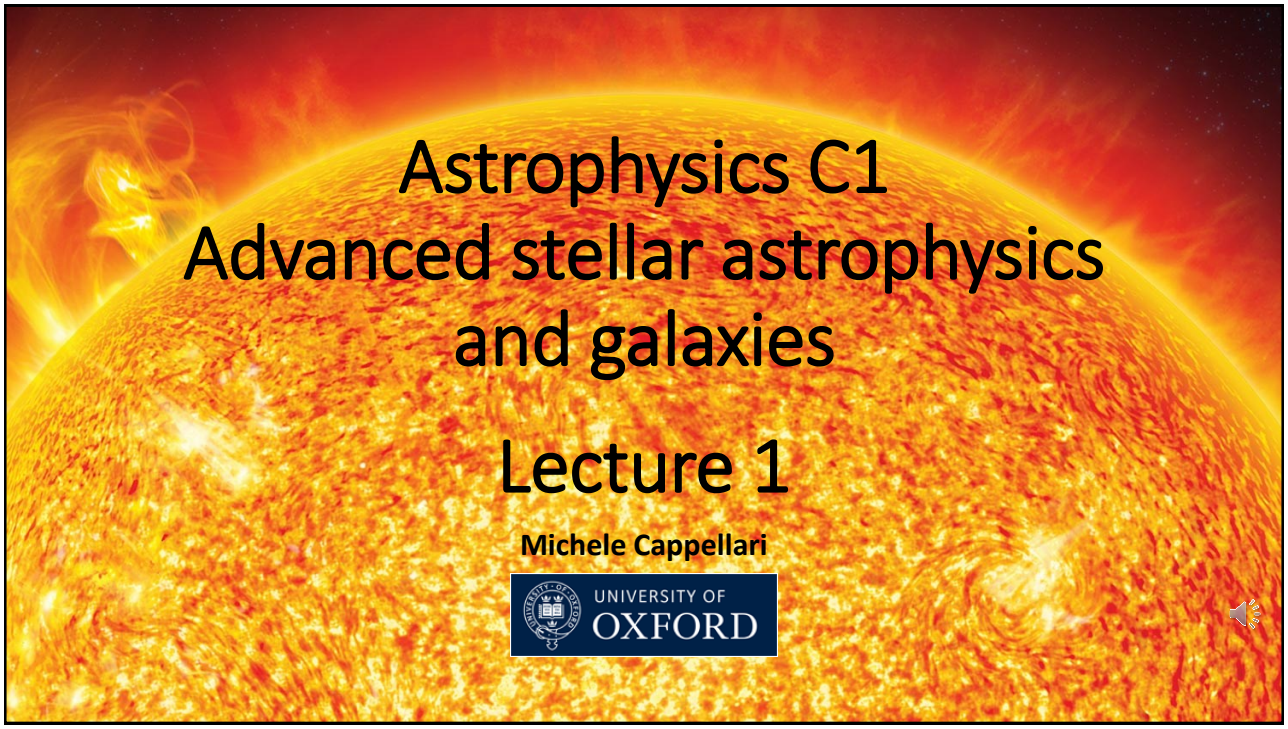



1

## Outline of the lectures

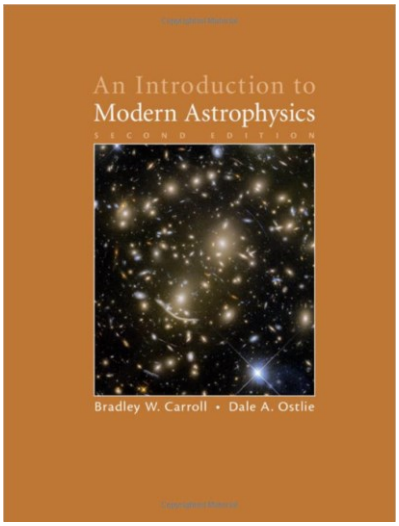
- Part I: stars
  - What can we observe
  - Stellar equations
  - Stellar evolution
  - Composite stellar population
- Part II: galaxies
  - What can we learn from photometry
  - Galactic dynamics
  - Stellar kinematics
  - Scaling relations of galaxies
  - Galaxy evolution

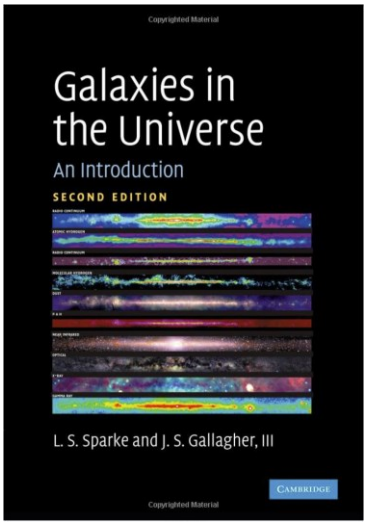


Credit: NASA, ESA, and the Hubble Heritage Team (and B. Whitmore et al.)

2

# Suggested books




They both cover much more than I will be able to cover in this part of the course

3

# Part I
# Stellar structure and evolution

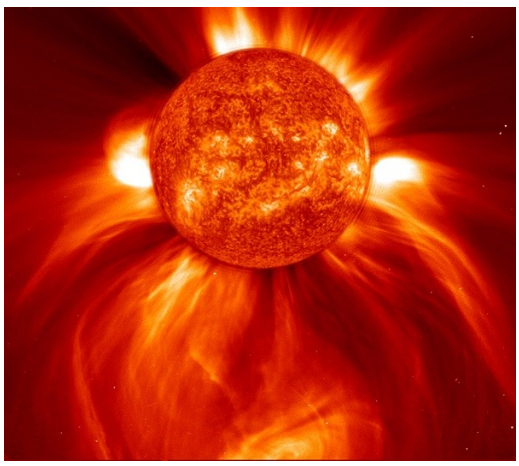
4

# Observable stellar parameters

5

## The Sun as a star

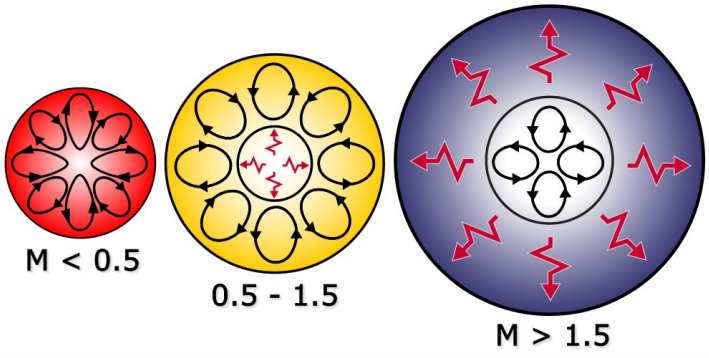
- We do not know any star as well as the Sun
- Crucial to our understanding of stars properties
- Mass: $\mathrm{M} = 2 \times 10^{30}$ kg
- Radius: $R = 7 \times 10^{8}$ m
- Luminosity: $L = 4 \times 10^{26}$ W
- Temperature: $T = 5800$ K



SOHO (NASA and ESA)

6

# The two defining properties of stars

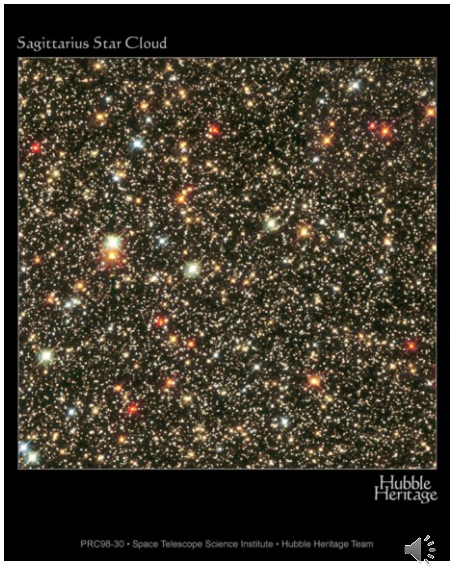



- 1 - Mass
- 2 - Chemical composition
- Just two properties define the structure and evolution of stars!
- These can be accurately computed using the laws of physics
- But chemical composition is the same in a region of space → mass is key

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# Intrinsic properties of stars

- Luminosity: key intrinsic property: Energy produced in the core
- Radius: size of the photosphere
- Surface temperature: determines shape of observed spectrum
- Chemical composition: drives stellar evolution




8

# Apparent properties of stars

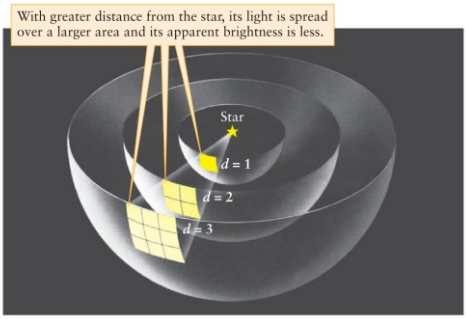
- Distance: needed to know absolute energy production

$$L_s = \int_0^\infty L_\lambda d\lambda = 4\pi R_s^2 \int_0^\infty F_\lambda d\lambda$$

- Astronomers measure at Earth surface

$$f_\lambda = \left(\frac{R_s}{D}\right)^2 F_\lambda$$

- To measure $L_\lambda$ we need to know D and correct for
  - Absorption of Earth atmosphere
  - Absorption in interstellar medium

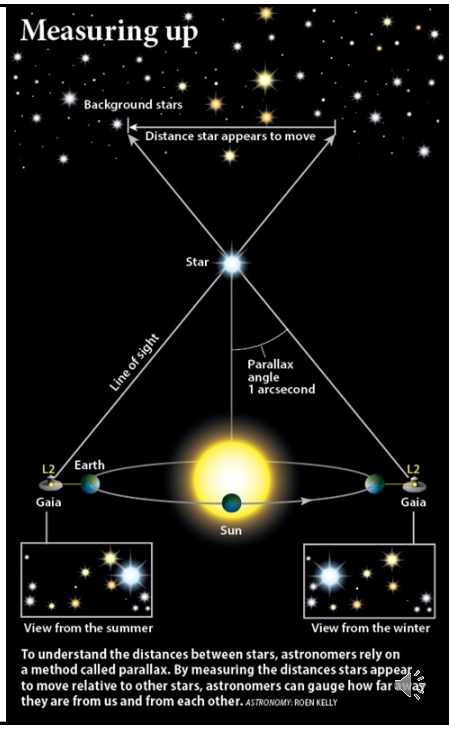



$$I \propto 1/D^2$$

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# Measuring stellar distances

- Most reliable method trigonometric parallax $p$
- Is apparent motion of nearby object with respect to distant ones
- Use Earth orbit as baseline over 6 months
- $\tan\theta \approx \theta \approx \frac{1\text{AU}}{D} \Rightarrow D = \frac{1\text{AU}}{\theta}$
- Def: Parsec is D when $p = 1$ arcsec
- $\text{pc} \approx 3 \times 10^{16}\text{m} \approx 3\ \text{lyr}$
- Distance in parsecs $D = 1/p$ with $p$ in arcsec




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# The Gaia distance-measuring spacecraft

- Sky positions, parallaxes and proper motions for 1.5 billion stars ($1.5 \times 10^9$)
- Radial velocities for a few $10^6$ stars
- Parallax error 10 μ arcsec = $10^{-5}$ arcsec
- 1 arcsec = 1/60/60=1/3600 degree
- Launched in 2013 and still ongoing
- Orbits around L2 Lagrange unstable equilibrium point
- Baseline planned for 5 years
- Currently extended to 2025

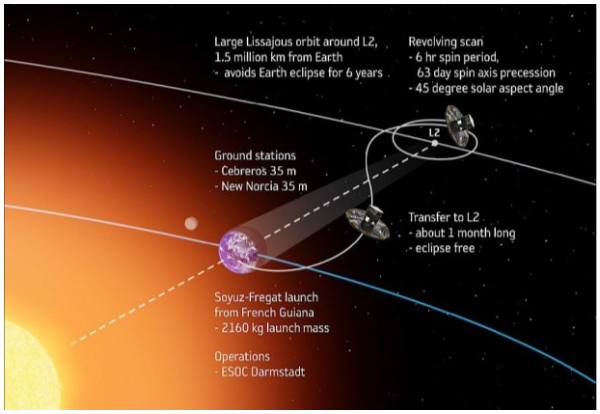
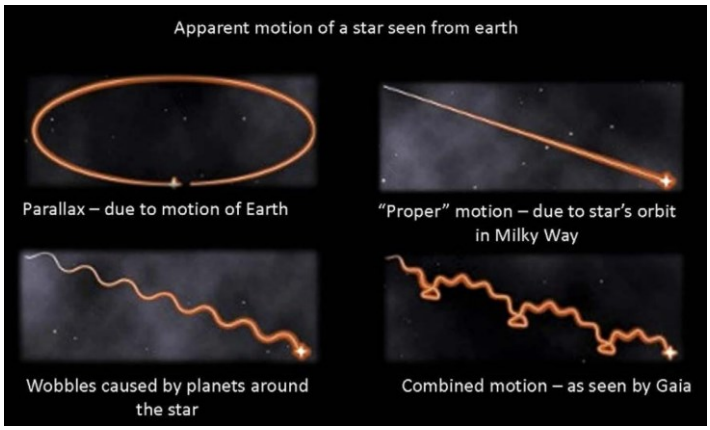



image credit: ESA

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# What Gaia actually observes

- Parallax not directly observable by itself
- One observes combination of proper motions and planet wobbles
- At discrete times
- Need time-dependent model to extract parallax

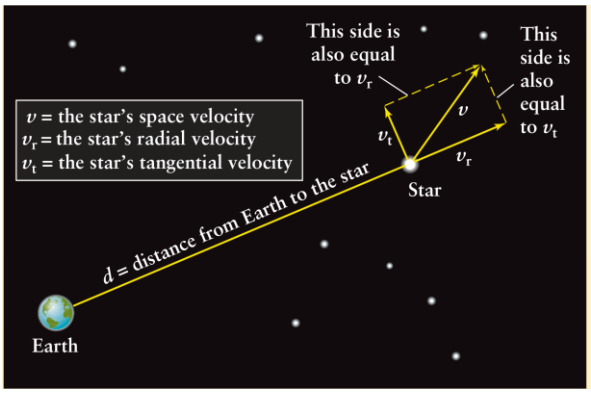



Gilmore 2018

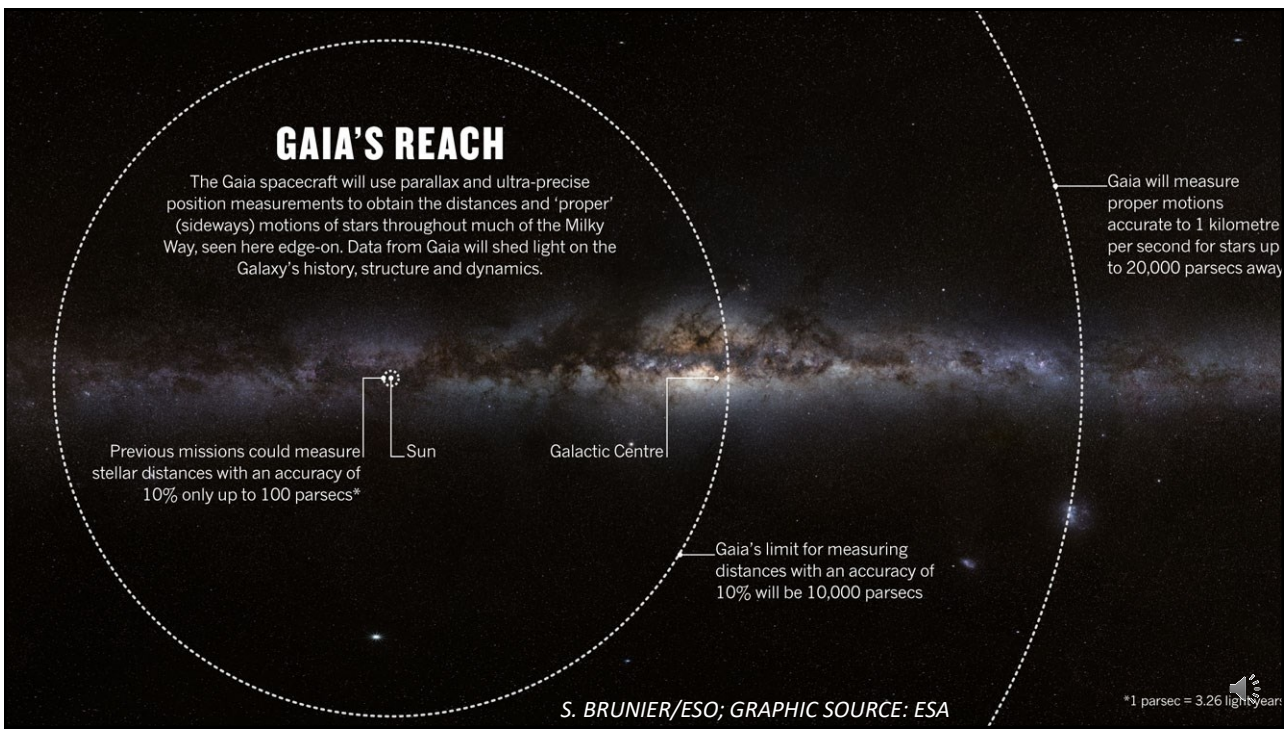
12

# Stellar proper motion

- For reference: circular velocity at Sun
- $v_{\odot c} = 236$ km/s
- 3D velocity $v = \sqrt{v_t^2 + v_r^2}$
- Obtain $v_t$ from proper motion $\mu$
- $v_t = 4.74\ \mu\ D$
- With $\mu$ in arcsec/yr and $D$ in parsecs
- $v_r$ via Doppler shift (for $v \ll c$)
- $\frac{v_r}{c} = \frac{\lambda - \lambda_0}{\lambda_0}$



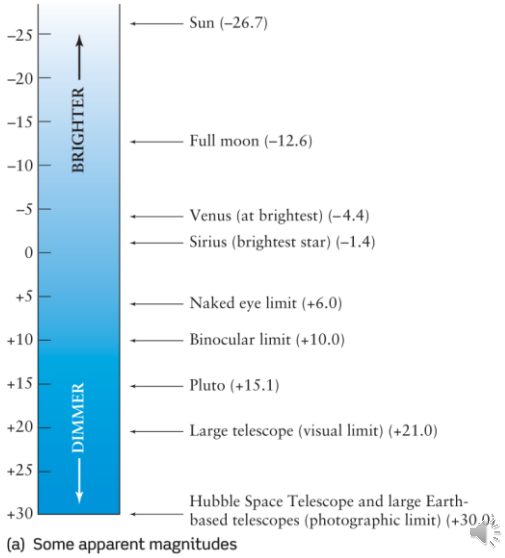

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# Apparent and absolute magnitudes (mag)

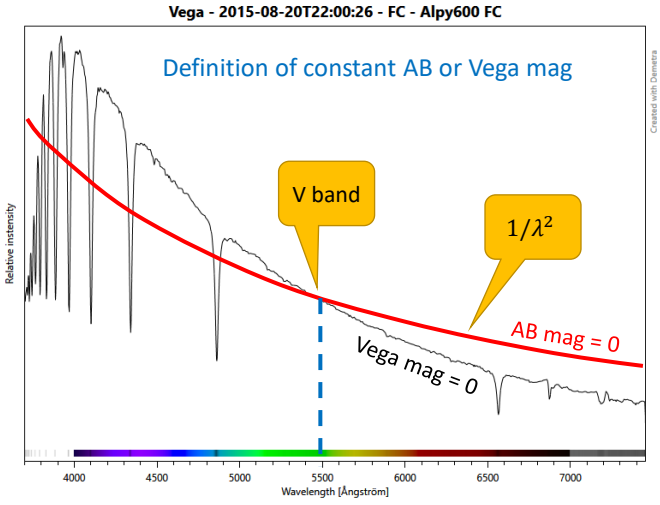
- Magnitude difference defined as
  $m_2 - m_1 = 2.5\log(b_1/b_2)$
- 5 mag differ by 100x
- Zero point defined using either
  - Standard stars in the Vega system
    - $m_{\mathrm{Vega}} = 0$ in all filters
  - Standard flux in the AB system
    - $m_{AB} = 0 \Rightarrow f_\nu = 3631$ Jy
    - $(1\mathrm{Jy} \equiv 10^{-26} W m^{-2} Hz^{-1})$



(a) Some apparent magnitudes

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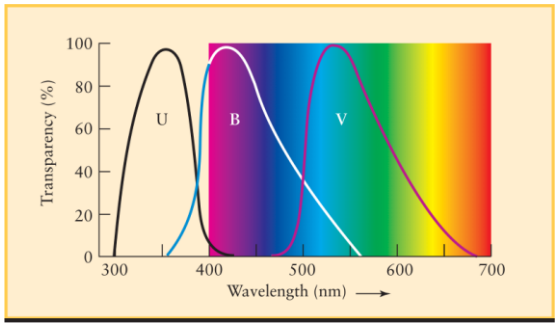
# Vega vs AB magnitude systems




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# Magnitude system

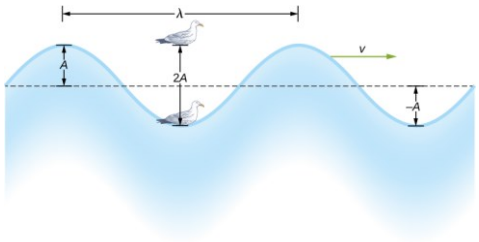
- Total luminosity (all wavelengths) $L$ not directly measurable
- Real observations use filters
- Main systems are Johnson (UBVRI) and Sloan Digital Sky Survey SDSS (ugriz) but more exist
- Def: Stellar colour ≡ difference of two magnitudes (e.g. $B - V$)

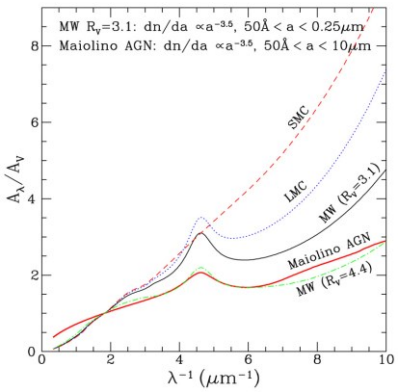



| approximate region | notation for magnitudes apparent | absolute | solar value |
|---|---|---|---|
| ultraviolet | U or $m_U$ | $M_U$ | 5.61 |
| blue | B or $m_B$ | $M_B$ | 5.48 |
| visual (near yellow) | V or $m_V$ | $M_V$ | 4.83 |

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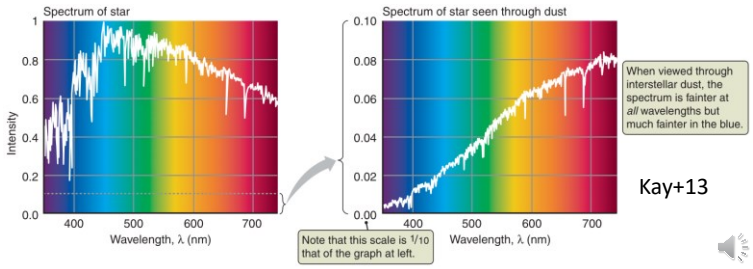
# Interstellar extinction




Li-06

- Little absorption when $\lambda \gg L$ dust-grain size
- Spectrum is both dimmed and reddened
- Visual extinction $A_V$ in mag
- Varies from 0—30 mag

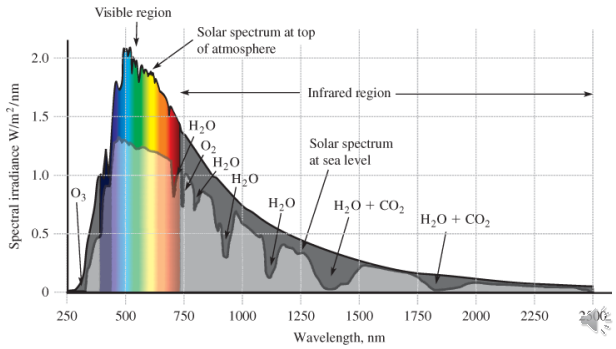



Kay+13

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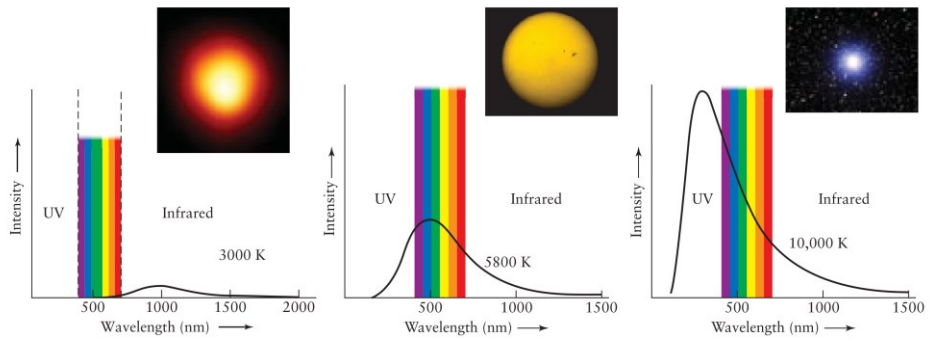
# Absolute and bolometric magnitudes (mag)

- Def: absolute mag is the apparent mag at D=10 pc
- $m - M = 2.5 \lg \left( \frac{D}{10\ \mathrm{pc}} \right)^2 = 5 \lg D - 5$
- Def: bolometric mag is the absolute mag integrated over all wavelength
- For the Sun $M_{\odot}^{\mathrm{bol}} = 4.72$
- Def: Bolometric correction
- $BC \equiv M_{\mathrm{bol}} - M_{\mathrm{V}}$ [is fun(colour)]
- In summary:
- $M_{\mathrm{bol}} = M_{\mathrm{V}} + \mathrm{BC} - A_{\mathrm{V}}$




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# Black body at different temperatures



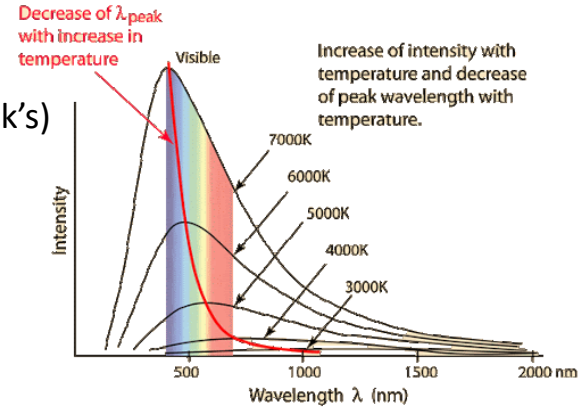

(Freedman Kaufmann Universe)

- For increasing temperature
- Wavelength shifts to the blue
- Flux strongly increased for hotter stars

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# Black body radiation

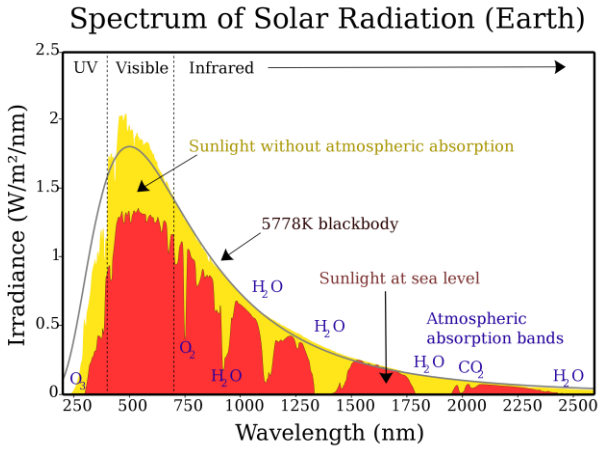
- Wien's law (before Planck's)
- $\lambda_{\max} = \frac{2.9\times10^6 \text{ nm } K}{T}$
- Stefan-Boltzmann law (before Planck's)
- $F = \sigma T^4$
- Planck's law (includes both)
- $B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/(\lambda k_B T)}-1}$




