the products.

β⁻ decay

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + e^- + \bar{\nu}_e $$

$$ \begin{aligned} Q &= m_X c^2 - m_Y c^2 - m_e c^2 \\ &= (m_n - m_p - m_e)c^2 + B_Y - B_X \\ &= 0.8\ \text{MeV} + B_Y - B_X \end{aligned} $$

The initial state is unstable and decay occurs if $Q > 0$ for any process

β⁺ decay

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}Y + e^+ + \nu_e $$

$$ \begin{aligned} Q &= m_X c^2 - m_Y c^2 - m_e c^2 \\ &= (m_p - m_n - m_e)c^2 + B_Y - B_X \\ &= -1.8\ \text{MeV} + B_Y - B_X \end{aligned} $$

EC

$$ {}^{A}_{Z}X + e^- \rightarrow {}^{A}_{Z-1}Y + \nu_e $$

$$ \begin{aligned} Q &= m_X c^2 - m_Y c^2 + m_e c^2 \\ &= (m_p - m_n + m_e)c^2 + B_Y - B_X \\ &= -0.8\ \text{MeV} + B_Y - B_X \end{aligned} $$

- B_X (B_Y) is nuclear binding energy for X (Y);
- we have assumed $m_{e^-} = m_{e^+} = m_e$ which is true, as far as we know;
- we have assumed (anti-)neutrino mass is negligible, which is correct ($m_\nu < 1\ \text{eV}/c^2$);
- we have neglected differences in atomic binding energy between X & Y.

Stability against beta decay

For beta decay to occur, the mass of the final state particles must be less than that of the initial state particles. The difference between the two, expressed as an energy, is called the *Q value*, and is shared between the kinetic energies of the products.

β⁻

β⁺

EC

Something to watch out for: nuclear or atomic mass ?

In writing, *e.g.* $Q = m_X c^2 - m_Y c^2 - m_e c^2$ (β⁺ decay)
we have working in terms of the **nuclear masses** m_X and m_Y .

The calculation may be done with **atomic masses** (lets denote these M_X and M_Y). This is convenient as there is then no need to look up nuclear binding energies, but we must remember that M_Y will have a contribution from one more (β⁺ decay or EC) or less (β⁻ decay) electron than is present in our final state atom, so formulae must be adjusted accordingly.

So in *e.g.* β⁺ case we will have $Q = MXc^2 - M_Y c^2 - 2m_e c^2$ and so atomic masses must differ by at least $2m_e c^2$ for the decay to occur.

r X (Y);

m_e

& Y.

$$Q = m_X c^2 - m_Y c^2 + m_e c^2$$
$$= (m_p - m_n + m_e)c^2 + B_Y - B_X$$
$$= -0.8 \text{ MeV} + B_Y - B_X$$

Stability against beta decay: simple cases

β^- decay	$Q = \quad 0.8\ \text{MeV} + B_Y - B_X$
β^+ decay	$Q = -1.8\ \text{MeV} + B_Y - B_X$
Electron capture	$Q = -0.8\ \text{MeV} + B_Y - B_X$

(no binding energy to consider in these examples)

Is a free neutron stable ? Only possibility is β^- decay.

$$n \rightarrow p + e^- + \bar{\nu}_e \quad ?$$

Q = 0.8 MeV > 0, so decay occurs (lifetime 879 s).

Is a free proton stable ? Only possibility is β^+ decay.

$$p \rightarrow n + e^+ + \nu_e \quad ?$$

Q = -1.8 MeV < 0, so decay forbidden.

This is fortunate !

So universe's existence is dependent on the relative mass differences of the *n*, *p* and *e*.

Is hydrogen atom stable against electron capture ?

$$p + e^- \rightarrow n + \nu_e \quad ?$$

Q = -0.8 MeV < 0, so hydrogen is stable.

Stability against beta decay: beryllium

Let's examine a few isotopes of beryllium and calculate their stability against beta decay, either into boron (β^-) or lithium (β^+ or EC). First use binding energies calculated from SEMF.

Binding energies calculated from SEMF, with typical set of coefficients

	A=7	A=9	A=10	A=11
Li (Z=3)	39.3 MeV	39.2 MeV	31.7 MeV	30.2 MeV
Be (Z=4)	36.7 MeV	57.5 MeV	64.9 MeV	61.9 MeV
B (Z=5)	6.8 MeV	54.5 MeV	63.7 MeV	76.0 MeV

Use these inputs to determine Q values for possible decays:

	β^- decay Q = 0.8 MeV + B_B - B_{Be}	β^+ decay Q = -1.8 MeV + B_{Li} - B_{Be}	EC Q = -0.8 MeV + B_{Li} - B_{Be}	So SEMF predicts:
^{7}Be	-29.1 MeV	0.8 MeV	1.8 MeV	Unstable via β^+ & EC
^{9}Be	-2.2 MeV	-20.1 MeV	-19.1 MeV	Stable
^{10}Be	-0.4 MeV	-35.0 MeV	-34.0 MeV	Stable
^{11}Be	14.9 MeV	-33.5 MeV	-32.5 MeV	Unstable via β^-

Stability against beta decay: beryllium

Let's examine a few isotopes of beryllium and calculate their stability against beta decay, either into boron (β^-) or lithium (β^+ or EC). First use binding energies calculated from SEMF.

Binding energies calculated from SEMF, with typical set of coefficients

	A=7	A=9	A=10	A=11
Li (Z=3)	39.3 MeV	39.2 MeV	31.7 MeV	30.2 MeV
Be (Z=4)	36.7 MeV	57.5 MeV	64.9 MeV	61.9 MeV
B (Z=5)	6.8 MeV	54.5 MeV	63.7 MeV	76.0 MeV

Use these inputs to determine Q values for possible decays:

	β^- decay Q = 0.8 MeV + B_B - B_{Be}	β^+ decay Q = -1.8 MeV + B_{Li} - B_{Be}	EC Q = -0.8 MeV + B_{Li} - B_{Be}	
^{7}Be	-29.1 MeV	0.8 MeV	1.8 MeV	Unstable via β^+ ✘ & EC ✓
^{9}Be	-2.2 MeV	-20.1 MeV	-19.1 MeV	Stable ✓
^{10}Be	-0.4 MeV	-35.0 MeV	-34.0 MeV	Stable ✘ decays via β^-
^{11}Be	14.9 MeV	-33.5 MeV	-32.5 MeV	Unstable via β^- ✓

So SEMF predicts:

However, what is found in nature does not quite agree.

Stability against beta decay: beryllium

So let's recalculate using measured values of binding energies…

Measured values of binding energies

	A=7	A=9	A=10	A=11
Li (Z=3)	39.2 MeV	45.3 MeV	45.3 MeV	45.6 MeV
Be (Z=4)	37.6 MeV	58.1 MeV	65.0 MeV	65.5 MeV
B (Z=5)	24.7 MeV	56.3 MeV	64.8 MeV	76.2 MeV

…we note that some are very different. Shouldn't surprise us as we know SEMF is not to be trusted below A=20. We were stupid to use it ! Use measurements to recalculate Q values:

	β^- decay Q = 0.8 MeV + B_B - B_{Be}	β^+ decay Q = -1.8 MeV + B_{Li} - B_{Be}	EC Q = -0.8 MeV + B_{Li} - B_{Be}	Updated predictions:
^{7}Be	-12.1 MeV	-0.2 MeV	0.8 MeV	Unstable via EC ✓
^{9}Be	-1.0 MeV	-14.6 MeV	-13.6 MeV	Stable ✓
^{10}Be	0.6 MeV	-21.5 MeV	-20.5 MeV	Unstable via β^- ✓
^{11}Be	11.5 MeV	-21.7 MeV	-20.7 MeV	Unstable via β^- ✓

Some values close to threshold have flipped sign. Now perfect agreement with nature.

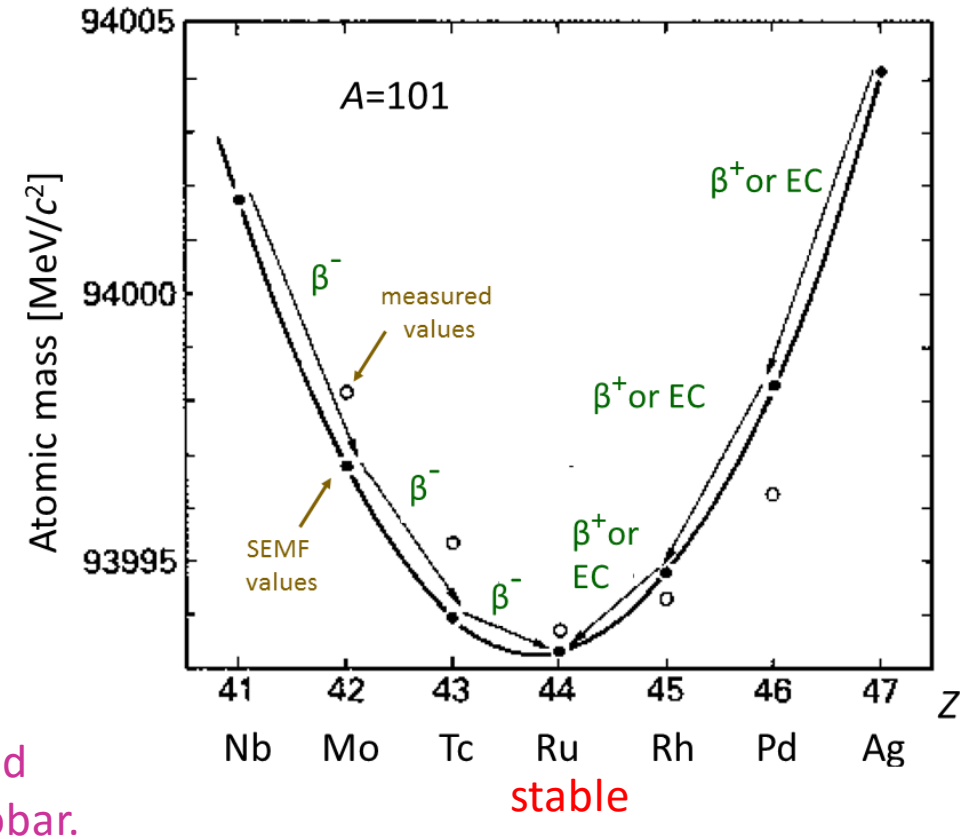
Mass parabolae: odd-*A* isobars

For fixed value of *A*, *i.e.* an isobar, SEMF says nuclear mass has a quadratic dependence on *Z*:

$$m(Z,A) = Zmp + (A-Z)m_n - [\, a_V A \; - a_S A^{2/3} - a_c Z^2/A^{1/3} - a_A(A-2Z)^2/A + \delta(Z,A)\,]/c^2$$

This means that the isobars should lie on (or close to) a parabola in *Z*.

Let's plot this for an odd-*A* value, *e.g.* *A*=101. Remember for odd-*A* the pairing term δ is zero (NB this plot uses atomic mass).



Z value of minimum given by:

$$Z = \frac{A\left(4a_A + \left(m_n - m_p\right)c^2\right)}{2a_C A^{2/3} + 8a_A}$$

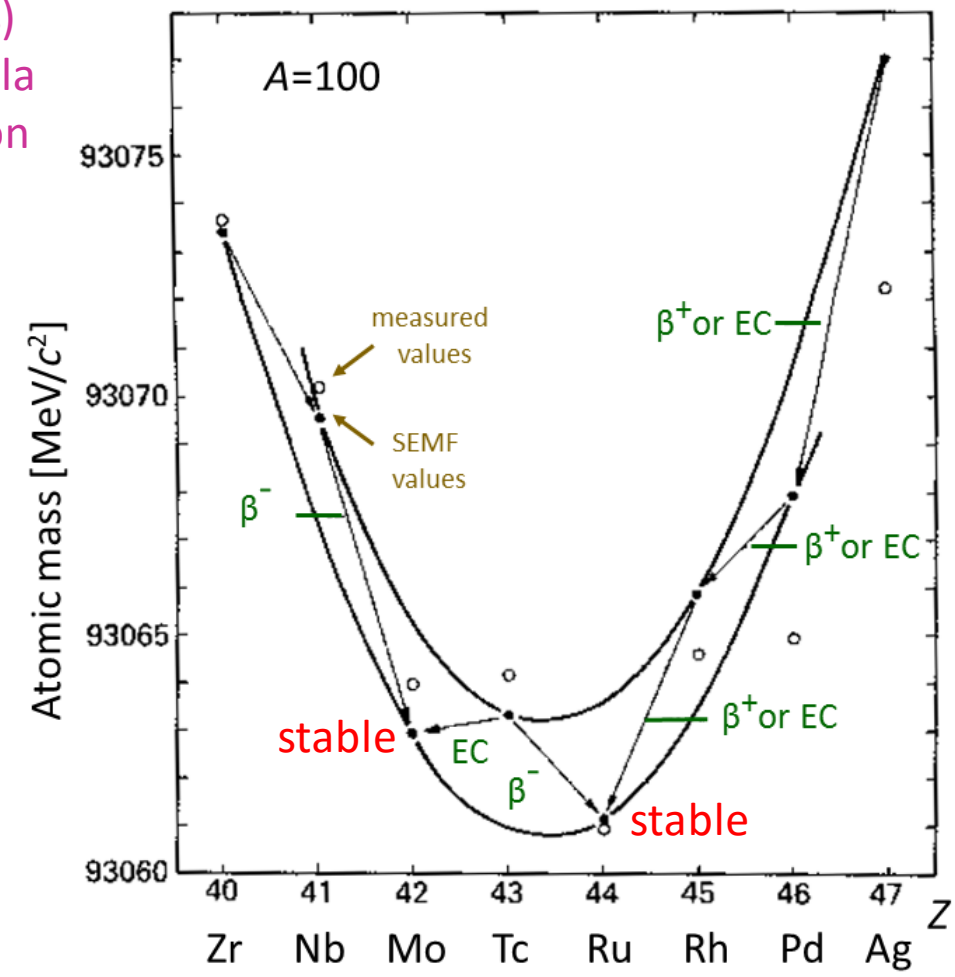
which in this case gives *Z*=44.1, & indeed ruthenium (*Z*=44) is the lowest mass isobar.

For a given odd value of *A* there can only be a single isobar stable against beta decay.

Mass parabolae: even-*A* isobars

In the case of even-A isobars the pairing term is non-zero and raises the mass for oo isobars, and lowers it for ee isobars. As a consequence there are two parabola, *e.g.* for *A*=100:

In this case beta decays (of all categories) will involve a transition from one parabola to the other. In some cases this transition will be kinematically forbidden, with the consequence that there will be another stable state above minimum.



General statements for even-*A* isobars:

- There are no stable oo nuclei (several exceptions to this:

  $^{2}_{1}$H, $^{6}_{3}$Li, $^{10}_{5}$B and $^{14}_{7}$N

  for which shell effects mean SEMF predictions are badly wrong);
- Many ee nuclei can have more than one stable isobar.

Double beta decay

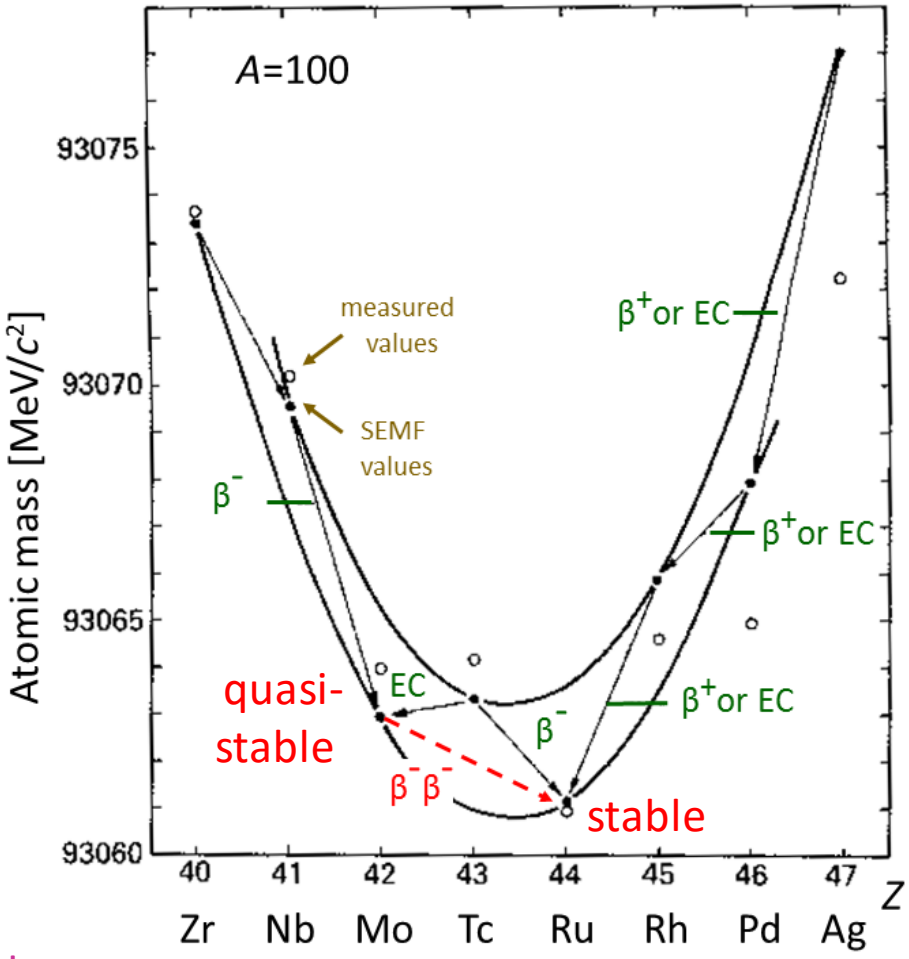
We have explained that ^{100}Mo is stable (as well as ^{100}Ru). This is not *entirely* true.

Double beta decay (ββ) is a rare process where two beta decays occur simultaneously.

In case where the intermediate state is kinematically inaccessible (here ^{100}Tc), then ββ provides a possible route to the bottom of the parabola.

There are 12 isotopes that have been observed to undergo ββ decay and 3 to undergo double EC.

Rare, *e.g.* ^{100}Mo lives ~10^{20} years.



Study of this process has an important role in particle physics for understanding nature of the neutrino – see later.

Fermi theory of beta decay

A theory to describe beta decay was provided by Fermi in 1933. Originally rejected by Nature as too speculative ('Nature's biggest blunder'), it was then published in Italian and German journals.

It describes β^- decay, and associated phenomena, remarkably well. We now know it is not a complete explanation of weak processes. For that we need the W boson and the electroweak (EW) theory of the Standard Model, but that does not make Fermi's proposal 'wrong' ! It's an *effective field theory* low-energy approximation of the EW concept.



Enrico Fermi (1901-1954); Nobel Prize 1938; Italian, USA citizenship.

Key assumptions (there are other, less fundamental, ones we will later invoke, but only to facilitate our calculations):

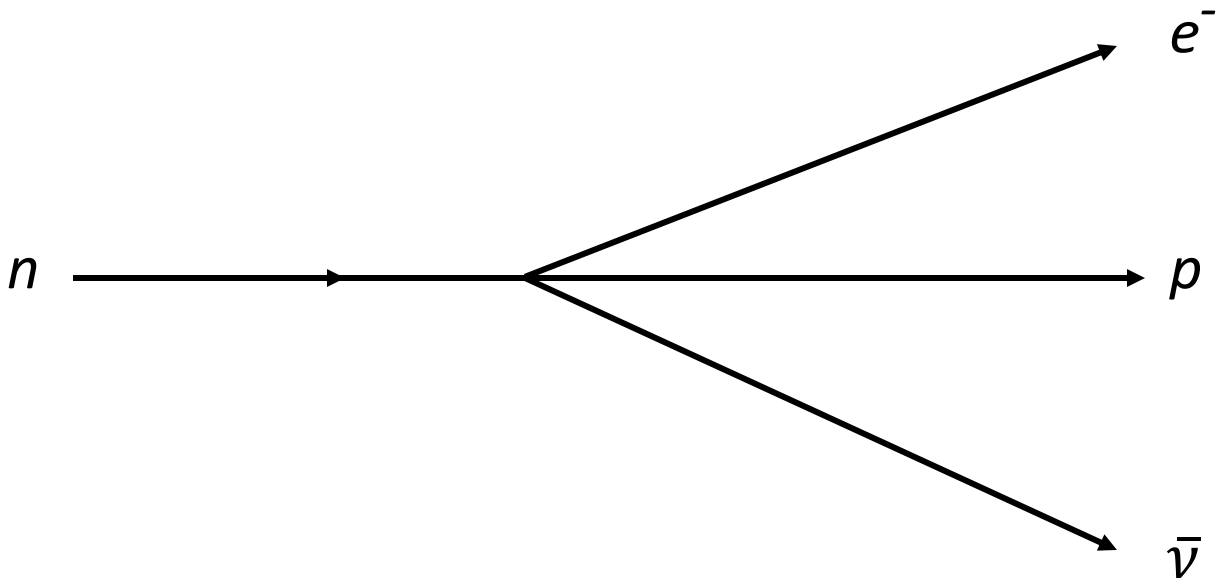
- The neutron transforms into a proton, electron and neutrino* (the electron and neutrino do not exist already inside the nucleus);
- The process occurs through a *four-point interaction*;
- The process is characterised by a coupling constant associated with the weak interaction, which we now call G_F (the *Fermi constant*).

We will not consider spin or parity when building the theory, but these can be included.

* When not important to specify between the two we will use 'neutrino' to mean neutrino &/or anti-neutrino

Four-point interaction

Using a Feynman-diagram-like representation, with time running left to right, we have:



The fact that all particles interact at a point is important. It tells us that the underlying physics of the interaction has ***no dependence*** on the energy of the particles or distances.

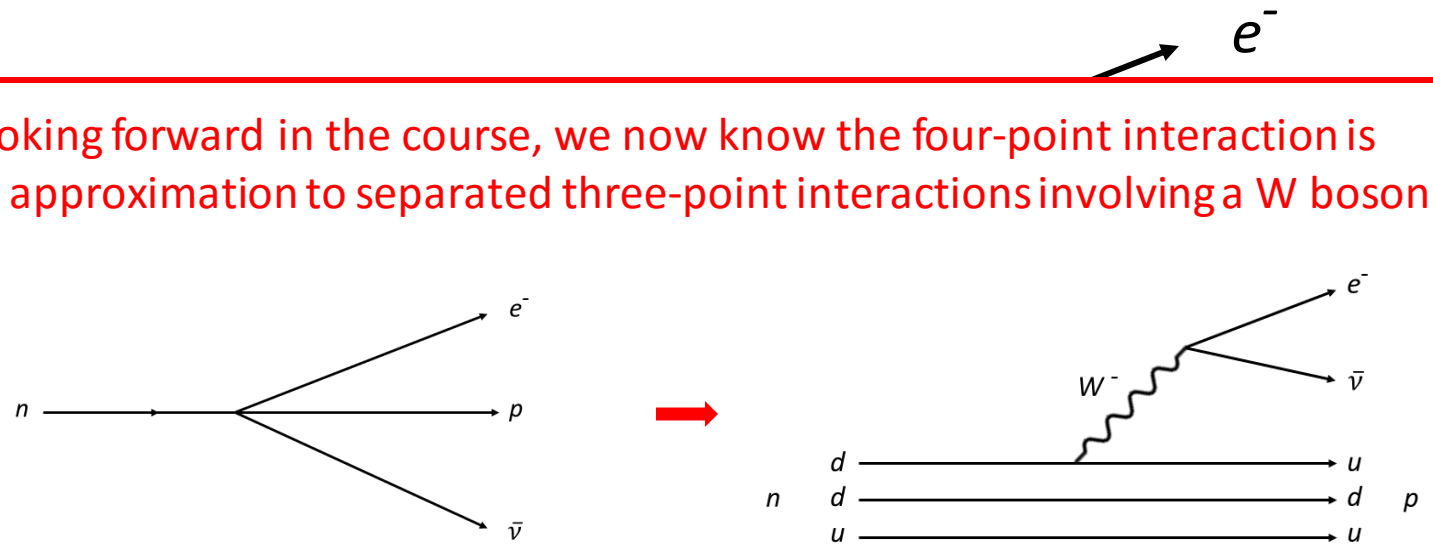
The Fermi constant has a value:

$$G_F = 1.436 \times 10^{-62}\ \mathrm{J\,m^3}$$
$$= 8.962 \times 10^{-5}\ \mathrm{MeV\,fm^3}$$
$$G_F/(\hbar c)^3 = 1.166 \times 10^{-5}\ \mathrm{GeV^{-2}}$$

Note that this is not dimensionless…

Four-point interaction

Using a Feynman-diagram-like representation, with time running left to right, we have:



Looking forward in the course, we now know the four-point interaction is an approximation to separated three-point interactions involving a W boson

The
phy

(with the diagram also benefitting from our knowledge of quarks), but Fermi's picture is perfectly adequate for describing low-energy phenomena.

es.

The Fermi constant has a value:

$$G_F = 1.436 \times 10^{-62}\ \mathrm{J\,m^3}$$
$$= 8.962 \times 10^{-5}\ \mathrm{MeV\,fm^3}$$
$$G_F/(\hbar c)^3 = 1.166 \times 10^{-5}\ \mathrm{GeV^{-2}}$$

Note that this is not dimensionless…

Calculating decay rates in Fermi theory

We start from Fermi's Golden Rule (FGR) which comes from first-order perturbation theory.

Transition probability / unit time from i to f

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}\left|M_{if}\right|^2 \rho$$

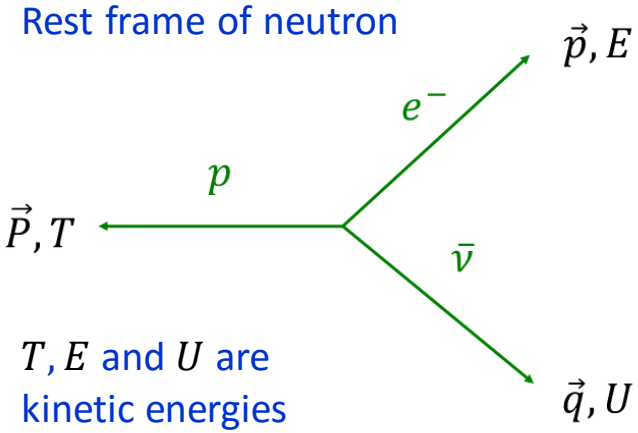
with ρ a density-of-states factor

and M_{if} is the matrix element that contains the nuclear/particle physics. Following Fermi's four-point ansatz we take M_{if} to have no kinematic dependence, & just be G_F. (In fact it does have some additional dependence on other, non-kinematic, factors, but we ignore these...).

We will first evaluate this rate in a bin of electron momentum, to enable us to predict the spectrum of decays, and then integrate to find the total decay rate.

Calculating decay rates in Fermi theory

Consideration of the kinematics is required, however, when calculating the density-of-states.



Q is available energy liberated in decay

$$\vec{P} + \vec{q} + \vec{p} = 0$$

$$T + E + U = Q$$

$U = q/c$, assuming $m_\nu = 0$

We are in the regime where $Q \sim 1$ MeV, which means the momenta are ~ 1 MeV/c.
For the proton then $T \sim P^2/2m_p \sim 10^{-3}$ MeV, which is negligible, so $E + U \approx Q$.

Density of states factor receives contributions from just two particles (conservation laws Then fix the kinematics of the third), so choosing the electron and neutrino:

$$\mathrm{d}^2 N = \frac{V \mathrm{d}\Omega\, p^2 \mathrm{d}p}{h^3} \frac{V \mathrm{d}\Omega\, q^2 \mathrm{d}q}{h^3}$$

We can integrate over the full solid angle, and also drop the volume factors V, as these will cancel with the normalisation factor in the wave-functions in the matrix element…

Calculating decay rates in Fermi theory

Let's evaluate $\frac{\mathrm{d}^2N}{\mathrm{d}Q}$ in a bin of electron momentum p to $p+\mathrm{d}p$. We have:

$$\mathrm{d}^2N = \frac{4\pi\, p^2\mathrm{d}p}{h^3}\,\frac{4\pi\, q^2\mathrm{d}q}{h^3} = \frac{1}{4\pi^4}\frac{1}{\hbar^6}p^2q^2\mathrm{d}p\mathrm{d}q\,.$$

The neutrino momentum is

$$q = (Q-E)/c\,,$$

and within the given bin of electron momentum we have $dq = dQ/c$. Hence

$$\frac{\mathrm{d}^2N}{\mathrm{d}Q} = \frac{1}{4\pi^4}\frac{1}{\hbar^6c^3}(Q-E)^2p^2\,\mathrm{d}p$$

and so the rate of decays with electron momentum within p to $p+\mathrm{d}p$ is $\Gamma(p)\,\mathrm{d}p$:

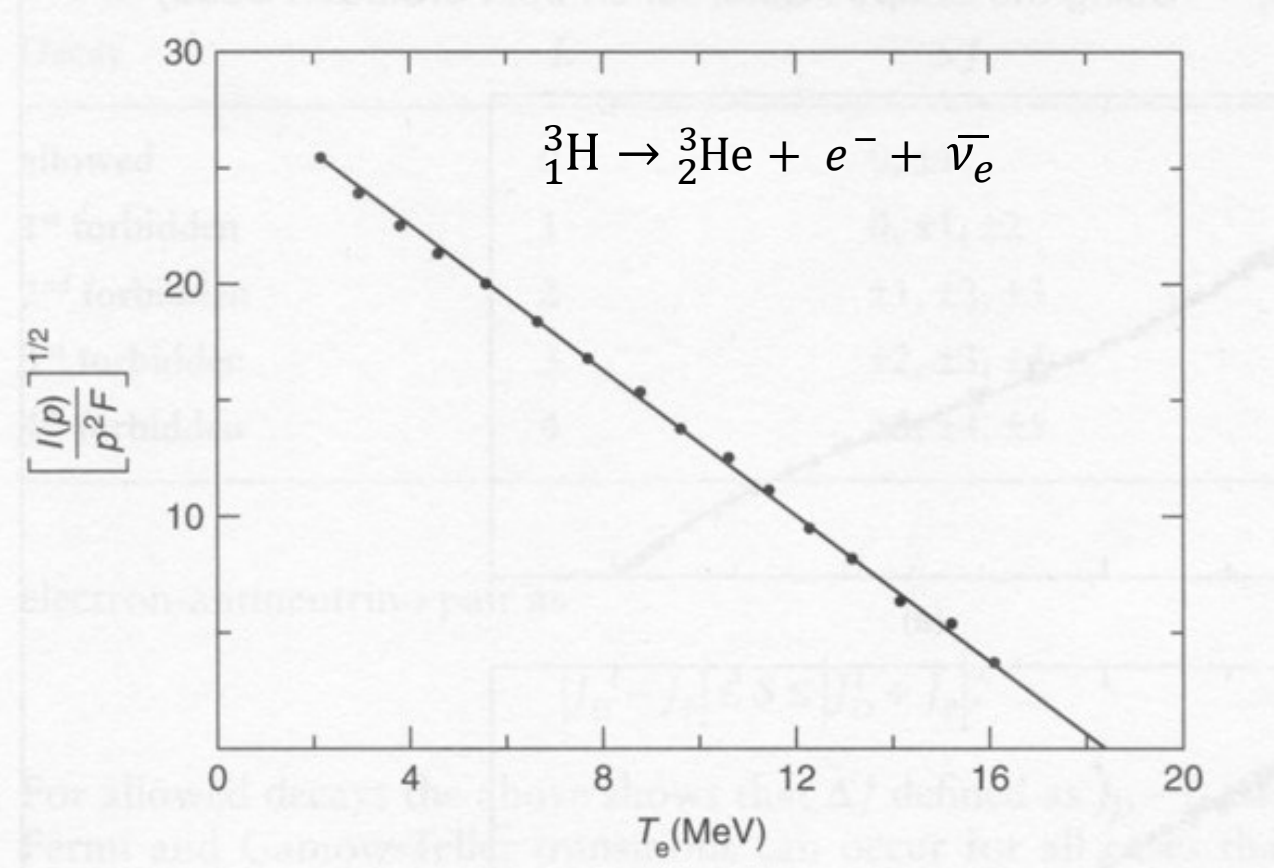
$$\boxed{\Gamma(p)\,\mathrm{d}p = \frac{G_F^2}{2\pi^3\hbar^7c^3}(Q-E)^2p^2\,\mathrm{d}p}$$

Kurie plot

According to Fermi theory $\sqrt{\left(\frac{I(p)}{p^2 K(Z,p)}\right)} = Q - E$ where

p, E	momentum and kinetic energy of electron
$I(p)$	counts in bin
$K(Z,p)$	constants and correction factor for Coulombic effects
Q	energy liberated in decay

Very well satisfied by measurement !

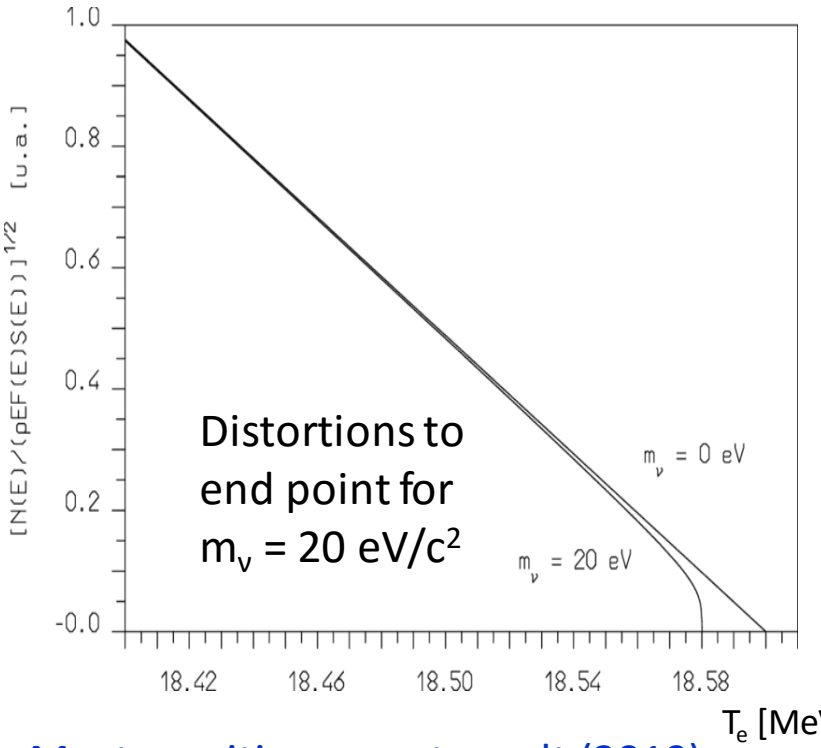


Kurie plot and neutrino mass

(not on syllabus)

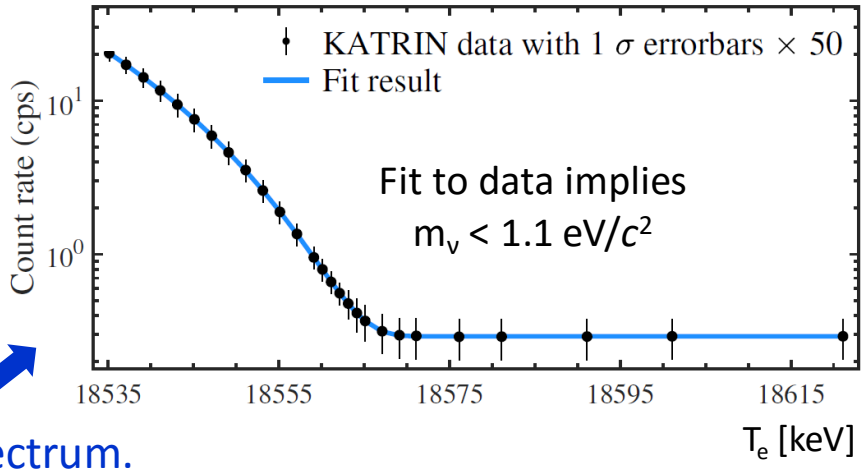
All expressions, and expected form of Kurie plot, assumed zero neutrino mass.
If neutrino has finite mass we expect deviations at end point of spectrum.

Best sensitivity in tritium decay,
as *Q*-value very low.



KATRIN on its way to installation





[PRL 123 (2019) 221802]

Most sensitive recent result (2019)
from KATRIN experiment, Karlsruhe.
Here shown as a fit to counts vs energy spectrum.

Sargent's rule

Starting from the decay rate in a small bin of momentum:

$$\Gamma(p)\mathrm{d}p = \frac{G_F^2}{2\pi^3\hbar^7 c^3}(Q-E)^2 p^2 \mathrm{d}p$$

we can integrate over all momenta, or energies if we prefer, to get the total decay rate.

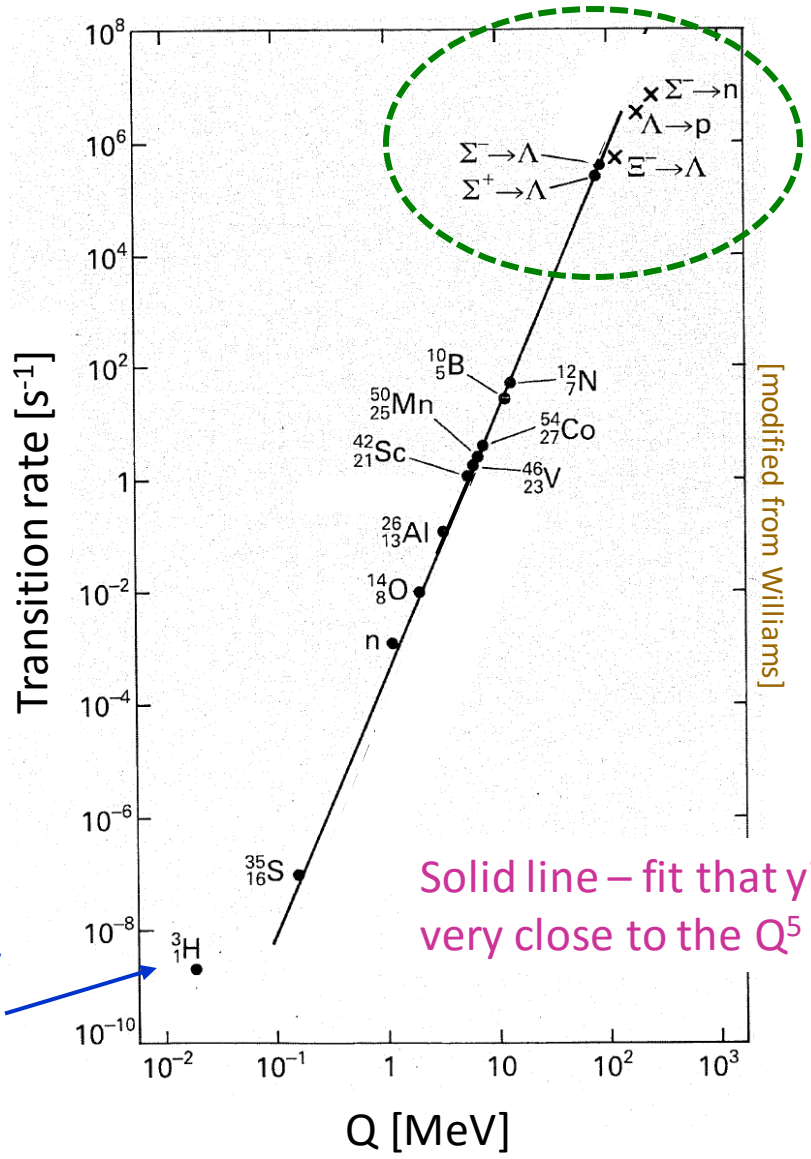
Let us do this for the case where $Q \gg m_e^2$, so that $p_e \approx E/c$ and

$$\Gamma(E)\mathrm{d}E = \frac{G_F^2}{2\pi^3\hbar^7 c^6}(Q-E)^2 E^2 \mathrm{dE}$$

$$\Longrightarrow \quad \boxed{\Gamma = \int_0^Q F(E)dE = \frac{G_F^2}{2\pi^3\hbar^7 c^6}\frac{Q^5}{30}}$$

This Q^5 dependence of the decay rate is known as Sargent's rule. This steep kinematic variation comes from the density-of-state alone, not the underlying weak physics !

Sargent's rule – validation from measurement



Decays of baryons containing strange quarks.

As we will later see, Sargent's rule also applies to weak decays in particle physics (some [X] with additional suppression factors that are understood).

Solid line – fit that yields a $Q^{4.7}$ dependence, very close to the Q^5 predicted by Sargent's rule.

A 'superallowed' decay that violates the rule (soon to be discussed).

Additional enhancement / suppression effects

We said the matrix element is constant. This is not true. Although there are no kinematical effects that can modify it, there are angular momentum considerations.

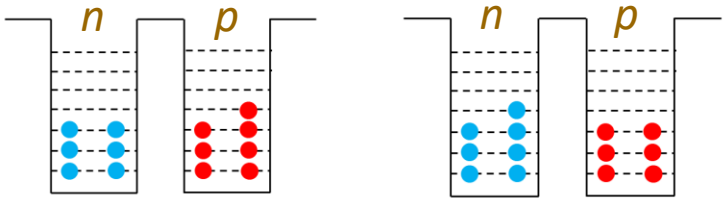
- If the change in the spin of the nucleus from initial to final state is $\Delta J = 0 \; or \; \pm 1$, with no change in parity then this can be accommodated by electron & anti-neutrino (both spin-1/2 fermions) being in a S=0 singlet (`*Fermi transition*') or S=1 triplet (`*Gamow-Teller transition*') with no orbital angular momentum (l=0):

  → 'allowed transition': no additional suppression; the default case.

- Some decays can be particularly favoured → 'superallowed transitions'.

For example, decays between mirror nuclei, where the wave function of the final state is ~ identical to that of the parent.

e.g. $^{13}_{7}\mathrm{N} \rightarrow {}^{13}_{6}\mathrm{C} + e^{+} + \nu_e$



Q=1.2 MeV
$\tau_{1/2}$ = 10 mins, around 100x shorter than predicted by Sargent's rule

- If $|\Delta J| > 1$ &/or a change of parity, then there must also be l>0, which is disfavoured:

  → 'forbidden transition', which really means suppressed – but by orders of magnitude.

Nuclear physics: topic overview

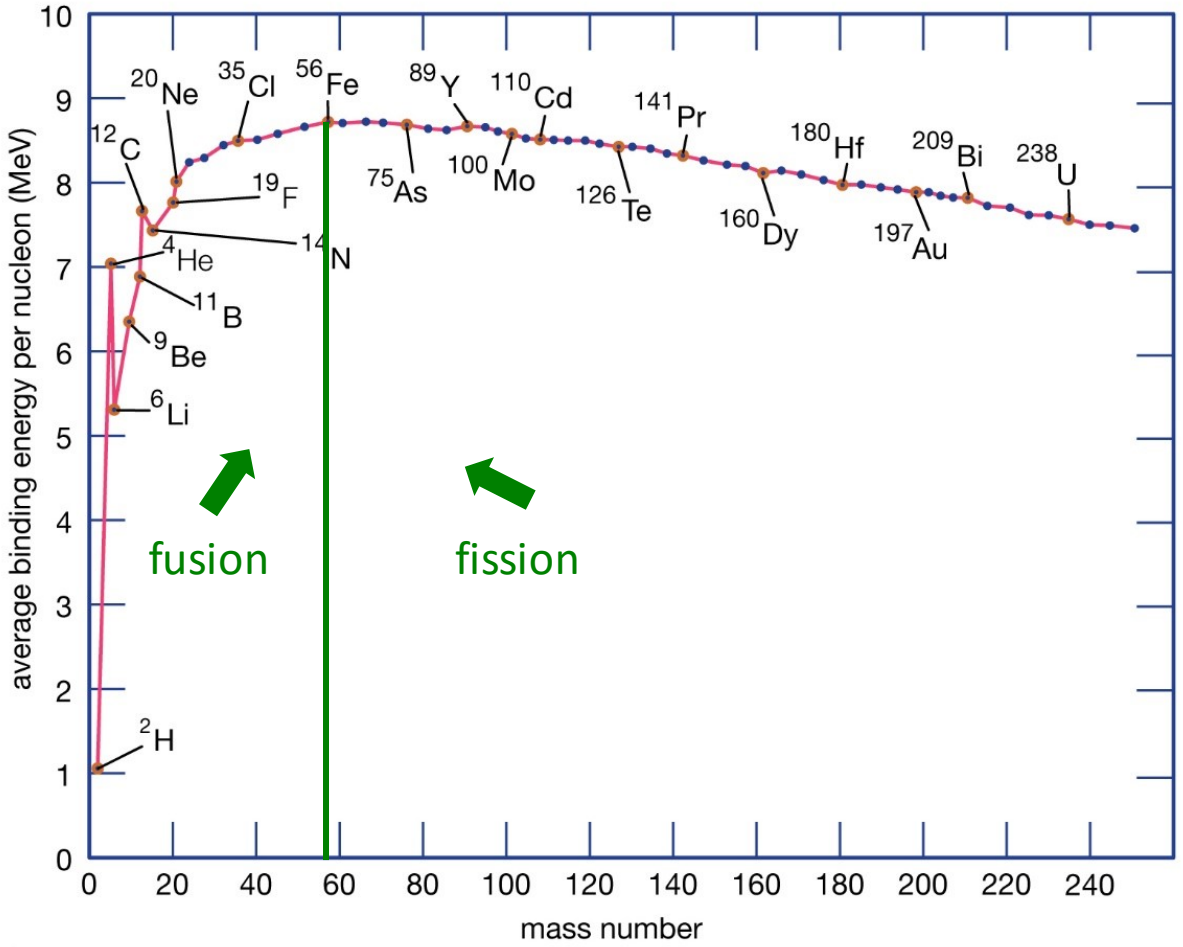
- Definitions and initial observations
- The Semi-Empirical Mass Formula (SEMF)
- The true nature of the nuclear interaction

- Nuclear decay and radioactivity: an overview
- α decay
- β decay and Fermi theory

- Fission and fusion: an overview
- Induced fission
- Fusion reactions
- Solar fusion and nucleosynthesis

Fission and fusion

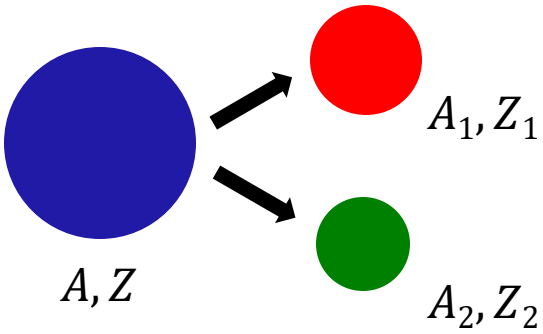
Depending on mass, nuclei can undergo fusion or fission reactions, and liberate energy through the increase in *B*. Fission can be spontaneous or induced. Fusion must be induced.



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Spontaneous fission

Binding energy curve suggests it could be advantageous for a heavy nucleus to fission. Ignoring special case of α decay, what are the expected relative sizes of the daughters and energy release? To get an approximate idea let's take a simple model and use SEMF.



Require $Q > 0$ with $Q = B(A_1, Z_1) + B(A_2, Z_2) - B(A, Z)$.

Let's assume $\frac{A_1}{A} = \frac{Z_1}{Z} = y$ and so $\frac{A_2}{A} = \frac{Z_2}{Z} = 1 - y$.

From SEMF:

$$Q = a_S A^{2/3}\left(1 - y^{2/3} - (1-y)^{2/3}\right) + a_c \frac{Z^2}{A^{1/3}}\left(1 - y^{5/3} - (1-y)^{5/3}\right)$$

(neglecting any possible small contributions from pairing term differences). Maximise:

$$\frac{dQ}{dy} = a_S A^{2/3}\left(-\frac{2}{3}y^{-1/3} + \frac{2}{3}(1-y)^{-1/3}\right) + a_c \frac{Z^2}{A^{1/3}}\left(-\frac{5}{3}y^{2/3} + \frac{5}{3}(1-y)^{2/3}\right) = 0$$

This is satisfied when $y = 1/2$, *i.e.* equal sized fission fragments.

Then $Q = 0.37 a_C \frac{Z^2}{A^{1/3}} - 0.26 a_S A^{2/3}$. Evaluated for $^{238}_{92}$U this gives $Q = 180$ MeV.

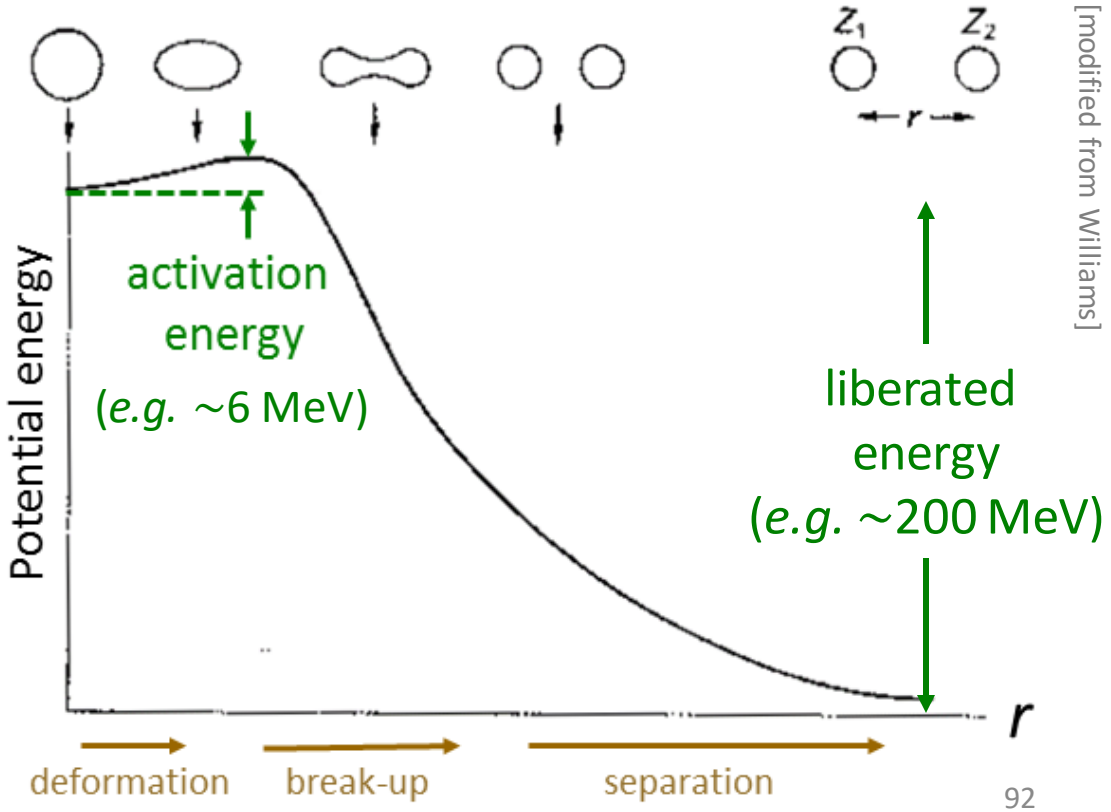
Spontaneous fission

If it's kinematically advantageous for a nucleus to split into two, why doesn't it just happen ?

Distortion → Increase in surface area

- → Mass increases through surface term — Wins for small distortions and inhibits fission
- → Mass decreases through Coulomb term — Wins at larger distortions

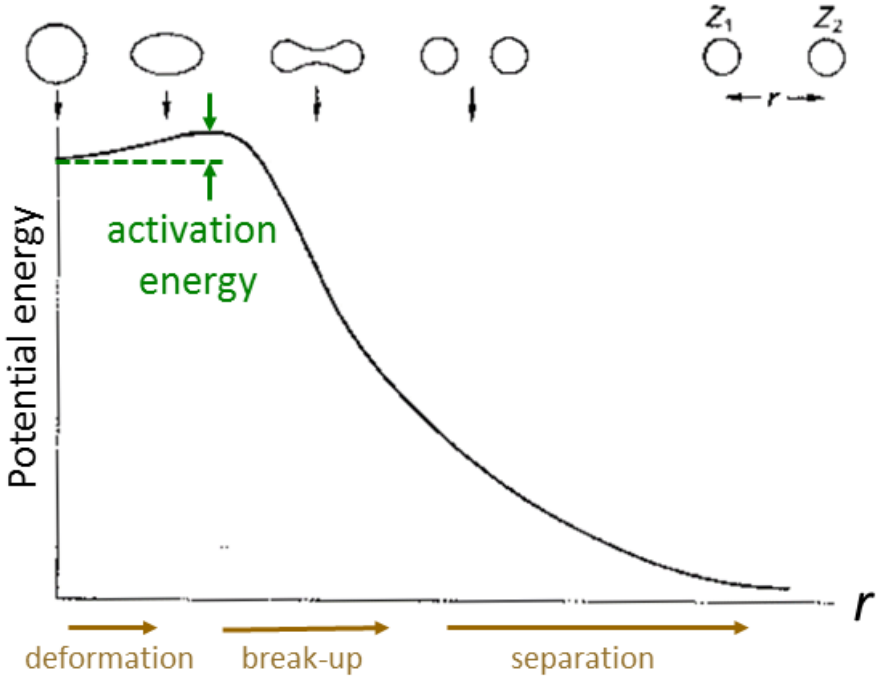
Viewed as a potential in which system sits these effects provide a Coulomb potential barrier through which the system must tunnel, as with alpha decay.

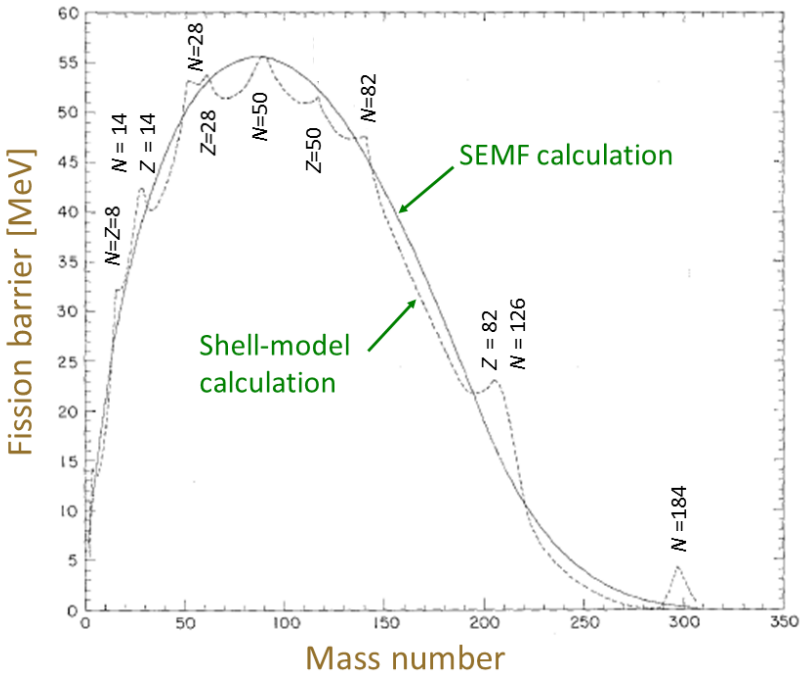


[modified from Williams]

Spontaneous fission

Size of barrier ('*activation energy*') decreases as *A* rises. A simple SEMF-based calculation (*e.g.* Krane) suggests it should vanish at $Z^2/A > 47$, or $A \sim 300$. But shell effects lower this.





[modified from Myers and Swiatecki, Nucl. Phys. 81 (1966) 1]

If barrier not too high then quantum tunnelling becomes feasible, however the probability of transition e^{-2G} is much lower than in alpha-decay. Recall from Geiger-Nuttal derivation:

$$G = \int_R^b \frac{\sqrt{2M_\alpha}}{\hbar} \sqrt{[V(r) - T_\alpha)}\, \mathrm{d}r$$

…and here the effective alpha particle is a fission fragment of ~25 times M_α.

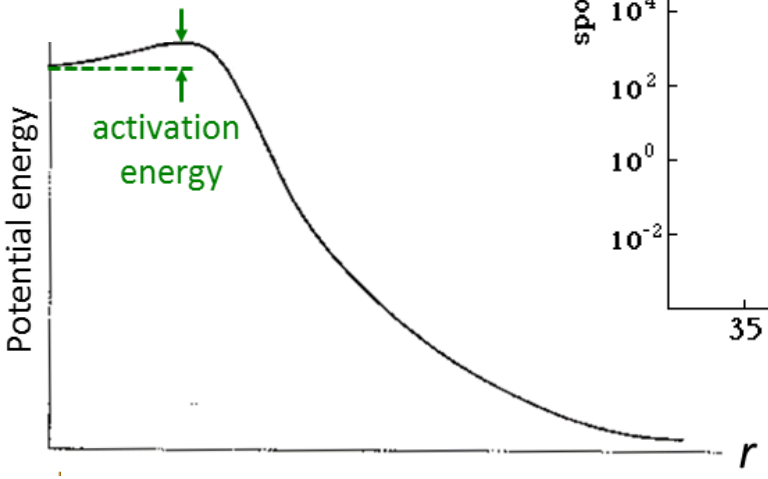
Spontaneous fission

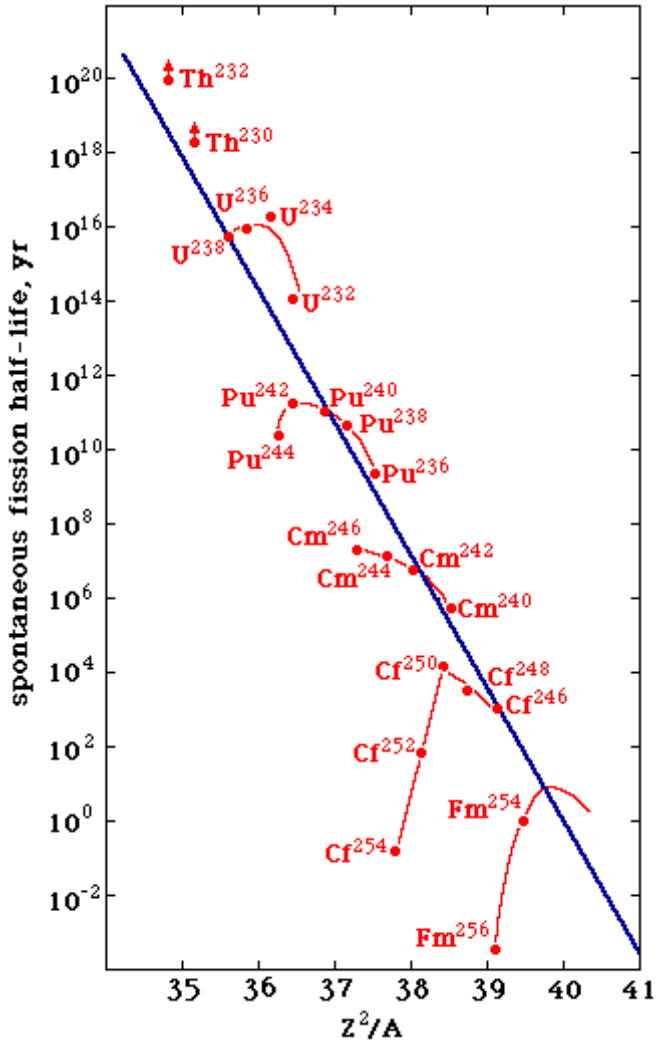
Spontaneous fission does happen, but is rather rare and confined to very heavy nuclei.

Even when it occurs, the decay probability may be much less than that of alpha emission.

The steep dependence of the tunnelling probability on barrier height means there is a very wide range of decay rate, with an approximate $\log \omega \propto Z^2/A$ dependence.

However, for many of these nuclei *induced fission* is possible, when an external intervention gives system a boost > activation energy, and helps it over barrier.

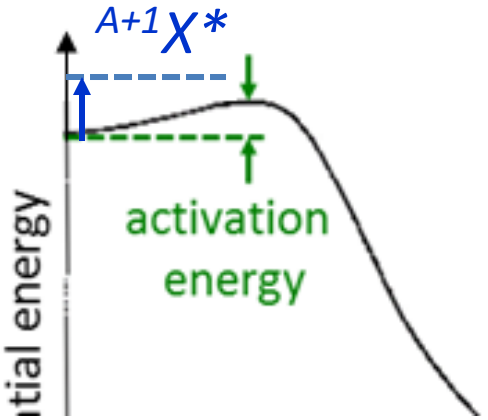




Induced fission

Bombard heavy element with neutrons, until one is captured to form heavier isotope in excited state:

$$^{A}X + n \rightarrow {}^{A+1}X^{*}$$



If excitation energy is higher than, or close to activation energy, fission will occur.

Consider first the case that the incident neutron energy is negligible, then if state ^{A+1}X is more tightly bound than ^{A}X, then there will be an excess of energy given by

$$\Delta B = {}^{A+1}X - {}^{A}X$$

which will be the energy then new nucleus has above ground state. In general the most favourable case is when the new nucleus is even-even, & we benefit from pairing term, *e.g.*

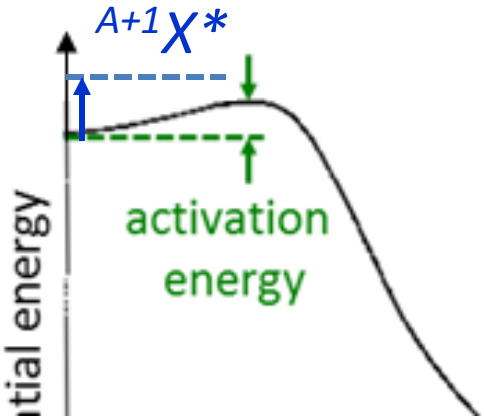
	$^{235}_{92}\mathrm{U} + n \rightarrow {}^{236}_{92}\mathrm{U}^{*}$	$^{239}_{94}\mathrm{Pu} + n \rightarrow {}^{240}_{94}\mathrm{Pu}^{*}$
ΔB	6.5 MeV	6.5 MeV
Activation energy	6.2 MeV	6.0 MeV

Since ΔB > activation energy of the $A+1$ nucleus then fission can occur.

Induced fission

Bombard heavy element with neutrons, until one is captured to form heavier isotope in excited state:

$$^{A}X + n \rightarrow {}^{A+1}X^{*}$$



If excitation energy is higher than, or close to activation energy, fission will occur.

Now, focusing on uranium as a good example, consider the case when the final-state isotope is even-odd. Then, if neutron energy is negligible, the excess energy this state has will be less than activation energy, and it can only de-excite through γ emission.

$$^{238}_{92}\mathrm{U} + n \rightarrow {}^{239}_{92}\mathrm{U}^{*}$$

ΔB	4.8 MeV
Activation energy	6.6 MeV

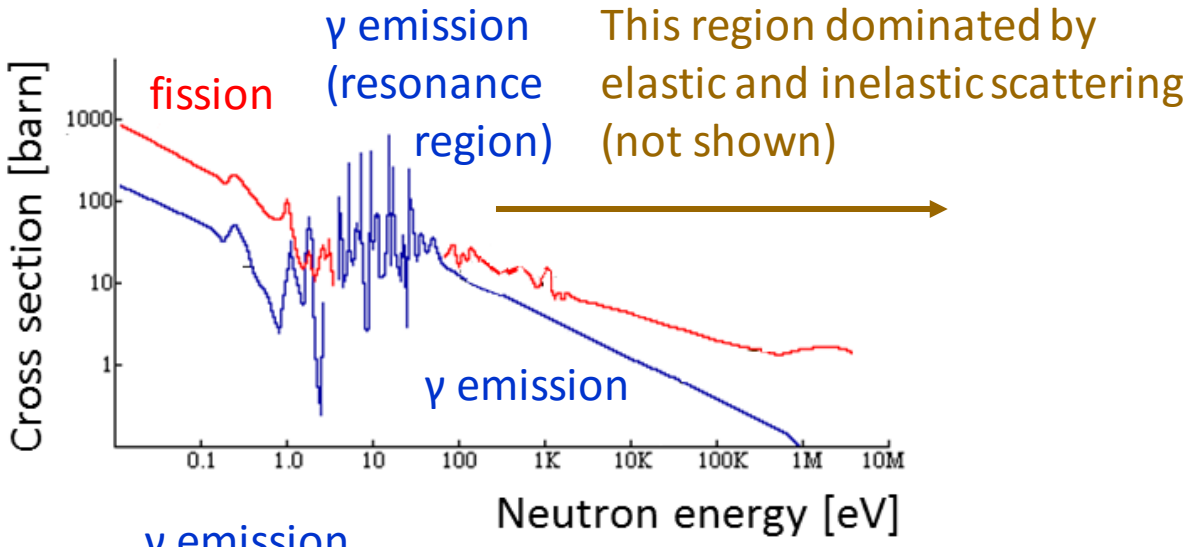
So only way for fission to occur is when the captured neutron has energy of a few MeV.

With these possibilities in mind, let's study *n*-absorption cross-section behaviour of uranium.

Neutron-absorption cross sections for uranium

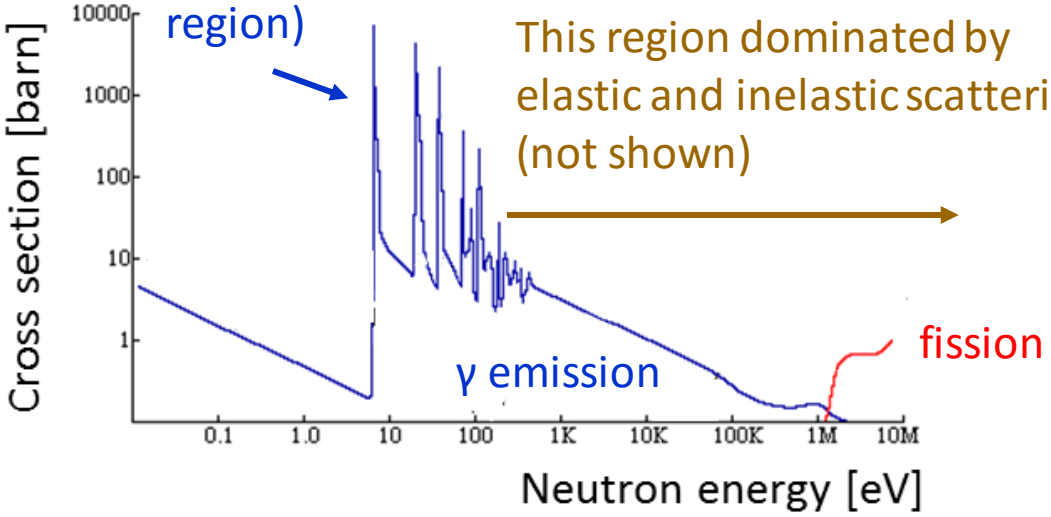
^{235}U

Natural abundance 0.7%

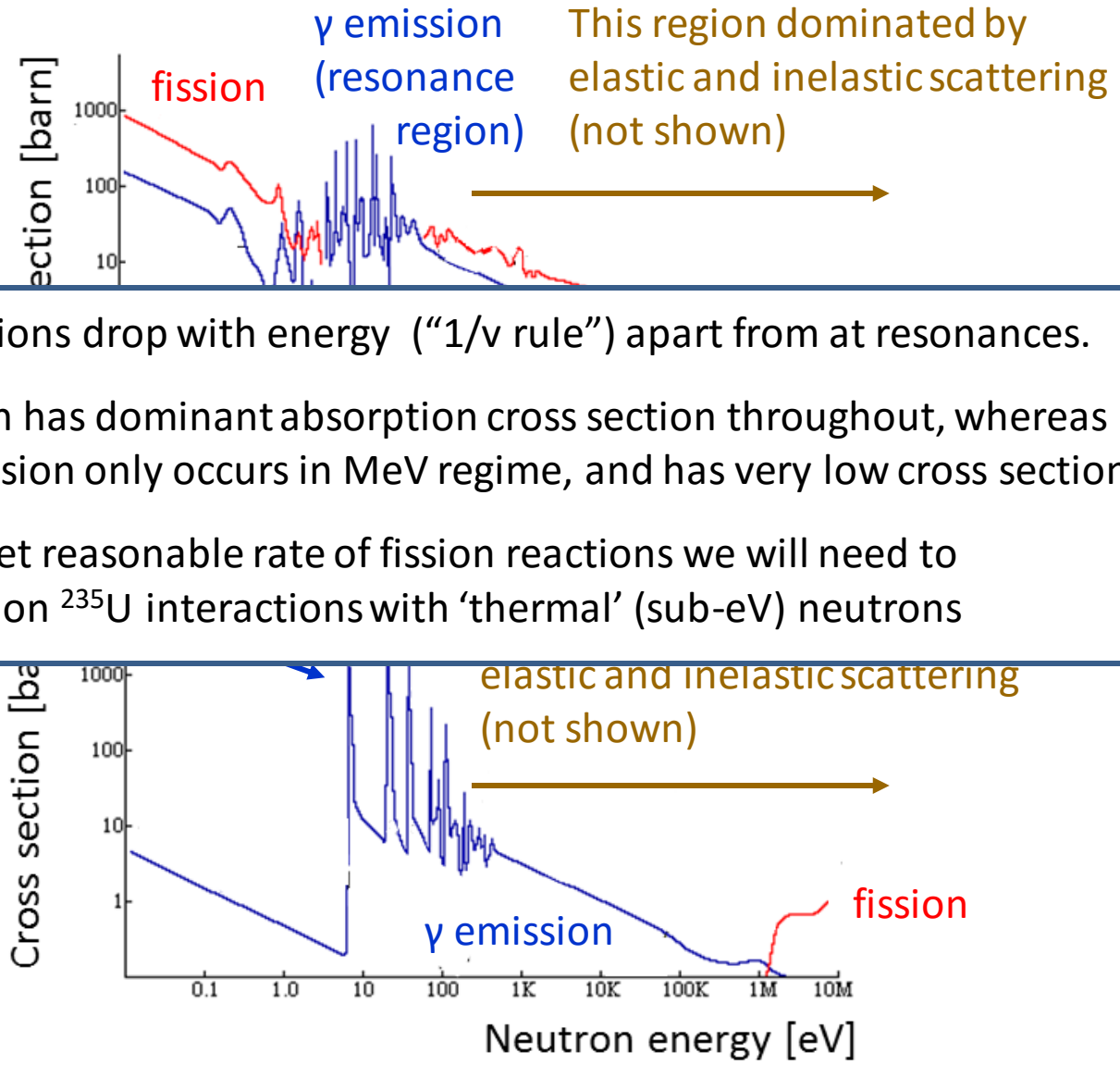


^{238}U

Natural abundance 99.3%



Neutron-absorption cross sections for uranium



Cross-sections drop with energy ("1/v rule") apart from at resonances.

