Special Relativity Handout — 2021

Recommended reading:

- *The Wonderful World of Relativity*, Andrew M. Steane, (OUP, 2011. A local author!)
- *Relativity made Relatively Easy*, Andrew M. Steane, (OUP, 2012. A local author!)
- *Special Relativity*, A P French, (MIT, Physics Series) [Nelson, 1968]
- *Spacetime Physics*, E F Taylor & J A Wheeler (Freeman, 1992) Several publishers including Nelson, Chapman & Hall.
- *Introducing Special Relativity*, W S C Williams (Taylor & Francis, 2002)

Lecture 1 outline:

0. Recommended reading

1. What is a *frame of reference*?

2. What is a *inertial* frame of reference?

3. Galilean relativity

4. *"We have a problem Houston...!"* Is the aether the solution...?

5 Einstein's two postulates

Figure H1: An event viewed in two different frames (labelled S and S′):

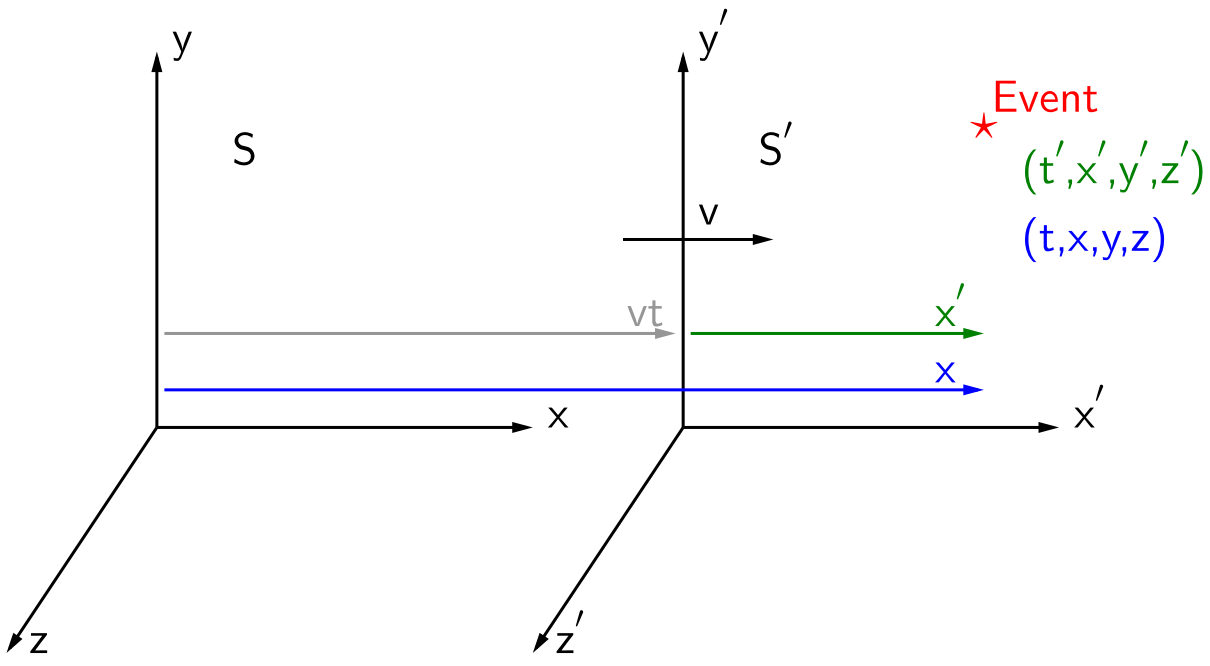
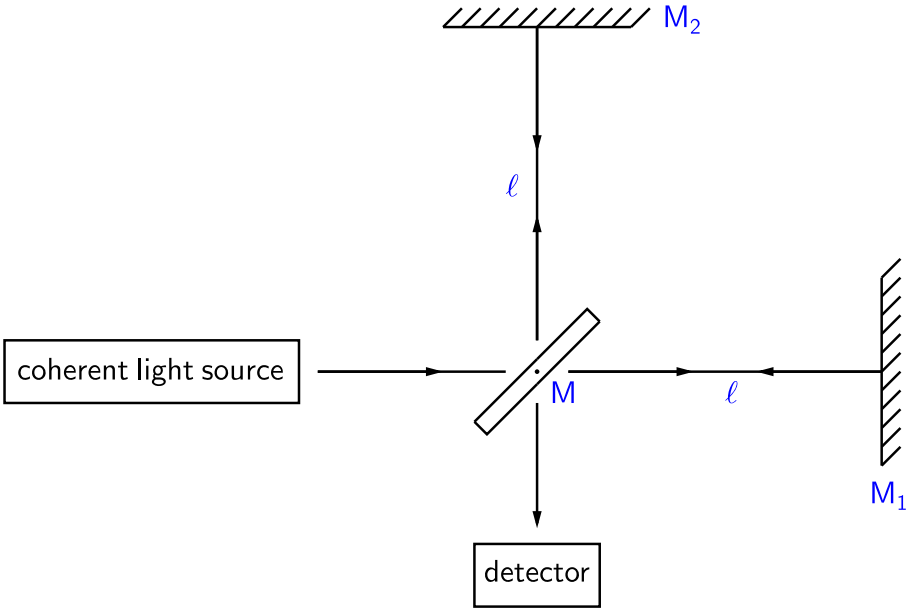