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# Effective and colour temperature (T)

- Effective temperature is T of black-body emitting same flux
- $L_s = 4\pi R_s^2 \int F_\lambda d\lambda = 4\pi R_s^2 \sigma T_{\text{eff}}^4$
- Cannot be measured directly as $R_s$ unknown
- Can be inferred using model atmospheres
- Colour temperature matches the shape of the continuum to a black body law




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# Astrophysics C1
# Advanced stellar astrophysics and galaxies

## Lecture 2

**Michele Cappellari**

UNIVERSITY OF OXFORD

1

## Absorption and emission in stellar spectra

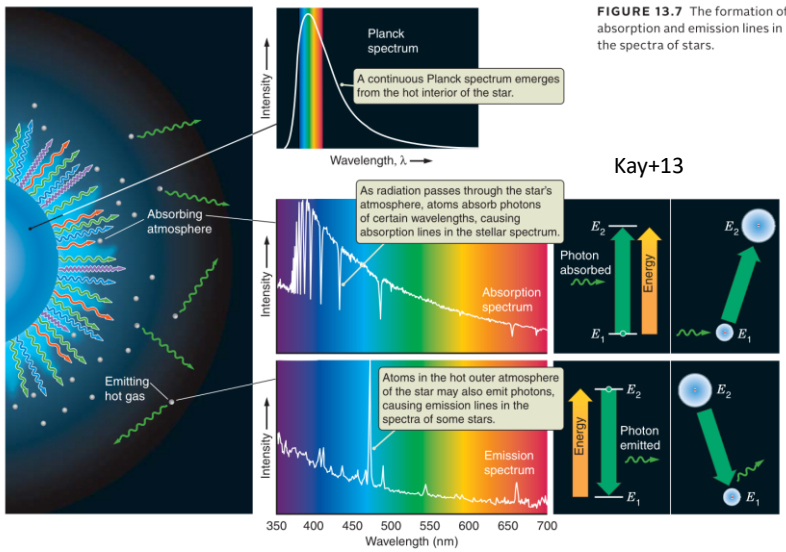


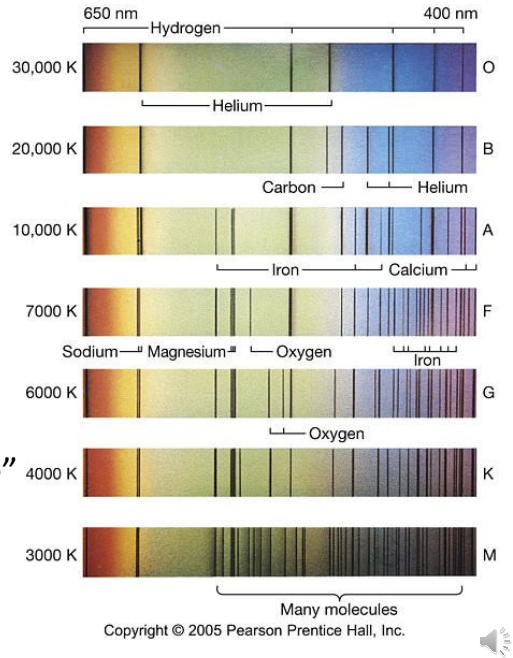

**FIGURE 13.7** The formation of absorption and emission lines in the spectra of stars.

Kay+13

2

# Stars spectral classes

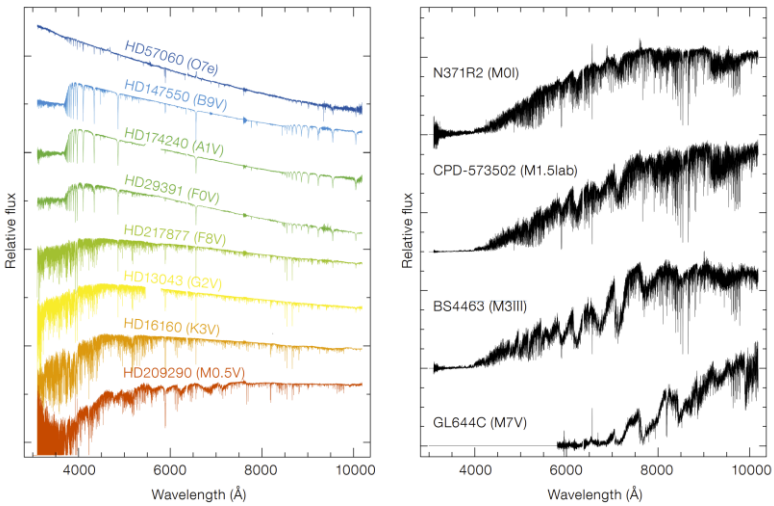
- Strength of absorption lines changes systematically
- Original alphabetic classes based on appearance
- Reordered after physical understanding
- OBAFGKM
- Mnemonic: “Oh be a find guy/girl kiss me”




3

# Spectra of spectral classes

- Temperature decreases with class
- Early spectral types close to black body
- Later types dominated by absorption lines
- Why are spectral lines sensitive to temperature?

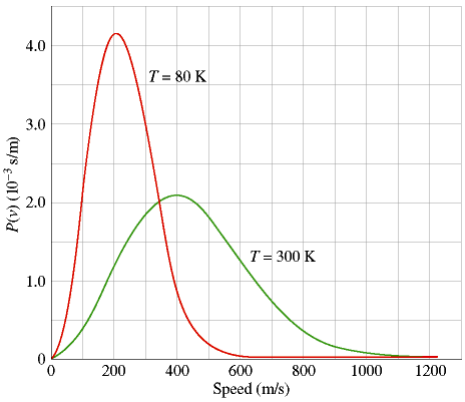



ESO X-shooter Spectral Library (Chen+14)

4

# Atoms Maxwellian velocity distributions

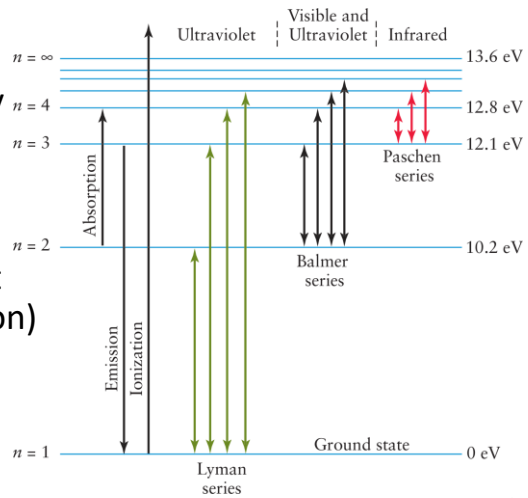
- In thermal equilibrium (approximately true at fixed radius inside a star)
- Number density of particles in $[v, v+dv]$ follow Maxwell-Boltzmann distribution
- $n_v dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv$
- $kT$ is characteristic thermal energy of gas
- Most probable speed $v_{\rm mp} = \left(\frac{2kT}{m}\right)^{1/2}$




5

# Boltzmann equation for energy levels

- In thermal equilibrium
- Relative number of atoms in each energy state is given by the Boltzmann equation
- $\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/kT}$
- $g$ = multiplicity (how many $e$ one can put in the level before violating Pauli exclusion)
- For H atom $g_n = 2n^2$




Hydrogen energy levels

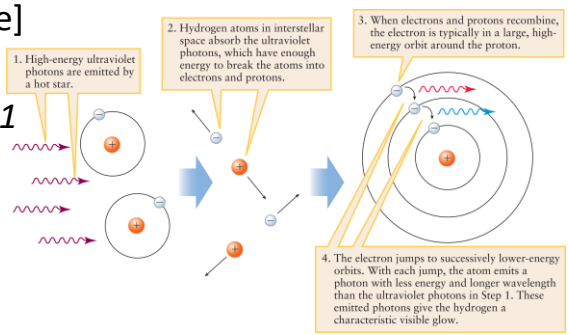
6

# Saha equation for ionization

- In equilibrium
- [ionization rate] = [recombination rate]

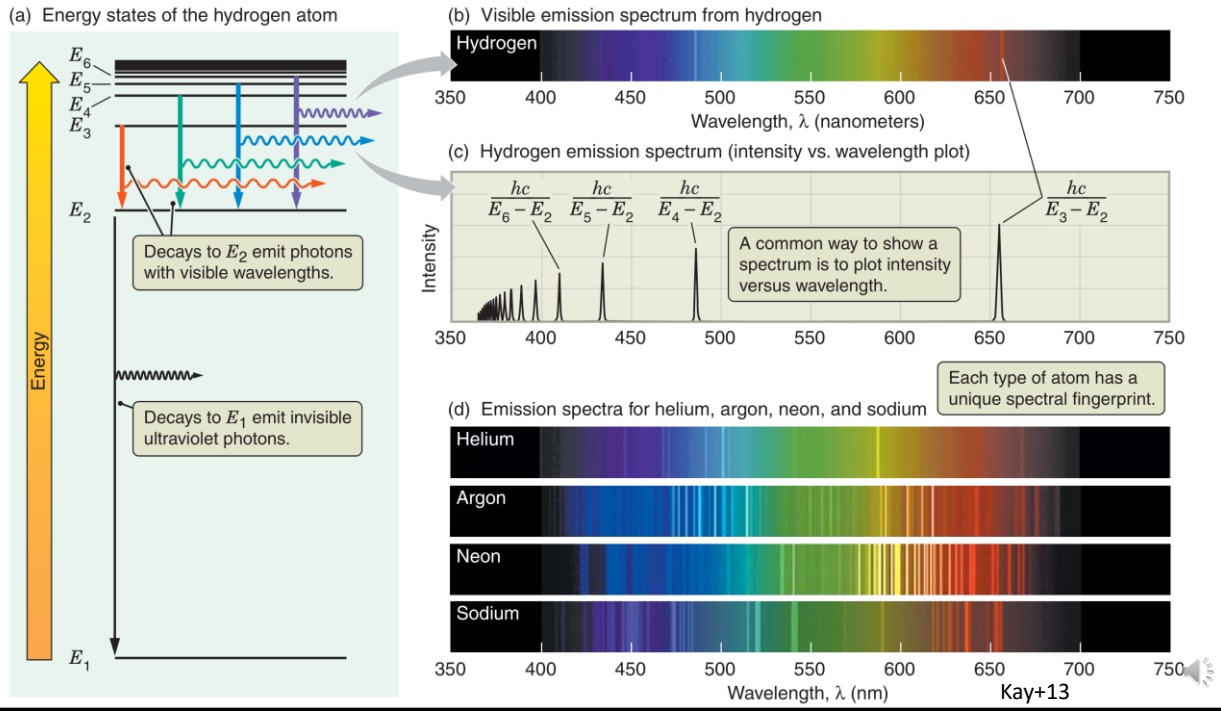
$$X \leftrightarrow X^+ + e^-$$

- The ratio of atoms in ionization *i* to *i+1*
- Is given by the Saha equation
- $\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}$
- For *H* atom there is only one electron
- Either $Z_I$ neutral $Z_{II}$ ionized




Freedmann Kaufmann (2008)

7

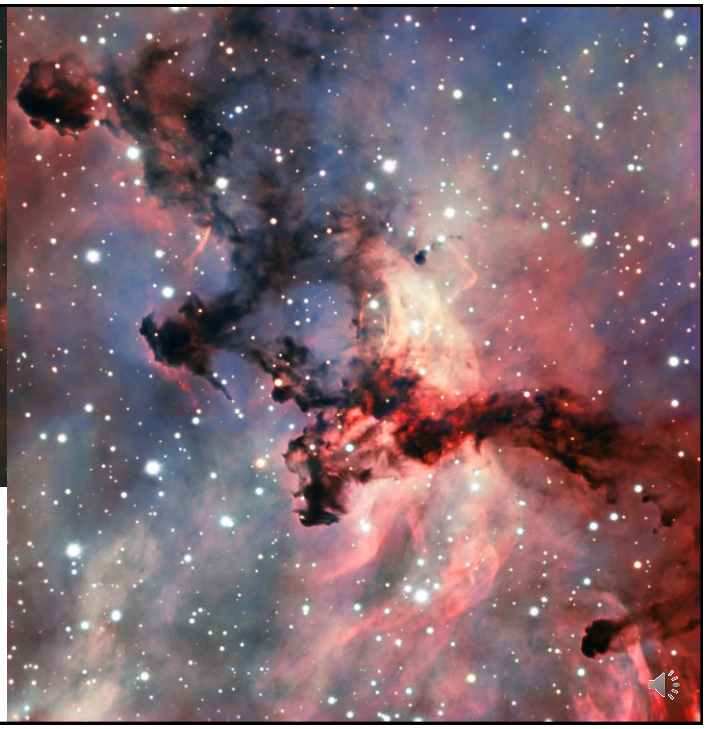



Kay+13

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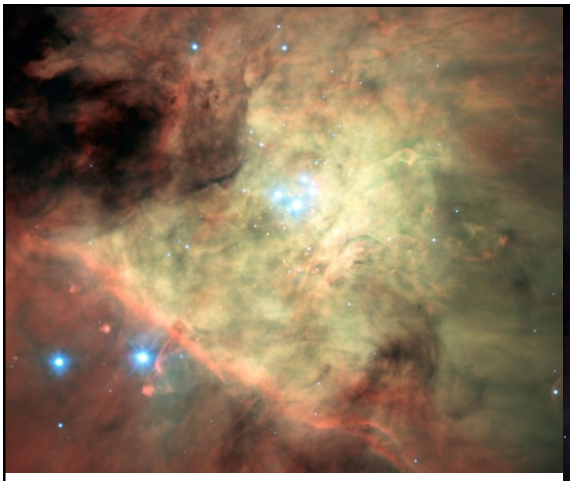

Rosette nebula ESO/VLT
Star forming region.
UV photons ionize gas
recombining photons
re-emitted in Balmer lines



9




10



ESO/MUSE



Orion nebula ESO/VST
Star forming region
UV photons ionize gas
Decaying photons
emitted in Balmer lines

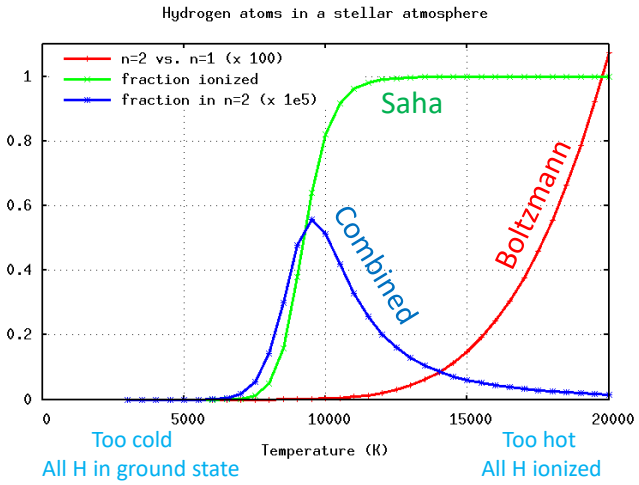
11

# Boltzmann + Saha explain Balmer peak

- For Balmer we want high T to push H atoms in $n = 2$ level
- But when T too high H is ionized
- Most H are in $n = 1,2$
- Implies $N_I \approx N_1 + N_2$

$$\frac{N_2}{N_{\text{tot}}} = \frac{N_2}{N_1 + N_2} \times \frac{N_I}{N_{\text{tot}}}$$

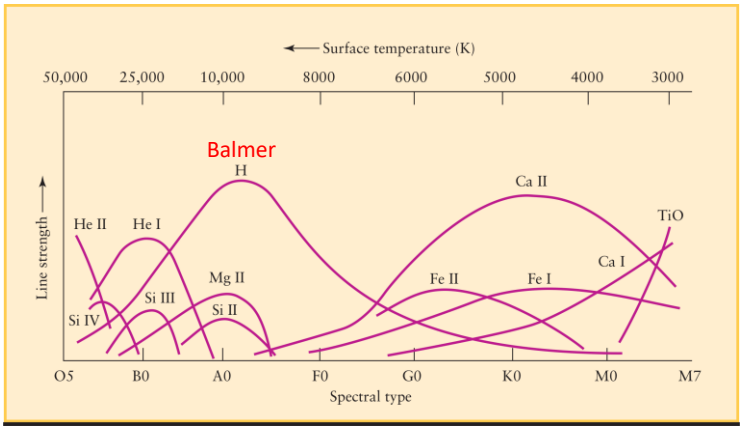
- Peak of Balmer lines is compromise between Boltzmann and Saha eq.




12

# Strength of absorption lines vs temperature

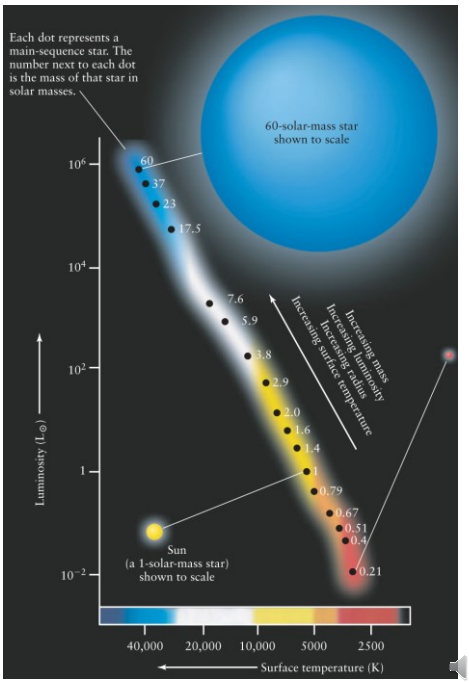
- Similar reasoning as for Balmer applies to other species
- Each line peaks at characteristic temperature
- Explains empirical temperature dependence of spectral types




13

# Herzprung-Russel diagram

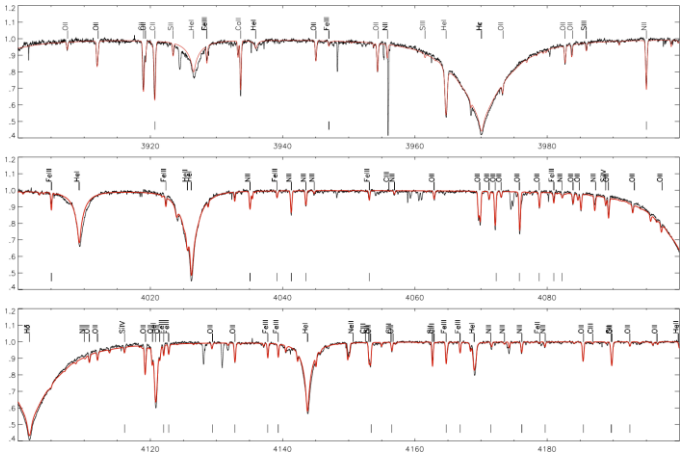
- Herzprung-Russel (HR) diagram
- Fundamental to understand stellar evolution
- Relates temperature and luminosity
- Most stars lie on tight main sequence
- Fainter stars are colder




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# Detailed chemical composition of stars

- Detailed elemental abundance can be obtained with models
- Most stars have composition similar to the Sun
- 74% H, 24% He, 2% rest ≡ metals
- But interestingly some stars are deficient of “metals”
- Metal expressed respect to Sun:
- We will discuss this aspect in a future class



Nieva & Przybilla (2012)

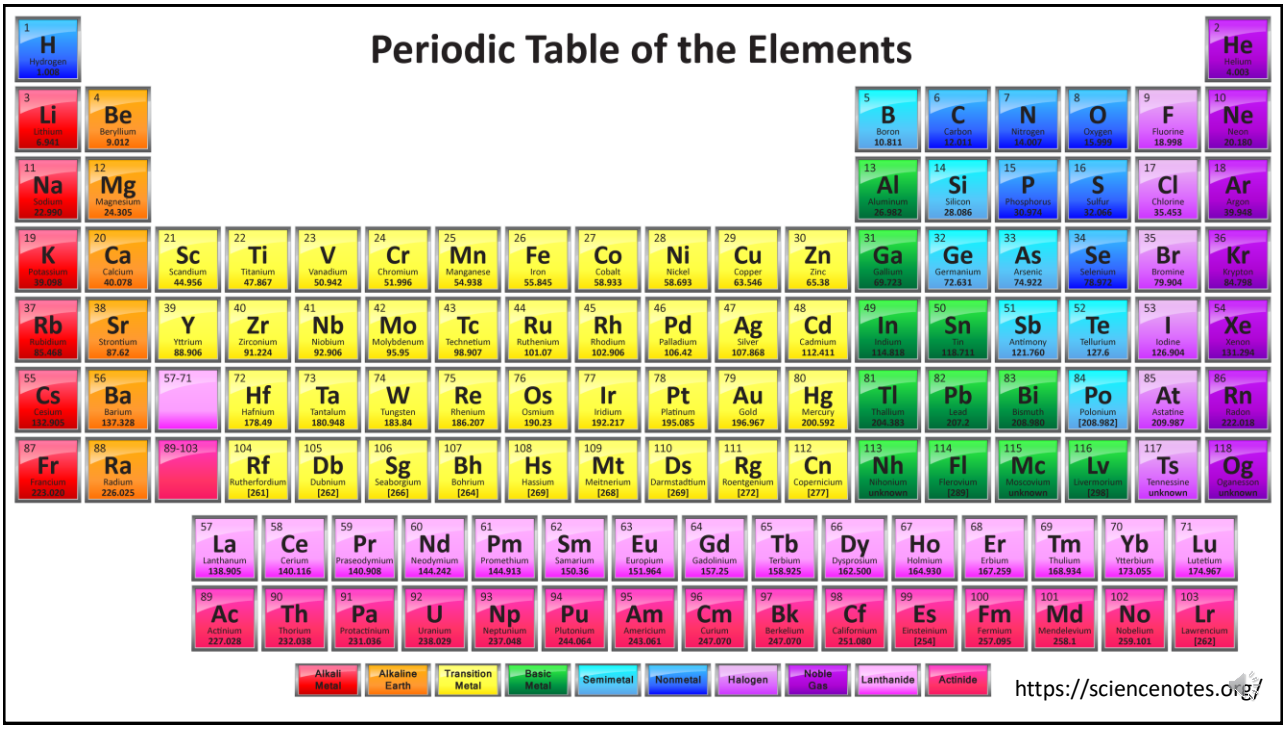
15

# Chemical composition of the Sun

| Element | Percentage by Number† | Percentage by Mass†† |
|---|---|---|
| Hydrogen | 92.5 | 74.5 |
| Helium | 7.4 | 23.7 |
| Oxygen | 0.064 | 0.82 |
| Carbon | 0.039 | 0.37 |
| Neon | 0.012 | 0.19 |
| Nitrogen | 0.008 | 0.09 |

Kay+13

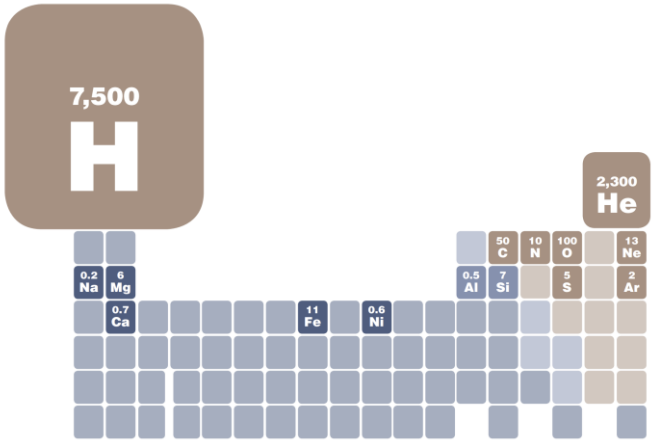
16




17

# The periodic table for astronomy

- After Big Bang: H, He (Li traces)
- ¾ H; ¼ He
- Major successful prediction of Big Bang theory
- Nearly all heavier elements formed in stars
- We call “metal” anything that is not H, He
- Def: metallicity is fraction of metals respect to H atoms

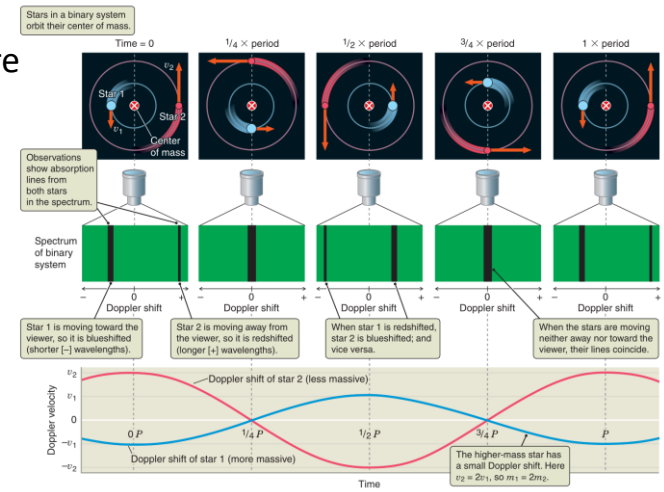



Credit: Chandra/NASA

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# Spectroscopic binaries

- Binary orbits common barycentre
- Generally low period for observational selection effects
- Orbits close to circular due to tidal circularization
- $\frac{m_1}{m_2} = \frac{r_1}{r_2} = \frac{v_1}{v_2}$
- In terms of observables
- $\frac{m_1}{m_2} = \frac{v_{1r}/\sin i}{v_{2r}/\sin i} = \frac{v_{1r}}{v_{2r}}$

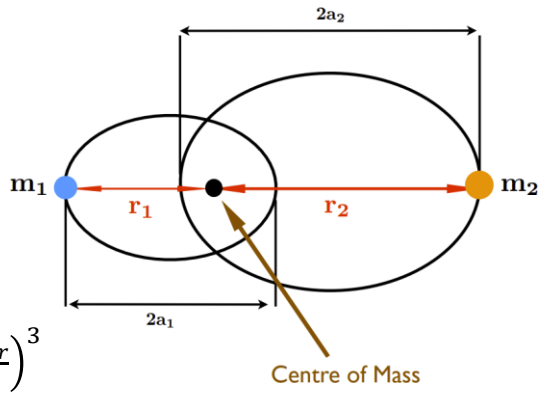


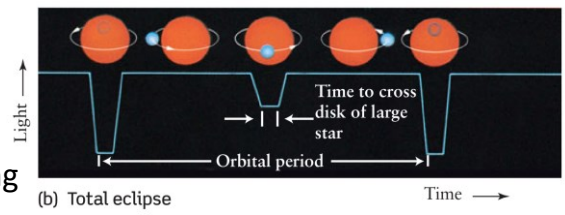

Kay+13

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# Measuring stellar masses

- Newton form of Kepler law
- $\left(\frac{P}{2\pi}\right)^2 = \frac{(a_1+a_2)^3}{G(m_1+m_2)}$; $P = \frac{2\pi a}{v}$ (for $e \ll 1$)
- $a = \frac{Pv}{2\pi} \Rightarrow a_1 + a_2 = \frac{P}{2\pi}(v_1 + v_2)$
- $m_1 + m_2 = \frac{P}{2\pi G}(v_1 + v_2)^3 = \frac{P}{2\pi G}\left(\frac{v_{1r}+v_{2r}}{\sin i}\right)^3$
- Replacing $m_2 = \frac{m_1 v_2}{v_1}$
- $m_1 = \frac{Pv_1(v_1+v_2)^2}{2G\pi} = \frac{Pv_{1r}(v_{1r}+v_{2r})^2}{2G\pi \sin^2 i}$
- Inclination can be treated statistically
- Inclination known if binary is also eclipsing