^{235}U fission has dominant absorption cross section throughout, whereas for ^{238}U fission only occurs in MeV regime, and has very low cross section.

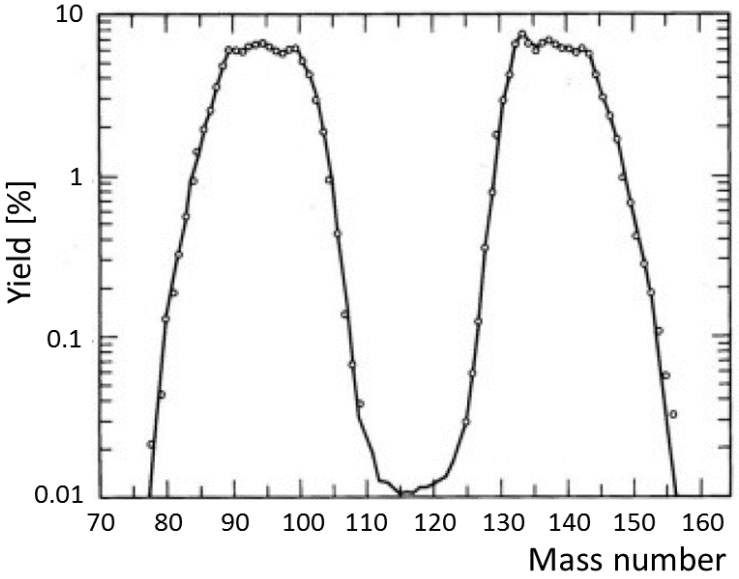
⇒ To get reasonable rate of fission reactions we will need to rely on ^{235}U interactions with 'thermal' (sub-eV) neutrons

Fission products

Continuing to take ^{235}U as the benchmark case (although situation is very similar for other nuclear fuels, *e.g.* ^{233}U and ^{239}Pu):

$$^{235}U + n \rightarrow ^{236}U^* \rightarrow X^* + Y^*$$

The fission fragments are in fact rarely identical in mass, but tend to cluster around magic number nuclei.



Lower peak: strontium to ruthenium;
higher peak: tellurium to neodymium.

X^* and Y^* are neutron rich w.r.t. stable nuclei with their Z values. Thus the fragments eject neutrons as the fission occurs ('prompt neutrons').

$$^{235}U + n \rightarrow ^{236}U^* \rightarrow ^{93}Rb + ^{141}Cs + 2n$$

Average number of prompt neutrons from $^{236}U^*$ is 2.42 (2.86 for $^{240}Pu^*$).

The fragments, still neutron rich, will decay through sequence of β and γ emission. Spontaneous neutron emission can occur, on timescales of seconds (& more) after fission, hence 'delayed neutrons', *e.g.* $^{93}Rb \rightarrow ^{92}Sr + n$, around 1% of time.

delayed neutrons / # prompt neutrons $\sim 10^{-2}$, but *very* important for reactor control.

Fission products: energy release

Typical fission reaction:

$$^{235}\mathrm{U} + n \rightarrow {}^{236}\mathrm{U}^{*} \rightarrow {}^{93}\mathrm{Rb} + {}^{141}\mathrm{Cs} + 2n$$

which has *Q*-value of 181 MeV.

Total energy release comes from fission, and later radioactive decays. Indicative values:

Energy of two fragments	160 MeV
Energy of neutrons	5 MeV
Energy from prompt γ emission	8 MeV
Energy from delayed β emission	19 MeV
Energy from delayed γ emission	7 MeV

Mean energy of each neutron ~ 2 MeV

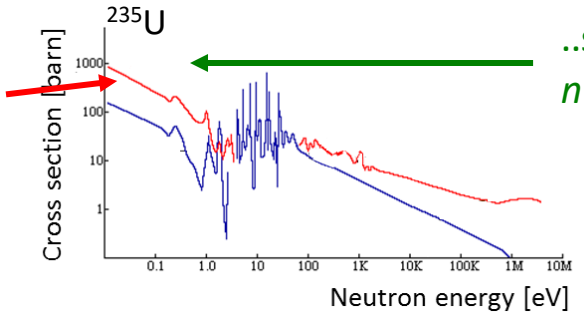
i.e. total energy produced per fission ≈ 200 MeV.

So a power plant producing 1600 MW with, say, 40% efficiency will have 1.25×10^{20} fissions / s, which corresponds to 0.05 g/s of ^{235}U.

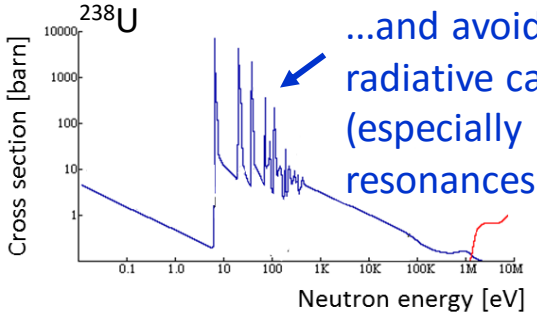
Moderation and enrichment

The ~2.5 neutrons per $^{236}U^*$ fission have mean energy of ~2 MeV. To increase chance of initiating another fission they must be *moderated*, *i.e.* reduced to sub-eV temperatures.

want to maximise ^{235}U fission-capture cross section…



..so to reduce *n* energy thus…



…and avoid ^{238}U radiative capture (especially resonances)

Achieved by allowing neutrons to have successive elastic collisions with small A material with low capture cross-section. They lose $2A/(1+A)^2$ of KE per collision, so for carbon (A=12)

Start at 2 MeV ➡ Aim to reach room-temperature energies = 3/2 x k_B x 293 K = 3/2 x 8.6 x 10^{-5} eV K^{-1} x 273 K = 0.032 eV ➡ Energy reduced by factor 0.86 per collision ➡ Requires ~120 collisions

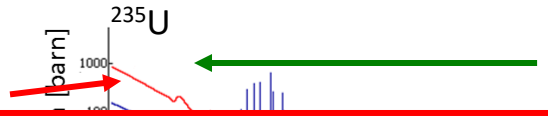
Therefore neutrons should spend much more time in moderator than fuel, so most of reactor volume must be taken up by moderator. Moderator choices in reactors:

H_2O: cheap, but absorbs neutrons through $n + p \rightarrow d + \gamma$;
D_2O: low neutron absorption cross-section, but expensive & forms radioactive tritium ^3H ;
^{12}C (graphite): need more of it, because A is larger. Must be pure (no boron!).

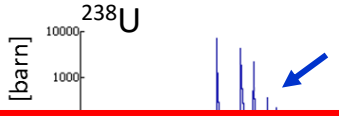
Also fuel will be *enriched* with ^{235}U proportion raised to 2-3%, to increase chance of fission.

Moderation and enrichment

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The f reactor volume must be taken up by moderator. Moderator choices in reactors:

Fast reactors exist which have no moderator and rely on fast-neutron capture. For these to work they need higher levels of enrichment (20% →), and a high fuel mass. But as it's the moderator that dominates the space, their overall volume is smaller. Suitable for submarines.



USS Nautilus, the first nuclear-powered submarine (1954).

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D_2O: low neutron absorption cross-section, but expensive & forms radioactive tritium ^3H ;
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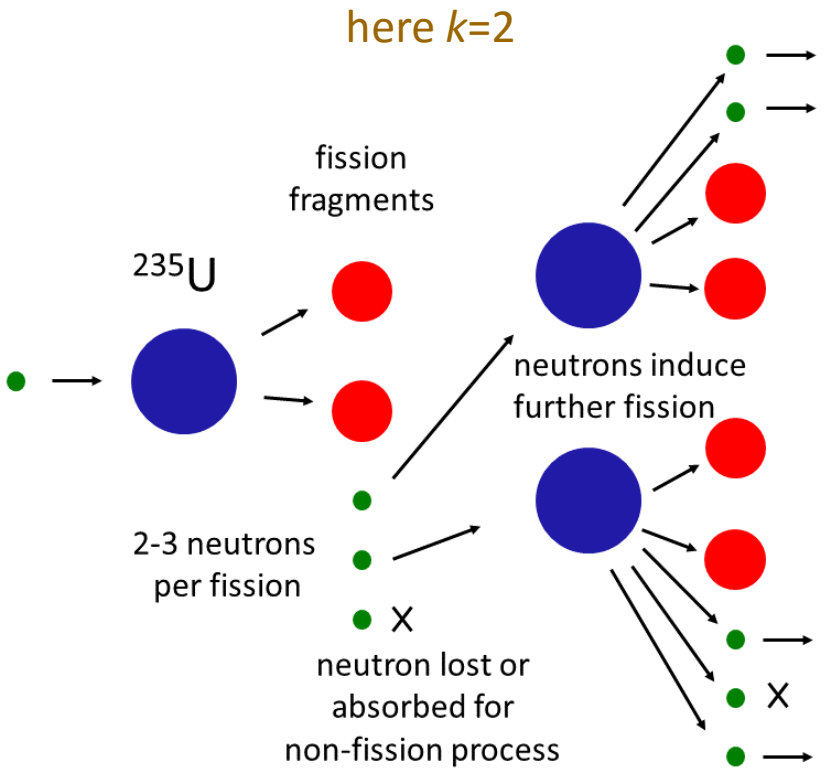
Also fuel will be *enriched* with ^{235}U proportion raised to 2-3%, to increase chance of fission.

Chain reactions and reactor control

Reproduction factor k =
number of neutrons available
to cause fission at one generation
compared to previous

Maximum possible value
in ^{235}U case is k=2.42.

$k \geq 1$ required for chain reaction to continue. In a reactor we want k=1 ($k>1$ is *super-critical*).



Reproduction factor is adjusted by inserting/withdrawing *control rods* made of material with high neutron-absorption cross section, *e.g.* cadmium or boron.

(Interesting question: what starts the chain reaction? In practice, usually a neutron source is inserted, but one can rely on spontaneous fission (rare, but not insignificant for ^{238}U) or neutron emitters in waste products in reactors that have been used before.)

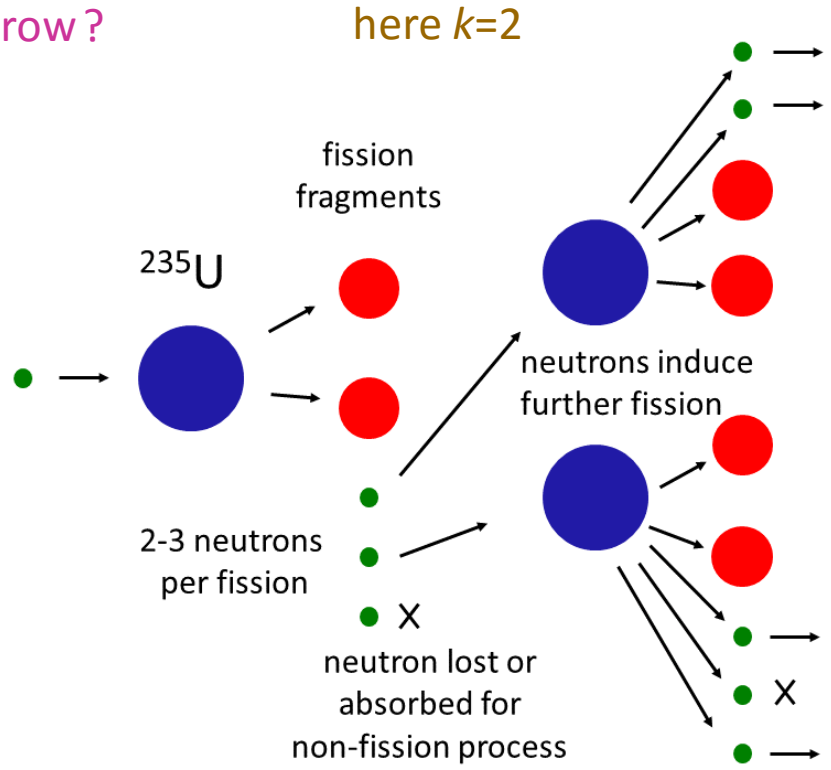
Chain reactions and reactor control

How quickly does neutron population grow ?

Taking a time constant $\tau \sim 10^{-3}$ s between neutron birth and absorption in fissile nucleus, and *N* the number of neutrons, then

$$dN = (kN - N)\frac{dt}{\tau}$$

$$\Rightarrow N(t) = N_0 e^{(k-1)t/\tau}$$



So assuming, for example a slightly super-critical situation with *k*=1.01, then the effective time constant is $\tau/(k-1)$ = 0.1 s, and the neutron population will grow by 22,000 in 1 s ! It is very challenging for a mechanical system to respond on this timescale.

Solution: the delayed neutrons arrive much later, so the trick is to keep the reactor sub-critical with the prompt neutrons, and bring it to *k*=1 with the delayed component.

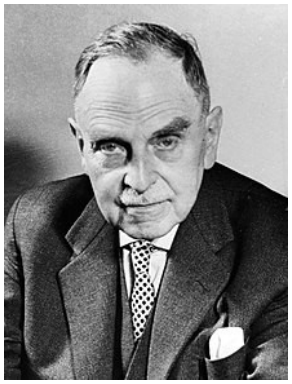
History of fission: discovery

Discovery of fission was made possible by Fermi's pioneering work in bombarding heavy elements with neutrons, and inducing radioactivity (for which he won his Nobel prize).

In Berlin in 1938 Hahn and Strassman exposed uranium to slow neutrons and discovered that barium was produced.

Meitner and Frisch, in Sweden, explained the process as fission (a term coined by Frisch) and calculated correctly the energy liberated.

Hahn alone was awarded the Nobel Prize, news that he heard whilst incarcerated at Farm Hall near Cambridge. The decision of the committee not to co-award the prize to Meitner was, and remains, controversial.



Otto Hahn (1879-1968); Nobel Prize in Chemistry 1944; German.



Fritz Strassmann (1902-1980); German.



Lise Meitner (1878-1968); nominated 19 times for Nobel Prize in Chemistry, 29 times for Nobel Prize in Physics; 'the German [sic] Marie-Curie' (Einstein) Austrian-Swedish.



Otto Frisch (1904-1979); nephew of Meitner; Austrian-British.

History of fission: chain reactions

In 1933 Leo Szilard read an article by Rutherford, describing how his students Cockcroft and Wilson had produced energy by disintegrating lithium through proton bombardment.



Leo Szilard (1898-1964); Hungarian-German-USA citizenship

Rutherford said, though interesting, nuclear disintegration was not a practical way to generate power. Szilard, however, speculated that reactions might exist that also produced the recently discovered neutron, and that this neutron could initiate further reactions…

With the discovery of fission, this idea became realisable. In 1939 Szilard showed that fissioning uranium produced neutrons and immediately understood the significance of his discovery ("That night, there was very little doubt in my mind that the world was headed for grief."). Nonetheless, he drafted the letter to President Roosevelt that he persuaded Einstein to sign, and which ultimately led to the Manhattan Project.

History of fission: the Einstein-Szilard letter

Albert Einstein
Old Grove Rd.
Nassau Point
Peconic, Long Island

August 2nd, 1939

F.D. Roosevelt,
President of the United States,
White House
Washington, D.C.

Sir:

Some recent work by E.Fermi and L. Szilard, which has been communicated to me in manuscript, leads me to expect that the element uranium may be turned into a new and important source of energy in the immediate future. Certain aspects of the situation which has arisen seem to call for watchfulness and, if necessary, quick action on the part of the Administration. I believe therefore that it is my duty to bring to your attention the following facts and recommendations:

In the course of the last four months it has been made probable - through the work of Joliot in France as well as Fermi and Szilard in America - that it may become possible to set up a nuclear chain reaction in a large mass of uranium,by which vast amounts of power and large quantities of new radium-like elements would be generated. Now it appears almost certain that this could be achieved in the immediate future.

This new phenomenon would also lead to the construction of bombs, and it is conceivable - though much less certain - that extremely powerful bombs of a new type may thus be constructed. A single bomb of this type, carried by boat and exploded in a port, might very well destroy the whole port together with some of the surrounding territory. However, such bombs might very well prove to be too heavy for transportation by air.

-2-

The United States has only very poor ores of uranium in moderate quantities. There is some good ore in Canada and the former Czechoslovakia, while the most important source of uranium is Belgian Congo.

In view of this situation you may think it desirable to have some permanent contact maintained between the Administration and the group of physicists working on chain reactions in America. One possible way of achieving this might be for you to entrust with this task a person who has your confidence and who could perhaps serve in an inofficial capacity. His task might comprise the following:

a) to approach Government Departments, keep them informed of the further development, and put forward recommendations for Government action, giving particular attention to the problem of securing a supply of uranium ore for the United States;

b) to speed up the experimental work,which is at present being carried on within the limits of the budgets of University laboratories, by providing funds, if such funds be required, through his contacts with private persons who are willing to make contributions for this cause, and perhaps also by obtaining the co-operation of industrial laboratories which have the necessary equipment.

I understand that Germany has actually stopped the sale of uranium from the Czechoslovakian mines which she has taken over. That she should have taken such early action might perhaps be understood on the ground that the son of the German Under-Secretary of State, von Weizsäcker, is attached to the Kaiser-Wilhelm-Institut in Berlin where some of the American work on uranium is now being repeated.

Yours very truly,
A. Einstein
(Albert Einstein)

History of fission: the first nuclear reactor

The initial motivation for a uranium nuclear reactor was to manufacture ^{239}Pu for a bomb.

$$^{238}\mathrm{U} + \mathrm{n} \rightarrow {}^{239}\mathrm{U} + \gamma \quad \underset{\text{mins}}{23} \quad {}^{239}\mathrm{U} \rightarrow {}^{239}\mathrm{Np} + e^- + \bar{\nu}_e \quad \underset{\text{days}}{2.3} \quad {}^{239}\mathrm{Np} \rightarrow {}^{239}\mathrm{Pu} + e^- + \bar{\nu}_e$$

'Chicago Pile 1' was overseen by Fermi, Szilard *et al.*

It was situated under a stand of a sports ground of the University of Chicago.

It used natural uranium, so a lot was needed, and a graphite modulator. 330 t of graphite; 4.9 of uranium metal; and 41 t of uranium oxide.

2 Dec 1942: the world's first human-made self-sustaining chain reaction.

It delivered a power of ~0.5 W (sic).



One of 29 'piles' that made up reactor.

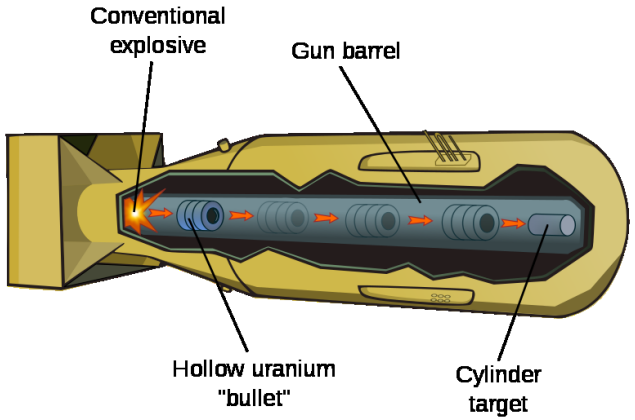
First reactor for civil purposes: Obninsk in Soviet Union, launched in 1954 (5 MW).

History of fission: the bomb

The Manhattan Project (1942-1946) was an enormous undertaking involving many of the most prominent Allied scientists of that era. The grim outcome was the dropping of the Hiroshima (6/8/45) and Nagasaki (9/8/45) bombs.

Hiroshima bomb ('Little Boy')

Built from 80% enriched uranium.
64 kg surrounded by neutron reflector, comprising 2.5 critical masses.

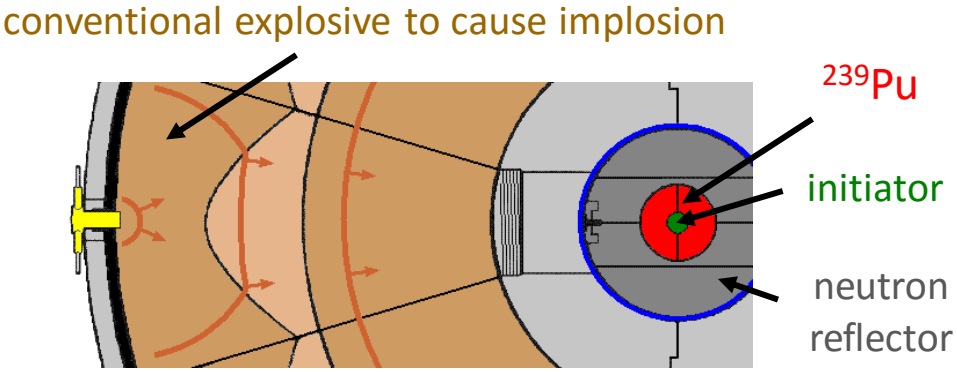


Based on 'gun' assembly principle, where 'bullet' & target are individually sub-critical.

< 1 kg fissioned; energy output 63 TJ.

Nagasaki bomb ('Fat Man')

Contained 6.2 kg of ^{239}Pu produced in reactor, surrounded by ^{238}U 'neutron reflector' or 'tamper'.



Based on implosion principle, where inner core of plutonium is crushed to critical density, while initiator produces neutrons to start reaction.

~1 kg fissioned; energy output 88 TJ.

Fusion reactions

Fusion reactions are inhibited by the Coulomb repulsion – the same barrier encountered in α decay, but now to be penetrated from outside. In most favourable case (d + t) this is 0.4 MeV high. Easy to surmount this in an accelerator, but no so in reactor where we must rely on temperature. *e.g.* at 10^8K the mean energy is 3/2 $k_B T$= 0.13 MeV.

With this in mind, here are the most interesting possibilities:

'D-D' reactions

$$d + d \rightarrow {}^{3}_{2}\mathrm{He} + n + 3.3\ \mathrm{MeV}$$

$$d + d \rightarrow t + p + 4.0\ \mathrm{MeV}$$

'D-T' reaction

$$d + t \rightarrow n + {}^{4}_{2}\mathrm{He} + 17.6\ \mathrm{MeV}$$

$$^{6}_{3}\mathrm{Li} + n \rightarrow t + {}^{4}_{2}\mathrm{He} + 4.8\ \mathrm{MeV}$$

d deuteron, *i.e. np* (^{2}H)
t triton, *i.e. nnp* (^{3}H)

(Note that a reaction such as $p + p \rightarrow {}^{2}\mathrm{He}$ is not viable because ^{2}He is incredibly weakly bound and there is also a higher Coulomb barrier for this process.)

The D-T reaction is the favoured one. In current experiments tritium is injected as fuel, but it is hoped in future this can be 'bred' through injecting $^{6}_{3}\mathrm{Li}$ & exploiting the $^{6}_{3}\mathrm{Li} + n$ process.

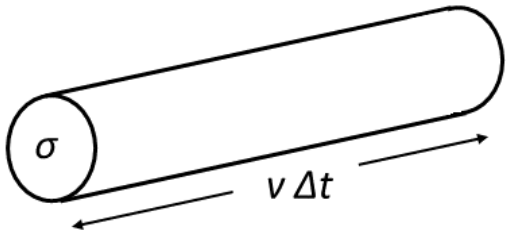
A reactor must keep fuel hot and dense enough to enable D-T reaction to occur, and then to sustain more reactions through the energy liberated.

Fusion cross section

Quantum tunnelling is required for nuclei to fuse. In a fusion reactor / star the bulk of nuclei, close to the mean energy, will have little chance of managing this. Those that fuse come from the higher tails of the Maxwell-Boltzmann distribution. Indeed, there is a 'sweet spot', called the Gamow peak, where probability is maximal.

Say we have n_1 particles / unit volume of type 1 (*e.g.* deuterons) and n_2 of type 2 (*e.g.* tritium), that have a relative velocity v, & a cross-section σ.

interaction volume swept out by particle 1 in Δt in particle 2 rest frame



Then the volumetric reaction rate *r* (number of interactions / unit volume / s) is:

$$r = n_1 n_2 v \sigma$$

with $v = \sqrt{\frac{2E}{\mu}}$, where $\sqrt{E}$ is energy in this frame and μ is the reduced mass of system.

But cross-section has itself a velocity / energy dependence….

Fusion cross section

Naively, we can consider cross section to be made up of three factors:

intrinsic 'size' of interacting nuclei, which is approximately de Broglie λ squared

$$\left(\frac{h}{p}\right)^2 \propto \frac{1}{E}$$

probability of tunnelling, which is same in form as for α decay

$$e^{-2G}$$

a term expressing probability of interaction, once barrier passed

$S(E)$ this is some slowly varying function of *E* whose form we won't have to worry about, and into which we will subsume other constants

For α decay we had

$$G \approx \frac{\sqrt{2M_\alpha T_\alpha}}{\hbar}\frac{2(Z-2)e^2}{4\pi\varepsilon_0 T_\alpha}\frac{\pi}{2}$$

making the substitutions

$$M_\alpha \rightarrow \mu \quad \text{(reduced mass)}$$
$$2 \rightarrow Z_1$$
$$(Z-2) \rightarrow Z_2$$
$$T_\alpha \rightarrow E$$

yields

$$e^{-C/\sqrt{E}} \quad \text{with} \quad C = Z_1 Z_2 \sqrt{\frac{A_1 A_2}{A_1 + A_2}}\pi\alpha\sqrt{2m_p c^2} = Z_1 Z_2 \sqrt{\frac{A_1 A_2}{A_1 + A_2}}\ 31.4\ (\text{keV})^{1/2}$$

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So we end up with:

$$\sigma(E) = \frac{S(E)}{E} e^{-C/\sqrt{E}}$$

For α decay we had

$$G \approx \frac{\sqrt{2}}{\hbar} \frac{}{4\pi\varepsilon_0 T_\alpha} \frac{}{2}$$

$\rightarrow \mu$ (reduced mass)

$\rightarrow Z_1$

$(Z-2) \rightarrow Z_2$

$T_\alpha \rightarrow E$

yields

$$e^{-C/\sqrt{E}} \quad \text{with} \quad C = Z_1 Z_2 \sqrt{\frac{A_1 A_2}{A_1 + A_2}} \pi\alpha \sqrt{2 m_p c^2} = Z_1 Z_2 \sqrt{\frac{A_1 A_2}{A_1 + A_2}}\ 31.4\ (\text{keV})^{1/2}$$

Gamow peak

To account for energy dependence in initial state particles must include Maxwell-Boltzmann:

$$r(E)\mathrm{d}E = n_1 n_2 \sqrt{\frac{2E}{\mu}} \frac{S(E)}{E} e^{-C/\sqrt{E}} \sqrt{\frac{4}{\pi}} \left(\frac{1}{k_B T}\right)^{3/2} \sqrt{E}\, e^{-E/k_B T}\, \mathrm{d}E$$

$$= n_1 n_2 \sqrt{\frac{8}{\pi\mu}} \left(\frac{1}{k_B T}\right)^{\frac{3}{2}} S(E) e^{-C/\sqrt{E}}\, e^{-E/k_B T}\, \mathrm{d}E$$

At what energy, E_0, is this differential rate maximal? $S(E)$ is slowly varying, so approx. by $S(E_0)$.

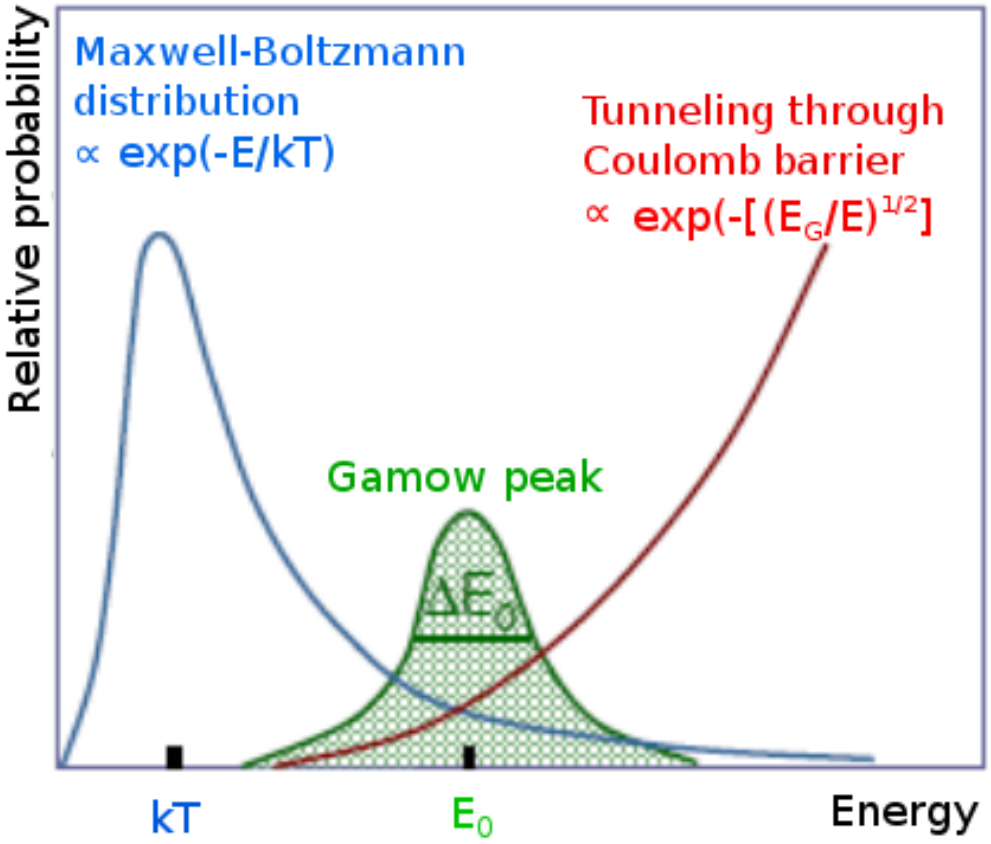
$$r = n_1 n_2 \sqrt{\frac{8}{\pi\mu}} \left(\frac{1}{k_B T}\right)^{\frac{3}{2}} S(E_0) \int_0^\infty e^{-C/\sqrt{E}}\, e^{-E/k_B T}\, \mathrm{d}E$$

Then we must find the stationary point of the function in the exponent. This gives:

$$E_o = \left(\frac{C k_B T}{2}\right)^{2/3}$$

which is the location of the Gamow peak, being the 'sweet spot' allowed by Maxwell-Boltzmann tail and quantum-tunnelling.

Gamow peak



Position of peak:

$$E_o = \left(\frac{Ck_BT}{2}\right)^{2/3}$$

Relative height of peak:

$$e^{-1.89\left(C^2/k_BT\right)^{\frac{1}{3}}}$$

So for case of *d* + *t* at, say, T=10^8 K, we have C=34.5 (keV)$^{1/2}$ and E_0=28 keV, to be compared to mean energy of $\frac{3}{2}k_BT$ = 12 keV. At this energy the tunnelling probability $e^{-C/\sqrt{E}}$ is 1 x 10^{-3}. This is not bad. The real challenge, however, is keeping n_1 and n_2 high at these temperatures.

Controlled fusion: confinement

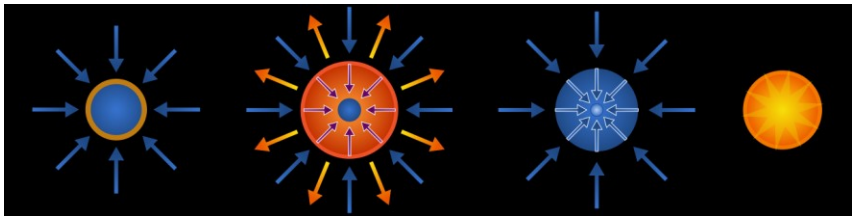
Challenge of controlled fusion is to attain temperatures of ~10^8K while keeping high enough density for long enough for sufficient reactions to occur. Two main strategies:

Inertial confinement

Expose pellet containing *d* + ^{3}H to laser beam, or X-rays from laser.

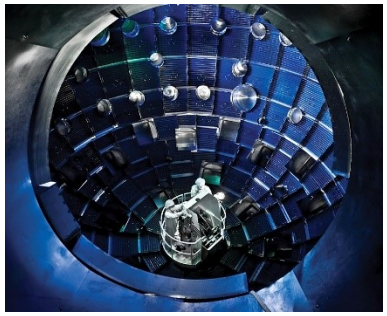
laser beams heat surface

capsule implodes, density 20x lead



fuel compressed by blow-off of surface

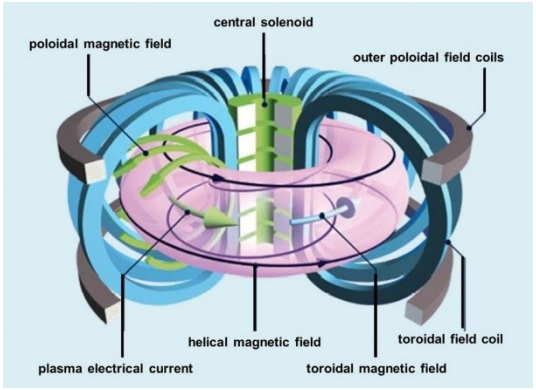
core ignites at 10^8 K, burn spreads through rest of fuel



e.g. National Ignition Facility (NIF), USA

Magnetic confinement

Use a configuration of magnetic fields to control plasma and minimise wall collisions, *e.g.* 'tokamak'.





e.g. ITER, France (to begin late 2025)

Controlled fusion: confinement

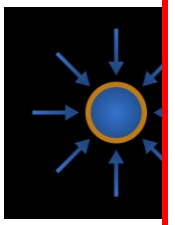
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Inertial confinement

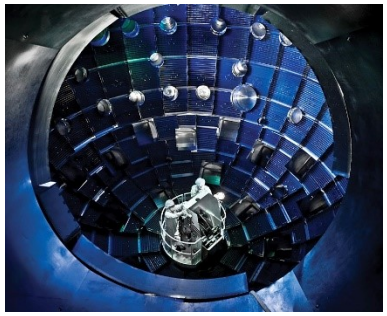
Exp… pellet containing *d* + ^{3}H…

to l…

laser bea…
heat surfa…



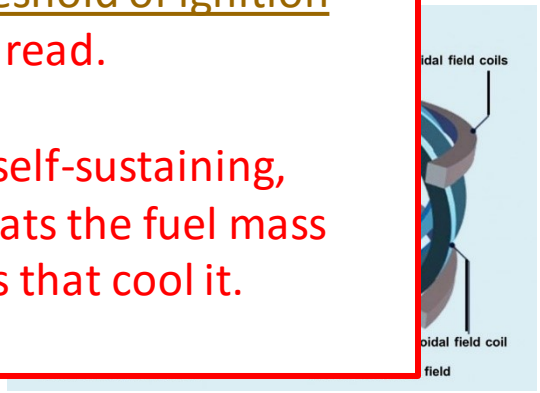
…blow-off of surface… 10^8 K, burn spreads through rest of fuel



e.g. National Ignition Facility (NIF), USA

Magnetic confinement

Use a configuration of magnetic fields …nimise …mak'.





e.g. ITER, France (to begin late 2025)

Recent developments

August 2021: NIF reported 'ignition' or 'threshold of ignition' depending on which report you read.

Ignition is when nuclear fusion becomes self-sustaining, and the energy given off by the reaction heats the fuel mass more rapidly than the loss mechanisms that cool it.

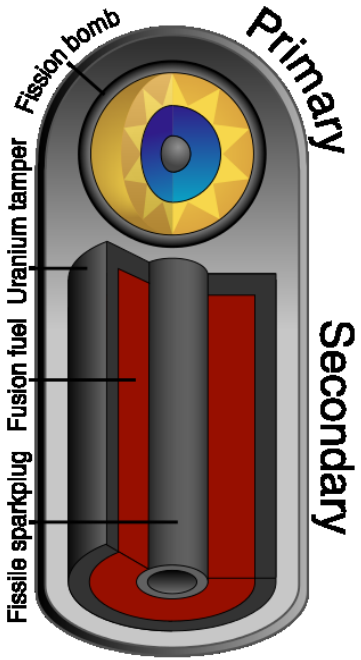
The thermonuclear ('H') bomb

In contrast to the continued struggles to achieve self-sustaining fusion reactions for civilian use, such reactions have been initiated since 1951 in thermonuclear ('hydrogen') bombs. Horrible objects.

Edward Teller often considered 'father of the H bomb' (as Oppenheimer is called 'father of the atomic [*i.e.* fission] bomb').



Edward Teller (1908-2003); Hungarian-USA citizenship



Teller-Ulam configuration

A fission bomb compresses and heats the fusion fuel, usually lithium deuteride, from which tritium is produced from the neutron flux of the fission, which then reacts with the deuterons and explodes. The neutron flux liberated in this reaction induces fission in the ^{238}U tamper.

In the 'neutron bomb' variant, the uranium tamper is omitted to minimise blast, but maximise killing power through neutron flux ("kills people and spares buildings"). Vile!

Can be built to any size. The Soviet 'Tsar Bomba' of the early 1960s had an output of 209 PJ, *i.e.* >2000x Nagasaki bomb.

Solar fusion



Star forms from gas/dust collapsing under gravity, until temperature at core reaches that of $\sim 1.5 \times 10^7$ K (centre of sun) and nuclear burning begins.

At first, the dominant process comes from interactions involving *p*, *d* and ^{3}He in the so-called *pp* chain. Why don't heavier elements interact ?

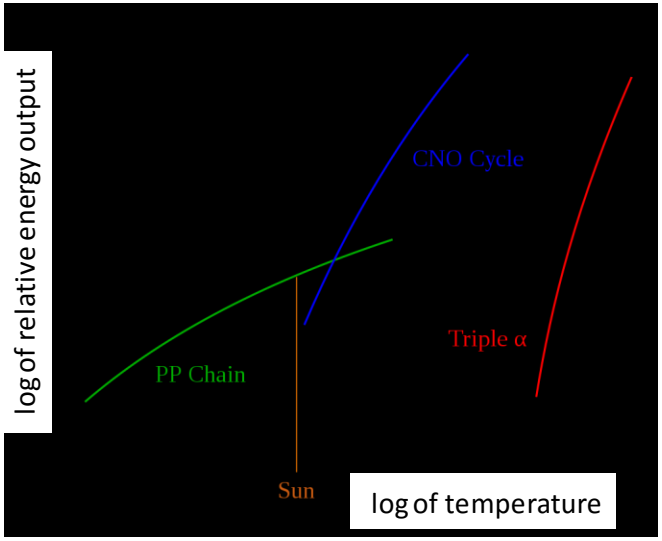
1. They are not yet present (doh!)
2. Temperature of star needs to rise so that Gamow peak is within reach, *e.g.*

Process	$p + p$	${}^3_2\mathrm{He} + {}^3_2\mathrm{He}$	$p + {}^{14}_7\mathrm{N}$	${}^{16}_8\mathrm{O} + {}^{16}_8\mathrm{O}$
E_0	5.9 keV	21.5 keV	26.6 keV	238.3 keV

Energy output, and predominance of each process, will then vary with temperature.

We will only discuss *pp* chain.

As stellar evolution progresses, heavier and heavier elements are produced until the endpoint of iron is reached, after which collapse ensure.



pp chain

The *pp* chain is made up of the following steps:

$$p + p \rightarrow d + e^{+} + \upsilon_e + 0.4\ \text{MeV} \qquad (1)$$

$$p + d \rightarrow {}^{3}_{2}\text{He} + \gamma + 5.5\ \text{MeV} \qquad (2)$$

$${}^{3}_{2}\text{He} + {}^{3}_{2}\text{He} \rightarrow p + p + {}^{4}_{2}\text{He} + 12.9\ \text{MeV} \qquad (3)$$

$$e^{+} + e^{-} \rightarrow \gamma + \gamma + 1.0\ \text{MeV} \qquad (4)$$

2 x (1) + 2 x (2) + 1 x (3) + 2 x (4) →

$$4p + 4e^{-} \rightarrow {}^{4}_{2}\text{He} + 2e^{-} + 2\bar{\nu}_e + 6\gamma + 26.7\ \text{MeV}$$

Of particular interest is reaction (1). After the Coulomb barrier is overcome a weak interaction β^{+} decay is required. Because of the low energy scale involved, this reaction proceeds very slowly, with each proton taking 9000 years to suffer such a fate. Sometimes called the 'bottleneck' it limits the overall rate of the chain.

Note that some of the liberated energy is 'lost' to neutrinos. For this chain it is only 2%, but there are other variants where the proportion is higher.

pp chain

The *pp* chain is made up of the following steps:

There are other routes whereby the two $^4_2\mathrm{He}$ nuclei can be produced, for example:

$$^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma + 1.6\ \mathrm{MeV}$$

$$^7_4\mathrm{Be} + p \rightarrow {}^8_5\mathrm{B} + \gamma + 0.1\ \mathrm{MeV}$$

$$^8_5\mathrm{B} \rightarrow {}^8_4\mathrm{Be}^{*} + e^{+} + \nu_e + 14.0\ \mathrm{MeV}$$

$$^8_4\mathrm{Be}^{*} \rightarrow {}^4_2\mathrm{He} + {}^4_2\mathrm{He} + 3.0\ \mathrm{MeV}$$

$$e^{-} + e^{+} \rightarrow \gamma + \gamma + 1.0\ \mathrm{MeV}$$

Here, on average, the neutrino carries away 7.2 MeV. Studying these higher energy neutrinos (*much* easier than the sub-MeV case) provides a valuable check of solar models, and also of the nature of neutrinos themselves (*i.e.* oscillations and mass).

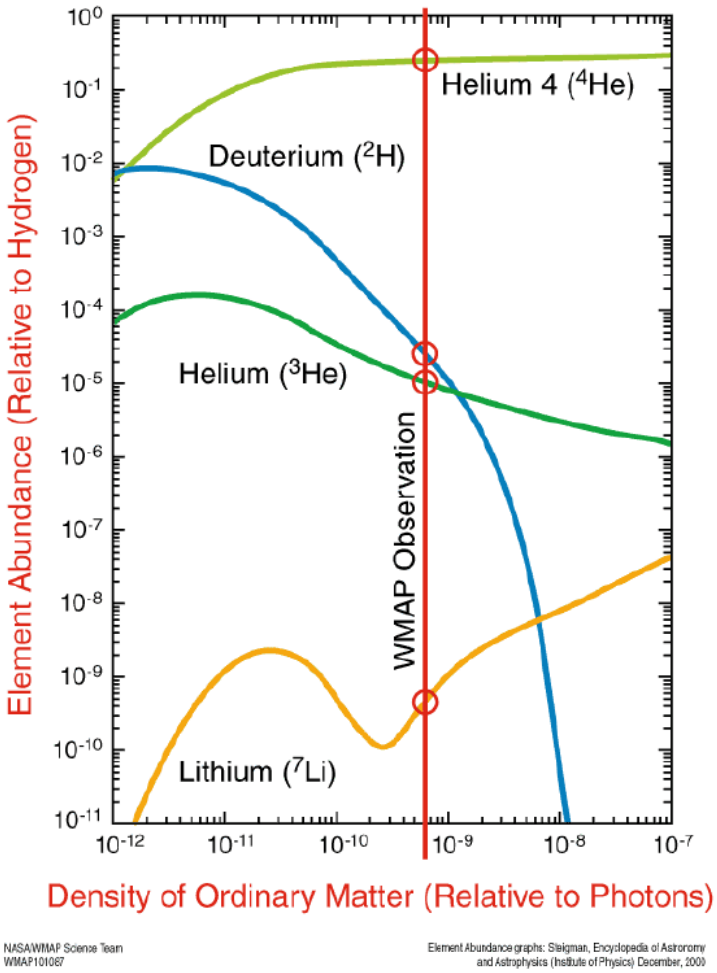
Of pa… e a weak intera… , this reacti… such a fate. …in.

Note that some of the liberated energy is 'lost' to neutrinos. For this chain it is only 2%, but there are other variants where the proportion is higher.

Nucleosynthesis – multiple sources

Big bang produced very light nuclei: 75% H, 24% He, 1% other (lithium, deuterium…).

Comparing the exact proportions, with other cosmic parameters, is a very sensitive test of Big Bang models.



Nucleosynthesis – multiple sources

Big bang produced very light nuclei: 75% H, 24% He, 1% other (lithium, deuterium…).

We have learned that most elements up to iron are produced by fusion process in stars.

There are however, certain light elements that get by-passed in this process, *e.g.* Li, Be and B. These can be manufactured by cosmic-ray spallation, *e.g.*

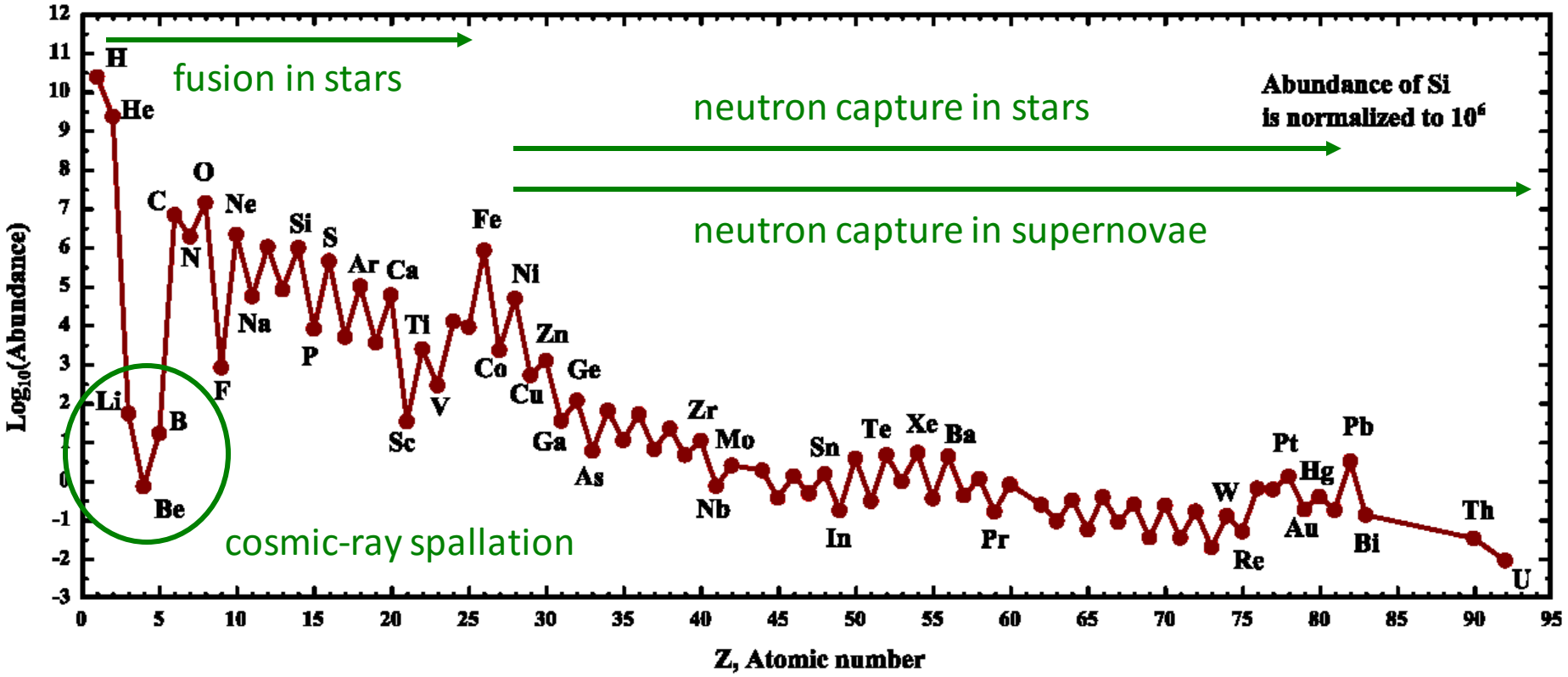
$$p + {}^{12}_{16}\mathrm{C} \rightarrow {}^{9}_{4}\mathrm{Be} + {}^{3}_{2}\mathrm{He} + p$$

can occur for cosmic-ray protons in the GeV regime

Nuclei heavier than iron must be produced by neutron capture. Some of these can be built by the relatively slow accumulation of neutrons that are present within the star ('s-process' for slow), but the very heavy elements (A>209) requite rapid accumulation of multiple neutrons ('r-process'), so that they can be formed before β decay occurs. The very high fluxes of neutrons for the r-process are only available in supernovae.

Nucleosynthesis

Abundance of elements in the solar system, normalised to Si.



Scattering theory

Scattering theory: topic overview

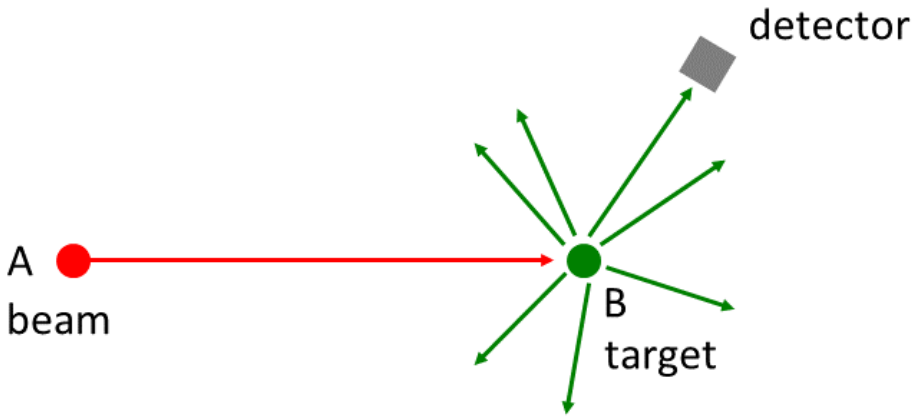
- Cross section: concept and applications
- Classical cross-section calculations
- Cross-section calculations in quantum mechanics
- Scattering off extended distributions: form factors
- Resonances and the Breit-Wigner function

Scattering theory: topic overview

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Differential cross section

Consider elastic scattering of a beam particle *A* off a target particle *B*. Incoming flux is j_A particles / unit area / unit time. Rate of scattered particles / solid angle / unit time varies with angle & is $n(\theta, \phi)$. Counted by a detector that subtends a small solid angle $\mathrm{d}\Omega$.



The number of detected particles $n(\theta, \phi)\mathrm{d}\Omega$ must be proportional to the incoming flux. We define the proportionality constant as $\mathrm{d}\sigma$ and the *differential cross section* $\mathrm{d}\sigma/\mathrm{d}\Omega$:

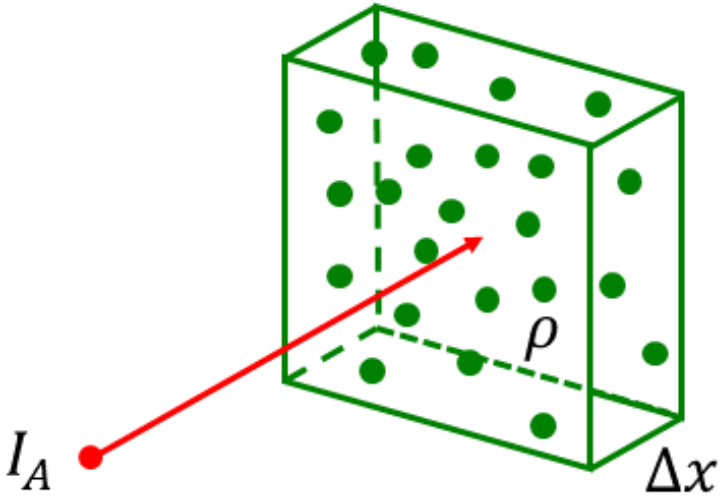
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{n(\theta, \phi)}{j_A}$$

with units m^2/sr.

Many targets and cross section

Now generalise to the case when we have many targets, N_t. Then $n(\theta,\phi)d\Omega \propto N_t\, j_A$. Given a block of thickness Δx with target density ρ we have $N_t = \rho\, \Delta x \times \text{area}$, and so, now referring to an incoming beam current I_A, the differential cross section becomes:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{n(\theta,\phi)}{\rho\, \Delta x\, I_A}$$



To get the total cross section we integrate:

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\, \mathrm{d}\Omega$$

So in this case:

$$\sigma = \frac{1}{\rho I_A} \frac{\int n(\theta,\phi)d\Omega}{\Delta x} = -\frac{1}{\rho I_A} \frac{\Delta I_A}{\Delta x}$$

where we have used $\int n(\theta,\phi)d\Omega = -\Delta I_A$, the reduction in current from scattering.

Other processes and choice of units

We introduced discussion for elastic collisions, *i.e.*

$$A + B \rightarrow A + B$$

but a cross section may also be similarly defined for an inelastic collision, *e.g.*

$$A + B \rightarrow C + D$$

or absorption, where a single state is formed, *i.e.*

$$A + B \rightarrow C$$

The total cross section will be the sum of these:

$$\sigma_{\text{total}} = \sigma_{\text{elastic}} + \sigma_{\text{inelastic}} + \sigma_{\text{absorption}}$$

We will find that these cross sections are generally energy dependent (already seen – recall neutron-nuclear cross sections in fission discussion.)

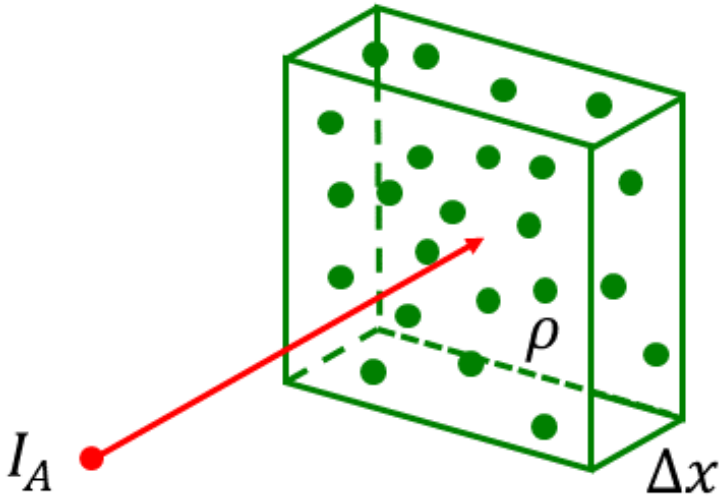
Unsurprisingly cross section has dimensions of area. Typically in nuclear and particle physics we use the unit of barn, where $1 \text{ barn} = 10^{-28} \text{ m}^2 = 100 \text{ fm}^2$.

In nuclear physics interactions can have cross sections of 1, 10, 100… barns. In particle physics, especially for the weak interaction, they can be *much* smaller.

Making use of the cross section

Trivially rearranging these expressions we see that if the cross section is known we may calculate the expected total rate of scattered particles $\mathrm{d}N/\mathrm{d}t$:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \rho\, \Delta x\, I_A\, \sigma$$



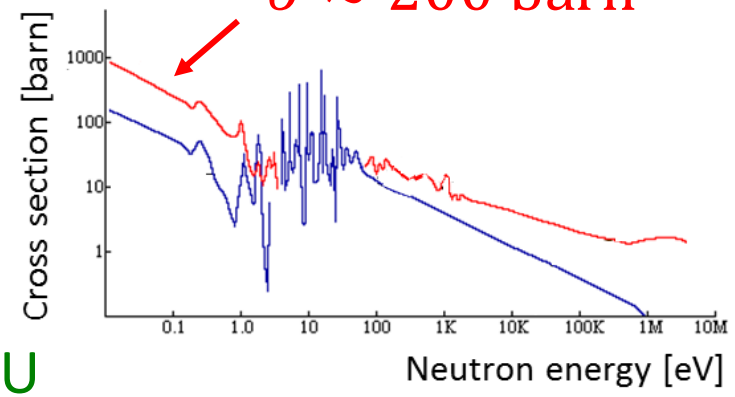
This is a useful expression to remember !

Or, by measuring the rate, and knowing our experimental efficiency, we can determine σ.

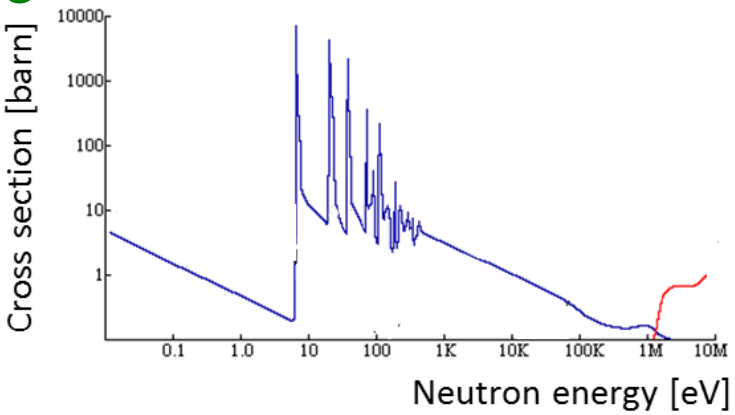
Making use of the cross section: example

Let's calculate mean distance a thermal neutron of 0.1 eV travels in reactor fuel before inducing a fission. Let's assume 3% enriched ^{235}U & that other cross-sections are negligible.

^{235}U



^{238}U



Starting from this:

$$\sigma = -\frac{1}{\rho I_A}\frac{\Delta I_A}{\Delta x}$$

We see the 'current' of a neutron beam will attenuate thus:

$$I_A(x) = I_A(0)e^{-\rho\sigma\, x}$$

with a mean-free path of $\lambda = 1/\rho\sigma$, which in this case is the mean distance a neutron will go before inducing a fission.

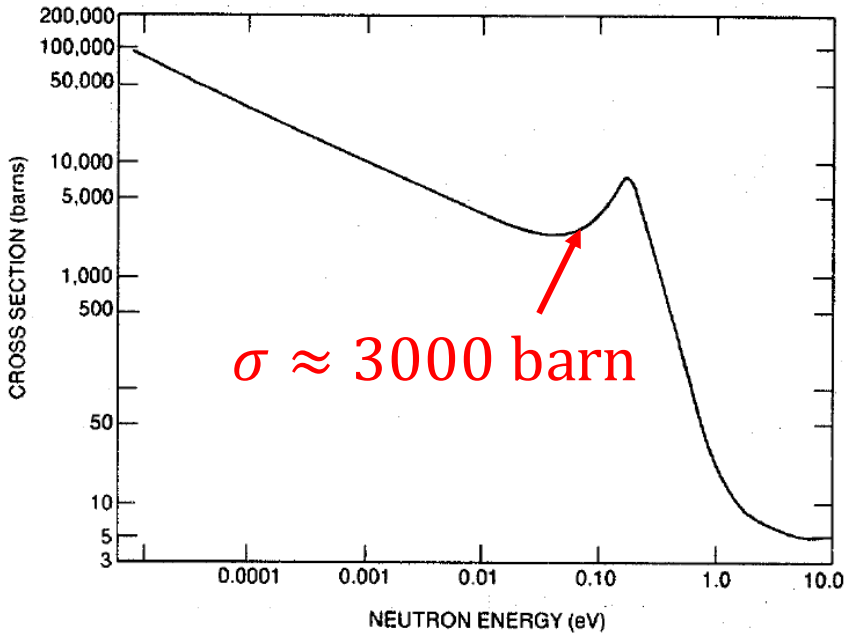
Density of uranium = 1.9×10^4 kg m^{-3}, which means density of ^{235}U targets is

$\sim 0.03\ (1.9 \times 10^4 / 238 \times m_p)$
$= 1.4 \times 10^{27}$ targets m^{-3}.

So $\lambda = 1/\rho\sigma = 1/(1.4 \times 10^{27} \times 200 \times 10^{-28}) =$ 3.6 cm.

Making use of the cross section: example

Now let's calculate how far the 0.1 eV travels in the cadmium control rod before absorption.



[Garber and Kinsey, `Neutron cross sections, Vol II, curves', BNL 325 (1976)]

This is total cross section, but in regime of interest it is dominated by absorption, so we will assume them to be same.

Density of cadmium 8.7×10^3 kg m^{-3}. So using an atomic mass of 112 gives a target density of

$$\sim (8.7 \times 10^3 / 112 \times m_p)$$
$$= 4.7 \times 10^{28} \text{ targets m}^{-3}.$$

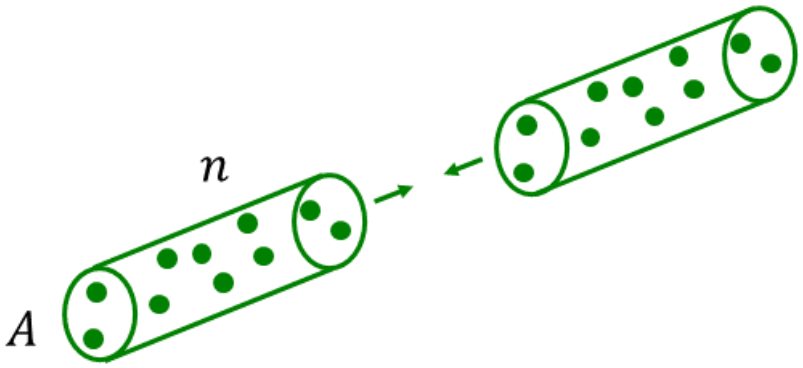
The mean-free path is then

$$\lambda = 1/\rho\sigma$$
$$= 1/(4.7 \times 10^{28} \times 3000 \times 10^{-28})$$
$$\sim 7 \times 10^{-5} \text{ m}$$

i.e. much shorter than corresponding number for fuel, which is good news for reactor control.

Making use of the cross section: colliders

Let's look ahead to particle physics, where colliders are an important tool. Here there is no distinction between beam & target, rather the colliding bunches function as both.



A is area of bunch that makes up beams.

n is number of particles in bunch.

Here it is usual to write $\frac{\mathrm{d}N}{\mathrm{d}t} = L\,\sigma$ where L is a quantity known as the *luminosity*, measured in $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. For the symmetric bunches shown $L \propto n^2/A^2$.

Maximising luminosity is often just as important as maximising collision energy In optimising discovery potential, as it increases the event rate for rare processes with low cross section.

Making use of the cross section: example

How many collisions per year of normal LHC operation will produce at least one beauty (*b*) quark in the LHCb detector at the LHC proton-proton collider at CERN?



Making use of the cross section: example

How many collisions per year of normal LHC operation will produce at least one beauty (b) quark in the LHCb detector at the LHC proton-proton collider at CERN?

You may assume: $\sigma = 150\ \mu b$ at the collision energy of 14 TeV — small number !

$L = 4\ \times 10^{32}$ cm^{-2}s^{-1} — large number !

Geometric acceptance of detector, ε = 40 %

Period of LHC operation $\sim 1/3$ of a year

$$\frac{\mathrm{d}N}{\mathrm{d}t} = L\,\sigma \quad \Rightarrow \quad N = \varepsilon\sigma \int L\,\mathrm{d}t$$

(including acceptance in integrated expression)

Let's first evaluate $\int L\,\mathrm{d}t$. This will have units of inverse area. It is common for to express this quantity in terms of inverse femtobarns = 10^{39} cm^{-2}, so lets do the same.

$$\int L\,\mathrm{d}t = 4 \times 10^{32}\ \times\ 10^{-39} \times\ 1 \times 10^{7} = 4\ \mathrm{fb}^{-1}$$

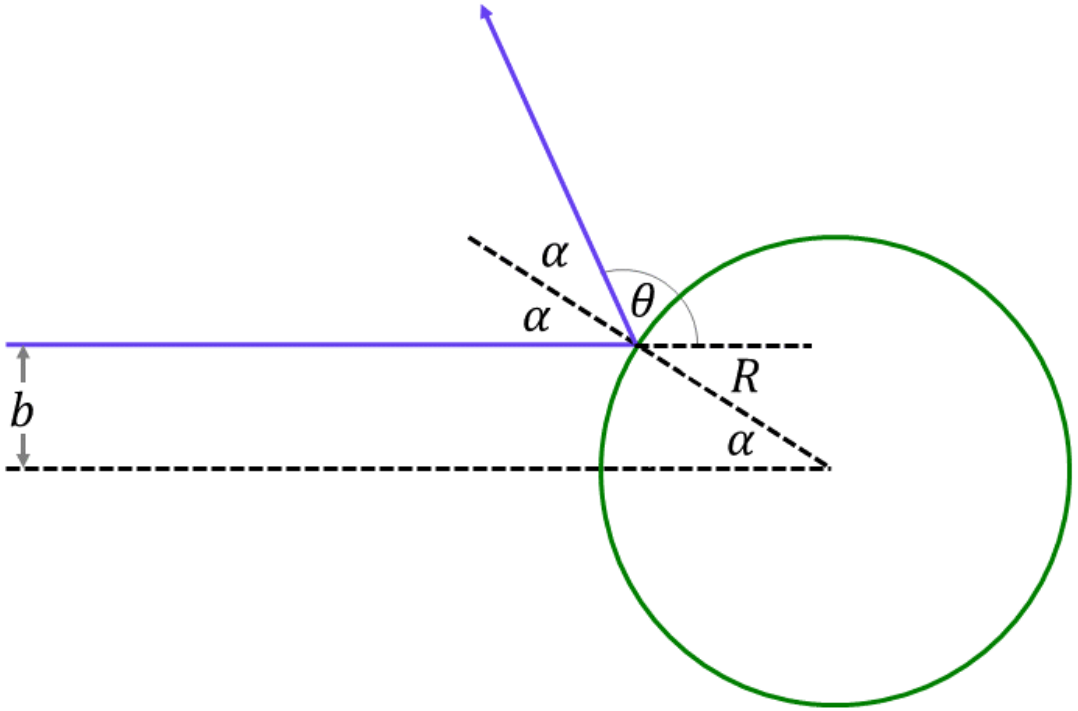
$$\sigma = 150\ \times 10^{9}\ \mathrm{fb} \quad \Longrightarrow \quad N = 2.4 \times 10^{11}$$ a lot !

Scattering theory: topic overview

- Cross section: concept and applications
- Classical cross-section calculations
- Cross-section calculations in quantum mechanics
- Scattering off extended distributions: form factors
- Resonances and the Breit-Wigner function

Classical cross-section calculation: hard-sphere scattering

Let us now calculate some cross sections. We will start using classical physics, and begin by considering scattering of a point-like projectile off a hard sphere of radius R.



b is known as the *impact parameter*

The projectile is scattered through θ

By making use of

$$2\alpha + \theta = \pi$$
$$b = R \sin \alpha$$

we can see that

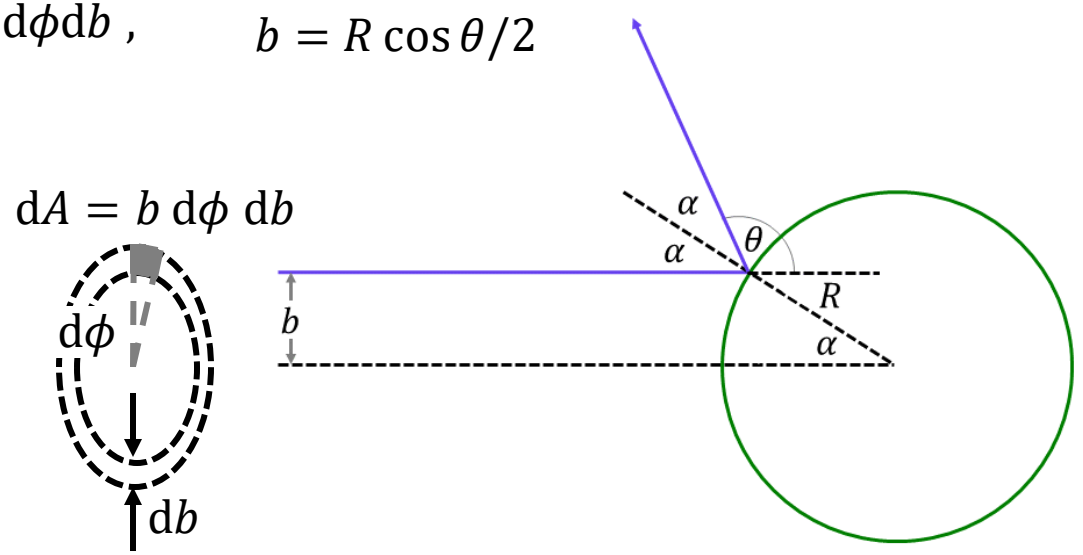
$$b = R \cos \theta/2$$

i.e. there is a direct relationship between impact parameter and scattering angle.