Figure H2: Michelson-Morley Interferometer



Lecture 2 outline:

6 Nomenclature

- **6.i** an Observer
- **6.ii** Frame co-ordinates
- **6.iii** Events
- **6.iv** Proper time
- **6.v** Proper length

7 Co-ordinates of an event viewed in two different frames

7.i Space-time diagrams

8 Three implications from Einstein's two postulates

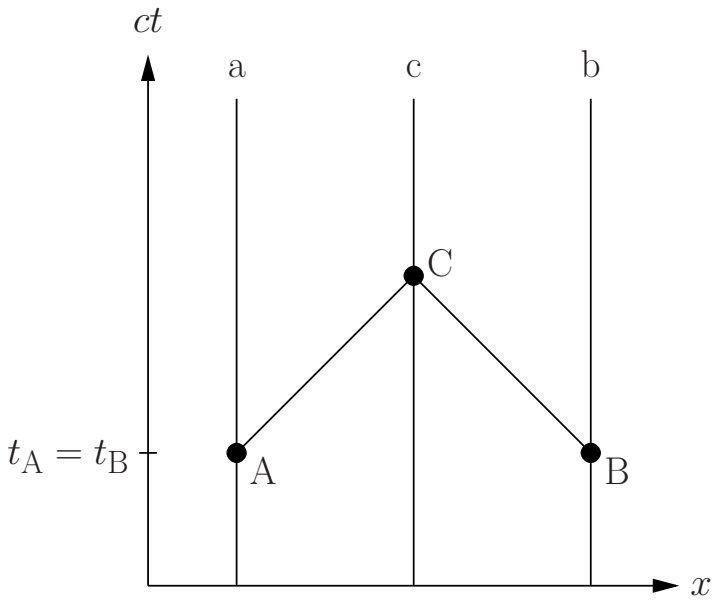
- **8.1** the Relativity of Simultaneity
- **8.2** Time Dilation
- **8.3** Lorentz Contraction *(next lecture)*

Some notes on Section 8.1, the **Relativity of Simultaneity**:

Consider three positions in space labelled by the letters a, b and c.

- Suppose at some initial time a flash of light is produced at position a. We will call this **event A**.
- At *exactly the same time*, a flash of light is produced at position b. We will call this **event B**.
- A person sitting at position c, which is equidistant from a and b, detects the two flashes of light arriving simultaneously at her detector. We call the arrival of the two flashes of light **event C**.

This situation is shown in the following spacetime diagram, **Figure H3a**:



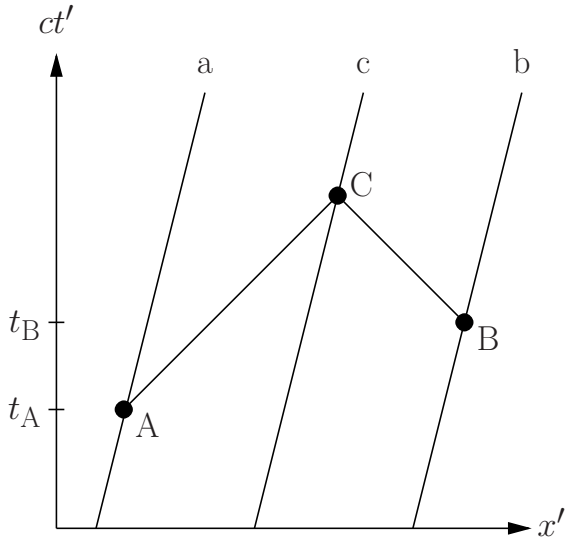
Notice that since positions a, b and c are fixed in space, they are drawn as vertical lines. The flashes of light travel at speed c, and so on a spacetime diagram they are shown as diagonal lines travelling at 45°. (This is because the time axis is plotted as ct and not t.)

The key question to ask is what would this situation look like to an observer moving at a constant velocity with respect to this frame of reference? The answer is shown in the following diagram, Figure H3b. Note first of all that the positions a, b and c are now slightly inclined to the vertical. This is because they are moving with respect to the new frame of reference (which we denote with the primed coordinates x' and ct').

Next, note that the two beams of light still travel at 45°. This is because light travels at the same speed with respect to all observers.

Notice also that the two beams of light intersect at position C. This must happen since it happened in the other frame of reference, and observers in either frame must agree that the two flashes of light were detected at the same time. Perhaps there was a 'coincidence detector' located at c, and this left a permanent record in a computer memory.

Figure H3b:



We have now applied the two principles of Einstein's theory: (i) that all observers in inertial frames of reference must have a consistent explanation of physical events and (ii) light travels at c in every frame of reference. However, what we have now found is that events A and B must necessarily occur at **different times**.

Therefore, all observers cannot agree on whether two events happened simultaneously. An observer in the unprimed frame will think that $t_{\mathrm{A}} = t_{\mathrm{B}}$ as the first diagram shows. An observer in the primed frame will think that $t_{\mathrm{A}} < t_{\mathrm{B}}$ as the second diagram shows. An observer in another frame travelling in the opposite direction to the primed frame could conclude that $t_{\mathrm{A}} > t_{\mathrm{B}}$.

This demonstrates the **relativity of simultaneity**. We might think that saying whether two spatially separated events occur at the same time (such as A and B) is an absolute statement. In fact, it depends on the frame of reference of the observer.