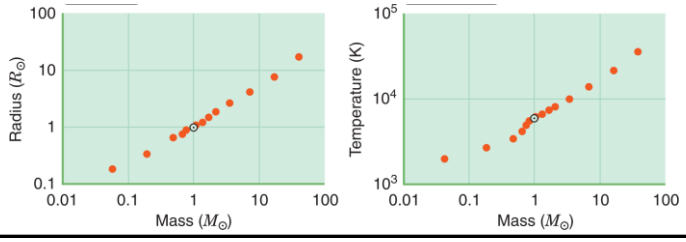

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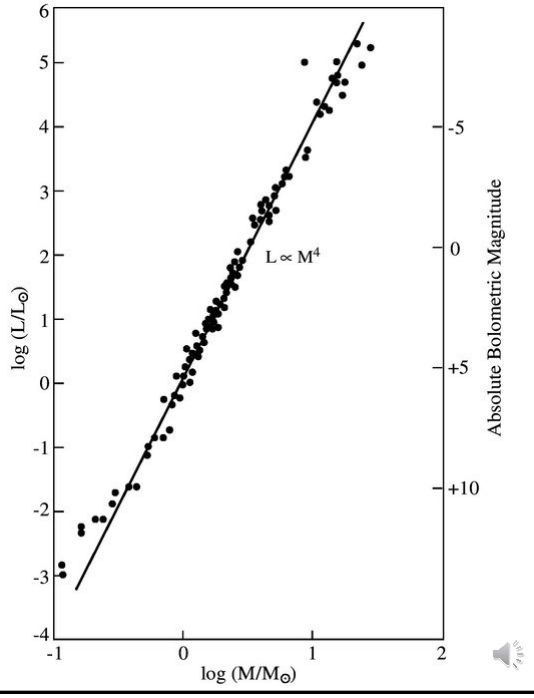
# Mass-luminosity relation

- Many years of binary-stars observations
- For stars on the main sequence
- Empirical tight relation between mass and luminosity

$$L/L_{\odot} \approx \left(M/M_{\odot}\right)^4$$

- Size and temp. also unique $f(M)$

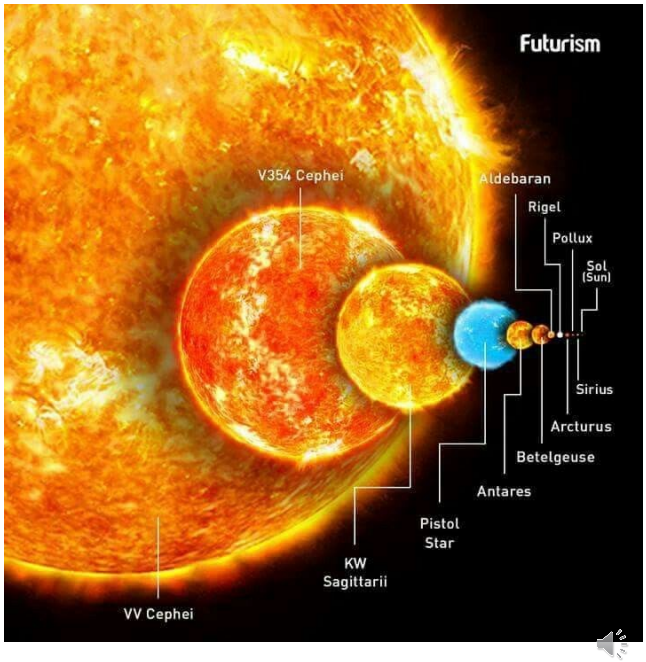



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# Estimating stellar radii

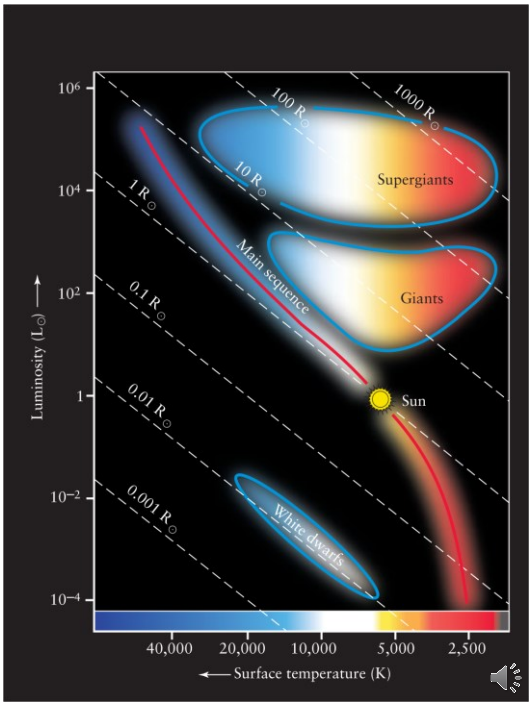
- Use Stefan-Boltzmann law
- $L_s = 4\pi R_s^2 \sigma T_{\text{eff}}^4$
- $\frac{L}{L_{\odot}} = \left(\frac{R}{R_{\odot}}\right)^2 \left(\frac{T}{T_{\odot}}\right)^4$
- $\Rightarrow \frac{R}{R_{\odot}} = \left(\frac{T}{T_{\odot}}\right)^2 \left(\frac{L}{L_{\odot}}\right)^{1/2}$
- $R = 0.01\ R_{\odot} \rightarrow 1000\ R_{\odot}$




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# Radii on HR diagram

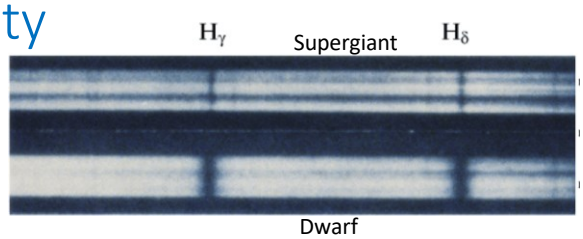
- Originally thought stars evolve along main sequence by cooling!
- Radii more similar on main sequence
- But other classes of stars have very large/small radii
- Outside main sequence $M - L$ relation is not unimodal

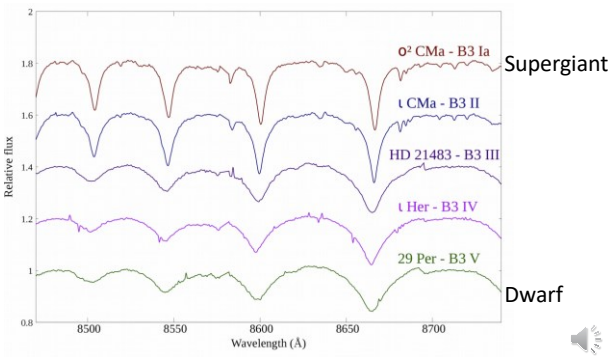



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# Measuring surface gravity

- Heisenberg's uncertainty principle for energy level $E$ with lifetime $\Delta t$
- $\Delta E \Delta t \approx \frac{\hbar}{2} \Rightarrow \Delta E \approx \frac{\hbar}{2\Delta t}$
- Photons emitted in transition from this level $E$ have frequency
- $\nu \approx \frac{E}{h}$ with range $\Delta\nu \approx \frac{\Delta E}{h} \approx \frac{1}{4\pi\Delta t}$
- Larger surface gravity → larger pressure
- Frequent collisions define $E$ level lifetime
- Energy level lifetime $\Delta t$ shorter
- Lines broader by pressure broadening

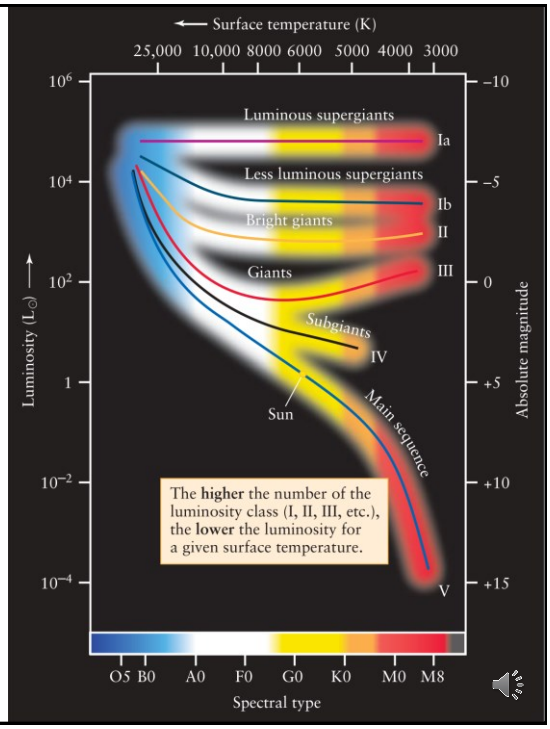


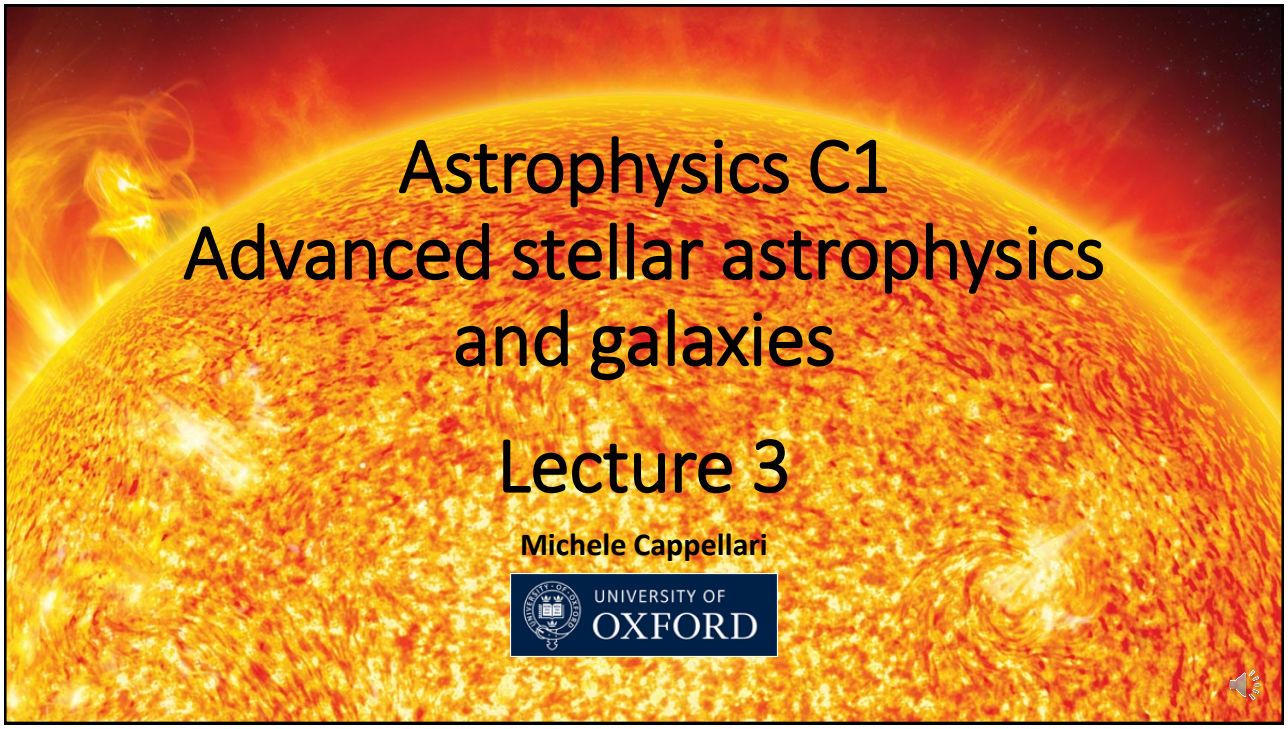

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# Luminosity classes on HR

- Five luminosity classes
- Roman numerals: I, II, III, IV, V
- Main-sequence stars = dwarf V
- Smallest luminosity class for largest stars




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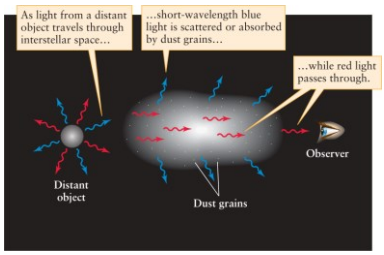



1

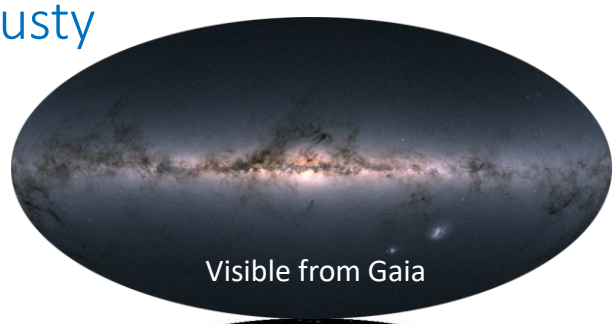
# Star formation

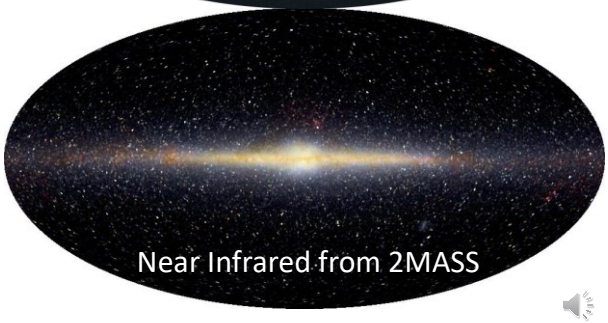
2

# Interstellar medium is dusty




- Clear evidence of dust in Milky Way
- Infrared light more weakly affected by dust extinction
- Allow us to see inside dusty regions



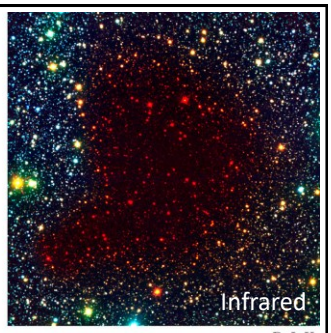

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# Molecular clouds

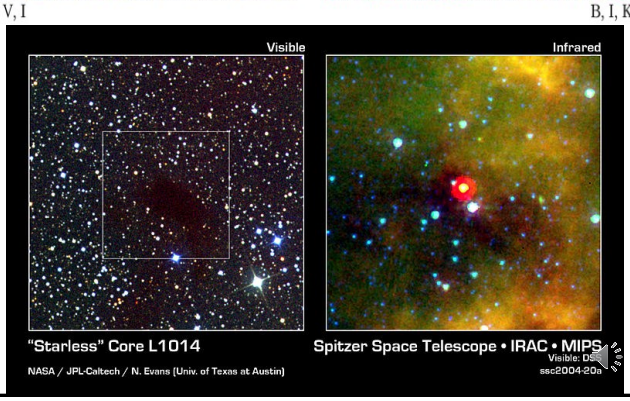
- To start gravitational collapse
- Gas pressure must be low
- → temperature must be low
- Only cold places are dark nebulae
- Cold enough for molecules to form and not being dissociated by collisions
- Called molecular clouds
- They are the cradles of star formation




B, V, I

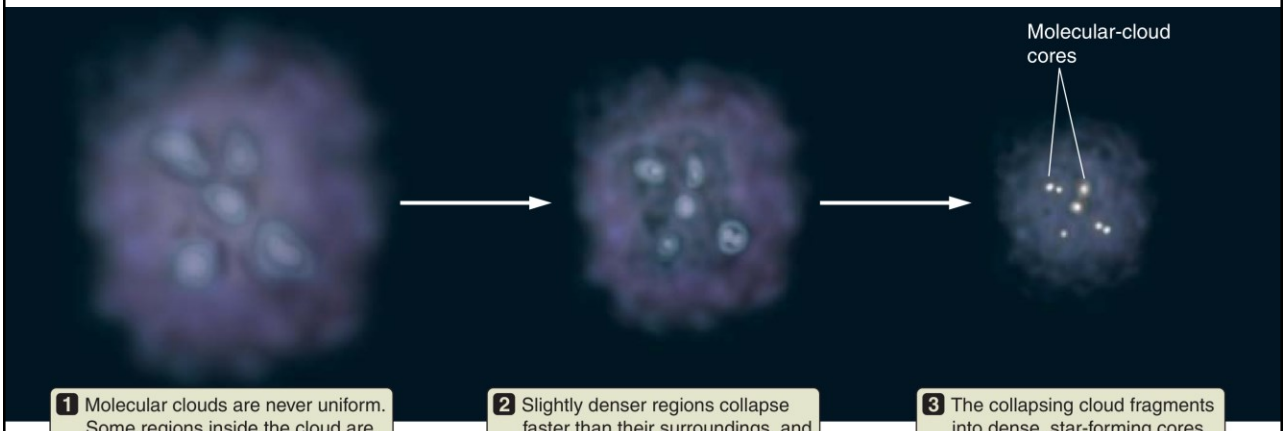



B, I, K




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# Gravitational collapse in molecular clouds

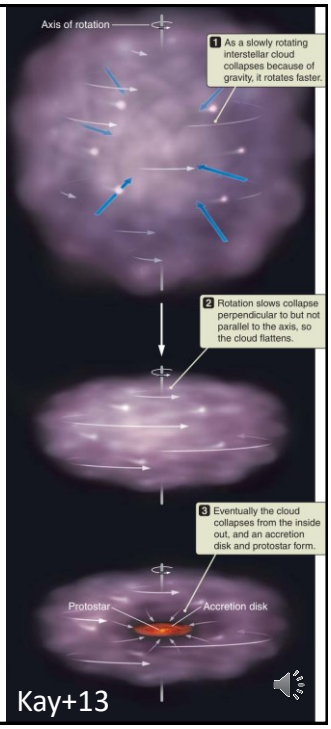



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# Stars and planets formation

- Inside giant molecular clouds (GMCs) of $\sim 10^5 M_\odot$
- Molecular-cloud cores contract by gravity
- Angular momentum is conserved
- Rotation speeds up
- Centrifugal force opposes gravity in equatorial direction
- Cloud shrinks and flattens
- Eventually gravitational instability takes over
- Protostar and accretion disk form
- Planets will form out of accretion disk

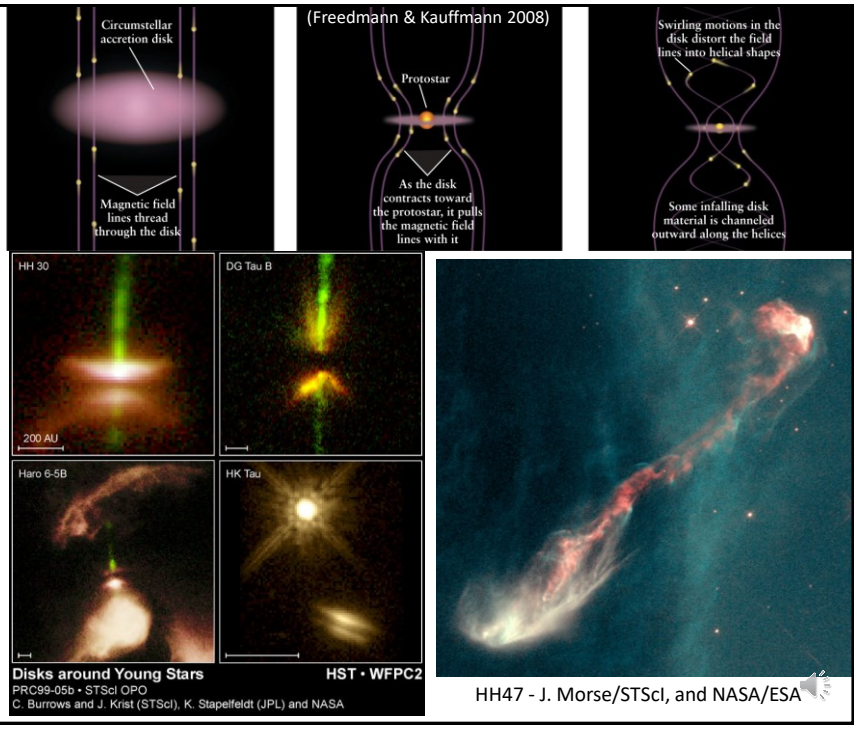



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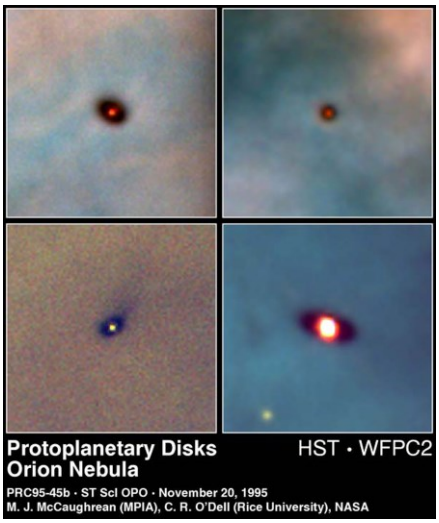
# Protostars

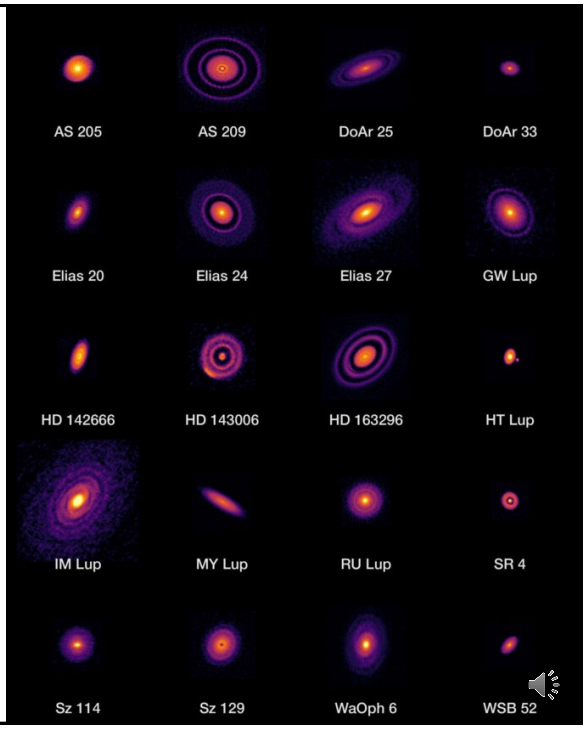
- Herbig-Haro objects
- Bipolar jets
- Hubble Space Telescope resolved dusty disks
- Jests are short lived $10^4 - 10^5 \mathrm{yr}$
- Not found in most massive stars




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# Protostars and protoplanetary disks

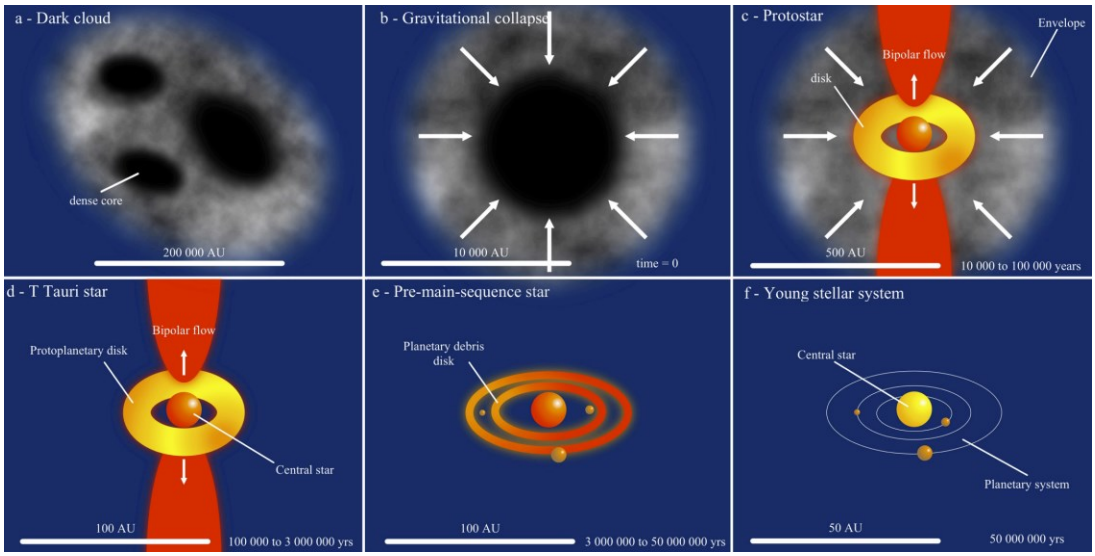



Credit: ALMA (ESO/NAOJ/NRAO), S. Andrews et al.; N. Lira (2018)

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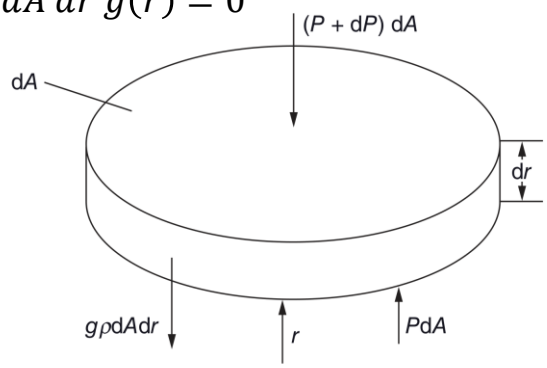
# Summary of stellar formation




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# Hydrostatic equilibrium 1

- Consider an infinitesimal volume element inside star atmosphere
- The condition for equilibrium is
- $P(r)dA - [P(r) + dP]dA - \rho(r)dA\,dr\,g(r) = 0$
- Simplifies to
- $\frac{dP(r)}{dr} = -\rho(r)g(r)$




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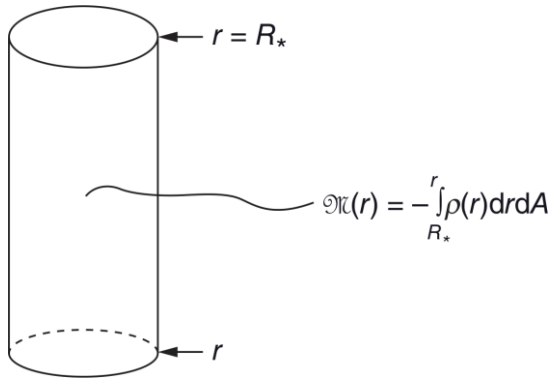
# Hydrostatic equilibrium 2

- Define mass inside cylinder

$$\mathfrak{M}(r) = -\int_{R_*}^{r} \rho(r)\,\mathrm{d}r\mathrm{d}A$$

- Then column mass is defined

$$m(r) = \frac{\mathfrak{M}(r)}{\mathrm{d}A} = -\int_{R_*}^{r} \rho(r)\,\mathrm{d}r = \int \mathrm{d}m$$