Classical cross-section calculation: hard-sphere scattering

Consider a flux j_A of projectiles passing through an area element $\mathrm{d}A = b\mathrm{d}\phi\mathrm{d}b$, these will be scattered into a solid-angle interval $\mathrm{d}\Omega$.

$b = R\cos\theta/2$



The number scattered into $\mathrm{d}\Omega$ / unit time is:

$$n(\theta,\phi)\mathrm{d}\Omega = |j_A\, b\, \mathrm{d}\phi\, \mathrm{d}b|$$

$$= \left|-j_A \frac{R^2}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\,\mathrm{d}\theta\,\mathrm{d}\phi\right|$$

(areas and solid angles intrinsically positive, so we have taken absolute values).

As $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ we have $n(\theta,\phi) = j_A\frac{R^2}{4}$ and from $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{n(\theta,\phi)}{j_A}$ we deduce

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{4}$$ and so $\boxed{\sigma = \pi R^2}$

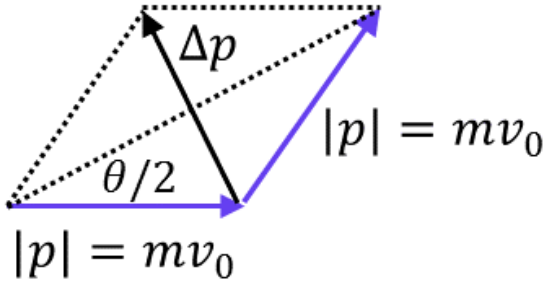
i.e. here the scattering cross section *is* the geometrical cross section (indeed we can identify $\mathrm{d}A$ with $\mathrm{d}\sigma$).

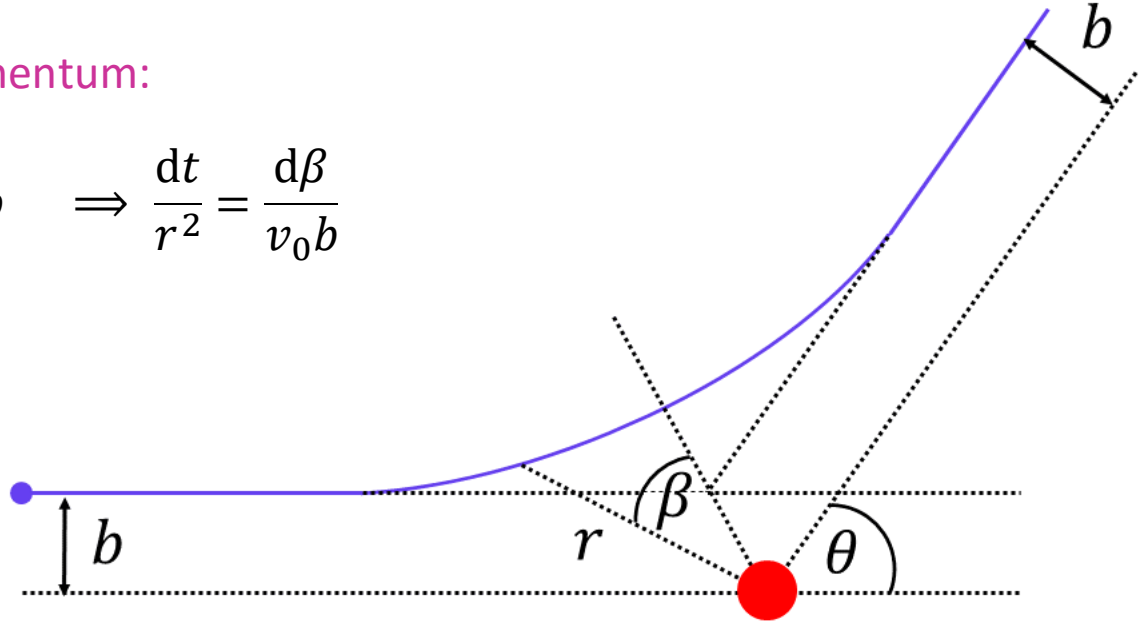
Classical cross-section calculation: Rutherford scattering

Model α particle and nucleus of charge *Ze* as point-like particles. Neglect recoil of nucleus. α particle has an impact parameter of *b*, initial velocity v_o, & is scattered through angle θ.

Conservation of angular momentum:

$$mr^2\frac{\mathrm{d}\beta}{\mathrm{d}t} = mv_0 b \quad \Longrightarrow \quad \frac{\mathrm{d}t}{r^2} = \frac{\mathrm{d}\beta}{v_0 b}$$





Evaluate impulse through momentum triangle and also integrating force over time.

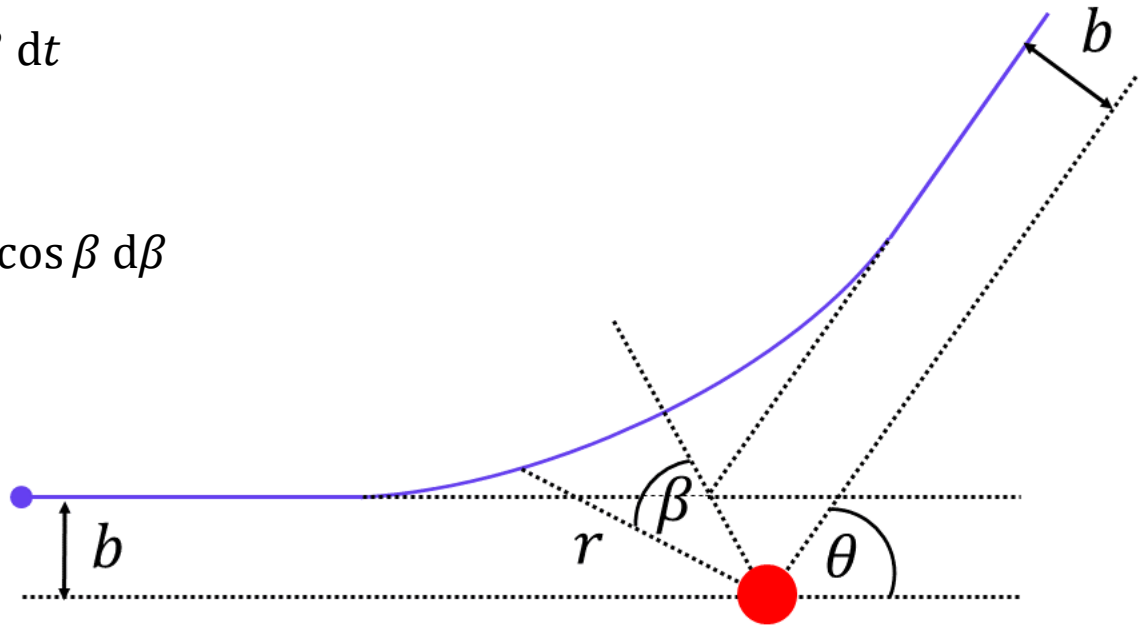
So $\Delta p = 2mv_0 \sin\frac{\theta}{2}$, but also $\Delta p = \int_{-\infty}^{+\infty} F\ \mathrm{d}t = \int_{-\infty}^{+\infty} \frac{2Ze^2}{4\pi\varepsilon_0 r^2}\cos\beta\ \mathrm{d}t$.

Classical cross-section calculation: Rutherford scattering

$$\Delta p = \int_{-\infty}^{+\infty} \frac{2Ze^2}{4\pi\varepsilon_0 r^2} \cos\beta \, \mathrm{d}t$$

$$= \int_{-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}^{\frac{\pi}{2}-\frac{\theta}{2}} \frac{2Ze^2}{4\pi\varepsilon_0 v_0 b} \cos\beta \, \mathrm{d}\beta$$

$$= \frac{Ze^2}{\pi\varepsilon_0 v_0 b} \cos\frac{\theta}{2}$$



but we also have

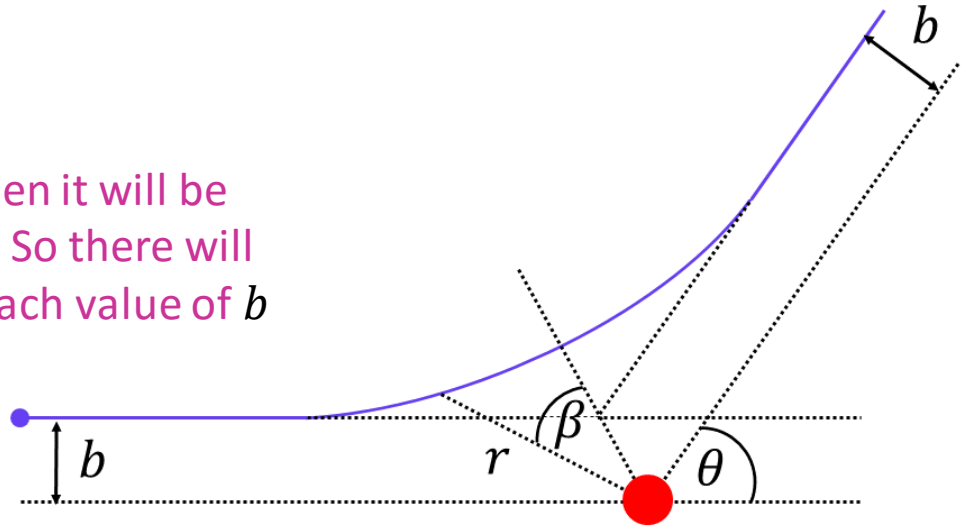
$$\Delta p = 2mv_0 \sin\frac{\theta}{2}$$

$$\Longrightarrow \quad b = \frac{Ze^2}{4\pi\varepsilon_0 T} \cot\frac{\theta}{2} \qquad \text{where } T = \frac{1}{2}mv_0^2.$$

Classical cross-section calculation: Rutherford scattering

$$b = \frac{Ze^2}{4\pi\varepsilon_0 T}\cot\frac{\theta}{2}$$

If a particle has a value of $b < b_0$ then it will be scattered through an angle $\theta > \theta_0$, So there will be a cross section associated with each value of b



$$\sigma(\theta > \theta_0) = \pi b_0^2$$

$$= \pi\left(\frac{Ze^2}{4\pi\varepsilon_0 T}\right)^2 \cot^2\frac{\theta}{2}$$

and we can write $$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \pi\left(\frac{Ze^2}{4\pi\varepsilon_0 T}\right)^2 \frac{1}{2}\frac{\sin\theta}{\sin^4\frac{\theta}{2}}$$ (note, there is a sign flip, as cross section decreases as angle increases).

Classical cross-section calculation: Rutherford scattering

So we have

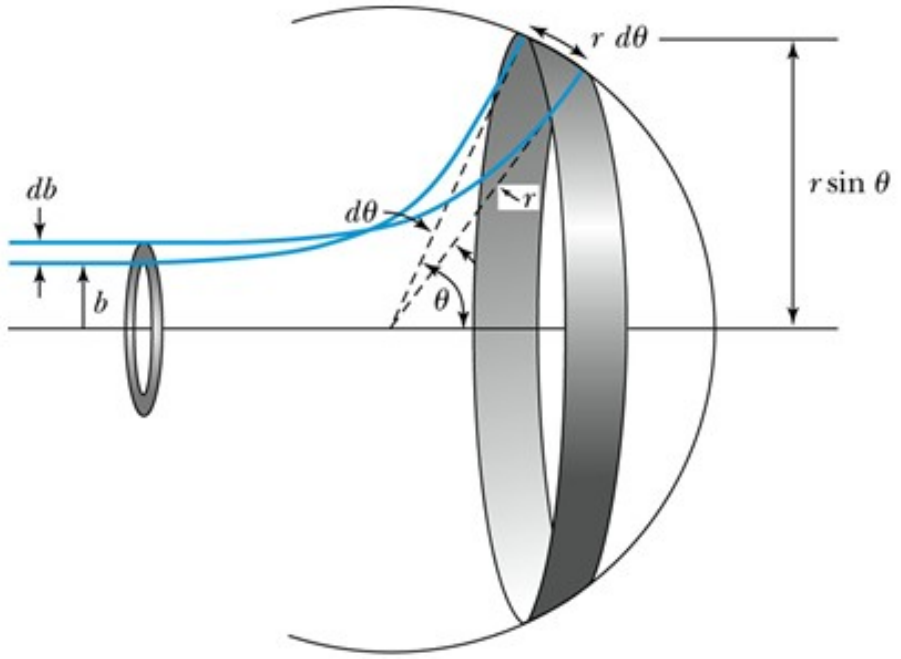
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \pi \left(\frac{Ze^2}{4\pi\varepsilon_0 T} \right)^2 \frac{1}{2} \frac{\sin\theta}{\sin^4 \frac{\theta}{2}}$$

and the scatter goes into an element of solid angle

$$\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta$$

yielding

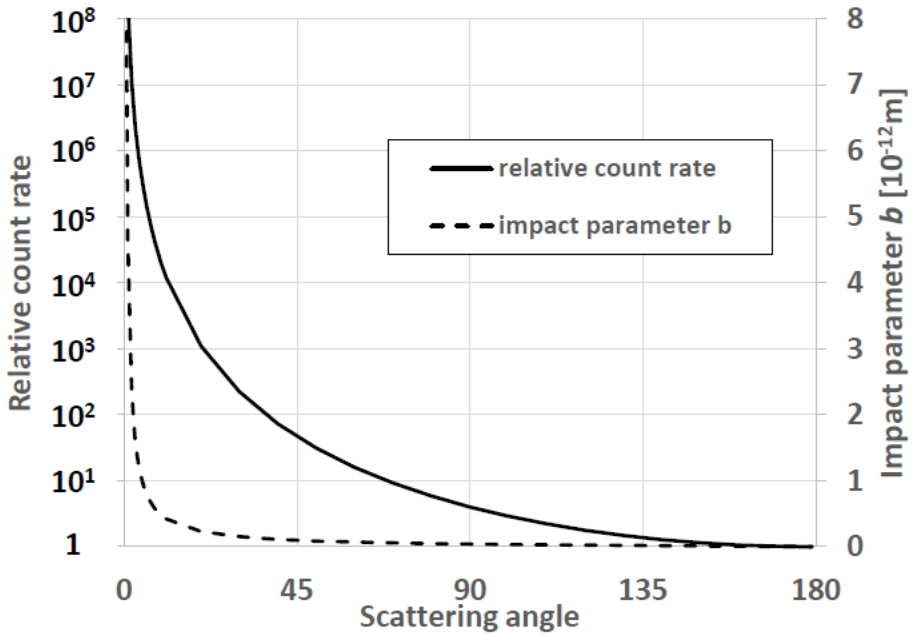
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4} \left(\frac{Ze^2}{4\pi\varepsilon_0 T} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}}$$



Classical cross-section calculation: Rutherford scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{Ze^2}{4\pi\varepsilon_0 T}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

Scattering of 5.59 MeV α particles off gold nucleus



- Even point-like particles have a cross section – geometrical picture breaks down.
- There is a singularity, so cross section only meaningful if integrated above cut off.

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Calculating the differential cross section in QM

We start from Fermi's Golden Rule (FGR) which comes from first-order perturbation theory.

Transition probability / unit time from *i* to *f*

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}\left|M_{if}\right|^2 \rho$$

with ρ a density-of-states factor

and M_{if} is the matrix element that contains the nuclear/particle physics

$$M_{if} = \langle \psi_f | \hat{H} | \psi_i \rangle = \int \psi_f^* \hat{H} \psi_i \mathrm{d}^3\vec{r}$$

with $\hat{H}$ the perturbing Hamiltonian, and $\psi_{i(f)}$ initial (final) plane waves

$$\psi_i = \frac{1}{\sqrt{V}} e^{i(\vec{k}.\vec{r} - \frac{Et}{\hbar})} \qquad \psi_f = \frac{1}{\sqrt{V}} e^{i(\overrightarrow{k'}.\vec{r} - \frac{Et}{\hbar})}$$

where we will assume *elastic scattering* so that $|\overrightarrow{k'}| = |\vec{k}|$ (and $E = E'$), and the wave functions are normalised to a volume *V* (this will drop out in final expression…).

Calculating the differential cross section in QM

So for a scattering potential $V(\vec{r})$ the matrix element becomes:

$$M_{if} = \frac{1}{V}\int e^{-i\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r}$$

with $\hbar\vec{q} = \hbar(\overrightarrow{k'} - \overrightarrow{k})$ the momentum transfer

Now the density-of-states factor for the final wave to be scattered into solid angle $d\Omega$ is

$$\rho = \frac{dN}{dE'} = \frac{dN}{dk}\frac{dk}{dE} = \frac{V}{(2\pi)^3}k^2\, d\Omega\, \frac{m}{\hbar^2 k}$$

$$= \frac{V}{(2\pi)^3}\frac{m}{\hbar^2}k\, d\Omega$$

where *N* is the number of states, recalling $E' = E$ and $k' = k$, and also considering non-relativistic case where $E=p^2/2m$.

giving the following for the rate of scattering into solid angle $\mathrm{d}\Omega$:

$$\Gamma_{i\to f} = \frac{1}{V}\frac{1}{(2\pi)^2}\frac{1}{\hbar^3}mk\left|\int e^{-i\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r}\right|^2 \mathrm{d}\Omega$$

Calculating the differential cross section in QM

Our master formula for the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{n(\theta,\phi)}{j}$$

So for the numerator we may use

$$\Gamma_{i\to f} = \frac{1}{V}\frac{1}{(2\pi)^2}\frac{1}{\hbar^3}mk\left|\int e^{-i\vec{q}.\vec{r}}\,V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2$$

For the denominator, with j the incident flux, we use the probability-current density

$$j = -\frac{i\hbar}{2m}(\psi_i^*\nabla\psi_i - \psi_i\nabla\psi_i^*) = \frac{\hbar k}{mV}$$

which then yields the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{(2\pi\hbar^2)^2}\left|\int e^{-i\vec{q}.\vec{r}}\,V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2$$

Born approximation to the differential cross section

Our result is sometimes expressed as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\phi)|^2 \qquad \text{with} \qquad f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r}$$

where $f(\theta,\phi)$ is called the *scattering amplitude*. This reflects an alternative derivation by Max Born as an approximation of the Lippmann-Schwinger equation.

Recall this is first order, so the problem should be such that the higher order corrections are negligible. Other assumptions:

- Potential is *localised*, that is there is a cut off, or it drops off sufficiently rapidly for the initial and final waves to be considered as free states;
- Connected to above, time to traverse scattering potential is 'short';
- No resonant state produced (see later).



Max Born (1882-1970); pioneer of quantum mechanics; Nobel Prize in 1954; German-British



Olivia Newton-John (1945-); British-Austrian singer; grand-daughter of Max Born

Born approximation to the differential cross section

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with

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Recall this i… should be such … are negli…

Interesting observation: this is essentially the Fourier transform of the scattering potential

- Poten… there is a cut off, … drops off sufficiently rapidly for the initial and final waves to be considered as free states;
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Calculating QM cross sections: the Yukawa potential

We might now want to calculate scattering from a Coulomb potential, but this does not satisfy our locality requirement, so let's first calculate scattering from a Yukawa potential:

$$\mathrm{V}(r) = V_0 \frac{e^{-r/a}}{r}$$

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r}$$

Now introduce spherical coordinates relative to $\vec{q}$:

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\iiint e^{-iqr\cos\theta} V_0 \frac{e^{-r/a}}{r} r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}r$$

$$= \frac{m}{2\pi\hbar^2}\frac{4\pi V_0}{q}\int_0^\infty e^{-r/a}\sin qr\, \mathrm{d}r$$

$$= \frac{m}{2\pi\hbar^2}\frac{4\pi V_0}{q^2+(1/a)^2} = \frac{2mV_0}{\hbar^2}\frac{1}{q^2+(1/a)^2}$$

Calculating QM cross sections: the Yukawa potential

So we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\phi)|^2 = \left(\frac{2mV_0}{\hbar^2}\right)^2 \left(\frac{1}{q^2 + (1/a)^2}\right)^2$$

Writing $q = 2k\sin\frac{\theta}{2}$ then $q^2 = 2k^2(1-\cos\theta)$ and

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2}\right)^2 \left(\frac{1}{2k^2(1-\cos\theta) + (1/a)^2}\right)^2$$

which when integrated gives the total cross section

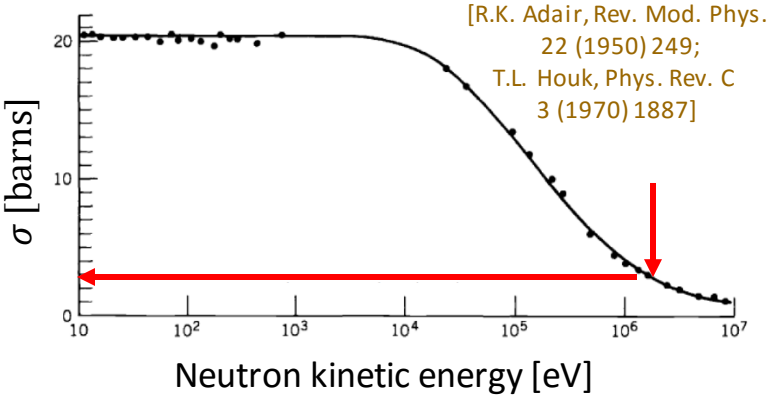
$$\sigma = \left(\frac{4maV_0}{\hbar^2}\right)^2 \frac{\pi}{4k^2 + (1/a)^2}$$

Yukawa potential: observations

$$\sigma = \left(\frac{4maV_0}{\hbar^2}\right)^2 \frac{\pi}{4k^2 + (1/a)^2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2}\right)^2 \left(\frac{1}{4k^2 \sin^4\theta/2 + (1/a)^2}\right)^2$$

- If we evaluate the cross section for nucleons ($m \sim 1000$ MeV/c^2), a typical momentum of 200 MeV/c (T=20 MeV), V_0 = 35 MeV fm (the depth of the potential well for the deuteron), and $a \sim 1$ fm then we get $\sigma \sim 0.1$ b. Not bad for a very simple model (no spin effects *etc.*) !



- We can use our result to find the differential cross-section for Rutherford scattering. Put $V_0 = (2Ze^2/4\pi\varepsilon_0)$ and give force infinite range, *i.e.* $a \to \infty$. Then we recover the previous result.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4}\left(\frac{Ze^2}{4\pi\varepsilon_0 T}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

- Finally, note that when $k^2 \ll (1/a)^2$, which is equivalent to saying that the momentum scale ≪ mass of exchange particle, then energy dependence vanishes. We will return to this when we discuss the weak interaction.

Scattering theory: topic overview

- Cross section: concept and applications
- Classical cross-section calculations
- Cross-section calculations in quantum mechanics
- Scattering off extended distributions: form factors
- Resonances and the Breit-Wigner function

Extended charge distributions: form factors

Until now we have considered scattering from a point-like object. Let us now consider scattering from an extended object. To be concrete let's describe electron scattering from an extended charge distribution (*e.g.* in the nucleus), but the argument is general.

So the potential is now

$$V(\vec{r}) = \int \frac{Ze^2}{4\pi\varepsilon_0} \frac{1}{\left|\vec{r} - \overrightarrow{r'}\right|} \rho(\overrightarrow{r'}) \mathrm{d}^3\overrightarrow{r'}$$

and the scattering amplitude becomes

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2} \iint e^{-i\vec{q}.\vec{r}} \frac{Ze^2}{4\pi\varepsilon_0} \frac{1}{\left|\vec{r} - \overrightarrow{r'}\right|} \rho(\overrightarrow{r'}) \mathrm{d}^3\overrightarrow{r'} \mathrm{d}^3\vec{r}$$

Writing $\vec{x} = \vec{r} - \overrightarrow{r'}$ yields

$$f(\theta,\phi) = \underbrace{\frac{m}{2\pi\hbar^2} \int e^{-i\vec{q}.\vec{x}} \frac{Ze^2}{4\pi\varepsilon_0} \frac{1}{|\vec{x}|} \mathrm{d}^3\vec{x}}_{\text{point-like amplitude}} \underbrace{\int e^{-\overrightarrow{iq.r'}} \rho(\overrightarrow{r'}) \, \mathrm{d}^3\overrightarrow{r'}}_{\text{form factor}}$$

So the scattering amplitude is now the product of the amplitude we would have for a point-like particle and something else, which we call the form factor.

Extended charge distributions: form factors

Until now we have considered scattering from a point-like object. Let us now consider scattering from an extended object. To be concrete let's describe electron scattering from an extended charge distribution (*e.g.* in the nucleus), but the argument is general.

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The form factor *F(q)* is the Fourier transform of the charge distribution:

$$F(q) = \int e^{-i\vec{q}.\vec{r}}\,\rho(\vec{r})\,\mathrm{d}^3\vec{r}$$

It modifies the scattering amplitude:

$$f(\theta,\phi) = f_{\text{point}}(\theta,\phi) \times F(q)$$

and hence changes the differential cross section by a factor $|F(q)|^2$.

$$f(\theta,\phi) = \underbrace{\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}.\vec{x}}\,\frac{Ze^2}{4\pi\varepsilon_0}\,\frac{1}{|\vec{x}|}\,\mathrm{d}^3\vec{x}}_{\text{point-like amplitude}}\ \underbrace{\int e^{-i\vec{q}.\vec{r'}}\,\rho(\vec{r'})\,\mathrm{d}^3\vec{r'}}_{\text{form factor}}$$

So the scattering amplitude is now the product of the amplitude we would have for a point-like particle and something else, which we call the form factor.

Form factor of sphere with uniform charge distribution

Let's calculate form factor for the simple case of a sphere of radius a and uniform charge density ρ_0 such that $Ze = \frac{4\pi}{3} a^3 \rho_0$. This can serve as a naïve model of the nucleus.

$$F(q) = \frac{1}{Ze} \int e^{-i\vec{q}.\vec{r}} \rho_0 \, \mathrm{d}^3\vec{r}$$

$$= \frac{1}{Ze} \iiint \rho_0 e^{-iqr\cos\theta} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= \frac{3}{qa^3} \int_0^a r \sin qr \, \mathrm{d}r$$

$$= \frac{3[\sin(qa) - (qa)\cos(qa)]}{(qa)^3}$$

So the differential cross-section will be that of a point particle multiplied by this squared.

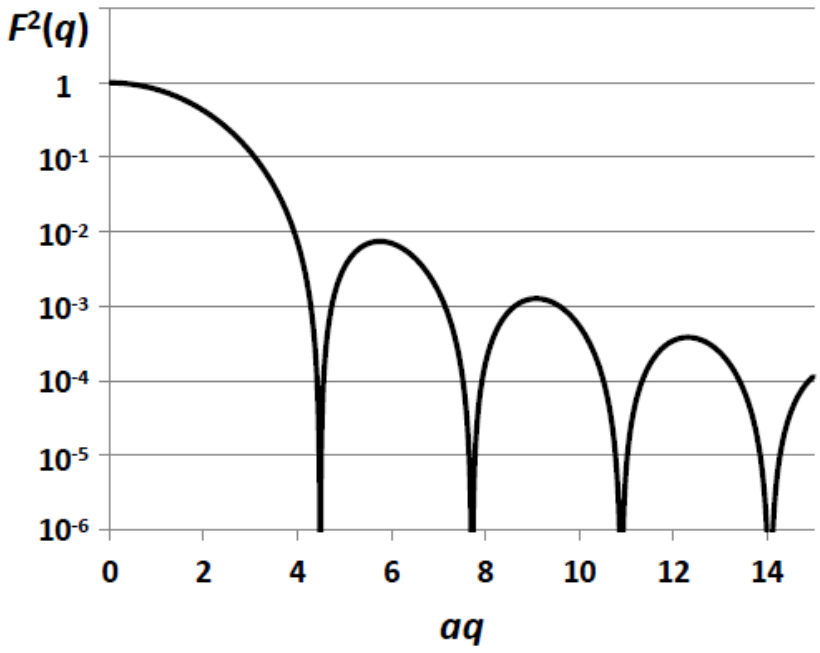
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$$|F(q)|^2 = \left(\frac{3[\sin(qa) - (qa)\cos(qa)}{(qa)^3} \right)^2$$

(this is a spherical Bessel function)

Reminiscent of a Fraunhofer diffraction pattern in optics



[figures courtesy Viehausser & Barr]

Now since $aq = a \times 2k \sin\frac{\theta}{2}$, studying this scattering pattern, in particular the position of minima vs. θ, can tell us about value of *a*, provided *k*, *i.e.* momentum is known.

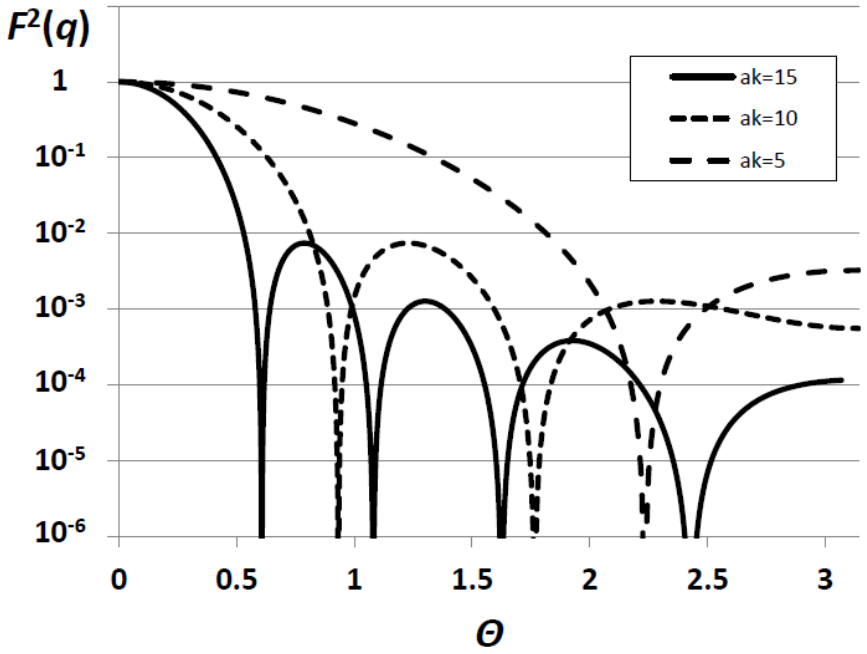
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$$|F(q)|^2 = \left(\frac{3[\sin(qa) - (qa)\cos(qa)}{(qa)^3} \right)^2$$

(this is a spherical Bessel function)

Same thing plotted against scattering angle, for different values of *ak* ,*i.e.* radius x momentum



[figures courtesy Viehausser & Barr]

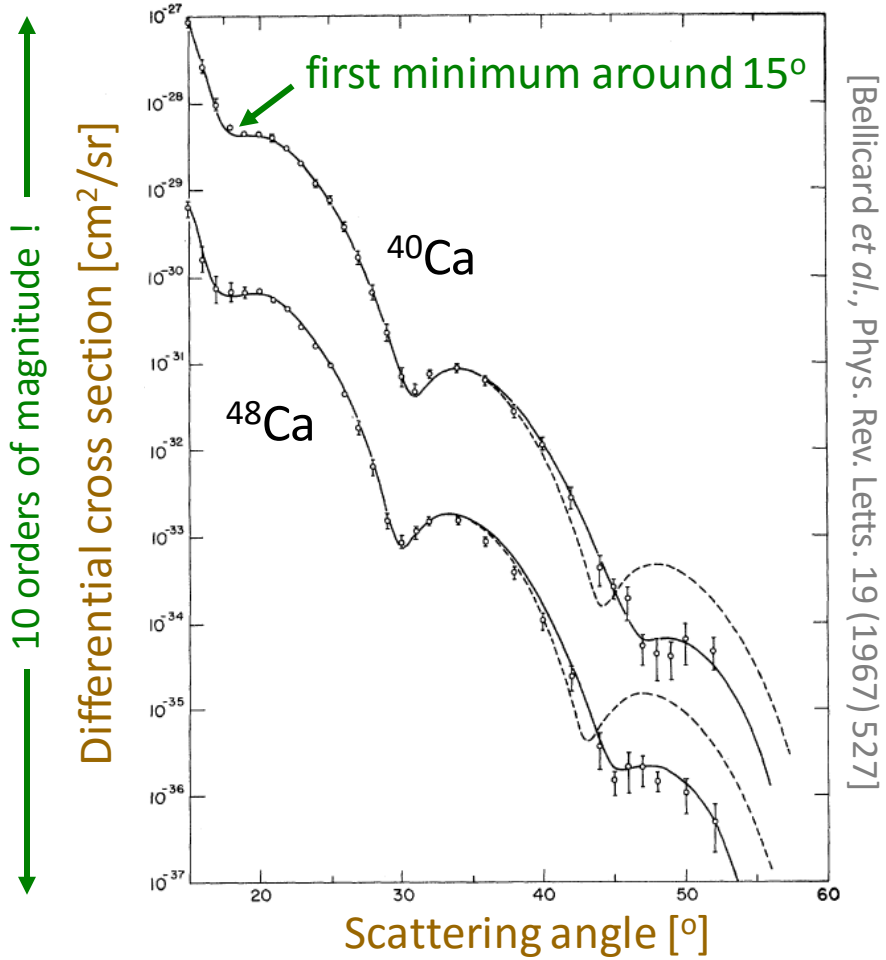
Note distribution widens as *ak* decreases. Value of probe momentum vs. radius must be well chosen to resolve structure.

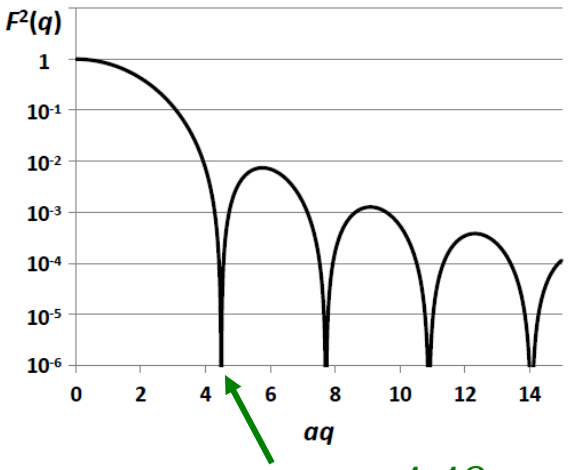
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Measuring nuclear radii

Measured differential cross-sections vs. scattering angle indeed shows clear minima. The minima don't go to zero. This is because our model, which assumes a sharp edge to the nucleus, is not realistic. Nonetheless we can use the data and our model to estimate a.

Scattering of p=757 MeV/c electrons off Ca





$aq \approx 4.49$

$$aq = 2ak \sin\theta/2$$

$$4.49 = (2a\, p[\mathrm{MeV}/c]\, \sin\frac{\theta}{2})/\hbar c$$

$$\Longrightarrow \quad a = 4.5 \text{ fm} \quad \text{for } {}^{40}\text{Ca}$$

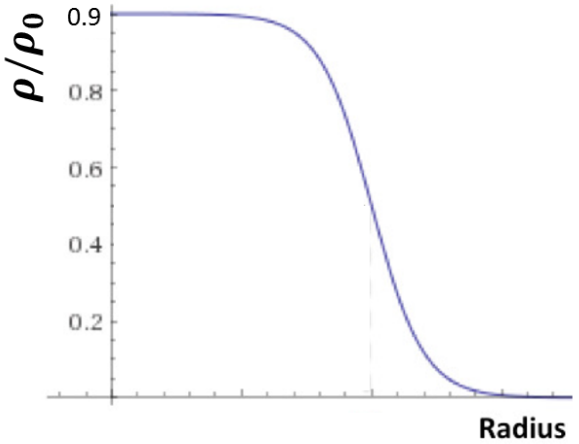
Compare to expectation

$$a = r_0 A^{1/3} = 4.1 \text{ fm}$$ ✓

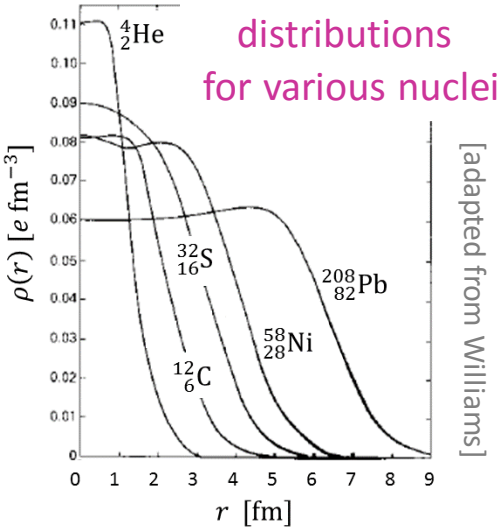
More on nuclear radii and shapes

A better description of the charge distribution is provided by the Woods-Saxon potential:

$$\rho = \frac{\rho_0}{1 + e^{\left(\frac{r-a}{d}\right)}}$$



Fitted Woods-Saxon distributions for various nuclei

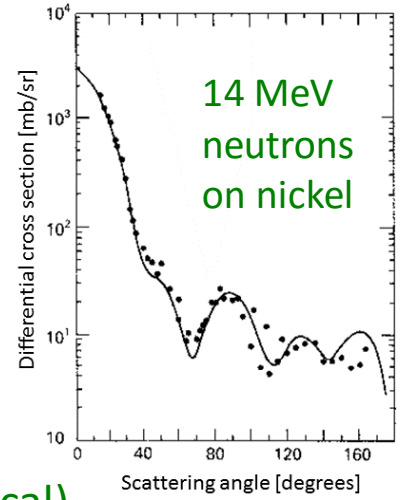


Studies of nuclear structure through electron scattering were pioneered by Robert Hofstadter.



Robert Hofstadter (1915-1990); pioneer of electron scattering; Nobel Prize in 1961; American

Scattering experiments can also be done with neutrons to probe nuclear distribution. Results similar (but not identical), to charged findings.



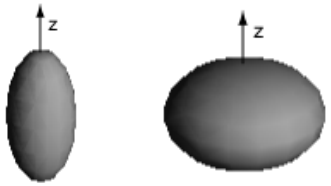
[Bauer *et al.*, 1963]

Other information on nuclear charge distribution from:

- Isotope shifts;
- X-ray spectroscopy of mu-mesic atoms.

Non-zero electric quadrupoles indicate deformations in shape.

Q>0 prolate



Q>0 oblate

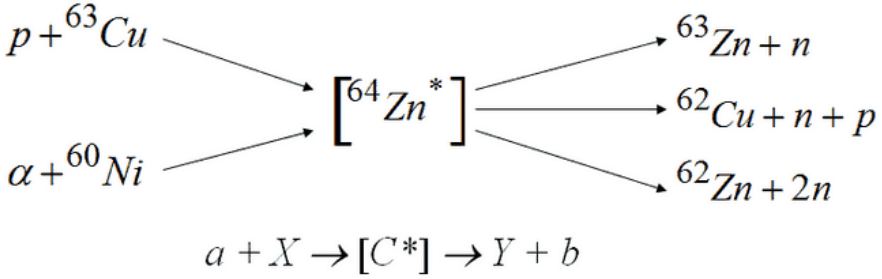
Scattering theory: topic overview

- Cross section: concept and applications
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- Cross-section calculations in quantum mechanics
- Scattering off extended distributions: form factors
- Resonances and the Breit-Wigner function

Short-lived unstable states: compound nuclei

In nuclear reactions an intermediate excited state can be formed: a *compound nucleus*. This then can decay into the same particles that formed it, or into other states.

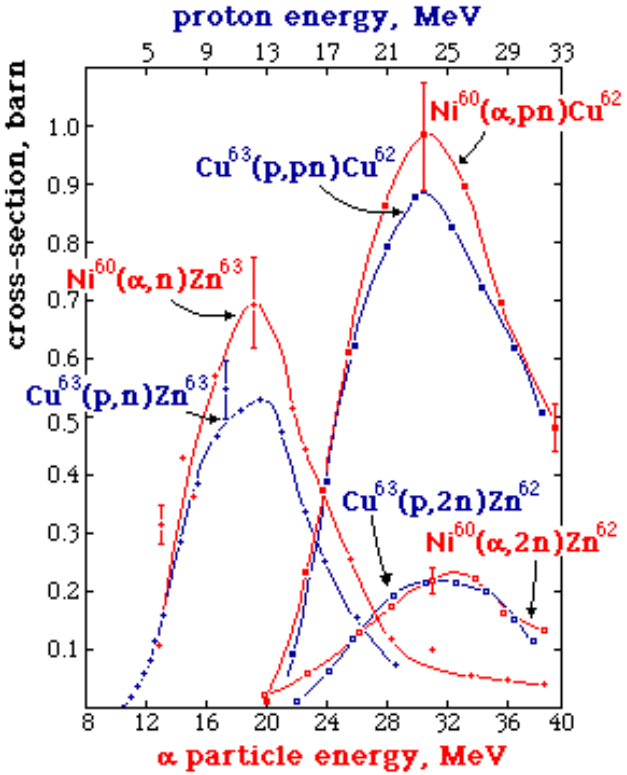
Note same compound nucleus can be produced by different initial states.



$$a + X \rightarrow [C^*] \rightarrow Y + b$$

Note in nuclear reaction notation this reaction is usually written:

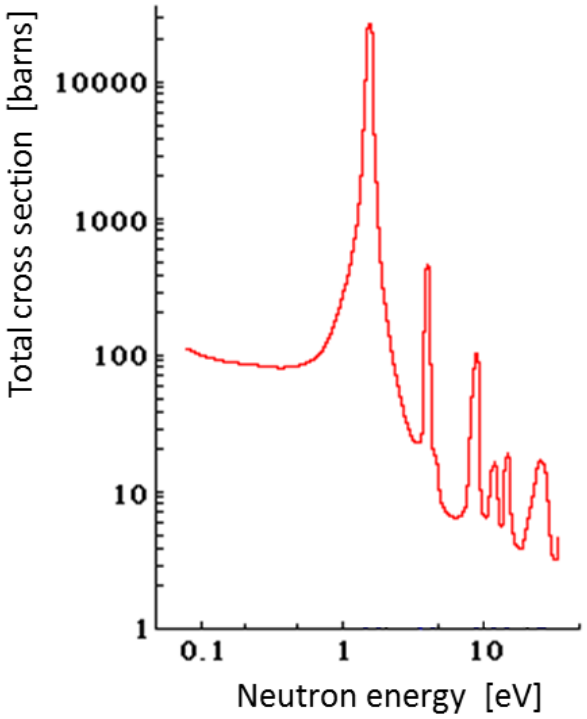
$$X(a,b)Y$$



Compound nucleus is short-lived (10^{-18} --10^{-16} s), but still exists long enough that it decays with no 'memory' of how it was formed, so the final-state decay probabilities do not depend on initial state. Manifests itself as a peak in cross-section vs. energy: *a resonance*.

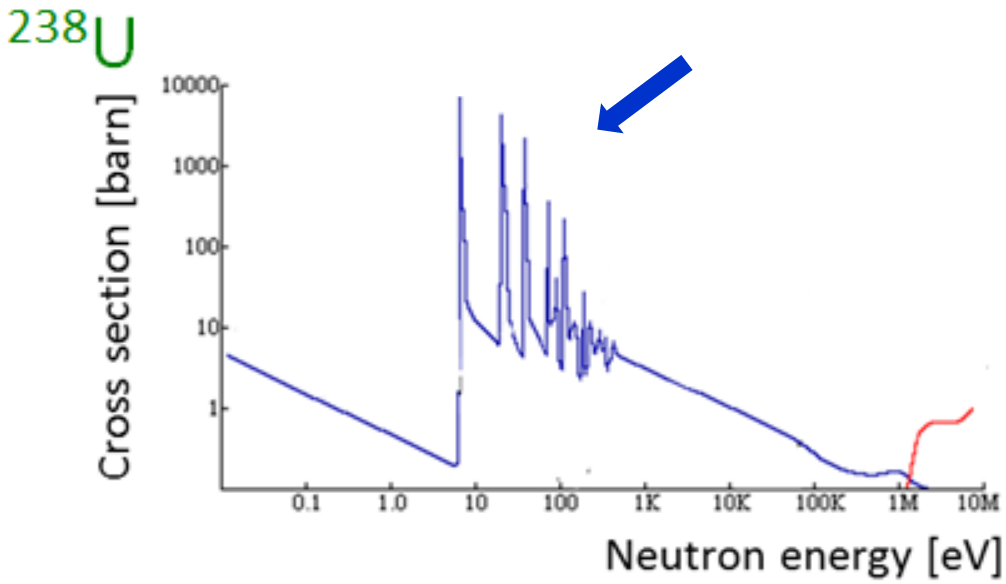
Other examples of resonances in nuclear physics

Neutron scattering on indium



Resonance region for

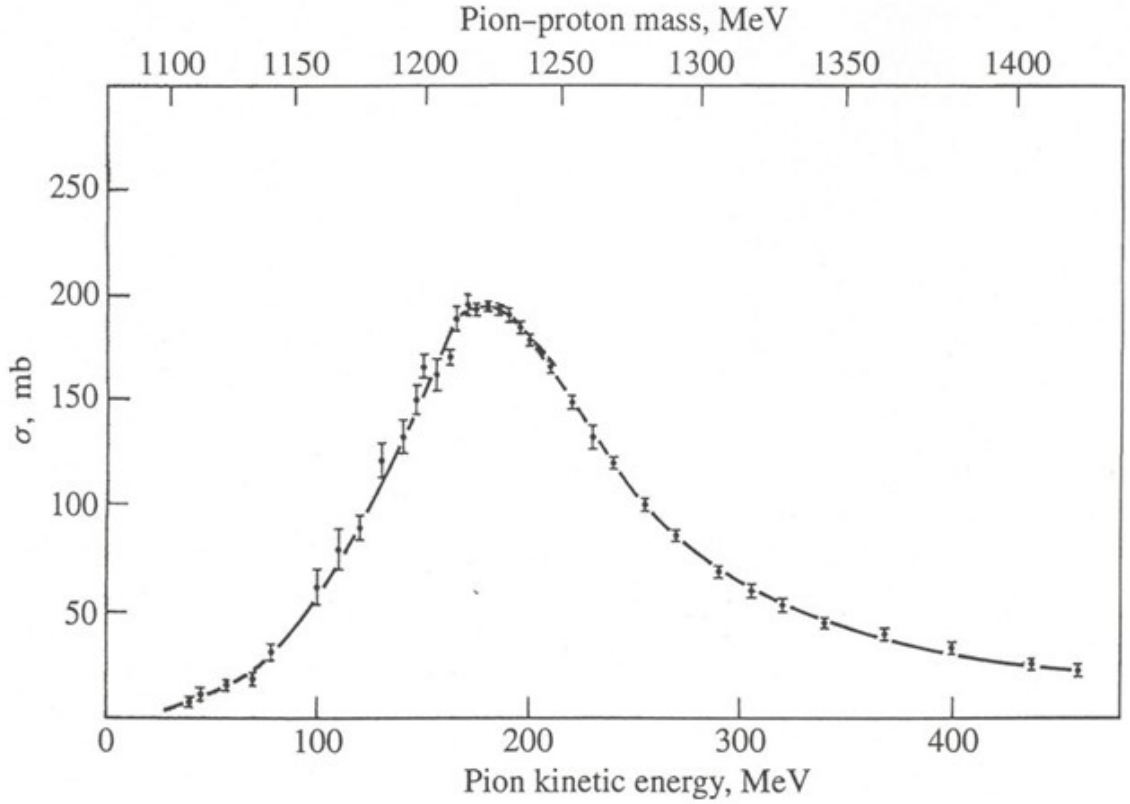
$$n + {}^{238}_{92}U \rightarrow {}^{238}_{92}U^{*} \rightarrow {}^{238}_{92}U + \gamma$$



Examples of resonances in particle physics

Many examples of resonances in particle physics, which will be discussed in coming lectures.

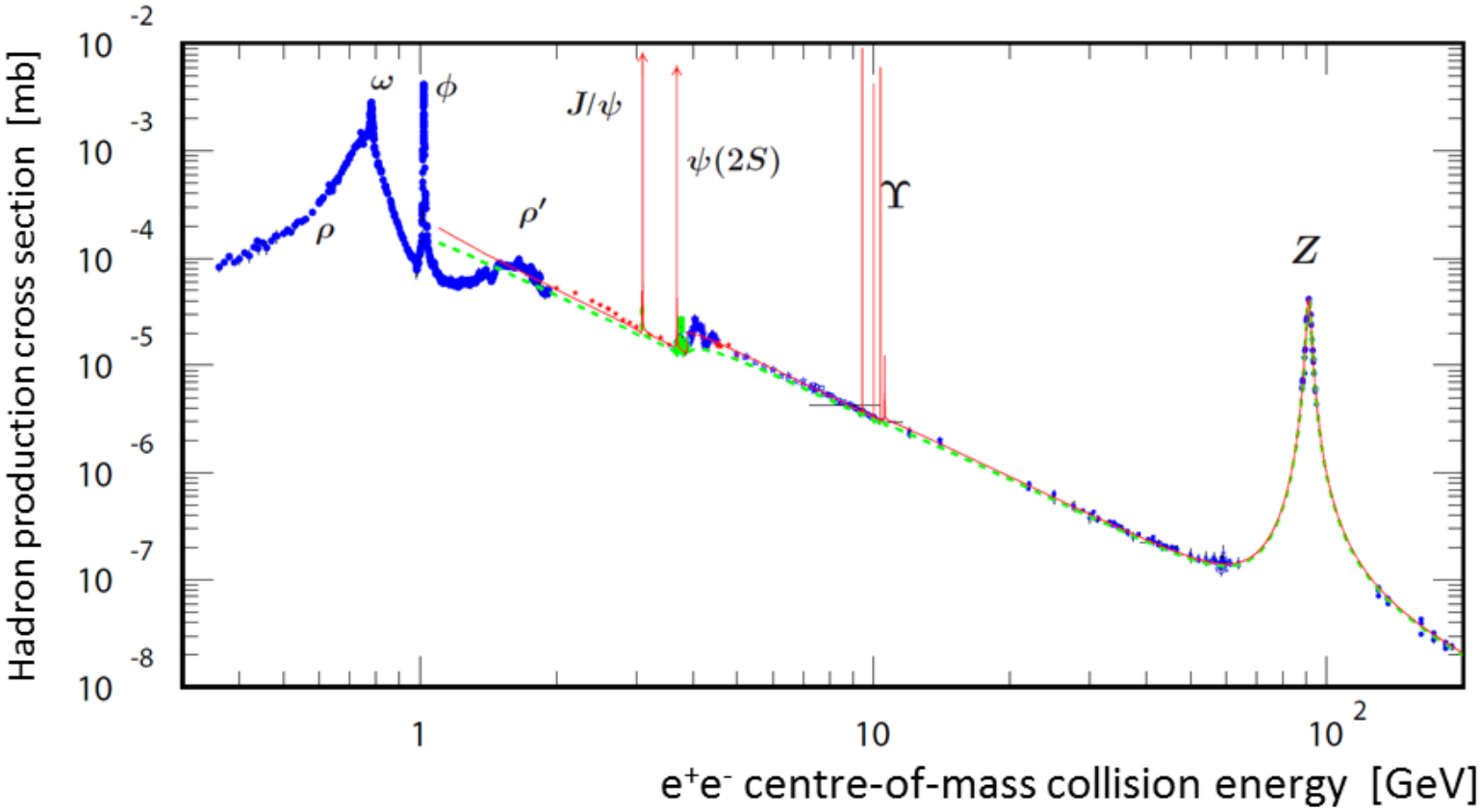
e.g. peak in the cross section of pions scattering on protons, corresponding to the $\Delta(1236)$ resonance: a short-lived intermediate hadron different to both the initial state particles.



Examples of resonances in particle physics

Many examples of resonances in particle physics, which will be discussed in coming lectures.

e.g. peaks in electron-positron cross section vs energy, especially the J/ψ and the Z^0.



Let us develop an expression for the scattering cross section for such resonances.

Decay of an unstable state

Consider a state (*e.g.* could be either a compound nucleus or an unstable sub-nuclear particle that decays strongly [see later]) formed at t=0 with energy E_0 & mean lifetime τ.

$$\psi(t) = \psi(0)e^{-itE_0/\hbar}e^{-t/2\tau} \Rightarrow |\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau}$$

i.e. obeys radioactive decay law as required

To see how this appears in energy space we must perform a Fourier transform

$$f(E) = \int_0^\infty \psi(t)e^{itE/\hbar}\mathrm{d}t = \int_0^\infty \psi(0)e^{-t[i(E_0-E)/\hbar+\frac{1}{2\tau}]}\mathrm{d}t = \frac{-i\hbar\psi(0)}{(E_0-E)-i\hbar/2\tau}$$

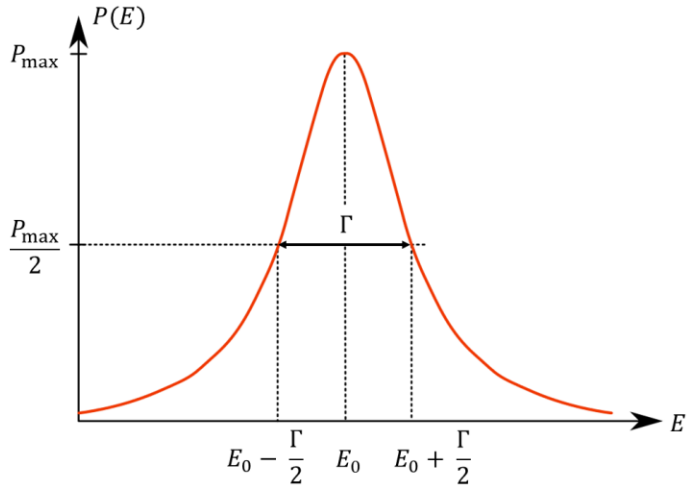
and so probability $P(E)$ of finding state at energy E is

$$P(E) \propto f(E)f^*(E) \propto \frac{1}{(E_0-E)^2+\hbar^2/4\tau^2}$$

Sketch this & determine the full-width-half-max Γ:

$$P(E_0) \propto \frac{1}{\hbar^2/4\tau^2} \text{ and } P(E_0 \pm \Gamma/2) \propto \frac{1}{\Gamma^2/4+\hbar^2/4\tau^2}$$

$$\Rightarrow \boxed{\Gamma = \hbar/\tau} \qquad \boxed{P(E) \propto \frac{1}{(E_0-E)^2+\Gamma^2/4}}$$



Cross section for resonance production: the Breit-Wigner function

Consider the elastic cross section for production of this state C^*, *i.e.* $a + b \rightarrow C^* \rightarrow a + b$. It is clear from conservation of probability that the cross-section, which measures the probability of forming the state, must be proportional to the probability of decay.

$$\sigma(E) \propto \frac{1}{(E_0 - E)^2 + \Gamma^2/4}$$

Knowing the distribution to have the shape it does, let's write this as the product of some maximum cross-section multiplied by $K/(E_0 - E)^2 + \Gamma^2/4$, where this function has a maximum value of 1. This requirement implies that $K = \Gamma^2/4$. Hence we have:

$$\sigma(E) = \sigma_{\max} \frac{\Gamma^2/4}{(E_0 - E)^2 + \Gamma^2/4}$$

We will state, but not justify, that $\sigma_{\max} = 4\pi/k^2$, where $\hbar k$ is momentum of incident particle in centre-of-mass system. This, and full function, can be derived using perturbation theory (see, *e.g.* Viehhauser notes), or in wave-optical formalism (see, *e.g.* Perkins or Krane).

Cross section for resonance production: the Breit-Wigner function

So we have for the elastic cross section $$\sigma(E) = \frac{\pi g}{k^2} \frac{\Gamma^2}{(E_0 - E)^2 + \Gamma^2/4}$$

Next we need to allow for the effects of spin by introducing a factor g that Is the ratio of the number of spin states for the resonance $(2J+1)$ averaged by the number of spin states of the initial system $(2s_a + 1)$ & $(2s_b + 1)$ (really part of the density of states contribution).

$$\mathrm{g} = \frac{(2J+1)}{(2s_a+1)(2s_b+1)}$$

i.e. $g = 1$ in spinless case

Finally, we generalise to the case where the state can *also* decay inelastically, i.e. $a + b \rightarrow C^* \longrightarrow c + d$. In this case we have to specify the cross section when going from some initial state i, to some final state f, and introduce the *partial widths* Γ_i and Γ_f.

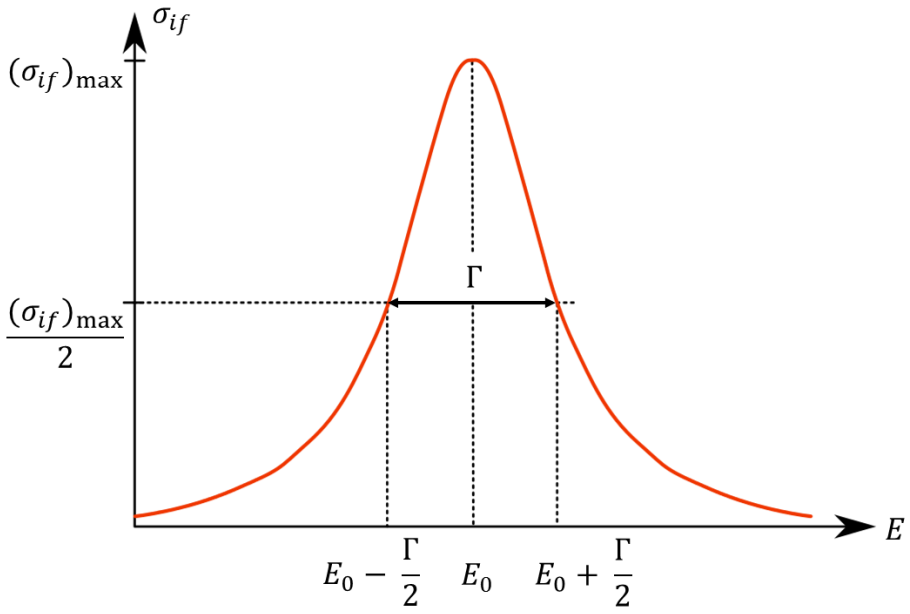
$$\sigma_{i \rightarrow f}(E) = \frac{\pi g}{k^2} \frac{\Gamma_i \Gamma_f}{(E_0 - E)^2 + \Gamma^2/4}$$

This is the so-called *Breit-Wigner function*, that describes the cross-section for production of a resonant state, *e.g.* compound nucleus, or, in particle physics strongly (or equally fastly decaying) particle. Mathematically identical to the Lorentzian / Cauchy-Lorentz distribution.

Breit-Wigner distribution

Formulated by Breit and Wigner in 1936

$$\sigma_{i\to f}(E) = \frac{\pi g}{k^2}\frac{\Gamma_i\Gamma_f}{(E_0 - E)^2 + \Gamma^2/4}$$





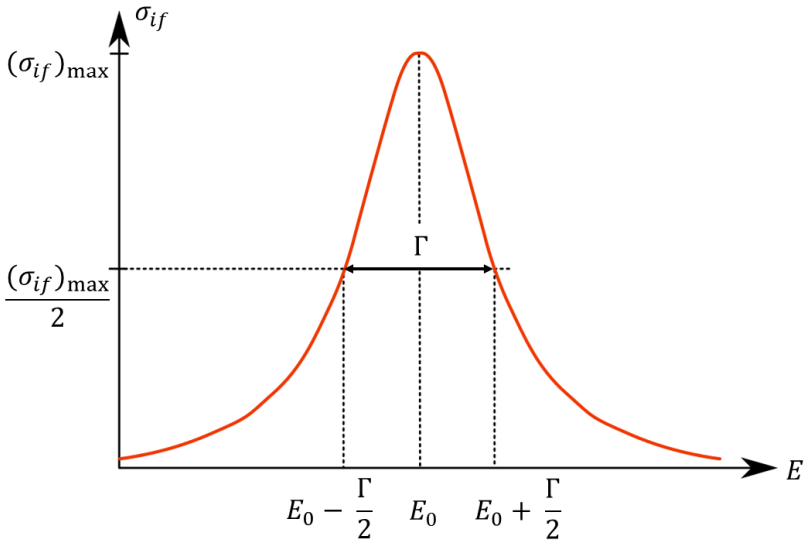
Eugene Wigner (1902-1995); theoretical nuclear/particle physicist; Nobel Prize in 1963; Hungarian-American; brother-in-law of Paul Dirac



Gregory Breit (1899-1981); theoretical nuclear/particle physicist; Russian-American

Breit-Wigner distribution: essential properties

$$\sigma_{i\to f}(E) = \frac{\pi g}{k^2}\frac{\Gamma_i\Gamma_f}{(E_0 - E)^2 + \Gamma^2/4}$$



- If $i = f$ we have elastic scattering
- Total cross section $\sigma_{\text{tot}} = \sum_f \sigma_{i\to f}$.
- Sum of partial widths = full width, *i.e.* $\Gamma = \sum_f \Gamma_f$
- As already shown, $\Gamma = \hbar/\tau$. In the relativistic case, relevant in particle physics, where E is the total centre-of-mass energy, we can associate E_0, the pole of the resonance, with the mass M of the short-lived particle through $M = E_0/c^2$.
- At peak of resonance, $\sigma_{i\to f}(E_0) = \frac{\pi g}{k^2}\frac{4\Gamma_i\Gamma_f}{\Gamma^2}$

  Note *height* of resonance *will vary* depending on final state through different Γ_f in numerator, but *width will not* ! Lifetime of particle does *not* depend on decay mode !
- Probability of decay into a final state f is the *branching fraction* (or *branching ratio*) B_f.

  $B_f = \sigma_{i\to f}/\sigma_{i,\text{tot}} = \Gamma_f/\Gamma$.

Breit-Wigner: a toy example

A neutron scatters off a heavy nucleus of total angular momentum $J = \frac{1}{2}$. The scattering rate is seen to peak at an energy of 200 keV, & is around half this at 190 keV & 210 keV.

Other measurements performed suggest that the compound nucleus formed in the reaction has a total angular momentum of $J = 1$.

The peak cross section for elastic scattering is measured to be 0.38 b, and is found to be the same value for production of a single proton in the final state.

The only other reaction detected is instantaneous production of alpha particles.

Determine:
- The lifetime of the compound nucleus;
- The peak cross section for alpha production.

and sketch the cross-section curves vs. neutron kinetic energy for the three reactions. You may assume the struck nucleus is sufficiently massive that the lab approximates to the centre-of-mass frame, that the neutron partial width is constant across the resonance*.

(Caution: this is an invented scenario of the lecturer, who is not a nuclear physicist...)

* Note this quantity is energy dependent, which leads to the '1/v law' growth in cross section for slow neutrons – see problem sheets.

Breit-Wigner: a toy example

Width $\Gamma = 20$ keV $\Rightarrow$ $\tau = \hbar/(0.02\ \text{MeV})$

$= \hbar c\ /\ (c\ \times 0.02\ \text{MeV})$

$\approx 200\ \text{MeV fm}/\ (c \times 0.02\ \text{MeV})$

$\approx 200\ \text{MeV fm}\ /\ (3 \times 10^{23}\ \text{fm s}^{-1}\ 0.02\ \text{MeV})$

$\approx 3.3\ \times\ 10^{-20}\ \text{s}$

(Note that this result is rather short-lived for a compound nucleus…)

Turning to cross section, $\sigma_{i\to f}(E_0) = \dfrac{\pi g}{k^2}\dfrac{4\Gamma_i\Gamma_f}{\Gamma^2}$, let's first calculate k:

$$p = \sqrt{2mE} \approx \sqrt{2 \times 0.200 \times 1000} = 20\ \text{MeV}/c$$

$$\hbar k = p \ \Rightarrow\ k = \frac{pc}{\hbar c} \approx \frac{20\ \text{MeV}}{200\ \text{MeV fm}} \approx 0.1\ \text{fm}^{-1} \ \Rightarrow\ \frac{1}{k^2} \approx 100\ \text{fm}^2 = 1\ \text{barn}$$

We also need *g* factor for each reaction:

so numerator is 3

$$\mathrm{g} = \frac{(2J+1)}{(2s_a+1)(2s_b+1)}$$

this is 2

this is also 2

Hence $\mathrm{g} = 3/4$.

Breit-Wigner: a toy example

We have $\sigma_{i\to f}(E_0) = \frac{\pi g}{k^2}\frac{4\Gamma_i\Gamma_f}{\Gamma^2}$, so for elastic scattering where $\Gamma_i = \Gamma_n$ we have

$$\Gamma_n = \sqrt{\frac{k^2\sigma_{n\to n}(E_0)}{\pi g}}\frac{\Gamma}{2}$$

$$\Gamma_n = \sqrt{\frac{4\times 0.38}{3\pi}}\frac{20\text{ keV}}{2} \approx 4\text{ keV}$$

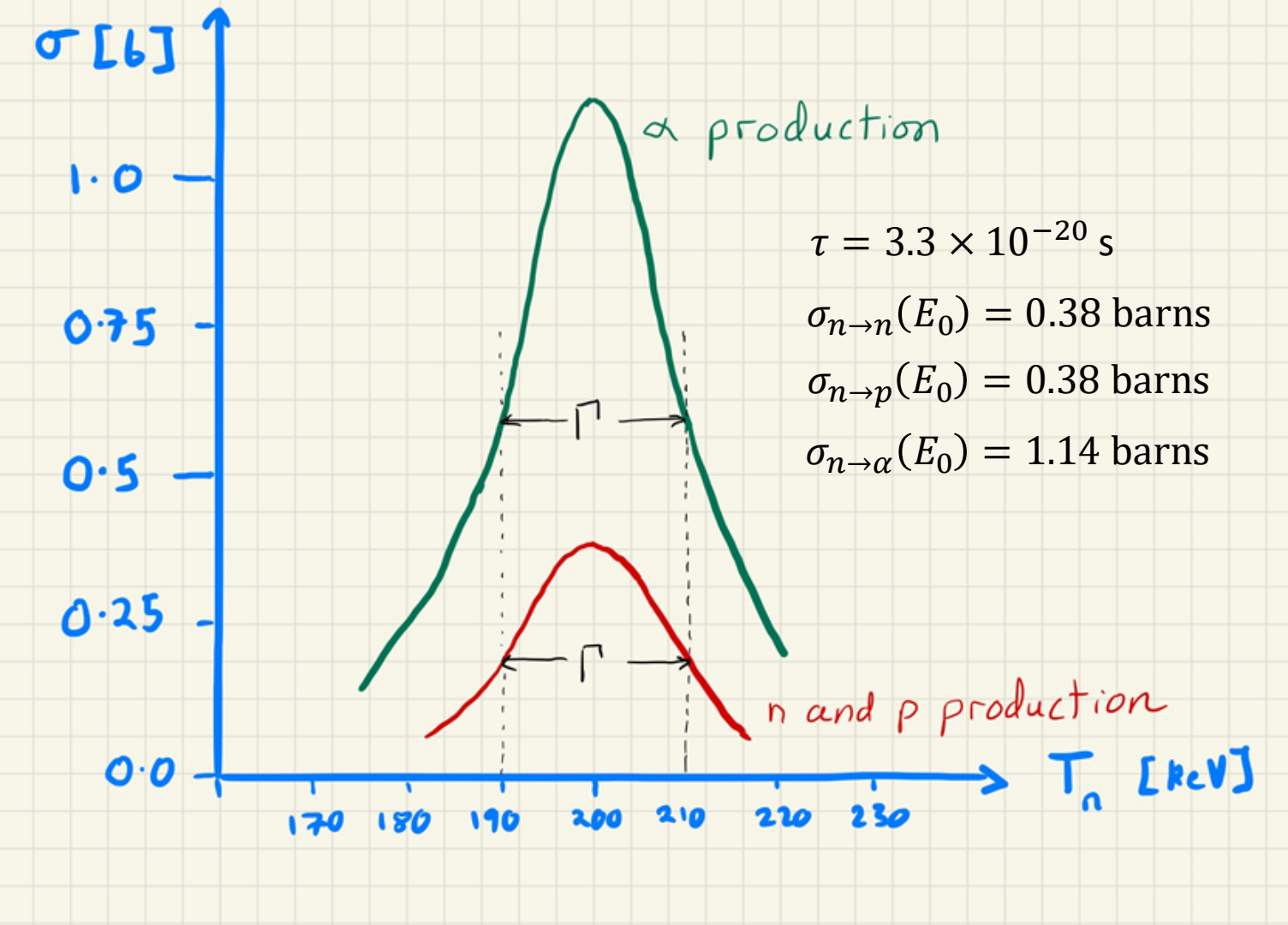
Partial width into protons is the same, *i.e.* $\Gamma_p = 4$ keV.