Figure H4:
Time dilation — frame S'

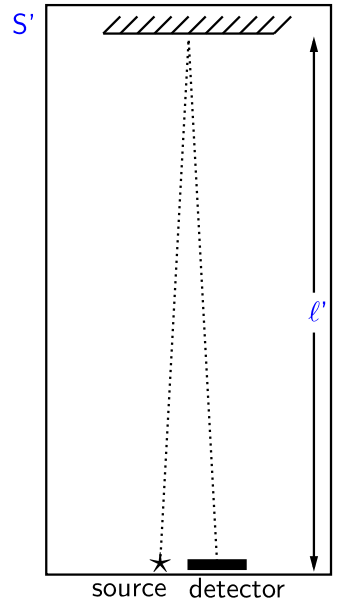
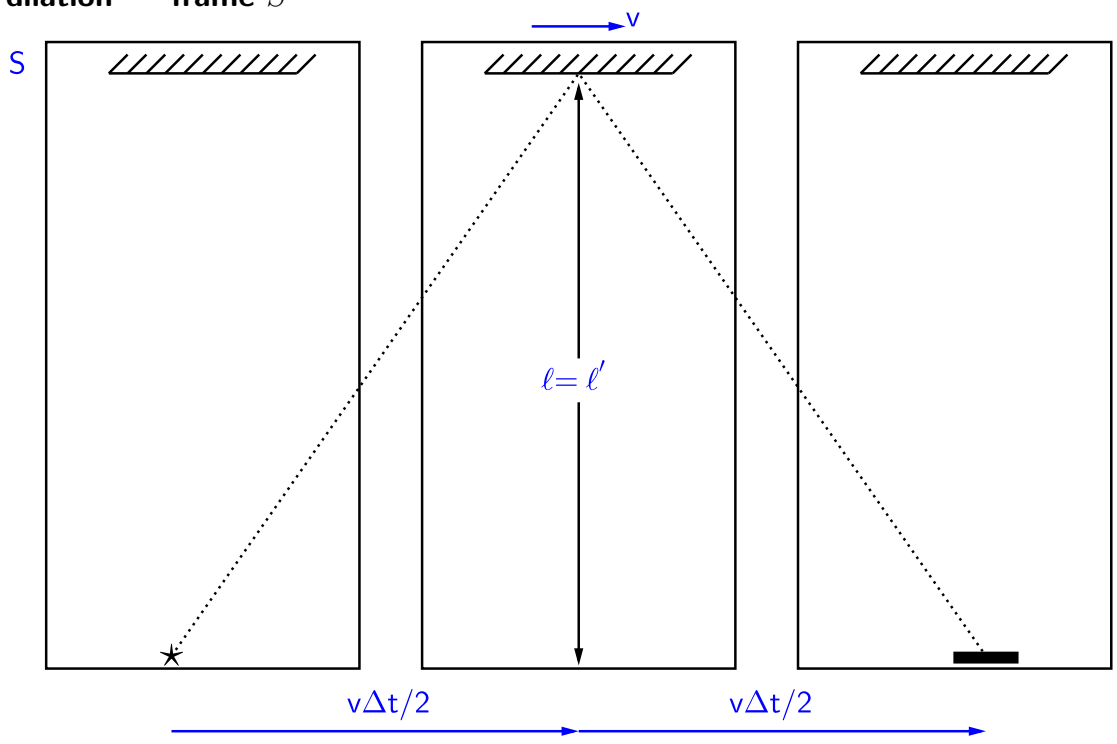


Figure H5:
Time dilation — frame S



Lecture 3 outline:

8 Three implications from Einstein's two postulates

- **8.1** the Relativity of Simultaneity *(previous lecture)*
- **8.2** Time Dilation *(previous lecture)*
- **8.3** Lorentz Contraction

9 the Lorentz factor

10 Minkowski space-time

11 Lorentz transformation

Figure H6: Length contraction — frame S:

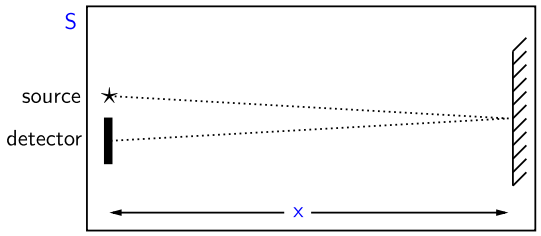
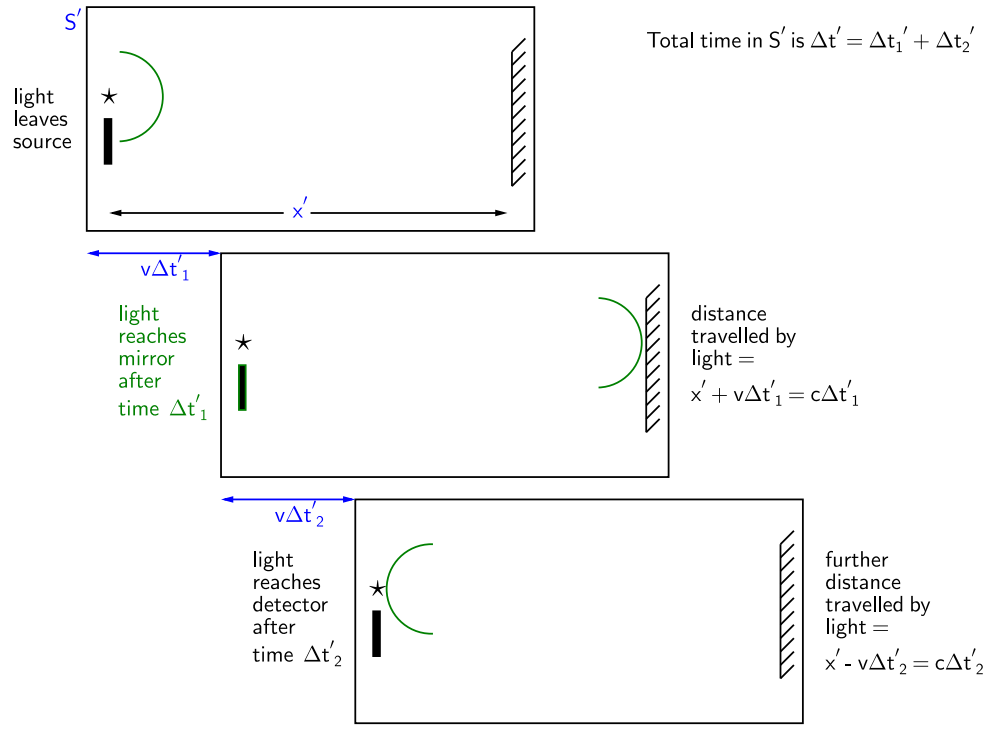