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# Hydrostatic equilibrium 3

- Then hydrostatic equilibrium

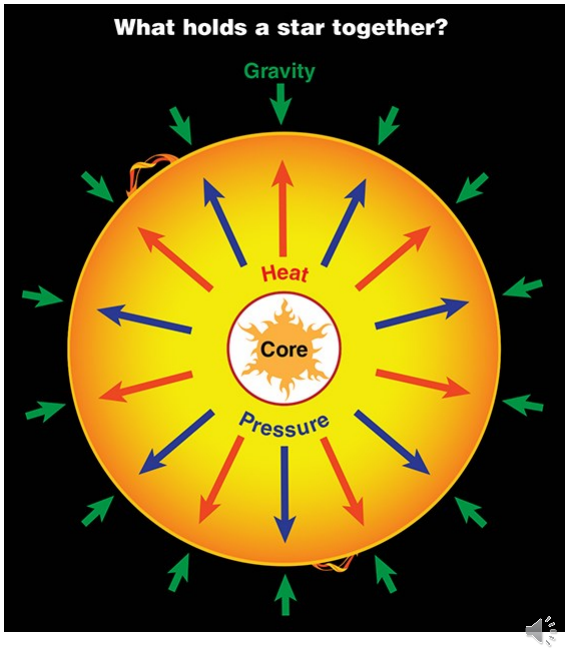
$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\rho(r)g(r)$$

- Can be rewritten with column mass

$$\frac{\mathrm{d}P}{\mathrm{d}m} = g(r)$$

- Pressure increases with column mass
- Integrating near the surface $P(m) \approx mg$
- Escaping radiation is absorbed by stellar plasma and photon momentum transferred to matter causing radiation pressure

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\rho(r)[g(r) - g_{\mathrm{rad}}(r)]$$



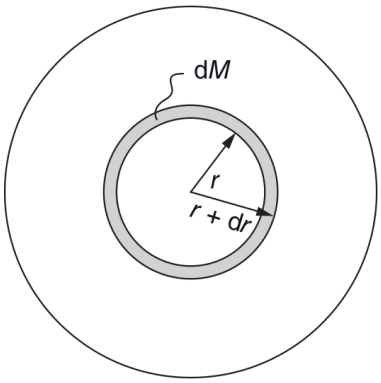

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# Equation of mass conservation

$$dM(r) = 4\pi r^2 \rho(r) dr$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$M(r) = \int_0^r dM(r) = \int_0^r 4\pi r^2 \rho(r) dr$$




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# Virial theorem 1

- Start from hydrostatic equilibrium

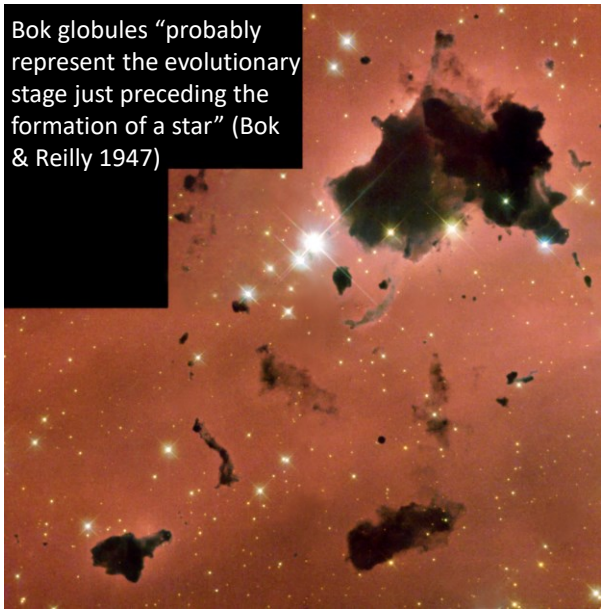
$$\frac{dP}{dr} = -\rho(r) g(r)$$

- Insert gravity for spherical symmetry

$$\frac{dP}{dr} = -\frac{\rho(r) G M(r)}{r^2}$$

- Multiply by volume $V(r) = 4\pi r^3/3$

$$\frac{4}{3}\pi r^3 dP = -\frac{4}{3}\pi r \rho(r) G M(r) dr$$




NASA and The Hubble Heritage Team (STScI/AURA)

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# Virial theorem 2

- Use mass element $dM = 4\pi r^2 \rho(r) dr$

$$V\mathrm{d}P = -\frac{GM(r)}{3r}\mathrm{d}M$$

- Integrate over the entire stellar volume
- Integrate left side by parts

$$\int_{P_{\mathrm{centre}}}^{0} V\mathrm{d}P = PV\Big|_{\substack{\mathrm{surface}\\ \mathrm{centre}}} - \int_{0}^{V_*} P\mathrm{d}V$$

- Top expression becomes

$$\int_{0}^{V_*} P\mathrm{d}V = \frac{1}{3}\int_{0}^{M_*} \frac{GM(r)}{r}\mathrm{d}M$$




Detail of Rosette Nebula

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# Virial theorem 3

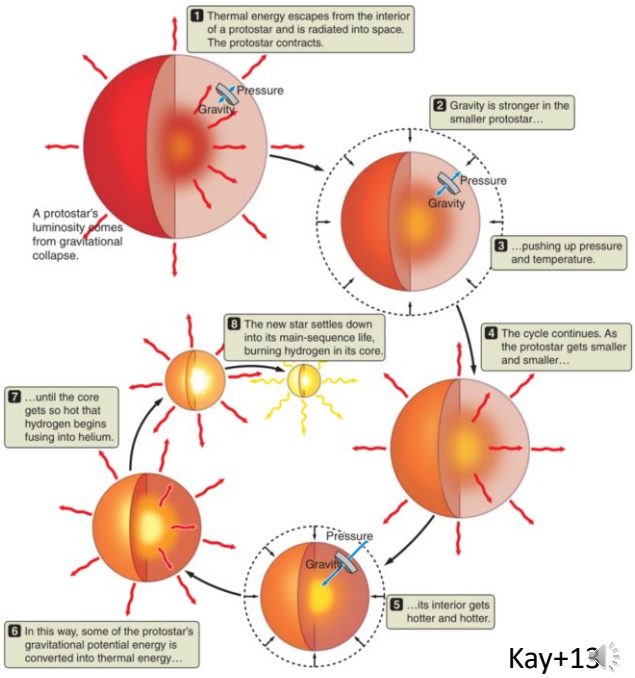
- By definition the potential energy
- $\Omega = -\int_0^{M_*} \frac{GM(r)}{r} dM$
- Implies $3\int_0^{V_*} PdV = \int_0^{M_*} \frac{GM(r)}{r} dM = -\Omega$
- From statistical physics average kinetic energy of gas particles is $\frac{3}{2}kT$
- Inside $V$ volume with N particles energy density per unit volume $\varepsilon = \frac{N}{V}\frac{3}{2}kT$
- For ideal gas $P = \frac{NkT}{V} = \frac{2}{3}\epsilon$
- $3\int_0^{V_*} PdV = 2\int_0^{V_*} \varepsilon dV = -\Omega$
- Given that thermal energy $U = \int_0^{V_*} \varepsilon dV$
- we obtain the virial theorem $U = -\frac{\Omega}{2}$




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# Virial theorem 4

- $U = -\Omega/2$
- During contraction $-\Omega$ and $U$ increase
- $\Delta U = -\Delta\Omega/2$
- ½ gravitational energy Ω heats gas
- ½ is radiated
- Star contracts in hydrostatic equilibrium
- Radiating energy by virial theorem
- Until central temperature is enough to start hydrogen $H$ fusion
- $H$ fusion only happens if $M_* \geq 0.08M_\odot$




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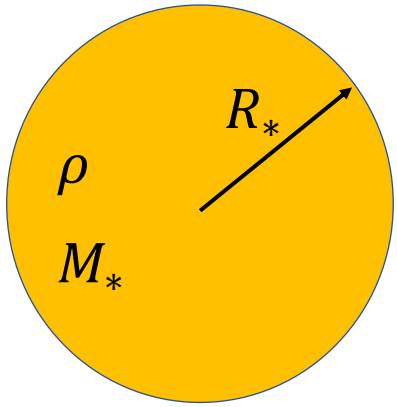
# Example: min central stellar pressure

- $\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$ hydrostatic equilibrium
- $\frac{dM(r)}{dr} = 4\pi r^2\rho(r)$ mass conservation
- $\frac{dP}{dr}/\frac{dM}{dr} = \frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4}$
- $-\int_0^{M_*}\frac{dP}{dM}dM = P_c - P_s = \int_0^{M_*}\frac{GM(r)}{4\pi r^4}dM$
- Clearly $r < r_s \rightarrow \frac{1}{r^4} > \frac{1}{r_s^4}$
- Implies $\int_0^{M_s}\frac{GM(r)}{4\pi r^4}dM > \int_0^{M_s}\frac{GM(r)}{4\pi r_s^4}dM = \frac{GM_s^2}{8\pi r_s^4}$
- $P_c > P_s + \frac{GM_s^2}{8\pi r_s^4} > \frac{GM_s^2}{8\pi r_s^4}$
- For the Sun $P_{c\odot} > 4.5 \times 10^{13} Nm^{-2} = 4.5 \times 10^8 \text{atm}$

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# Potential energy $\Omega$ of a uniform sphere

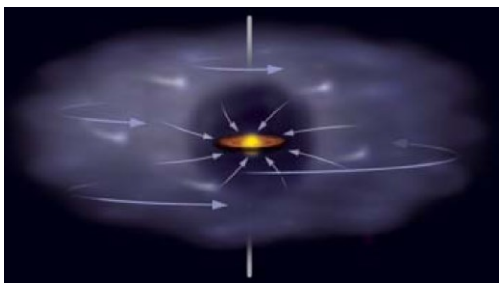
- $\Omega = -\int_0^{M_*} \frac{GM(r)}{r} dM$
- As the density is constant
- $\rho = M(r) / \left(\frac{4\pi r^3}{3}\right)$
- Solving for radius $r = \left[M(r) / \left(\frac{4\pi\rho}{3}\right)\right]^{1/3}$
- Replacing in the top equation
- $\Omega = -G \left(\frac{4\pi\rho}{3}\right)^{1/3} \int_0^{M_*} M^{2/3} dM$
- $\Omega = -G \left(\frac{4\pi\rho}{3}\right)^{1/3} \left(\frac{3}{5} M_*^{5/3}\right)$ but $\rho = M_* / \left(\frac{4\pi R_*^3}{3}\right)$
- $\Omega = -\frac{3}{5} \frac{GM_*^2}{R_*}$




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# Jeans criterion for cloud collapse

- If $-\Omega > 2U$ the cloud is unstable to collapse
- Assume clouds of constant density
- $\Omega = -\frac{3}{5} \frac{GM^2}{R}$
- Assuming isothermal $T$ cloud $U = \frac{3}{2} NkT$
- $N = M / \mu m_H$ ($\mu m_H$ = mean mass per particle)
- Cloud unstable when $\frac{3}{5} \frac{GM^2}{R} > \frac{3MkT}{\mu m_H}$
- With R $= \left[M / \left(\frac{4\pi\rho}{3}\right)\right]^{1/3}$
- Cloud unstable above Jeans mass
- $M > M_J = \left(\frac{5kT}{\mu m_H G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2}$ or equivalently $\rho > \rho_j = \left(\frac{5kT}{\mu m_H G}\right)^3 \frac{3}{4\pi M^2}$



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# Jeans criterion for molecular clouds

- Jeans mass of average molecular cloud
- $M \sim 10^3 M_\odot, T \sim 10K, \rho \sim 10^3\ H_2/\mathrm{cm}^3$
- Mainly composed of $H_2$ molecules → $\mu = 2$
- Plugging numbers in Jeans's criterion
- $\rho = 2m_H \times 10^3 \mathrm{cm}^{-3} \approx 3 \times 10^{-21} \mathrm{g\ cm}^{-3}$
- $M_J = \left(\frac{5kT}{\mu m_H G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} \approx 20 M_\odot$
- This mass is smaller than the clouds
- Implies cloud <u>will</u> fragment and collapse into stars

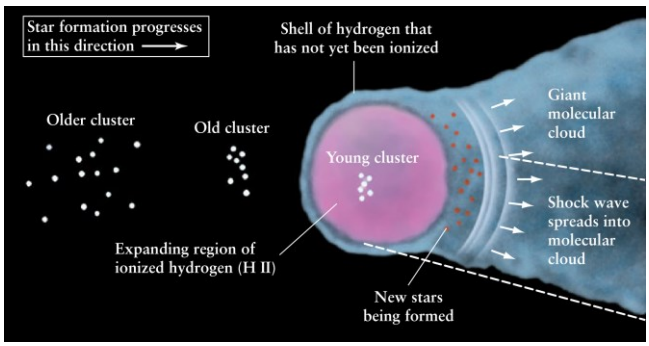


Credits: NASA, ESA and the Hubble Heritage Team (STScI/AURA)

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# Understanding "pillars of creation"



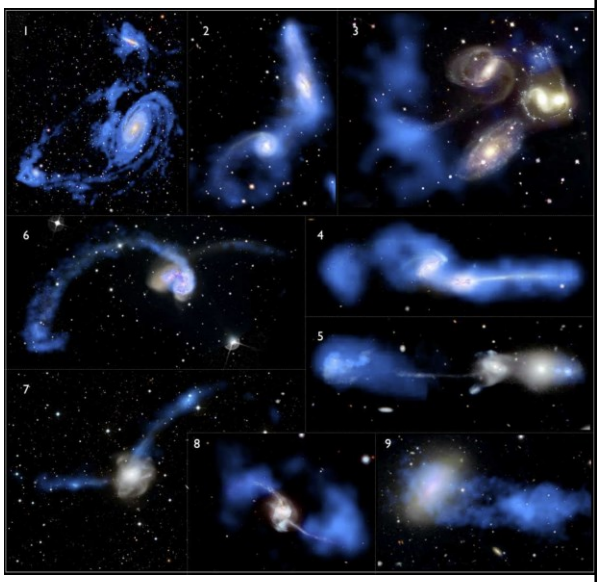




(Freedman & Kaufmann 2008)

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# Jeans criterion for diffuse HI clouds

- Jeans mass of average HI cloud (=neutral H)
- $M \lesssim 100 M_{\odot}, T \sim 100K, \rho \lesssim 10^3\ H/\mathrm{cm}^3$
- Mainly composed of neutral $H \rightarrow \mu = 1$
- $\rho \lesssim m_H \times 10^3 \mathrm{cm}^{-3} \approx 2 \times 10^{-21} \mathrm{g\ cm}^{-3}$
- Plugging numbers in Jeans's criterion
- $\rho_j = \left(\frac{5kT}{\mu m_H G}\right)^3 \frac{3}{4\pi M^2} \approx 1 \times 10^{-18} \mathrm{g\ cm}^{-3}$
- This density is larger than the clouds
- Implies cloud will <u>not</u> fragment and collapse



Galactic tidal tails - Duc & Renaud (2012)

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# Stellar Initial Mass Function (IMF)

- Gives the mass $\chi(m)$ distribution of coeval stars formed in a cloud
- $\chi(m) = \frac{dN}{dm} \propto m^{-\alpha}$
- Only measurable directly in the Milky Way
- Need to resolve individual stars
- Measurable in globular/open clusters
- Need coeval formation and same distance
- Salpeter (1955) $\alpha \sim 2.35$ for $m > 1 M_{\odot}$
- Low-mass stars much more numerous than massive ones

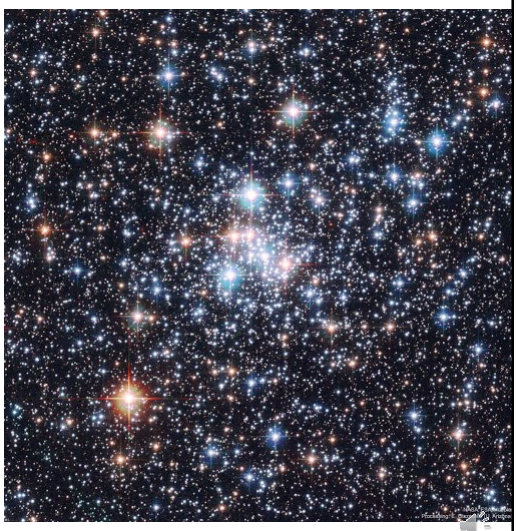
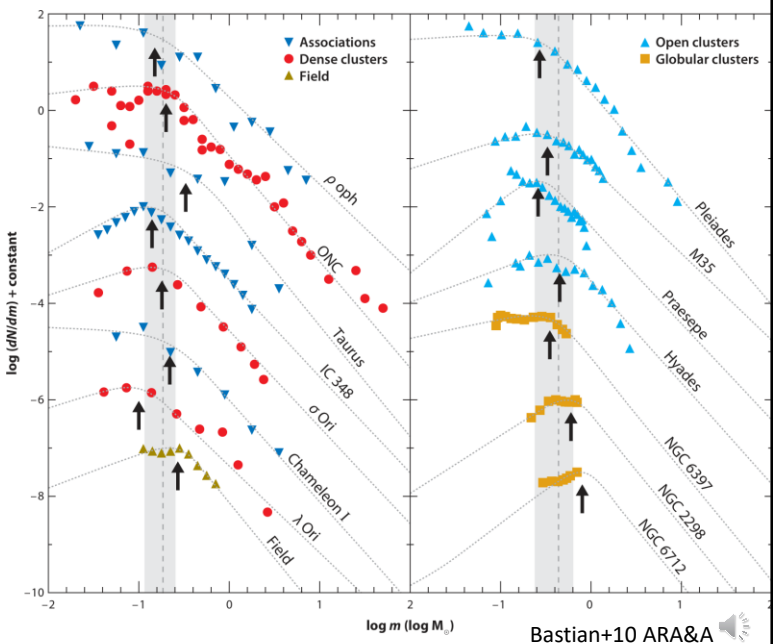


Image Credit: NASA, ESA, Hubble; E. Olzewski

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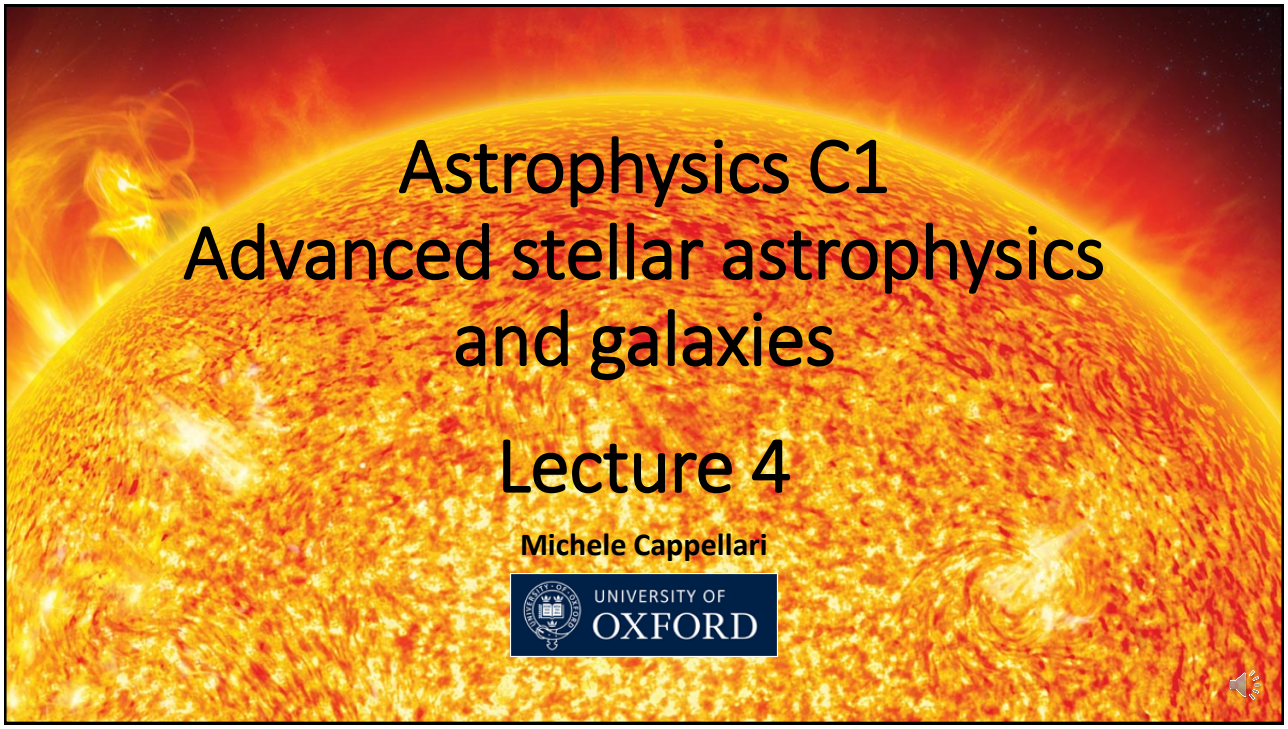
# IMF universality in the Milky Way

- Stellar counts in many Milky Way clusters
- Affected by crowding, incompleteness and dynamical evolution
- Convert luminosity into masses using models
- Slope consistent with Salpeter for $m \gtrsim 1M_{\odot}$
- Exponent changes at lower stellar masses




Bastian+10 ARA&A

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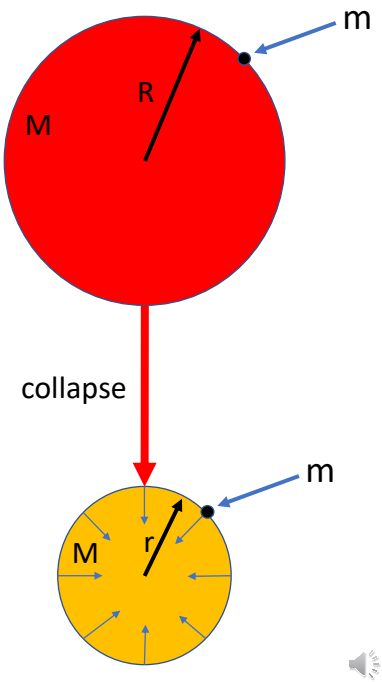

1

# Physics of stellar evolution

2

# Free-fall time scale

- Start with tracer mass $m$ at rest on surface of spherical cloud
- Tracer mass follows free-fall collapse of the cloud
- When surface contract from $R \rightarrow r$
- Kinetic energy is difference of potential energies
- $K = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 = \frac{GMm}{r} - \frac{GMm}{R}$.
- The particle velocity becomes $\frac{dr}{dt} = -\left(\frac{2GM}{r} - \frac{2GM}{R}\right)^{1/2}$
- $t_{\rm ff} = -\int_R^0 \left(\frac{2GM}{r} - \frac{2GM}{R}\right)^{-1/2} dr$
- With change $x = r/R$ then $t_{\rm ff} = \left(\frac{R^3}{2GM}\right)^{1/2} \int_0^1 \left(\frac{x}{1-x}\right)^{1/2} dx$
- $t_{\rm ff} = \frac{\pi}{2}\left(\frac{R^3}{2GM}\right)^{1/2} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}$
- Where $M = \frac{4}{3}\pi R^3 \rho$




3

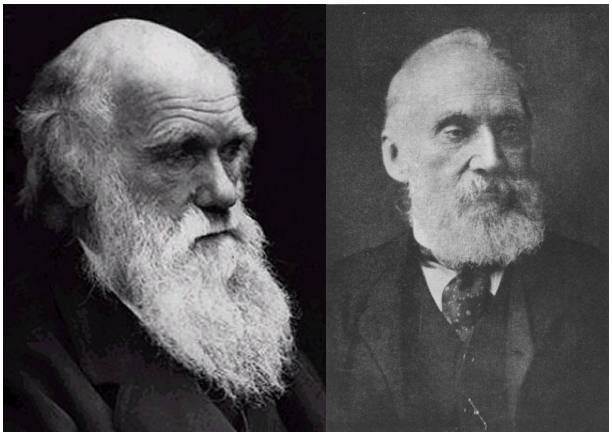
# Example: Free-fall time for $1 M_\odot$

- Assume a $1 M_\odot$ molecular cloud
- Assume collapse starts at Jeans density
- $M \sim 1 M_\odot, T \sim 10 K, \rho \sim 10^3\ H_2/\mathrm{cm}^3$
- $\rho_j = \left(\frac{5kT}{\mu m_H G}\right)^3 \frac{3}{4\pi M^2} \approx 2 \times 10^{-18} \mathrm{g\ cm}^{-3}$
- $t_{\rm ff} = \left(\frac{3\pi}{32G\rho_J}\right)^{1/2} \approx 50{,}000\ \mathrm{yr}$
- Very short time respect to stellar evolution

4

# Thermal time scale (Kelvin–Helmholtz)

- Star has $R_\mathrm{i} \gg R_* \Rightarrow \Delta\Omega = \Omega(R_\mathrm{i}) - \Omega(R_*) \approx \Omega(R_*)$
- During contraction, by virial theorem
- ½ of gravitational energy is radiated
- $\Omega = -\frac{3}{5}\frac{GM_*^2}{R_*}$ for constant density
- $\Delta U = -\frac{\Delta\Omega}{2} \approx \frac{3}{10}\frac{GM_*^2}{R_*}$
- Assuming constant luminosity
- Kelvin–Helmholtz time scale
- $t_\mathrm{KH} = \frac{\Delta \mathrm{U}}{\mathrm{L}_*} = \frac{3}{10}\frac{GM_*^2}{L_* R_*}$
- For the Sun $t_\mathrm{KH} \sim 10^7 \mathrm{yr}$
- Thought to be source of Sun energy
- But age of Earth $t_\oplus \approx 4.5 \times 10^9 \mathrm{yr}$



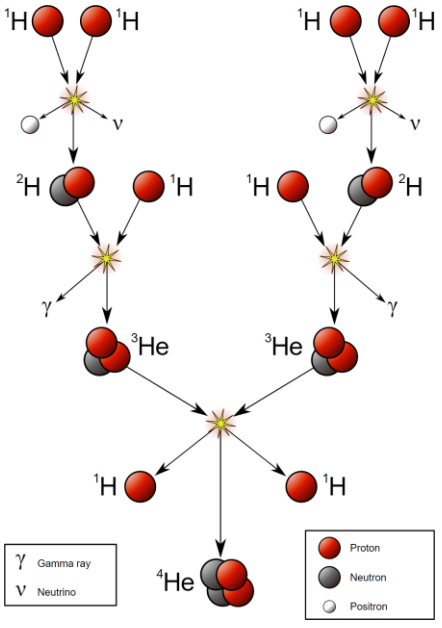
Darwin Kelvin

Intense but unsolved debate

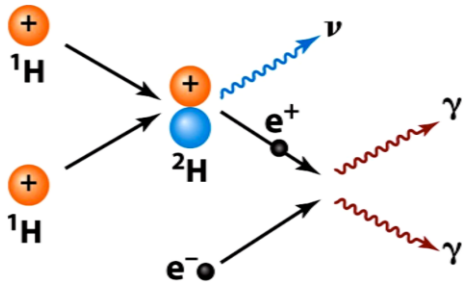
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# Nuclear time scale

- Not known at the time of Kelvin
- Atomic fusion in stellar interior
- $4\ {}^1\mathrm{H} \rightarrow {}^4\mathrm{He} + \mathrm{energy}$
- $E = \Delta m c^2$ with $\Delta m \approx 0.7\% \equiv \epsilon$
- Nuclear time scale
- $t_\mathrm{nuc} \approx \frac{\epsilon M c^2}{L}$
- For the Sun $\mathrm{t_{nuc}} \sim 10^{10}\mathrm{yr} = 10\ \mathrm{Gyr}$
- $t_\mathrm{nuc} \gg t_\mathrm{KH} \gg t_\mathrm{ff}$
- Is the only process long enough to power stars



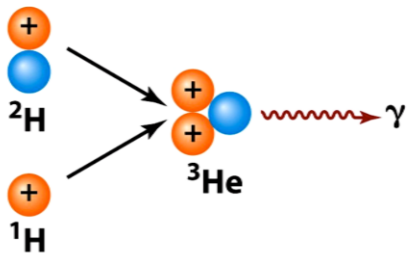

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**Step 1:**

- Two protons (hydrogen nuclei, $^1$H) collide.
- One of the protons changes into a neutron (shown in blue), a neutral, nearly massless neutrino ($\nu$), and a positively charged electron, or positron ($e^+$).
- The proton and neutron form a hydrogen isotope ($^2$H).
- The positron encounters an ordinary electron ($e^-$), annihilating both particles and converting them into gamma-ray photons ($\gamma$).