Remainder of decays is taken up by α particles, so $\Gamma_\alpha = \Gamma - \Gamma_n - \Gamma_p = 12$ keV.

So peak cross section with α production is $3\times$ that of *n* or *p* case:

$$\sigma_{n\to\alpha}(E_0) = 1.14\text{ barns.}$$

Breit-Wigner: a toy example

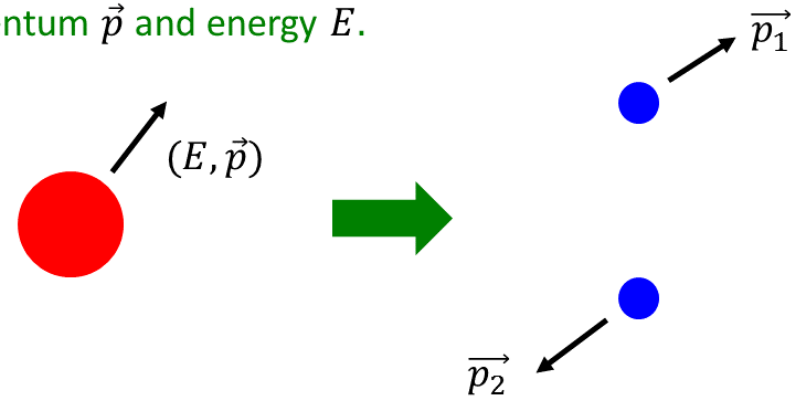


(Note in the sketch we have not bothered to account for the $1/k^2$ dependence, which is a 20% effect, as in reality it will also be suppressed by energy dependence in Γ_n.)

Aside on reconstructing short-lived states

We are discussing scattering, but the Breit-Wigner characteristics of short-lived particles also manifest themselves if they are reconstructed through the invariant mass of the decay products. This is a widely used technique in particle physics.

Mother particle X of unknown mass M, produced moving in lab frame with unknown momentum $\vec{p}$ and energy E.



NB method works equally well for non-identical particles and for final states of > 2 particles.

Decays into two identical particles of known rest masses m and measured momenta $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$.

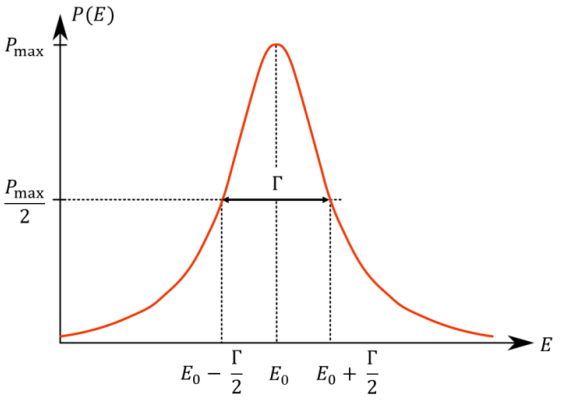
The quantity $\left(\left[(m^2c^4 + p_1^2c^2)^{1/2} + (m^2c^4 + p_2^2c^2)^{1/2}\right]^2 - (\overrightarrow{p_1} + \overrightarrow{p_2})^2c^2\right)^{1/2}$ is the invariant mass of the final state system and corresponds to the rest mass M of the initial state. No need to know E or $\vec{p}$ of the decaying particle !

Aside on reconstructing short-lived states: the invariant mass

Invariant mass: $$\left(\left[(m^2c^4 + p_1^2c^2)^{1/2} + (m^2c^4 + p_2^2c^2)^{1/2}\right]^2 - (\overrightarrow{p_1} + \overrightarrow{p_2})^2c^2\right)^{1/2}$$

Now, we showed that the probability of finding a decaying particle of nominal mass E_0/c^2 at energy E is given by:

$$P(E) \propto \frac{1}{(E_0 - E)^2 + \Gamma^2/4}$$



Hence this if we reconstruct this invariant mass for many decays, and fill a histogram with the results, we do not in general expect a delta-function. Rather, we will obtain a Breit-Wigner distribution of width Γ that tells us the lifetime of the decaying particle !

Two (related) practical caveats:

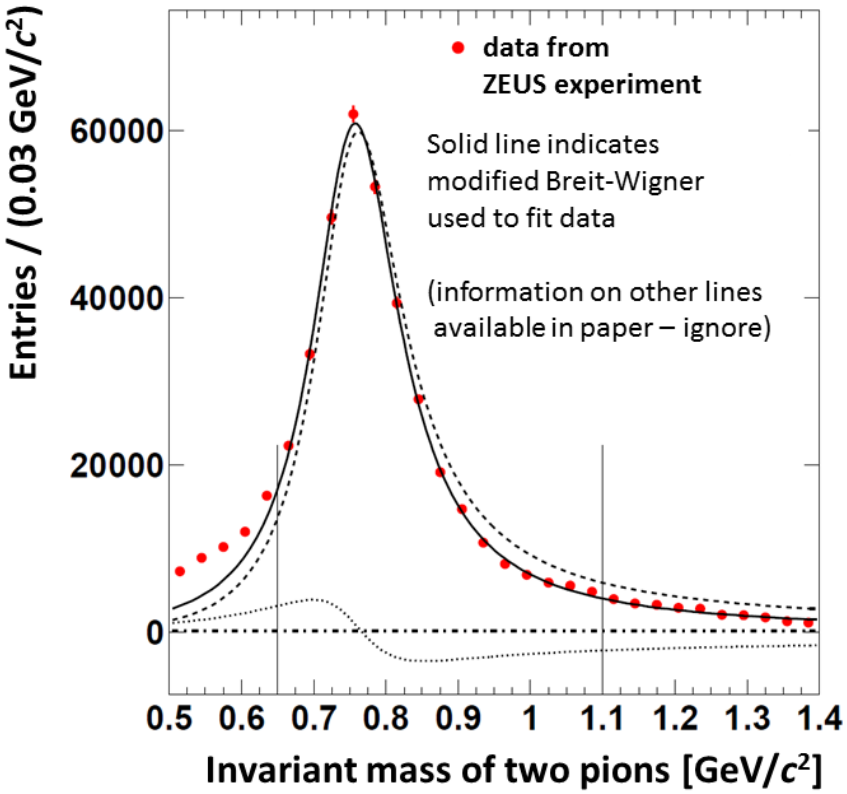
- Experimental resolution on invariant mass is not perfect, and is a few MeV/c^2 for most current detectors, so any underlying physics distribution will appear convolved with the experimental resolution;
- Hence with this method we are only sensitive to decay widths of a few 10s to 100s of MeV.

Aside on reconstructing short-lived states: the invariant mass

Here is a vivid example from the ZEUS experiment.

A ρ meson, which has a mass of around 0.77 GeV/c^2, has been reconstructed by histogramming the invariant mass of its two decay products (two charged pions).

A clear Breit-Wigner is seen, with a width much larger than the experimental resolution, meaning we have real sensitivity to the particle's lifetime.



[ZEUS, arXiv:0708.1478]

The fitted width is 154 MeV (now expressing the mass scale as an energy)

$$\tau = \frac{\hbar}{\Gamma} \sim 4 \times 10^{-24}\, s$$

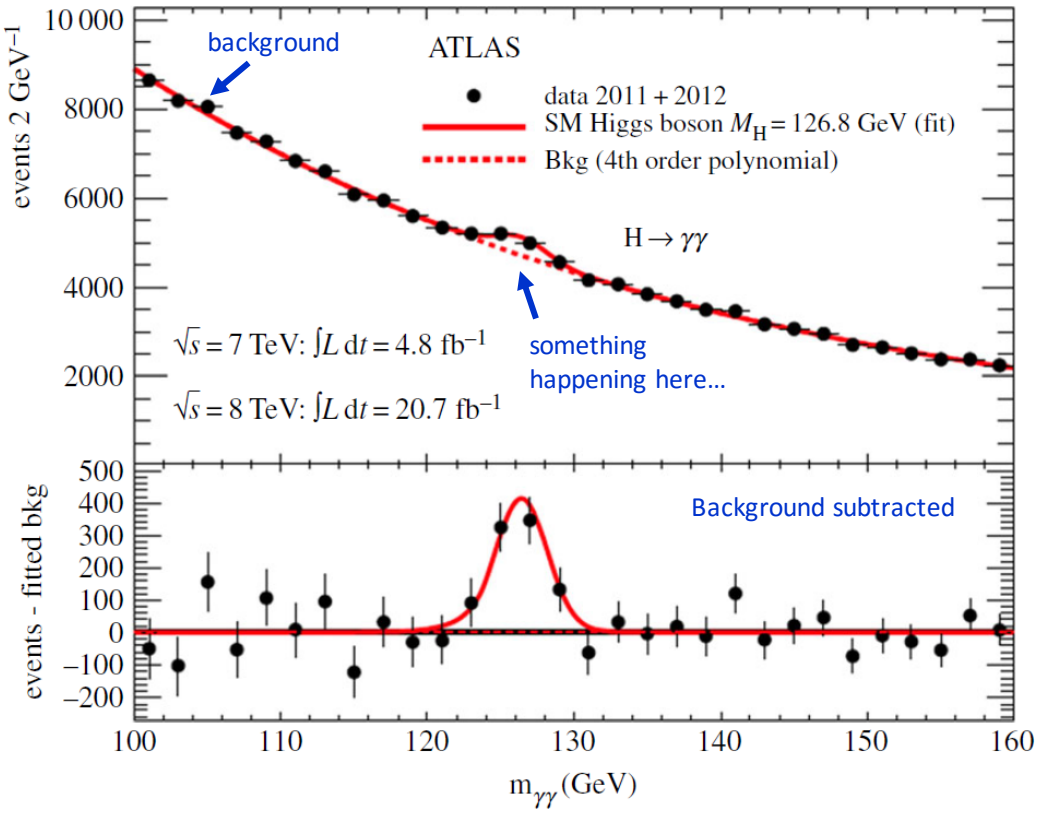
Aside on reconstructing short-lived states: the invariant mass

Another example: histogramming the invariant mass of high-energy photon pairs in the ATLAS experiment at the LHC.

The falling exponential that dominates the plot is contamination arising from random combinations of photons that come from background processes.

A small, but clear, bump is seen around 125 GeV. This comes from decays of the type:

$$H \rightarrow \gamma\gamma$$



[ATLAS, Phys. Lett. B 716 (2012) 1; also Gianotti & Virdee, Phil. Trans. R. Soc. A 737: 20140384]

So this technique has discovered the Higgs boson ! The peak, once background is subtracted, has a definite width. But this width is dominated by experimental resolution, not lifetime.

Particle physics

Particle physics: topic overview

Introduction

- The Standard Model
- Natural units
- Feynman diagrams

Hadrons, quarks & the strong interaction

- Hadrons & their quantum numbers
- The light quark model
- Colour & QCD
- Quarkonia
- The *R* ratio

Leptons and the weak interaction

- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

Final words

Particle physics: topic overview

Introduction

- The Standard Model
- Natural units
- Feynman diagrams

Standard Model of Particle Physics

Spoiler alert! We begin with a few slides that summarise our current understanding of the fundamental interactions and particles (apart from gravity), which comprises the Standard Model of particle physics. We will then step through those sub-topics of the Standard Model which are on the syllabus, discussing each in (roughly) historical order.

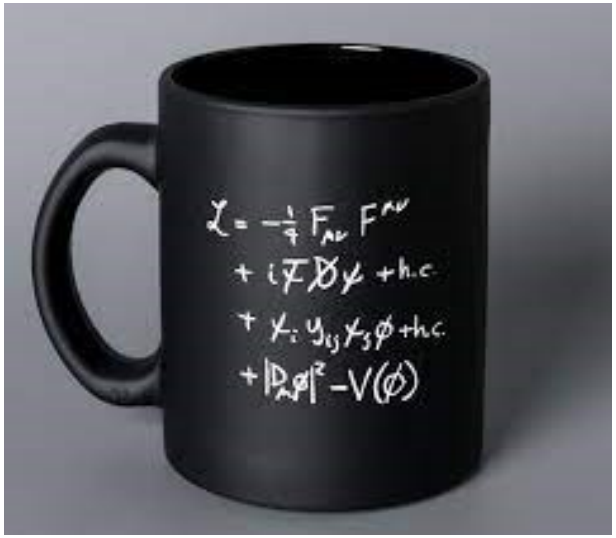
Interactions described by the Standard Model:

- Electromagnetism
- Weak interaction

  Unified within a single *electroweak interaction*, in which the Higgs mechanism plays an important part, giving masses to the *W* and *Z* bosons, as well as to the quarks and charged leptons.

  Cabibbo mixing, parity (*P*) violation and *CP* violation are all important attributes of the weak interaction.

The Standard Model Lagrangian density



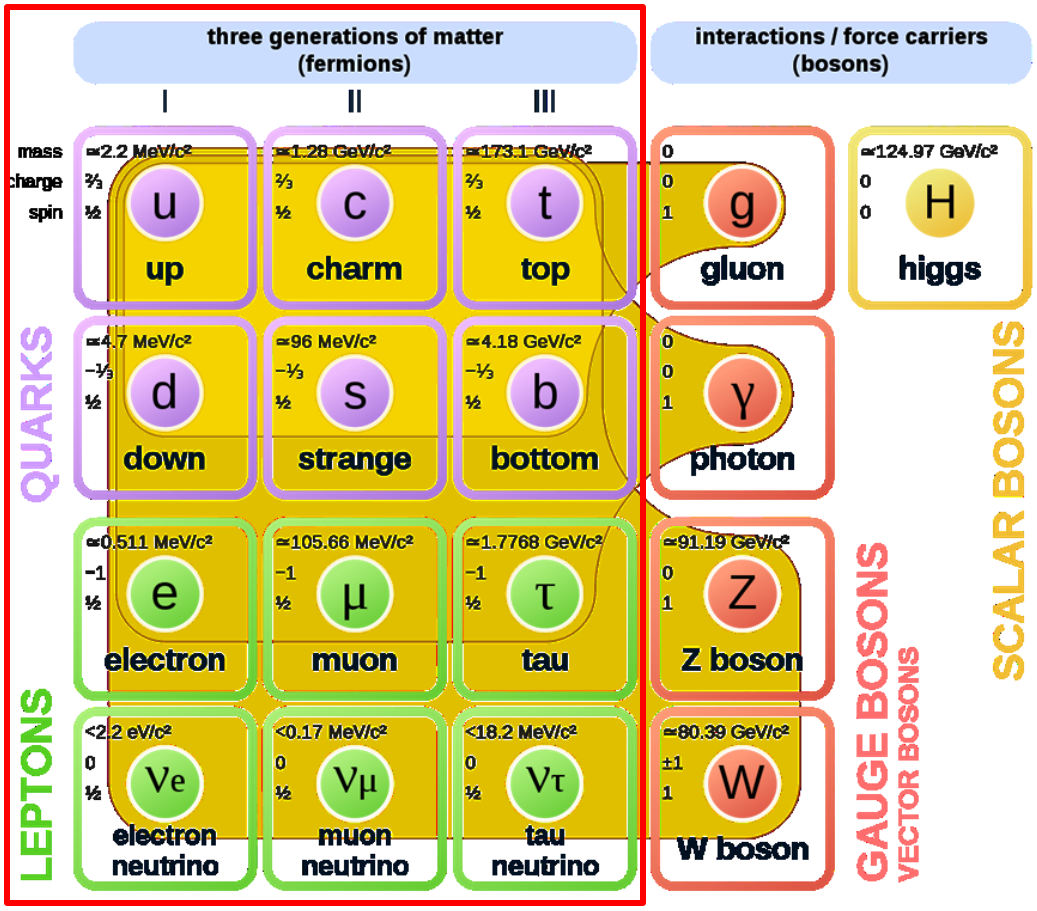
- Strong interaction

  Described by *quantum chromodynamics* (QCD), which explains the existence of hadrons.

Mathematically, the interactions can be shown to arise from the invariance of the Lagrangian under so-called *gauge symmetries*: SU(3) for the strong interaction, and SU(2) x U(1) for electroweak. Hence the associated force-carrying particles are called *gauge bosons*.

Particle content of the Standard Model

Three generations of fermions, involving an 'up-type' (charge +2/3e) and a `down-type' (charge -1/3e) quark, a charged lepton and a neutral neutrino. Leptons do not experience the strong interaction. Note that anti-fermion counterparts also exist.



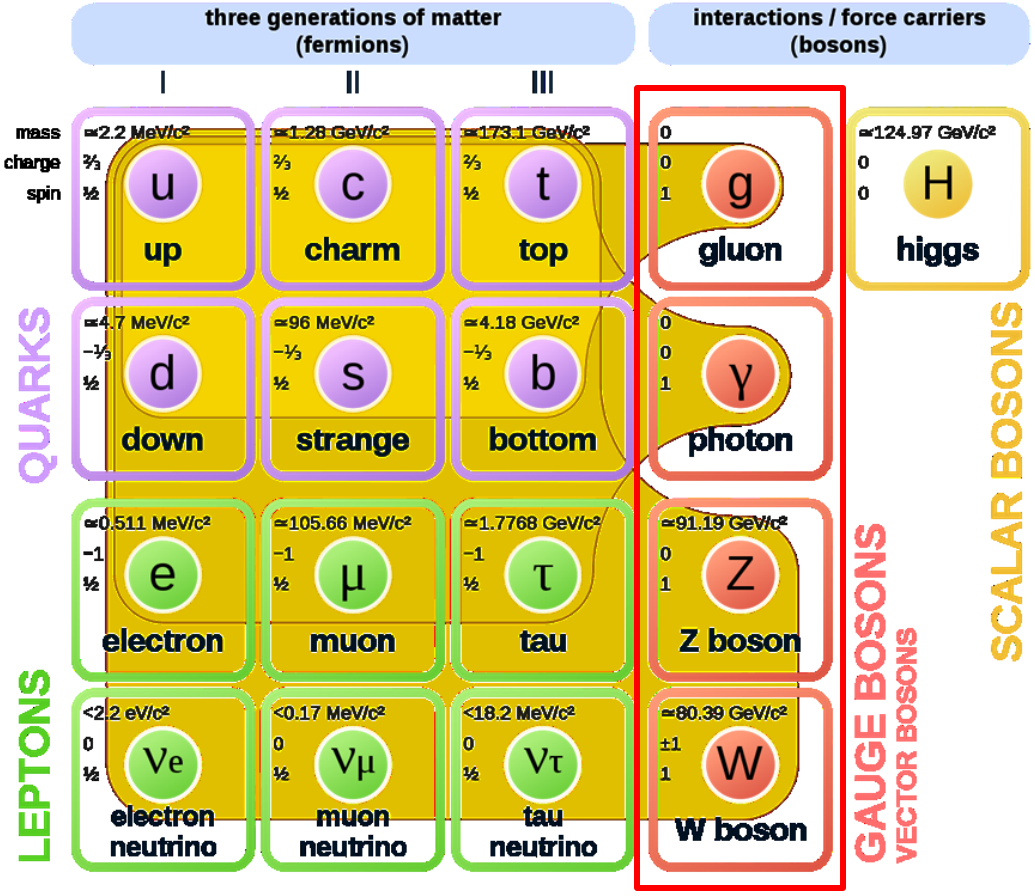
We don't know:

- why nature replicates this fermion structure over three generations;
- the reason for the extreme variation in mass (the up and down quarks `weigh' essentially nothing; the top quark has the mass of a gold nucleus), and the striking hierarchy of the elements in the matrix that couples them to the weak interaction;
- how the neutrinos got their masses, & why they are so small.

These different varieties of quarks and leptons are called `flavours'.

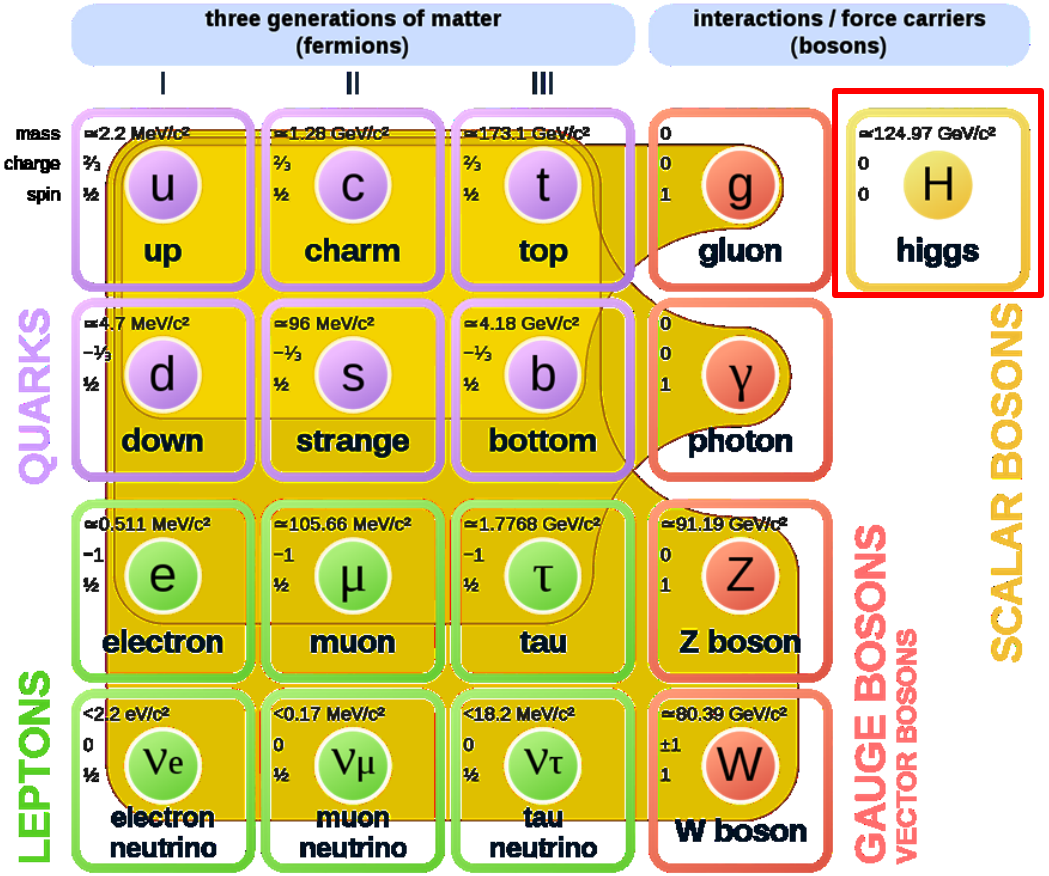
Particle content of the Standard Model

Four categories of vector (spin-1) gauge bosons, which mediate the strong & electroweak interactions. The gluon & photon are massless. The gluon comes in eight states carrying different superpositions of colour charge. The neutral Z^0 & the charged W^+/ W^- are massive.



Particle content of the Standard Model

The Higgs boson is the only known fundamental scalar (spin 0) particle in nature. It is an excitation of the Higgs field that gives mass to the quarks, charged leptons, & the *W* & *Z*. Although the second most massive of the fundamental particles, it is *much* lighter than a priori arguments would predict, which constitutes another profound mystery.



Unanswered questions in the Standard Model

Not on syllabus !

The Standard Model was fully formulated by the 1970s. Since then the existence of its remaining particle contents have been successively verified, culminating with the discovery of the Higgs boson in 2012. It has withstood all experimental tests in the laboratory (mostly at colliders), albeit with a few outstanding anomalies, which may or may not persist.

However, there are many problems and unanswered questions.
Those encountered already:

- The Higgs is too light;
- Too many free parameters;
- Striking and unexplained patterns in the flavour sector;
- No explanation for neutrino mass.



And many additional ones, *e.g.*:

- No candidate for a dark matter particle;
- It does not include gravity;
- It does not explain the baryon-antibaryon asymmetry in the universe (*i.e.* why the universe is overwhelming made of matter, not anti-matter);
- No explanation for dark energy.

We won't ponder on the answers to these in our lectures, but the search for 'New Physics' to address at least some of these issues is what drives current studies at the LHC & beyond.

Particle physics: topic overview

Introduction

- The Standard Model
- Natural units
- Feynman diagrams

A word about natural units

In particle physics it is common to work in natural units, where $c = \hbar = 1$ (also often with $\varepsilon_0 = \mu_0 = 1$, which implies $e = \sqrt{4\pi\alpha} = 0.303$). The most prominent consequence is:

energy and mass have same units: GeV

(or MeV, but in particle physics we are usually in the GeV regime)

It also means that length and time have units of GeV^{-1}, which can certainly be confusing.

We can calculate the conversion factors for length and area thus:

$$\hbar c = 0.197 \text{ GeV fm}$$

$$1 = 0.197 \text{ GeV fm}$$

$$\Rightarrow \quad \boxed{\begin{aligned} 1 \text{ fm} &= 5.08 \text{ GeV}^{-1} \\ 1 \text{ barn} &= 2580 \text{ GeV}^{-2} \end{aligned}}$$

and also for time:

$$\hbar = 6.58 \times 10^{-25} \text{ GeV s}$$

$$1 = 6.58 \times 10^{-25} \text{ GeV s}$$

$$\Rightarrow \quad \boxed{1 \text{ s} = 1.52 \times 10^{24} \text{ GeV}^{-1}}$$

In this course we will generally be sticking to SI units, apart from the energy / mass case.

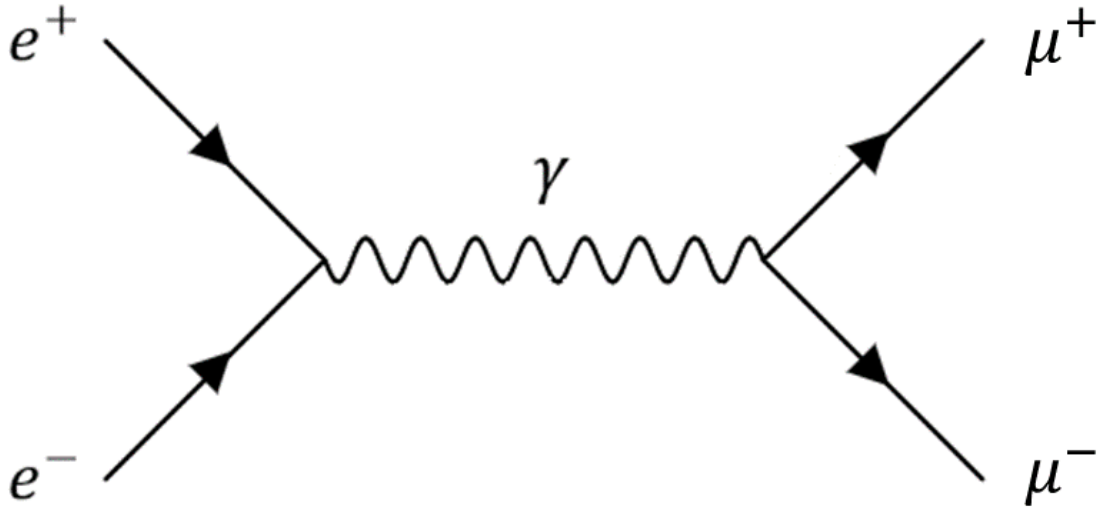
Particle physics: topic overview

Introduction

- The Standard Model
- Natural units
- Feynman diagrams

Feynman diagrams

First introduced in 1948 by Richard Feynman, as a tool in his studies of QED, Feynman diagrams are an exceeding useful visual and calculational tool in quantum field theory.





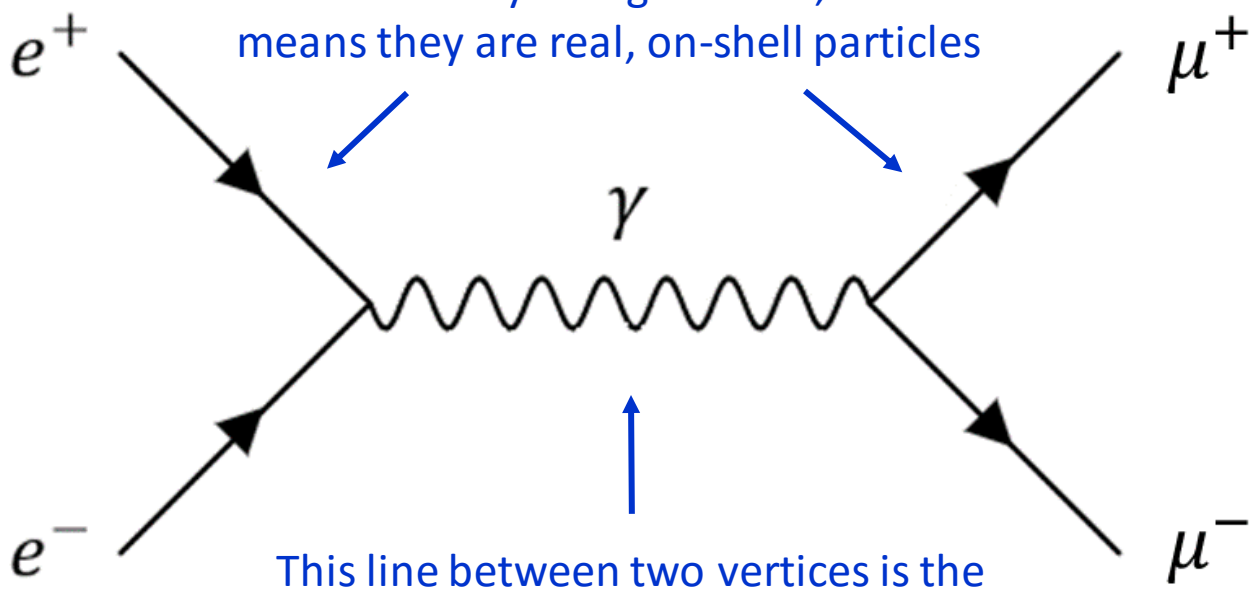
Richard Feynman,
1918-1988; American;
Nobel Prize for Physics 1965

Let us first concentrate on the visual aspects & conventions, before making some remarks on their role in calculations. We will cover all of the Standard Model, apart from the Higgs.

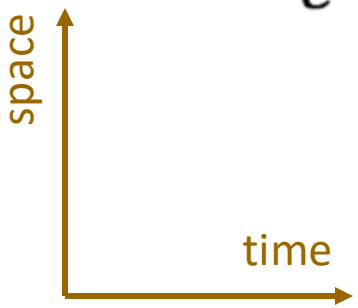
Feynman diagrams

Leading order diagram for $e^+e^- \rightarrow \mu^+\mu^-$

Initial and final state fermions. These lines have only a single vertex, which means they are real, on-shell particles



This line between two vertices is the *propagator* (here the photon propagator). The propagator is in general a virtual particle, *i.e.* not on mass shell, by which we mean $E^2 - p^2c^2 \neq m^2c^4$.



Dramatis personae



fermion



photon, $W^{\pm}$ or Z



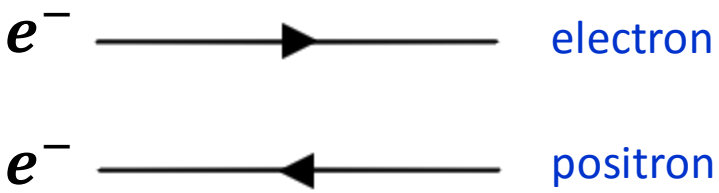
gluon

scalar boson, of which the only known fundamental example is the Higgs

Remarks on arrows

Feynman appreciated that mathematically an antiparticle travelling forwards in time is the same as an particle travelling backwards. This is important for the calculations and can be represented in the choice of arrow direction and labels.

e.g., with time running left to right:



But many people find this labelling of the antiparticle confusing, so instead write

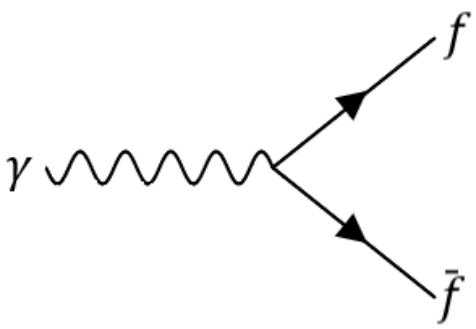


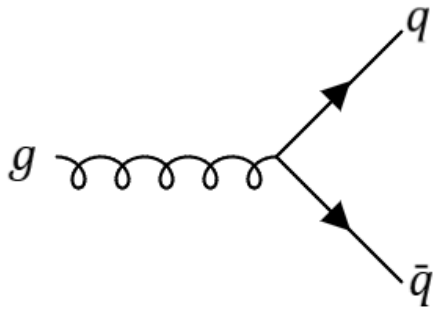
This is clearer, but logically make no sense. So others (including me) prefer the simpler

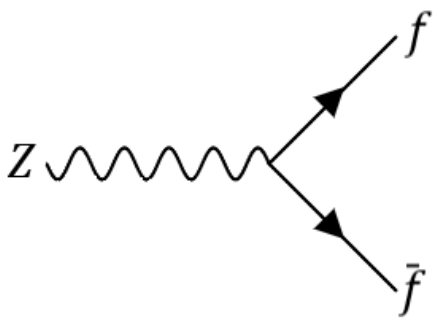


Purists (and possibly examiners) will sneer and say this is incorrect, and there are certainly diagrams where this is a bad choice, but it is the one I shall be generally using.

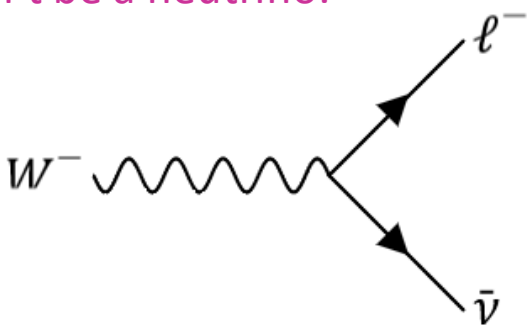
Allowed vertices involving fermions



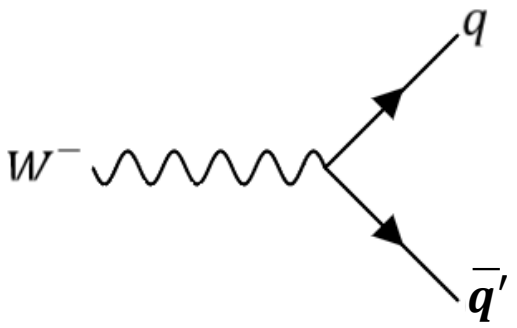




Here f must be charged, so can't be a neutrino.



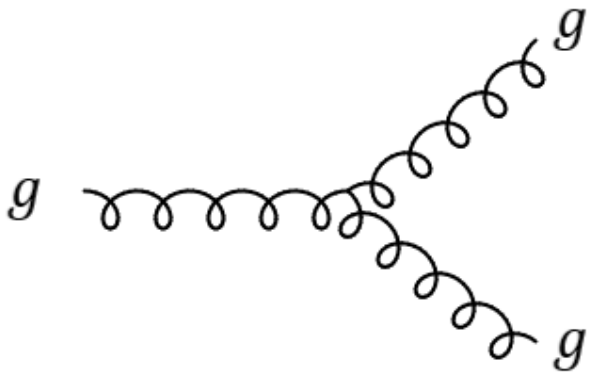
Corresponding diagrams exist for W^+ case.

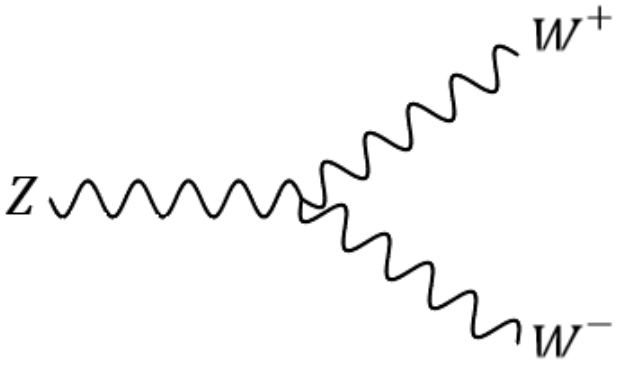


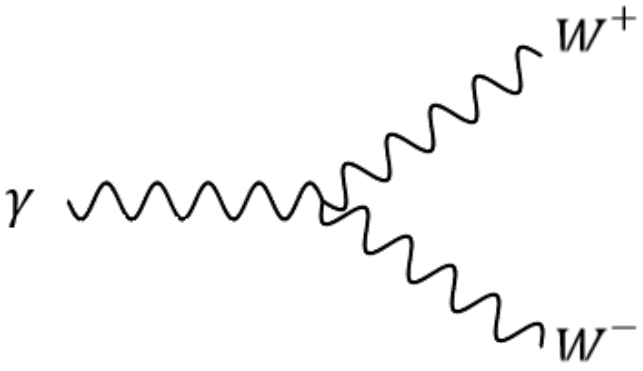
This is the *only* diagram in the Standard Model that violates flavour.

All vertices involve three particles: one boson & two fermions or two quarks, never a quark & a lepton. All vertices satisfy charge conservation. Higgs vertices not shown.

Allowed vertices involving bosons alone





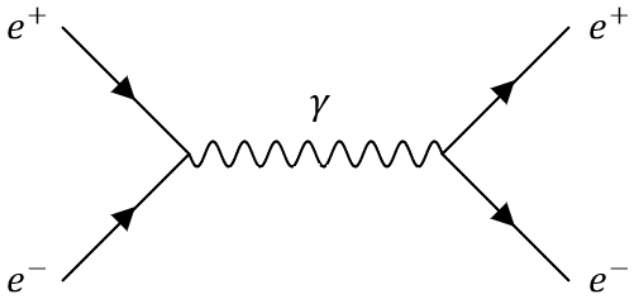


Here we have restricted ourselves to three-boson vertices, but four-boson coupling is also allowed. As on previous slide we do not show Higgs cases here.

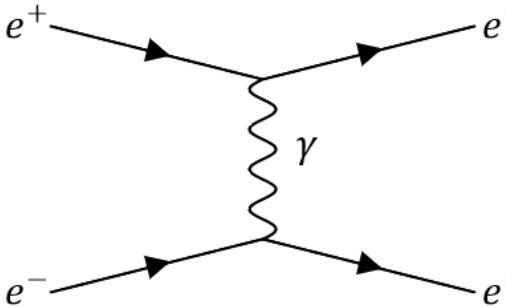
Rotated vertices, scattering and annihilation

All the vertices may be drawn rotated, provided they satisfy the conservation laws.

$$e^+e^- \rightarrow e^+e^-$$



Annihilation
's-channel'



Scattering
't-channel'

Here the same vertices appear in both diagrams, but rotated, to give two different topologies, which both contribute as separate amplitudes to the matrix element.

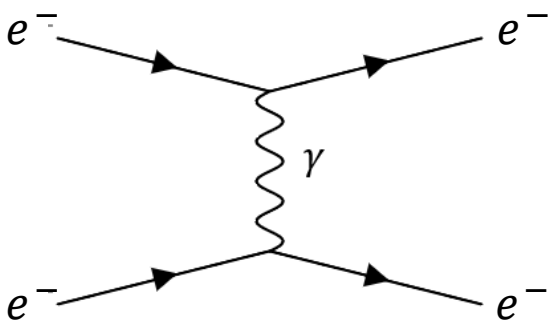
So, to first order $|M_{if}|^2 = |M_s + M_t|^2$.

(In general, diagrams may have different phases, so destructive interference is possible.)

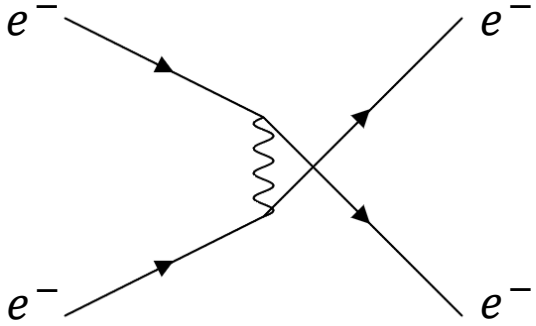
Rotated vertices, scattering and annihilation

All the vertices may be drawn rotated, provided they satisfy the conservation laws.

$$e^- e^- \rightarrow e^- e^-$$



Scattering
't-channel'



Scattering
'u-channel'

Initial state particles are identical means no annihilation diagram, but we have another scattering diagram as we don't know which vertices the final state electrons come from.

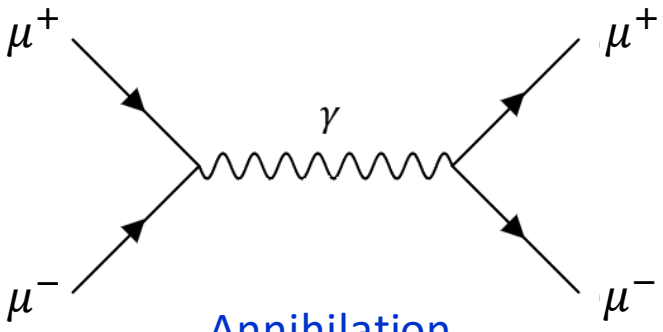
So, to first order $|M_{if}|^2 = |M_t + M_u|^2$.

(In general, diagrams may have different phases, so destructive interference is possible.)

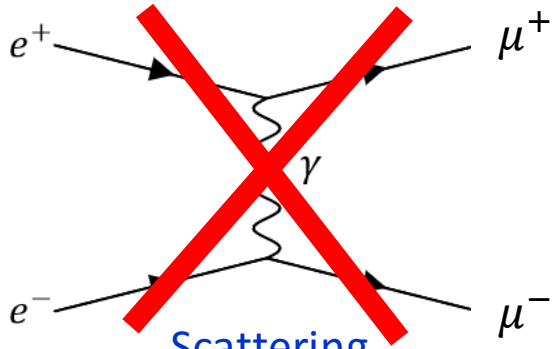
Rotated vertices, scattering and annihilation

All the vertices may be drawn rotated, provided they satisfy the conservation laws.

$$e^+e^- \rightarrow \mu^+\mu^-$$



Annihilation
's-channel'



Scattering
't-channel'

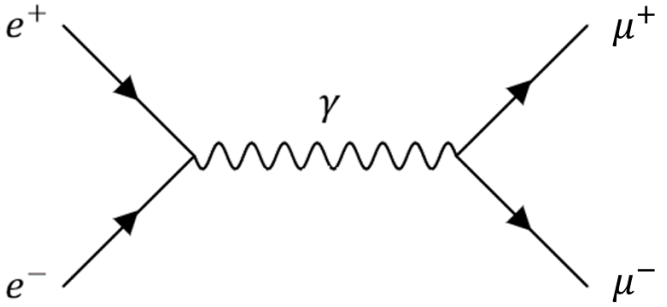
In this case the scattering diagram is forbidden as it involves flavour changes at both vertices (not allowed for a photon). This is a consequence of the initial and final-state particles being different, which was not the case for $e^+e^- \longrightarrow e^+e^-$.

So, here, at first order $|M_{if}|^2 = |M_s|^2$.

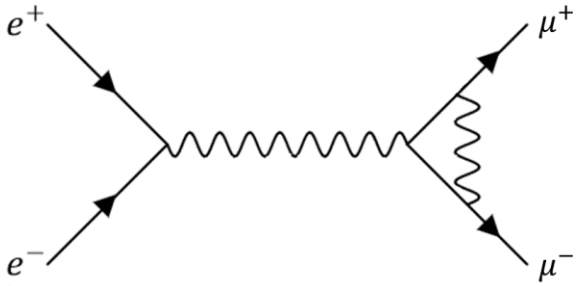
Higher-order diagrams

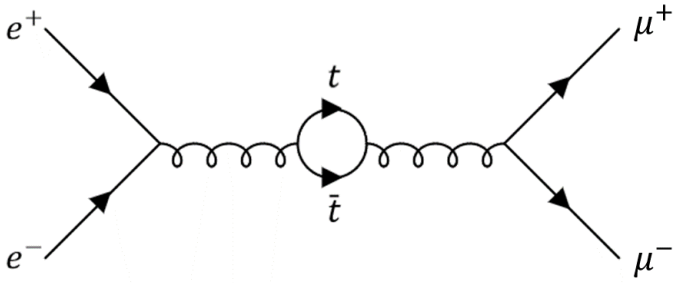
All the diagrams we have drawn so far have been leading order (sometimes called 'tree level'). But there will be an infinite number of higher-order diagrams. These will involve more vertices, which makes them higher order in the perturbation series.

Leading order diagram for $e^+e^- \to \mu^+\mu^-$



Two higher-order diagrams

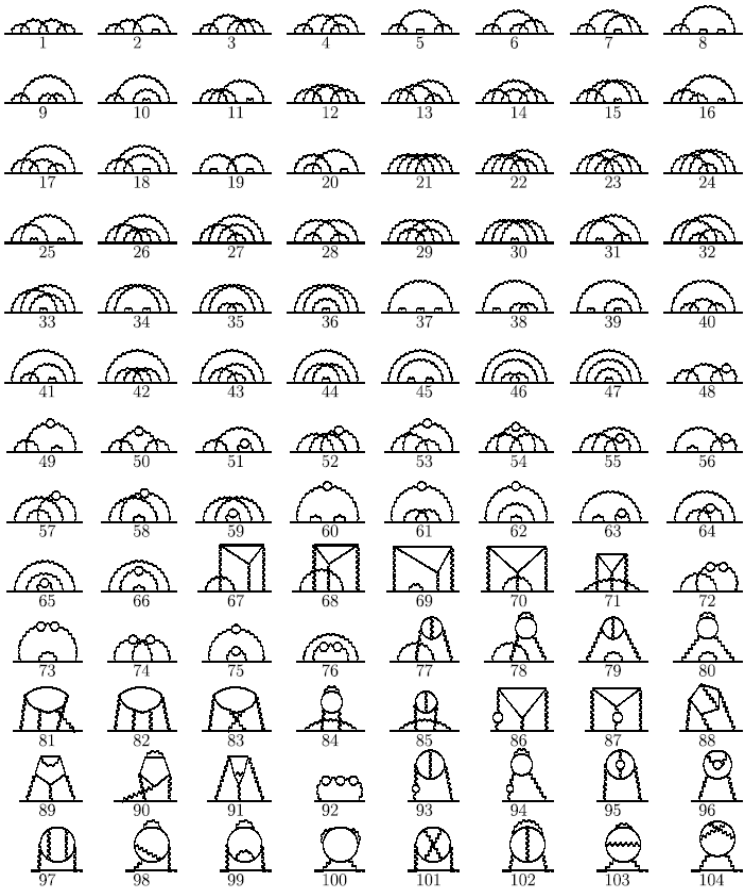




Current experiments are sensitive to these higher order diagrams. Note that they involve loops (hence 'loop diagrams'), in which the mass of the virtual particles can be higher than the available energy. Hence careful comparison between measurement and prediction can tell us about particles beyond the reach of our accelerators !

Higher-order diagrams

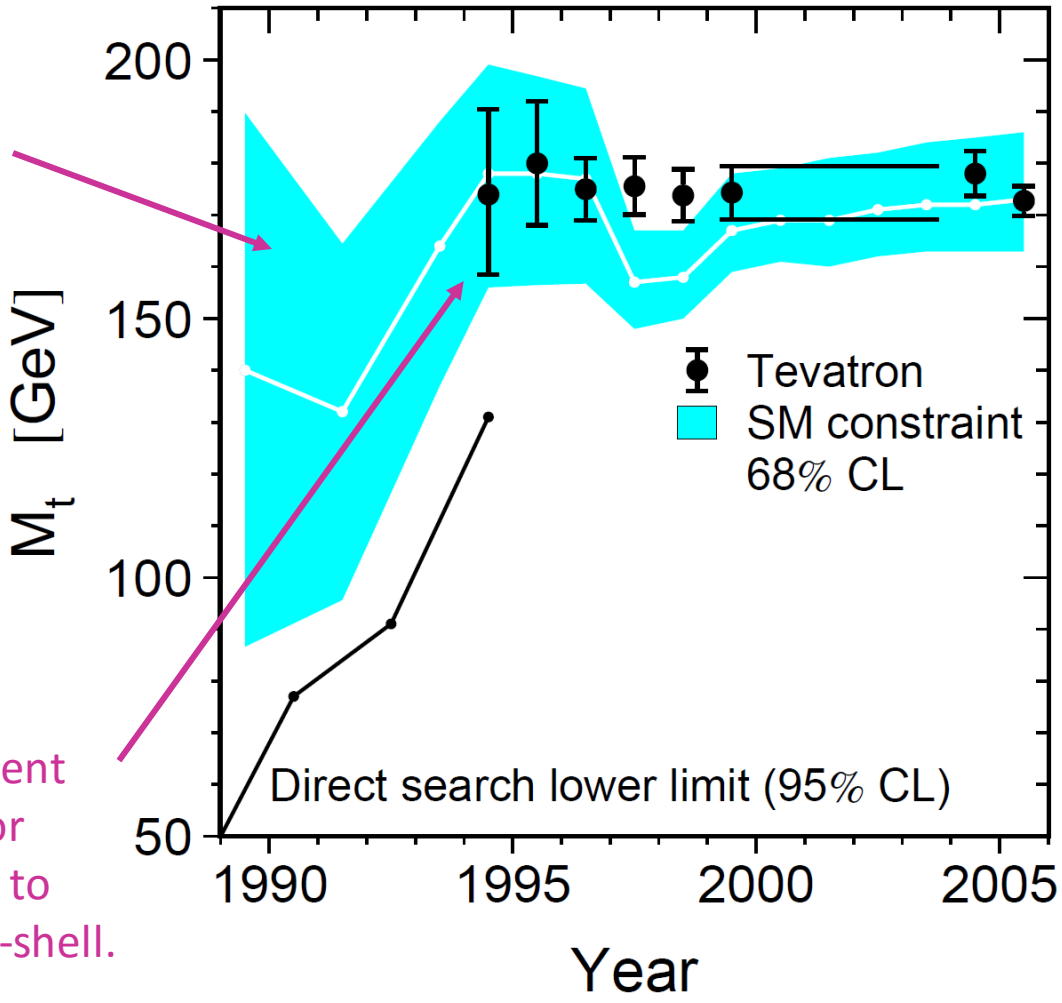
Some of the higher-order four-loop diagrams contributing to the calculation of the anomalous magnetic moment of the electron.



[Laporta, Phys. Letts. B 772 (2017) 232]

The power of higher-order diagrams: the top mass

Prediction of top-quark mass m_t from measurements of $e^+e^- \to \mu^+\mu^-$ and other processes at energies below m_t.

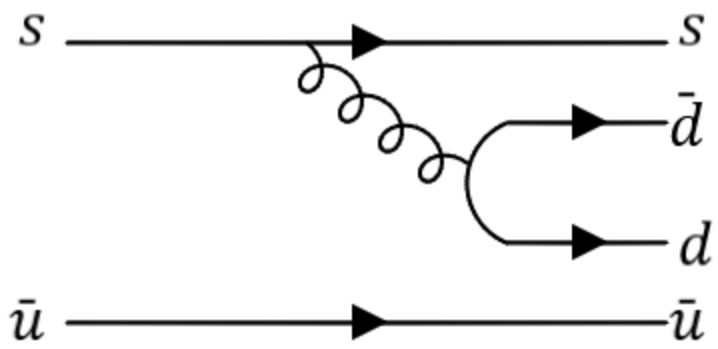


First direct measurement of m_t at an accelerator with sufficient energy to produce top quark on-shell.

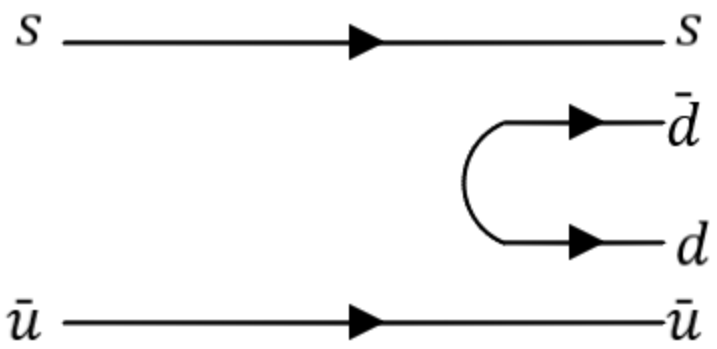
Quark-flow diagrams

Quark-flow diagrams are simplified versions of Feynman diagrams for strong decays, in which the gluons are not shown.

Feynman diagram

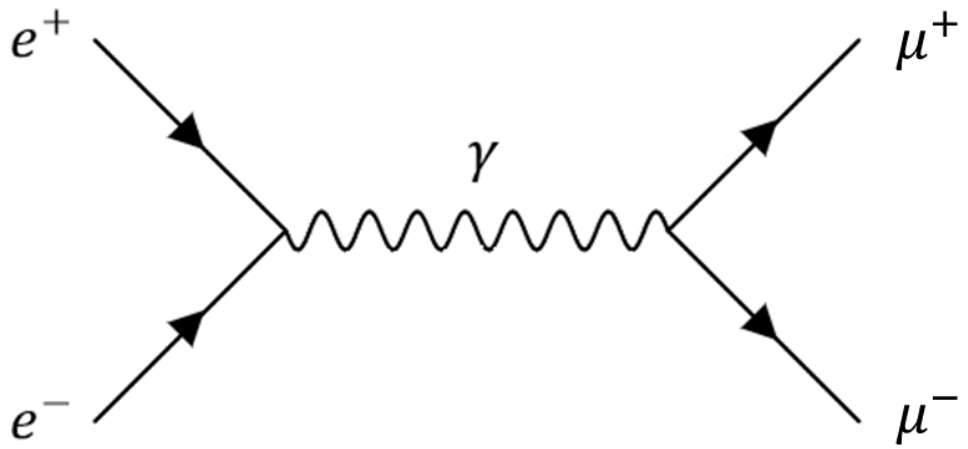


Quark-flow diagram



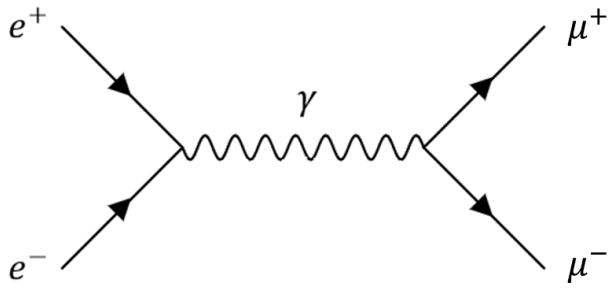
An example calculation: $e^+e^- \rightarrow \mu^+\mu^-$

Feynman diagrams come with a full set of rules that enable each diagram to be written as an amplitude in field theory. Here we will sketch out the most important elements for $e^+e^- \rightarrow \mu^+\mu^-$. We will not concern ourselves with constants and so will work in natural units. We will assume the relativistic case $E_{CM} \gg m_\mu$.



matrix element = vertex factor × propagator factor × vertex factor

An example calculation: $e^+e^- \rightarrow \mu^+\mu^-$



matrix element

=

vertex factor × propagator factor × vertex factor

Vertex factor

Strength of the interaction at that vertex. For an EM process this is $Qe = Q\sqrt{4\pi\alpha}$ in natural units, with here $Q = 1$ at both vertices. (For a weak interaction we would have g_W and for a strong $\sqrt{\alpha_S}$).

Propagator factor

This goes as $\dfrac{1}{\tilde{P}^2 - m^2}$, where $\tilde{P}$ is the four momentum of the propagator, and m is its mass. For e^+e^- annihilation into a photon in the centre-of-mass frame we have $\tilde{P}=(E_{\mathrm{CM}}, 0)$ and $m = 0$.

So, the matrix element goes as $M_{if} \sim \dfrac{\alpha}{E_{\mathrm{CM}}^2}$.

An example calculation: $e^+e^- \to \mu^+\mu^-$

To obtain cross section we recall $\sigma = \Gamma_{i\to f} / \text{incident flux}$ and $\Gamma_{i\to f} = \frac{2\pi}{\hbar}\left|M_{if}\right|^2 \rho$.

In the relativistic limit the incident flux is c, *i.e.* a constant. What about density of states ?

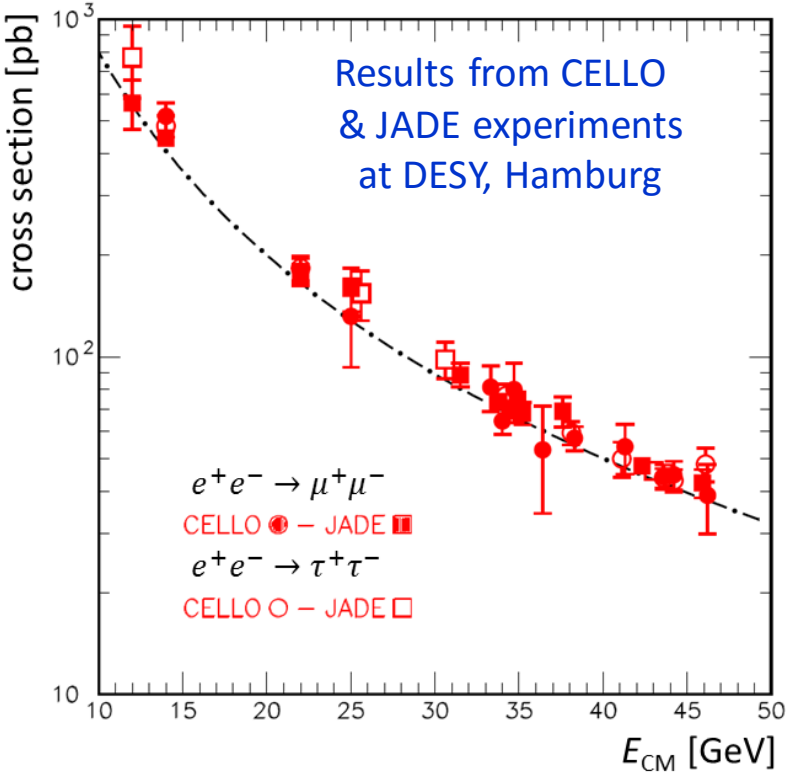
$$\rho = \frac{\mathrm{d}N}{\mathrm{d}E_{CM}} = \frac{\mathrm{d}N}{\mathrm{d}p_\mu}\frac{\mathrm{d}p_\mu}{\mathrm{d}E_{\mathrm{CM}}} \sim p_\mu^2 \frac{1}{2} \sim E_{\mathrm{CM}}^2 \qquad \text{since} \qquad p_\mu = E_{\mathrm{CM}}/2$$

Hence we expect:

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\hbar^2 c^2 \frac{\alpha^2}{E_{\mathrm{CM}}^2}$$

(here including constants from full calculation involving spin effects).

And indeed, this is the energy dependence that is observed (also for $e^+e^- \to \tau^+\tau^-$, which is expected to be identical) !



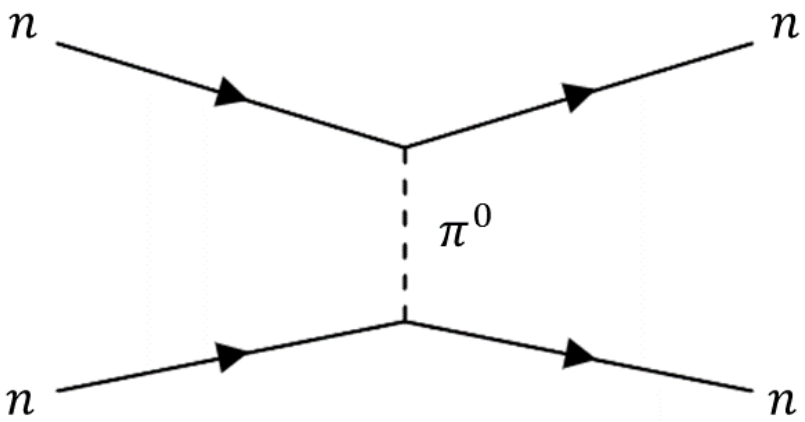
A final remark on propagators

Let us also consider the propagator $\frac{1}{\tilde{P}^2 - m^2}$ in the case of Yukawa scattering.

There we had a potential given by:

$$V(r) = V_0 \frac{e^{-r/a}}{r} = V_0 \frac{e^{-m_\pi r}}{r}$$

with a the length scale associated with the boson (pion) exchange. In natural units this is also the inverse mass of the exchanged pion



This is elastic scattering, so the four-vector of the propagator is given by $\tilde{P} = (0, q)$.

Thus the propagator factor becomes $\frac{-1}{q^2 - m_\pi^2}$ (here the overall minus sign is irrelevant).

And indeed, this factor appeared in our earlier derived expression for the differential cross section !

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mnV_0}{\hbar^2}\right)^2 \left(\frac{1}{q^2 + {m_\pi}^2}\right)^2$$

Particle physics: topic overview

Hadrons, quarks & the strong interaction

- Hadrons & their quantum numbers
- The light quark model
- Colour & QCD
- Quarkonia
- The *R* ratio

The hadron zoo

The positron was discovered in 1932 by Anderson, & the muon in 1936 by Anderson & Neddermeyer (`who ordered that?' Rabi), but the first hadron, aside from the proton & neutron, was not found until 1947. A huge number were discovered over the following $\sim$15 years, classified into ***mesons*** with ***integer spin***, & *baryons* with ***half-integer spin***.

From observation, ***baryon number is conserved*** (*i.e.* the net number of baryons and anti-baryons in a reaction must remain the same), but there is no such requirement on mesons.

Some early examples of discoveries (many more followed):

$\pi^{\pm}$	1947	Cosmic rays	Powell (Bristol)
$K^{\pm}$	1947	Cosmic rays	Rochester and Butler (Manchester)
π^{0}	1950	Accelerator	Bjorklund *et al.* (Berkley)
Λ^{0}	1950	Cosmic rays	Hopper and Biswas (Melbourne)
$\bar{p}$	1955	Accelerator	Segrè (Berkley)

The early discoveries were made using emulsion or cloud chamber photographs of cosmic rays, the later ones, from around 1950 onwards, in accelerators.

Sometimes the same particle was mislabelled as two separate ones on account of its decay into separate decay modes, *e.g.* the `$\tau^{+}$' and `θ^{+}', now known to be the K^{+}.

Strangeness

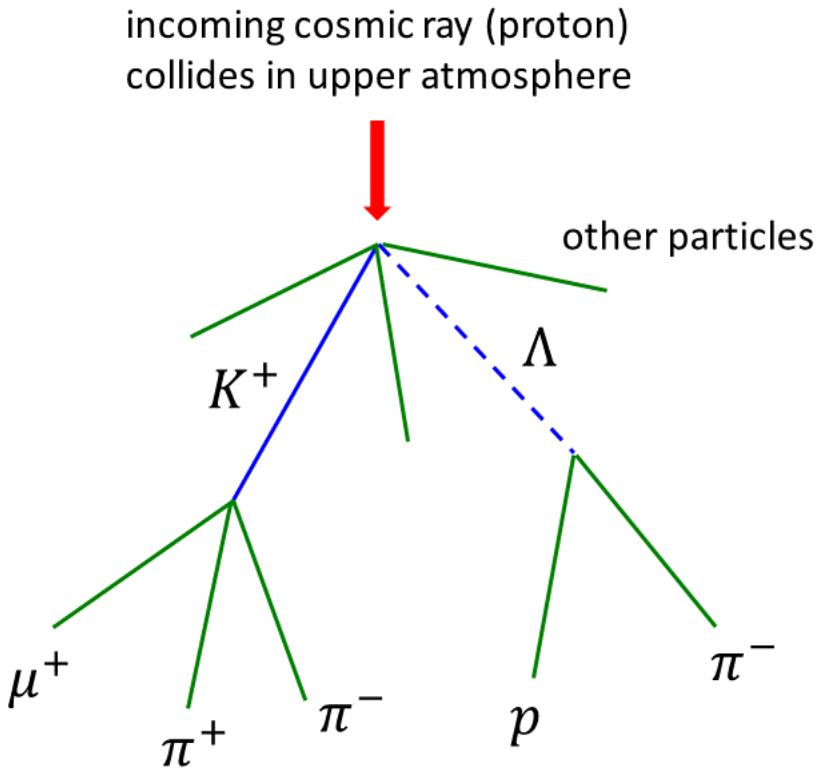
New quantum numbers were assigned to label the particles according to their properties.

`Strange' particles were so-named as they had the seemingly unusual properties of:

- Being produced in pairs – known as 'associated production'.

  (Here it's important to appreciate that the only production processes being studied were all via strong interaction.)

- Relatively long lived, which is explained by the fact that the particles can only decay via the weak interaction.



A quantum number of strange was assigned, which is conserved in strong (& electromagnetic) processes, but not in weak interactions.

We now know that strangeness is just one of six individual quark flavours, and that the strong (and electromagnetic) interaction conserves flavour and the weak interaction (or more specifically the weak-charged interaction) does not.

Isospin

We have noted the charge independence of the nuclear interaction, and the very small difference in mass between the proton and neutron. This led Heisenberg to propose the concept of isospin, where the p (n) is the spin up (down) member of a isospin doublet.

i.e. the nucleon has isospin $I = \frac{1}{2}$, and the third component I_3 has eigenvalues $+\frac{1}{2}$ (proton) and $-\frac{1}{2}$ (neutron).

Furthermore isospin I is conserved in strong interactions (but not EM or weak).

Other groups of particles can be classified similarly.

e.g. the π^+, π^0 and π^- form an isospin triplet with $I = 1$, and $I_3 = -1, 0$ and 1;
the K^0 and K^+ form an isospin doublet with $I = \frac{1}{2}$, and $I_3 = -\frac{1}{2}$ and $\frac{1}{2}$;
the deuteron is an isospin singlet with $I = 0$.

Isospin is not an exact symmetry, so is not quite on the same footing as other quantum numbers that we will encounter, but it is a useful tool of classification, and also can be employed to make predictions about relative cross-sections. The maths that it obeys is highly analogous to that of quantum spin, hence the name.

Isospin and nucleon-nucleon scattering

Consider the following reactions:

$$p + p \rightarrow d + \pi^+$$

$$p + n \rightarrow d + \pi^0$$

$$n + n \rightarrow d + \pi^-$$

The $|I, I_3\rangle$ of the final states are $|1, 1\rangle$ for $d + \pi^+$, $|1, 0\rangle$ for $d + \pi^0$ and $|1, -1\rangle$ for $d + \pi^-$. For the initial states we have $|1, 1\rangle$ for $p + p$ and $|1, -1\rangle$ for $n + n$. However, $p + n$, which has $I_3 = 0$, can be produced by either $|1, 0\rangle$ or $|0, 0\rangle$, and so is a superposition of both.

	$\lvert\psi(I, I_3)_{\text{initial}}\rangle$	$\lvert\psi(I, I_3)_{\text{final}}\rangle$
$p + p \rightarrow d + \pi^+$	$\lvert 1, 1\rangle$	$\lvert 1, 1\rangle$
$p + n \rightarrow d + \pi^0$	$\sqrt{\frac{1}{2}}(\lvert 1, 0\rangle + \lvert 0, 0\rangle)$	$\lvert 1, 0\rangle$
$n + n \rightarrow d + \pi^-$	$\lvert 1, -1\rangle$	$\lvert 1, -1\rangle$

Isospin and nucleon-nucleon scattering

The cross section is $\sigma \propto \langle \psi_{\text{final}} | H | \psi_{\text{initial}} \rangle^2$ with H an isospin operator that gives an amplitude M is connecting an $I = 1$ and $I = 1$ state, but zero if connecting $I = 0$ and $I = 1$, since Isospin is conserved. Therefore the cross sections are:

$$\sigma_{pp \to d\pi^+} \propto |M|^2$$
$$\sigma_{pn \to d\pi^0} \propto \frac{1}{2}|M|^2$$
$$\sigma_{nn \to d\pi^-} \propto |M|^2$$

	$\lvert\psi(I, I_3)_{\text{initial}}\rangle$	$\lvert\psi(I, I_3)_{\text{final}}\rangle$
$p + p \to d + \pi^+$	$\lvert 1, 1\rangle$	$\lvert 1, 1\rangle$
$p + n \to d + \pi^0$	$\sqrt{\frac{1}{2}}(\lvert 1, 0\rangle + \lvert 0, 0\rangle)$	$\lvert 1, 0\rangle$
$n + n \to d + \pi^-$	$\lvert 1, -1\rangle$	$\lvert 1, -1\rangle$

And the following ratio is expected

$$\frac{\sigma(p + n \to d + \pi^0)}{\sigma(p + p \to d + \pi^+)} = \frac{1}{2}$$

This is what is seen, at least for $p + p$ (the $n + n$ study is experimentally challenging…).

Spin and parity

Particles are labelled by their spin J and intrinsic parity P, where the latter can be positive or negative, *i.e.* $J^P = 0^-, 1^-, 3/2^+$ *etc.* You are very familiar with the concept of spin. For particles it is determined through angular distribution of decay products, and insisting that angular momentum is conserved in a reaction. Parity needs more discussion.