Figure H7: Length contraction — frame S':



Lorentz Transformation for a single event

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) = \gamma\left(t - \frac{\beta x}{c}\right) \\ x' &= \gamma(x - vt) = \gamma(x - c\beta t) \\ y' &= y \\ z' &= z \end{aligned}$$

Lorentz Transformation for intervals between two events, a and b

$$\begin{aligned} t'_a &= \gamma\left(t_a - \frac{\beta x_a}{c}\right) \\ t'_b &= \gamma\left(t_b - \frac{\beta x_b}{c}\right) \\ \text{Differencing the above 2 equations} : \Delta t' &= \gamma\left(\Delta t - \frac{\beta \Delta x}{c}\right) \end{aligned}$$

$$\begin{aligned} x'_a &= \gamma(x_a - c\beta t_a) \\ x'_b &= \gamma(x_b - c\beta t_b) \\ \text{Differencing the above 2 equations} : \Delta x' &= \gamma(\Delta x - c\beta \Delta t) \end{aligned}$$

$$\begin{aligned} y'_a &= y_a \\ y'_b &= y_b \\ \text{Differencing the above 2 equations} : \Delta y' &= \Delta y \end{aligned}$$

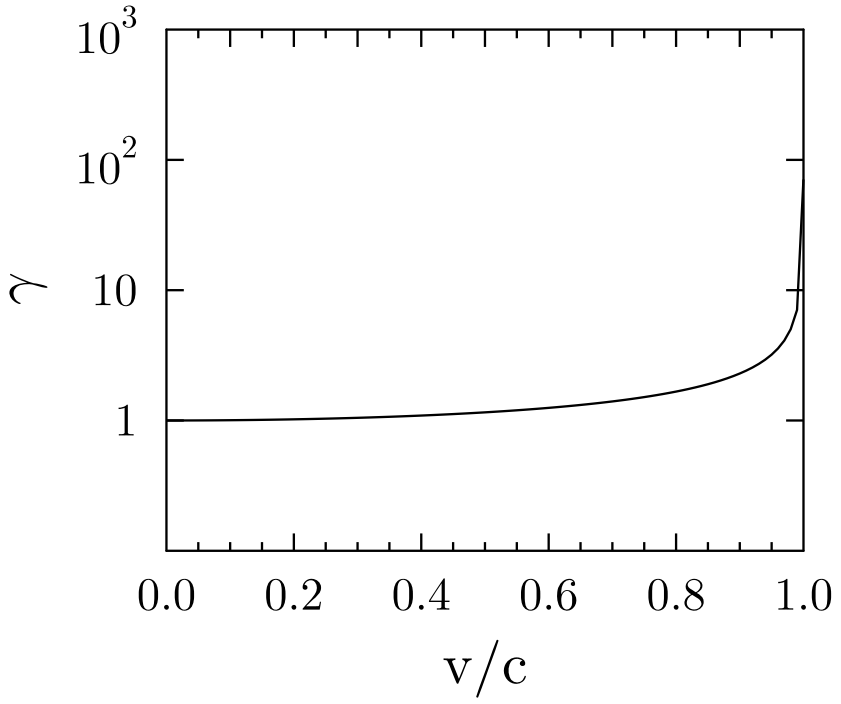
$$\begin{aligned} z'_1 &= z_1 \\ z'_2 &= z_2 \\ \text{Differencing the above 2 equations} : \Delta z' &= \Delta z \end{aligned}$$

A particularly convenient representation of the Lorentz transformation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Figure H8: The relation between γ and v/c



Lecture 4 outline:

11 Lorentz transformation

- **11.4** Properties of the Lorentz transform
- **11.5** Lorentz transformation as a rotation of 4-vectors
- **11.6** "The Interval"