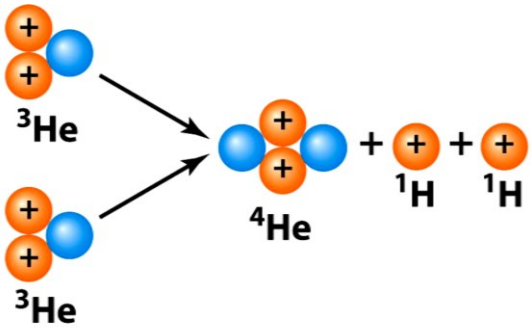
7




**Step 2:**

- The $^2$H nucleus from the first step collides with a third proton.
- A helium isotope ($^3$He) is formed and another gamma-ray photon is released.

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**Step 3:**

- **Two $^3$He nuclei collide.**
- **A different helium isotope with two protons and two neutrons ($^4$He) is formed and two protons are released.**

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# The P-P cycle

$$p + p \rightarrow d + e^+ + \nu_e$$

$$d + p \rightarrow {}^3He + \gamma$$

$${}^3He + {}^3He \rightarrow {}^4He + 2p$$

**CHAIN I**
Q = 26.20 MeV

$${}^3He + {}^4He \rightarrow {}^7Be + \gamma$$

$${}^7Be + e^+ \rightarrow {}^7Li + \nu_e$$
$${}^7Li + p \rightarrow {}^4He + {}^4He$$

**CHAIN II**
Q = 25.66 MeV

$${}^7Be + p \rightarrow {}^8B + \gamma$$
$${}^8B \rightarrow {}^8Be + e^+ + \nu_e$$
$${}^8Be \rightarrow {}^4He + \alpha$$

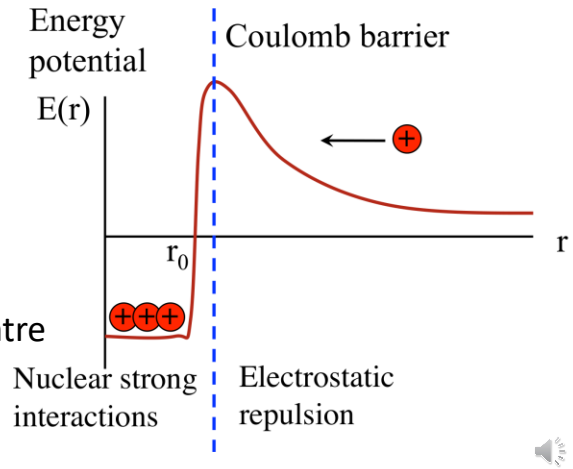
**CHAIN III**
Q = 19.17 MeV

Second independent H → He channel also exists
CNO cycle uses carbon, nitrogen and oxygen as catalysts

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# Coulomb barrier

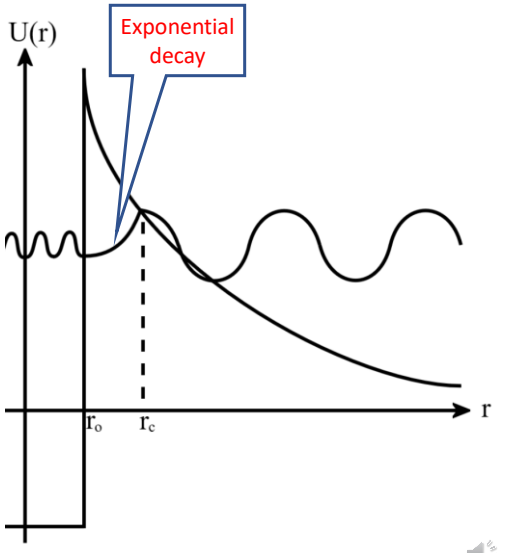
- But can fusion actually happen in stars?
- Consider classical Coulomb repulsion between protons
- $\frac{1}{2}\mu_m \overline{v^2} = \frac{3}{2}kT_{\text{class}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r_0}$
- $T = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 k r_0}$
- For $r \sim 10^{-15} m \rightarrow T_{\text{class}} \sim 10^{10}\text{K}$
- $T_{\text{class}} \gg T_\odot \sim 1.6 \times 10^7 \text{K}$ at Sun centre




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# Quantum mechanical tunnelling

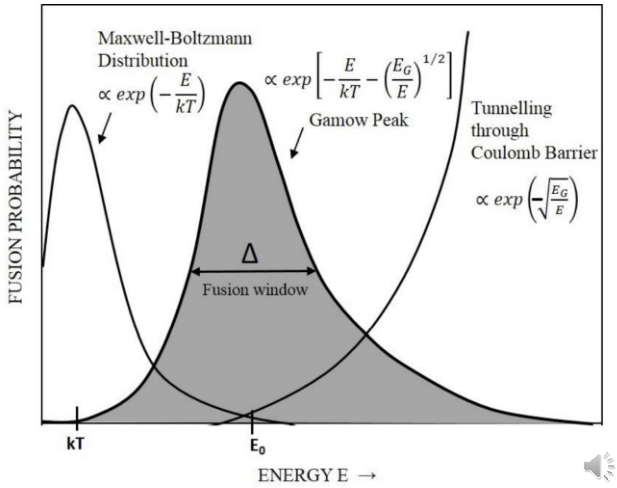
- Nuclear fusion happens via quantum mechanical tunnelling
- Heisenberg principle implies particle position not defined better than its wavelength $\lambda$
- Atoms can tunnel when de Broglie wavelength
- $\lambda = h/p \gtrsim r_0$
- $\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m} = \frac{(h/\lambda)^2}{2\mu_m} = E(\lambda) = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda}$
- $\rightarrow \lambda = 2h^2\pi\epsilon/(e^2\mu Z_1 Z_2)$
- $T_{\text{quant}} = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 k\lambda} = \frac{e^4 \mu Z_1^2 Z_2^2}{12h^2 k\pi^2\epsilon_0^2} \sim 10^7 K$
- For 2 protons, consistent with Sun temperature




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# Gamow peak

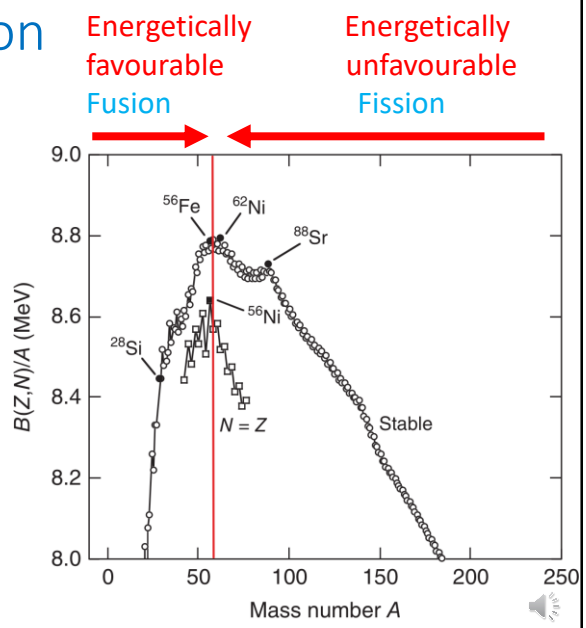
- Tunnelling probability steeply increases with particle energy
- But number of particles at given energy decreases exponentially at large energies (Maxwell-Boltzmann tail)
- Nonzero probability for a restricted range of energies/temperatures
- The allowed range is the Gamow peak




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# Binding energy per nucleon

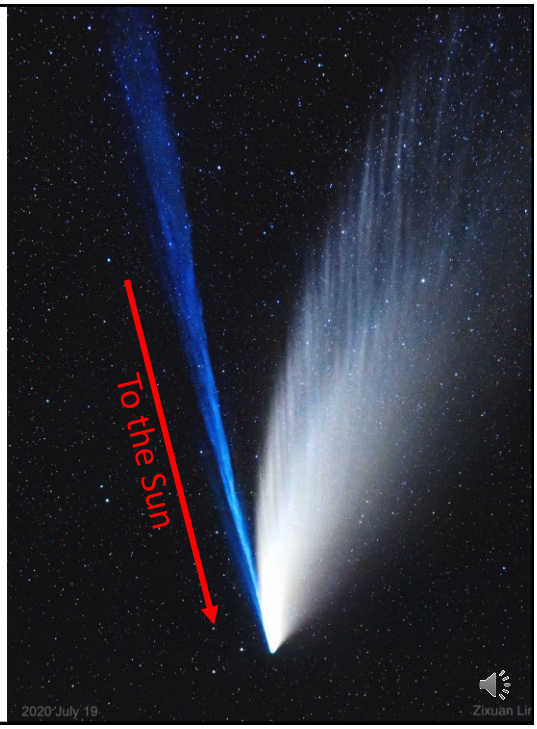
- Total nuclear mass < mass nucleons
- $m_{\mathrm{nuc}} = Zm_p + Nm_n - \Delta m$
  - Z: number of protons
  - N: number of neutrons
- Binding energy from $E = mc^2$
- $B(Z,N) = \left(Zm_p + Nm_n - m_{nuc}\right)c^2$
- Most bound nucleus is $^{56}$Fe
- Fusion up to $^{56}$Fe produces energy
- Producing heavier elements requires energy input




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# Radiation pressure

- General pressure integral for particles
- $P = \frac{1}{3}\int_0^\infty pv\, n_p dp$
- For photons $p_\nu = \frac{h\nu}{c}$ and $v = c$
- Change of variable $n_p dp = n_\nu d\nu$
- $P_{\rm rad} = \frac{1}{3}\int_0^\infty p_\nu c\, n_\nu d\nu = \frac{1}{3}\int_0^\infty h\nu\, n_\nu d\nu$
- $P_{\rm rad} = \frac{1}{3}\int_0^\infty u_\nu\, d\nu = \frac{1}{3}aT^4$




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# Energy-transport equation

- From radiative-transfer solution at large optical depth the Eddington flux
- $\frac{dP_{\rm rad}}{dr} = -\frac{k_R\rho}{c}F_{\rm rad}$ with $F_{\rm rad}$ the outward radiative flux
- Using the radiation-pressure equation $P_{\rm rad} = \frac{1}{3}aT^4$
- $\frac{dP_{\rm rad}}{dr} = \frac{\partial}{\partial T}\left(\frac{1}{3}aT^4\right)\frac{dT}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}$
- Equating the two expressions
- $\frac{dT}{dr} = -\frac{3k_R\rho}{4acT^3}F_{\rm rad}$ with $F_{\rm rad} = \frac{L(r)}{4\pi r^2}$
- $\frac{dT}{dr} = -\frac{3k_R\rho}{64\pi r^2\sigma T^3}L(r)$
- Luminosity increase → larger T gradient to transfer extra energy
- Opacity $k_R$ increase → larger T gradient to compensate difficulty of photons to cross the medium

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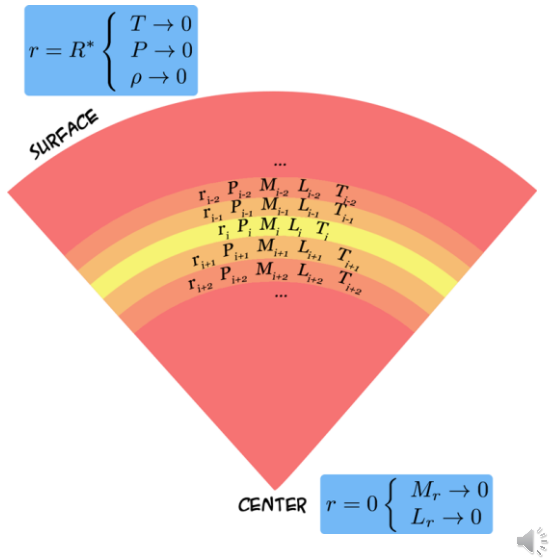
# Summary: equations of stellar structure

- $\frac{dP}{dr} = -G\frac{\rho(r)M(r)}{r^2}$ hydrostatic equilibrium
- $\frac{dM(r)}{dr} = 4\pi r^2\rho(r)$ mass conservation
- $\frac{dL(r)}{dr} = 4\pi r^2\rho(r)\epsilon$ thermal equilibrium
- $\frac{dT}{dr} = -\frac{3k_R\rho}{64\pi r^2\sigma T^3}L(r)$ energy transport
- Supplemented by relations for
  - pressure $P(\rho, T, \text{composition})$
  - Energy production $\epsilon(\rho, T, \text{composition})$
  - Opacity $k_R(\rho, T, \text{composition})$
- These apply in the absence of convection

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# Vogt-Russell non-theorem

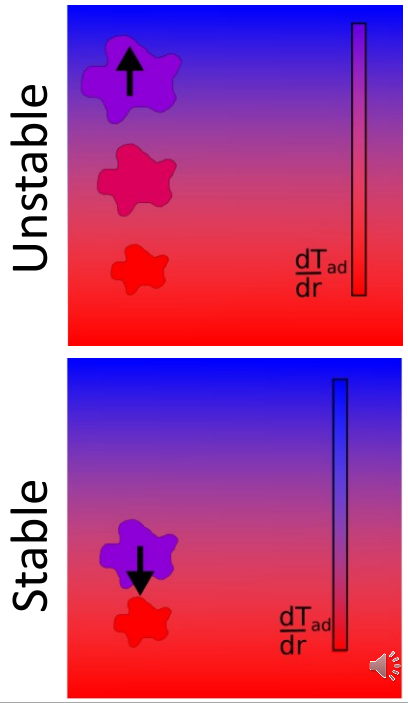
- The mass and composition of a star uniquely determine its radius, luminosity, and internal structure, as well as its subsequent evolution
- Historically wrongly called “theorem”
- Actually represent general outcome of the equations solution and observations



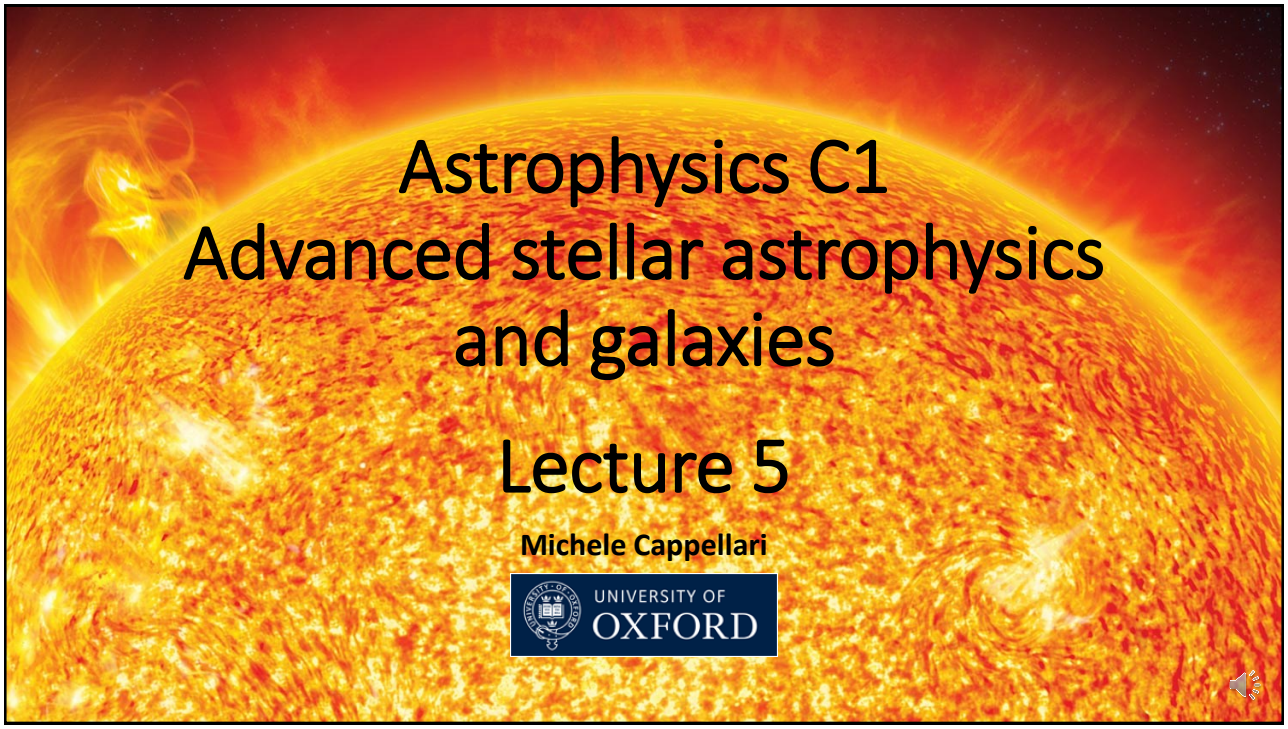

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# Convective instability

- Consider a rising bubble of gas
- Without energy exchange it quickly expands adiabatically at the speed of sound
- If adiabatic temperature change is smaller than the surrounding, it will be hotter and keeps rising
- $\left(\frac{dT}{dr}\right)_{\mathrm{ad}} < \left(\frac{dT}{dr}\right)_{\mathrm{rad}}$ → convectively unstable
- $\left(\frac{dT}{dr}\right)_{\mathrm{ad}} = \left(\frac{dT}{dP}\frac{dP}{dr}\right)_{\mathrm{ad}} = \frac{dP}{dr}\left(\frac{dT}{dP}\right)_{\mathrm{ad}}$
- $\left(\frac{dT}{dP}\right)_{\mathrm{ad}} < \left(\frac{dT}{dP}\right)_{\mathrm{rad}} \Rightarrow \frac{P}{T}\left(\frac{dT}{dP}\right)_{\mathrm{ad}} < \frac{P}{T}\left(\frac{dT}{dP}\right)_{\mathrm{rad}}$
- Schwarzschild criterion for convective instability for layers in radiative equilibrium
- $\nabla_{\mathrm{ad}} = \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{ad}} < \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{rad}} = \nabla_{\mathrm{rad}}$




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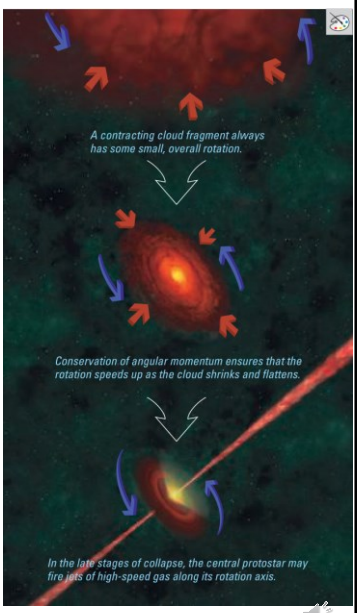

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# Stellar evolution

2

# From cloud to proto-star

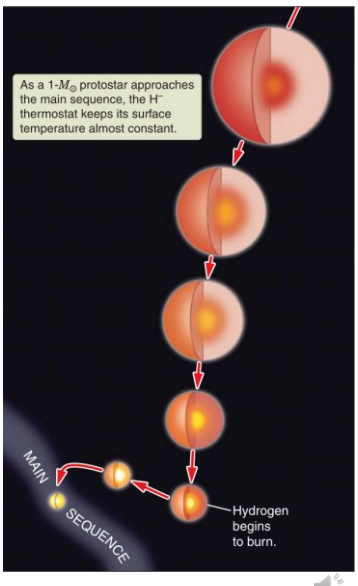
- Consider $1M_{\odot}$ molecular cloud
- Initially radiation has long free path before absorption
- Initial collapse is isothermal (energy radiated, T constant)
- Happens of free-fall time scale
- Above $\rho \sim 10^{-10}\mathrm{kg/m^3}$ becomes optically thick (dusty)
- Collapse more adiabatic (limited external radiation)
- Protostar in near hydrostatic equilibrium with $R \sim 5\mathrm{AU}$
- Contracts on Kelvin–Helmholtz timescale (40 Myr)
- One can start following evolutionary track




3

# Pre-main-sequence evolution

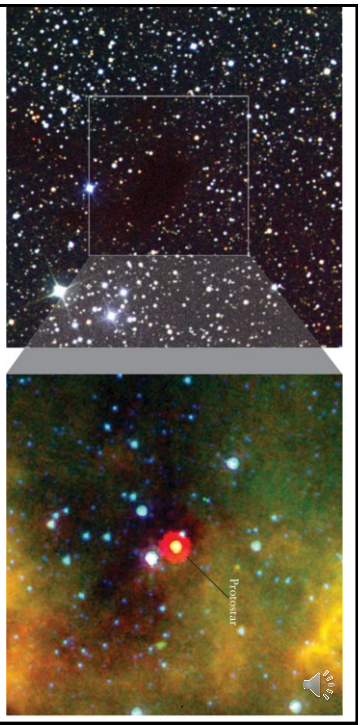
- Consider $1M_{\odot}$ prot-star
- Interior optically thick due to $H^-$
  - → Produced by heavier elements with lower ionization potential
  - → Energy transported by convection keeping star well stirred
- $H^-$ acts as a thermostat
  - Colder star allows for more $H^-$ → less energy escape and star gets hotter
  - Hotter star ionizes $H^-$ → energy radiated efficiently and stars cools down
- As star shrinks the interior grows hotter by virial theorem
- Equal surface temperature but shrinking radius
  - → Decreasing luminosity by Stefan-Boltzmann
- But surface temperature nearly constant
  - → Moves along nearly vertical Hayashi tracks on H-R
- Star is defined as the start of H burning




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# Example: proto-star luminosity evolution

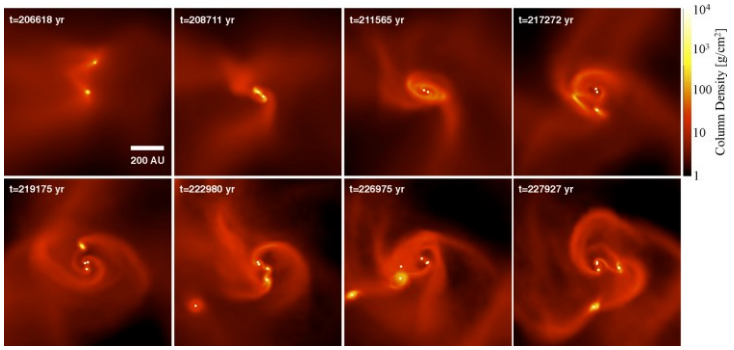
- When the Sun was 100x larger it had $T \approx 3500$ K
- The current surface temperature is $T \approx 5800$ K
- $L = 4\pi R^2 \sigma T^4$
- $\frac{L_{\mathrm{prot}}}{L_\odot} = 100^2 \left(\frac{3500}{5800}\right)^4 \approx 1{,}300$
- Protostars are very bright but hidden in dust
- Generally only visible in the infrared




5

# Many/most stars are multiple

- Proto-stellar disk fragment
- Produces multiple cores
- Result in multiple co-eval stellar systems
- Solves the angular momentum problem
- Most of the cloud angular momentum end up in orbital one
- Stars do not rotate too fast
- Theory and observation predict many/most stars are multiple
- Important effect on stellar evolution

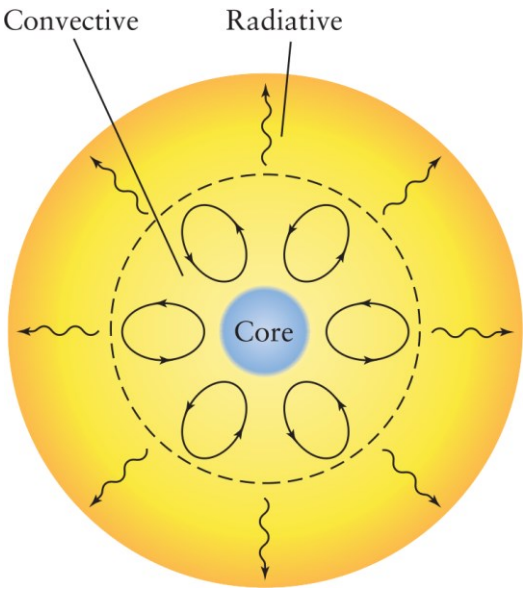



Bathe 2018

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# High-mass protostar

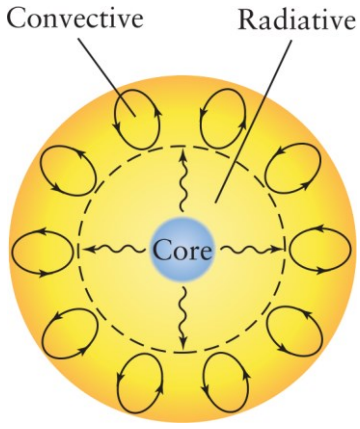
- Contracts and heats up rapidly
- Fusion begins early
- Luminosity quickly stabilizes
- T keeps rising during collapse
- Moves from right to left on H-R
- Large mass → large luminosity
- Large luminosity → large temperature gradient
- $\nabla_{\mathrm{adi}} < \nabla_{\mathrm{rad}}$ in the centre → central convection




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# Solar mass protostar

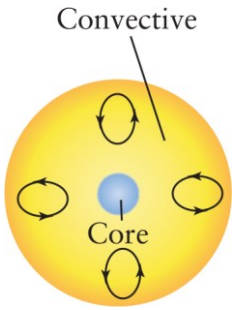
- Inner layers fully ionized
- Outer layers quite cool
- Dominated by $H^-$
- Most photons absorbed by $H^-$
- Makes outer region opaque
- Inefficient radiation transport
- Large opacity → large temperature gradient
- $\nabla_{\mathrm{adi}} < \nabla_{\mathrm{rad}}$ in outer parts → outer convection




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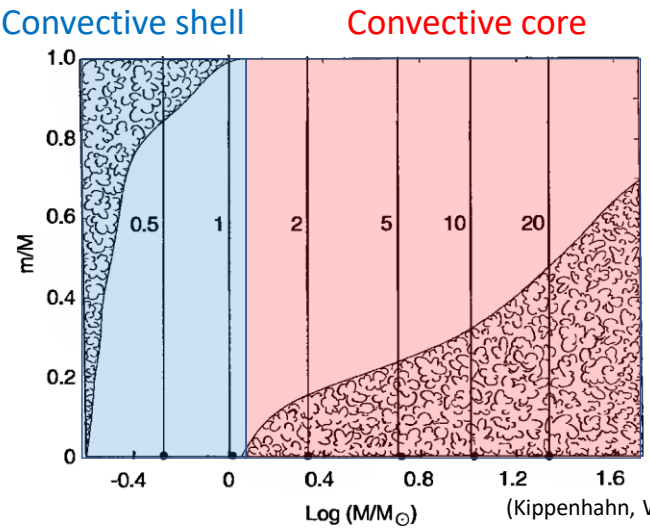
# Low-mass protostar

- Interior never fully ionized
- Very opaque to radiation
- Inefficient radiation transport
- Large opacity → large temperature gradient
- $\nabla_{adi} < \nabla_{rad}$ everywhere
- Energy transported by convection throughout the star volume




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# Convective zones vs stellar mass




(Kippenhahn, Weigert 1990)