A parity operation is a *discrete transformation* that inverts spatial coordinates:

$$\hat{P}\psi(\vec{r}) = \psi(-\vec{r}) \qquad \textit{i.e. } (x, y, z) \rightarrow (-x, -y, -z)$$

Vectors, *e.g.* position $\vec{r}$ or momentum $\vec{p}$ invert sign under $\hat{P}$ whereas *axial vectors*, such as angular momentum $\vec{L} = \vec{r} \times \vec{p}$, remain unchanged. The laws of classical physics are invariant under a parity transformation. Now let us consider a quantum system.

The spatial part of the wavefunction is that describing the orbital angular momentum. Consider, *e.g.,* the spherical-harmonic wavefunction in a hydrogen atom $Y_l^m(\theta, \phi)$. We note that the spatial inversion $\vec{r} \rightarrow -\vec{r}$ corresponds to $\theta \rightarrow \pi - \theta$ and $\phi \rightarrow \pi + \phi$.

$$\hat{P}Y_l^m(\theta, \phi) = (-1)^l Y_l^m(\theta, \phi)$$

The orbital angular momentum wavefunction is an eigenstate of $\hat{P}$ with eigenvalues $(-1)^l$.

The $l = 0,2,4, \ldots$ (*s*,*d*,*g*...) states have even parity & the $l = 1,3,5 \ldots$ (*p*,*f*,*h*...) odd parity.

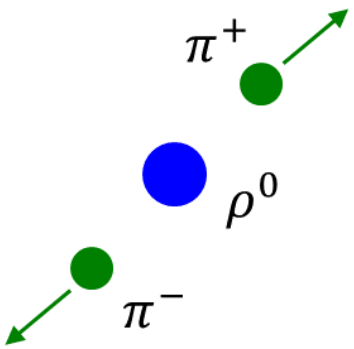
Intrinsic and total parity

It was assumed that parity must be conserved in the strong interaction, as in electromagnetism and all the laws of classical physics. This means that that the total parity of the initial state of a reaction must equal that of the final state. For this to work, we need to assign *intrinsic parity* to particles and calculate the total parity thus:

$$P = \prod_i P_i (-1)^l$$

where P_i is the intrinsic parity of particle i.

For example, consider the observed decay $\rho^0 \to \pi^+\pi^-$. What is the parity of the ρ^0 ?



The ρ^0 has $J = 1$, the pions have $J = 0$. Furthermore, the pions have negative parity* (see problem sheet IV).

In order to conserve angular momentum the two pions must be in an $l = 1$ state. Therefore final-state parity is:

$$P_\rho = P_\pi \times P_\pi \times (-1)^1 = -1$$

Hence the ρ^0 has negative parity, and so has $J^P = 1^-$.

* In fact here we even don't need to know this; all we need to know is that the parity of an anti-particle is unchanged from that of the particle.

Intrinsic and total parity

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$$P = \prod_i P_i (-1)^l$$

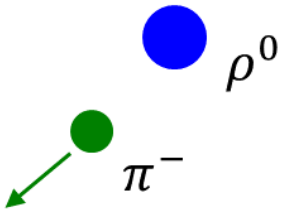
where P_i is the intrinsic parity of particle i.

It will be the case that the intrinsic parity of almost all mesons we encounter will be negative, and that of baryons positive, but there certainly exist examples where this is not so.

Also useful to note that the photon has negative parity.

For exa… of the ρ^0 ?

…hermore, …sheet IV).



In order to conserve angular momentum the two pions must be in an $l = 1$ state. Therefore final-state parity is:

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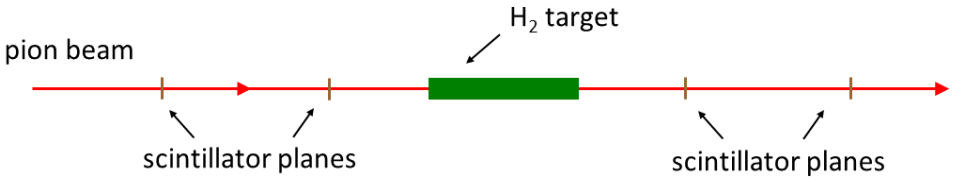
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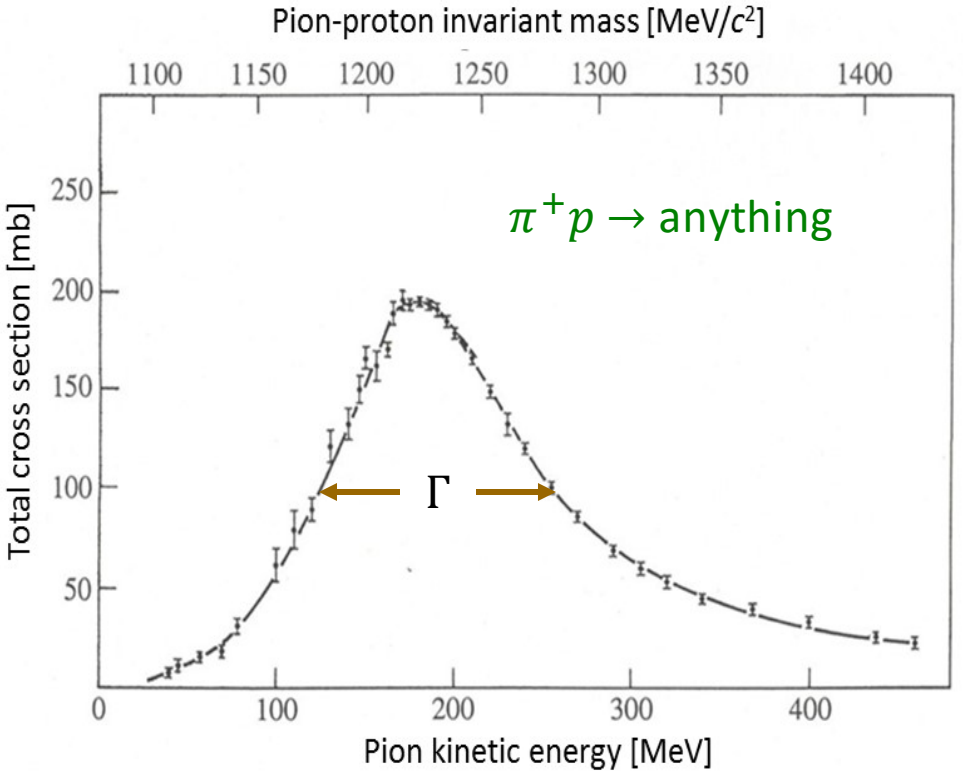
* In fact here we even don't need to know this; all we need to know is that the parity of an anti-particle is unchanged from that of the particle.

The Δ resonances

As well as particles observed directly, other extremely short-lived states were discovered as *resonances*, the first of which was the Δ at 1236 MeV/c^2, produced in πN scattering (*e.g.* $\pi^+ + p$, $\pi^- + p$ *etc.*), and first seen by Fermi in 1952 using a new cyclotron at Chicago.

Pion energy varied, and cross section variation inferred from change of activity in downstream scintillator planes.





FWHM of resonance indicated that the intermediate state was very short lived indeed:

$$\Gamma \approx 100 \text{ MeV} \Rightarrow \tau \sim 10^{-23} \text{s} !$$

A peak at the same location and width, but different cross section was also seen in $\pi^- p$ scattering.

Calculating pion-nucleon scattering with isospin

Consider the three reactions $\pi^+ + p \to \pi^+ + p$, $\pi^- + p \to \pi^- + p$ & $\pi^- + p \to \pi^0 + n$. The initial & final states are superpositions of $I = 3/2$ and $I = 1/2$ isospin amplitudes, with the coefficients coming from the Clebsch-Gordan coefficients of spin algebra.

	$\lvert\psi(I, I_3)_{\text{initial}}\rangle$	$\lvert\psi(I, I_3)_{\text{final}}\rangle$
$\pi^+ + p \to \pi^+ + p$	$\left\lvert \frac{3}{2}, \frac{3}{2} \right\rangle$	$\left\lvert \frac{3}{2}, \frac{3}{2} \right\rangle$
$\pi^- + p \to \pi^- + p$	$\sqrt{\frac{1}{3}} \left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{2}{3}} \left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$	$\sqrt{\frac{1}{3}} \left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{2}{3}} \left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$
$\pi^- + p \to \pi^0 + n$	$\sqrt{\frac{1}{3}} \left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{2}{3}} \left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$	$\sqrt{\frac{2}{3}} \left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle + \sqrt{\frac{1}{3}} \left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$

Calculating pion-nucleon scattering with isospin

The cross section is

$$\sigma \propto \langle \psi_{\text{final}} | H | \psi_{\text{initial}} \rangle^2$$

with H an isospin operator that is H_1 if operating on an I=1/2 state & H_2 if operating on an I = 3/2 state. Isospin conservation means there is no operator connecting I=1/2 & I=3/2.

	$\lvert\psi(I,I_3)_{\text{initial}}\rangle$	$\lvert\psi(I,I_3)_{\text{final}}\rangle$
$\pi^+ + p \to \pi^+ + p$	$\left\lvert \frac{3}{2}, \frac{3}{2} \right\rangle$	$\left\lvert \frac{3}{2}, \frac{3}{2} \right\rangle$
$\pi^- + p \to \pi^- + p$	$\sqrt{\frac{1}{3}}\left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{2}{3}}\left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$	$\sqrt{\frac{1}{3}}\left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{2}{3}}\left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$
$\pi^- + p \to \pi^0 + n$	$\sqrt{\frac{1}{3}}\left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{2}{3}}\left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$	$\sqrt{\frac{2}{3}}\left\lvert \frac{3}{2}, -\frac{1}{2} \right\rangle + \sqrt{\frac{1}{3}}\left\lvert \frac{1}{2}, -\frac{1}{2} \right\rangle$

Writing

$$M_1 = \left\langle \psi(I = \frac{1}{2})_{\text{final}} | H_1 | \psi(I = \frac{1}{2})_{\text{initial}} \right\rangle \quad \text{and} \quad M_3 = \left\langle \psi(I = \frac{3}{2})_{\text{final}} | H_3 | \psi(I = \frac{3}{2})_{\text{initial}} \right\rangle$$

yields the following results for the three cross sections:

$$\sigma_{\pi^+ p \to \pi^+ p} \propto |M_3|^2$$

$$\sigma_{\pi^- p \to \pi^- p} \propto \frac{1}{9}|M_3 + 2M_1|^2$$

$$\sigma_{\pi^- p \to \pi^0 n} \propto \frac{2}{9}|M_3 - M_1|^2$$

Calculating pion-nucleon scattering with isospin

$$\sigma_{\pi^+ p \to \pi^+ p} \propto |M_3|^2$$

$$\sigma_{\pi^- p \to \pi^- p} \propto \frac{1}{9}|M_3 + 2M_1|^2$$

$$\sigma_{\pi^- p \to \pi^0 n} \propto \frac{2}{9}|M_3 - M_1|^2$$

Thus, if the Δ is a pure I=3/2 state then we would expect

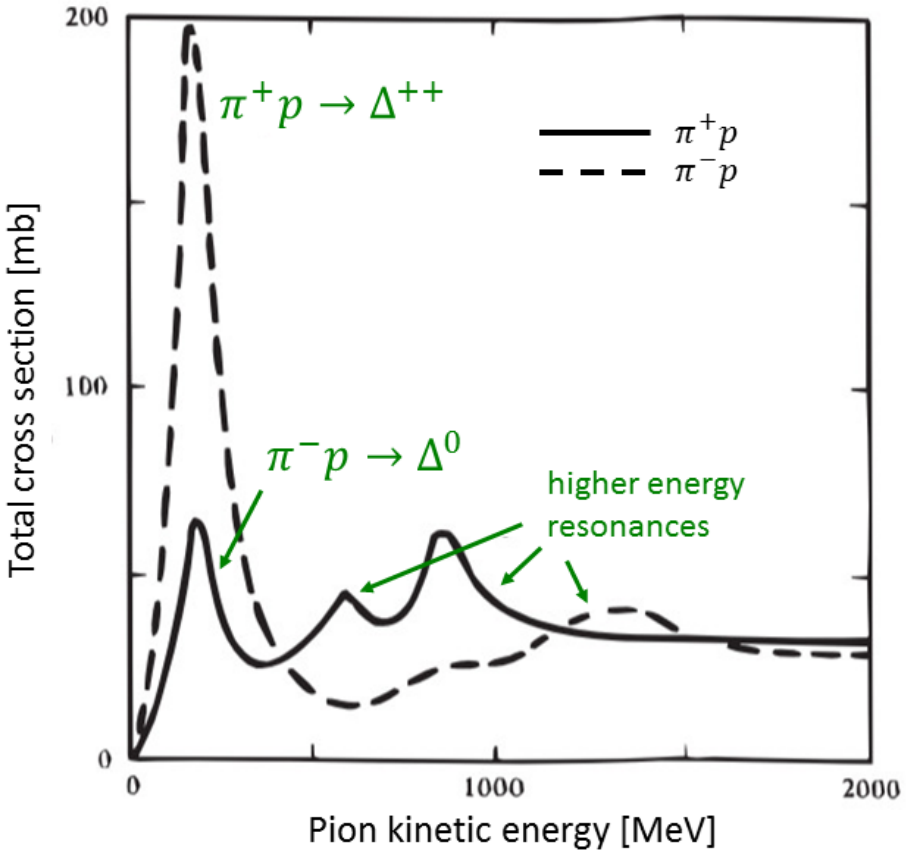
$$\frac{(\sigma_{\pi^+ p})_{\text{total}}}{(\sigma_{\pi^- p})_{\text{total}}} = 3$$

and if $I = 1/2$

$$\frac{(\sigma_{\pi^+ p})_{\text{total}}}{(\sigma_{\pi^- p})_{\text{total}}} = 0$$

The data clearly support the former.

$$\frac{(\sigma_{\pi^+ p})_{\text{total}}}{(\sigma_{\pi^- p})_{\text{total}}} = \frac{9|M_3|^2}{|M_3 + 2M_1|^2 + 2|M_3 - M_1|^2}$$



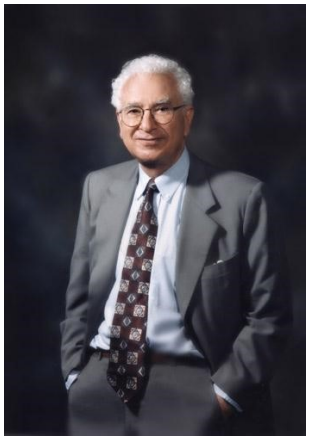
Particle physics: topic overview

Hadrons, quarks & the strong interaction

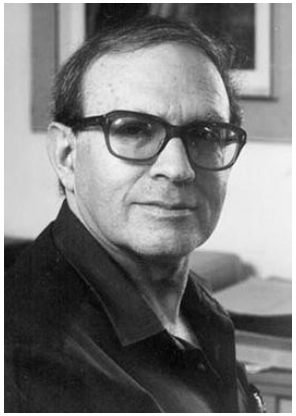
- Hadrons & their quantum numbers
- The light quark model
- Colour & QCD
- Quarkonia
- The *R* ratio

The eightfold way and the quark model

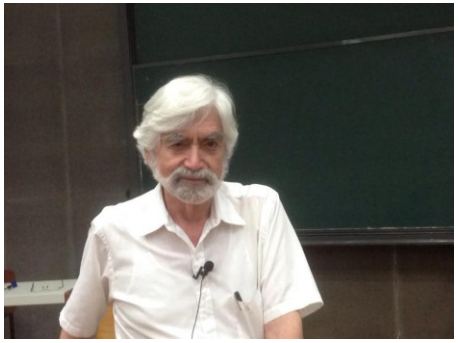
An organisational principle for the hadrons was developed independently by Murray Gell-Mann & Yuval Ne'eman in 1961, and dubbed the `eightfold way'. Built on SU(3) group theory, it revealed the clear patterns that emerge when classifying hadrons in I_3 versus S space, & allowed predictions of then missing particles in these patterns, *e.g.* the Ω^- baryon.



Murray Gell-Mann,
1929-2019; American;
Nobel Prize for Physics 1969



Yuval Ne'eman, 1925-2006;
Israeli physicist and politician



George Zweig, 1937-;
Russian-born American;
physicist and financier

The quark model was proposed independently by Gell-Mann and George Zweig in 1964. It explained the eightfold way by proposing hadrons to be made of combinations of (then only) three quarks: up (*u*), down (*d*) & strange (*s*). The model was recognised as a powerful tool, but quarks were not widely accepted as being 'real' until November 1974 (see later).

The eightfold way and the quark model

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The constituents of the light-quark model

	I	I_3	S	Q/e
u	$1/2$	$+1/2$	0	$+2/3$
d	$1/2$	$-1/2$	0	$-1/3$
s	0	0	-1	$-1/3$

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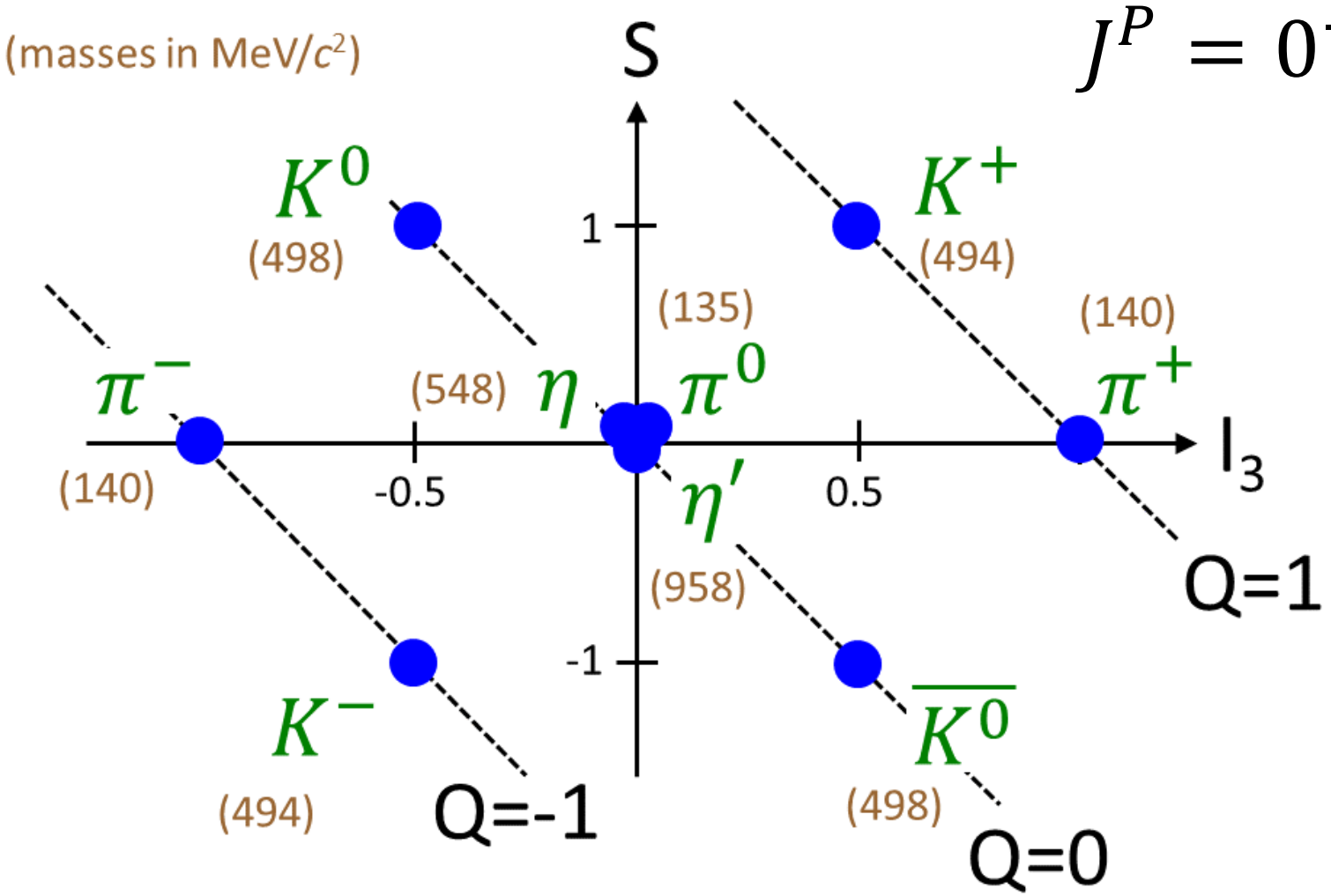
The pseudoscalar meson nonet

Consider the lightest mesons, all with $J^P = 0^-$. These are sometimes called *pseudoscalars* & there are nine of them. They can be classified according to isospin & strangeness.

	I	I_3	S
π^+	1	+1	0
π^0	1	0	0
π^-	1	-1	0
K^+	1/2	+1/2	+1
K^0	1/2	-1/2	+1
$\overline{K^0}$	1/2	+1/2	-1
K^-	1/2	-1/2	-1
η	0	0	0
η'	0	0	0

When plotted, these states display a characteristic pattern…

The pseudoscalar meson nonet



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η	0	0	0
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The quark model allows us to explain the J^P values of these states.

Quarks have spin ½. We may put them in a spin singlet state to give $S = 0$. We also give them orbital angular momentum $l = 0$, so $J = 0$.

$\uparrow\downarrow$

According to field theory, fermions and anti-fermions have opposite parity. The overall parity is

$$p_q p_{\overline{q'}} (-1)^l,$$

which here gives $P = -1$.

Hence the quark model gives $J^P = 0^-$.

We may now write down the quark contents and wavefunctions for each state.

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	I	I_3	S	Quark wavefunction
π^+	1	+1	0	$\lvert u\bar{d}\rangle$
π^0	1	0	0	$(1/\sqrt{2})\lvert d\bar{d} - u\bar{u}\rangle$
π^-	1	-1	0	$\lvert d\bar{u}\rangle$
K^+	1/2	+1/2	+1	$\lvert u\bar{s}\rangle$
K^0	1/2	-1/2	+1	$\lvert d\bar{s}\rangle$
$\overline{K^0}$	1/2	+1/2	-1	$\lvert s\bar{d}\rangle$
K^-	1/2	-1/2	-1	$\lvert s\bar{u}\rangle$
η	0	0	0	$(1/\sqrt{6})\lvert d\bar{d} + u\bar{u} - 2s\bar{s}\rangle$
η'	0	0	0	$(1/\sqrt{3})\lvert d\bar{d} + u\bar{u} + s\bar{s}\rangle$

The quark assignments nicely explain the charge, isospin and strangeness.

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η'	0	0	0	$(1/\sqrt{3})\|d\bar{d} + u\bar{u} + s\bar{s}\rangle$

The minus sign in this wavefunction can be justified in various ways. For instance, they are generated in operating on the π^+ or π^- states with isospin ladder operators, analogous to spin ladder operators.

The quark assignments nicely explain the charge, isospin and strangeness.

The pseudoscalar meson nonet

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				$\|d\bar{u}\rangle$
				$\|u\bar{s}\rangle$
				$\|d\bar{s}\rangle$
$\overline{K^0}$	1/2	+1/2		$\|s\bar{d}\rangle$
K^-	1/2	-1/2	-1	$\|s\bar{u}\rangle$
η	0	0	0	$(1/\sqrt{6})\|d\bar{d} + u\bar{u} - 2s\bar{s}\rangle$
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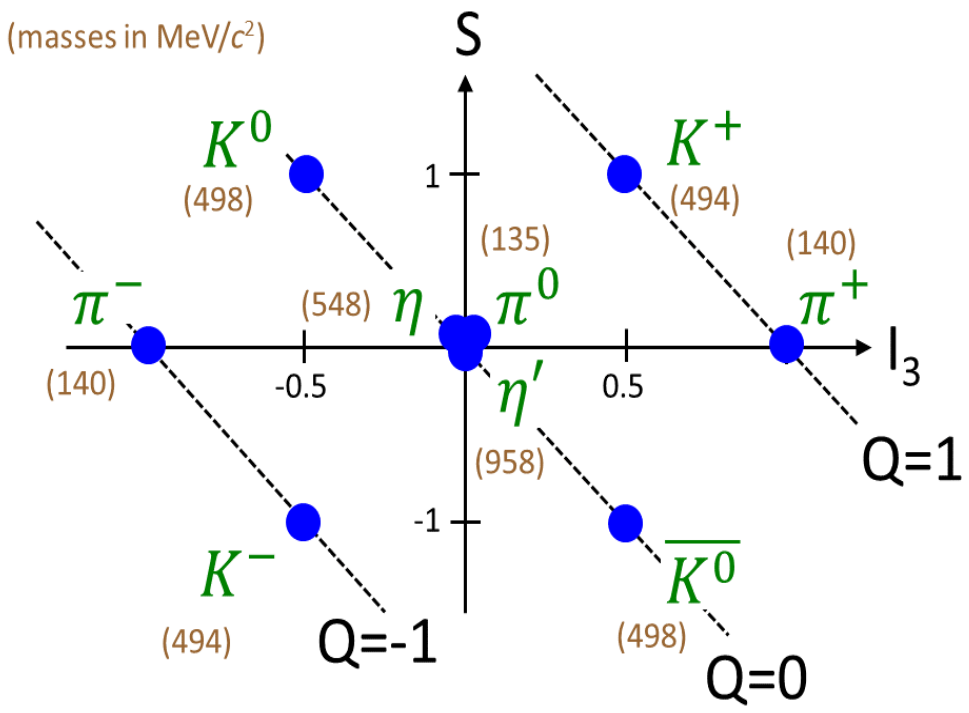
Note, that these two states, having identical quantum numbers, can mix. And indeed the physical manifestations is a superposition of both.

One other remark: even though the I_3=0 mesons are superpositions of different $q\bar{q}$ states, only a single state (whichever the most convenient) need be drawn on a Feynman diagram.

The pseudoscalar meson nonet

Other remarks on the properties of the pseudoscalar mesons:

- Surveying the variation in masses indicates that those with strange quark content are a little heavier, *i.e.* the *s* quark is more massive than the *u* and *d* quarks.
- The mesons have very different lifetimes reflecting the interaction by which they decay:



$\pi^{\pm}$ decay weakly: $\pi \to \mu\nu$;

π^0 decays through EM: $\pi^0 \to \gamma\gamma$;

K all decay weakly, to pions (and, sometimes, leptons);

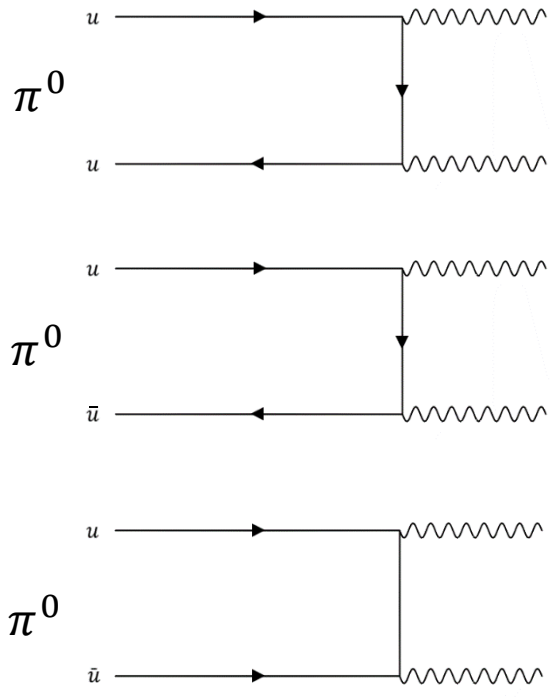
η decays through EM ($\eta \to \gamma\gamma$) or strongly ($\eta \to \pi^0\pi^0\pi^0$);

η' decays strongly ($\eta' \to \pi^+\pi^-\eta$) or through EM ($\eta' \to \pi^0\pi^0\gamma$).

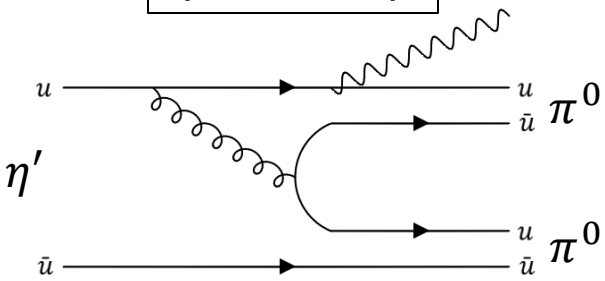
Pseudoscalar meson decays

$\pi^0 \to \gamma\gamma$

Drawn according to three separate arrow / showing / not-showing the bar conventions. Top one is probably the most correct, if not necessarily the clearest in all circumstances.

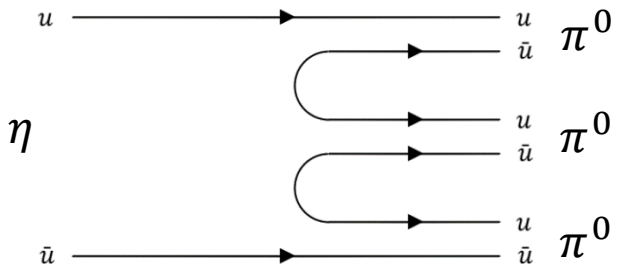


$\eta' \to \pi^0\pi^0\gamma$

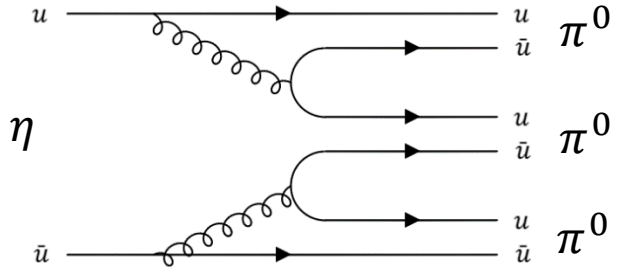


$\eta \to \pi^0\pi^0\pi^0$

Drawn as a quark-flow diagram…



…and then a Feynman diagram.

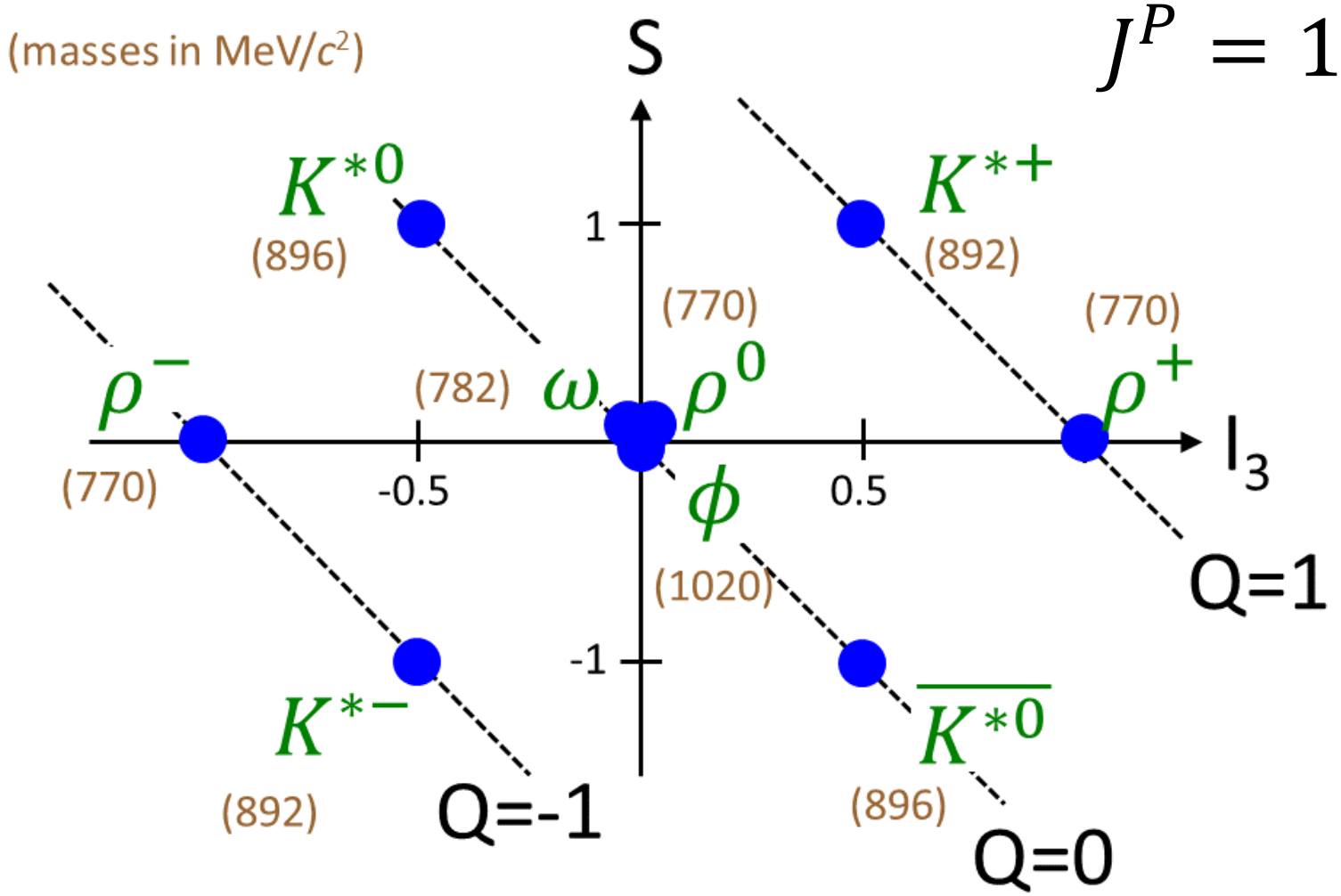


We postpone diagrams of the $\pi^\pm$ weak decays until later.

Note that $\eta \to \gamma\gamma$ would look identical.

The vector meson nonet

There is a corresponding set of nine *vector mesons* with $J^P = 1^-$, all significantly heavier.



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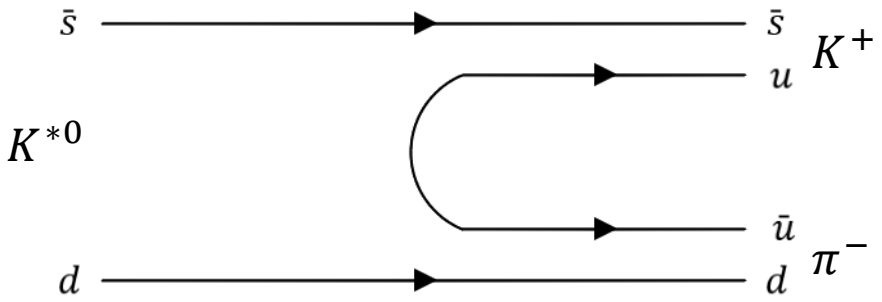
	I	I_3	S	Quark wavefunction
ρ^+	1	+1	0	$\left\|u\bar{d}\right\rangle$
ρ^0	1	0	0	$(1/\sqrt{2})\left\|d\bar{d} - u\bar{u}\right\rangle$
ρ^-	1	-1	0	$\left\|d\bar{u}\right\rangle$
K^{*+}	1/2	+1/2	+1	$\left\|u\bar{s}\right\rangle$
K^{*0}	1/2	-1/2	+1	$\left\|d\bar{s}\right\rangle$
$\overline{K^{*0}}$	1/2	+1/2	-1	$\left\|s\bar{d}\right\rangle$
K^{*-}	1/2	-1/2	-1	$\left\|s\bar{u}\right\rangle$
ω	0	0	0	$(1/\sqrt{6})\left\|d\bar{d} + u\bar{u} - 2s\bar{s}\right\rangle$
ϕ	0	0	0	$(1/\sqrt{3})\left\|d\bar{d} + u\bar{u} + s\bar{s}\right\rangle$

- $J = 1$ is explained by putting the $q\bar{q}'$ in a $S = 1$ triplet state ($\uparrow\uparrow$) with $l = 0$ as before.
- Parity is explained in identical manner to pseudoscalar meson case.
- Again the $I = 0$ states mix. In practice the ϕ meson is almost completely $s\bar{s}$.
- We need further discussion to understand the higher masses.

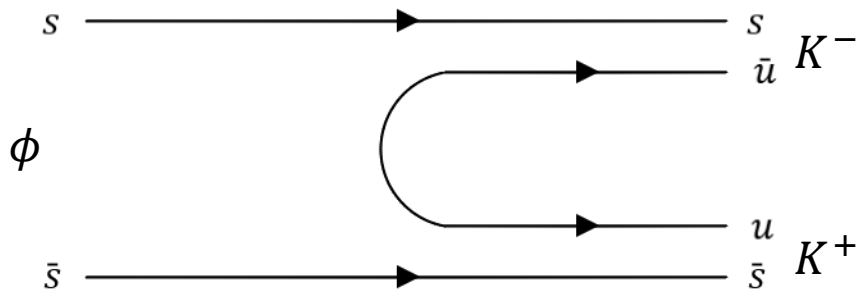
Vector meson decays

All of the vector mesons can decay strongly, and hence do this predominantly. Here are example quark flow diagrams for some leading decay modes.

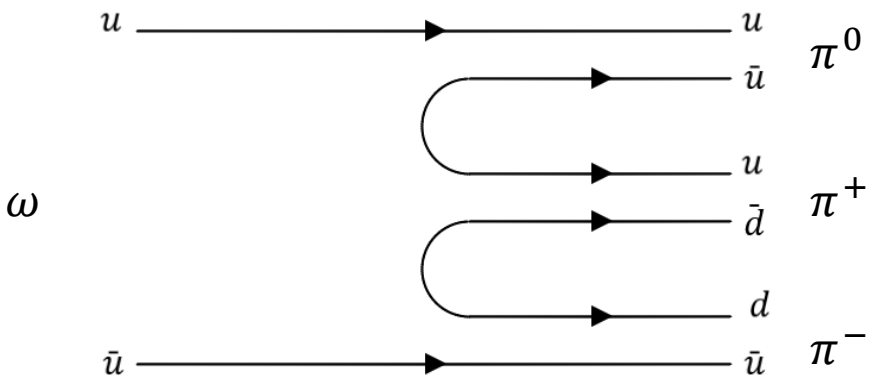
$K^{*0} \to K^+\pi^-$



$\phi \to K^+K^-$



$\omega \to \pi^0\pi^+\pi^-$



Hyperfine interactions and the hadron masses

In the hydrogen spectrum the energy levels are affected by hyperfine interactions between the magnetic moments of the electron and the proton. In hadrons there is a similar effect involving the interaction between the strong-force ('colour') magnetic moments of the quarks.

The magnetic moment is proportional to the spin and inversely proportional to the mass. The hyperfine interaction correction is proportional to the dot product of the two moments:

$$\Delta E_{ij} = k\frac{\vec{s_i}\cdot\vec{s_j}}{m_i m_j}$$

with $\vec{s_i}$ and m_i the spin and mass of quark i, respectively, and k a constant.

So the mass of the meson is

$$m = m_i + m_j + \Delta E_{ij}$$

Now the total spin of the meson is $\vec{S} = \vec{s_i} + \vec{s_j}$ and the square of this is given by

$$S^2 = s_i^2 + s_j^2 + 2\vec{s_i}\cdot\vec{s_j}$$

and thus

$$\vec{s_i}\cdot\vec{s_j} = \frac{1}{2}\left(S^2 - s_i^2 - s_j^2\right)$$

Hyperfine interactions and the hadron masses

So we have $m = m_i + mj + k\dfrac{\vec{s_i}\cdot\vec{s_j}}{m_i m_j}$ and $\vec{s_i}\cdot\vec{s_j} = \dfrac{1}{2}\left(S^2 - s_i^2 - s_j^2\right)$.

Recalling that the eigenvalues of the $\vec{L^2}$ operator are $l(l+1)\hbar^2$ we get

$$\vec{s_i}\cdot\vec{s_j} = \frac{1}{2}\left(S(S+1) - s_i(s_i+1) - s_j(s_j+1)\right)\hbar^2$$

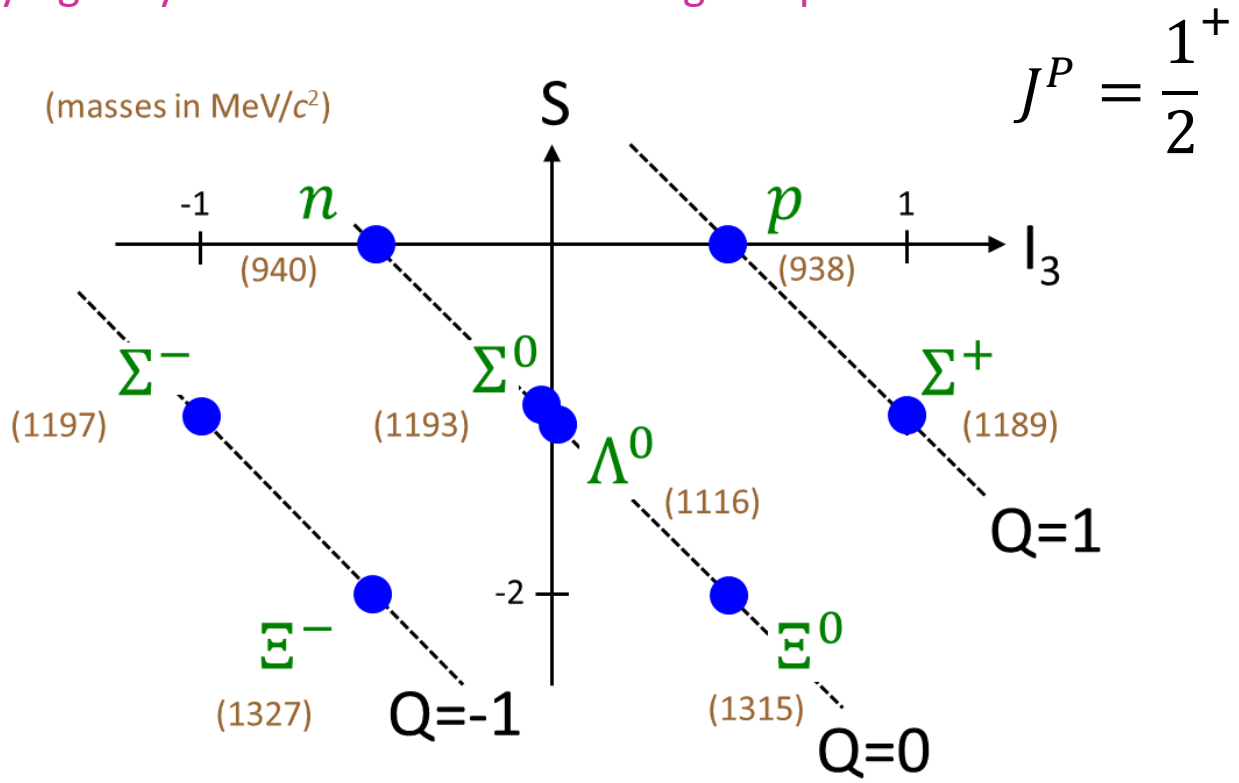
$$= \begin{cases} \dfrac{+\hbar^2}{4} & \text{for } S = 1 \text{ vector mesons} \\ \dfrac{-3\hbar^2}{4} & \text{for } S = 0 \text{ pseudoscalar mesons} \end{cases}$$

Hence the vector mesons are expected to be heavier than the pseudoscalars. Indeed fits to the measured meson masses can be accommodated within this ansatz, with consistent values for the quark masses and the constant. Note that unlike in hydrogen the correction is a first order effect (the strong coupling constant is large) !

The same considerations apply to the baryons, which we shall look at next.

The baryon octet

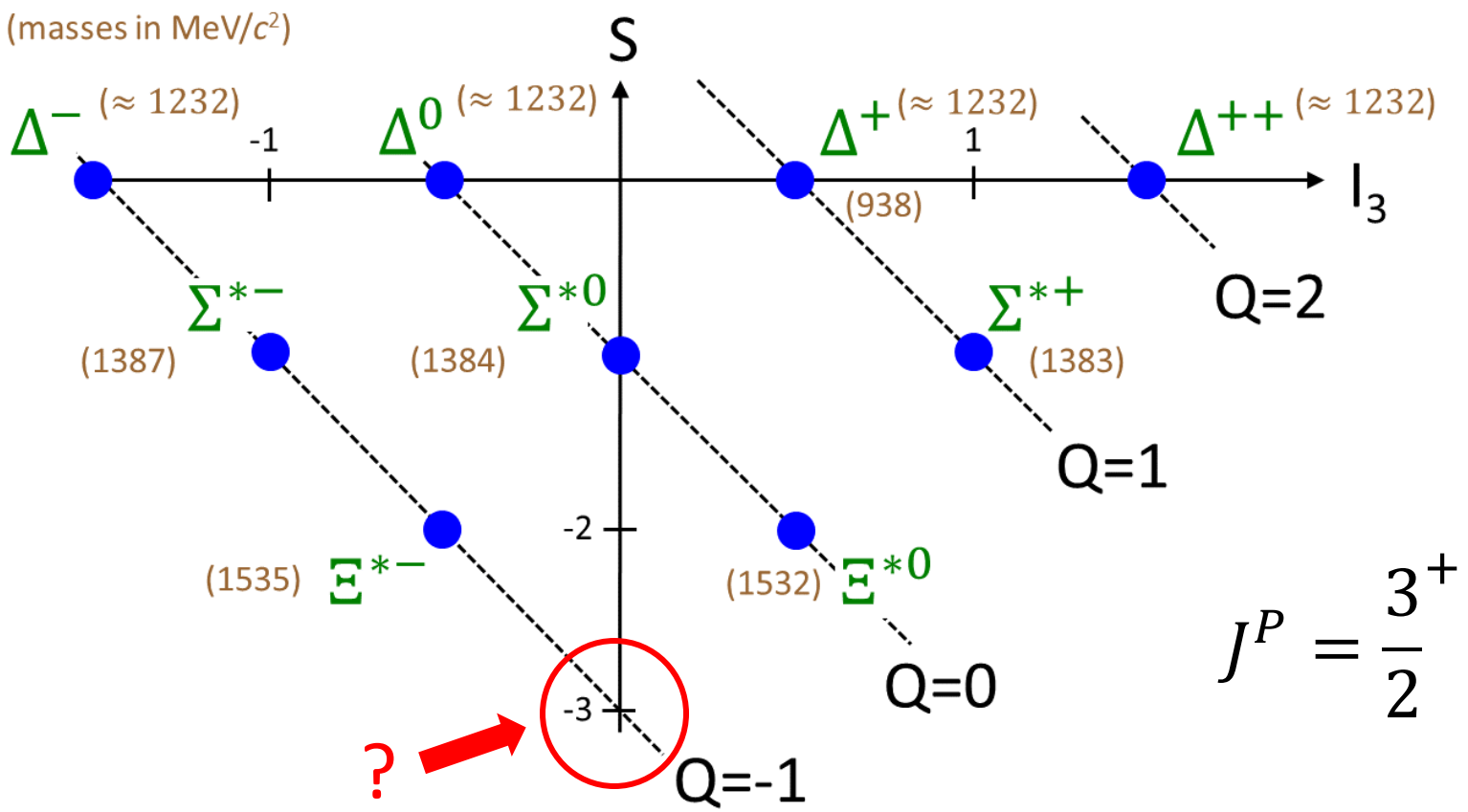
The lowest lying baryon states also form a hexagonal pattern.



- $J^P = \frac{1}{2}^+$ follows from $S = \frac{1}{2}$, *i.e.* ($\uparrow\uparrow\downarrow$), $l = 0$ & three quarks with positive parity.
- Quark content, and reason for progression of masses, should be self-evident.
- However, there are certain combinations (uuu, ddd, sss) not present. Interesting…
- We don't show wavefunctions, as they are non-trivial (see, *e.g.* Perkins or Griffiths).
- All of these baryons decay weakly, apart from the p, which is stable, and $\Sigma^0 \rightarrow \Lambda^0\ \gamma$.

The (incomplete) baryon decuplet

When Gell-Man and Ne'eman plotted the $J^P = \frac{3^+}{2}$ baryons they found a different pattern…



…and also a hole. The pattern is missing a $S = -3$ entry. Gell-Mann and Ne'eman were able to predict its mass, lifetime and decay properties. Then, three years later….

The discovery of the Ω^- baryon

The Ω^- was discovered in a bubble-chamber experiment at Brookhaven lab in 1964. As there is nothing lighter than it with the same quark content, it decays weakly.

VOLUME 12, NUMBER 8 PHYSICAL REVIEW LETTERS 24 FEBRUARY 1964

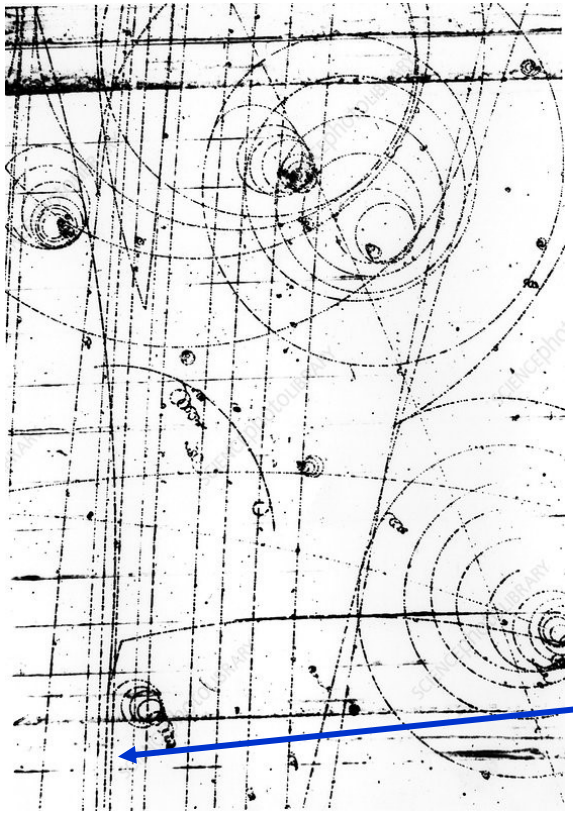
OBSERVATION OF A HYPERON WITH STRANGENESS MINUS THREE*

V. E. Barnes, P. L. Connolly, D. J. Crennell, B. B. Culwick, W. C. Delaney, W. B. Fowler, P. E. Hagerty,† E. L. Hart, N. Horwitz,† P. V. C. Hough, J. E. Jensen, J. K. Kopp, K. W. Lai, J. Leitner,† J. L. Lloyd, G. W. London,‡ T. W. Morris, Y. Oren, R. B. Palmer, A. G. Prodell, D. Radojičić, D. C. Rahm, C. R. Richardson, N. P. Samios, J. R. Sanford, R. P. Shutt, J. R. Smith, D. L. Stonehill, R. C. Strand, A. M. Thorndike, M. S. Webster, W. J. Willis, and S. S. Yamamoto
Brookhaven National Laboratory, Upton, New York
(Received 11 February 1964)

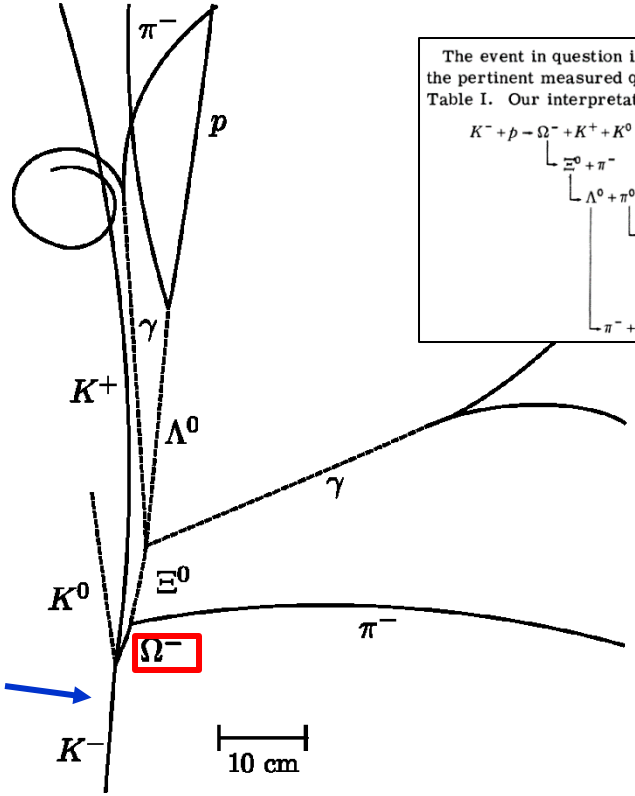
It has been pointed out[1] that among the multitude of resonances which have been discovered recently, the $N_{3/2}^*(1238)$, $Y_1^*(1385)$, and $\Xi_{1/2}^*(1532)$ can be arranged as a decuplet with one member still missing. Figure 1 illustrates the position

length of ~10^6 feet. These pictures have been partially analyzed to search for the more characteristic decay modes of the Ω^-.

The event in question is shown in Fig. 2, and the pertinent measured quantities are given in



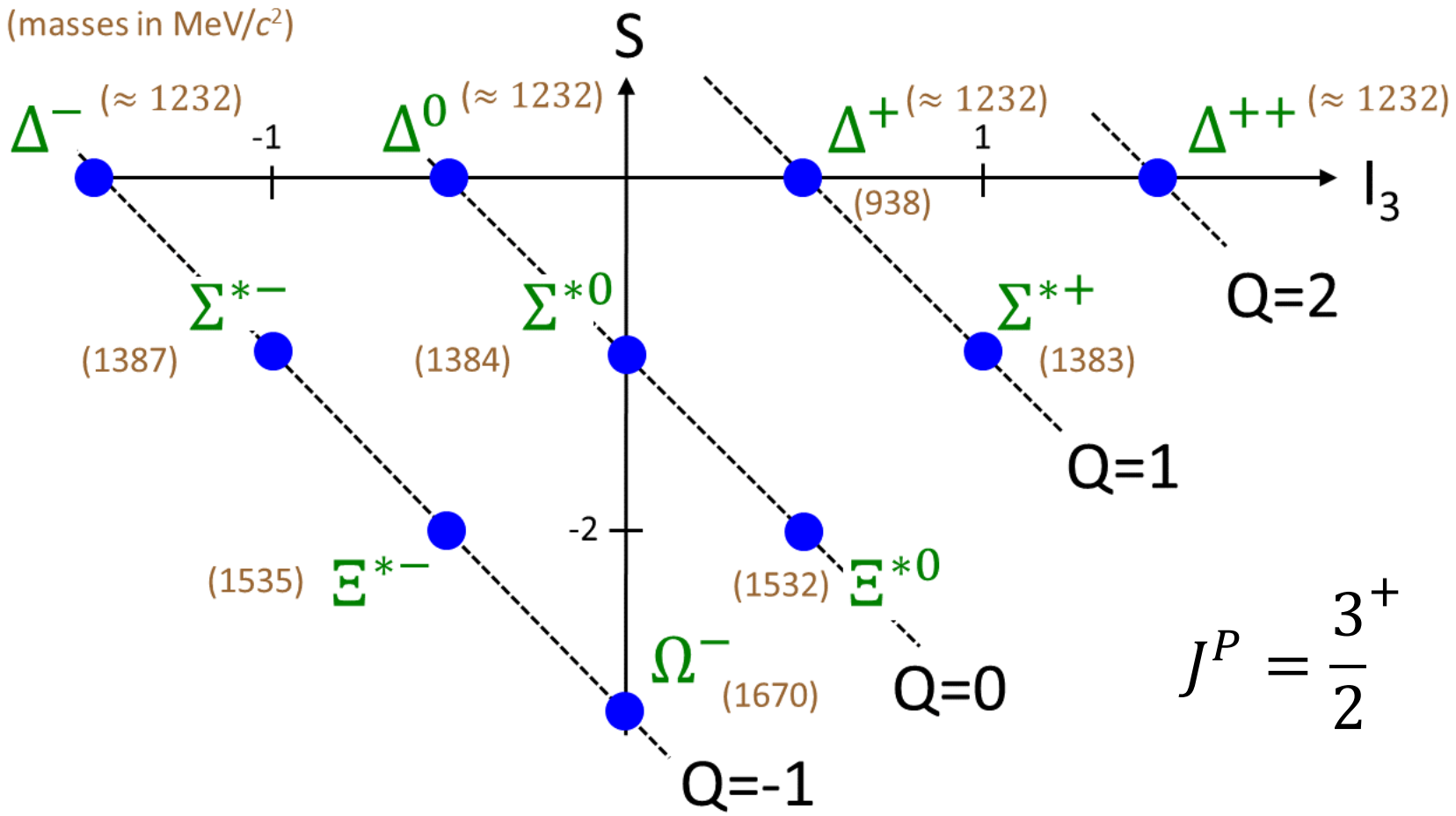
incoming kaon beam



The event in question is shown in Fig. 2, and the pertinent measured quantities are given in Table I. Our interpretation of this event is

$K^- + p \to \Omega^- + K^+ + K^0$
$\Xi^0 + \pi^-$
$\Lambda^0 + \pi^0$
$\gamma_1 + \gamma_2$
$e^+ + e^-$
$e^+ + e^-$
$\pi^- + p.$

The baryon decuplet



The baryon decuplet was now complete. All members, apart from the Ω^- decay strongly. However the very existence of the Ω^- (and indeed the Δ^- and Δ^{++}) poses a puzzle.

Particle physics: topic overview

Hadrons, quarks & the strong interaction

- Hadrons & their quantum numbers
- The light quark model
- Colour & QCD
- Quarkonia
- The *R* ratio

The need for colour

The $J^P = \frac{3}{2}^+$ baryons have quark spins aligned identically (*e.g.* $\uparrow\uparrow\uparrow$), but for those constituted of identical quarks, such as the Ω^- this appears a violation of Pauli's exclusion principle, which forbids two or more identical fermions to be in the same quantum state.

Looking at this in more detail, we require the total wavefunction, which we take to be

$$\psi = \psi(\text{space})\ \psi(\text{spin})\ \psi(\text{flavour})$$

to be anti-symmetric under the exchange of any two quarks. But for the $J^P = \frac{3}{2}^+$ baryons $\psi(\text{space})$ is symmetric, because $l = 0$, and $\psi(\text{spin})$ is clearly symmetric.
Moreover, for the Ω^- we have $\psi(\text{flavour})$ is symmetric also. We clearly have a problem !

The solution is to introduce another contribution $\psi(\text{colour})$ to the wavefunction:

$$\psi = \psi(\text{space})\ \psi(\text{spin})\ \psi(\text{flavour})\ \psi(\text{colour})$$

If this is anti-symmetric, then the Ω^- can exist & all is well. In fact $\psi(\text{colour})$ is always antisymmetric for baryons, & we now can see why the baryon octet has no uuu, ddd or sss states, as here $\psi(\text{spin})$ is antisymmetric & so such a configuration would have ψ symmetric.

Colour and colour requirements on hadrons

Colour is the 'charge' of the strong interaction with possibilities: red (r), green (g) or blue (b). An individual quark carries a single colour. Anti-quarks carry the anti-colour $\bar{r}$, $\bar{g}$ or $\bar{b}$.

Any hadron must be a colour singlet, that is it must be colour neutral overall.

This is achieved in mesons by having, *e.g.* a red quark and an anti-red anti-quark.
It is achieved in baryons by having a red, green and blue quark.

The baryon colour wavefunction is antisymmetric:

$$\psi(\text{color}) = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

which explains the existence of the Ω^- and the absence of any $J^P = \frac{1}{2}^+$ counterpart.

One also sees that it should be possible to form colour singlet hadrons out of more exotic combinations than we have so far encountered. For example:

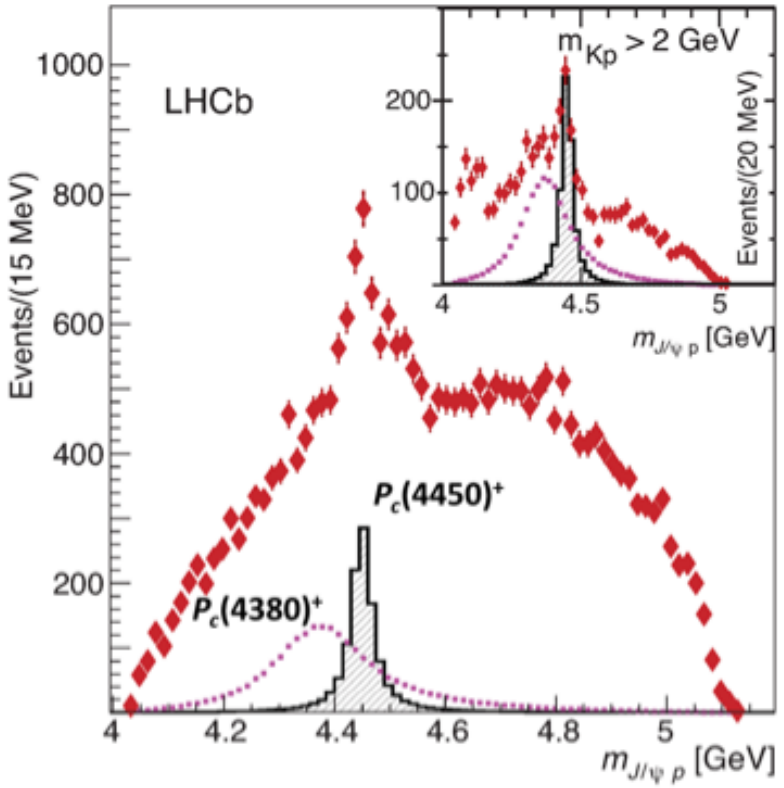
Tetraquark mesons: $q_1\overline{q_2}q_3\overline{q_4}$ can be built colour neutral *e.g.* $r\bar{r}g\bar{g}$

Pentaquark baryons: $q_1q_2q_3q_4\overline{q_5}$ can also satisfy this requirement, *e.g.* $rgbr\bar{r}$

These possibilities were already appreciated by Gell-Mann and Zweig in 1964, but no candidates for such 'exotic' hadrons were found by experiment for many decades.

Tetraquarks and pentaquarks

A convincing tetraquark candidate, the $X(3872)$, was first observed in Japan by the Belle experiment in 2003. Since then many more have been found. The first pentaquarks were only seen in 2015, by the LHCb experiment at CERN, with $uudc\bar{c}$ quark content.



[LHCb, PRL 115 (2015) 072001]

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Quantum Chromodynamics (QCD)

The strong force that binds the quarks together in hadrons is described by a theory called *Quantum Chromodynamics* (*QCD*), which is based on an underlying SU(3) gauge symmetry.

It has a dimensionless coupling constant α_S, analogous to α in QED. We may estimate their relative values by looking at the strong & electromagnetic decay of related states.

	Lifetime	*Q* value
$\Sigma^0(1385) \rightarrow \Lambda + \pi^0$	10^{-23} s	130 MeV
$\Sigma^0(1192) \rightarrow \Lambda + \gamma$	10^{-19} s	77 MeV

If we take these *Q* values to be approximately the same, and assume the lifetime of the state to depend on 1/(coupling constant)^2, then we deduce the following ratio:

$$\frac{\alpha_s}{\alpha} = \left(\frac{10^{-19}}{10^{-23}}\right)^{1/2} \approx 100$$

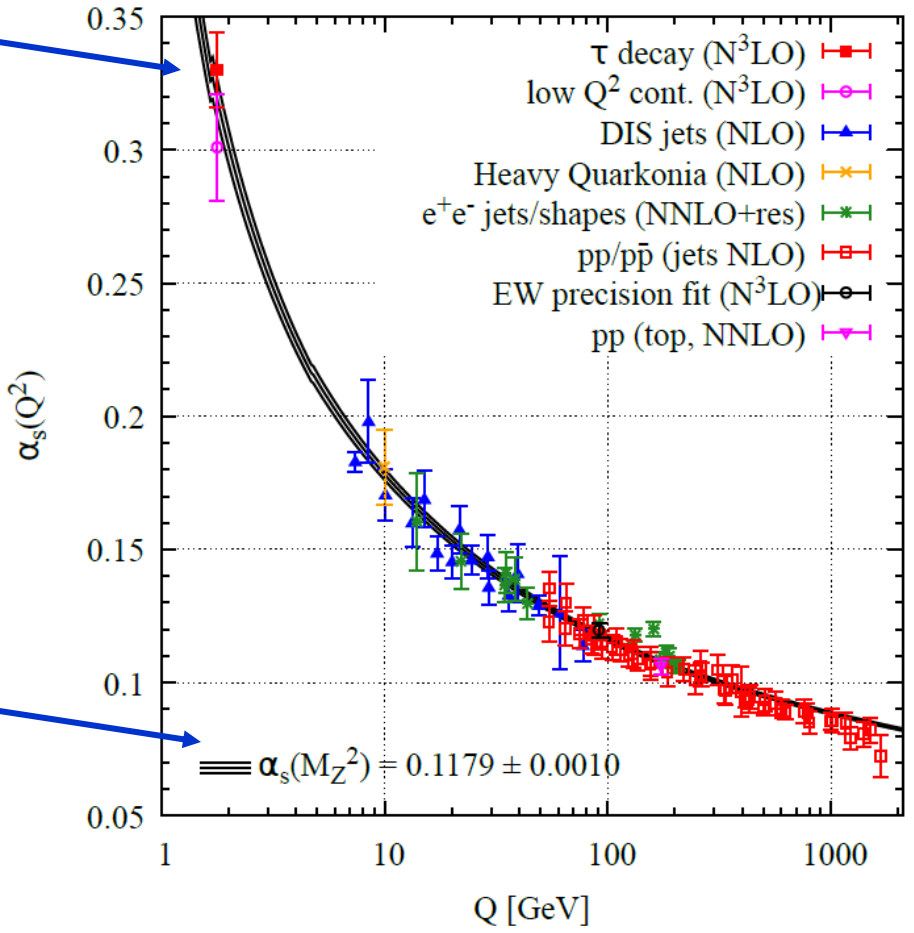
and since $\alpha = 1/137$ we deduce that $\alpha_s \sim 1$. This is indeed the case, although is important to appreciate that α_S `runs' with energy (as, indeed, do all fundamental coupling constants).

Energy variation ('running') of α_S

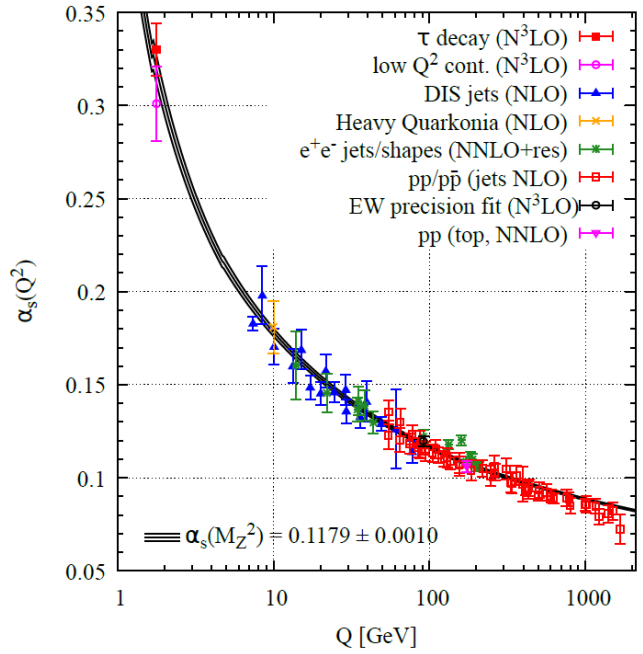
Coupling constants of fundamental interactions are in fact not constant, but vary with energy, as seen here for measurements of α_S performed at different energy scales.

approaching ~1
at very low energies

fitted value at an
energy of ~100 GeV



Asymptotic freedom and confinement

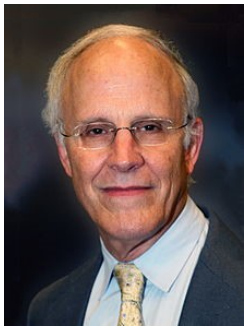


This variation with energy can also be regarded as a variation with 1 / length scale.

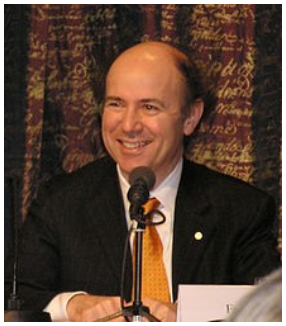
So as the energy becomes greater, or the length scale becomes smaller, then the interaction becomes weaker and weaker. This is the property of *asymptotic freedom*, which allows the quarks inside the hadrons not to suffer ~infinite force, as early models of the strong interaction suggested.

Conversely, at larger distances, The strength of the interaction grows. This property, known as *confinement*, prevents quarks from existing as free objects outside hadrons.

This understanding, and their work on QCD, led to Gross, Wilczek and Politzer being awarded the Nobel Prize in 2004.