12 Einstein's train problem

Figure H9: Frame S

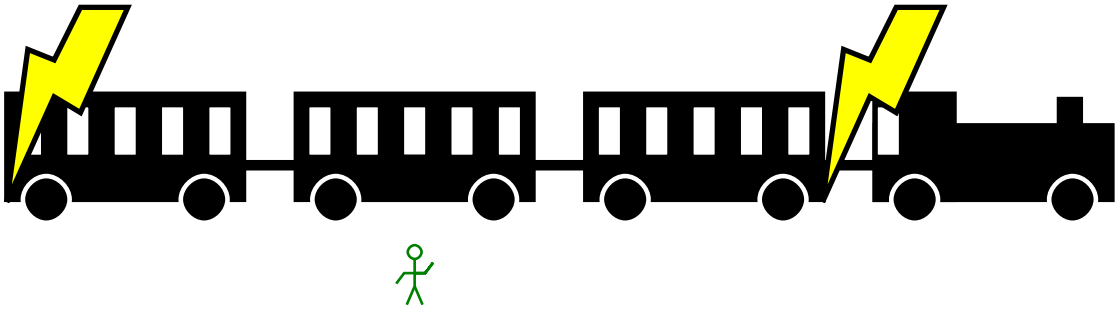
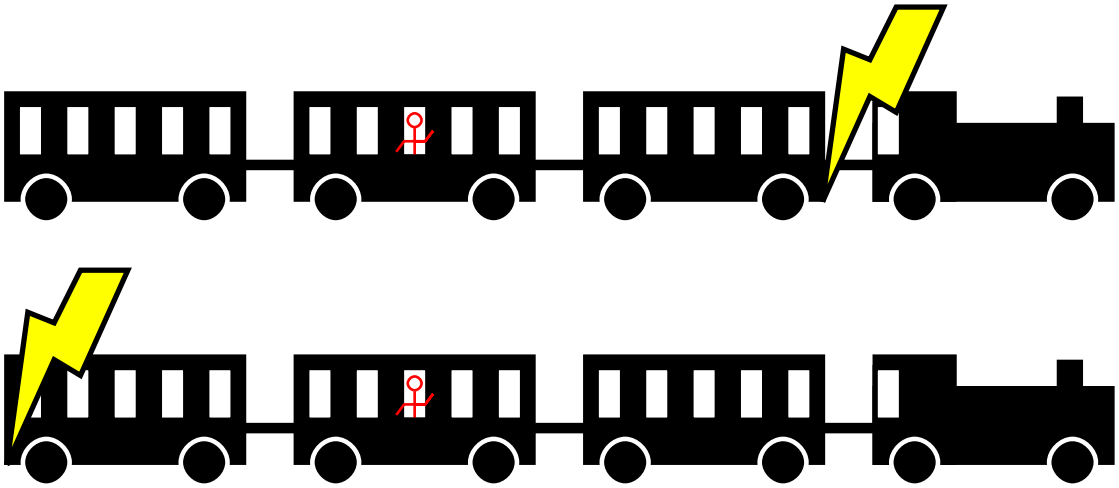


Figure H10: Frame S′



Lecture 5 outline:

13 The "twin paradox" - is it really a paradox?

14 Cosmic ray decay - real world examples of time dilation and Lorentz contraction

15 Relativistic velocity addition formulae; mutual velocity and relative velocity.

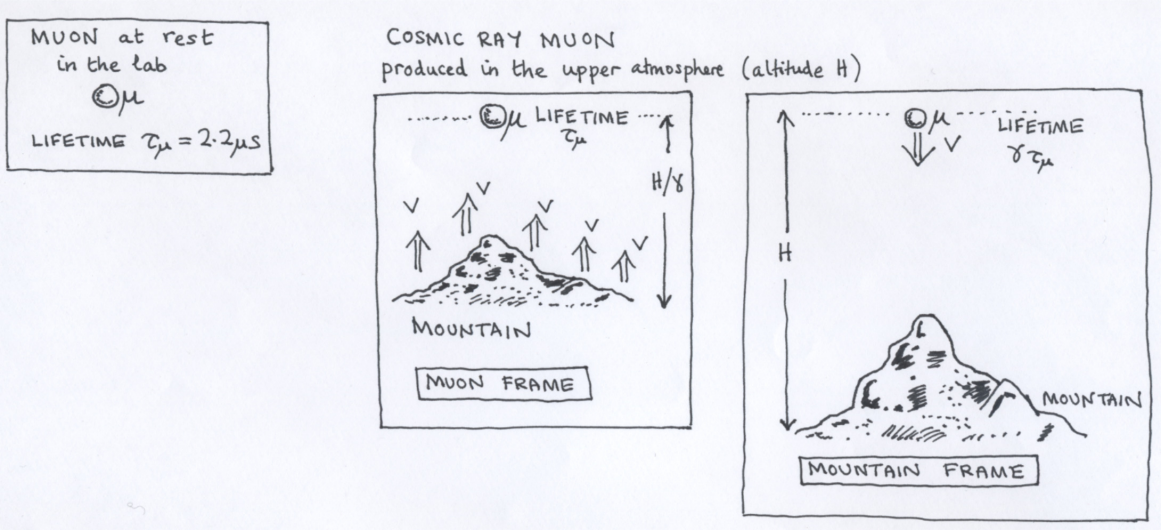
The "twin paradox" - is it really a paradox?

Consider two synchronised clocks, A and B, at rest at a point P in an inertial frame S. Clock A remains at point P, while B is briefly accelerated to speed v and travels to a distant point Q in S. There it is briefly deaccelerated and made to return to A with velocity $-v$.