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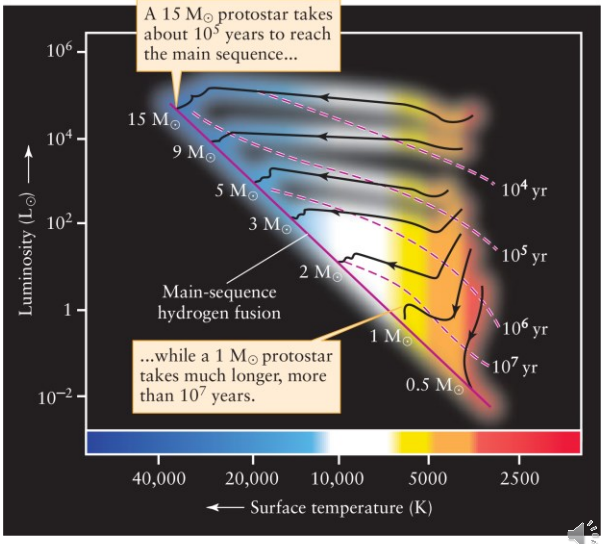
# Proto-star evolution vs mass

- Low-mass stars take much longer to reach main sequence
- Nearly-constant temperature

→ Nearly vertical tracks

- High-mass stars must radically increase surface temperature
- Much larger luminosities

→ Nearly horizontal tracks



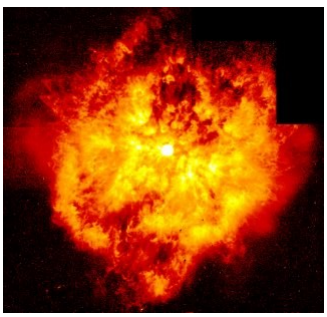

Freedman & Kaufmann 2008

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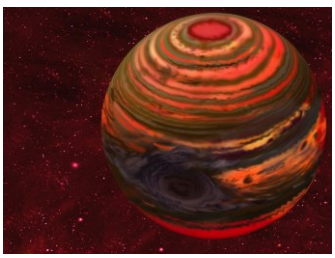
gtrsim

# Limits on stellar masses

- Proto-stars with $M \gtrsim 200\ M_\odot$ do not reach the main sequence
- They explode releasing their mass before reaching equilibrium
- Proto-stars with $M \lesssim 0.08\ M_\odot$ do not reach high-enough temperature to star H fusion
- They are called brown dwarfs
- They cool down by Kelvin-Helmholtz contraction



*WR124: Grosdidier, Moffat, Joncas, Acker, Morse and NASA.*

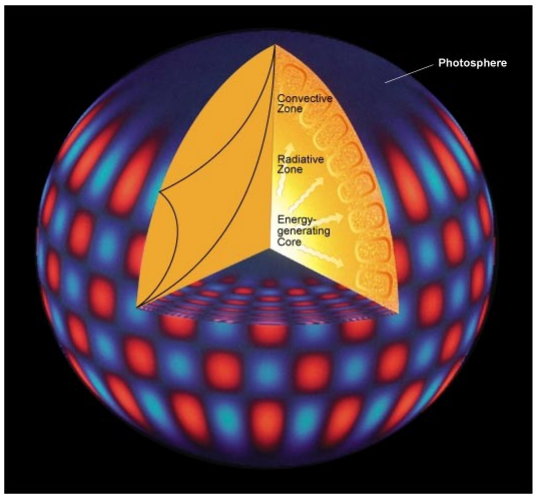


JON LOMBERG artist impression

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# Helioseismology

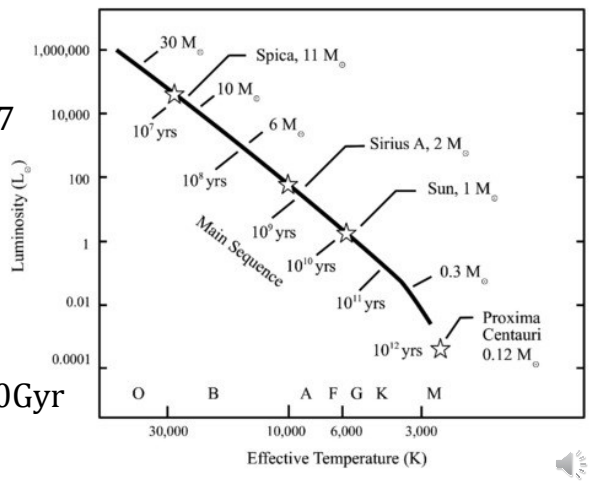
- Very accurate surface motions can be measured for the Sun
- Using doppler shift
- Analogue to using Earthquake to study Earth interior
- Gives direct measure of Sun density profile
- Convection layers can be detected directly
- Crucial test for stellar models




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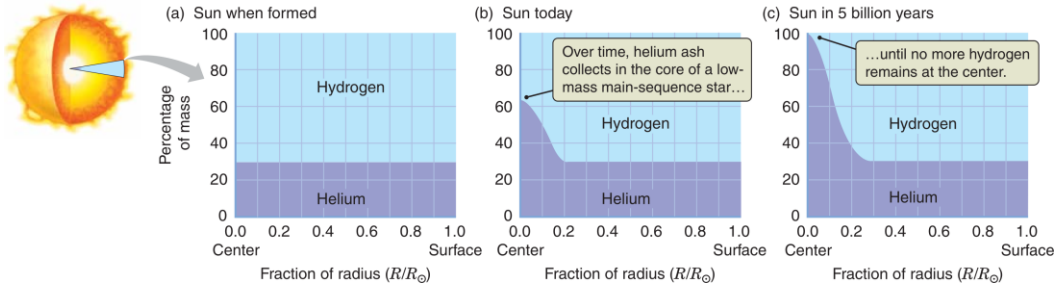
# Main-sequence lifetimes

- Total energy by H fusion in a star
- $E = fMc^2$
- $f = \epsilon\, f_{\mathrm{H-fus}} \approx (0.7\%) \times 0.1 = 0.0007$
- $E = Lt_{\mathrm{MS}}$
- → $t_{\mathrm{MS}} = \frac{fMc^2}{L} \propto \frac{M}{L}$
- Empirically $L \propto M^{3.5}$
- → $t_{\mathrm{MS}} \propto M^{-2.5}$
- For reference $t_{MS\odot} \approx 1 \times 10^{10}\mathrm{yr} = 10\mathrm{Gyr}$




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# Chemical evolution $1M_{\odot}$ on main sequence




- Radiative energy transfer in the centre: no mixing
- $\mathrm{H} \rightarrow \mathrm{He}$ conversion accumulates He in the core
- H fusion moves outwards
- Central temperature not enough to fuse He
- Decreased radiation pressure in the core to support gravity
- → density increases until electron degeneracy

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# Electron degeneracy

- By Heisenberg principle a particle location and momentum satisfy

$$\Delta V \Delta^3 \mathrm{p} \geq \mathrm{h}^3$$

- When gas is compressed the particle's volume $\Delta V \propto \rho^{-1}$
- If density large enough quantum effects become important for electrons
- Writing the momentum element $\Delta^3 \mathrm{p}$ in spherical coordinates

$$\Delta V \Delta^3 \mathrm{p} = \Delta V \, 4\pi p^2 dp \geq \mathrm{h}^3$$

- Pauli exclusion principle: two electrons (spin up/down) allowed per $\Delta V$ implies a limit in the electron density

$$n_e(p)dp = \frac{2}{\Delta V} \leq \frac{2}{h^3} 4\pi p^2 dp$$

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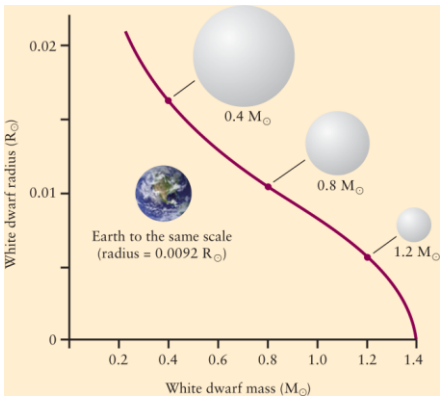
# Degenerate gas pressure

- In complete degeneracy, all cells below a threshold Fermi momentum $p_F$ are filled and the inequality becomes an equality
- $n_e(p)dp = \frac{2}{h^3}4\pi p^2 dp \Rightarrow N_e = \int_0^{p_F} \frac{2}{h^3}4\pi p^2 dp = \frac{2}{h^3}\frac{4\pi}{3}p_F^3 \Rightarrow p_F = \frac{h}{2}\left(\frac{3N_e}{\pi}\right)^{1/3}$
- From general kinetic theory the pressure integral is
- $P_e = \frac{1}{3}\int_0^{\infty} pv(p)n(p)dp$
- Non-relativistic case: $v(p) = p/m_e$
- $P_e = \frac{1}{3}\int_0^{p_F} pv(p)n_e(p)dp = \frac{1}{3}\int_0^{p_F} \frac{p^2}{m_e}\frac{2}{h^3}4\pi p^2 dp = \frac{8\pi}{15m_e h^3}p_F^5 = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3} N_e^{5/3}$
- Relativistic case: $v(p) = c$
- $P_e = \frac{1}{3}\int_0^{p_F} pv(p)n_e(p)dp = \frac{1}{3}\int_0^{p_F} pc\frac{2}{h^3}4\pi p^2 dp = \frac{2\pi c}{3h^3}p_F^4 = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} N_e^{4/3}$

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# Mass-radius of degenerate stars

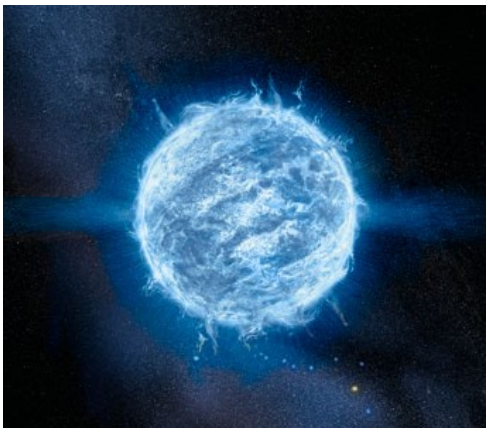
- From hydrostatic equilibrium with constant density
- $\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} = -\frac{G\left(\frac{4}{3}\pi r^3\rho\right)\rho}{r^2} = -\frac{4}{3}\pi G\rho^2 r$
- Integrating with boundary $P = 0$ at surface $r = R_*$
- $\int_0^{R_*} dP = P(R_*) - P(0) = -\int_0^{R_*}\frac{4}{3}\pi G\rho^2 r\, dr$
- $P(0) = \frac{2}{3}\pi G\rho^2 R_*^2$
- Equating hydrostatic and degeneracy $P(0) = P_e$ gives
- $\frac{2}{3}\pi G\rho^2 R_*^2 = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3} N_e^{5/3}$
- Use $N_e \approx \rho\ /\ m_H$ and $\rho \approx M_*\ /\ \left(\frac{4}{3}\pi R_*^3\right)$
- Obtain $R \propto M_*^{-1/3}$ or $\rho \propto M_*^2$
- Adding mass keeps shrinking star until electrons are relativistic $p_e \sim m_e c$




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# Chandrasekhar mass 1

- Consider star with N fermions (electrons or neutrons)
- Mass fermion $M_f = \mu_f m_H$
- Number density $n \sim N/R_*^3$
- Volume per fermion $\Delta V \sim 1/n$
- Heisenberg uncertainty principle $\Delta V \Delta^3 \mathrm{p} \sim \mathrm{h}^3$
- Typical momentum $p \sim hn^{1/3}$
- Fermi energy of relativistic particle
- $E_f = cp \sim chn^{1/3} \sim ch\frac{N^{1/3}}{R_*}$
- Gravitational energy per fermion
- $E_g \sim -\frac{GMM_f}{R_*} = -\frac{G(N\mu_f m_H)(\mu_f m_H)}{R_*}$

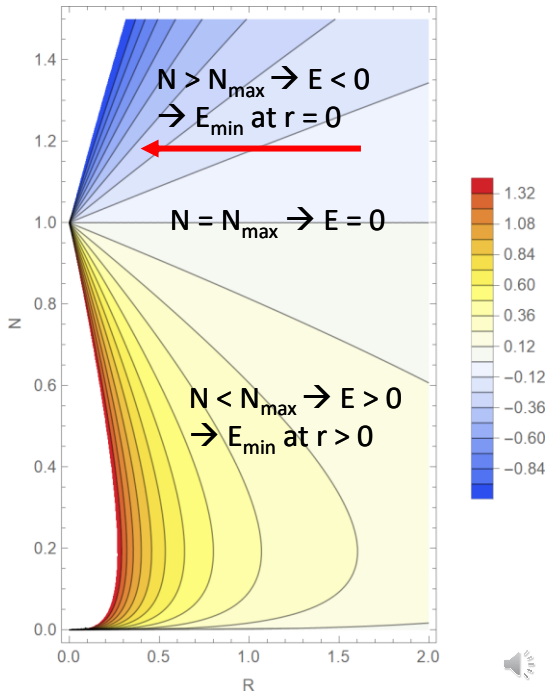


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# Chandrasekhar mass 2

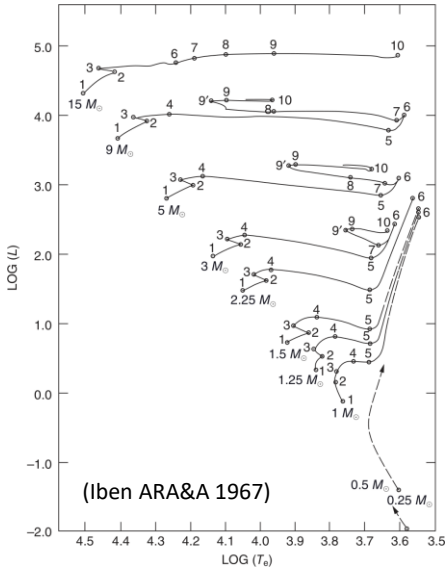
- Total energy per fermion
- $E = E_f + E_g \sim \frac{chN^{1/3}}{R_*} - \frac{GN(\mu_f m_H)^2}{R_*} = \frac{1}{R_*} f(N)$
- Stable configuration has minimum total energy
- Above $N_{\max}$ one has $E < 0$
- *E* can decrease by lowering $R_*$ without limit
- → gravitational collapse
- Below $N_{\max}$ the *E* is minimized at given *R*
- $N_{\max} \Rightarrow E = 0 \Rightarrow N_{\max} \sim \left[\frac{hc}{G(\mu_f m_H)^2}\right]^{3/2}$
- $M_{\max} \sim N_{\max}(\mu_f m_H) \sim 1.5\, M_\odot$
- Chandrasekhar mass for white dwarfs
- $M_{Ch} = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_\odot$ with $\mu_e \approx 2$
- Analogue limit for neutron stars maximum mass




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# Caveat on understanding evolutionary tracks

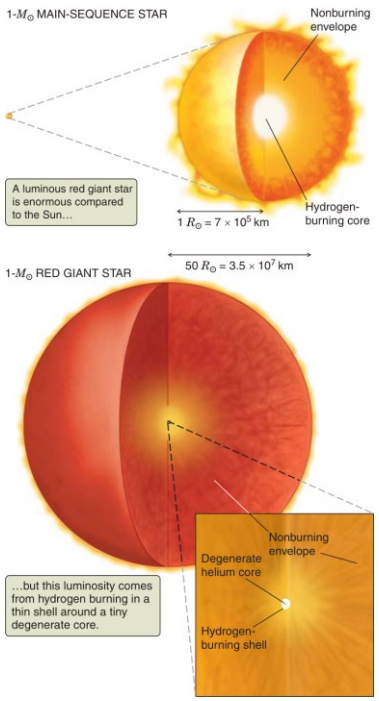
- We can use physical insight to understand the overall tracks on H-R
- But not all characteristics have simple qualitative explanations
- Equations to solve are coupled and non-linear
- Ultimately the tracks come from numerical computation
- What we really understand are the mathematical equations to solve




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# $1M_\odot$ star becoming red giant

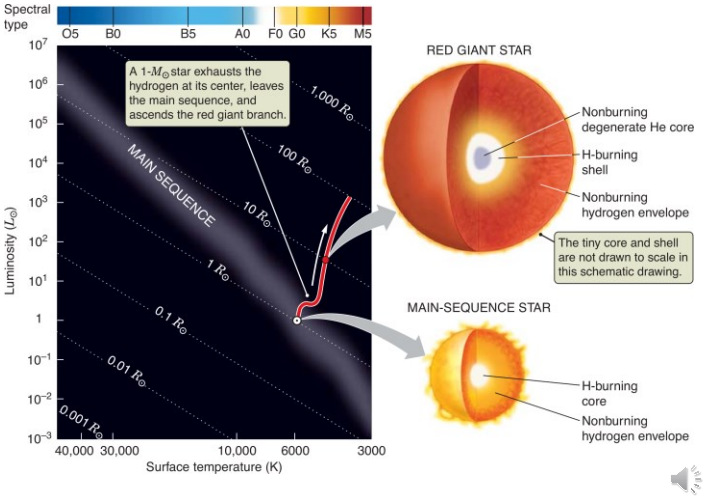
- When all H converted into He in the core
- H fusion stops in the He core
- No radiation pressure: He core contracts
- He core becomes electron degenerate
- $(\rho, T)$ increase in H-burning shell
- → Energy production $L$ increases
- Perfect gas $T \nearrow \Rightarrow P \nearrow \Rightarrow R \nearrow$
- Outer layers expand: larger $R_*$
- Surface temperature decreases $L = 4\pi\sigma R_*^2 T^4$



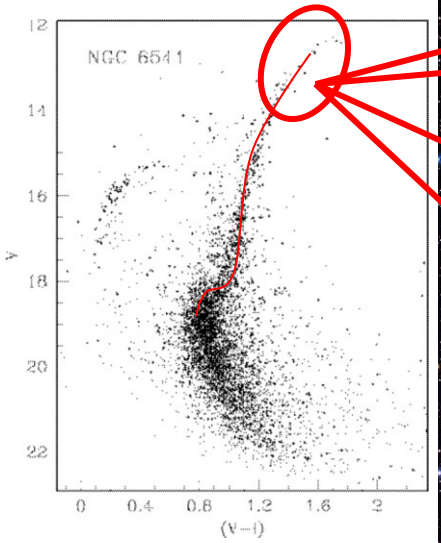

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# $1M_{\odot}$ rising on red-giant branch

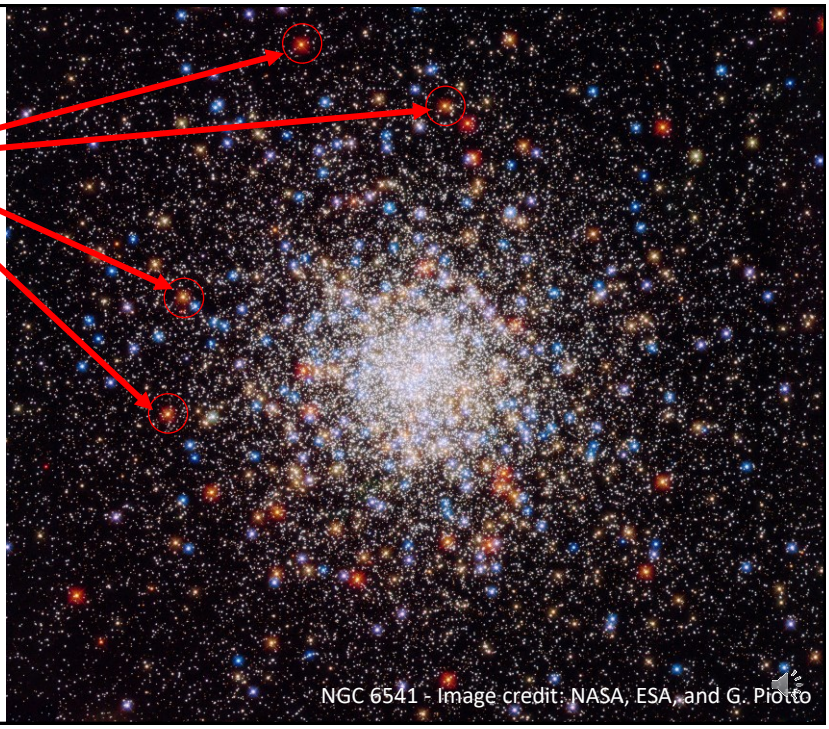
- When envelop expands it cools down until $H^-$ dominates
- Like during protostar phase $H^-$ acts as thermostat
- Giant approximately follows inverse of protostar path
- L rises at nearly-constant T
- Runaway process
  - Larger pressure
  - → More H burning in shell
  - → Faster He core growth
  - → More massive and dense core
  - → Even larger pressure… and so on
- Evolution speeds up
- Until He burning stars in core




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Rosenberg+2000



NGC 6541 - Image credit: NASA, ESA, and G. Piotto

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# Triple-$\alpha$ process

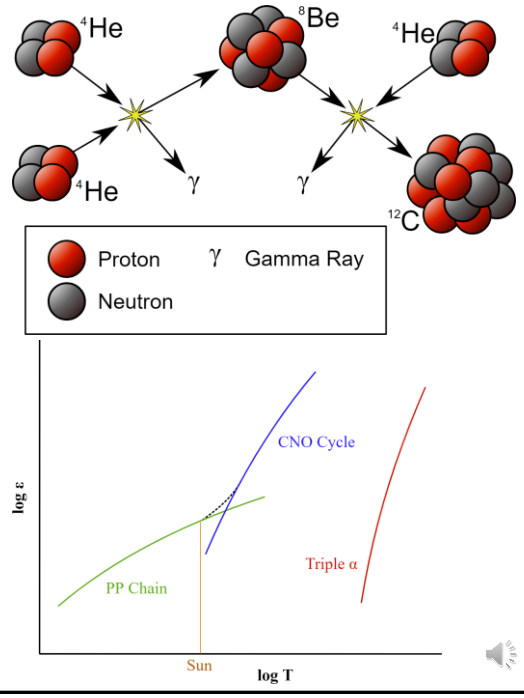
- Named from $\alpha$ particles = He
- Kind of three-body interaction
- $^{8}$Be is highly unstable and decays
- But collision with $^{4}$He sufficiently frequent at high density and temperature
- It has a very steep temperature dependence

$$\epsilon \propto \left(\frac{T}{10^8 K}\right)^{41}$$

- At He-burning temperature other minor competing reactions occur:

$$^{12}_{6}\mathrm{C} + ^{4}_{2}\mathrm{He} \rightarrow ^{16}_{8}\mathrm{O} + \gamma$$

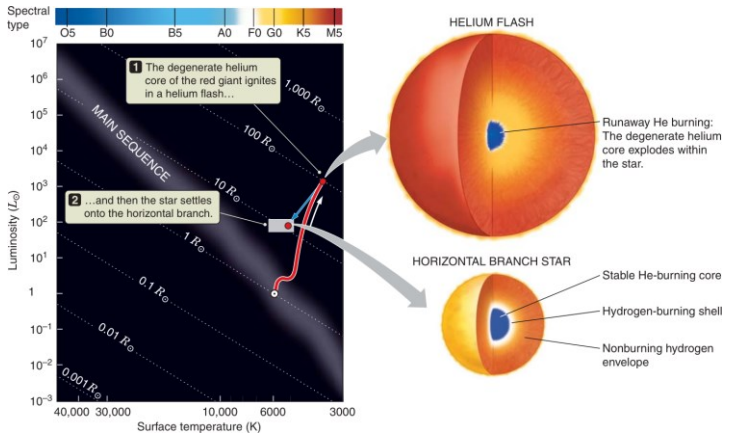
$$^{16}_{8}\mathrm{O} + ^{4}_{2}\mathrm{He} \rightarrow ^{20}_{10}\mathrm{Ne} + \gamma$$



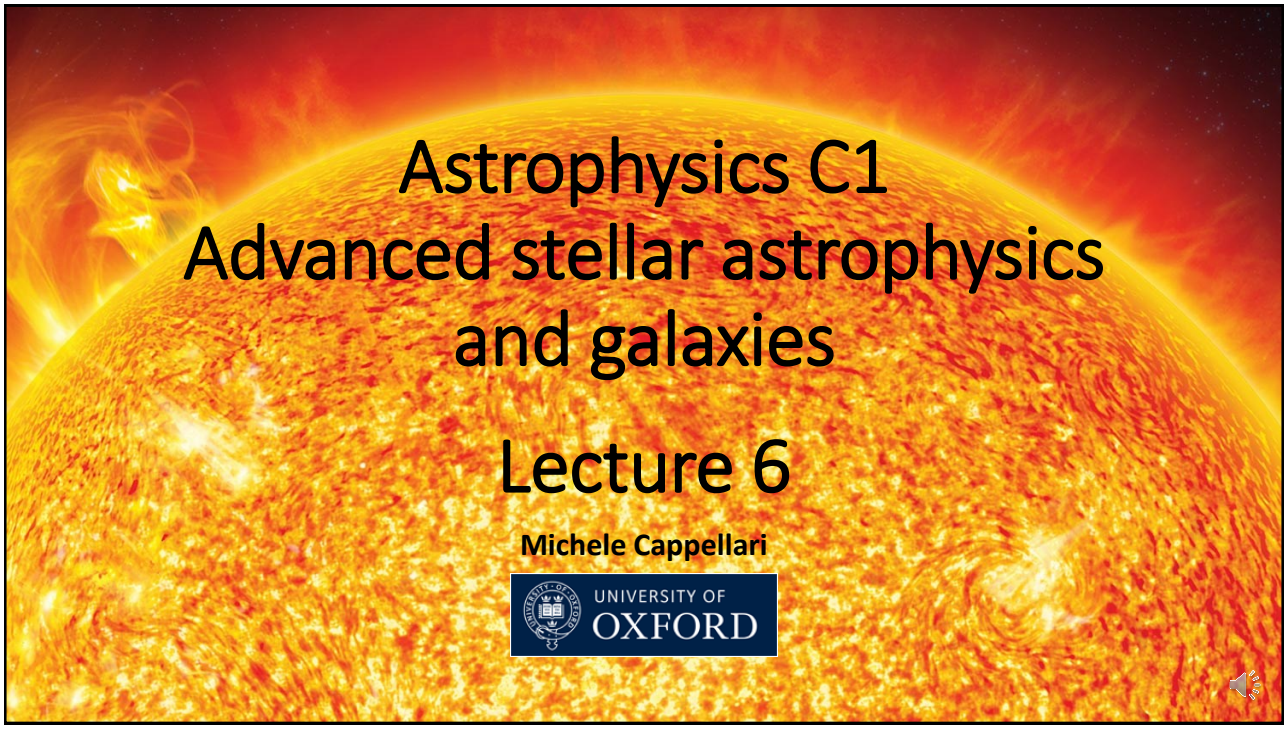

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# $1M_{\odot}$ from helium flash to horizontal branch

- When He burning starts
- Core temperature increases ($T \nearrow$)
- In perfect gas:
  - $T \nearrow \Rightarrow P \nearrow \Rightarrow R \nearrow$ (expansion)
- In degenerate gas
  - P independent of T
  - No self-regulation
- Helium flash runaway process
  - $T \nearrow$ implies faster reaction
  - With steep $\epsilon$ dependency
  - Implies more $T \nearrow$… and so on
- Core explodes inside the star
- Expansion finally cools down core
- Star settles on He burning at lower L



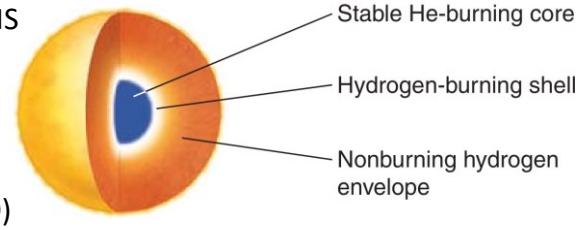

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1

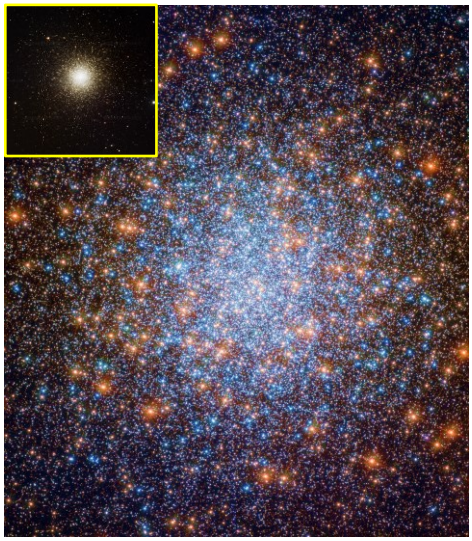
## $1M_{\odot}$ on the horizontal branch

- Horizontal branch is analogue to main sequence
- Star fusing $\mathrm{He} \rightarrow \mathrm{C}$ inside core
- $\mathrm{H} \rightarrow \mathrm{He}$ fusion continues in shell
- Life on horizontal branch $\sim 1/100$ of MS
  - Efficiency $\epsilon_{3\alpha} \approx 0.1 \times \epsilon_{p-p}$
  - Mass in He core is small
  - Luminosity larger than on main sequence
- Star leaves horizontal branch when He exhausted and core is made of C (and O)

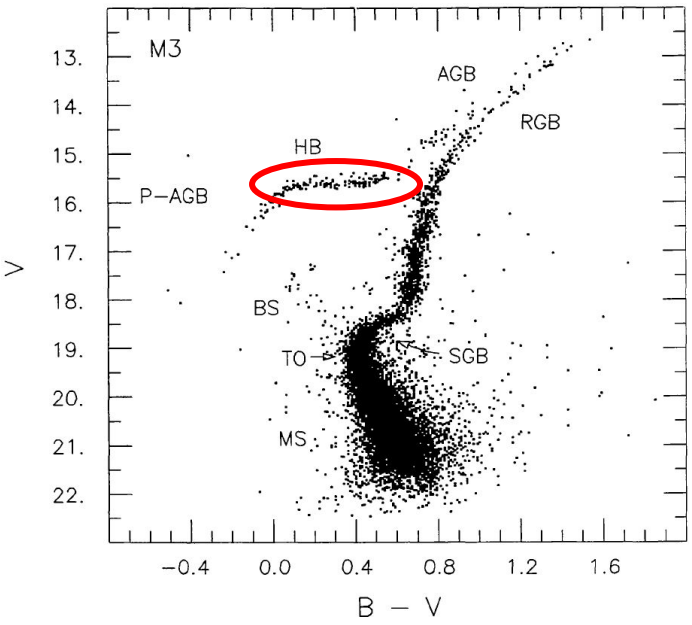



2

# Horizontal branch



M3 - credit: NASA / ESA / Hubble / G. Piotto *et al*.