David Gross 1941-, American; Nobel Prize for Physics 2004



Frank Wilczek 1951-, American; Nobel Prize for Physics 2004



Hugh David Politzer 1949-, American; Nobel Prize for Physics 2004

QCD potential

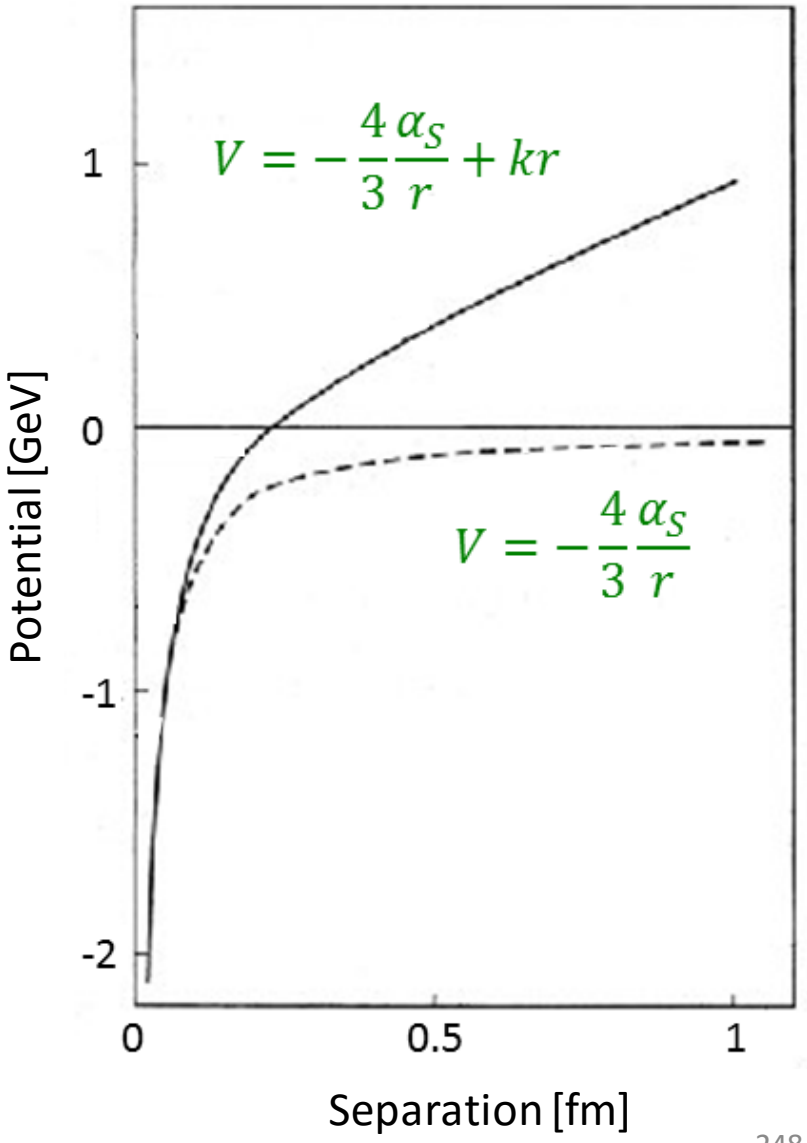
QCD is a complete theory, which in principle allows for exact calculations.

However the high value of the coupling constant (at low Q) makes perturbation theory of limited use, as does the complexity of all the possible Feynman diagrams, as the force-carrying particles of QCD can interact with each other.

Therefore models are still useful, and a simple one used to describe the qq potential is given by:

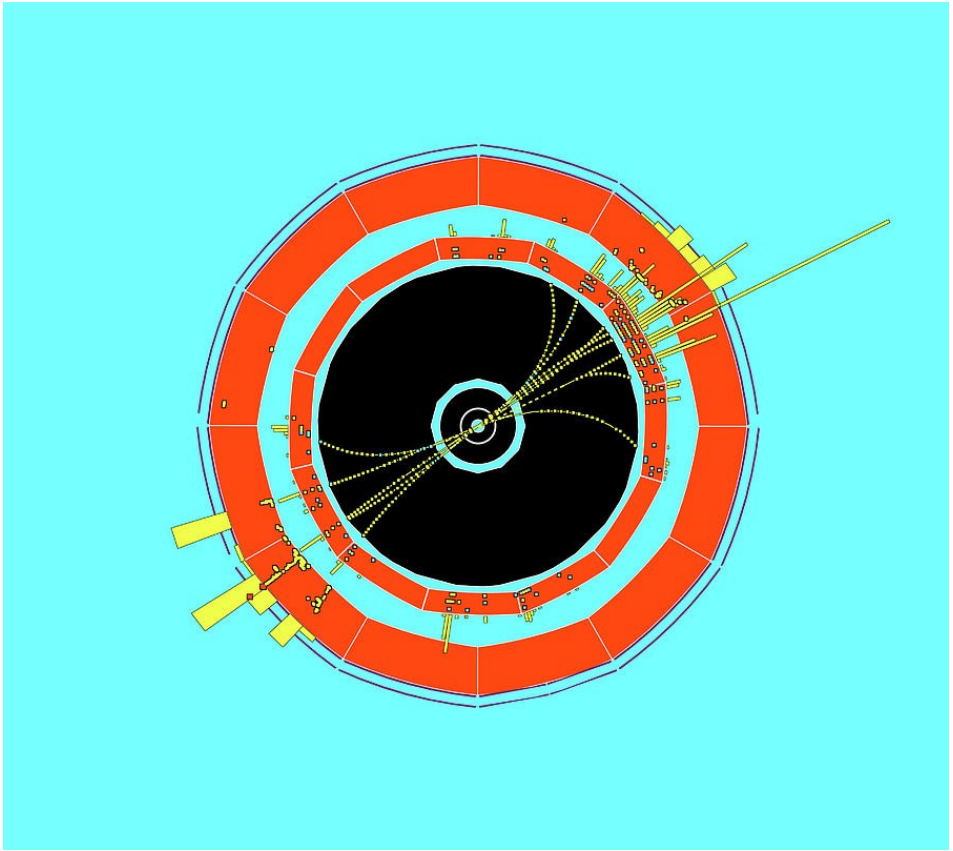
$$V = -\frac{4}{3}\frac{\alpha_S}{r} + kr$$

which is Coulomb-like at small distances (asymptotic freedom) and becomes infinite at large distances (confinement).



Confinement at work

In a collider experiment individual quarks are produced, *e.g.* $e^+e^- \to q\bar{q}$. As these two quarks move apart, the potential between them grows, and from this energy other $q\bar{q}$ pairs form, which leads to the formation of two jets of hadrons. This is known as *hadronisation*.



A two jet event at the ALEPH experiment at LEP, CERN.

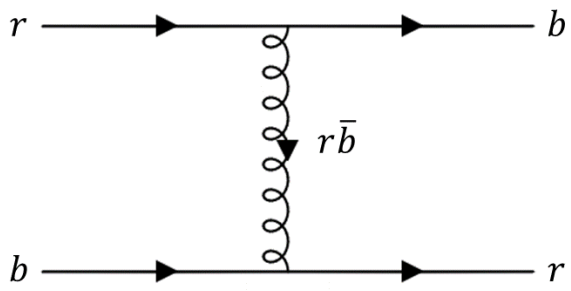
Gluons

The gluons have already unavoidably appeared in our discussion in various Feynman diagrams. They are the force-carrying spin-1 bosons of the strong interaction. In contrast to electromagnetism, where the photon is neutral, the gluons themselves carry colour charge.

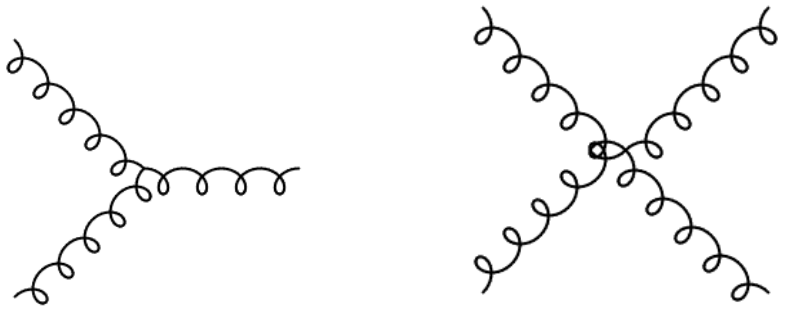
These are not just individual colours, but superpositions of a colour and anti-colour:

$$r\bar{b},\ r\bar{g},\ b\bar{g},\ b\bar{r},\ g\bar{r},\ g\bar{b},\ \frac{r\bar{r}-b\bar{b}}{\sqrt{2}} \text{ and } \frac{r\bar{r}+b\bar{b}-2g\bar{g}}{\sqrt{6}}$$

This means the gluons can change quark colours during interactions…



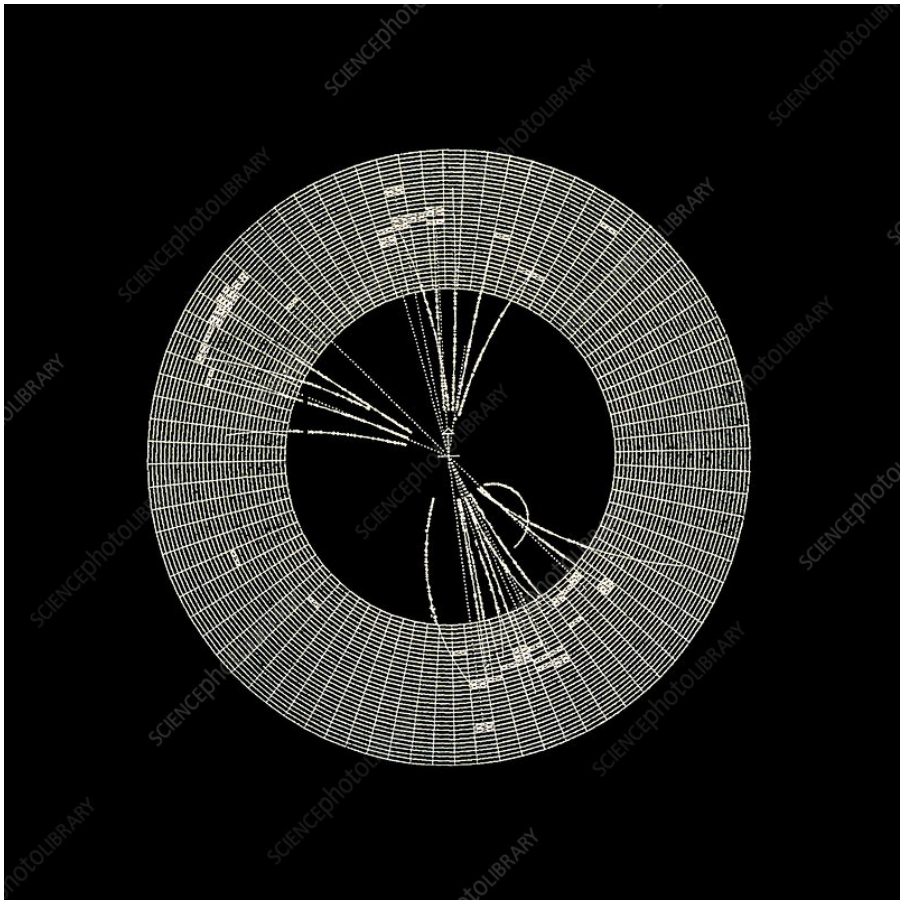
…as well as interact with each other in three and four-point vertices.



The gluon was ‘observed’ in summer 1979 through three-jet events at DESY, Germany.

The discovery of the gluon

Three-jet event seen in the JADE detector on the PETRA accelerator at DESY, Hamburg.



Search proposed by
Ellis, Gaillard and Ross
[Nucl. Phys. B 111 (1976) 253]



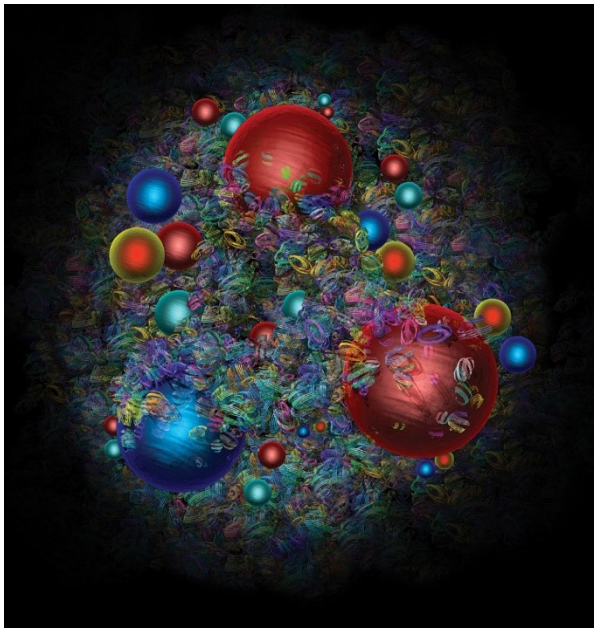
Graham Ross FRS,
1944-2021

One quark has emitted a gluon through strong bremsstrahlung, which has then split into a quark anti-quark pair, which has hadronised into a third jet.

The nucleon revisited

With our understanding of gluons, and the complexity of the strong interaction, we now realise that our model the nucleon as containing only three quarks is a little too simple.

These three constituents are more strictly called the *valence quarks*. They exist in a sea of virtual $q\bar{q}$ pairs created from gluons, called *sea quarks*, & indeed the gluons themselves.



[whacky artist's impression; courtesy CERN]

The role and importance of these other constituents depend on the experiment and energy, becoming more evident at higher energies. Note the LHC collides protons with protons. If the proton had only valence quarks then no matter anti-matter annihilation could occur !

Particle physics: topic overview

Hadrons, quarks & the strong interaction

- Hadrons & their quantum numbers
- The light quark model
- Colour & QCD
- Quarkonia
- The *R* ratio

The status of the quark model prior to 1974

In the early 1970s the quark model was taken seriously:

- It successfully described the spectrum of hadrons and decay properties;
- Recent results from electron-nucleon scattering experiments indicated that nucleons contained point like constituents (see later discussion in context of high-energy neutrino-nucleon scattering);
- Measurements of the `*R* ratio' (see next topic).

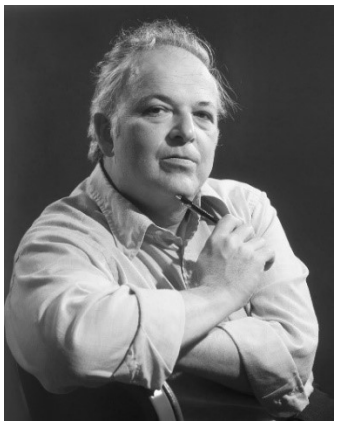
However, the model was not universally accepted as describing reality. Prior to the concept of asymptotic freedom it was difficult to explain why free quarks had not been seen in isolation. Furthermore alternative models existed.

In parallel, electroweak theory was taking shape (see later), which included quarks. In trying to understanding puzzling behaviour of strange mesons within this context, Glashow, Illiopoulos and Maiani proposed that a fourth quark was required.

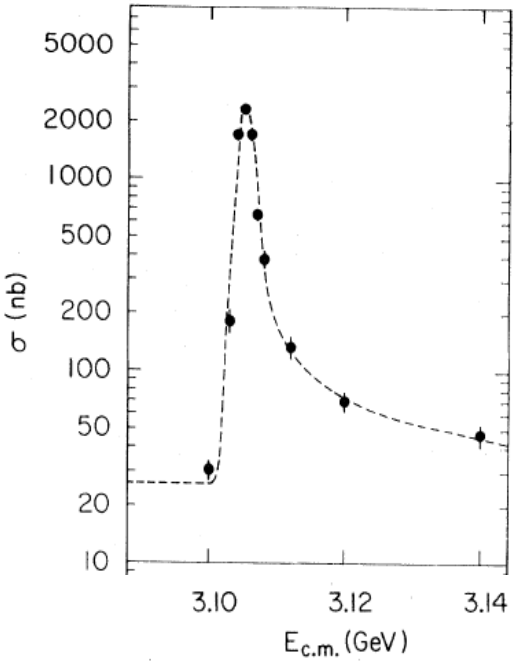
In November 1974 the discovery of a new particle, the J/ψ, convinced the community of the reality of quarks, and showed a fourth quark, charm, existed.

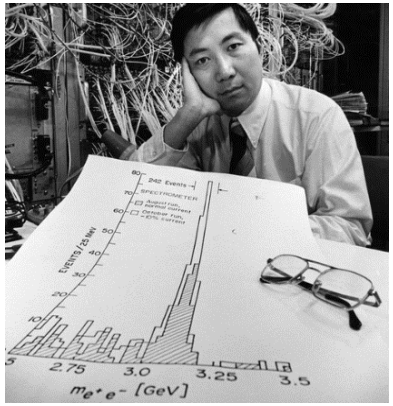
The November revolution: the J/ψ

On November 11, 1974, two groups simultaneously announced the discovery of a new resonance. The Stanford Linear Accelerator Centre team observed it through scanning the e^+e^- cross section vs. E_{CM}. The MIT team observed it at Brookhaven through reconstructing the invariant mass of e^+e^- pairs produced from proton collisions on a beryllium target.

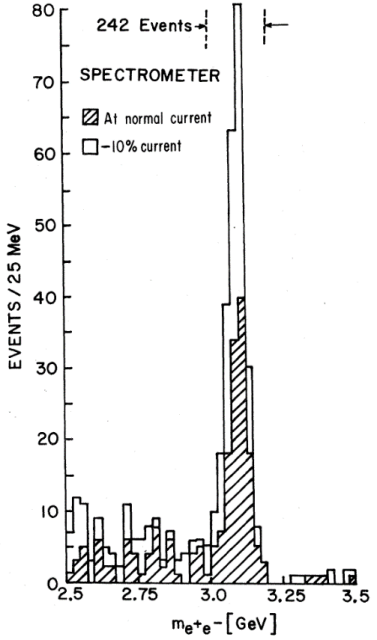


Burton Richter, 1931-2018; American; Nobel Prize 1976





Samuel C.C. Ting, 1936-; Chinese & American citizenship; Nobel Prize 1976



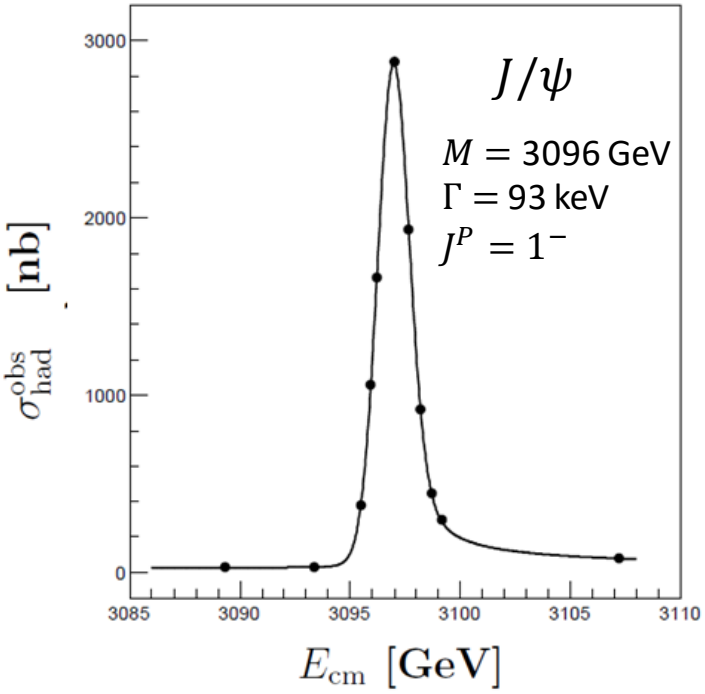
`Experimental observation of a heavy particle *J*', J.J. Aubert *et al.*, Phys. Rev. Lett. 33 (1974) 1404

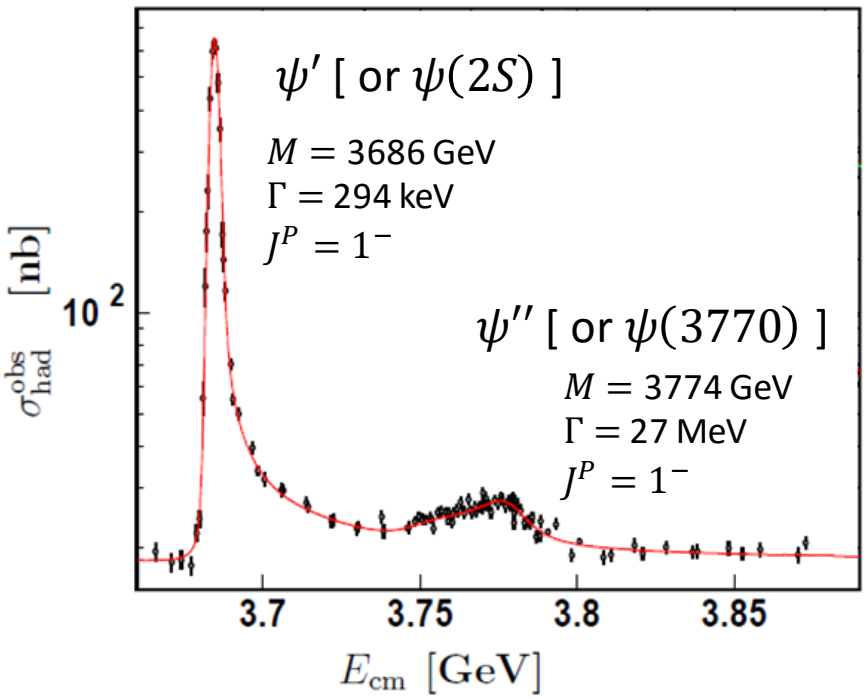
'Discovery of a narrow resonance in e^+e^- annihilation', J.-E. Augustin et al., Phys. Rev. Lett. 33 (1974) 1406

Quarkonia: the J/ψ, the ψ' and the ψ''

The most remarkable feature of the J/ψ was its extreme narrowness, indicating a lifetime several orders of magnitude longer than seen for any other then known strongly decay particle. Its discovery was immediately followed by that of another very narrow resonance, the ψ', and (a couple of years later) a much broader and 'normal' state the $\psi(3770)$.

[KEDR, JHEP 03 (2018) 119]



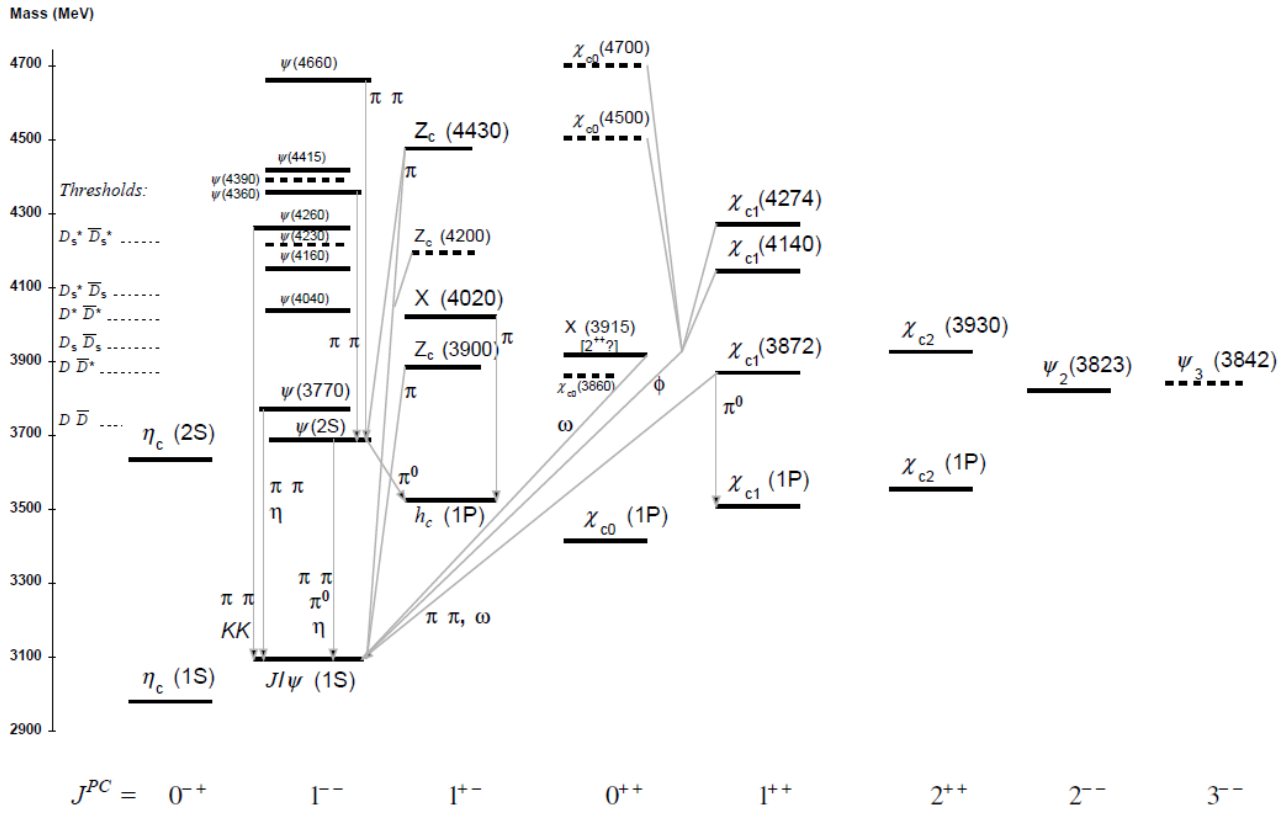


[BES, Phys. Rev. Lett. 101 (2008) 102004]

(NB the width of the narrow resonances seen here is dominated by experimental resolution. The measured value comes from $\Gamma_{\psi\to e^+e^-}$, $B(\psi \to e^+e^-)$ and the Breit-Wigner formula.)

Making sense of charmonia

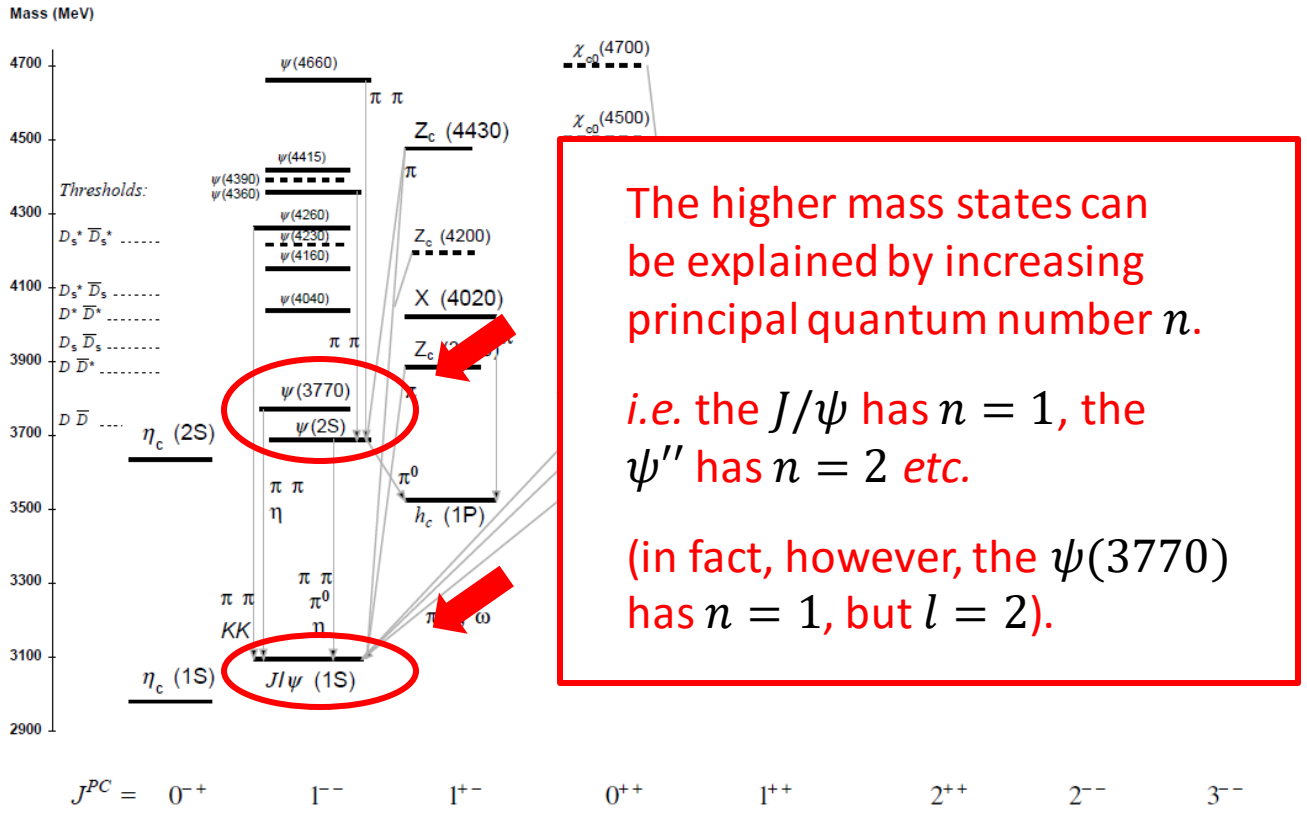
These states are conveniently understood as a non-relativistic system of a new heavy quark (charm), and its antiparticle moving in the strong potential. The energy levels can be understood through analogy with hydrogen, or better still positronium.



In e^+e^- collisions only these states (and a few others) can be directly produced, as they share the J^P of the photon, but more states are predicted and have now been observed.

Making sense of charmonia

These states are conveniently understood as a non-relativistic system of a new heavy quark (charm), and its antiparticle moving in the strong potential. The energy levels can be understood through analogy with hydrogen, or better still positronium.



The higher mass states can be explained by increasing principal quantum number n.

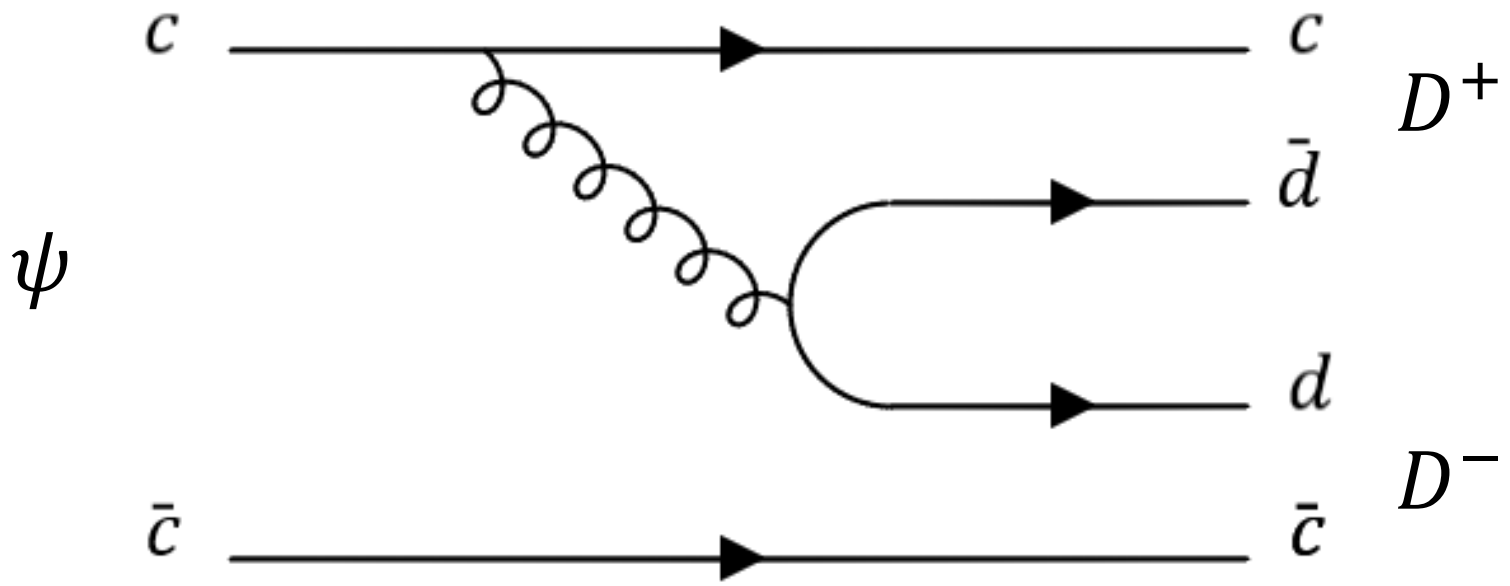
i.e. the J/ψ has $n = 1$, the ψ'' has $n = 2$ *etc.*

(in fact, however, the $\psi(3770)$ has $n = 1$, but $l = 2$).

In e^+e^- collisions only these states (and a few others) can be directly produced, as they share the J^P of the photon, but more states are predicted and have now been observed.

Charmonia decays

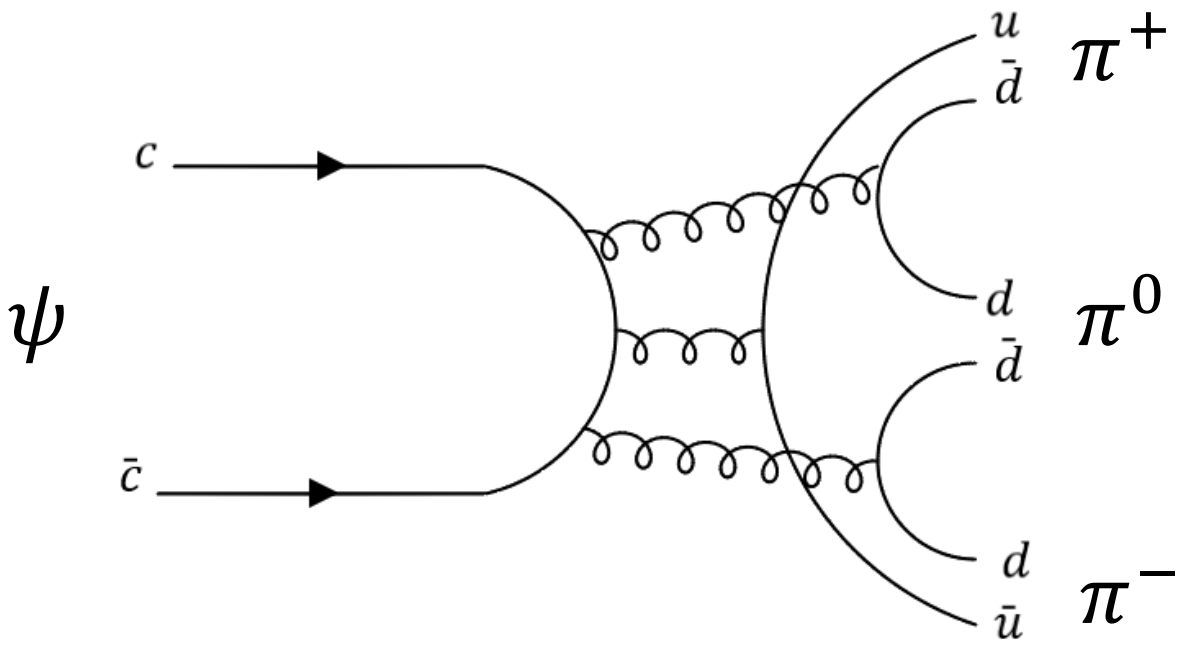
How though to explain the great variation in lifetime among the three resonances? Well, the charm quark must be able to form a meson with a lighter quark. It can – these are the D mesons (of D_s is an s quark is involved. So the most natural decay of charmonium is:



This is indeed the most common decay mode of the $\psi(3770)$. However the J/ψ and ψ' *cannot* decay like this. Lying very close to threshold, they have a mass $< 2 \times m_D$!

Zweig (OZI) suppression

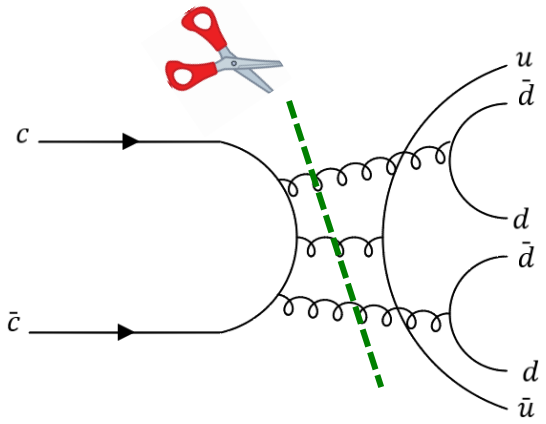
If the J/ψ and ψ' can't decay into D mesons then the other option is for the c and $\bar{c}$ quarks to annihilate. Then gluons must carry the full four-momenta and quantum numbers of The initial state. A single gluon cannot do this because *it is not a colour singlet*. Two gluons are no good, because this violates a quantum number called C parity. *So three are required* !



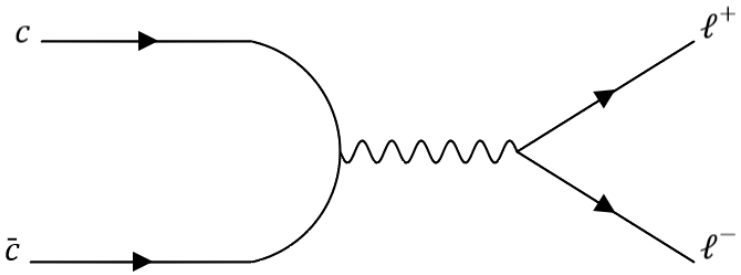
Decay probability must go as $|(\alpha_s)^3|^2$, and α_s must be evaluated for momentum transfers equivalent the charmonium mass, which gives $\alpha_s \sim 0.25$ and $|(\alpha_s)^3|^2 \sim O(10^{-4})$. This is to be contrasted to the decay into D mesons where $\alpha_s \sim 1$ and the decay probability $\sim |\alpha_s|^2 \sim 1$.

Zweig (OZI) suppression

So we now understand why the lower-mass resonances are so long lived. Decays with this property are said to be *Zweig* or *OZI* (Okubo Zwieg Iizuka) *suppressed*. They can be readily spotted if it is possible to divide the Feynman diagram without cutting a quark line.



Also noteworthy that for such decays the EM process competes with the strong amplitude.

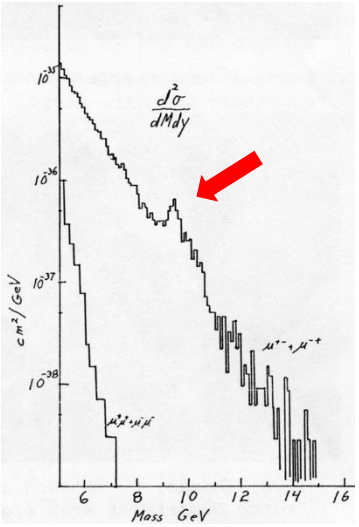


For this reason $B\,(J/\psi \to e^+e^-) \approx 6\%$, whereas $B\,(\psi(3770) \to \mu^+\mu^-) \approx 10^{-5}$.

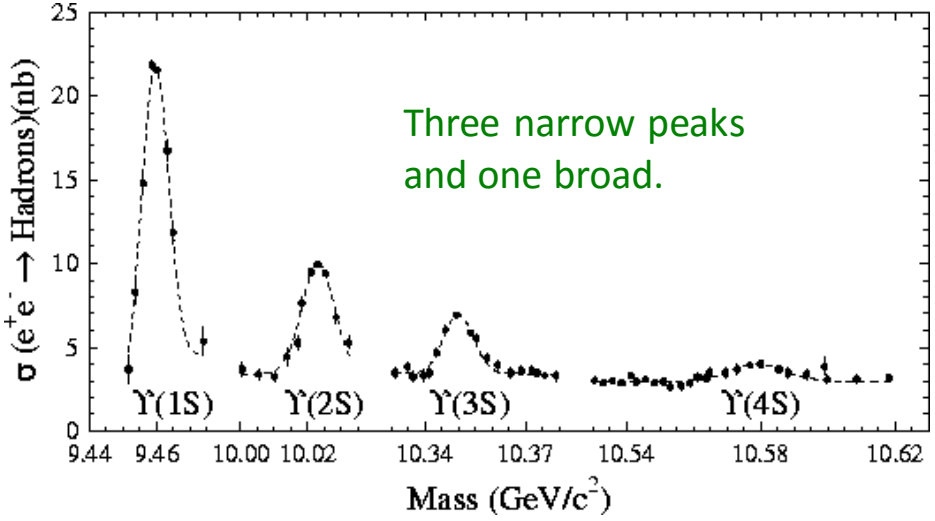
More quarkonia: the discovery of the *b* quark

In 1977 The E288 experiment at Fermilab under Leon Lederman discovered another narrow resonance at ~ 9.5 GeV. This was called the *upsilon*. Eventually three narrow peaks were found, followed by a fourth, much wider state.

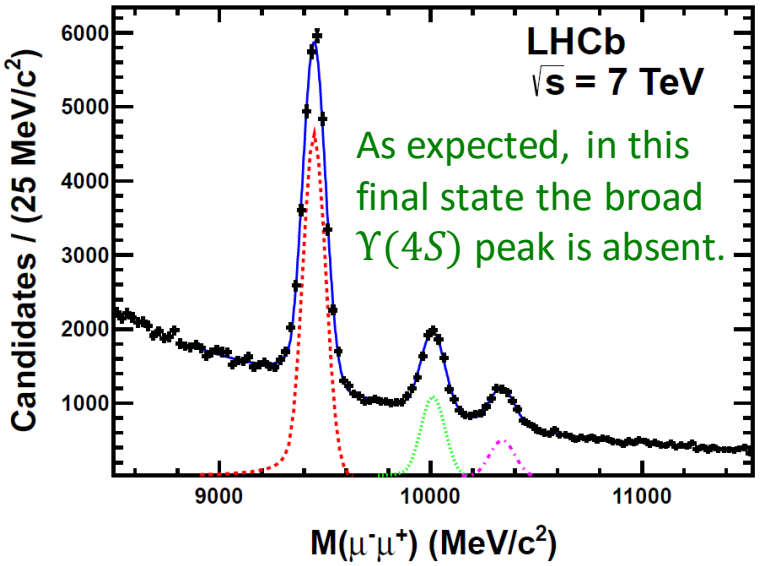
Analogous explanation as for charmonium. Here we are seeing a $b\bar{b}$ system, with b being the 5th quark, dubbed *beauty* or *bottom*, with a mass of around 5 GeV.



Upsilon spectrum seen by the CUSB experiment at Cornell, in decays to hadronic final states.



Upsilon spectrum seen by the LHCb experiment at CERN, in decays to $\mu^+\mu^-$.



[LHCb, EPJC 72 (2012) 2025]

Discovery of the top quark

There is now no doubt that hadrons are composed of quarks, which, together with the fundamental interactions, account for all their characteristics and behaviour.

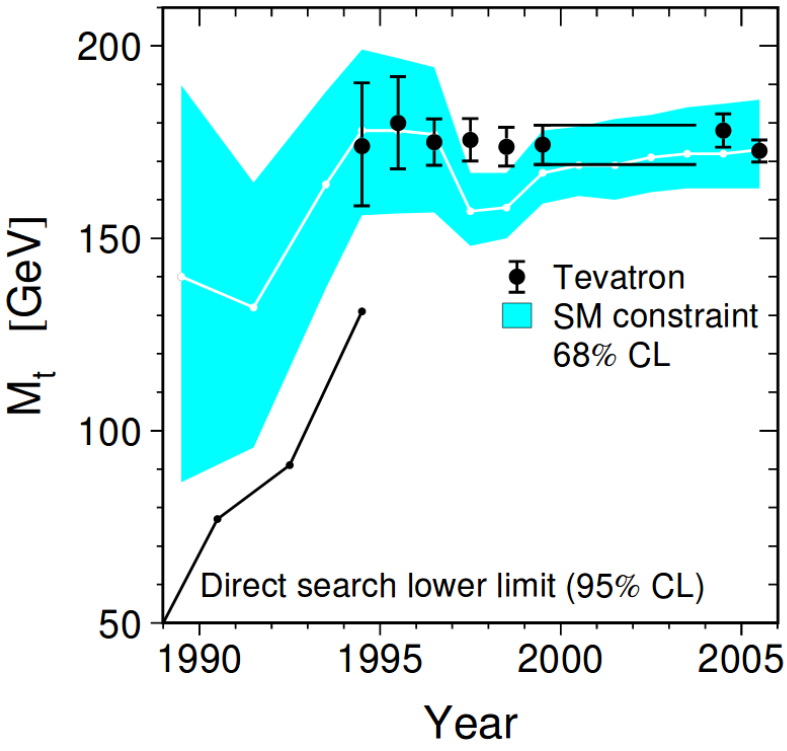
So far we have no indication that quarks themselves have any substructure.

The final (?) quark, top, was discovered by the CDF and D0 experiments at Fermilab in 1995.

As already remarked, it was found at a mass entirely consistent with that predicted by the theory, based on the ensemble of other precise measurements at lower energies.

The top is the most massive of all particles in the Standard Model, with m_t = 173 GeV.

With such a mass the top decays before it hadronises, and hence can be considered as the only 'free' quark observed in nature !



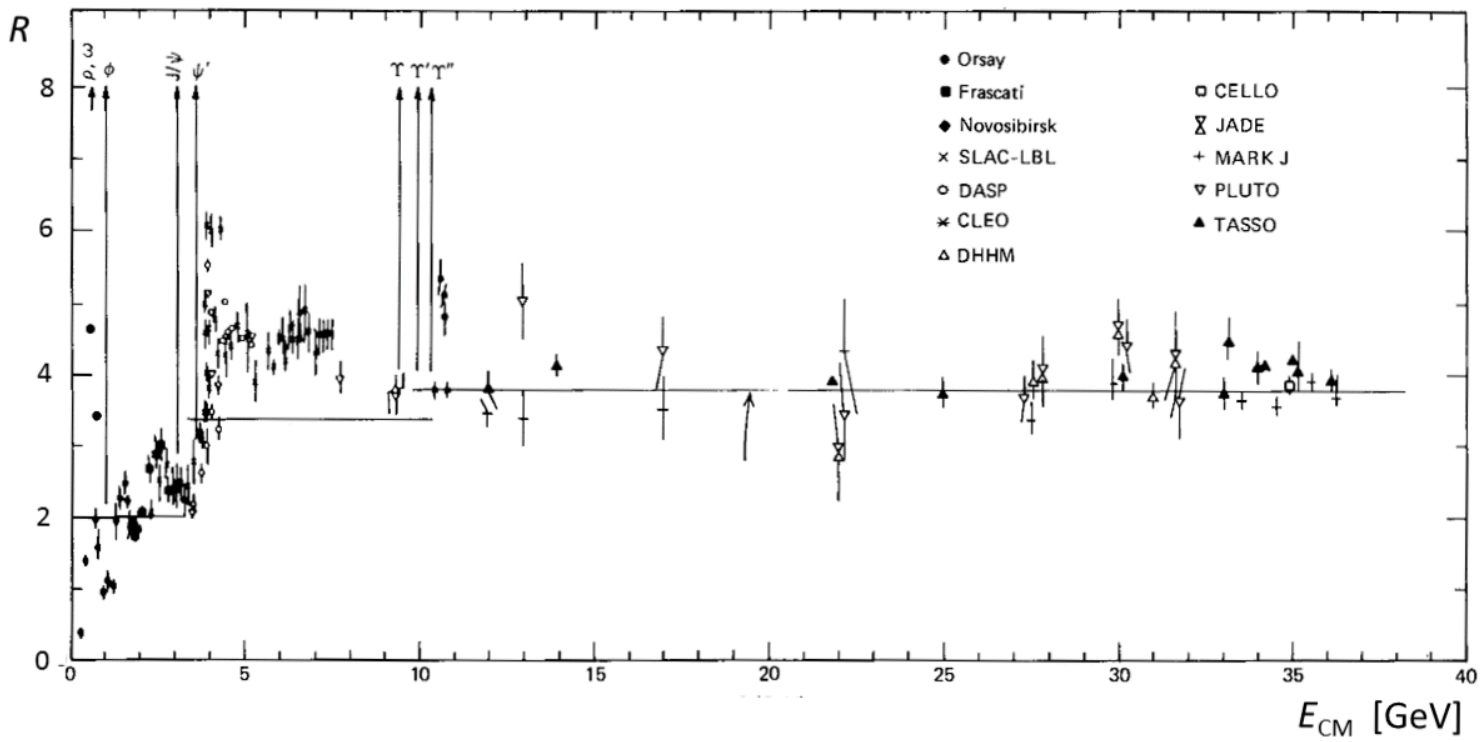
Particle physics: topic overview

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- The *R* ratio

R ratio in e^+e^- interactions

The ratio $R \equiv \dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ has the below dependence on collision energy.

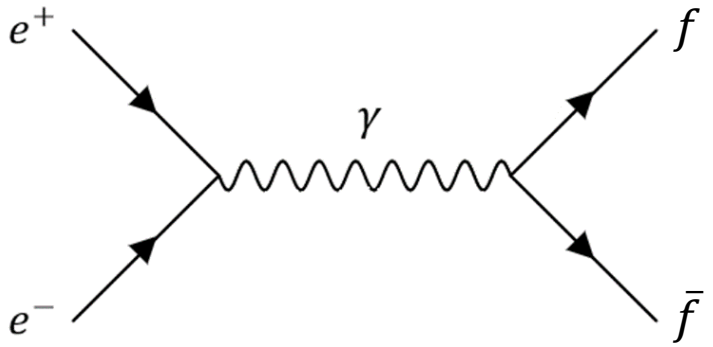


The data rise with energy in a succession of steps (although the points seen in the 3-10 GeV interval are above the suggestive line that has been superimposed on the plot). Furthermore, there are a succession of spikes that occur at the $J^P = 1^-$ resonances.

R ratio in e^+e^- interactions

Earlier we calculated:

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\hbar^2 c^2 \frac{\alpha^2}{E_{\mathrm{CM}}^2}$$



The cross section for all $e^+e^- \to f\bar{f}$ processes (apart from $e^+e^- \to e^+e^-$) should take the same form, provided E_{CM} well above $2m_f$.

However, there will be differences in the vertex factors, which go as $Q\sqrt{\alpha}$ in the matrix element. So for the charged leptons this will give α^2 in the cross section, whereas for

up-type quarks $\quad \sigma \sim \left(\frac{2}{3}\right)^2$

down-type quarks $\quad \sigma \sim \left(\frac{1}{3}\right)^2$

Furthermore, the quarks can be produced as $r\bar{r}$, $b\bar{b}$ or $g\bar{g}$, giving a colour factor of three.

$$\Rightarrow \quad \boxed{R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_i Q_{q_i\overline{q_i}}^2}$$

where the sum runs over the quarks accessible at that energy.

R ratio in e^+e^- interactions

We therefore can make the following predictions:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_i Q^2_{q_i\overline{q_i}} \quad \Rightarrow$$

2, when *u*,*d*,*s* produced

10/3, when *u*,*d*,*s*,*c* produced

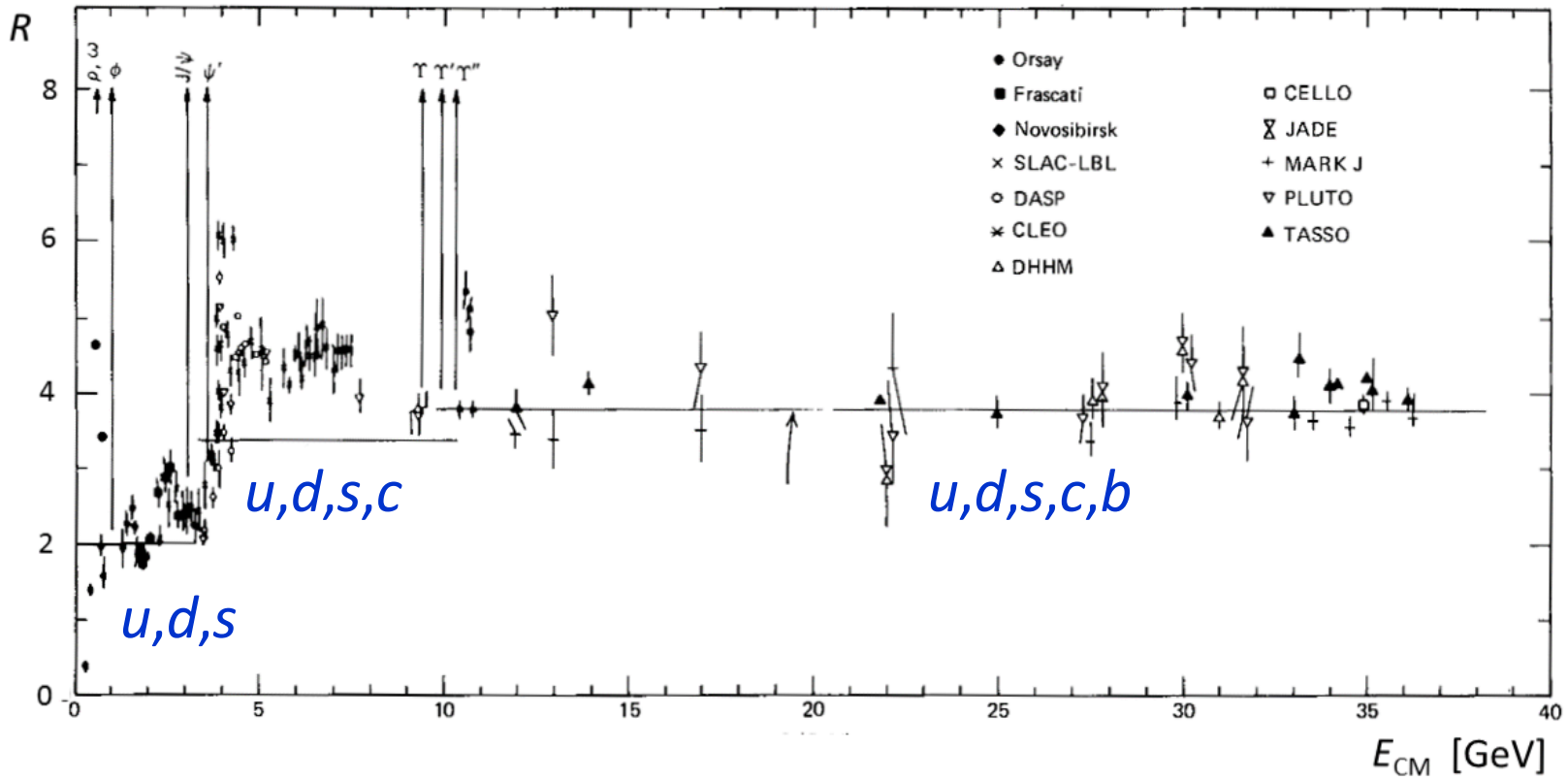
11/3, when *u*,*d*,*s*,*c*,*b* produced

There are modifications to these values expected when not so far above threshold coming from the effect of the quark masses, and also QCD corrections to the process.

Finally, just above the thresholds where the $J^P = 1^-$ resonances exist, the behaviour will be different, as the resonances enhance the cross section above continuum, and then the resonances decay into hadrons and muons according to their own rules.

R ratio in e^+e^- interactions

When all corrections are accounted for the observed ratio agrees very well with prediction.



This observable is very compelling in confirming the existence of quarks, as well as the values of their masses and charges, and also that there is a colour degree of freedom.

Particle physics: topic overview

Leptons and the weak interaction

- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

Weak interactions of leptons

We encountered the weak interaction when studying β decay. Let us now return to this topic in the context of particle physics, first considering the decay of leptons.

$$\mu^- \rightarrow e^- + \bar{\upsilon_e} + \nu_\mu$$

This is pretty much how the muon always decays, though there are some very rare alternatives.

$$\tau^- \rightarrow e^- + \bar{\upsilon_e} + \nu_\tau$$

This happens with a branching fraction of 17.8 %; there is an analogous decay $\tau^- \rightarrow \mu^- + \overline{\upsilon_\mu} + \upsilon_\tau$ and then a host of decay modes involving hadrons. The sole reason the τ has these options and the μ doesn't is its higher mass.

The following principles arise from studies of these decays & processes involving leptons:

Lepton flavour universality (LFU)

The weak force couples equally strongly to the charged leptons in each generation, and also the neutrinos. (NB tantalising current hints that this may be violated, but far from established.)

Lepton number conservation (LNC)

The number of leptons + anti-leptons is conserved.

Lepton flavour conservation (LFC)

Equivalent to LNC within a family. This forbids, for example, $\mu^- \rightarrow e^- + \gamma$ and $\tau^- \rightarrow \mu^-\mu^+\mu^-$. However, we now know this is violated for neutrinos, as they oscillate between generations.

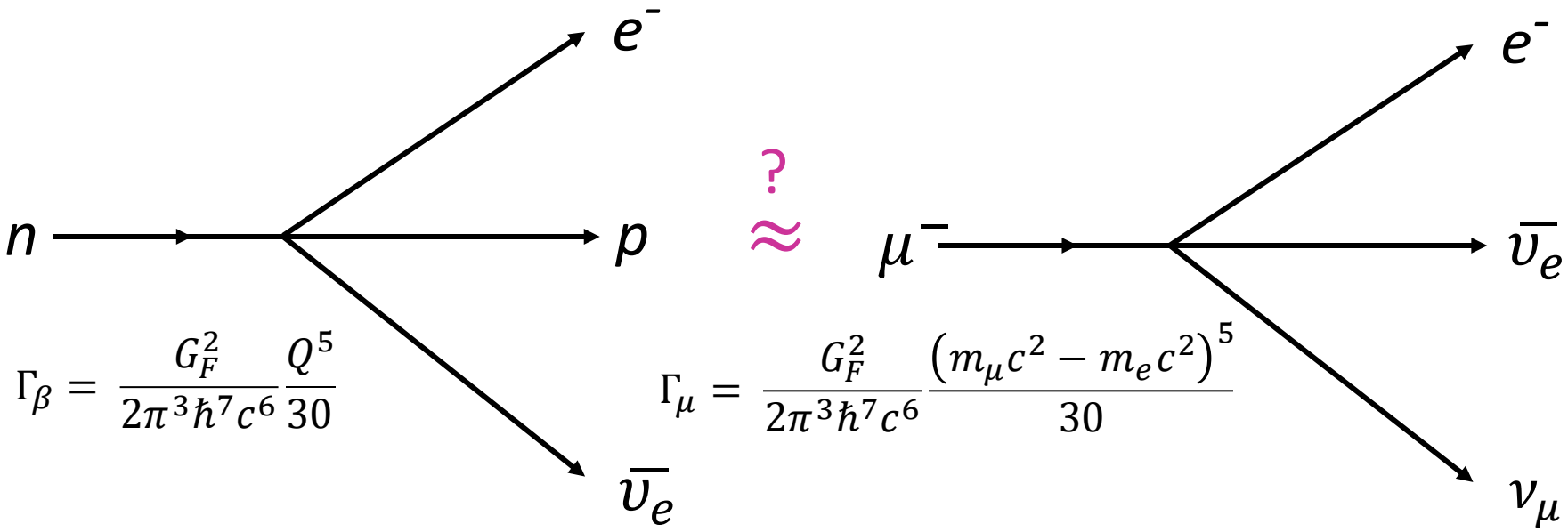
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Leptons and the weak interaction

- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

Fermi theory revisited: muon decay

We note that our lepton decays look very similar to β decay. Let's see if Fermi theory works.



$$\Gamma_\beta = \frac{G_F^2}{2\pi^3\hbar^7c^6}\frac{Q^5}{30} \qquad \Gamma_\mu = \frac{G_F^2}{2\pi^3\hbar^7c^6}\frac{\left(m_\mu c^2 - m_e c^2\right)^5}{30}$$

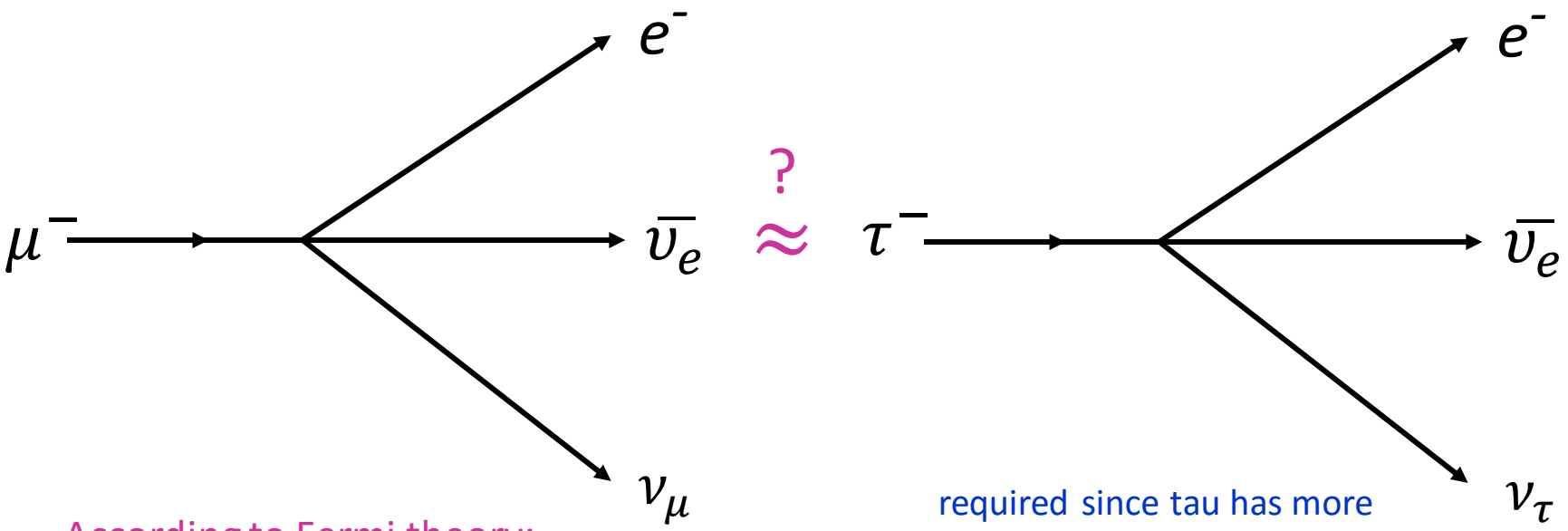
Recalling $\frac{G_F}{(\hbar c)^3} = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$ we write $\Gamma_\mu = \frac{1}{2\pi^3}\left(\frac{G_F}{(\hbar c)^3}\right)^2 \frac{c}{\hbar}\frac{\left(m_\mu c^2 - m_e c^2\right)^5}{30}$

and so $\Gamma_\mu = \frac{1}{2\pi^3}\left(1.166 \times 10^{-5}\ \text{GeV}^{-2}\right)^2 \frac{3\times10^{23}\,\text{fm s}^{-1}}{0.197\ \text{GeV fm}}\frac{(0.105\ \text{GeV})^5}{30} = 1.42 \times 10^6$

Therefore $\tau_\mu = 1/\Gamma_\mu = 0.7\ \mu\text{s}$, to be compared to true value of $2.2\ \mu\text{s}$. Not bad, particularly as our calculation takes no account of *e.g.* spin effects. We conclude Fermi theory applies !

Fermi theory revisited: tau decay

Let us be even more bold and see if we can scale from muon to tau decay.



According to Fermi theory:

required since tau has more decay paths open to it than muon

$$\tau_\tau = \left(\frac{m_\mu c^2 - m_e c^2}{m_\tau c^2 - m_e c^2}\right)^5 \mathrm{BF}(\tau^- \to e^- \bar{\nu}_e \nu_\tau)\, \tau_\mu$$

$$= \left(\frac{0.105}{1.776}\right)^5 \times 0.178 \times 2.197\ \mu\mathrm{s} = 0.282\ \mathrm{ps}$$

which agrees very well with the measured value of 0.290 ps. Awesome!

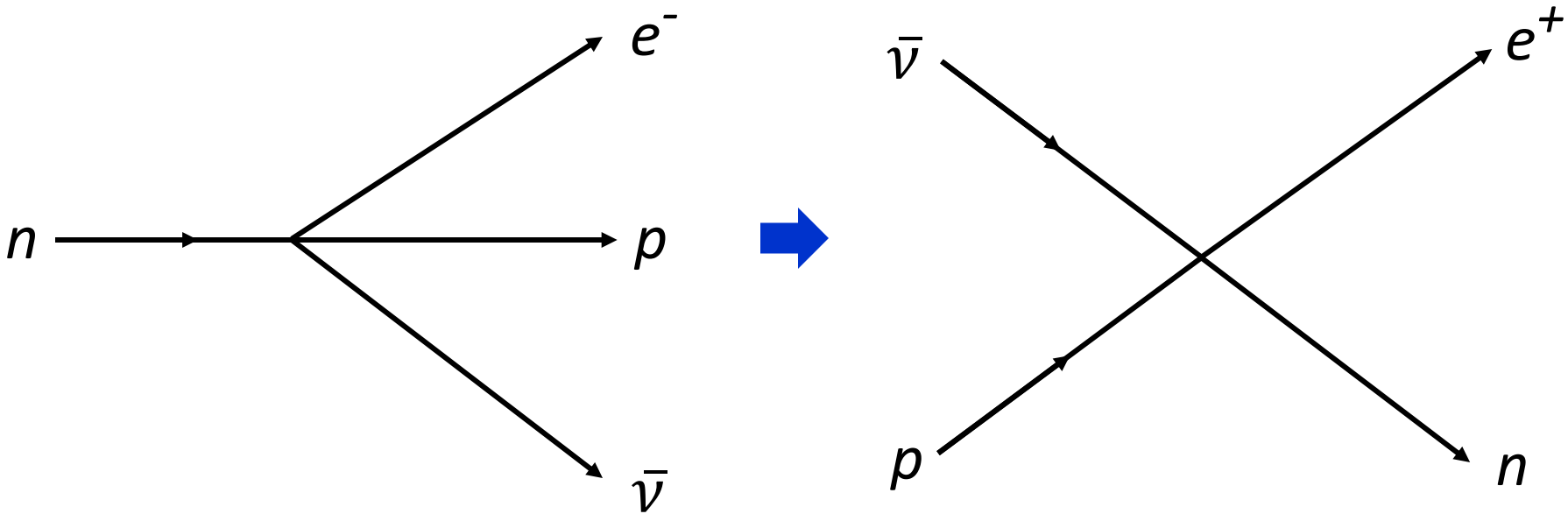
This beautiful consistency validates both Fermi theory and lepton universality.

Fermi theory revisited: inverse β decay

Let us now study the so-called inverse β decay reaction

$$\bar{\nu}_e + p \rightarrow n + e^+$$

We can take the four-point interaction diagram of β decay and merely rotate the legs:

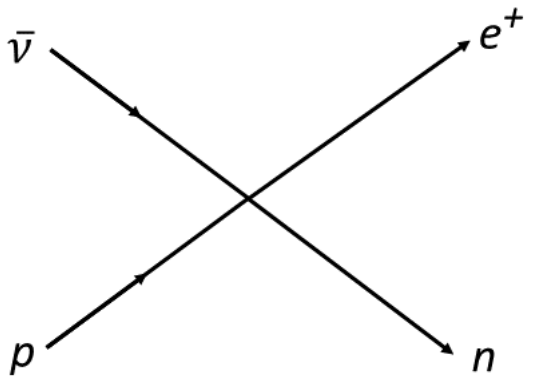


We will now apply Fermi theory to this new configuration to determine the cross section.

Inverse β decay

We have the process:

$$\bar{\nu_e} + p \rightarrow n + e^+$$



F.G.R. gives the transition rate:

$$\Gamma_{i\rightarrow f} = \frac{2\pi}{\hbar}\left|M_{if}\right|^2\rho$$

which is related to the cross-section thus

$$\sigma = \Gamma_{i\rightarrow f} / \text{incident flux}$$

Now flux = relative velocity in C-of-M, which is just *c*, divided by a volume *V*, which we will neglect as it cancels with the wave-function normalisation factors.

Obeying the key principle of Fermi theory we will take the matrix element to have no kinematic dependence ('four-point interaction') & write it just as G_F. We pre-multiply G_F^2 by a factor 4 which we will not justify, but comes from summing over possible spin states.

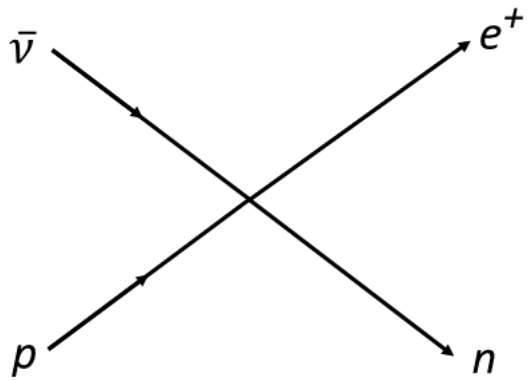
$$\sigma = \frac{2\pi}{\hbar c} 4{G_F}^2 \rho$$

We now must calculate the density-of-states factor.

