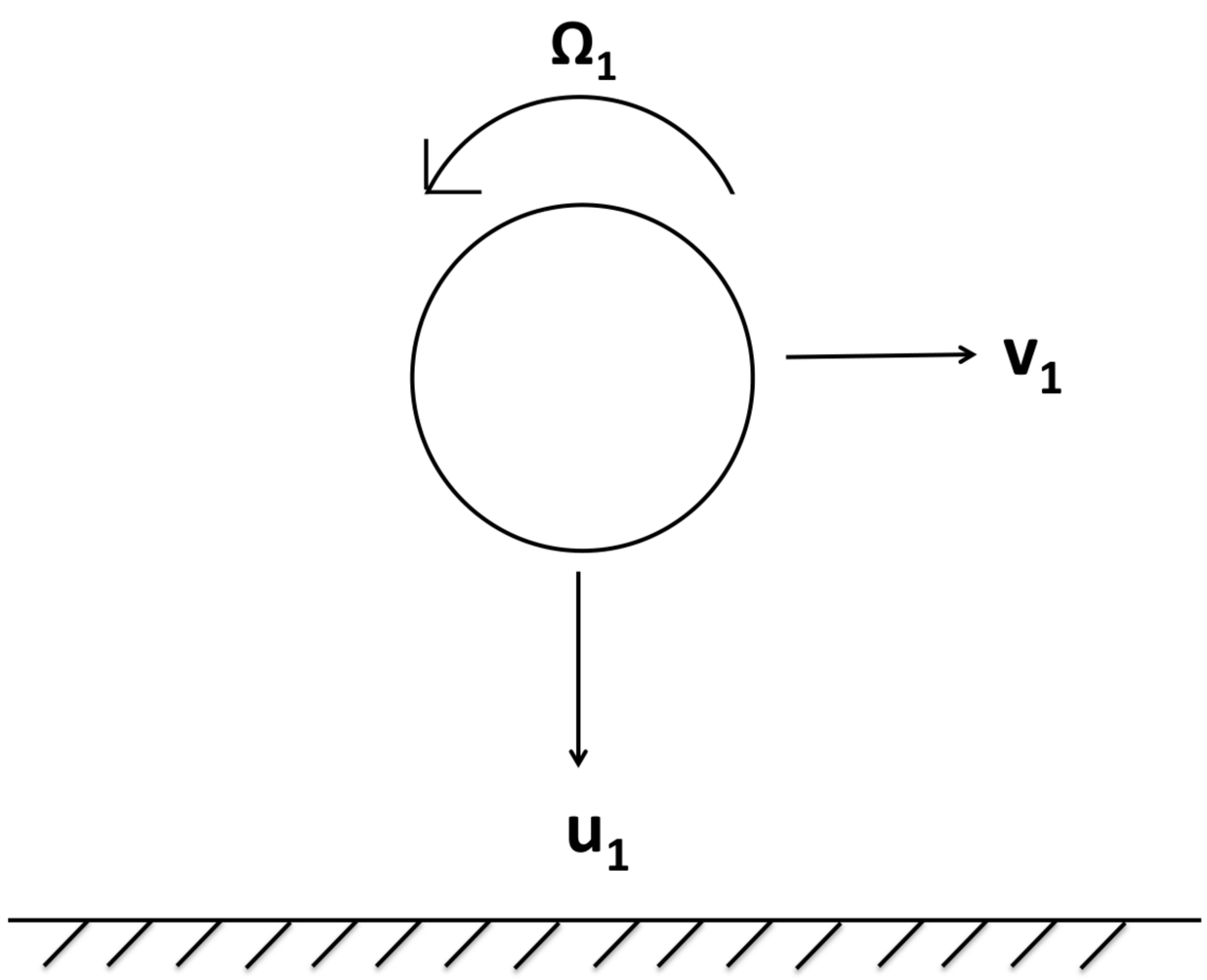
Section B

8. Show that the moment of inertia of a disk of radius a and mass m about an axis through its centre and perpendicular to the plane of the disk is $\frac{1}{2}ma^2$. [3]

By dividing a sphere into thin disks, or otherwise, calculate the moment of inertia of a uniform sphere, of mass M and radius a, about an axis through its centre. [7]

A spherical superball is perfectly elastic, incompressible and rough. In the figure below a superball of radius a is spinning with an angular velocity Ω_1 as shown. The superball hits a rough, horizontal surface while travelling with components of velocity u_1 normal to the surface and v_1 parallel to the surface.