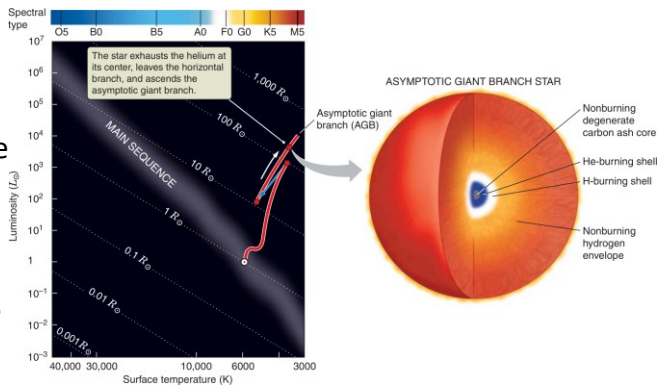



Renzini & Fusi Pecci 1988 ARA&A

3

# $1M_{\odot}$ up the asymptotic giant branch (AGB)

- Non-burning C core
- C core becomes degenerate
- $\mathrm{He} \rightarrow \mathrm{C}$ fusion in first shell
- $\mathrm{H} \rightarrow \mathrm{He}$ fusion in second shell
- Like earlier, the P in degenerate C core is insensitive to temperature
- Piling up of C increases P on He shell
- Energy production increases
- Star expands as in giant-branch phase
- But there is no “C flash”




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# $1M_{\odot}$ thermal-pulse asymptotic giant branch (TP-AGB)

- Detailed process simulated numerically
- He shells energy production is quasi periodic
- Repeated He flashes in degenerate shell
- H burning shell turns on/off
- Example: Mira-class variable stars
- Dredge-up phases with convection extending deep into star
- Bring heavy metal to surface
- Chemical surface abundance is an observable test of the theory

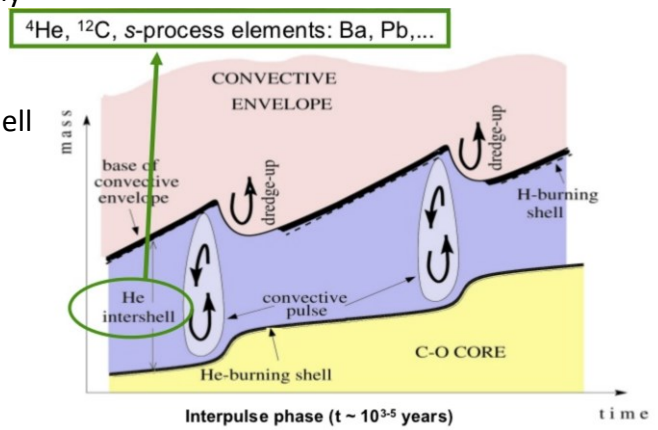



Image credit: Maria Lugaro

5

# Low escape velocities on giant branches

- Stars on AGB can have size $\mathrm{r} \sim 100R_{\odot}$
- Escape velocity is $v_{\mathrm{esc}} = \sqrt{2GM/r}$
- For the Sun $v_{\mathrm{esc},\odot} \sim 615\ \mathrm{km\ s^{-1}}$
- If mass was unchanged, the ABS star would have $v_{\mathrm{esc}}^{\mathrm{AGB}} \approx 0.1 v_{\mathrm{esc},\odot} \approx 60\ \mathrm{km\ s^{-1}}$
- But actual mass is even smaller
- Escape velocity is low → mass can escape



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# $1M_{\odot}$ mass loss on AGB

- During both giant-branch and AGB phases stars lose mass
- About 30% of mass is lost
- Giants stars often surrounded by dust
- TP-AGB pulsations lead to irregular mass ejection in shells
- Large variety of shapes & asymmetry likely driven by stellar companions

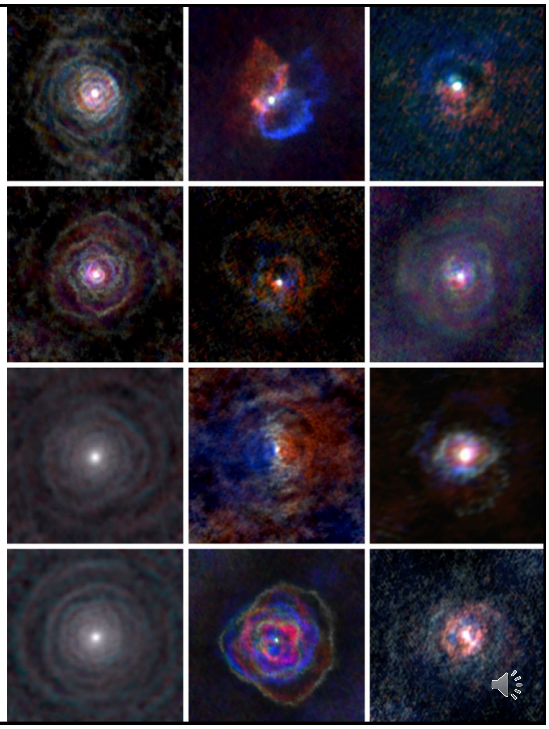
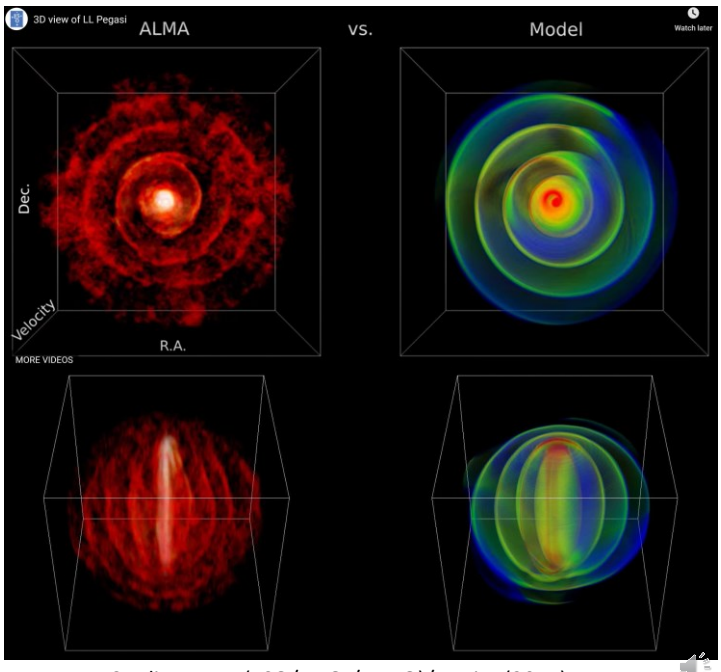


Image credit: L. Decin/ESO/ALMA (Decin et al. 2020)

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# Example LL Pegasi

- Spiral in ejecta
- Evidence for influence of orbiting companion
- Consistent with detailed models

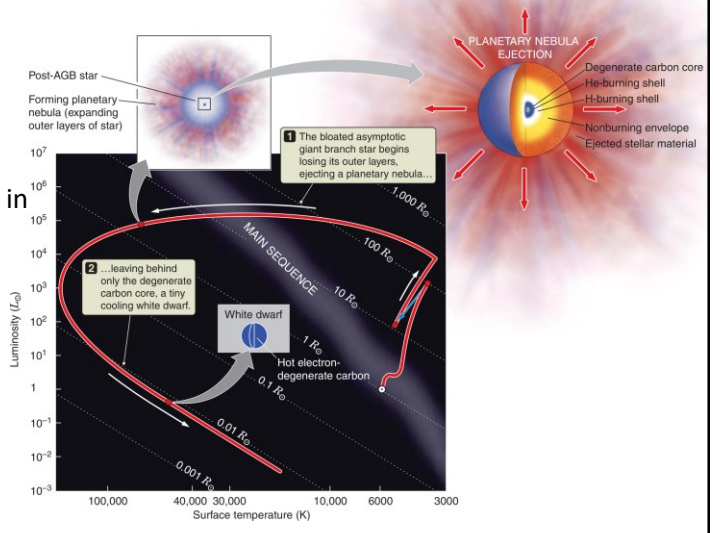



Credit: ALMA (ESO/NAOJ/NRAO)/H. Kim (2017)

8

# $1M_{\odot}$ planetary nebula formation

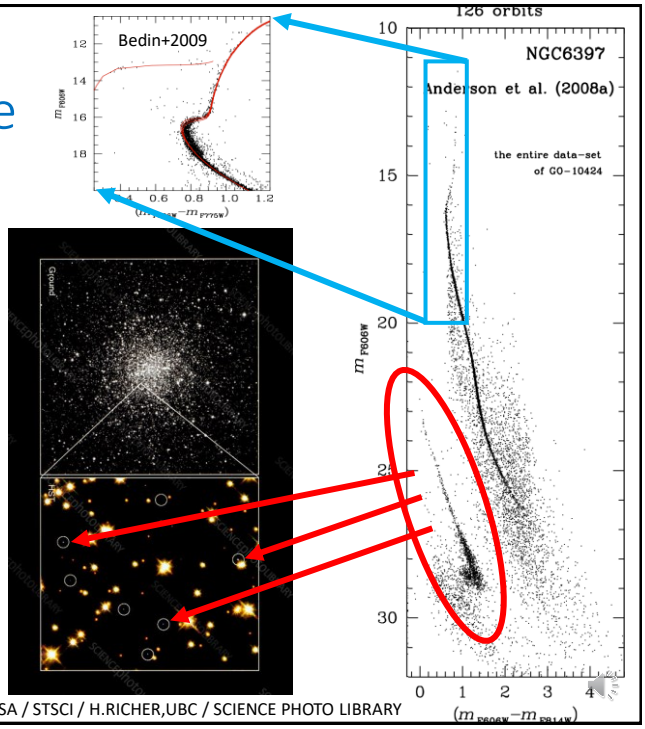
- C-O core becomes degenerate
- (Similar to evolution before He flash)
- $P \nearrow \Rightarrow T \nearrow \Rightarrow L \nearrow$ in He and H shells
- But not enough mass for C fusion to start in core → no C flash
- Planetary nebula is runaway process
  - Increased L speeds-up mass loss
  - Decreased mass lowers $v_{\rm esc}$
  - Lower $v_{\rm esc}$ increases mass loss… and so on
- Outer layers ejected reveal C-O core
- Much increased surface temperature
- Star ionizes ejected material → nebula
- Timescale $10^4$yr




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# White-dwarfs sequence

- White dwarfs are exposed degenerated C-O cores
- No energy production $L = 4\pi\sigma R^2 T^4$
- They slowly cool down
- May crystallize into stellar diamonds
- Maximum mass at Chandrasekhar limit of $1.4M_{\odot}$
- Narrow mass range $M_{\rm WD} \approx 0.6\ M_{\odot}$
- End-point of stellar evolution when $M \lesssim 7\ M_{\odot}$
- Very faint: require very deep observations from space




Credit NASA / ESA / STSCI / H.RICHER,UBC / SCIENCE PHOTO LIBRARY

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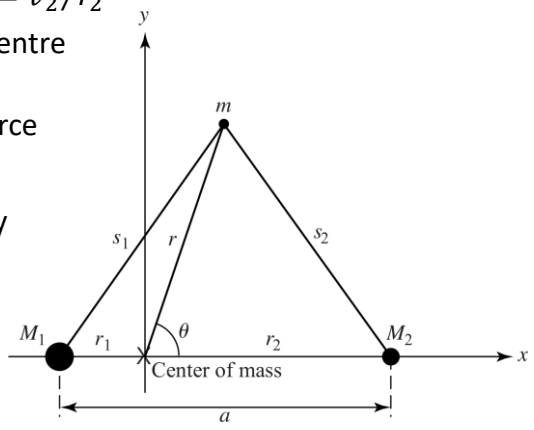


Hubble / ESA / NASA; processed by Judy Schmidt CC BY 2.0

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# Restricted three body problem 1

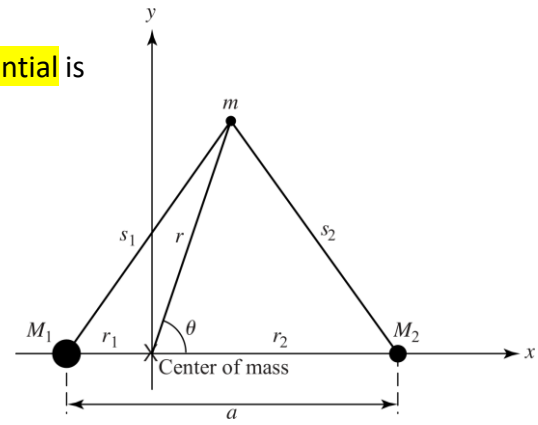
- Consider two bodies orbiting around the common barycentre with angular velocity $\omega = v_1/r_1 = v_2/r_2$
- Adopt rotating coordinates centred on barycentre and aligned with the bodies
- In this system a mass $m$ feels a centrifugal force
- $F_c = m\omega^2 r\,\hat{\mathbf{r}}$
- The centrifugal potential energy is defined by
- $U_f - U_i = \Delta U_c = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \boldsymbol{F}_c \cdot d\boldsymbol{r}$
- $\Delta U_c = -\int_{r_i}^{r_f} m\omega^2 r\, dr = \frac{1}{2}m\omega^2\left(r_f^2 - r_i^2\right)$
- Setting $U_c(0) = 0$ then $U_c = \frac{1}{2}m\omega^2 r^2$




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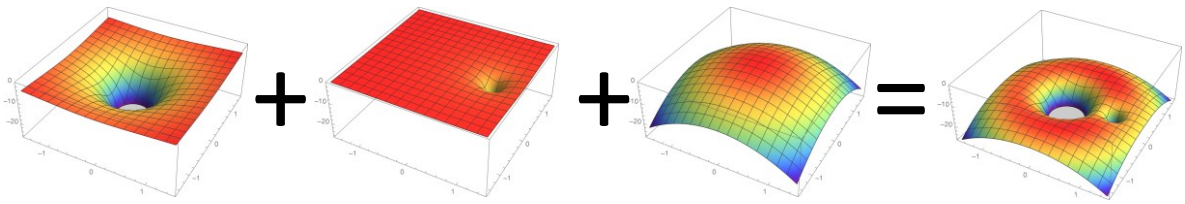
# Restricted three body problem 2

- The gravitational potential due to each mass is
- $U_g = -G\frac{Mm}{r}$
- The gravitational + centrifugal effective potential is
- $\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2$
- $\omega$ comes from Kepler's third law
- $\omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{G(M_1+M_2)}{a^3}$




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# Effective potential of restricted 3-body problem

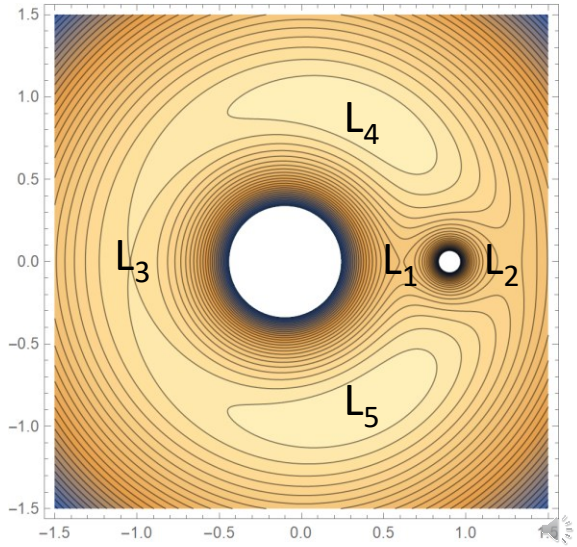


$U_g = -G\frac{M_1 m}{s_1}$ $U_g = -G\frac{M_2 m}{s_2}$ $U_c = -\frac{1}{2}\omega^2 r^2$ $U_{\text{eff}}$

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# Lagrange stationary points

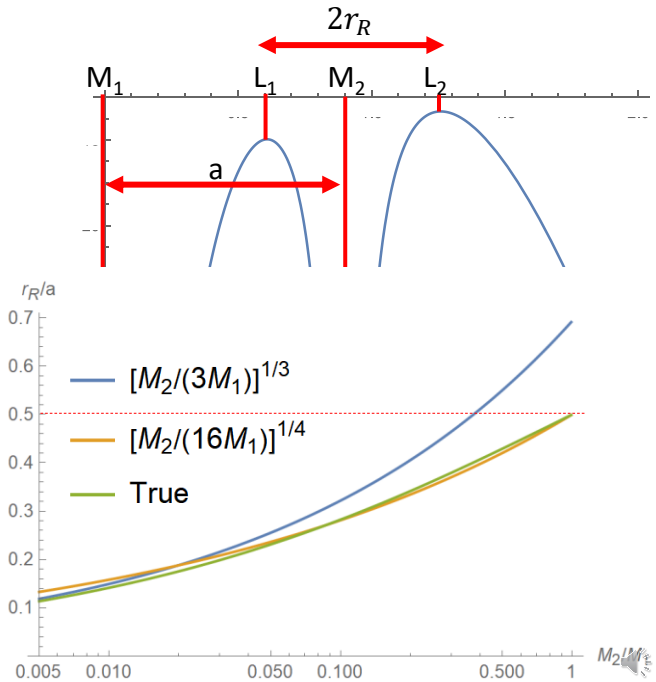
- Five stationary points in the effective potential
- None of them is stable
- Three $L_1$-$L_3$ saddle points along axis
- Two maxima $L_4$ and $L_5$




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# Roche radius

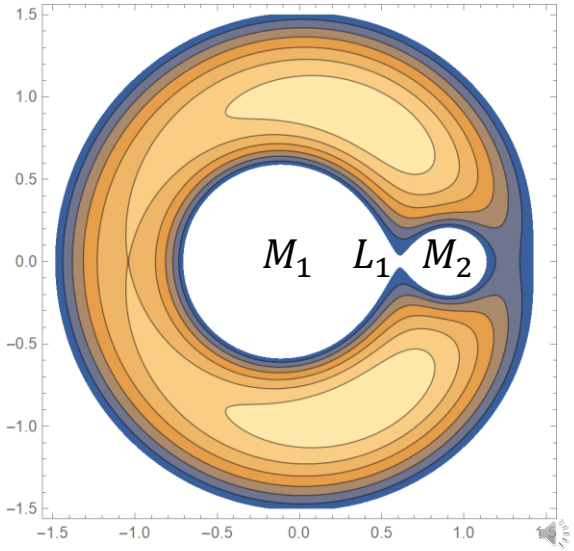
- Location of $L_1$ and $L_2$ easily found numerically
- Solving for $\frac{\partial \Phi_{\text{eff}}}{\partial x} = 0$
- When $M_2 = M_1 \rightarrow r_R = a/2$
- In the limit $M_2 \ll M_1$
- Roche radius $r_{\text{R}} \approx a \left(\frac{M_2}{3M_1}\right)^{1/3}$
- For $M_1/M_2 \lesssim 100$ to 1% accuracy
- $r_{\text{R}} \approx a \left(\frac{M_2}{16M_1}\right)^{1/4}$




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# Allowed space at given energy

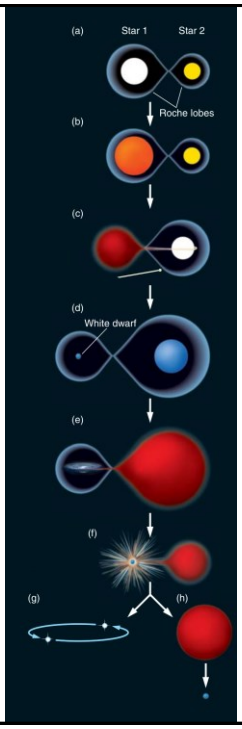
- Particle with total energy $E_{\mathrm{tot}}$
- Allowed region $\Phi_{\mathrm{eff}}\,(x, y) \leq E_{\mathrm{tot}}$
- When $E_{tot} \geq \Phi(L_1)$
- Mass transfer between $M_1$ and $M_2$
- Crucial to study binary stars evolution
- Also e.g. for travelling to the Moon
- Figure of eight called Roche lobes




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# Binary star evolution 1

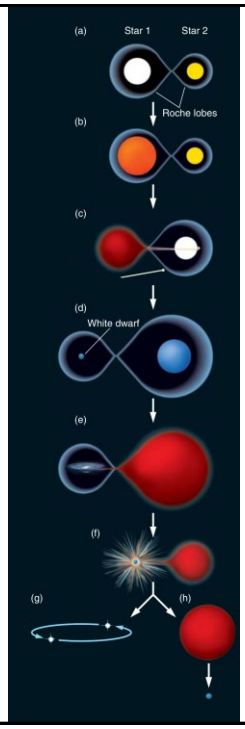
- (a) In general pairs will not have equal mass
- (b) More massive $M_1$ evolves faster and becomes giant
- If the pair is close enough it will fill the Roche lobes
- (c) Mass transfer from $M_1 \rightarrow M_2$
- $M_1$ skips full AGB phase
- (d) $M_1$ burns He leaving C-O core white dwarf
- $M_2$ gets more massive and evolves faster
- (e) $M_2$ becomes giant and fills Roche lobe




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# Binary star evolution 2

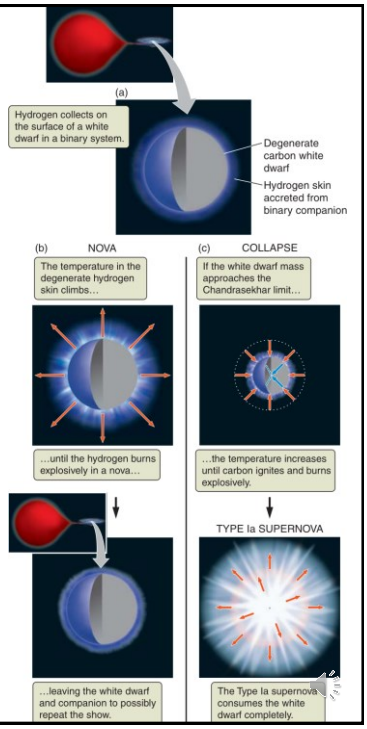
- (e) Mass transfer from $M_2 \rightarrow M_1$
- $M_1$ very compact: material misses star
- Angular momentum conserved $L = r\ v$
- Large reduction of radius → fast rotation
- Hot disk formation due to friction
- (f) Disk emits in X-ray or UV
- NB: detailed evolution depends on masses of 2 stars




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# Novae & Type Ia supernovae

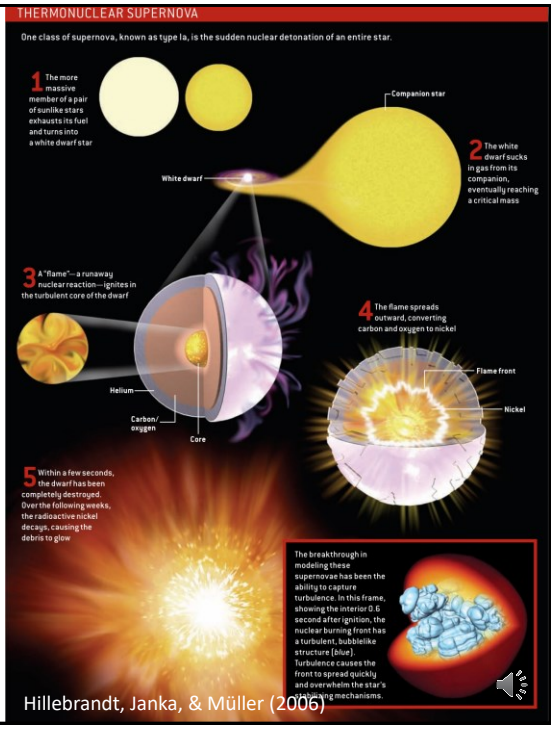
- Gas spirals down the $M_1$ hot accretion disk
- Ultimately H accreted on surface of C-O degenerate neutron star to very high density
- Like before He flash $P \nearrow \Rightarrow T \nearrow \Rightarrow L \nearrow$
- Until H burning starts in shell
- But H burning not contained by stellar envelope
- Results in explosive Nova and H-shell ejection
- Core remains and Nova can repeat multiple times
- May end with two white dwarfs




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# Type Ia supernovae (SN)

- During gas accretion onto white dwarf
- Chandrashekar mass limit approached
- No stable radius → Gravitational collapse
- Density and temperature become enough to fuse C and O into heaver elements
- Very energetic reaction due to high T and P causes supernova explosion
- No remnant left: heavy elements scattered into space enriching the interstellar medium
- Due to narrow mass range of progenitor luminosity production is a reproducible process

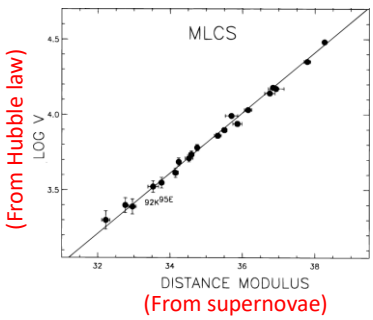


Hillebrandt, Janka, & Müller (2006)