Inverse β decay

We have the process:

$$\bar{\nu}_e + p \rightarrow n + e^+ \quad \text{with} \quad \sigma = \frac{2\pi}{\hbar c} 4{G_F}^2 \rho$$



We have $\rho = \dfrac{\mathrm{d}N}{\mathrm{d}E} = \dfrac{\mathrm{d}N}{\mathrm{d}p_e}\dfrac{\mathrm{d}p_e}{\mathrm{d}E}$ where E is the available energy in the C-of-M frame.

Note that in contrast to β decay we only have one contribution to the density-of-states factor as it is a two-body, rather than three-body final state.

Let's focus on the case of anti-neutrinos from nuclear reactors, which emerge with mean energies of ~1 MeV, *i.e.* $\ll m_{n,p}c^2$. Then we can assume the neutron is produced at rest.

$$E = E_e = E_\nu + m_p c^2 - m_n c^2$$

$$\Rightarrow \quad \frac{\mathrm{d}p_e}{\mathrm{d}E} = \frac{\mathrm{d}}{\mathrm{d}E_\nu}\frac{1}{c}(E_e^2 + m_e^2 c^4)^{1/2} = \frac{E_e}{p_e c^2}$$

and $\dfrac{dN}{dp_e} = \dfrac{4\pi p_e^2}{(2\pi\hbar)^3}$ so $\boxed{\dfrac{dN}{dE} = \dfrac{4\pi p_e E_e}{(2\pi\hbar)^3 c^2}}$

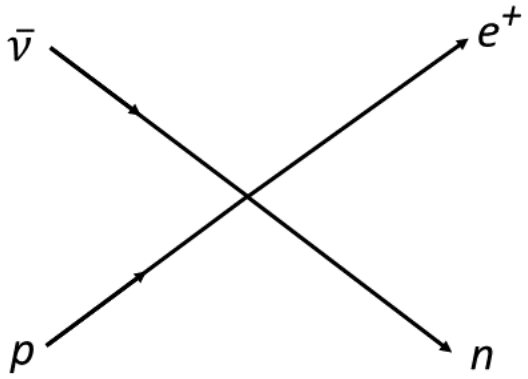
Inverse β decay

We have the process:

$\bar{\nu}_e + p \rightarrow n + e^+$ with

$$\sigma = \frac{2\pi}{\hbar c} 4{G_F}^2 \rho$$

$$\rho = \frac{dN}{dE} = \frac{4\pi p_e E_e}{(2\pi\hbar)^3 c^2}$$



and so $\boxed{\sigma = \frac{4G_F^2 c}{\pi(\hbar c)^4} p_e E_e}$, being the total cross section for inverse β decay.

In the case $E_e = 1$ MeV then $p_e = 0.86$ MeV/c . Recall $\frac{G_F}{(\hbar c)^3} = 1.166 \times 10^{-5}$ GeV^{-2}.

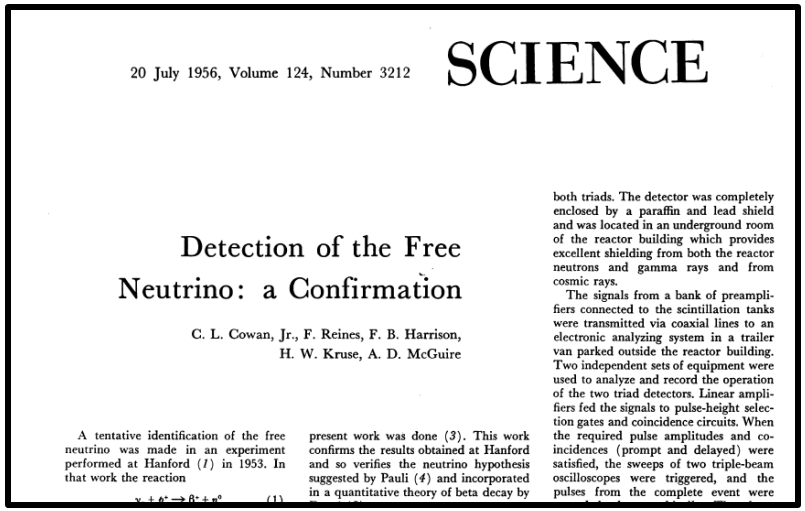
$$\sigma = \frac{4}{\pi}(1.166 \times 10^{-11})^2 (197)^2 \times 0.86 \times 1 \text{ fm}^2$$

$$= 5.8 \times 10^{-48} \text{ m}^2 \quad \text{or} \quad 5.8 \times 10^{-20} \text{ barn}$$

This is an unbelievably small cross section, on a scale completely different from what we have seen before. Corresponds to a mean free path in water of 100 light years !

Discovery of the (electron) neutrino

Such a ridiculously small cross section presumably makes detection impossible ?
No! As demonstrated by Cowan and Reines in 1956 at a reactor at Los Alamos.



20 July 1956, Volume 124, Number 3212 **SCIENCE**

Detection of the Free Neutrino: a Confirmation

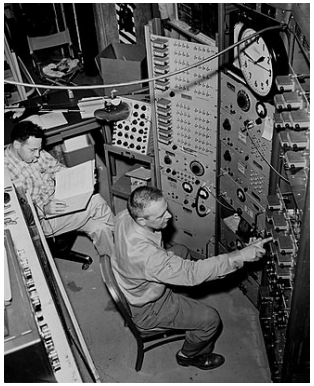
C. L. Cowan, Jr., F. Reines, F. B. Harrison, H. W. Kruse, A. D. McGuire

A tentative identification of the free neutrino was made in an experiment performed at Hanford (*1*) in 1953. In that work the reaction

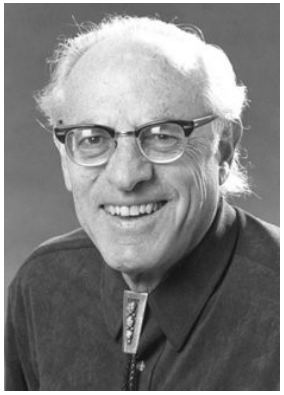
present work was done (*3*). This work confirms the results obtained at Hanford and so verifies the neutrino hypothesis suggested by Pauli (*4*) and incorporated in a quantitative theory of beta decay by

both triads. The detector was completely enclosed by a paraffin and lead shield and was located in an underground room of the reactor building which provides excellent shielding from both the reactor neutrons and gamma rays and from cosmic rays.

The signals from a bank of preamplifiers connected to the scintillation tanks were transmitted via coaxial lines to an electronic analyzing system in a trailer van parked outside the reactor building. Two independent sets of equipment were used to analyze and record the operation of the two triad detectors. Linear amplifiers fed the signals to pulse-height selection gates and coincidence circuits. When the required pulse amplitudes and coincidences (prompt and delayed) were satisfied, the sweeps of two triple-beam oscilloscopes were triggered, and the pulses from the complete event were



Clyde Cowan 1919-1974; American; Nobel Prize For Physics 1995



Frederick Reines 1918-1998; American; Nobel Prize For Physics 1995

Anti-neutrino flux = 5×10^{13} $s^{-1}cm^{-2}$. Target volume 200 litres of water (+ cadmium chloride). Protons of interest are those of the H atoms.

$\Longrightarrow$ 0.04 reactions / s Not bad !

Signal comes from two gamma rays emitted from the reaction $e^+ + e^- \rightarrow \gamma\gamma$, soon followed by another photon from the reaction $n + {}^{108}_{48}\mathrm{Cd} \longrightarrow {}^{109m}_{48}\mathrm{Cd} \rightarrow {}^{109}_{48}\mathrm{Cd} + \gamma$.

Pauli's 'desperate remedy' confirmed 26 years after first proposal.
$\overline{\nu_e} + p \rightarrow n + e^+$ cross section in perfect agreement with Fermi theory.

Discovery of the muon and tau neutrinos

The muon neutrino was discovered in 1962 by Lederman, Schwartz and Steinberger, who were awarded the 1988 Nobel Prize.

[Danby *et al.*, Phys Rev Lett 9 (1962) 36]

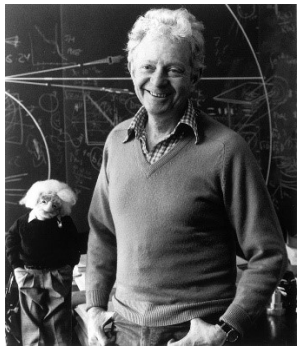
OBSERVATION OF HIGH-ENERGY NEUTRINO REACTIONS AND THE EXISTENCE OF TWO KINDS OF NEUTRINOS*

G. Danby, J-M. Gaillard, K. Goulianos, L. M. Lederman, N. Mistry, M. Schwartz,† and J. Steinberger†

Columbia University, New York, New York and Brookhaven National Laboratory, Upton, New York
(Received June 15, 1962)

In the course of an experiment at the Brookhaven AGS, we have observed the interaction of high-energy neutrinos with matter. These

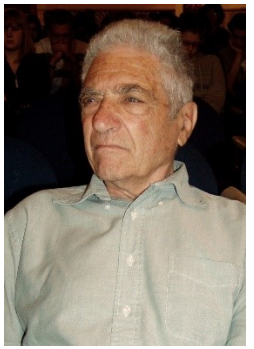
duce μ mesons but do not produce electrons, and hence are very likely different from the neutrinos involved in β decay and (2) approximate cross



Leon Lederman
1922-1988,
American citizen

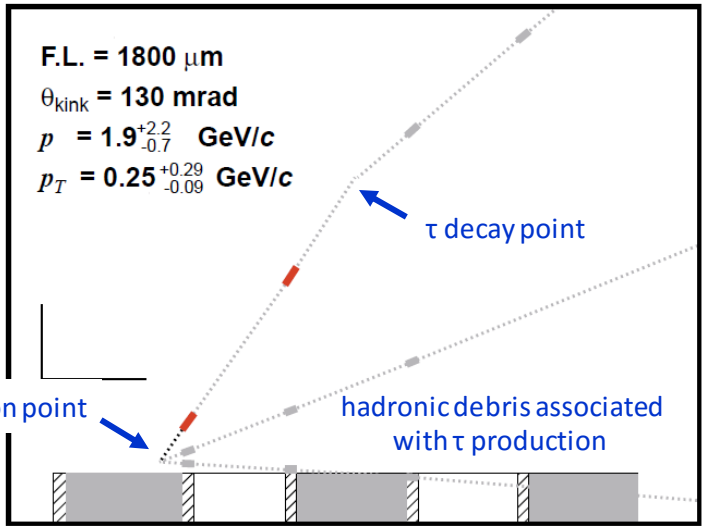


Melvin Schwartz,
1932-2006,
American citizen



Jack Steinberger,
1921-2020, German
and American citizen

The tau neutrino was first observed in 2000 by the DONUT ('Direct Observation of Nu Tau') who reported the observation of four clear tau creations and decays in an emulsion experiment exposed to a neutrino beam.



[DONUT, Phys. Lett. B 504 (2001) 218]

Both of these discoveries made use of neutrinos interacting at significantly higher energies (~GeV) than in inverse β-decay case.
Such interactions are also a sensitive probe of hadronic structure in `DIS' events…

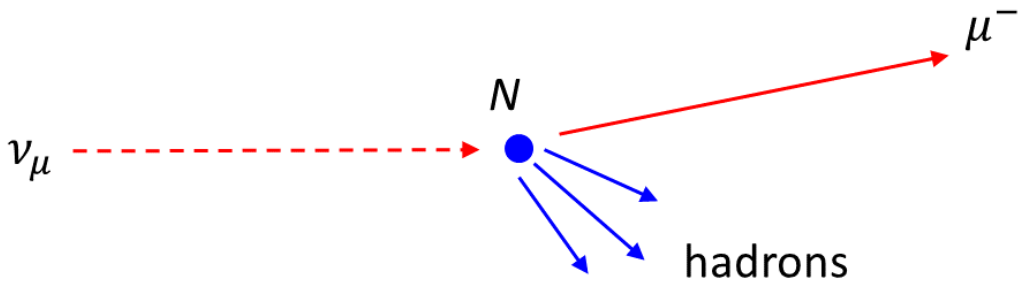
High-energy neutrino-nucleon scattering

Consider the neutrino scattering at energies of say $E_\nu > 1$ GeV (much above the inverse β decay case). Now we are sensitive to scales smaller than the nucleon as the de Broglie wavelength of the neutrino $\lambda = h/p < 2\pi\hbar c/pc \sim 1$ fm, & scatter off the quarks themselves!

It is convenient to perform these experiments with muon neutrino beams, as pions can be produced at high energy through proton collisions on targets, & pions decay to muons.

The following reaction is observed:

$$\nu_\mu + N \rightarrow \mu^- + \text{hadrons}$$



The momentum transfer is sufficient to split open the nucleon, from which hadronic debris forms. The muon emerges as a penetrating track, travelling much further than the hadrons.

These are the neutrino 'deep inelastic scattering' (DIS) events, study of which began in the early 1970s in bubble chambers at CERN, as a natural successor to the electron-nucleon DIS studies at SLAC, California. Together they provided important evidence for the quark model.

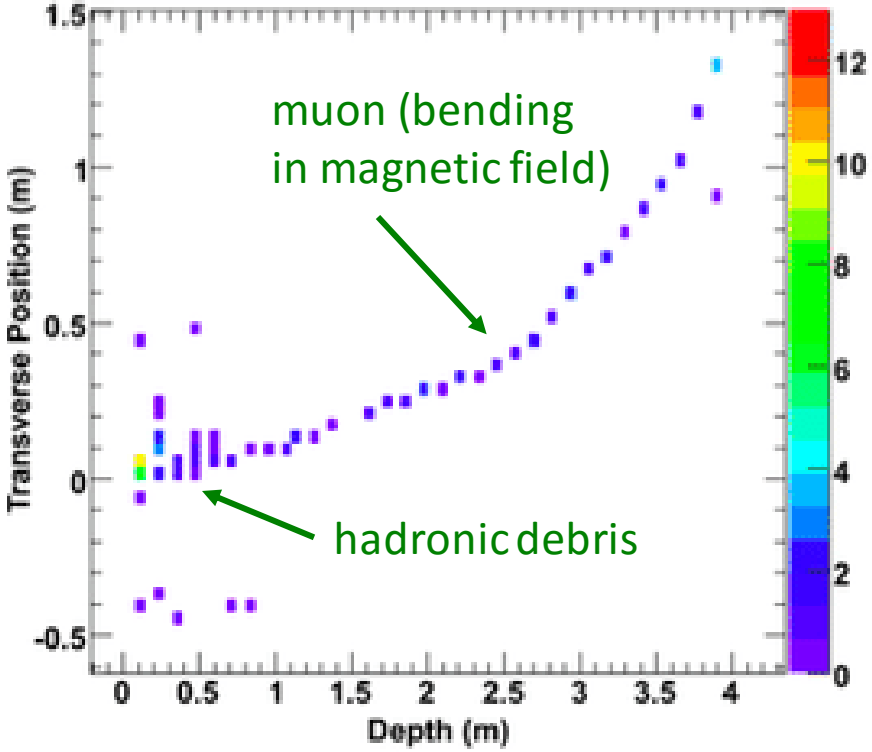
Neutrino-nucleon scattering event in MINOS

$\nu_\mu + N \rightarrow \mu^- +$ hadrons event as seen by MINOS detector in Minnesota.

MINOS: front view



Energy deposits:
depth vs longitudinal position

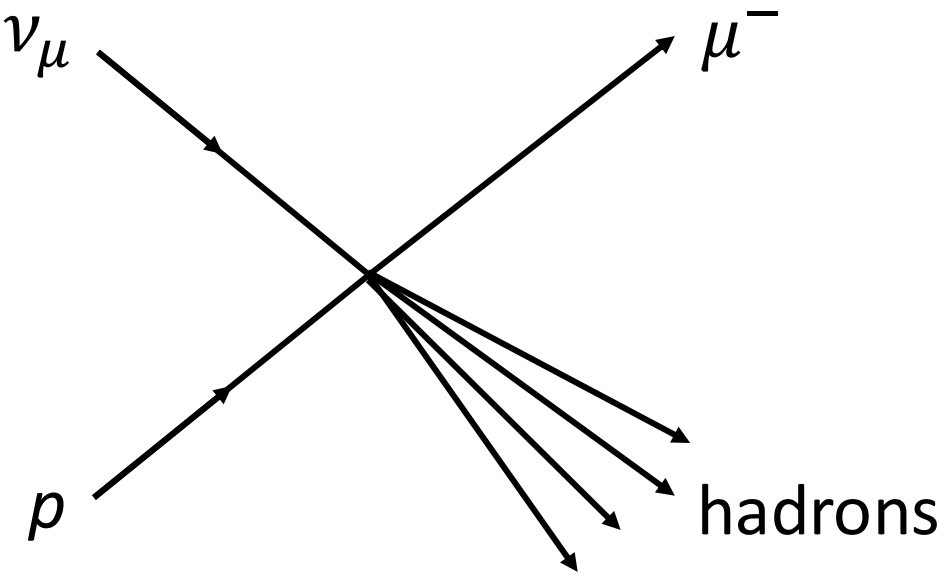


High-energy neutrino-nucleon scattering

We can therefore take previous expression for the cross section for inverse β decay:

$$\sigma = \frac{2\pi}{\hbar c} 4{G_F}^2 \rho$$

and re-evaluate the density-of-states factor for this new situation.



$$\rho = \frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}p_\mu}\frac{\mathrm{d}p_\mu}{\mathrm{d}E} = \frac{4\pi p_\mu^2}{(2\pi\hbar)^3}\frac{\mathrm{d}p_\mu}{\mathrm{d}E}$$

p_μ is muon momentum, and E the total energy in the C-of-M. E_ν is the neutrino momentum in lab.

We have $p_\mu c \approx E/2$ for $E_\nu \gg m_p$, and also

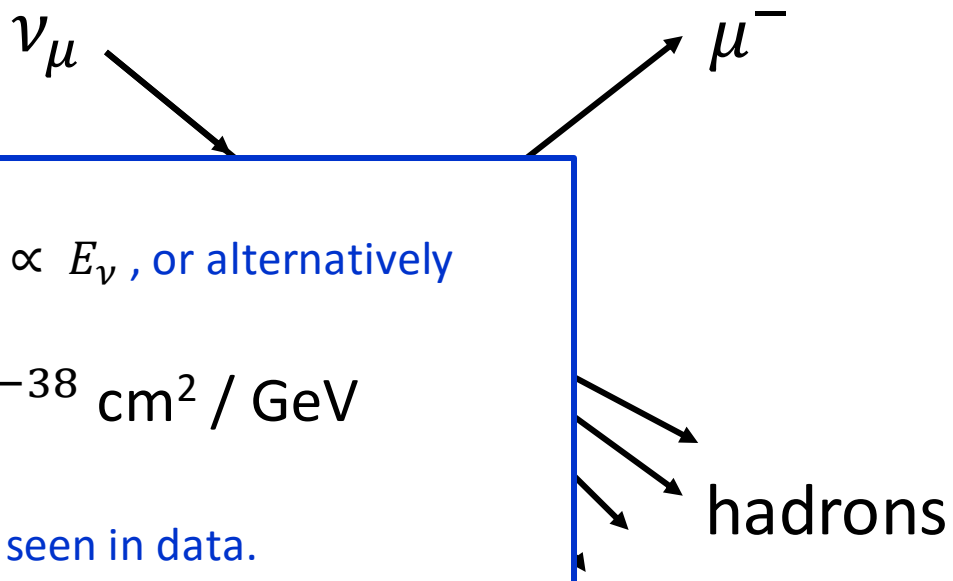
$$(E_\nu + m_p c^2)^2 - p_\nu^2 c^2 = E^2$$

$$2E_\nu m_p c^2 \approx E^2$$

$$\Longrightarrow \quad \boxed{\sigma = \frac{G_F^2 m_p c^2 E_\nu}{\pi(\hbar c)^4}}$$

High-energy neutrino-nucleon scattering

We can therefore take previous expression for the cross section for inverse β decay:



$$\sigma = \frac{2}{\hbar}$$

and re-evalu
factor for thi

This result tells us that $\sigma \propto E_\nu$, or alternatively

$$\frac{\sigma}{E_\nu} = 1.6 \times 10^{-38}\ \text{cm}^2 / \text{GeV}$$

Let's compare to what is seen in data.

$$\rho = \frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}p_\mu}\frac{\mathrm{d}p_\mu}{\mathrm{d}E} = \frac{4\pi p_\mu^2}{(2\pi\hbar)^3}\frac{\mathrm{d}p_\mu}{\mathrm{d}E}$$

p_μ is muon momentum, and E the total energy in the C-of-M. E_ν is the neutrino momentum in lab.

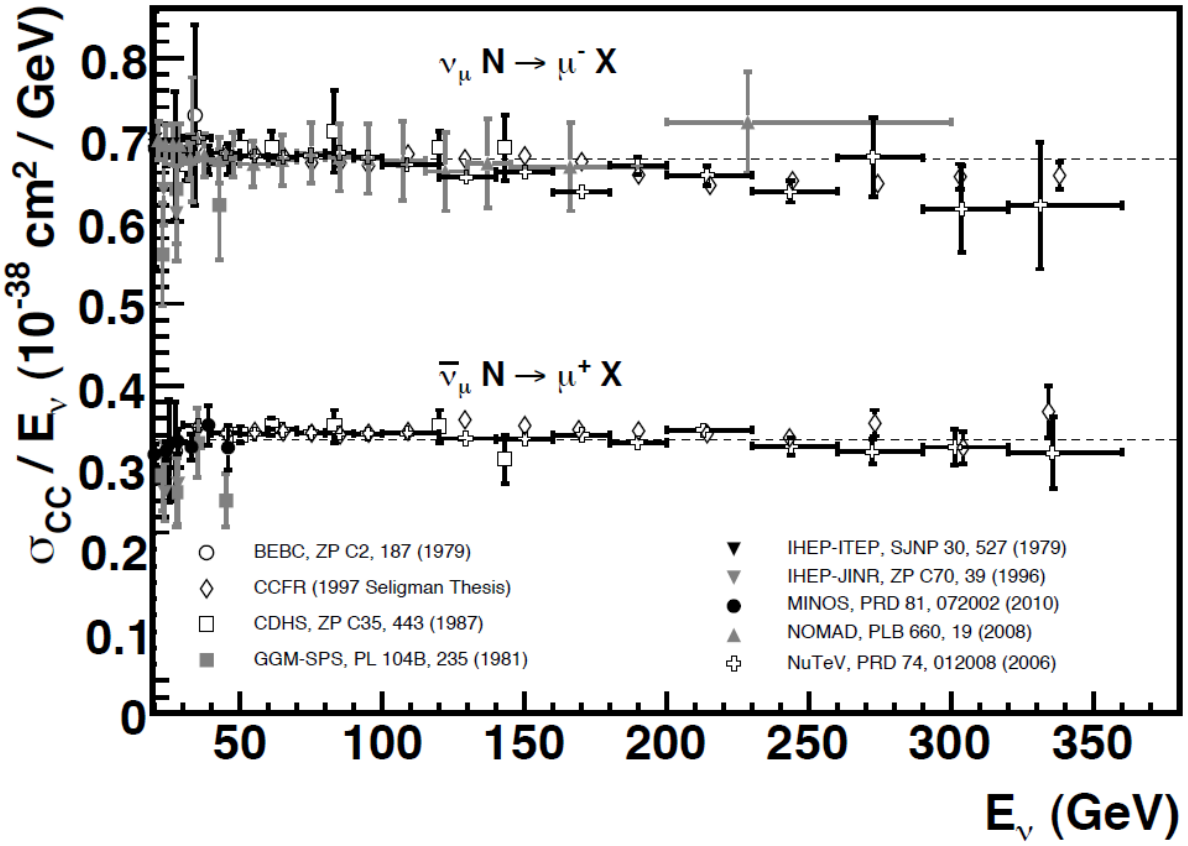
We have $p_\mu c \approx E/2$ for $E_\nu \gg m_p$,
and also

$$(E_\nu + m_p c^2)^2 = E^2$$
$$2E_\nu m_p c^2 \approx E^2$$

$$\Rightarrow \quad \sigma = \frac{G_F^2 m_p c^2 E_\nu}{\pi(\hbar c)^4}$$

Measured cross sections for $\nu_\mu + N \rightarrow \mu^- +$ hadrons

Here is a compilation of cross section / E_ν results plotted vs. E_ν.

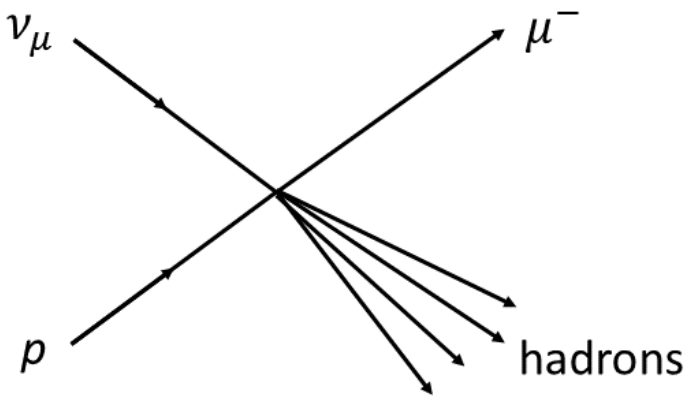


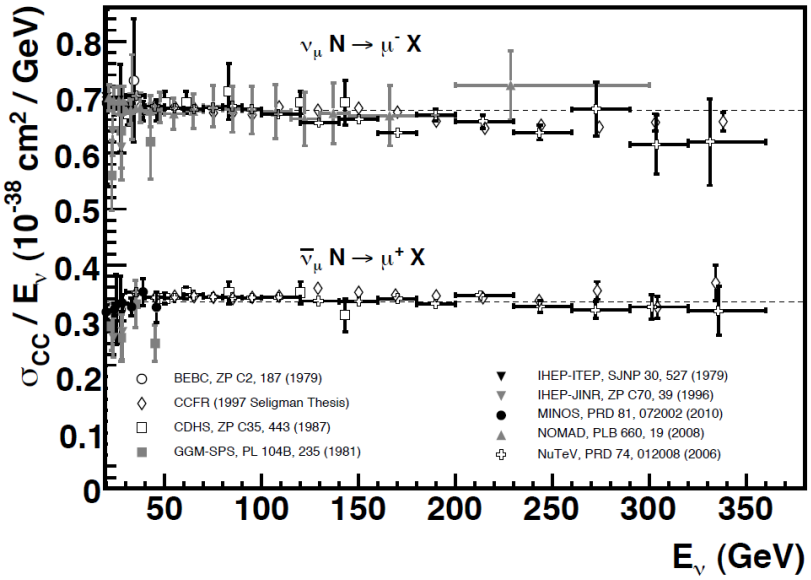
[Particle Data Group, Nakamura *et al.*, J. Phys G37, 075021]

The results are different for neutrinos and antineutrinos. Interesting and important, but not what we are discussing right now. The key point is that σ/E is a constant! Surprisingly, given the crudeness of our calculation, the constant is close to expectation.

High energy neutrino-nucleon scattering: take-home messages

There are two important take-home messages from this calculation and results.





1) The constant value is consistent with the predictions of Fermi theory, which assumes a point-like interaction. We conclude that, within the range probed, quarks are point like !

2) The suggestion is also that the cross section grows without limit. This is troubling, as in scattering theory there are general considerations that put upper bounds on cross sections (the *unitarity limit*). Although these bounds are beyond current experimental reach, they are violated by our result. Something must be wrong !

Particle physics: topic overview

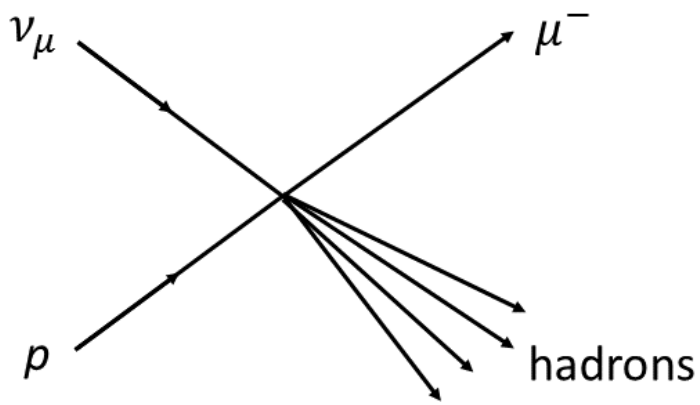
Leptons and the weak interaction

- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

Beyond Fermi theory

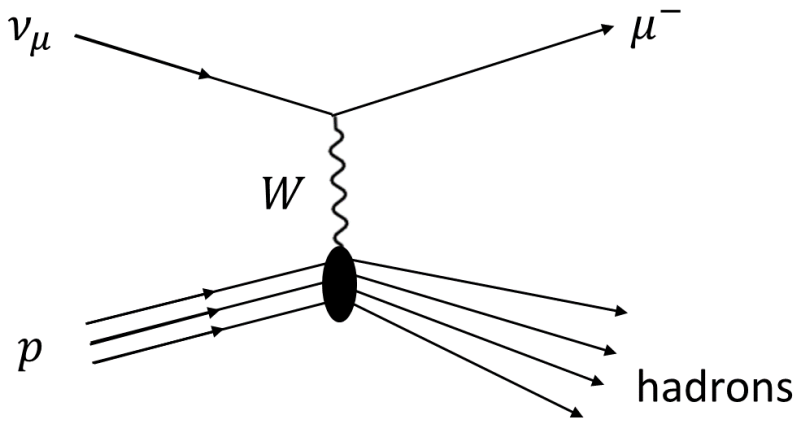
Of course the unitarity limit is not violated. In fact Fermi theory is an effective field theory, and what is really happening is that a massive boson, the *W*, is being exchanged

Fermi theory



Matrix element $\sim G_F$

Electroweak theory (part of the 'Standard Model')



Matrix element $\sim$ vertex factors $\times$ propagator

$$g_W^2 \times \frac{1}{q^2 + m_W^2 c^4}$$

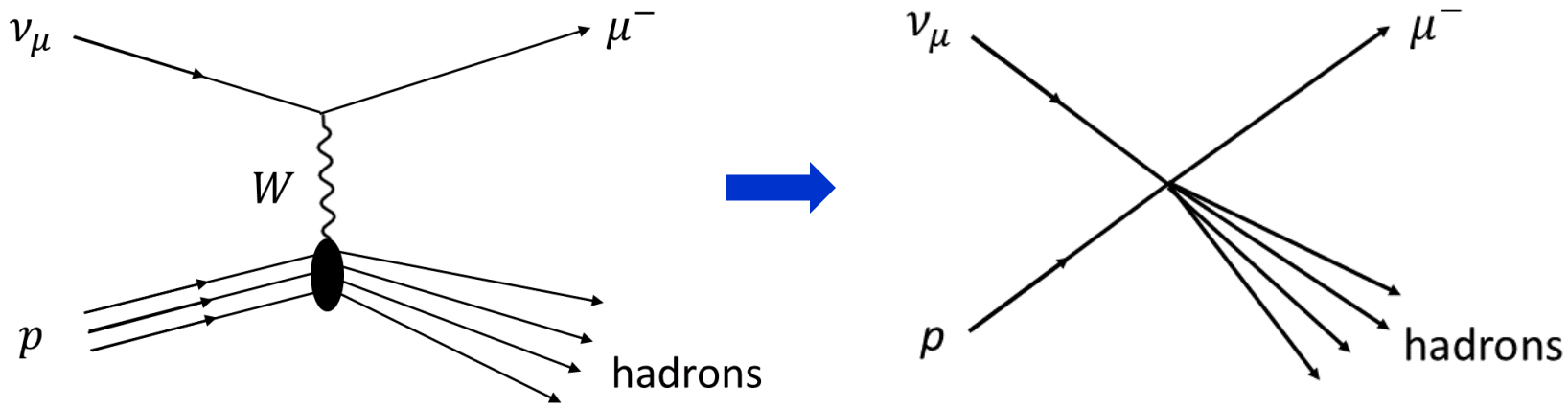
g_W is dimensionless, just like $\sqrt{\alpha}$ and *not* like G_F

Here q is the momentum exchange

m_W is the mass of the *W* boson, which is ~80 GeV/c^2

Beyond Fermi theory

So, in the limit that $q \ll m_W$ then the W exchange contracts to the four-point interaction.



$$g_W^2 \times \frac{1}{q^2 + m_W^2 c^4} \rightarrow \frac{g_W^2}{m_W^2 c^4} \sim G_F$$

including constants

$$\frac{G_F}{(\hbar c)^3} = \frac{\sqrt{2}}{8}\left(\frac{g_W}{m_W c^2}\right)^2$$

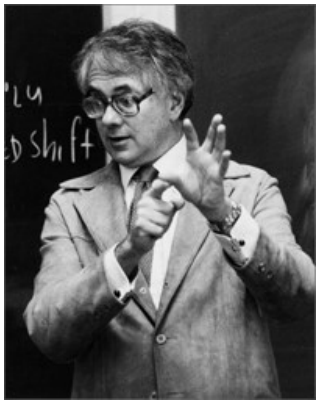
We also note that as q^2 approaches, and exceeds m_W then the matrix element acquires energy dependence, in contrast to Fermi theory, & the cross-section growth is suppressed.

Electroweak theory

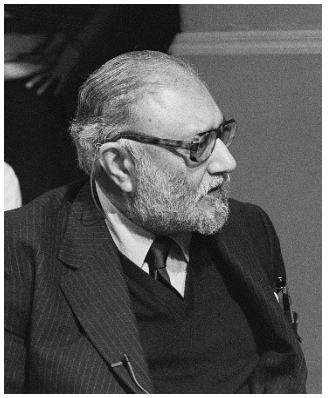
The idea of the weak force being mediated by a massive exchange boson was put on a solid theoretical footing by Weinberg, Glashow & Salam, recognised with the Nobel Prize in 1979.



Steven Weinberg 1933-2021; American citizen; Nobel Prize for Physics, 1979



Sheldon Lee Glashow 1932-; American citizen; Nobel Prize for Physics, 1979



Abdus Salam 1926-1996; Pakistani and British citizen; Nobel Prize for Physics, 1979

Key idea: the photon and the W arise from the mixing of two underlying set of fields that obey a particular symmetry ('gauge symmetry', hence 'gauge bosons'). This means they can be regarded as manifestations of a single underlying electroweak interaction.

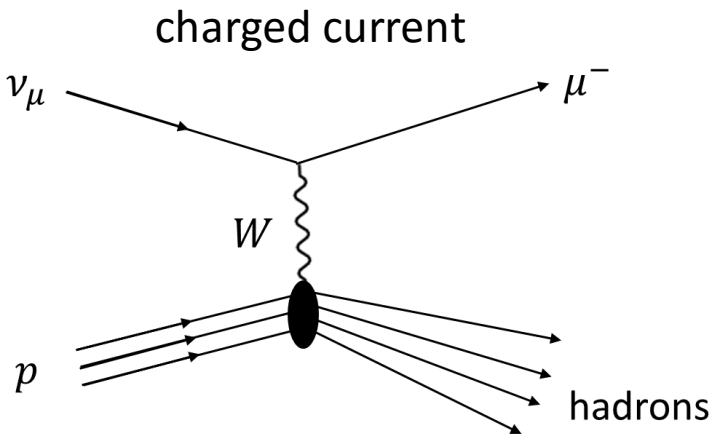
The W acquires mass through the Higgs mechanism. The theory was shown to be 'renormalisable' by 't Hooft and Veltman (Nobel Prize for Physics in 1999).

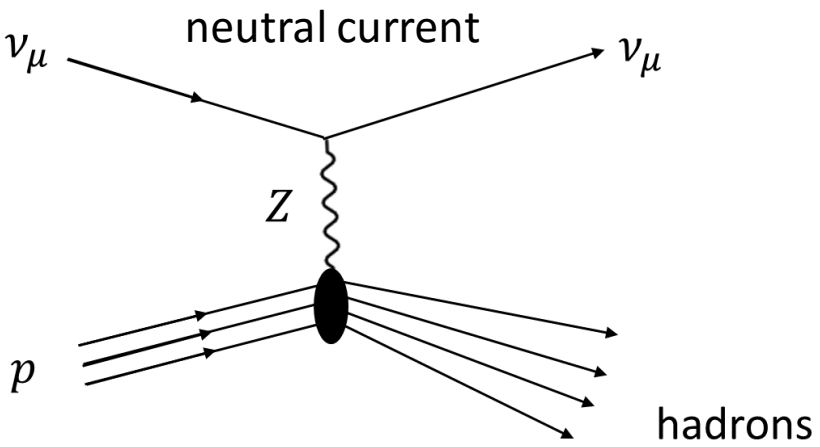
That's not all: the mixing of the fields predicted the existence of an *additional* massive weak boson, and one that was neutral. This was dubbed the Z^0.

Weak neutral currents

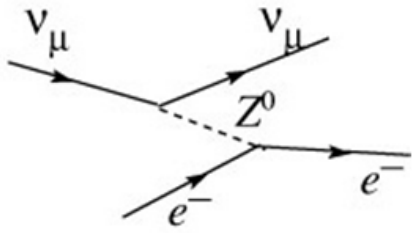
The Z^0 was predicted to have a similar mass to the W ($m_Z \approx 91.2$ GeV/c^2) and a similar strength of interaction. Its existence would lead to weak-interaction processes that, in an exchange, would lead to no particle change or exchange of charge: *weak neutral currents*.

For example, in neutrino interactions one would expect something similar to that caused by W-boson exchange, but with no charged track emerging, apart from the hadronic debris.





Other interactions are possible, such as scattering off atomic electrons. Rarer, but easier to observe above background.



Discovery of weak neutral currents

The weak neutral current events predicted by the Glashow-Weinberg-Salam theory were discovered in the Gargamelle bubble chamber at CERN in 1973.



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OBSERVATION OF NEUTRINO-LIKE INTERACTIONS WITHOUT MUON OR ELECTRON IN THE GARGAMELLE NEUTRINO EXPERIMENT

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Events induced by neutral particles and producing hadrons, but no muon or electron, have been observed in the CERN neutrino experiment. These events behave as expected if they arise from neutral current induced processes. The rates relative to the corresponding charged current processes are evaluated.

We have searched for the neutral current (NC) and charged current (CC) reactions:

NC $\nu_\mu/\bar{\nu}_\mu + N \to \nu_\mu/\bar{\nu}_\mu$ + hadrons, (1)

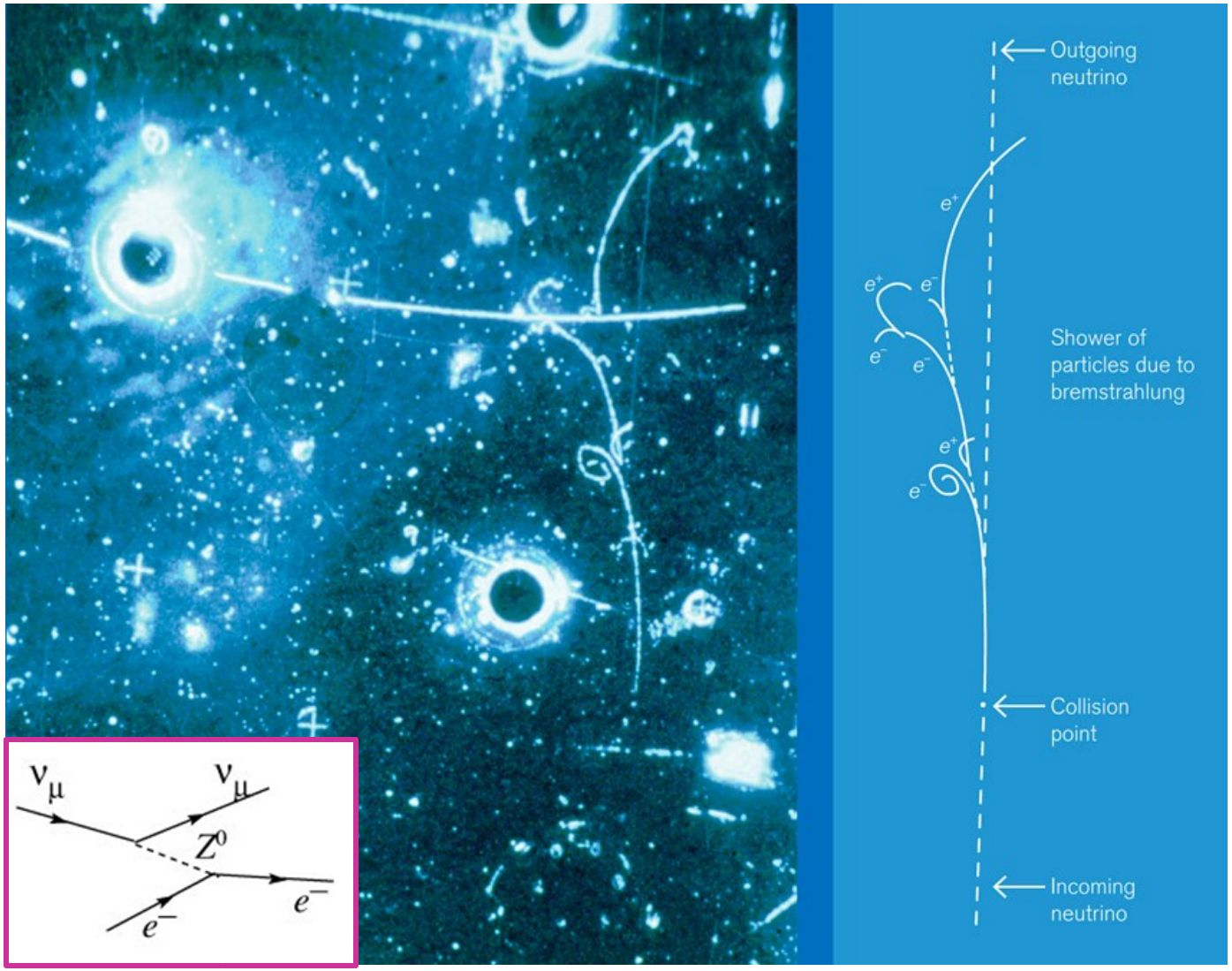
CC $\nu_\mu/\bar{\nu}_\mu + N \to \mu^-/\mu^+$ + hadrons (2)

which are distinguished respectively by the absence of any possible muon, or the presence of one, and only one, possible muon. A small contamination of $\nu_e/\bar{\nu}_e$ exists in the $\nu_\mu/\bar{\nu}_\mu$ beams giving some CC events which are easily recognised by the e^-/e^+ signature. The analysis is based on 83 000 ν pictures and 207 000 $\bar{\nu}$ pictures taken at CERN in the Gargamelle bubble chamber filled with freon of density 1.5×10^3 kg/m^3 ‡ The dimensions of this chamber are such that most

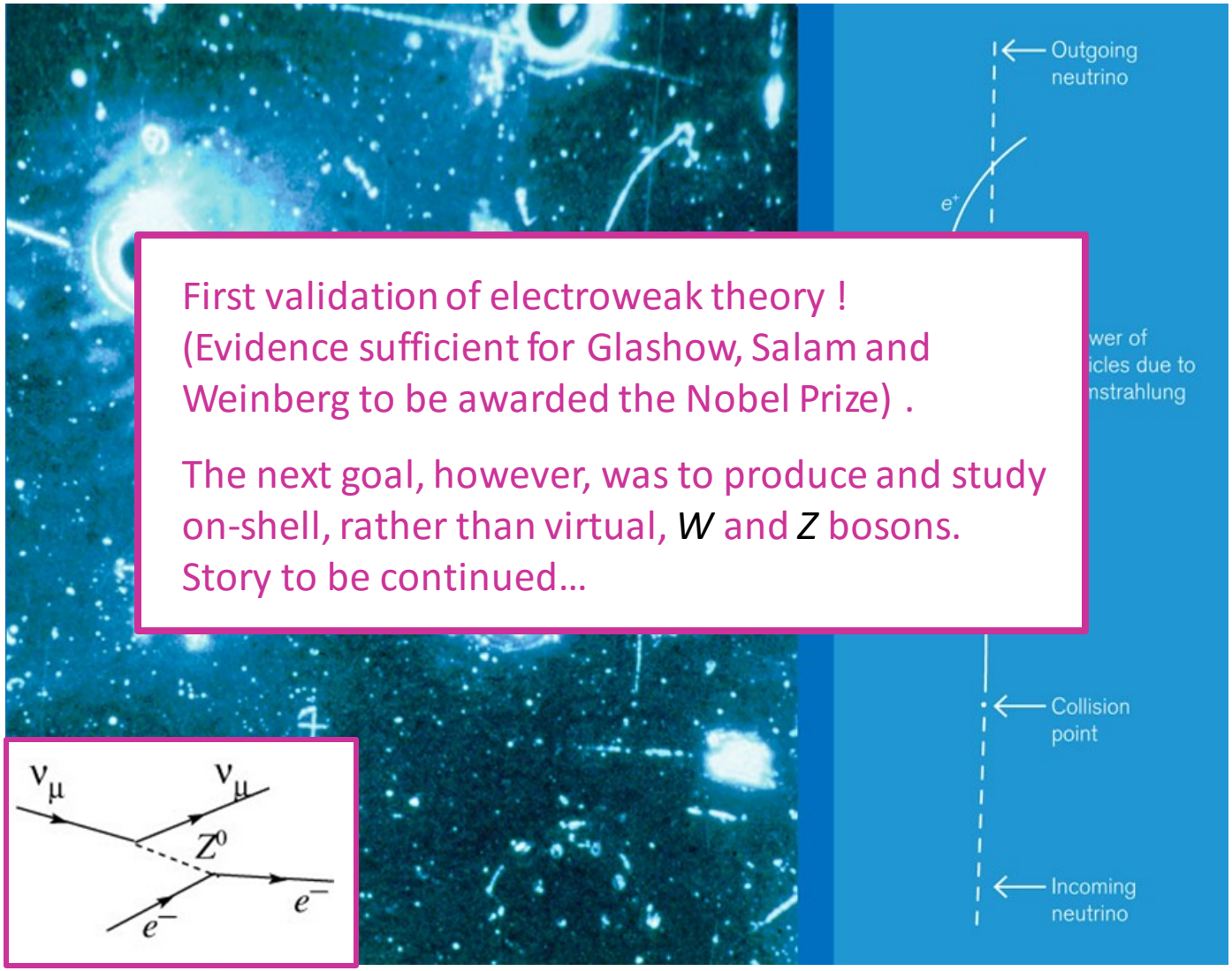
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*[9] Supported by Science Research Council grant.

‡ A more detailed account of the analysis of this experiment appears in a paper to be submitted to Nuclear Physics.

Discovery of weak neutral currents



Discovery of weak neutral currents

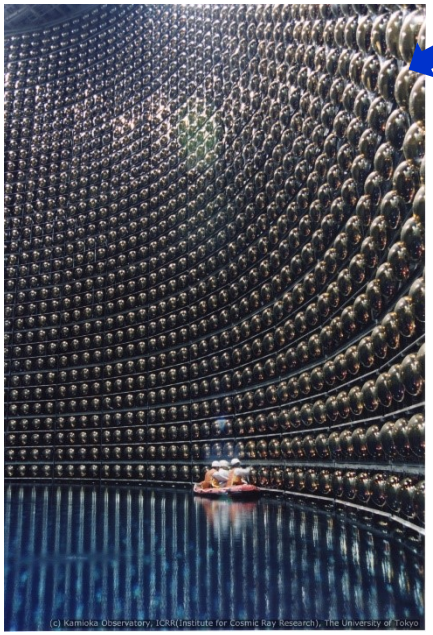


First validation of electroweak theory !
(Evidence sufficient for Glashow, Salam and Weinberg to be awarded the Nobel Prize) .

The next goal, however, was to produce and study on-shell, rather than virtual, *W* and *Z* bosons.
Story to be continued…

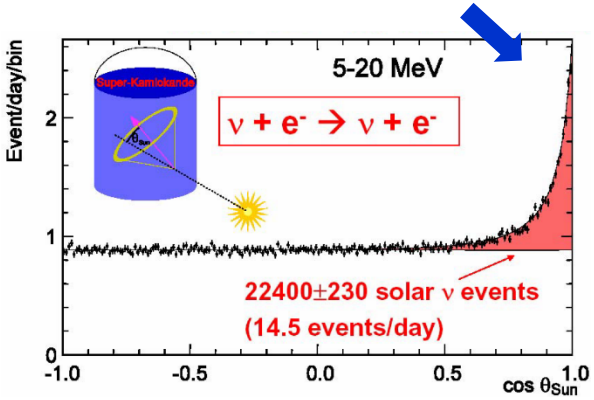
Last word (for now) on neutrinos

The study of neutrinos has yielded some of the most important results in particle physics over the past 20 years. Experiments such as SNO and Super Kamiokande have studied neutrinos produced in the sun and from cosmic rays and concluded that oscillations occur between neutrino flavours, as they propagate. This can only happen if neutrinos have mass !

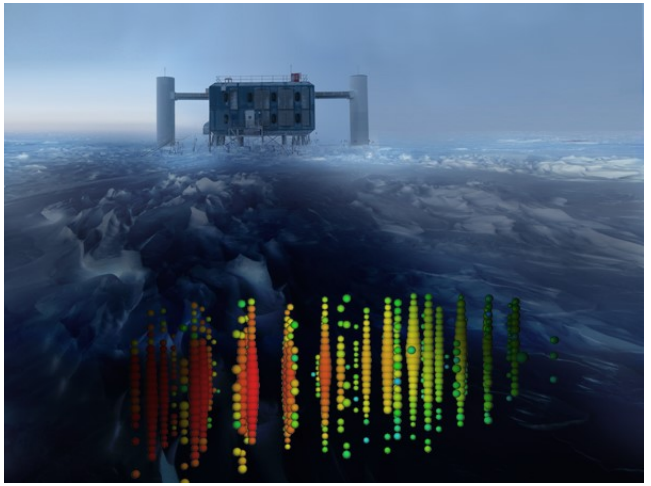


Checking phototubes at Super K.

Detected electron neutrinos vs Z-mediated elastic scattering vs. angle to sun. Clear signal seen !



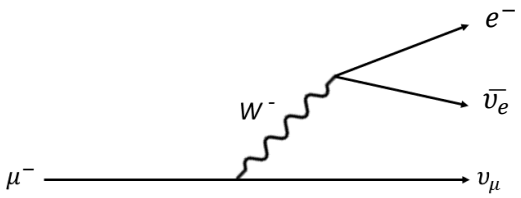
Superimposed picture of IceCube lab and its phototube strings responding to a high energy cosmic neutrino.

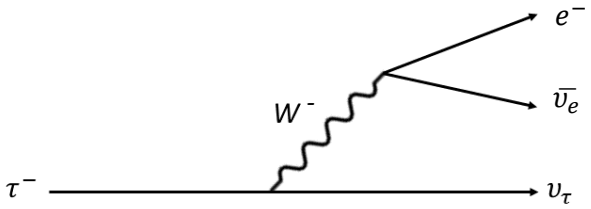


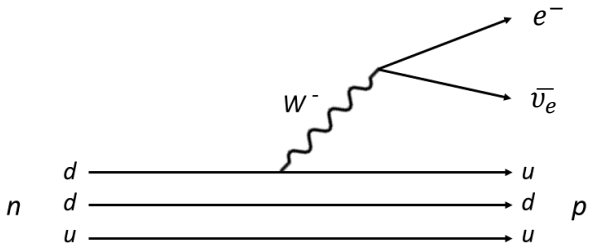
Many neutrino properties remain to be discovered, *e.g.* whether they exhibit *CP* violation, and whether the neutrino is its own anti-particle (a `Majorana particle'). Furthermore neutrino astronomy is a growing field – see for example the IceCube experiment in Antarctica.

Decays of leptons and hadrons as viewed in electroweak theory

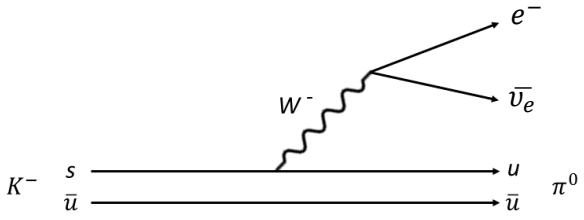
Returning to our decays of muons, taus, beta decay *etc.*, we see that in electroweak theory they can all be drawn involving two vertices connected by a *W* exchange.

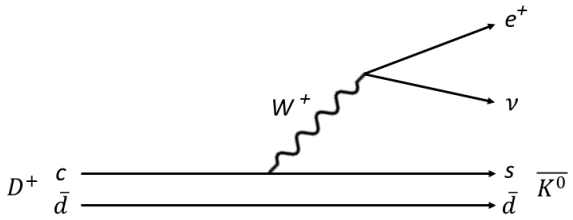


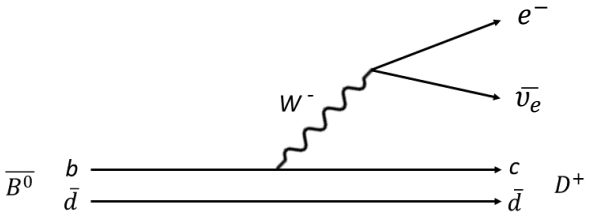




Similarly, we expect that any flavour-changing weak decay of a meson is the same.







This is correct. Therefore we should be able to apply Fermi theory to calculate the lifetimes of these mesons. Also correct. However, here, something new must be accounted for…

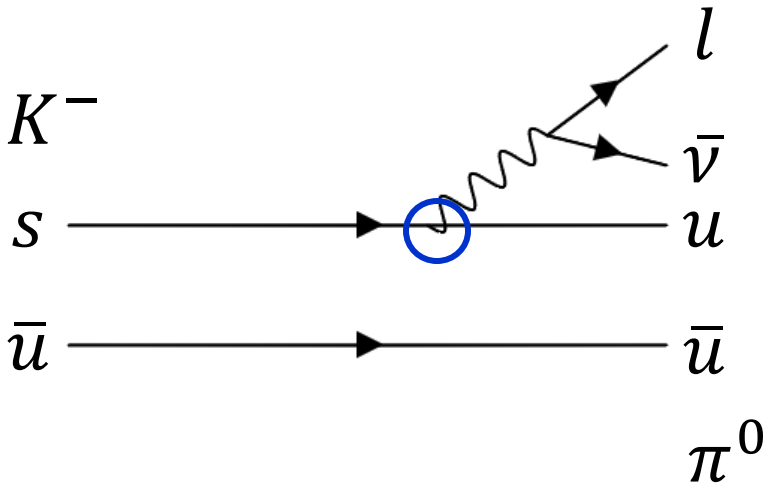
Particle physics: topic overview

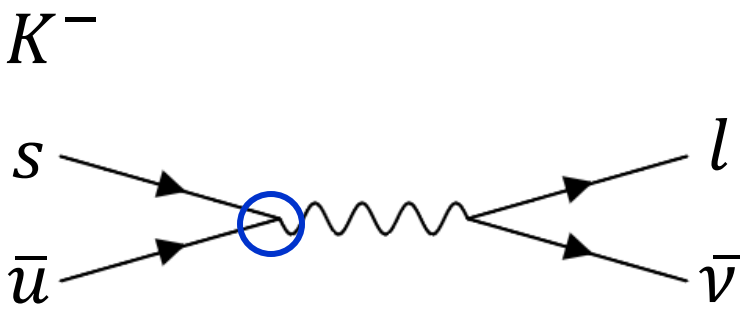
Leptons and the weak interaction

- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

Inter-generation hadronic decays

Any $|\Delta S| = 1$ decay of a strange hadron involves a $s \to u$ transition. This therefore means that the W couples generations, something that does not apply for charged leptons.



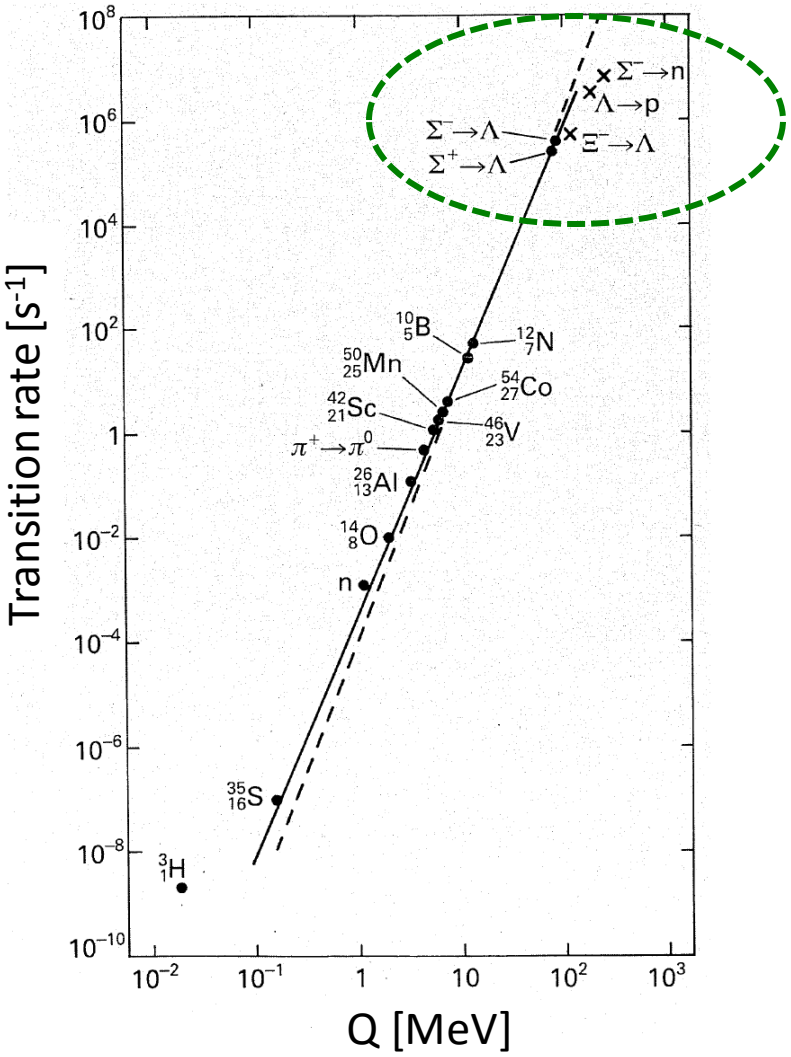


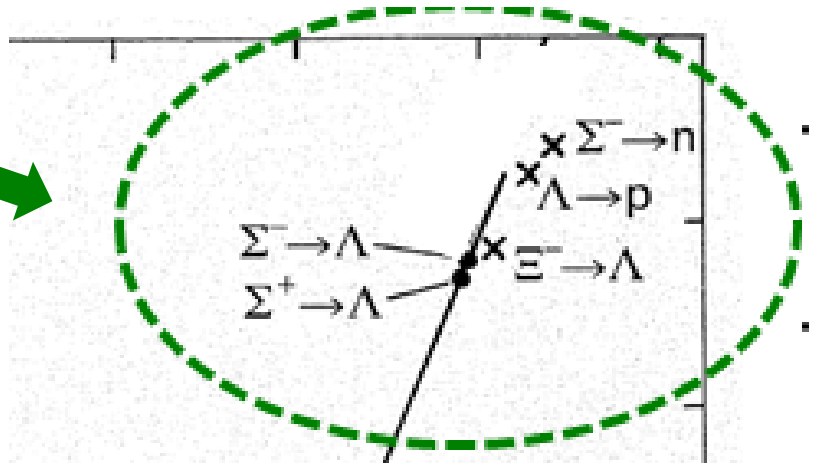
these vertices couple generations

It is possible, however, for certain strange hadrons to decay through transitions of the u or d quarks. Comparing these cases with the $|\Delta S| = 1$ decays is instructive.

Strange baryon ('hyperon') transition rates

Recall the plot of transition rate vs. Q value for β decay, with the fitted line that showed the broad validity of Sargent's rule. Let's look more closely at top-right corner.





These are in fact all decays of strange baryons.

The two that lie on the line involve decays where the u or d quark has transitioned to a d or a u quark, and seem quite well behaved.

The ones that lie off, and are happening more slowly than expected, all have an $s \to u$ decay.

Cabibbo mixing

An explanation for this behaviour in the context of u, d, s interactions was proposed by Nicola Cabibbo in 1963. It was then extended to include charm by Glashow, Iliopoulos and Maiani (`GIM') in 1970; indeed this extension constituted a prediction of charm's existence!

The W does not couple u and d, but rather u and d', where d' is a superposition of d and s. Similarly the W couples c and s', with s' a superposition of s and d.

The coupling is expressed by the Cabibbo mixing matrix

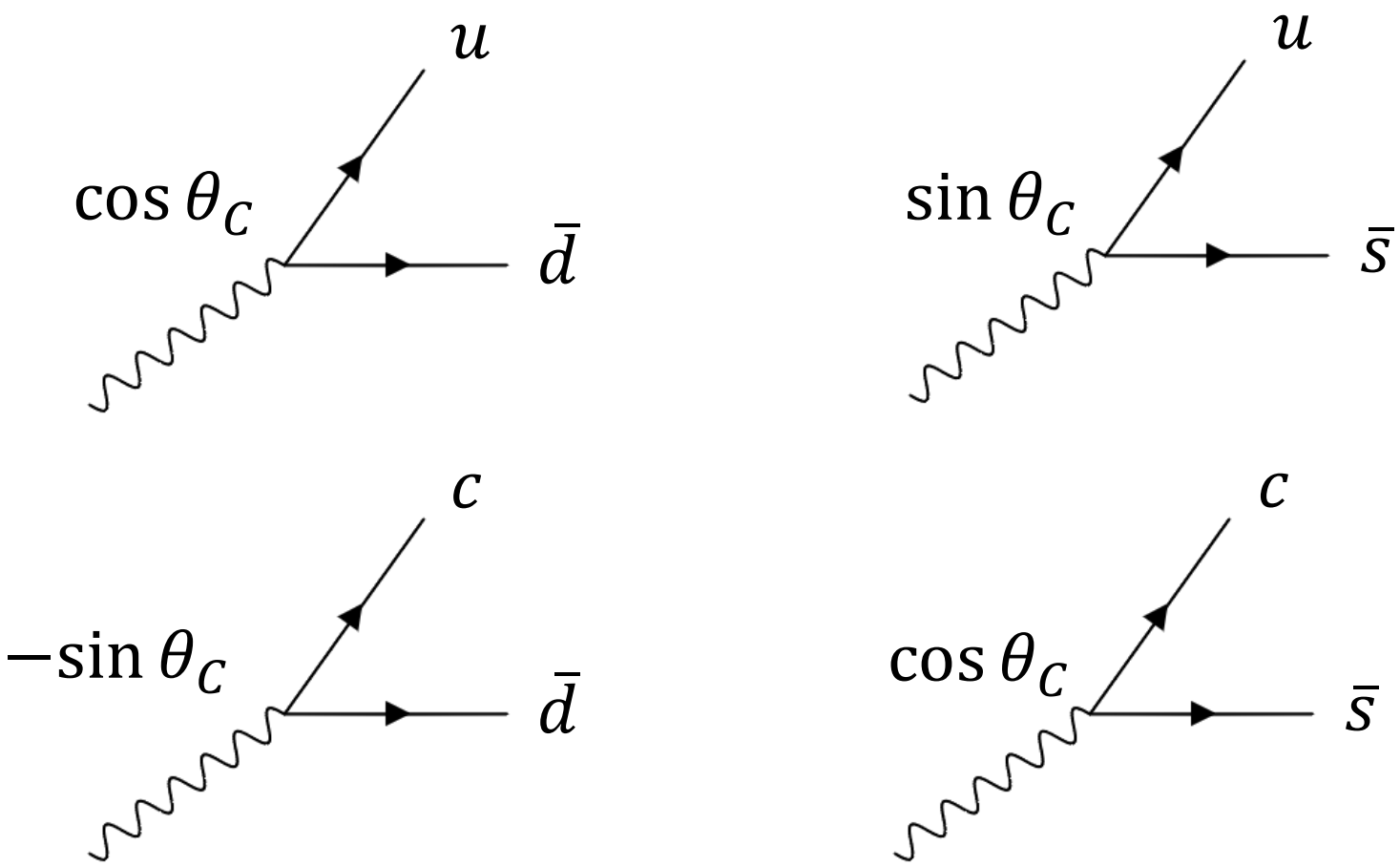
$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

where the Cabibbo angle $\theta_C \approx 13.0^o$. Another way to explain what is happening:

The weak-flavour eigenstates are rotated with respect to the mass eigenstates by a small angle θ_C.

Practical consequence: each vertex in the Feynman diagram comes with a Cabibbo factor.

$Wq\bar{q}'$ vertices with Cabibbo factors



The $\cos\theta_C$ vertices are Cabibbo favoured; the $\sin\theta_C$ vertices are Cabibbo suppressed.

Calculating relative decay rates with Cabibbo factors

Let's find the ratio of branching fractions

$$\frac{B(D^+ \to K^+\pi^-\pi^+)}{B(D^+ \to K^-\pi^+\pi^+)}$$

In the $D^+ \to K^+\pi^-\pi^+$ decay both vertices are suppressed, and so the overall amplitude

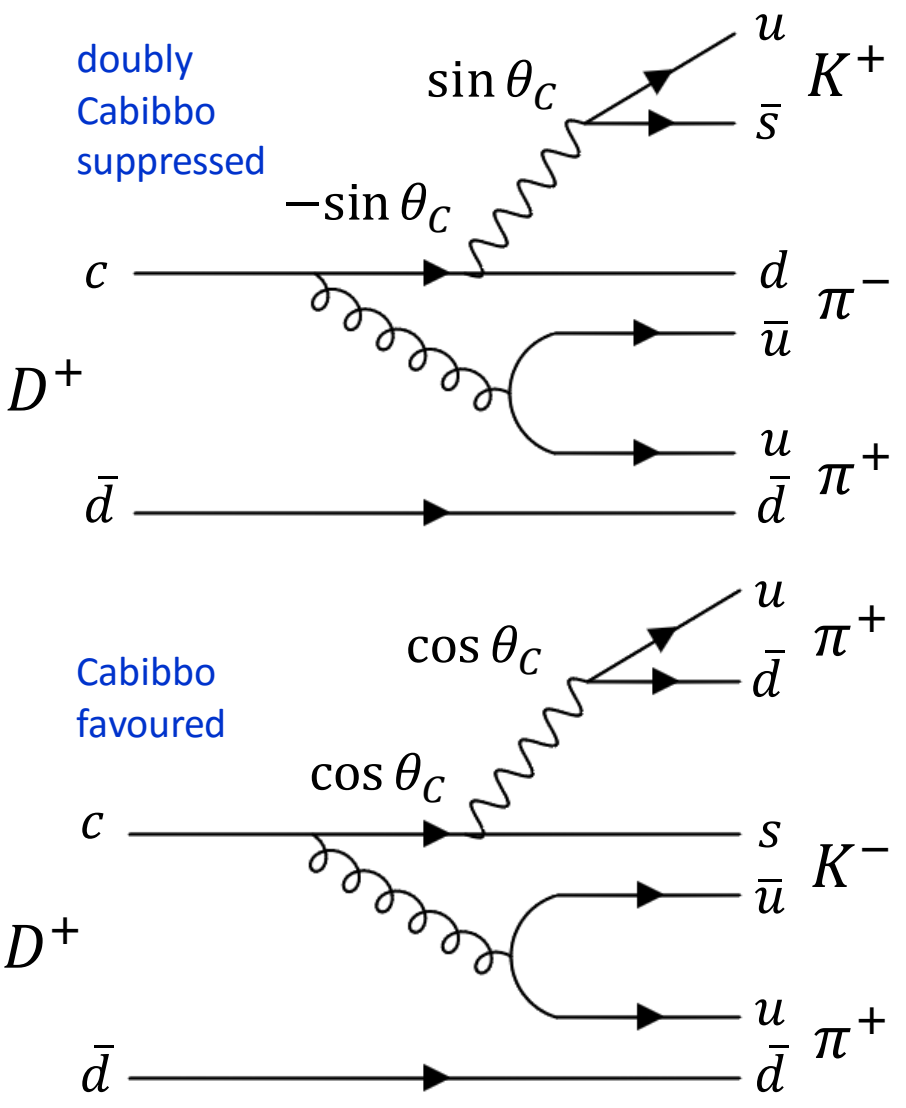
$$A_{K^+\pi^-\pi^+} \sim -\sin\theta_c \sin\theta_c$$

whereas for the other decay, both vertices are favoured, hence

$$A_{K^-\pi^+\pi^+} \sim \cos\theta_C \cos\theta_C$$

As the branching fraction depends on the amplitude squared, we have

$$\frac{B(D^+ \to K^+\pi^-\pi^+)}{B(D^+ \to K^-\pi^+\pi^+)} = (\tan\theta_c)^4 \approx 3 \times 10^{-3}$$



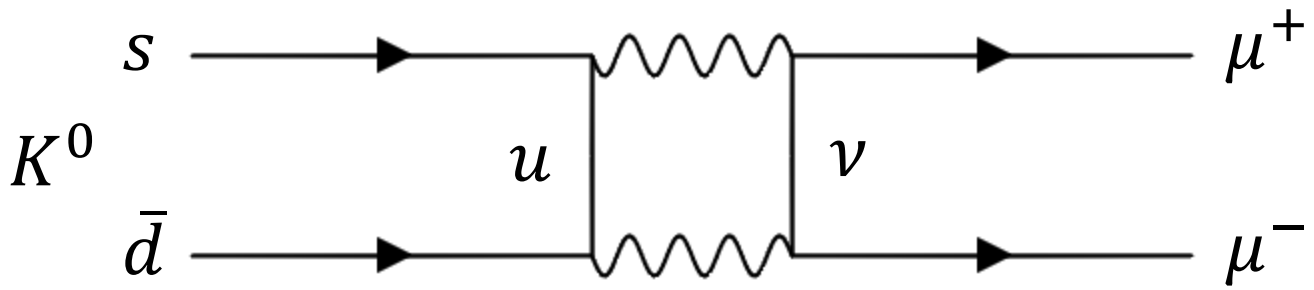
which is close to the measured value of 5×10^{-3}.