Now imagine that there are two twins, one of whom stays with the 'at rest' clock at A, while the other goes along for the ride, travelling with B. Both twins age at the same rate, according to their own individual clocks. However, when they meet at the end of the B's journey they find that twin A has aged more than twin B.

- *Argument from A's perspective:* B's clock is "running slow". In A's frame, B's time is dilated, so that one hour experienced by B is γ hours for A.
- *The apparent paradox:* Can't B argue from his/her frame in which B remained stationary that A did all the travelling, and so therefore (by the same argument as we used from A's frame) shouldn't B be the older one?
- *The resolution:* A and B do not have identical experiences. A has remained in a single inertial frame, while B has not, as the accelerometer in B's spacecraft will have recorded.

Figure H13: Decay of cosmic rays



Relativistic velocity addition

$$v'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\gamma(\mathrm{d}x - u\mathrm{d}t)}{\gamma(\mathrm{d}t - u\mathrm{d}x/c^2)} = \frac{(dx/dt) - u}{1 - u(dx/dt)/c^2} \tag{1}$$

$$= \frac{v_x - u}{1 - uv_x/c^2} \tag{2}$$

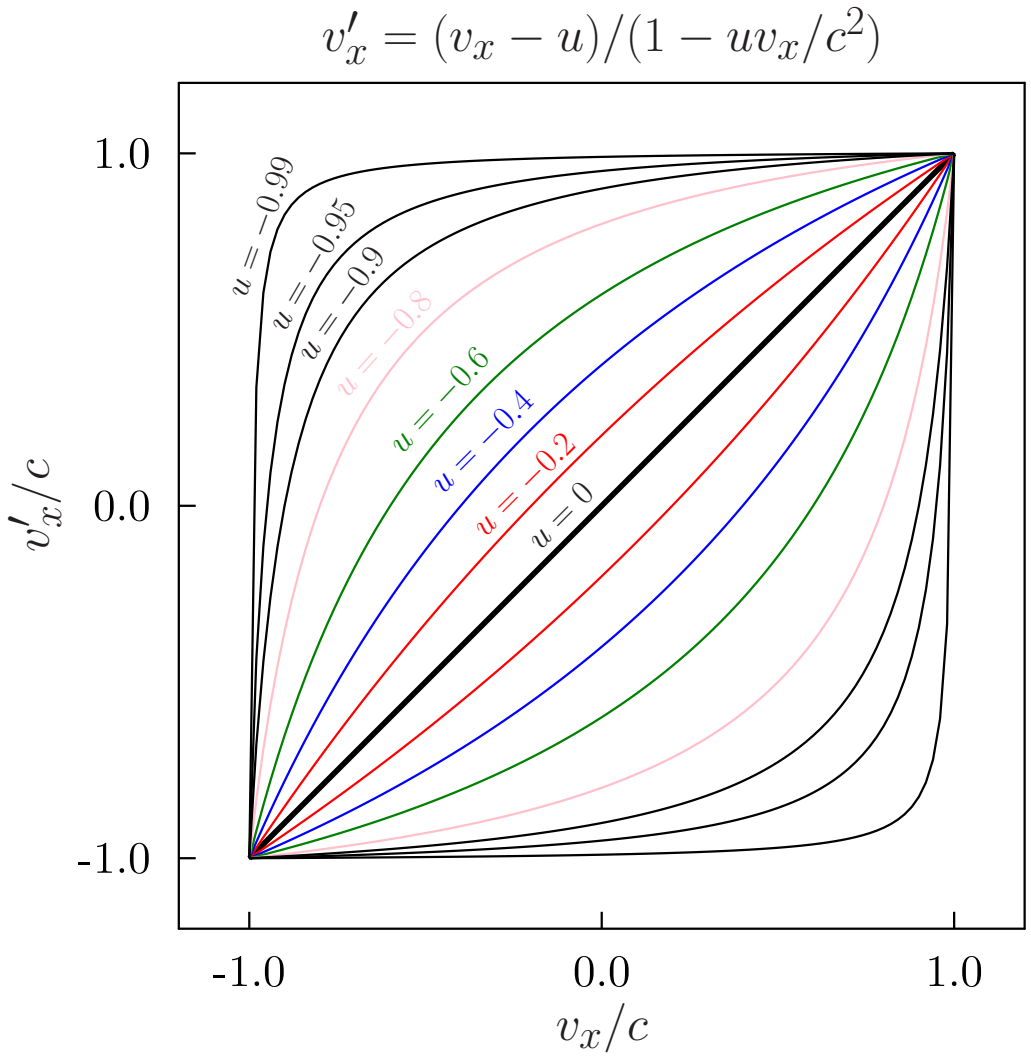
$$v'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\gamma(\mathrm{d}t - u\mathrm{d}x/c^2)} \tag{3}$$

$$= \frac{v_y}{\gamma(1 - uv_x/c^2)} \tag{4}$$

$$v'_z = \frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{\mathrm{d}z}{\gamma(\mathrm{d}t - u\mathrm{d}x/c^2)} \tag{5}$$

$$= \frac{v_z}{\gamma(1 - uv_x/c^2)} \tag{6}$$

Figure H14: General velocity transformation



Lecture 6 outline:

