Michaelmas Term 2021 1st Year Relativity - Problems 1

Special Relativity: Professor K M Blundell

*The problems marked with * are somewhat harder. Numerical answers are given to some of the problems.*

SECTION A – Lorentz Transformation and Elementary Consequences

1. The space and time coordinates of two events as measured in a frame S are as follows:

Event 1: $x_1 = x_0, \quad t_1 = x_0/c \quad (y_1 = 0,\ z_1 = 0)$

Event 2: $x_2 = 2x_0,\ t_2 = x_0/2c \quad (y_2 = 0,\ z_2 = 0)$

Show that there exists an inertial frame in which these events occur at the same time, and find the velocity of this frame relative to S. At what value of time in the new frame do both events occur?

[Ans: $\beta = -1/2$, $t' = \sqrt{3}x_0/c$.]

2. A small, powerful laser is placed on a turntable that rotates at 1200 rev/s. The laser, whose beam makes a 30° angle with the horizontal, shines on clouds 50 km away. Calculate the speed with which the light spot on the clouds moves. Does this speed violate the limitation of the speed of light? Explain.

[Ans: 3.3×10^8 m s^{-1}]

*3. Two observers A and B, in inertial frames, have identical clocks. B passes close to A with a relative velocity 0.5c. t seconds after B has passed, A flashes him a light signal. $2t$ seconds after receiving this signal (as measured on his own clock), B acknowledges by a return light signal to A. When does A receive it?

[Ans: $(3 + 2\sqrt{3})t$.]

4. Proxima Centauri, the star nearest our own, is some 4.2 ly away. (a) If a spaceship could travel at a speed of 0.80c, how long would it take to reach the star, according to the spaceship's pilot? (b) What would someone in the frame that moves with the spaceship measure as the distance to Proxima Centauri?

[Ans: (a) 3.1 yr, (b) 2.5 ly]

5. The Galaxy is about 10^5 light years across and the most energetic cosmic rays known have energies of the order of 10^{19} eV. How long would it take a proton with this energy to cross the Galaxy as measured in the rest frame of (i) the Galaxy and (ii) the proton.

The rest-mass of the proton is 938 MeV/c^2.

[Ans: (i) 10^5 years, (ii) 4.96 min.]

6. The proper mean lifetime of π^+ mesons is 26 ns.

(a) What is the mean lifetime of a burst of π^+ mesons travelling with $\beta = 0.73$?

(b) What distance is travelled at this velocity during one mean lifetime?

(c) What distance would be travelled without the effect of time dilation?

[Ans: (a) 38 ns, (b) 8.32 m, (c) 5.69 m.]

7. A burst of muons is produced by a cosmic ray interacting in the upper atmosphere. They travel towards the Earth's surface with an average speed of $0.99c$. If 1% of the burst survive to reach ground level, estimate the height of the burst and calculate the distance as measured in the muon's frame. The muon mean lifetime is 2.2 μsec.

[Ans: 2.13×10^4 m, 3.01×10^3 m.]

*8. Twin paradox for fun – C. Darwin, Nature **180**, 976 (1957). On New Year's day 2030 an astronaut (A) sets out from Earth at speed $0.8c$ to travel to the nearest star α-Centauri, about 4 light-years away (as measured in the Earth frame of reference). On reaching the star A immediately turns around and returns to Earth arriving back on New Year's day 2040 (Earth time). A has a sibling B who remains on Earth and they agree to send each other greetings by radio every New Year's day until A returns.

(a) Satisfy yourself that A sends only 6 messages (including the one on the last day of the trip, whereas B sends 10.

(b) Draw a space-time diagram of A's journey in the Earth frame (mark the scales of x and ct in light years). Draw the world lines of all the radio signals that B transmits. Use the diagram to verify that A (astronaut) has received only 1 signal up to the moment of reaching the star and turning back, and receives the other 9 during the return half of the trip.

(c) On a second space-time diagram (again in the Earth frame) show the world lines of the astronaut (A) and all the signals that A sends. Verify that B (on Earth) receives one message each 3 years of Earth time for the first 9 years of the journey, then receives 3 more messages in the last year to make the total of 6 - which is the expected total since according to A's time the trip takes 3 years out and 3 years back.

SECTION B – Addition of velocities; Energy & Mass

9. Show that two successive Lorentz transformations with velocities v_1 and v_2 in the same direction are equivalent to a single Lorentz transformation with velocity $(v_1+v_2)/(1+v_1v_2/c^2)$.

10. *Estimate* the mass lost when 1 millon tonnes of TNT explodes. Assume that each chemical reaction between individual molecules involves 10 eV of energy.

11. Energy from the Sun (distance 1.496×10^{11} m from Earth) arrives at the Earth (above the atmosphere) at a rate of about 1400 W m^{-2}. How fast is the sun losing mass due to energy radiation?

[Ans: 4.4×10^9 kg s^{-1}]

Michaelmas Term 2021 1st Year Relativity - Problems 2

Special Relativity: Professor K M Blundell

SECTION A – Collision problems, threshold energies, decays, recoils

1. The K^0 meson at rest can decay into a pair of charged π mesons. Given that the rest masses of the K and π are 498 and 140 MeV/c^2 respectively, show that the speeds of the pions are 0.83c in the rest frame of the K^0.

2. An electron accelerated from the Stanford Linear Accelerator (SLAC) in California has a total energy of 50 GeV. (a) How much of this is kinetic energy? (b) What is the momentum of the electron? (c) What is its speed?