Sketch and explain the forces acting on the superball at the moment of contact with the surface. [3]

Find the linear and angular velocities u_2, v_2 and Ω_2 with which it rebounds. [7]

9. (a) Write down expressions for the momentum p and the total energy E of a relativistic particle, and use them to verify that $E^2 - p^2c^2$ is an invariant. [3]

A particle of mass m travelling at speed v_1 collides head-on with a particle of mass $2m$ travelling at speed $v_2 = c/3$. They merge to form a single particle of mass M at rest in the same frame. Find expressions for v_1 and M. Comment on why M is not equal to $3m$. [6]

(b) Write down the Lorentz transforms for energy and momentum. Consider a photon of energy E' travelling in a frame S' which is moving at speed $v = \beta c$ with respect to a frame S. Show that the relativistic Doppler factor, D, is given by

$$D = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$$

where $D = E/E'$ is the ratio of the photon energies. [3]

In the case where the relative speed is close to the speed of light, c, i.e. $\beta = (1-\epsilon)$, where ϵ is small, show that the Doppler factor can be written as

$$D = \left(4\gamma^2 - 1\right)^{\frac{1}{2}}$$

where γ is the Lorentz factor. [2]

(c) The propagation of extremely high energy photons through the universe is limited by a reaction with the cosmic microwave background radiation in which electron-positron pairs are formed:

$$\gamma_1 + \gamma_2 \rightarrow e^+ + e^-.$$

Assuming that the background photon γ_2 has an energy of 10^{-3} eV, calculate the minimum energy that the other photon γ_1 must have in order for the above reaction to take place. [4]

What is the velocity of the resulting electron-positron pair in the laboratory frame? [2]

11. A spacecraft of mass m is approaching Jupiter at an initial relative speed of v_0 when far from the planet, and with an impact parameter b. Show that the energy of the orbit can be written as

$$E = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} - \frac{GmM_J}{r}$$

where M_J is the mass of Jupiter, and explain each term in the equation. [4]

Calculate an expression for the distance of closest approach, r_{min}, in terms of v_0, b, M_J and G. [4]

Show that the angle of deflection of the spacecraft's path relative to Jupiter, ϕ, is given by

$$\cot(\phi/2) = bv_0^2/GM_\mathrm{J}.$$

[6]

If $v_0 = 10\,\mathrm{km\,s^{-1}}$ and $M_\mathrm{J} = 2 \times 10^{27}\,\mathrm{kg}$, calculate the value of b such that the spacecraft trajectory is deflected by 90 degrees. [2]

The orbital velocity of Jupiter relative to the Sun is $13\,\mathrm{km\,s^{-1}}$. If after being deflected the spacecraft is travelling in the same direction as Jupiter, what are the velocities of the spacecraft relative to the Sun before and after the encounter? [4]

Section A

1. An old-fashioned television set has a screen 50 cm across and uses an evacuated cathode ray tube: an electron gun behind the screen fires electrons with initial energy 100 eV in the direction of the centre of the screen. Before reaching the screen these electrons pass between a pair of parallel conducting plates located 30 cm behind the screen and at right angles to the screen. If the plates are squares of side 1 cm, and are separated by 1 cm, calculate the range of voltages between the two plates which are required to produce a display across the full width of the television screen. [You may ignore edge effects for the plates and the influence of gravity.] [5]

2. A wire forms a closed circular ring of enclosed area A, resistance R and self-inductance of L. A spatially-uniform external magnetic field is applied normal to the plane of the ring, and the amplitude of the magnetic flux density varies with time as $B = B_0 \cos \omega t$, where B_0 and ω are constants. Find expressions for the e.m.f. and the current induced in the ring due to the applied field. Find an expression for the time-average of the power dissipated in the ring. [6]

3. Two infinite sheets of electric charge, one with surface charge density $+\sigma$ and the other with surface charge density $-\sigma$, intersect in the centre at right angles. Find the magnitude and direction of the **E**-field, and sketch the **E**-field lines. [5]

4. What are the fluxes of **E**-field ($\int\int_{\text{Area}} \mathbf{E} \cdot \mathrm{d}\mathbf{A}$) through each of the faces of a cube of side a in the following cases:
i) if a point charge q is placed at the centre of the cube;
ii) if a point charge q is placed at one of the corners of the cube?
[Hint: you do not need to evaluate the surface integral directly if you consider the symmetries.] [4]