16 The lightcone, causality and time-like and space-like intervals

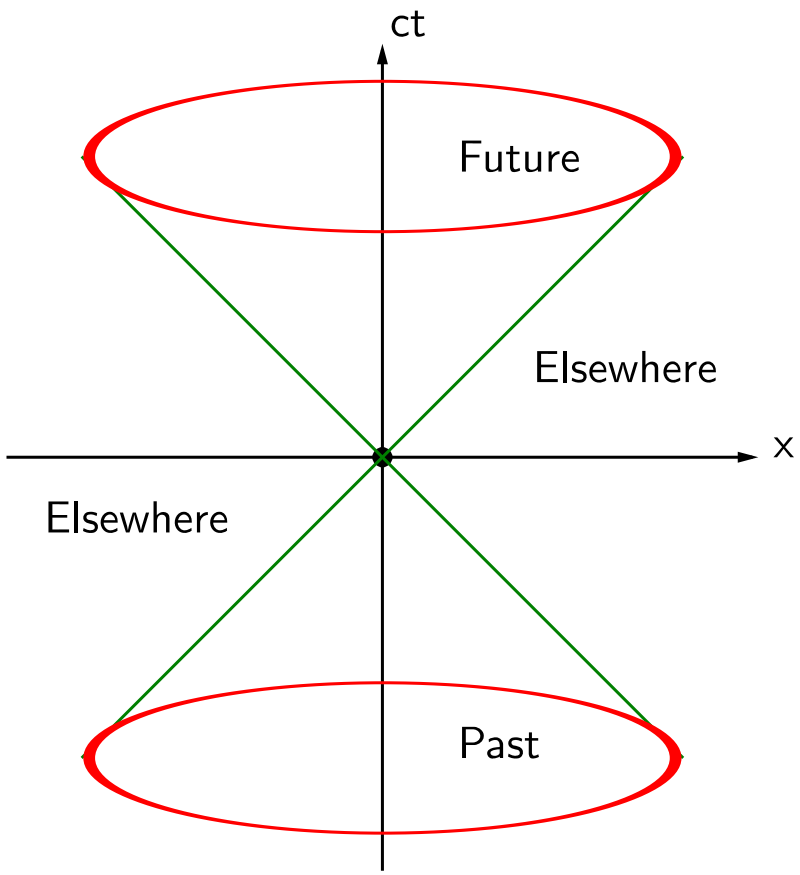
17 4-vectors and invariance

- **17.1** Displacement 4-vector
- **17.2** Velocity 4-vector

18 Relativistic dynamics

- **18.1** Relativistic momentum
- **18.2** Relativistic energy
- **18.3** Momentum 4-vector

Figure H15: Causality and the lightcone



Lecture 7 outline:

19 Mathematical objects in Special Relativity

20 Relativistic kinetic energy

21 Collisions & what gets conserved

21.1 Types of collisions

21.2 Is matter created in sticky collisions?

Matter can be created in sticky collisions but there are additional conservation laws that have to be obeyed. e.g. conservation of baryon number, lepton number, charge etc.

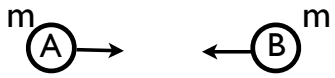
21.3 Zero-momentum frame (aka ZMF)

21.4 Example of 2 particles colliding in the ZMF & their KE

21.4 Example of 2 particles colliding in the ZMF & their KE

Two identical particles (each rest mass m, kinetic energy T) collide head on. What is the kinetic energy T' of one in the rest frame of the other?

Zero-momentum frame



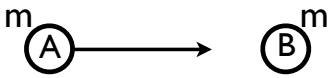
$$E = T + mc^2$$

$$p^{\mu}_{A} = \left(\frac{E}{c}, \underline{p}\right)$$

$$p^{\mu}_{B} = \left(\frac{E}{c}, -\underline{p}\right)$$

$$p^{\mu}_{TOT} = \left(\frac{2E}{c}, 0\right)$$

Rest frame of B



$$E' = T' + mc^2$$

$$p^{\mu'}_{A} = \left(\frac{E'}{c}, \underline{p'}\right)$$

$$p^{\mu'}_{B} = \left(\frac{mc^2}{c}, 0\right)$$

$$p^{\mu'}_{TOT} = \left(\frac{E' + mc^2}{c}, \underline{p'}\right)$$

Use invariant:

$$(p^{\mu}_{TOT})^2 = (p^{\mu'}_{TOT})^2$$

$$\left(\frac{2E}{c}\right)^2 \qquad \left(\frac{E' + mc^2}{c}\right)^2 - (p')^2$$

$$\implies 4E^2 = (E' + mc^2)^2 - (p')^2c^2$$

use $E'^2 = (p')^2c^2 + m^2c^4$ Energy momentum invariant

$$\implies 4E^2 = (E')^2 + 2mc^2E' + m^2c^4 - (E')^2 + m^2c^4$$

$$\implies 2E^2 = mc^2(E' + mc^2)$$

$$\implies 2(T + mc^2)^2 = mc^2(T' + 2mc^2)$$

$$\implies T' = 4T\left(1 + \frac{T}{2mc^2}\right)$$

22 Derivation of Lorentz transform *(non-examinable)*

Uniform motion for a body with no force in all frames. Therefore, Lorentz transform must preserve straight lines: it must be linear

$$\begin{aligned} t' &= At + Bx \\ x' &= Dt + Ex \\ y' &= y \\ z' &= z \end{aligned}$$