[Ans: (a) 0.99999E, (b) 2.7×10^{-17} kg m s^{-1}, (c) $(1-(5\times10^{-11}))c$]

3. A π^0 meson whose rest mass is 135 MeV/c^2 is moving with a kinetic energy of 1 GeV. It decays in flight into two photons whose paths are along the direction of motion of the meson. Find the energies of the two photons.

[Ans: 4 and 1131 MeV.]

4. Neutral π^0 mesons can be formed by the reaction

$$p + p \rightarrow p + p + \pi^0.$$

What is the threshold kinetic energy for π^0 production by this process if the target proton is stationary? (Masses of π^0 and proton are 135 and 938 MeV/c^2.)

[Ans: 280 MeV.]

5. What is the threshold $K.E.$ for production of an electron pair by collision of an incident electron with a stationary electron *i.e.* for

$$e^- + e^- \rightarrow e^- + e^- + (e^+ + e^-)?$$

[Ans: $6m_{\rm e}c^2$, where $m_{\rm e}$ is the rest mass of an electron.]

6. The Tevatron at the Fermi National Accelerator Laboratory in the USA carries two beams of 1 TeV (1000 GeV) protons moving in opposite directions, so that for proton-proton collisions the centre of mass frame is also the laboratory frame. What proton energy must an accelerator provide to get the same centre of mass energy when bombarding a stationary hydrogen target?

[Ans: 2.13×10^3 TeV.]

7. (i) Show that a free electron moving in a vacuum at velocity v cannot emit a single photon. (ii) A hydrogen atom in an excited state can emit a single photon, why is this different from (i) above?

8. An atom in an excited state of energy Q_0 (as measured in its rest frame) above the ground state moves towards a scintillation counter with speed v. The atom decays to its ground state by emitting a photon of energy Q (as recorded by the counter), coming completely to rest as it does so. If the rest mass of the atom is M, show that $Q = Q_0[1 + (Q_0/2Mc^2)]$.

9. A driver was caught running a red light. His defence is that he saw the light as green, as a result of the Doppler shift. He is arrested. What for? Estimate the seriousness of his transgression.

10. Light of frequency ν_0 is emitted from a source moving away from an observer with speed βc. If the light is emitted along the line of the motion show that the frequency measured by the observer is $\nu = \nu_0\gamma(1 - \beta)$. An absorption line in Ca^+ has a wavelength of 394 nm as measured for a stationary source. The same line in the spectrum of Hydra occurs at 475nm. What can be deduced about the motion of Hydra with respect to the Earth?

SECTION B – Some past Prelims questions

11. State the *Lorentz transformations* which relate the space and time coordinates in two inertial frames, S and S', travelling with relative speed v along the $x(x')$ axis. Use the transformations to derive expressions for the phenomena of *length contraction* and *time dilation*.

A rocket passes close to the earth with relative velocity $0.8c$, at which time observers on the earth and the rocket reset their clocks $(t_E = t_R = 0)$.

(a) After 30 minutes as measured by the rocket's clock, the rocket passes a space-station which is fixed relative to the earth. What time is recorded by the observer on the earth?

(b) How far is the space-station relative to the earth

(i) in earth coordinates,

(ii) in rocket coordinates?

(c) When the rocket passes the space-station, the rocket sends a radio signal back to earth. At what time (by earth time) does the signal arrive?

(d) The station on earth replies immediately. At what time (by rocket time) is the reply received by the rocket?

12. Write down the relativistic expressions for momentum p and energy E of a particle of rest mass m_0 in terms of its velocity v and show that they satisfy the equation

$$E^2 = p^2c^2 + m_0^2c^4$$

(a) An electron of energy $9.0\,\text{GeV}$ and a positron of energy E collide head-on to produce a B^0 meson and an anti-B^0 meson, each with a mass of $5.3\,\text{GeV/c}^2$. What is the minimum positron energy (threshold energy) required to produce the B^0-meson pair? [You may neglect the rest-mass energies of the electron and positron.]

(b) Electrons and positrons collide head-on in the laboratory with beam energies as in part (a), producing B^0 and anti-B^0 mesons at the threshold energy. The B^0 mesons undergo decay with a mean proper lifetime of $1.5 \times 10^{-12}\,\text{s}$. Calculate the mean distance that the B^0 mesons travel before decay, as observed in the rest frame of the laboratory.

13. State the *Lorentz transformations* which relate the space and time coordinates in two inertial frames, S and S', where S' is travelling with velocity v relative to S along the positive x direction. Use the transformations to derive an expression for the phenomenon of *length contraction*.

An observer in S measures the velocity of a body travelling in the positive x direction to be u_x. Show that the velocity of the body measured by an observer in S' is given by

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$

An observer on the earth measures a spaceship to be approaching at a speed of $0.7c$. The same observer measures a second spaceship to be approaching in the opposite direction at a speed of $0.8c$. Each spaceship has a length of $100\,\text{m}$ (as measured by observers in the respective spaceships).

(a) What are the lengths of each spaceship, as measured by the observer on the earth?

(b) What is the length of each spaceship, as measured by the observer on the other spaceship?

14. Write down the relativistic expressions for momentum p and energy E of a particle of rest mass m_0 in terms of its velocity v and show that they satisfy the equation

$$E^2 = p^2c^2 + m_0^2c^4$$

A D^0 meson decays at rest to a $\bar{K}^0$ meson and a π^0 meson. Calculate the energy and velocity of the π^0.

The π^0 produced in the D^0 decay subsequently decays into a pair of photons. Calculate the maximum and minimum energies of each photon in the D^0 rest frame.

[The rest masses of the D^0, $\bar{K}^0$ and π^0 mesons are $1.86\,\text{GeV/c}^2$, $0.50\,\text{GeV/c}^2$ and $0.13\,\text{GeV/c}^2$, respectively.]