The GIM mechanism

Not on syllabus !

In the early days of electroweak theory the Cabibbo mechanism was extended by Glashow, Illiopolous and Maiani (`GIM') to explain certain suppressed phenomena in kaon decays.

For example, consider the decay $K^0 \to \mu^+\mu^-$. In the Standard Model the leading-order diagram includes a loop, which in the light-quark model involves a virtual up quark (and neutrino). This tells us that the decay should be fairly rare…



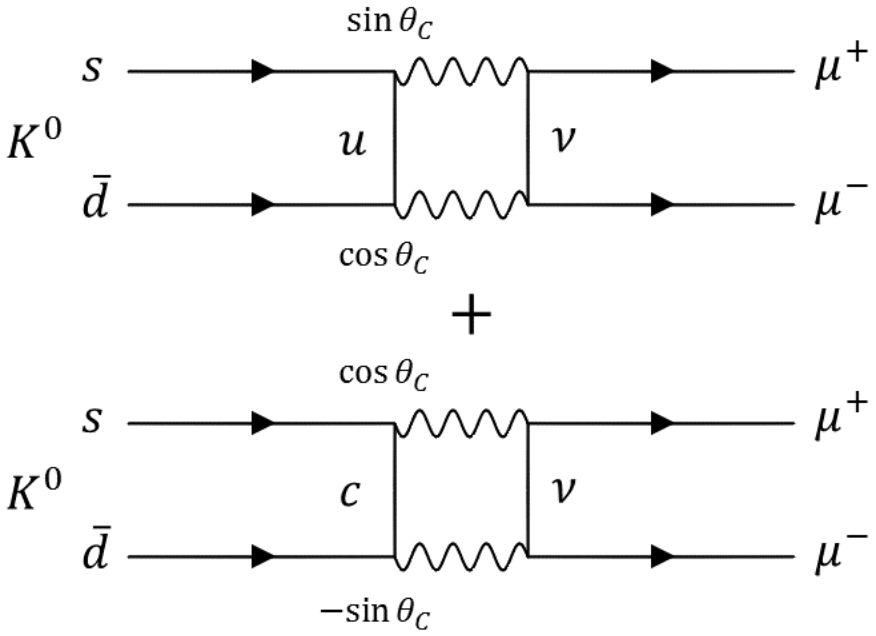
…however, it shouldn't be *too* rare, and from the perspective of 1970s it was a puzzle why the decay had not been observed. The GIM mechanism offered a solution.

(NB the decay of interest is $K_L^0 \to \mu^+\mu^-$, but we won't worry about the K^0/K_L^0 distinction)

The GIM mechanism

Not on syllabus !

GIM proposed the existence of the charm quark, which contributes an additional leading-order amplitude to the decay. In considering both, Cabibbo factors must be included.



With only one diagram $|M|^2 = |\sin\theta_c \cos\theta_C|^2$, which is none zero, but including both:

$$|M|^2 = |\sin\theta_C \cos\theta_C - \cos\theta_C \sin\theta_C|^2 = 0$$

i.e. charm and the Cabibbo mechanism provides perfect cancellation, thus explaining the non-observation of the decay ! In reality, the masses of the quarks enter, and the fact that $m_C \neq m_u$ means the cancellation is not perfect: $B(K^0 \to \mu^+\mu^-) = 7 \times 10^{-9}$.

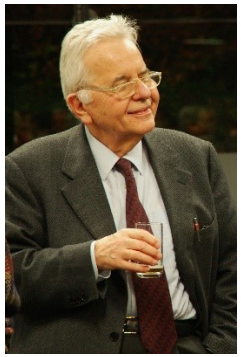
The CKM matrix

Not on syllabus !

The third generation means that the Cabibbo matrix must be extended from 2×2 to 3×3. This extension was proposed by the Japanese physicists Kobayashi and Maskawa in 1973. In fact, this proposal *predated* the discovery of both charm and beauty, and arose out of consideration of Cabibbo-related phenomena in kaons (*CP* violation).

The Cabibbo-Kobayashi-Maskawa (CKM) matrix:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$



Nicola Cabibbo, 1935-2010; Italian citizen



Makoto Kobayashi, 1944-; Japanese citizen; Nobel Prize for Physics, 2008



Toshihide Maskawa, 1940-2021; Japanese citizen; Nobel Prize for Physics, 2008

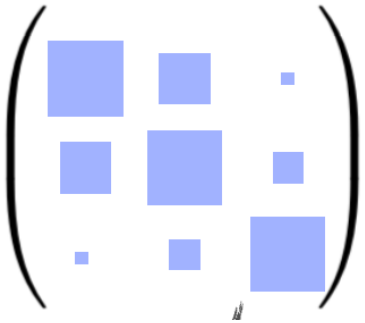
The CKM matrix: a puzzling hierarchy

Not on syllabus !

Measured magnitudes of CKM matrix elements (assuming unitarity of matrix):

$$V_{\rm CKM} = \begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\ 0.22438 \pm 0.00044 & 0.97359^{+0.00010}_{-0.00011} & 0.04214 \pm 0.00076 \\ 0.00896^{+0.00024}_{-0.00023} & 0.04133 \pm 0.00074 & 0.999105 \pm 0.000032 \end{pmatrix}$$

or represented graphically:



This, and the hierarch of quark masses, is presumably telling us something, but what?



In the Standard Model the CKM matrix & the quark masses arise from the Higgs mechanism.

Particle physics: topic overview

Leptons and the weak interaction

- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

Parity violation

In 1956 T.D. Lee and C.N. Yang surveyed results of weak decays, in particular:

$$\theta^+ \rightarrow \pi^+\pi^0$$

$$\tau^+ \rightarrow \pi^+\pi^+\pi^-$$

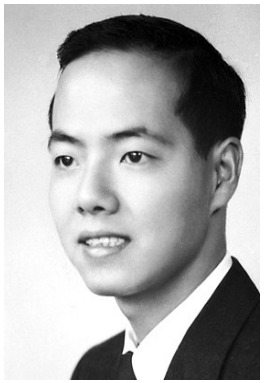
warning – this is a different particle from what we now call the τ !

They strongly suspected that the $\theta^+$and τ^+ were the same particle, now called the K^+. This, however, was deeply troubling, as analysis indicated that the parity of the two final states was different (see problem sheets). How could this be ?

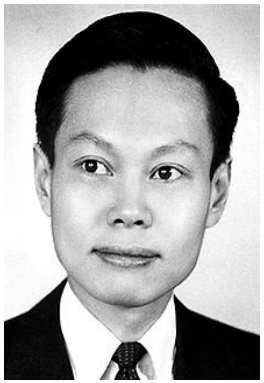
They realised that, in contrast to the strong interaction, there was little experimental evidence checking parity conservation in weak decays, and so proposed several.

Madame Wu of Colombia University, with extensive help from cryogenic experts from the US National Bureau of Standards, conducted the most famous experiment, announcing the results in 1957.

A complementary experiment, studying muon decay in an accelerator, obtained results supporting an identical conclusion at the same time.



T.D. Lee, 1926-; Chinese-American citizen; Nobel Prize for Physics, 1957



C.N. Yang, 1922-; Chinese citizen; Nobel Prize for Physics, 1957



C.S. Wu, 1912-1997; Chinese-American citizen.

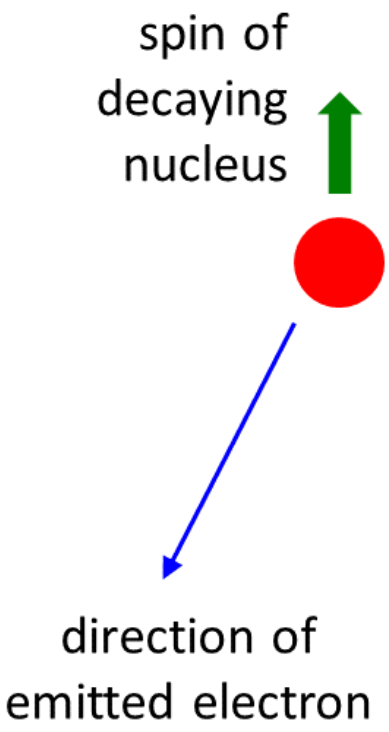
Parity violation in ^{60}Co decay

Wu's experiment studied the decay

$$^{60}_{27}\mathrm{Co} \rightarrow {}^{60}_{28}\mathrm{Ni} + e^- + \bar{\nu}_e$$

The ^{60}Co was placed in a magnetic field and kept cold (0.01^o K). The nuclei then became polarised such that the majority had their spin vector aligned with the applied field.

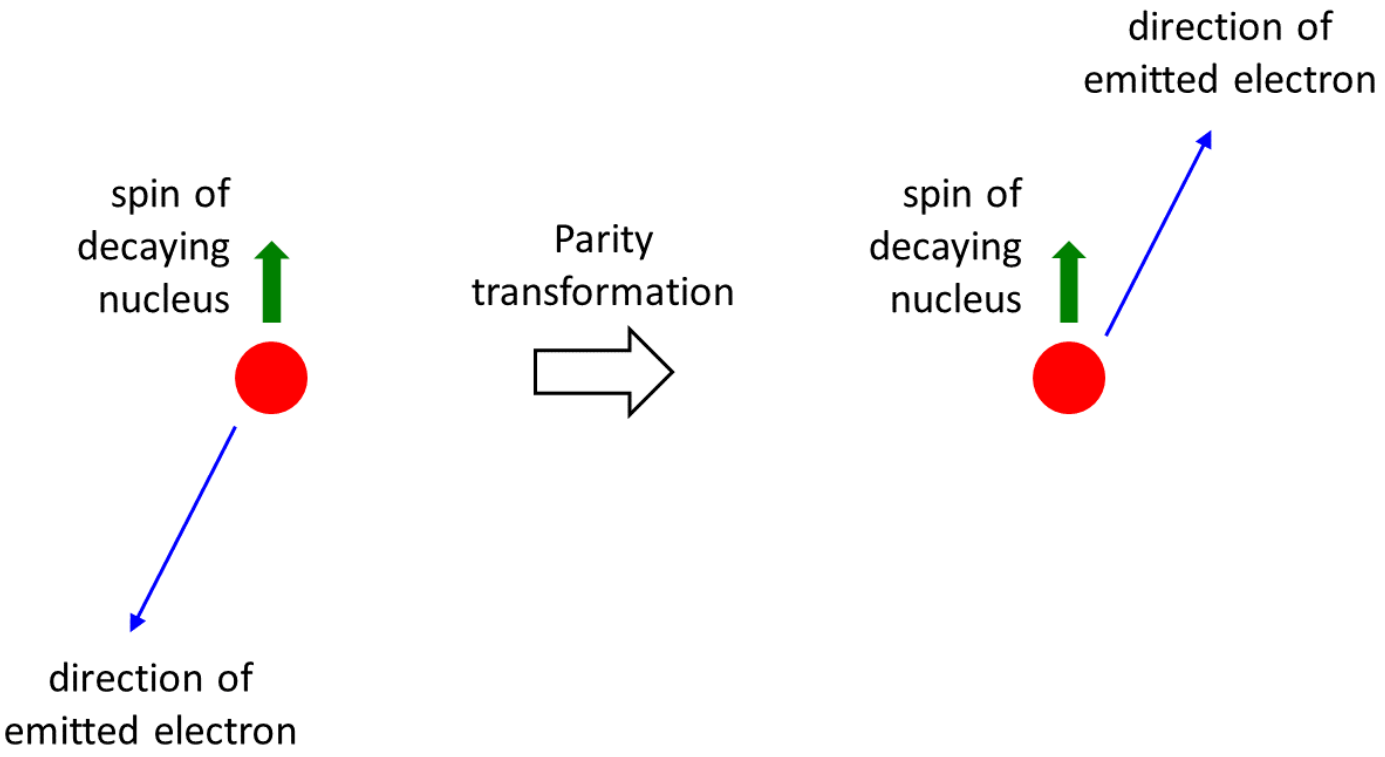
The direction of the emitted electrons was recorded. It was found that there was a clear preference for the electrons to be emitted in the 'southern' hemisphere as defined by the nucleus spin direction.



This observation indicated that the weak interaction does not conserve parity !

Parity violation in ^{60}Co decay

Under a parity transformation the electrum momentum vector changes direction ($\vec{p} \rightarrow -\vec{p}$), however the direction of spin remains unchanged, as angular momentum is an axial vector.



The resulting configuration corresponds to a situation that is different from what is observed (when averaged over many decays) in the real world. Parity has been violated !

Parity violation in ^{60}Co decay: a closer look

The electron intensity distribution was studied as a function of velocity. It was found:

$$I(\theta) \propto 1 \; - \frac{v}{c} \cos\theta$$

Hence the parity violation becomes maximal when $v \to c$ and vanishes when $v \to 0$.

The same expressing can be written

$$I(\theta) \propto 1 \; - \frac{\vec{\sigma} \cdot \vec{p} c}{E}$$

where $\vec{\sigma}$ is a unit spin vector in the direction of the ^{60}Co spin (define as *z*-axis). We note that this expression is parity violating since under a parity transformation $\vec{\sigma} \to \vec{\sigma}$ and $\vec{p} \to -\vec{p}$, hence $\vec{\sigma} \cdot \vec{p} \to -\vec{\sigma} \cdot \vec{p}$ (a pseudoscalar), hence $I(\theta)$ is not invariant.

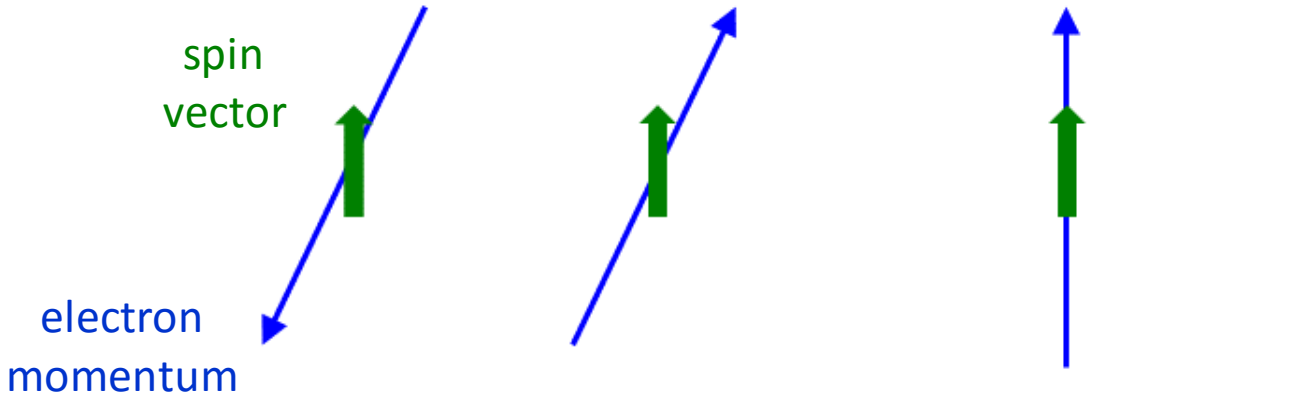
The change in nuclear spins is $\Delta J = -1$ in *z* direction, so angular-momentum conservation requires that both the electron and anti-neutrino have their spins aligned in this direction. Thus the intensity distribution can be interpreted as arising from a correlation between the spin direction, momentum direction and velocity of the fermions emitted in β decay.

Helicity and the weak interaction

This configuration is more likely…

…than this configuration…

…and in the limit $v \to c$ this configuration does not occur !



The net longitudinal polarisation is defined as *helicity*. In the charged weak interaction:

$$H = -\frac{v}{c} \quad \text{for fermions,} \quad H = +\frac{v}{c} \quad \text{for anti-fermions}$$

where a $-$ $(+)$ sign indicates spin anti-parallel (parallel) to momentum

In the case that $H = -1$ (1) the particle is said to be *left handed* (*right handed*). The anti-neutrino is right handed and travels in the direction of J. The electron 'prefers' negative helicity values and so will more often travel opposite to the anti-neutrino.

P, CP and T violation in the weak interaction

Not on syllabus !

The discovery of parity violation in the weak interaction came as a great surprise.

Pauli: "I do not believe that the Lord is a weak left-hander, and I am ready to bet a high sum that the experiment will give a symmetrical distribution."

Isador Rabi:

"A rather complete theoretical structure has been shattered at the base and we are not sure how the pieces will be put together."

L.D. Landau:

"Space *cannot* be asymmetric"

Feynman: lost $50 bet that parity would be conserved

Parity violation in the weak interaction is not a small, accidental effect, but is a central property ! A theoretical description, *V-A theory*, was developed by Marshak, Sudarshan, Feynman and Gell-Mann in 1957, which later was subsumed into electroweak theory.

In addition to P (parity), there also exist the discrete transformations of:

C - charge conjugation, in which the signs of all charges (and all additive quantum numbers) are swapped

T - time reversal

It was also assumed that CP would be a good symmetry. But in 1964 this too was found to be violated in the weak interaction ! (Cronin & Fitch, 1980 Nobel Prize). Indeed CP violation is an important requirement for generating the universe we live in, although the CP violation found in the weak interaction is inadequate to explain this. Other sources must exist !

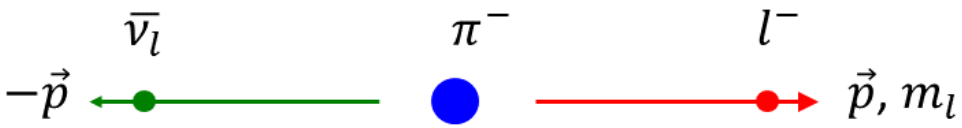
If CPT is conversed, and CP is violated, then T is also violated, which is also observed.

Parity violation and decay rates

Parity violation can have important consequences for decay rates. Consider the decays

$$\pi^- \to \mu^- \overline{\nu_\mu} \qquad\qquad \pi^- \to e^- \overline{\nu_e}$$

The decay rate $\Gamma_{\pi\to l} \sim G_F p^2 \frac{\mathrm{d}p}{\mathrm{d}E}$, with p the lepton momentum and E the available energy.



Evaluating the density of states factor implies that

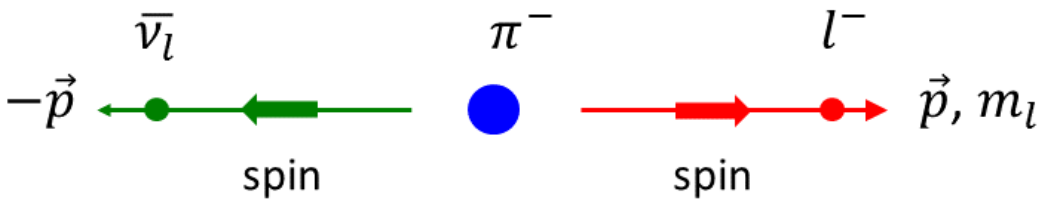
$$\frac{\Gamma_{\pi\to e\bar{\nu}}}{\Gamma_{\pi\to\mu\bar{\nu}}} = \frac{\left(m_\pi^2 - m_e^2\right)^2 \left(m_\pi^2 + m_e^2\right)}{\left(m_\pi^2 - m_\mu^2\right)^2 \left(m_\pi^2 + m_\mu^2\right)} \sim 3.5$$

$m_\pi = 139.6$ MeV
$m_\mu = 105.7$ MeV
$m_e = 0.51$ MeV

with the electron decay dominating because of the much larger phase space.
However, the measured ratio is 1.3×10^{-4} ! What's going on ?

Parity violation and decay rates

Recall the pion has zero spin. Also recall that the anti-neutrino must be right handed. Therefore, to conserve angular momentum, the charged lepton must be right handed also.



This is an unnatural helicity configuration. The probability goes as $1-\frac{v}{c}$, where

$$1-\frac{v}{c}=\frac{2m_l^2}{(m_\pi^2+m_l^2)}$$

So including this factor gives

$$\frac{\Gamma_{\pi\to e\bar{\nu}}}{\Gamma_{\pi\to\mu\bar{\nu}}}=\frac{m_e^2(1-(m_e/m_\pi)^2)^2}{m_\mu^2\left(1-\left(m_\mu/m_\pi\right)^2\right)^2}\sim 1.3\times 10^{-4}$$

In perfect agreement with measurement ! We say these modes are *helicity suppressed*. The electron decay is much more suppressed because it is ultra-relativistic and therefore much less likely to be found in the unnatural helicity state.

Particle physics: topic overview

Leptons and the weak interaction

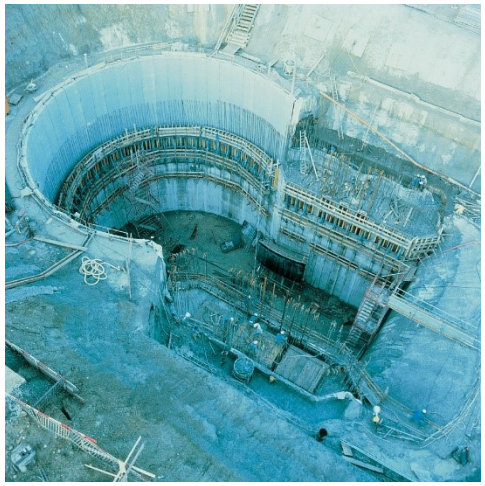
- Lepton overview & conservation laws
- Fermi theory revisited: lepton decays & neutrino scattering
- Electroweak theory
- Cabibbo mixing
- Parity violation
- Discovery & properties of the *W* & *Z*

The hunt for the W and Z bosons

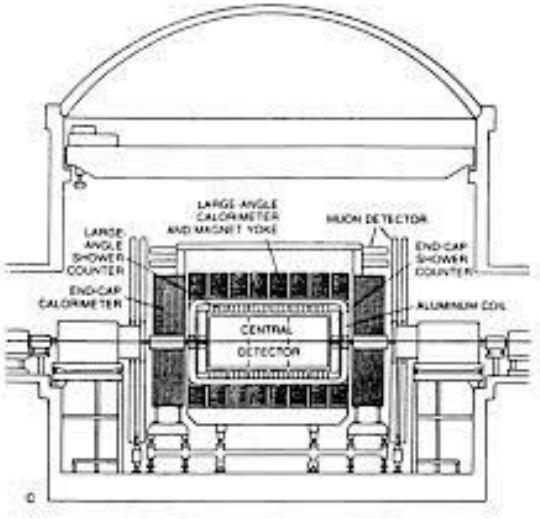
Full acceptance of electroweak theory required that the *W* and *Z* bosons be observed as real, on-shell particles, and their properties studied and compared to prediction.

In the 1980s no accelerator had sufficient energy or luminosity to make these particles. In 1976 Carlo Rubbia *et al.* had proposed converting the then new CERN SPS (super proton synchrotron) into a proton-antiproton ($p\bar{p}$) collider (S$p\bar{p}$S) with collision energy 540 GeV. The accelerator physicist Simon van der Meer developed the scheme of *stochastic cooling* that kept the bunches sufficiently dense for the necessary luminosities to be reached.

Two large experiments were built for the *W* and *Z* search: UA1 (`Underground Area One'), led by Rubbia, and UA2 under Pierre Darruilat, and began data taking in 1981.



Excavation of UA1 pit



Schematic of UA1



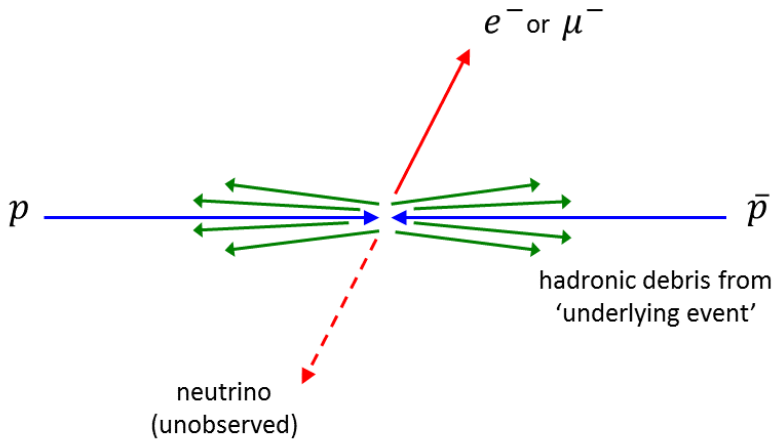
Carlo Rubbia, 1934-; Italian citizen; Nobel Prize for Physics, 1984

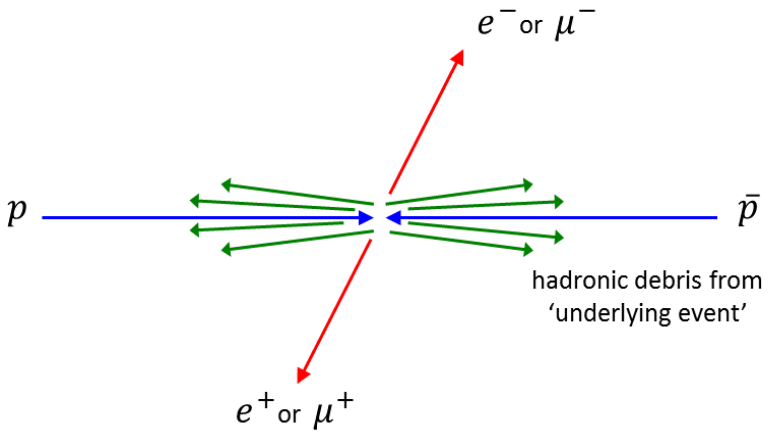


Simon van der Meer, 1925-2011; Dutch citizen; Nobel Prize for Physics, 1984

The hunt for the W and Z bosons

The bosons were searched for in the decay modes where background would be least. This meant looking for leptonic decays, as decays with hadrons suffer from background from other QCD processes, as well as the debris produced from the rest of the collision.





The high mass of the bosons (80.4 GeV for the *W*, 91.2 GeV for the *Z*) means that is the decay direction happens to be at an angle to the beamline then these leptons will have high 'transverse momentum / energy' (E_T), which provides a distinctive signature. In *W* decays there will also be 'missing transverse momentum / energy' (missing E_T).

The discovery of the *W* was announced in 1983, and the *Z* one year later.

W discovery

Volume 122B, number 1 PHYSICS LETTERS 24 February 1983

EXPERIMENTAL OBSERVATION OF ISOLATED LARGE TRANSVERSE ENERGY ELECTRONS WITH ASSOCIATED MISSING ENERGY AT $\sqrt{s}$ = 540 GeV

UA1 Collaboration, CERN, Geneva, Switzerland

Received 23 January 1983

We report the results of two searches made on data recorded at the CERN SPS Proton–Antiproton Collider: one for isolated large-E_T electrons, the other for large-E_T neutrinos using the technique of missing transverse energy. Both searches converge to the same events, which have the signature of a two-body decay of a particle of mass ~ 80 GeV/c^2. The topology as well as the number of events fits well the hypothesis that they are produced by the process $\bar{p} + p \rightarrow W^{\pm} + X$, with $W^{\pm} \rightarrow e^{\pm} + \nu$; where $W^{\pm}$ is the Intermediate Vector Boson postulated by the unified theory of weak and electromagnetic interactions.

Volume 122B, number 5,6 PHYSICS LETTERS 17 March 1983

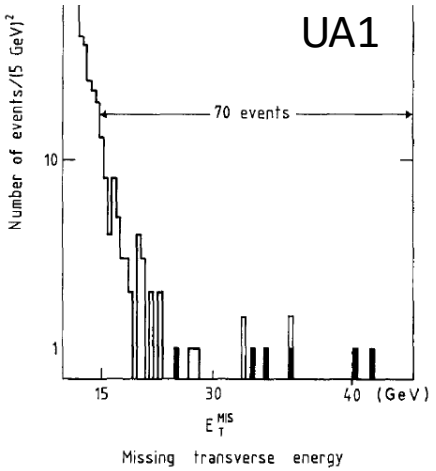
OBSERVATION OF SINGLE ISOLATED ELECTRONS OF HIGH TRANSVERSE MOMENTUM IN EVENTS WITH MISSING TRANSVERSE ENERGY AT THE CERN $\bar{p}p$ COLLIDER

The UA2 Collaboration

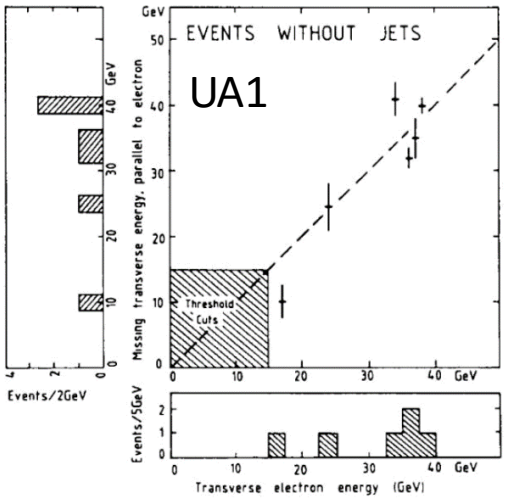
Received 15 February 1983

We report the results of a search for single isolated electrons of high transverse momentum at the CERN $\bar{p}p$ collider. Above 15 GeV/c, four events are found having large missing transverse energy along a direction opposite in azimuth to that of the high-p_T electron. Both the configuration of the events and their number are consistent with the expectations from the process $\bar{p} + p \rightarrow W^{\pm} +$ anything, with $W^{\pm} \rightarrow e^{\pm} + \nu$, where $W^{\pm}$ is the charged Intermediate Vector Boson postulated by the unified electroweak theory.

Events with high missing E_T far away from background distribution; shaded ones also contain electrons.

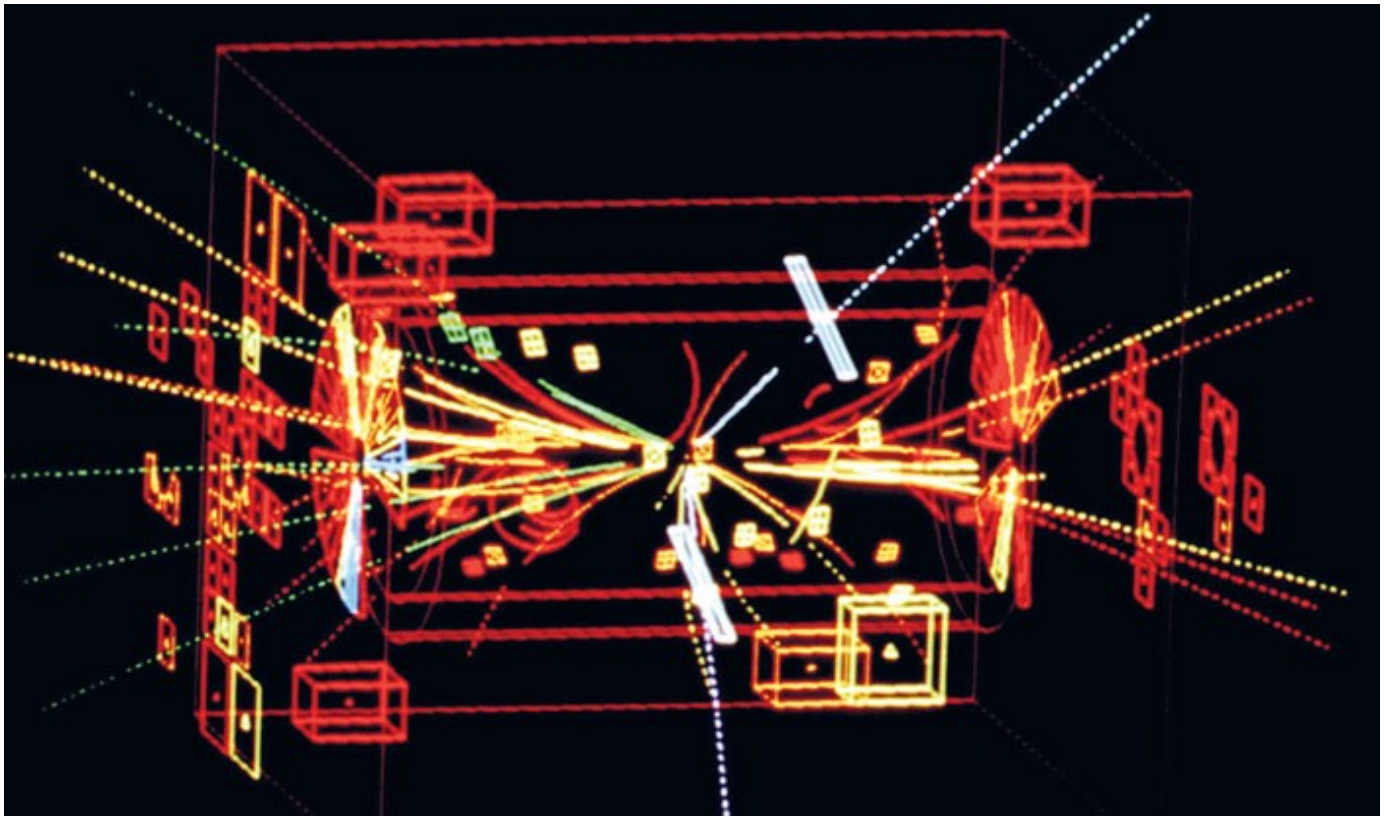


Correlation between electron E_T and missing E_T.



Z discovery

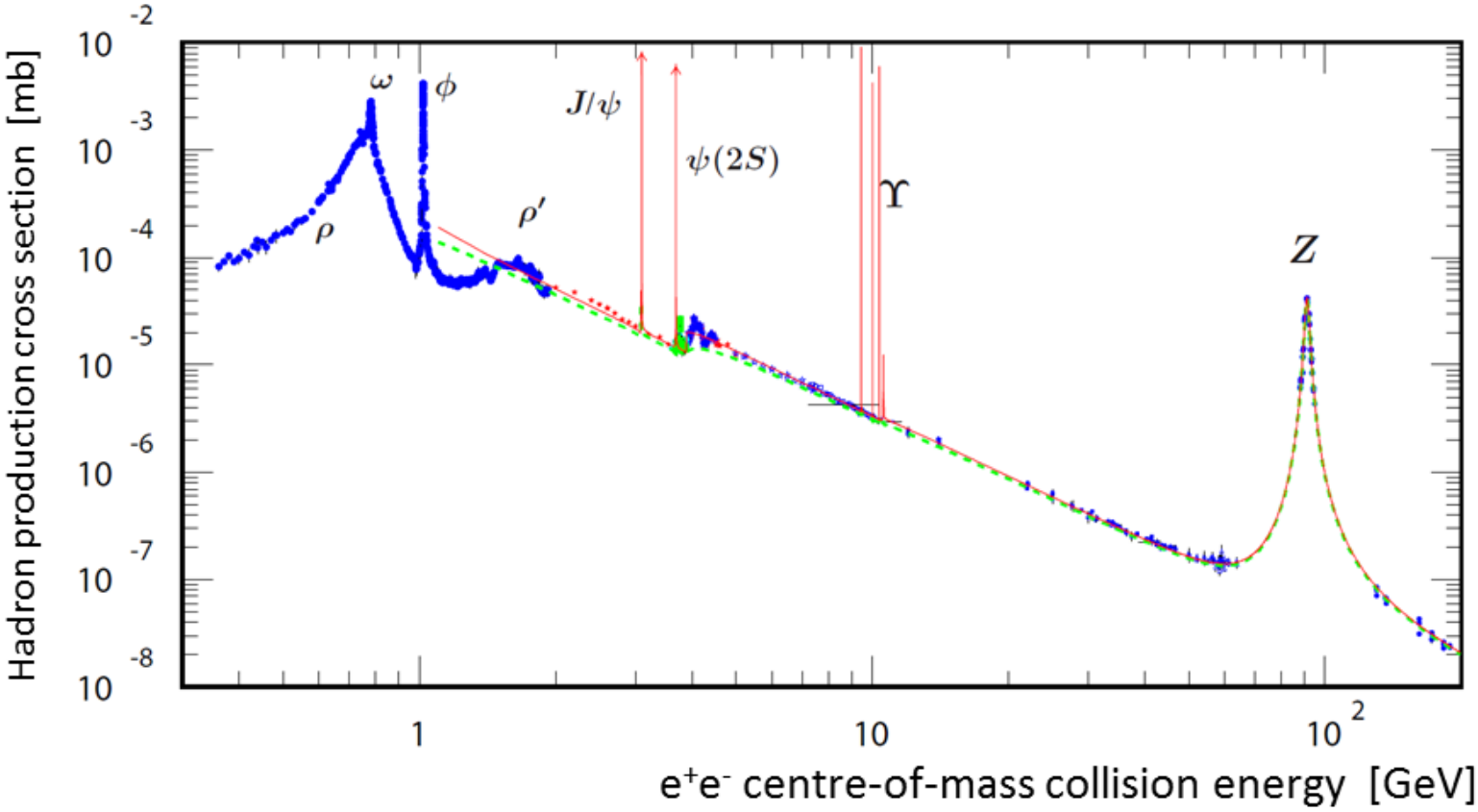
UA1 candidate Z event seen in 1983. The two white tracks are the electron-positron pair. (The Z decay has an even more distinctive signature, but the production rate is lower.)



The observed masses of the W and Z were consistent with the predictions of electroweak theory, made using inputs from the earlier measurements of the neutrino-scattering data.

Precise studies of the Z and W in e^+e^- collisions

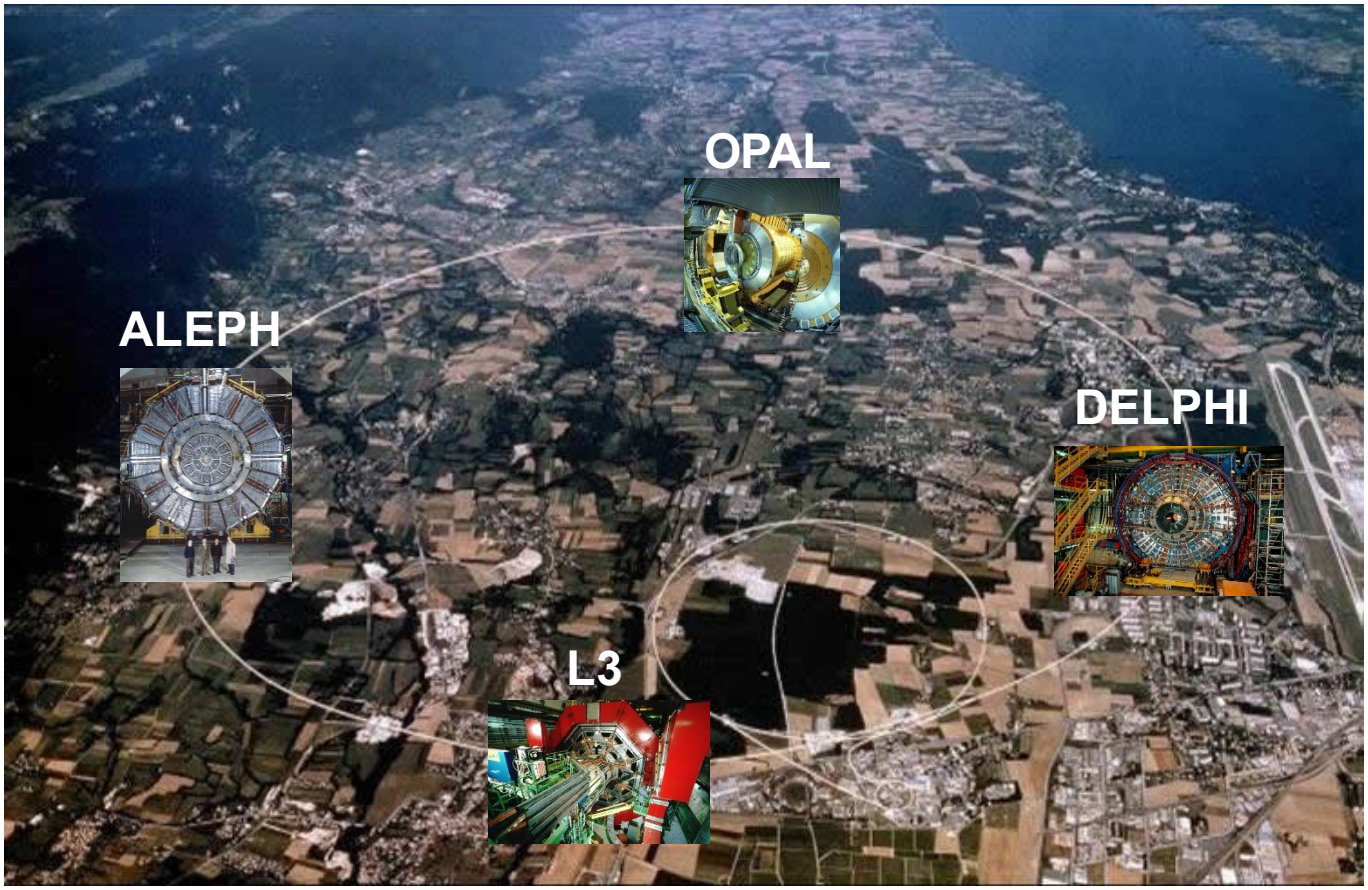
An e^+e^- collider is better suited to making precise studies than a hadron collider because there are fewer background processes and the colliding particles are fundamental (unlike the proton), which allows E_{CM} to be precisely known.



Furthermore the Z^0 manifests itself as a resonance with high cross section.

LEP: the Large Electron Positron collider

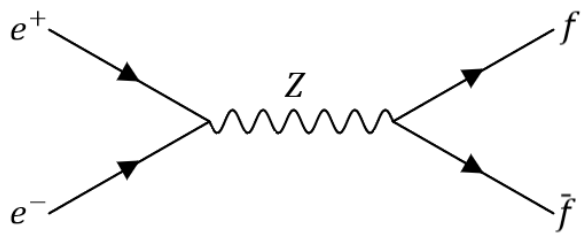
LEP operated in the same 27 km circumference tunnel currently used by the LHC and housed four experiments. Data taking took place between 1989 and 2000.



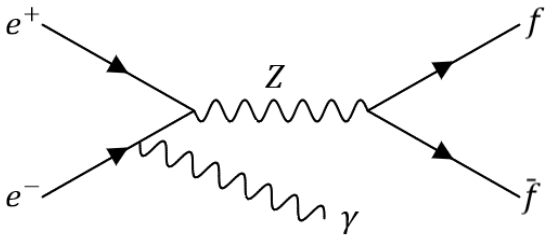
The large size is motivated by the fact that high energy electrons lose energy by synchrotron radiation, where the power loss $P \propto E^4/(m^4 r^2)$, where r = radius.

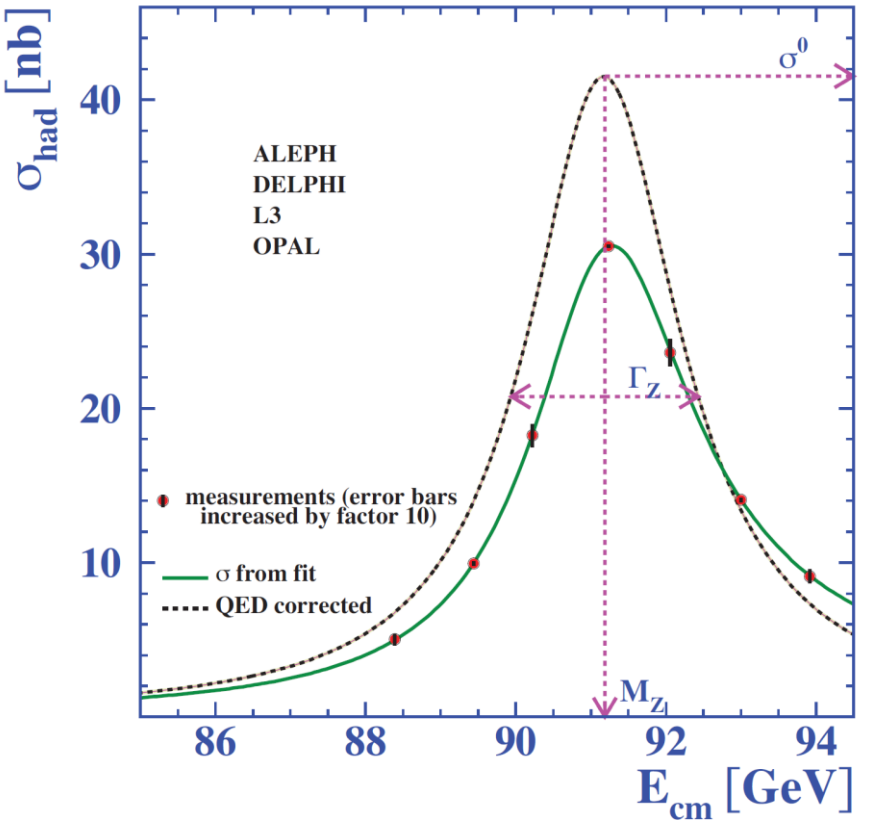
The Z^0 resonance

At $E_{\mathrm{CM}} \approx 91$ GeV the production of $f\bar{f}$ pairs is dominated by on-shell Z^0 exchange:



The underlying resonance follows a Breit-Wigner curve, but is distorted by initial-state bremsstrahlung.

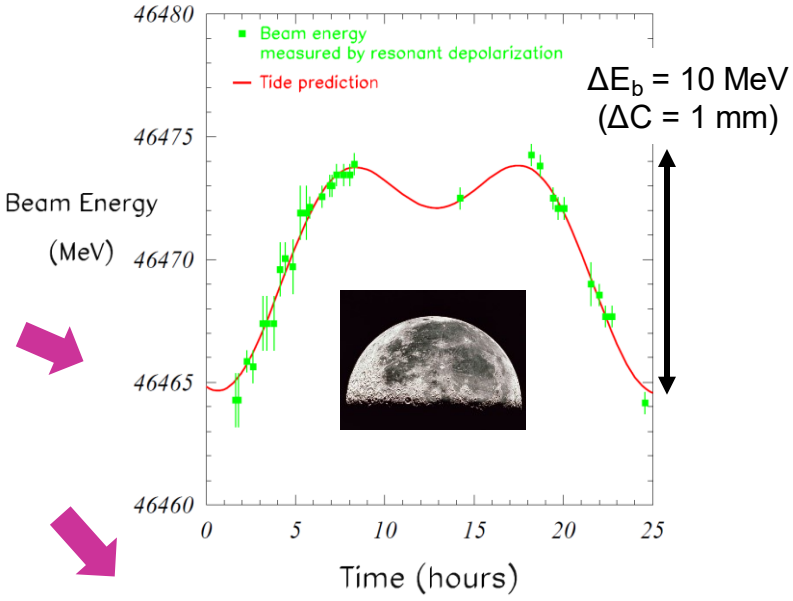




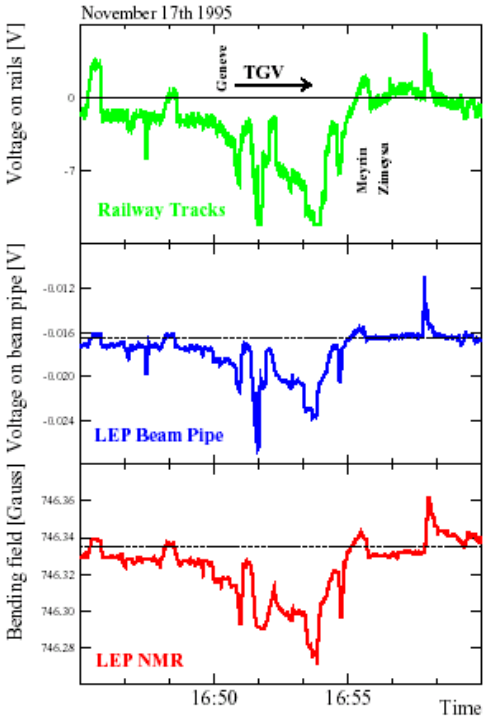
LEP collected 17 million Z^0 decays, which meant that high systematic control was required, for example of the value of E_{CM}, which sets the horizontal scale of the plot.

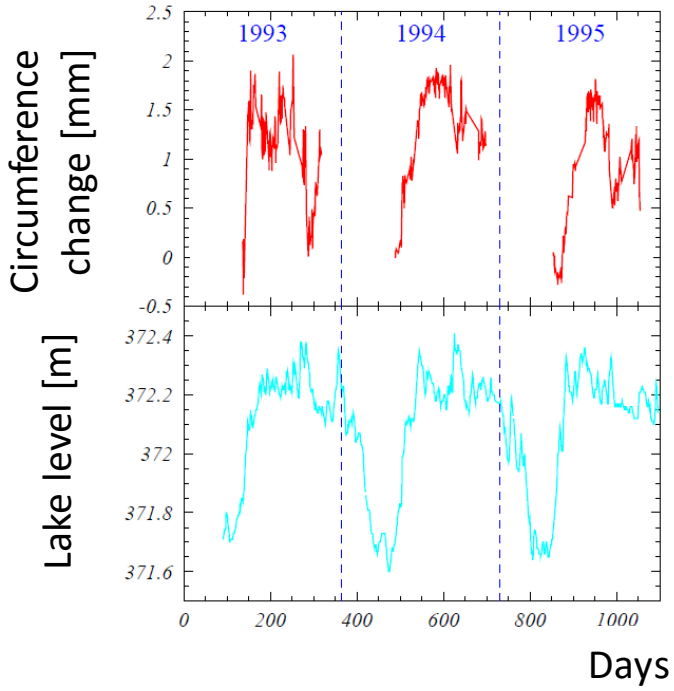
Understanding E_{CM}

The ring shape has short-term distortions driven by 'earth tides', caused by moon, and in the longer term by the water level in Lake Leman. This changes the e^+e^- trajectories in the quadrupoles, and leads to changes in the beam energy.



Other effects include currents from nearby electric trains (TGV) leaking on to the beam pipe of the accelerator and stimulating the dipole magnets to climb up their hysteresis curves.





Understanding these effects allowed M_Z to be measured to 2×10^{-5}, and Γ_Z to be measured to 1×10^{-3}.

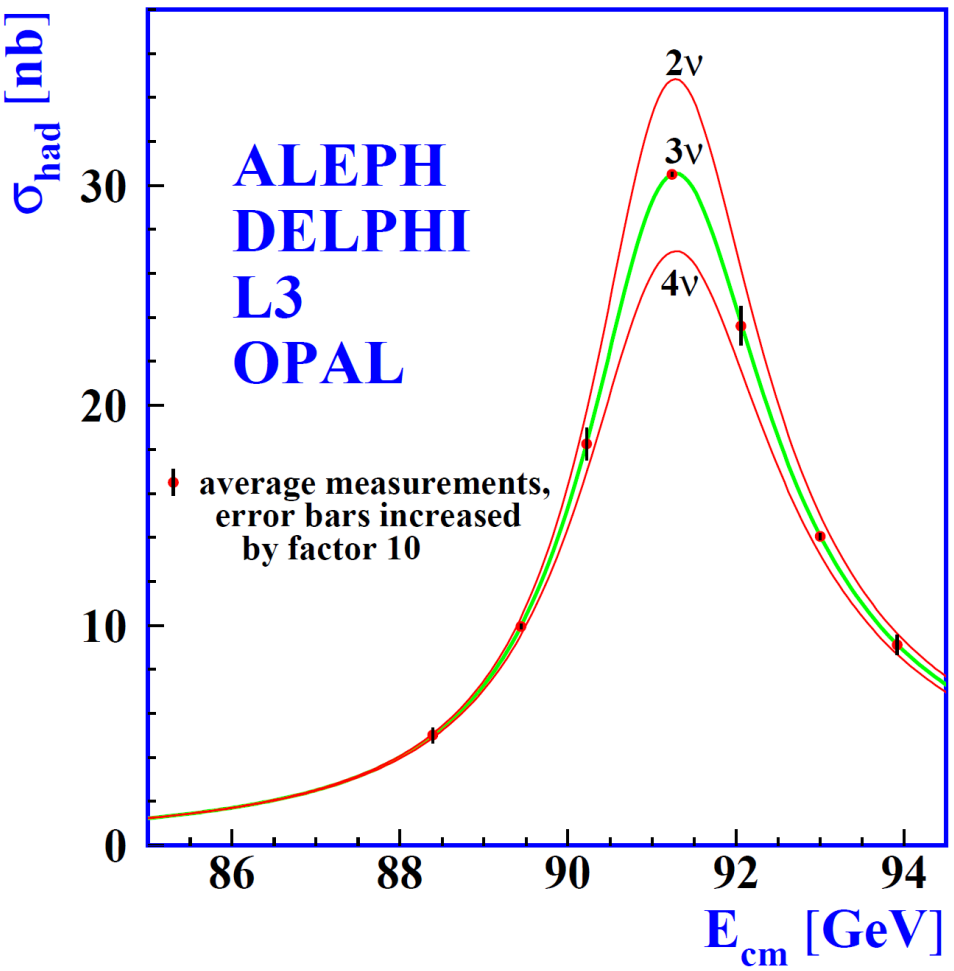
Number of neutrino generations

Many measurements were made of the properties of the Z^0 that allowed for precise comparisons with Standard Model predictions.

One important study was the determination of the number of neutrino generations (and therefore, presumably, generations of matter).

Neutrino production is not observed, but its presence is there in the shape of the lineshape, and so the number of generations can be inferred (see problem sheet). The final result is:

$$N_\nu = 2.9840 \pm 0.0082$$

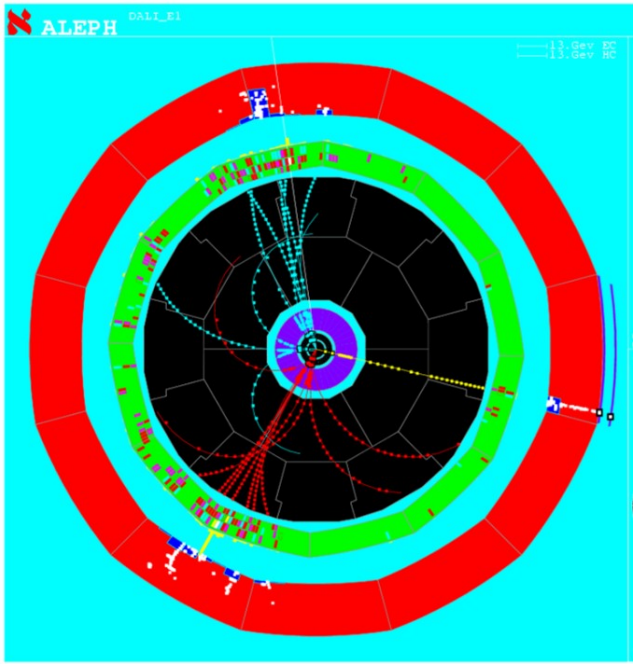


i.e. nature finds three to be a magic number !

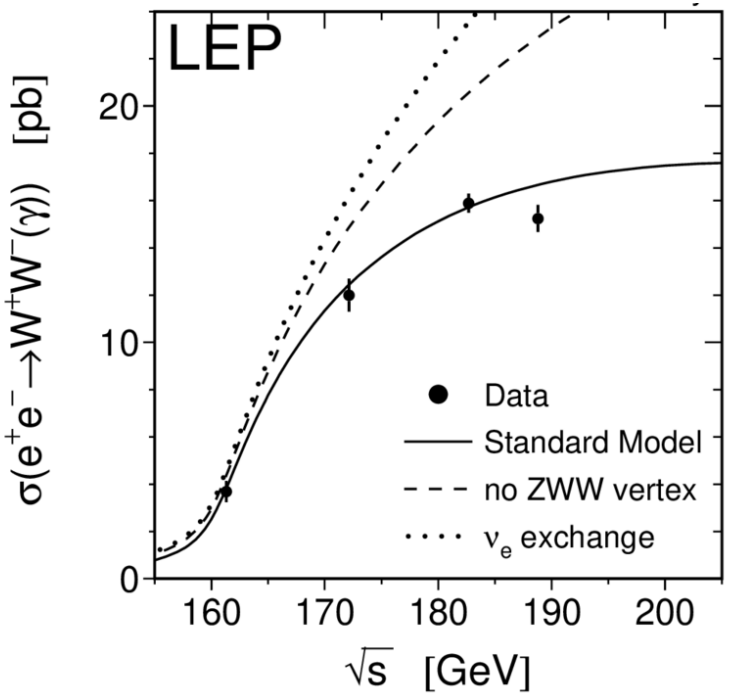
$e^+e^- \rightarrow W^+W^-$

W bosons are produced in pairs in e^+e^- collisions. This means that there is no resonance. At LEP many studies were performed of the production and decay properties and W mass.

W^+W^- event containing two jets and one muon.



Measured $e^+e^- \rightarrow W^+W^-$ cross section as a function of collision energy ($\sqrt{s}$).



The production process is the sum of three amplitudes (see problem sheet). Interestingly, these interfere destructively so that $|\sum M_{if}|^2 < \sum |M_{if}|^2$.

Particle physics: final words

We have completed our tour of the subatomic B4 syllabus. There are many particle physics topics of importance, but not in the syllabus, that were not touched upon, including:

- The Higgs mechanism;
- Neutrino oscillations and properties;
- *CP* violation;
- The search for dark matter;
- The search for new heavy particles at the energy frontier;
- The study of flavour-changing neutral currents in heavy quark decays;
- The principles and techniques of particle detection.

Unanswered questions in the Standard Model

The Standard Model was fully formulated by the 1970s. Since then the existence of its remaining particle contents have been successively verified, culminating with the discovery of the Higgs boson in 2012. It has withstood all experimental tests in the laboratory (mostly at colliders), albeit with a few outstanding anomalies, which may or may not persist.

However, there are many problems and unanswered questions.



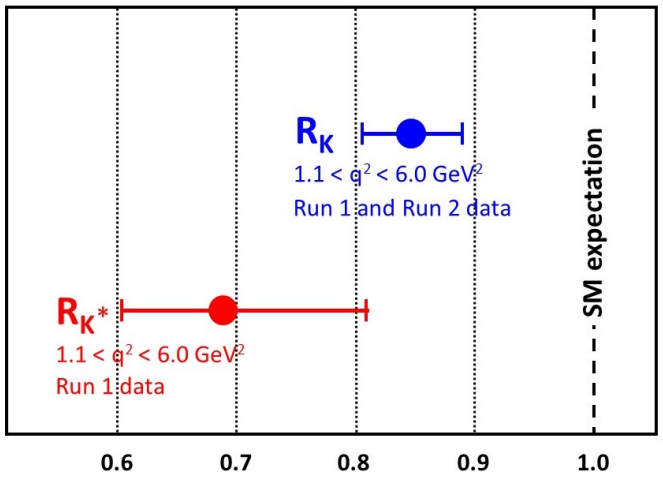
- The Higgs is too light;
- Too many free parameters;
- Striking and unexplained patterns in the flavour sector;
- No explanation for neutrino mass.
- No candidate for a dark matter particle;
- It does not include gravity;
- It does not explain the baryon-antibaryon asymmetry in the universe (*i.e.* why the universe is overwhelming made of matter, not anti-matter);
- No explanation for dark energy.

Progress on these questions are almost certainly contingent on experimental progress. These require more precise experiments, and higher luminosity and energy colliders.

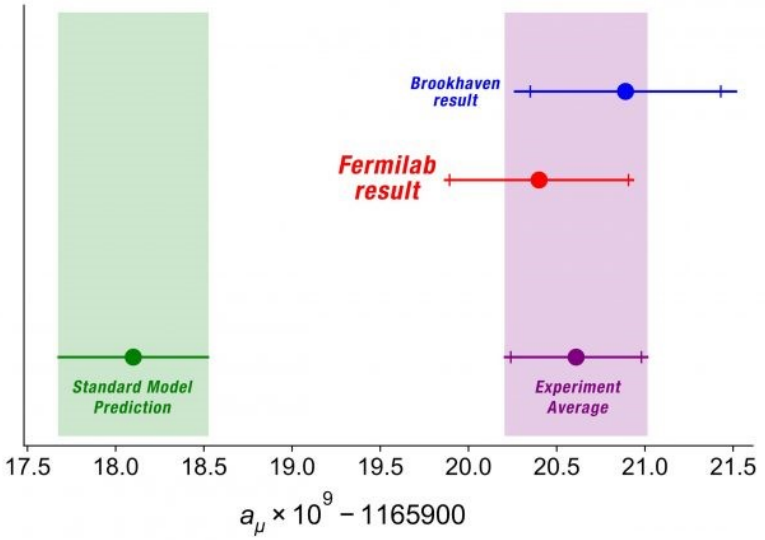
Intriguing hints

The search for evidence of physics beyond the Standard Model is entering an interesting phase. For instance, there are growing hints of violations of lepton universality.

Measurements of the ratios (R_K, R_{K^*}) of the branching ratio of processes such as $B \to K l^+ l^-$ and $B \to K^* l^+ l^-$ for muons and electrons seem to differ from unity.



Very precise measurements of the magnetic moment of the muon seem not to agree with Standard Model.

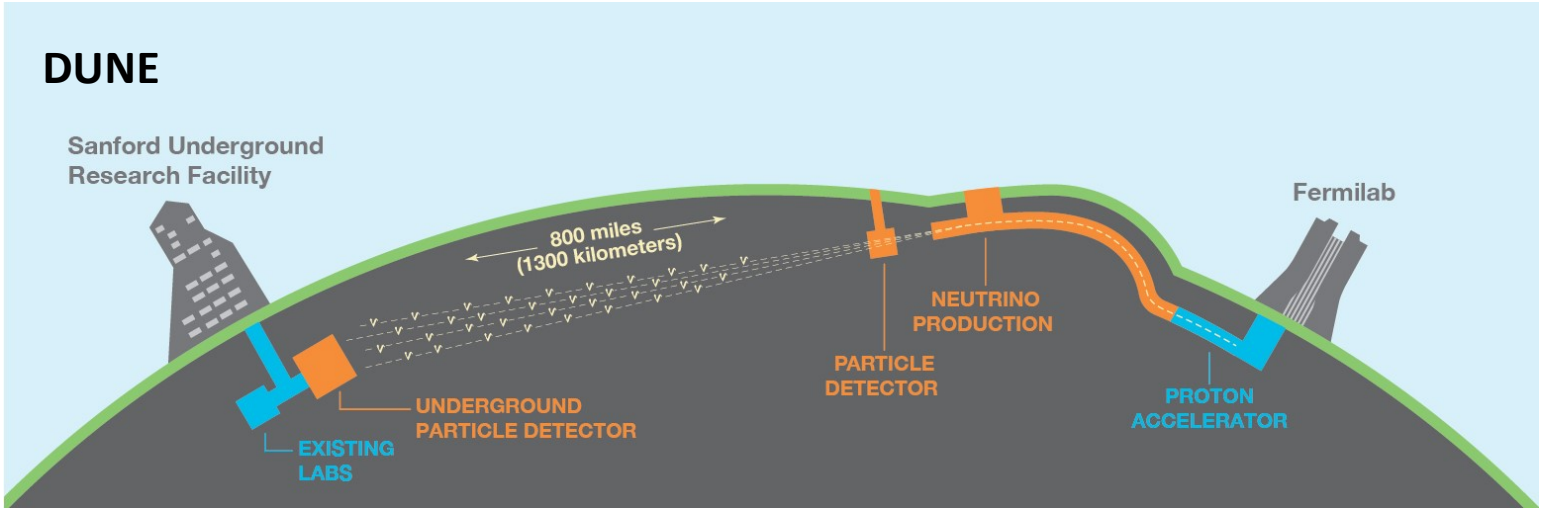


These anomalies are *not* yet at the level where we can claim a 'discovery', and may yet evaporate with more data. But we should know for certain rather soon, and these hints are important demonstrations of the need for ever-more precise measurements.

Near-term experimental prospects

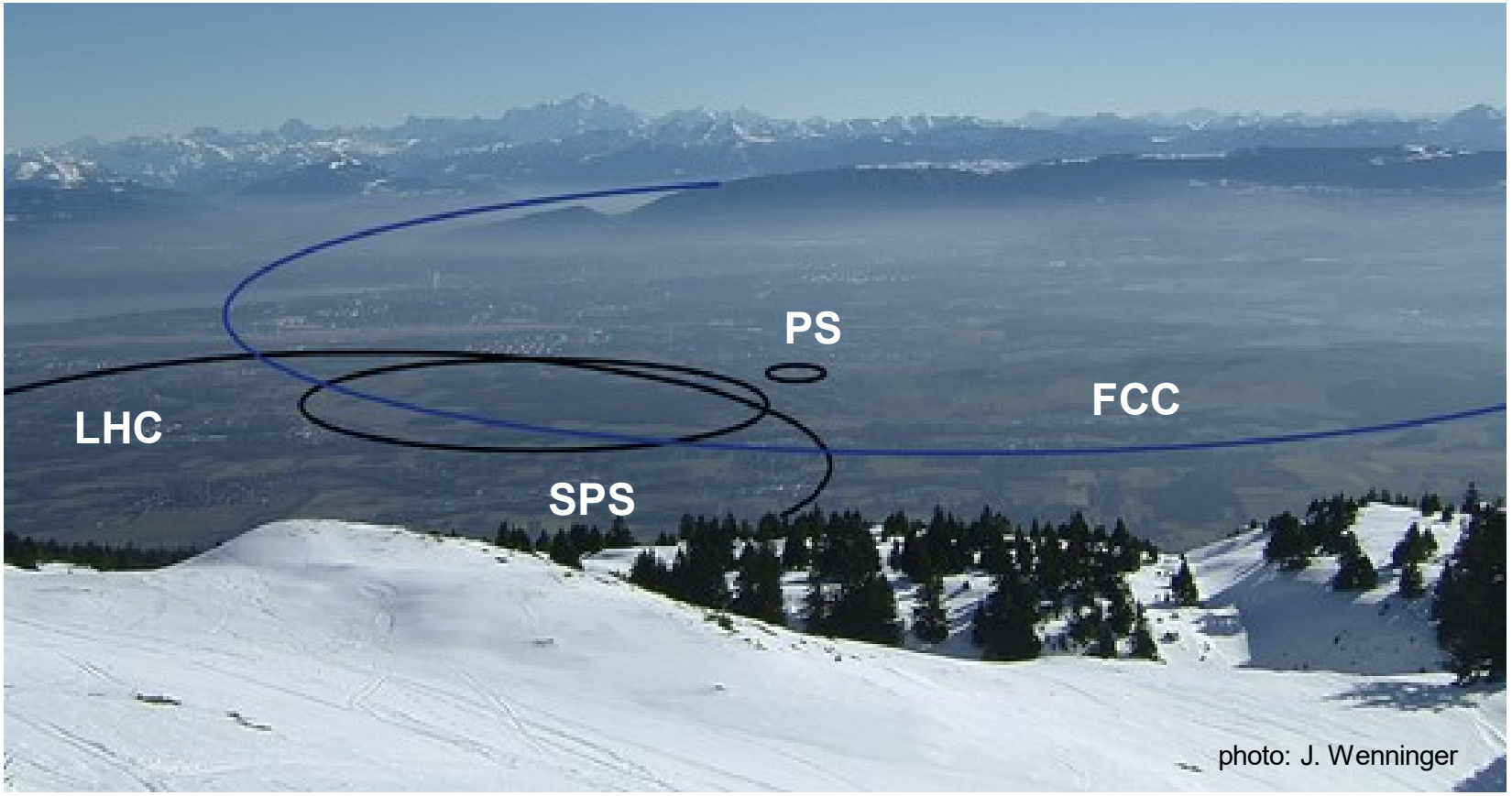
Over the coming years experimental prospects are very exciting:

- LHC is about to start taking data again ('Run 3') and is preparing for a new high-luminosity phase ('Runs 4 and 5'), which will stretch into late 2030s;
- Smaller experiments focused on super-precise measurements of particular observables (*e.g.* muon magnetic moment) will continue to take data and improve;
- In the next ~ 5 years two new massive neutrino experiments will begin in US (DUNE) and Japan (Hyper-Kamiokande), that will greatly increase knowledge of this sector.



The longer term

Various plans exist for future colliders for the era beyond the LHC. The most ambitious is the Future Circular Collider (FCC) at CERN, which will be > 90 km in circumference, and have a first incarnation as an e^+e^- machine, and then a second as a 100 TeV pp collider.



This science needs you !

Sign up for the C4 fourth-year option and think hard about enrolling for a DPhil on these highly important projects.

Thanks for your attention, and Merry Christmas and a Happy New Year to all !