A, B, D, E are constants

$$\begin{aligned} x' &= 0 \Longrightarrow x = vt \quad \therefore D = -Ev \\ x &= 0 \Longrightarrow x' = vt' \therefore D = -Av \\ \Longrightarrow A &= E \end{aligned}$$

$$\begin{aligned} t' &= At + Bx \\ x' &= A(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

Use $c^2t^2 - x^2 - \cancel{y^2} - \cancel{z^2} = c^2(t')^2 - (x')^2 - \cancel{(y')^2} - \cancel{(z')^2}$

$$\begin{aligned} \therefore c^2t^2 - x^2 &= c^2(A^2t^2 + 2ABxt + B^2x^2) - A^2x^2 + 2A^2xvt - A^2v^2t^2 \\ &= \underbrace{c^2t^2A^2\left(1 - \frac{v^2}{c^2}\right)}_{=1} \underbrace{- x^2(A^2 - c^2B^2)}_{=1} \underbrace{+ xt2A(Bc^2 + Av)}_{=0} \end{aligned}$$

$$\therefore A = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \equiv \gamma \qquad \text{agrees} \qquad \therefore B = -\frac{v}{c^2}A$$

$$\begin{aligned} \therefore t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

Lecture 8 outline:

23 The photon, and its momentum

24 The Compton effect

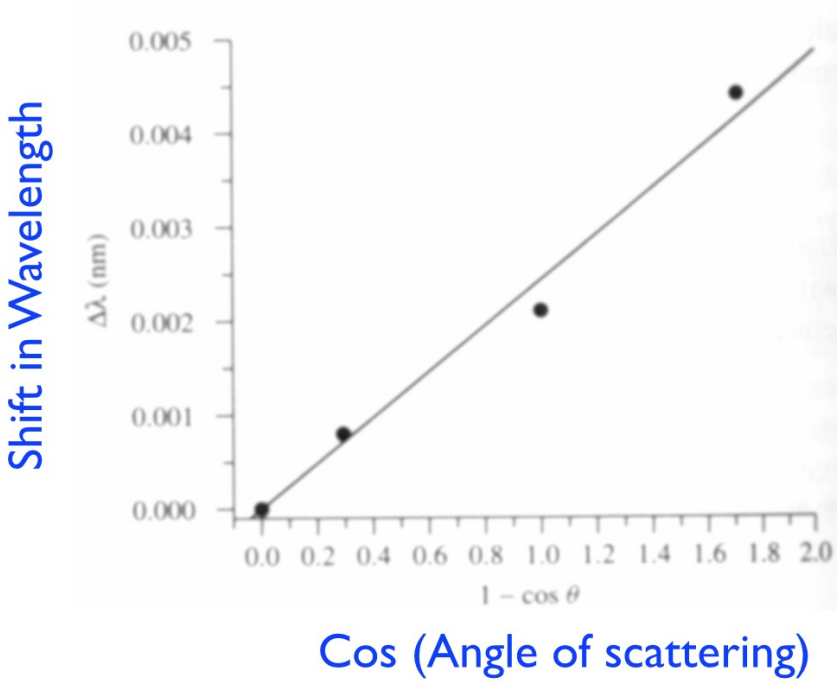


Figure H16: The Compton effect

24.1 Example (traditional method)

24.2 Example (4-vector method)

The Compton Effect - with 4-vectors!

Before:

$$p_\gamma = \frac{E}{c}(1,1,0,0)$$

$$p_\gamma = \frac{h}{\lambda}(1,1,0,0)$$

$$p_e = (mc,0,0,0)$$

After:

$$p'_\gamma = \frac{E'}{c}(1,\cos\theta,\sin\theta,0)$$

$$p'_\gamma = \frac{h}{\lambda'}(1,\cos\theta,\sin\theta,0)$$

Invariants:

$$p_\gamma \cdot p_\gamma = p'_\gamma \cdot p'_\gamma = 0 \qquad p_e \cdot p_e = p'_e \cdot p'_e = m_\circ^2\, c^2$$

Conservation of 4-momentum:

$$p_e + p_\gamma = p'_e + p'_\gamma$$

$$p_e + p_\gamma - p'_\gamma = p'_e$$

$$(p_e + p_\gamma - p'_\gamma)^2 = p'^2_e$$

by invariant

$$p_e \cdot p_e + p_\gamma \cdot p_\gamma + p'_\gamma \cdot p'_\gamma + 2p_e \cdot p_\gamma - 2p_e \cdot p'_\gamma - 2p_\gamma \cdot p'_\gamma = p'_e \cdot p'_e$$

= 0 for photon

$$p_e \cdot p_\gamma - p_e \cdot p'_\gamma - p_\gamma \cdot p'_\gamma = 0$$

$$\frac{hcm}{\lambda} - \frac{hcm}{\lambda'} - \frac{h^2}{\lambda\lambda'} + \frac{h^2}{\lambda\lambda'}\cos\theta = 0$$

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

Lecture 9 outline:

24.4 Decay example of pion into muon and neutrino

25 Doppler effect

- **25.0** Classical Doppler effect
- **25.1** Relativistic Doppler effect (longitudinal)
- **25.2** Frequency shifts, red shifts and blue shifts
- **25.3** Agreement with non-relativistic limit
- **25.4** Wavelength change under Doppler
- **25.5** Transverse Doppler effect (non-examinable)

Lecture 10 outline:

Summary of course and recap.