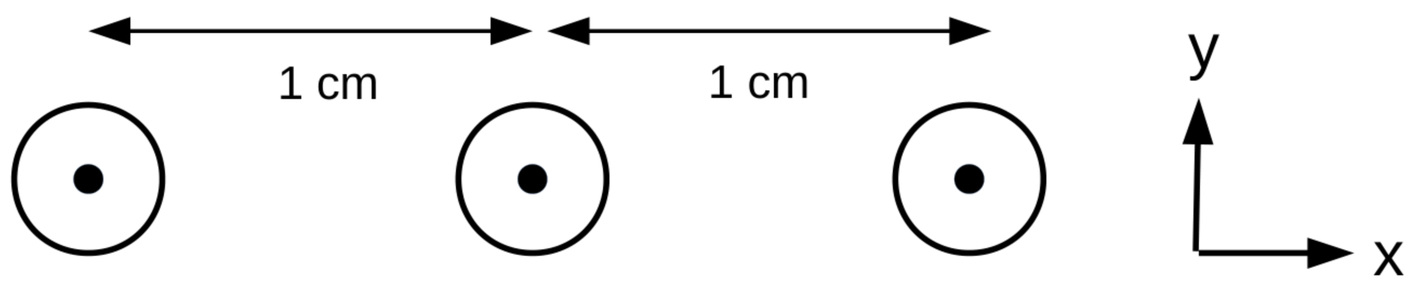
5. Starting from Maxwell's equations in a vacuum, show that the **E**-field of an electromagnetic wave obeys a wave equation, and find the velocity of propagation. If the **E**-field is polarized along the x-axis and varies as $\mathbf{E} = E_0 \cos(k\,z - \omega\,t)\,\hat{\mathbf{x}}$, where $E_0 = 100\,\mathrm{V\,m^{-1}}$, derive an expression for the magnetic flux density vector **B** and calculate the amplitude of the **B**-field. [7]

6. State the *lens maker's formula*, relating the focal length f to the positions of the source (u) and image (v) relative to the centre of the lens along the optical axis. If the human eye can focus on an object as close as 25 cm, what is the maximum magnification of a magnifying glass consisting of a converging lens with focal length 4 cm? Sketch a ray diagram for the case of maximum magnification, and state whether the image produced is real or virtual. [6]

9. Write down the *Biot-Savart law*, and use this to derive the magnitude and direction of the **B**-field produced by a thin wire of length $2a$ carrying a current I, at a point along the perpendicular bisector of the wire, at a distance s from the centre of the wire. Show that this is consistent with Ampère's law for an infinitely long wire. [6]

Consider two parallel very long wires separated by distance d each with a current I flowing in opposite directions. What is the force per unit length between the wires, and will they attract or repel each other? Calculate the self-inductance per unit length of this system, assuming that the wires are thin. [6]

Three parallel infinite wires of negligible width are co-planar, evenly spaced by 1 cm and each carry current of 1 A in the same direction, as in the diagram below. Each wire has a mass per unit length, μ_m, of $1\,\mathrm{g\,m^{-1}}$.



Draw the magnetic field lines around the wires, and calculate the position(s) where the magnetic flux density is zero. [3]

The entire middle wire is displaced a small amount in the y-direction (perpendicular to the initial plane of the three wires), while the other two wires are held fixed. Describe the subsequent motion of the middle wire when it is released, give an expression for the relevant timescale of its motion, and calculate a value for this timescale. [5]

10. Explain what is meant by *Fraunhofer diffraction*. [3]

A slit of width w is illuminated at normal incidence by a collimated beam of coherent light of wavelength λ. Derive an expression for the intensity distribution as a function of angle for the diffraction pattern in the Fraunhofer limit. [6]

What is meant by the *Rayleigh criterion*? Use this to find an expression for the diffraction-limited angular resolution of a single slit. [3]

A spy satellite orbits 200 km above the surface of the Earth and observes light at visible wavelengths. Estimate the size of the main mirror on the spy satellite if it is able to read the headlines of a tabloid newspaper, with a font thickness of 3 cm. You may assume that only diffraction affects the resolution. [4]

A digital camera is used to record the image at the focus of the spy satellite's main mirror. If the main mirror of the spy satellite has a focal length of 80 m, estimate the maximum distance between the centres of adjacent pixels for the camera to fully resolve the newspaper headlines. Using these size pixels, how large a format would the camera need to be (in terms of the number of pixels across) in order to image all of the High Street in Oxford with one exposure? [Oxford High Street is about 500 m long.] [4]