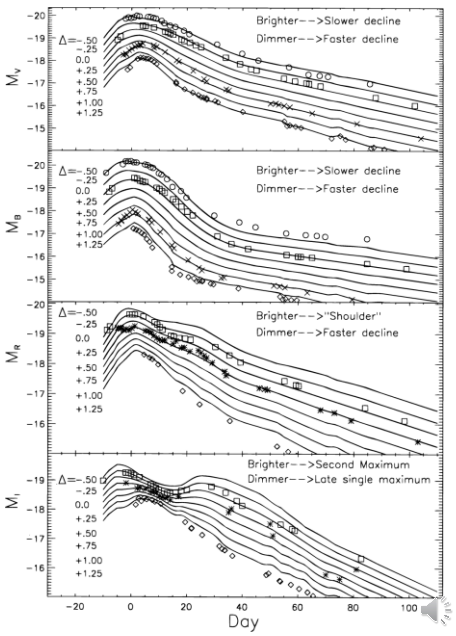
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# SN Ia as standard candles

- Supernovae type Ia have tight relation between luminosity and speed of decline
- Can be calibrated to be used as standard candles
- Most reliable distance indicator out to $z \gtrsim 1$
- Key for the discovery of the Universe acceleration




Riess et al. (1996)



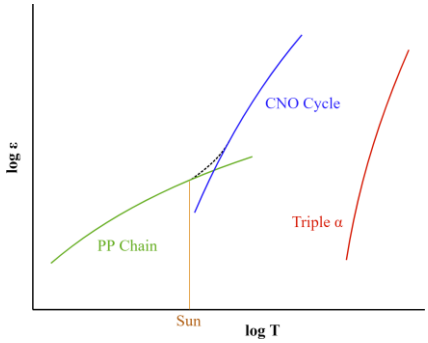

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# Evolution of massive stars

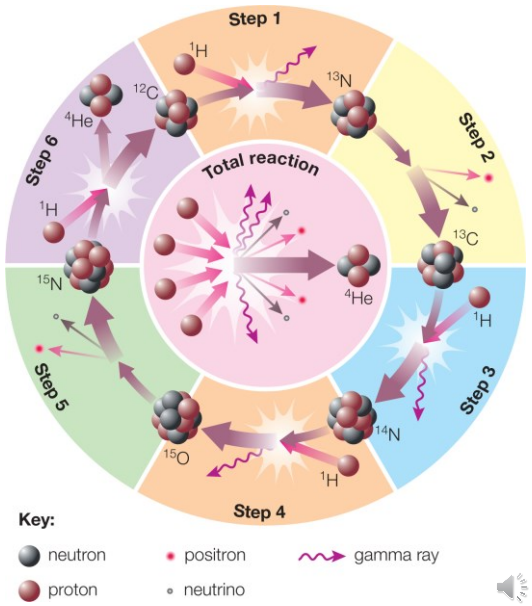
23

## CNO fusion cycle

- In more massive stars gravitational collapse produces larger temperatures
- H and He fusion via CNO cycle
- C, N and O only act as catalysts
- Steeper temperature dependence



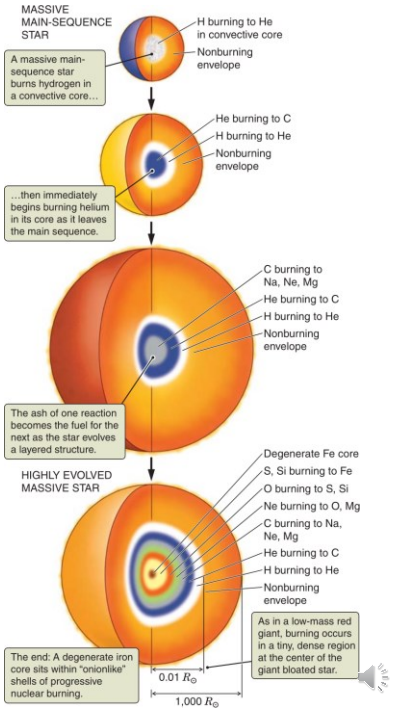

Credit: Bennett+18




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# Structure of massive stars

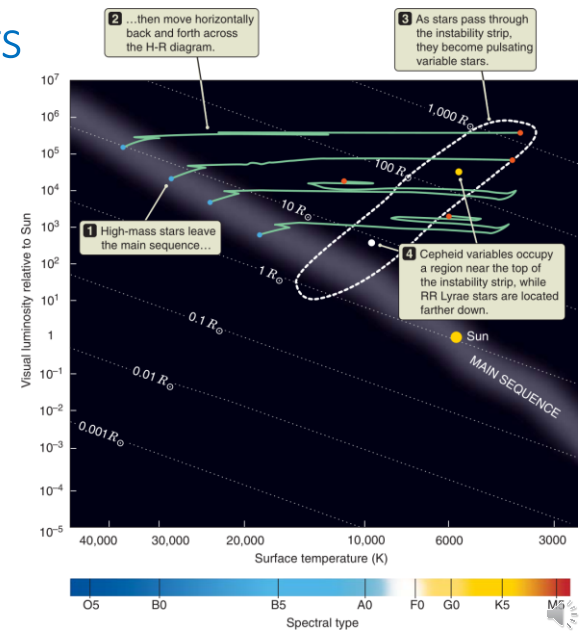
- Convection in central parts: H burning mixes He throughout the convective core
- Central temperature sufficient to burn elements heavier than C
- Atomic energies can overcome larger Coulomb barriers
- Produces an onion-like star with simultaneous fusion of elements in shells
- Heavier elements in inner shells
- Fusion sequence stops when innermost core becomes Fe: highest binding energy per nucleon




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# Evolution of massive stars

- Massive stars live less than the time humans walked on Earth!
- Outer layers too hot for $H^-$ thermostat
- Core temperature increases after main sequence to fuse heavier elements
- Outer envelope expands and cools
- Stars move to the right at constant L




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# Burning times of massive stars

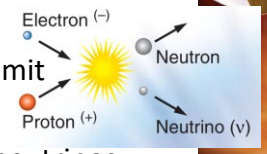
| Core Burning Stage | 15-$M_\odot$ Star | 25-$M_\odot$ Star | Typical Core Temperatures |
|---|---|---|---|
| Hydrogen (H) burning | 11 million years | 7 million years | $(3–10) \times 10^7$ K |
| Helium (He) burning | 2 million years | 800,000 years | $(1–7.5) \times 10^8$ K |
| Carbon (C) burning | 2,000 years | 500 years | $(0.8–1.4) \times 10^9$ K |
| Neon (Ne) burning | 8 months | 11 months | $(1.4–1.7) \times 10^9$ K |
| Oxygen (O) burning | 2.6 years | 5 months | $(1.8–2.8) \times 10^9$ K |
| Silicon (Si) burning | 18 days | 0.7 day | $(2.8–4) \times 10^9$ K |

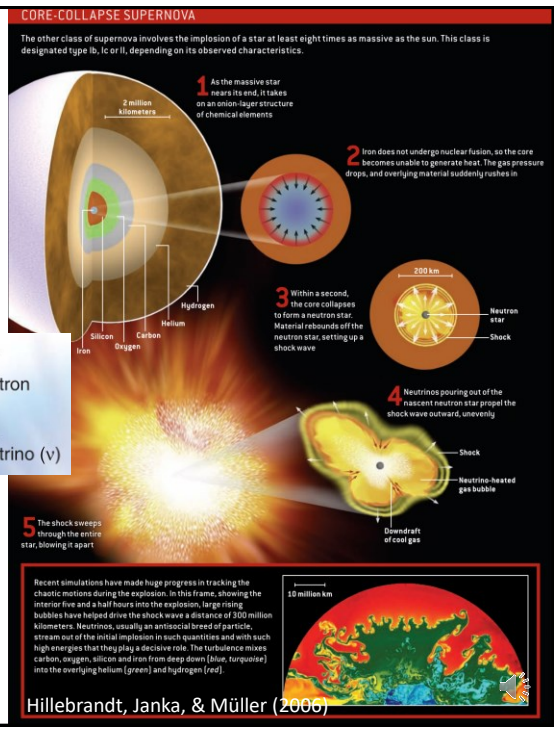
- Burning time shortens for heavier elements
- From C burning energy transport changes from radiation/convection to neutrinos
- Energy efficiently carried away mainly by neutrinos (no interactions)

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# Core-collapse supernova

- When the core is made of Fe
- No more fusion is possible
- Fe core becomes degenerate
- Size decreases but cannot start subsequent fusion reaction
- Outer layers keep falling in
- Core mass above Chandrasekhar limit
- Core collapses into a neutron star
- Electron capture by protons emit neutrinos
- Shock wave from material rebound
- Neutrinos production propels the gas
- Most of the stars is blow away
- Enriches medium of heavy metals



Hillebrandt, Janka, & Müller (2006)

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# $\alpha$-process ladder

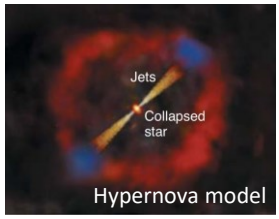
- α-process is one of the two channels to convert He into heavier elements (the other is triple-α)
- Can happen in the extreme temperature, when Si fusion starts in precursors of core-collapse supernovae
- Produces elements separated by 4 in atomic number
- These are called α-elements and tend to vary together
- For this reason, useful to define its abundance
- $\left[\frac{\alpha}{Fe}\right] \equiv \lg\left(\frac{N_\alpha}{N_e}\right)_{\text{star}} - \lg\left(\frac{N_\alpha}{N_e}\right)_{\text{Sun}}$
- $N_\alpha, N_{Fe}$: number of $\alpha$ elements or Fe per unit volume
- α-elements production happens early in stellar evolution of a population, when massive stars explode

$$^{12}_{6}\mathrm{C} + ^{4}_{2}\mathrm{He} \longrightarrow ^{16}_{8}\mathrm{O} + \gamma$$
$$^{16}_{8}\mathrm{O} + ^{4}_{2}\mathrm{He} \longrightarrow ^{20}_{10}\mathrm{Ne} + \gamma$$
$$^{20}_{10}\mathrm{Ne} + ^{4}_{2}\mathrm{He} \longrightarrow ^{24}_{12}\mathrm{Mg} + \gamma$$
$$^{24}_{12}\mathrm{Mg} + ^{4}_{2}\mathrm{He} \longrightarrow ^{28}_{14}\mathrm{Si} + \gamma$$
$$^{28}_{14}\mathrm{Si} + ^{4}_{2}\mathrm{He} \longrightarrow ^{32}_{16}\mathrm{S} + \gamma$$
$$^{32}_{16}\mathrm{S} + ^{4}_{2}\mathrm{He} \longrightarrow ^{36}_{18}\mathrm{Ar} + \gamma$$
$$^{36}_{18}\mathrm{Ar} + ^{4}_{2}\mathrm{He} \longrightarrow ^{40}_{20}\mathrm{Ca} + \gamma$$
$$^{40}_{20}\mathrm{Ca} + ^{4}_{2}\mathrm{He} \longrightarrow ^{44}_{22}\mathrm{Ti} + \gamma$$
$$^{44}_{22}\mathrm{Ti} + ^{4}_{2}\mathrm{He} \longrightarrow ^{48}_{24}\mathrm{Cr} + \gamma$$
$$^{48}_{24}\mathrm{Cr} + ^{4}_{2}\mathrm{He} \longrightarrow ^{52}_{26}\mathrm{Fe} + \gamma$$
$$^{52}_{26}\mathrm{Fe} + ^{4}_{2}\mathrm{He} \longrightarrow ^{56}_{28}\mathrm{Ni} + \gamma$$

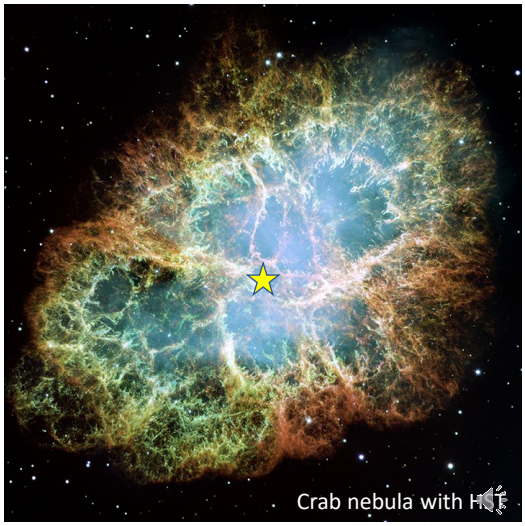
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# Remnant of core-collapse supernovae

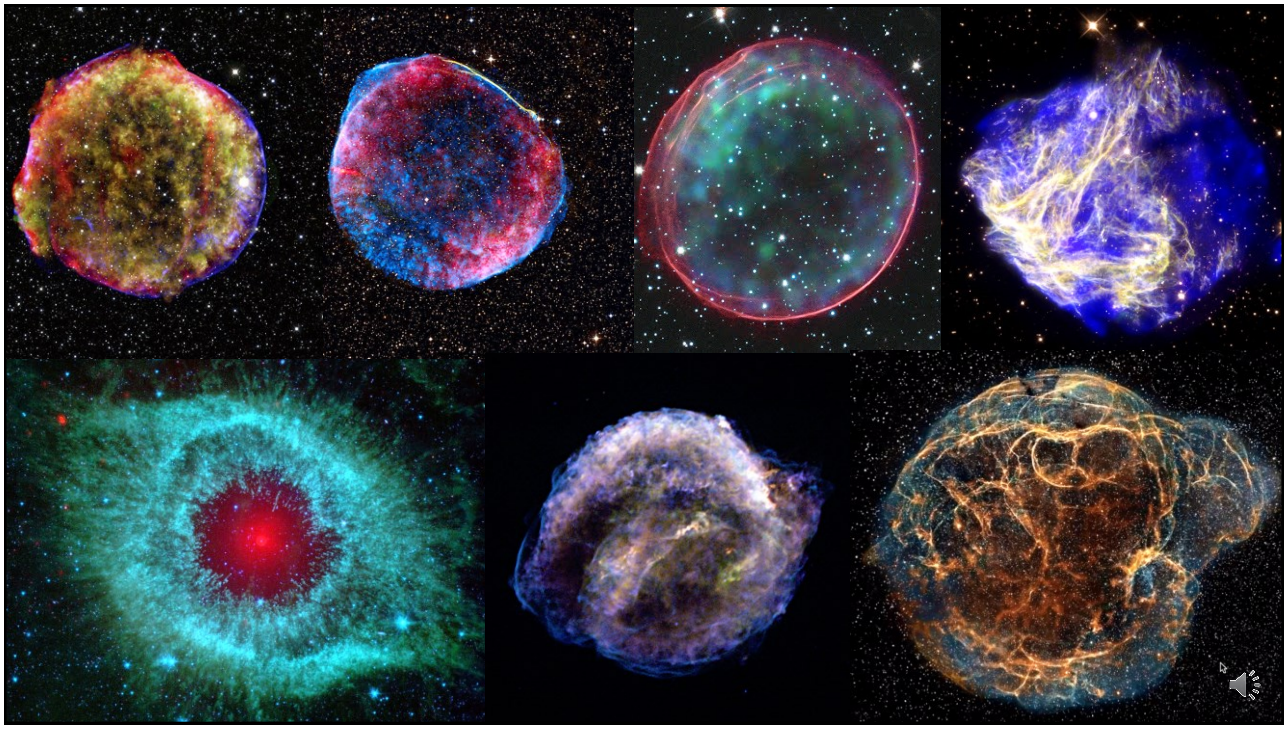
- If core mass $M \lesssim 3M_\odot$ remnant is neutron star
- Observable in the region of remnant gas
- Supported by neutron degeneracy pressure
- Upper mass limit analogue to e-degeneracy
- If $M \gtrsim 3M_\odot$ remnant is black hole
- In this case explosion called hypernova
- Most common explanation for long-duration gamma-ray bursts (GRBs)



Hypernova model


Crab nebula with HST

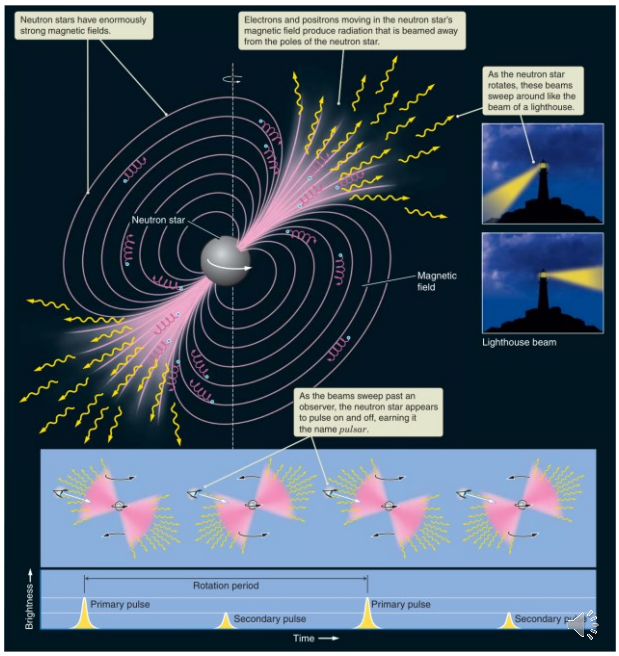
30



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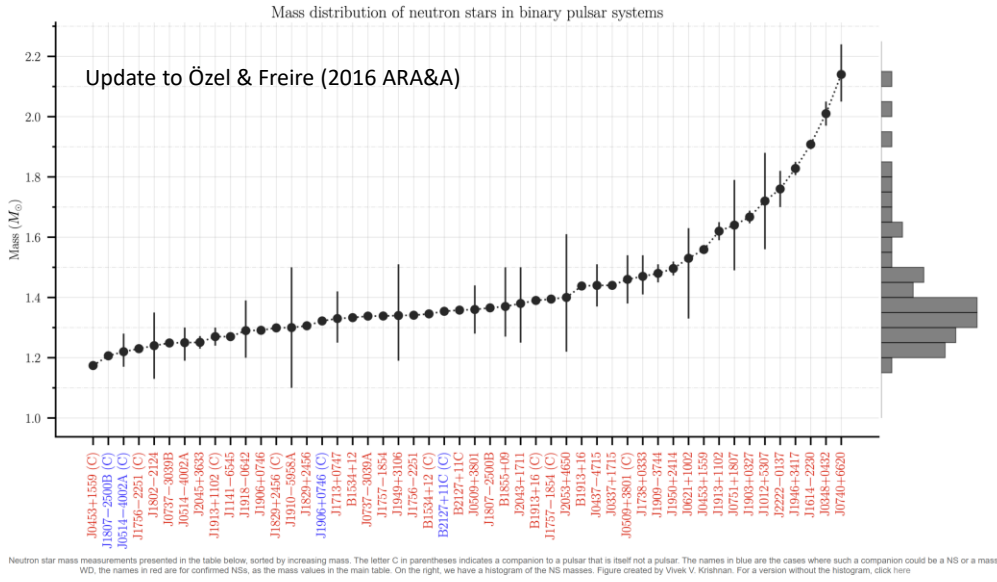
# Pulsars

- Assuming core had angular momentum $\mathcal{L}$ and period $\mathcal{P} = 27$d of the Sun
- $\mathcal{L} \approx M_c R_\odot v_\phi = 2\pi M_c R_\odot^2 / \mathcal{P}_\odot$
- Density of neutron stars requires radii $R_{\rm ns} \approx 10 - 20$ km!
- $\mathcal{L}$ conservation during collapse implies
- $\mathcal{P}_{\rm ns} = \mathcal{P}_\odot \left(\frac{R_{\rm ns}}{R_\odot}\right)^2 \approx 2 \times 10^{-3} s$
- Large magnetic fields with axes misaligned vs rotation (as on Earth)
- Charged particles produce collimated synchrotron radiation
- Discovered by Bell Burnell as PhD student by Hewish (1968) at Cambridge




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# Masses of neutron stars in binaries




Neutron star mass measurements presented in the table below, sorted by increasing mass. The letter C in parentheses indicates a companion to a pulsar that is itself not a pulsar. The names in blue are the cases where such a companion could be a NS or a massive WD, the names in red are for confirmed NSs, as the mass values in the main table. On the right, we have a histogram of the NS masses. Figure created by Vivek V. Krishnan. For a version without the histogram, click here

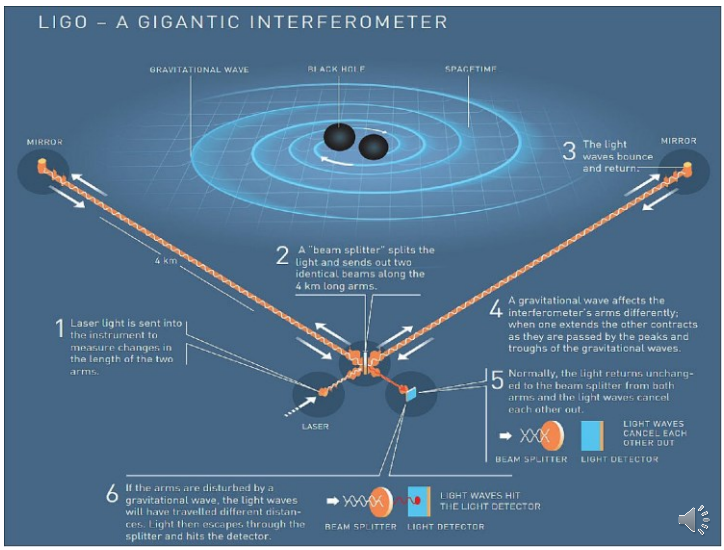
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# LIGO/Virgo gravitational-waves interferometers





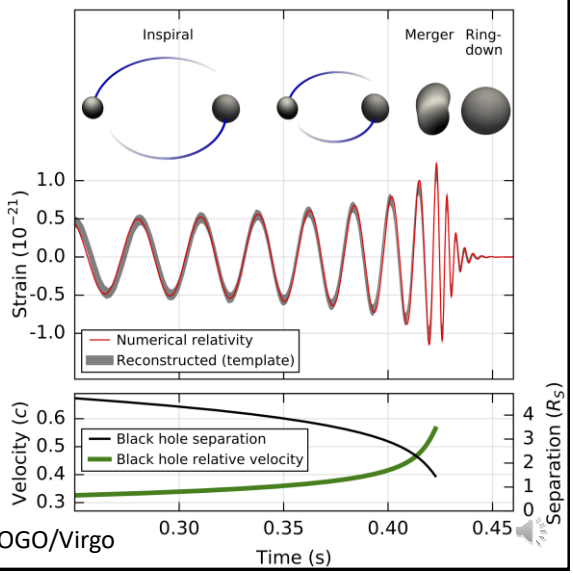





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# Gravitational waves from black-holes mergers

- Gravitational waves from black holes merger first detected in 2016
- Important test of general relativity
- Opens a new observable window on our Universe
- 50 black holes mergers detection up to 2020

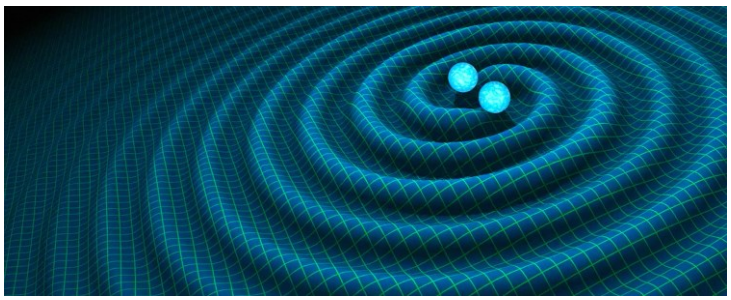


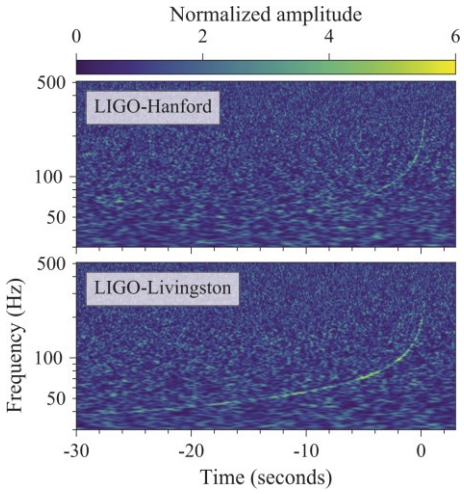

Abbott+16 LOGO/Virgo

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# Gravitational waves from neutron-stars mergers

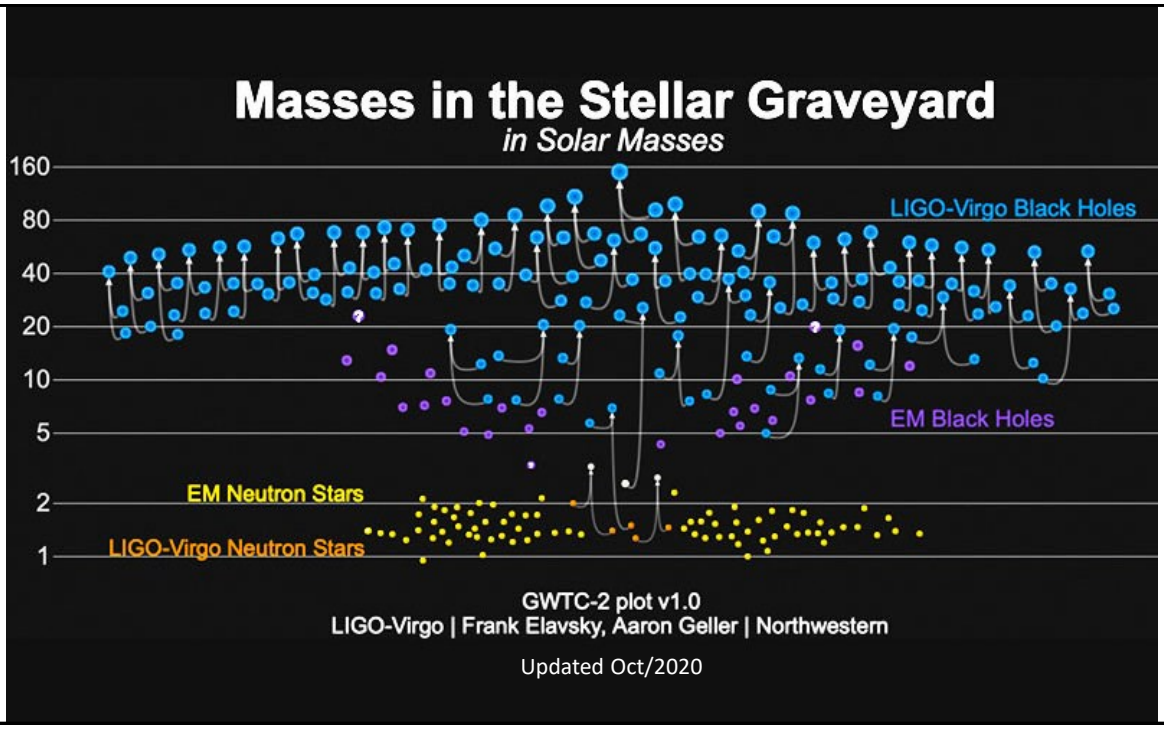
- Gravitational waves from neutron-stars merger first detected in 2017
- Using L-shaped LIGO interferometers
- Two neutron-stars mergers up to 2020






Abbott+17 LIGO/Virgo

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Masses in the Stellar Graveyard
in Solar Masses
160
80
40
20
10
5
2
1
LIGO-Virgo Black Holes
EM Black Holes
EM Neutron Stars
LIGO-Virgo Neutron Stars
GWTC-2 plot v1.0
LIGO-Virgo | Frank Elavsky, Aaron Geller | Northwestern
Updated Oct/2020

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