Section A

1. Define the Lagrangian and the Hamiltonian as used in classical mechanics. Prove that the Hamiltonian is conserved in a system for which the Lagrangian has no explicit time dependence. [6]

2. A spaceship is required to hover at a constant height just above the surface of a planet where the acceleration due to gravity is g. The exhaust expels fuel at a constant speed u, and the initial mass of the spaceship including fuel is M_0. While there is fuel remaining, what should be the rate of ejected mass $\alpha(t) = -\frac{\mathrm{d}M(t)}{\mathrm{d}t}$ to keep the ship hovering at rest at a time t? Express your answer in terms of M_0, u, g and t. [6]

3. A ball of mass M and radius r rests on top of a fixed hemisphere of radius R. There is friction between the ball and hemisphere. The ball is given an infinitesimal perturbation and rolls down the hemisphere without slipping. Assuming that $r \ll R$, find the angle to the vertical at which the ball loses contact with the hemisphere. [7]

[The moment of inertia of the ball about its principal axis is $\frac{2}{5}Mr^2$.]

4. A non-relativistic particle of mass m_1 and speed u_1 collides elastically and obliquely with another particle of mass m_2, which is initially at rest. Draw diagrams illustrating the situation after the collision in the laboratory frame and in the centre of mass frame.

Show that, after the collision, the scattering angle of m_2 in the laboratory frame, θ_2, is related to the scattering angle, θ^*, in the centre of mass frame by

$$\tan\theta_2 = \frac{\sin\theta^*}{1-\cos\theta^*} \ ,$$

where all angles are defined with respect to the initial direction of m_1. [8]

5. An object has velocity components u_x and u_y in an inertial frame S. Starting from the Lorentz transforms for space and time, show that the velocity components of the same object in a frame S', with speed v in the positive x direction relative to S, are

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \quad \text{and} \quad u'_y = \frac{u_y}{\gamma\,(1 - u_x v/c^2)} \ ,$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

As seen from Earth, two spaceships A and B are travelling along perpendicular directions. If A is observed by a stationary Earth observer to have velocity $+0.5c$ in the x direction and B to have velocity $-0.5c$ along y, determine the speed of ship B as measured by the pilot of ship A. [8]

Section A

1. Write down an expression for the electric field at a displacement $\mathbf{r}$ from a charge of magnitude Q. [2]

A vertical dipole consists of a charge of $+30\,\mathrm{C}$ held at a height of $8\,\mathrm{km}$ above an infinite conducting plane and a second charge of $-30\,\mathrm{C}$ held directly below the first charge but at a height of $5\,\mathrm{km}$ above the same infinite conducting plane. Calculate the electric field directly below the vertical dipole at a height of 0 km. [5]

2. Write down Ampère's law in the case of non-time-varying fields and explain each of the terms. Using this law derive an expression for the force per unit length between two parallel and infinitely long straight wires, each carrying a constant current I in the same direction and separated by a distance r. In which direction does the force act? [6]

3. Explain why a displacement current term has to be added to Ampère's law in the case of time-varying fields. [3]

A parallel-plate capacitor consists of two circular plates, each of radius a and separated by a distance d, where $d \ll a$. The space between the plates is filled with air. An alternating potential difference of $V_0 \exp(-j\omega t)$ is applied across the capacitor. Find an expression for the magnitude of the magnetic field between the plates. [5]

4. Estimate the electric field strength $100\,\mathrm{km}$ from a $100\,\mathrm{kW}$ television transmitter assuming isotropic radiation. [6]

5. Sketch and clearly label an experimental arrangement for using a diffraction grating as a spectrometer. Comment on the conditions required for proper illumination of the diffraction grating. [4]

If a diffraction grating of 2000 lines per cm is used to analyse the spectrum of mercury, find the angular separation in the second-order spectrum of the two lines of wavelengths $579.0\,\mathrm{nm}$ and $577.0\,\mathrm{nm}$. [3]

6. Draw and label a circuit using one operational amplifier and resistors which gives an output

$$V_{\mathrm{out}} = -2(V_{\mathrm{a}} + 2V_{\mathrm{b}})$$

for two voltage inputs V_{a} and V_{b}.

Describe the behaviour of the circuit if the feedback resistor is replaced by a capacitor. [6]

Section A

1. A particle is projected from the ground inside a horizontal tunnel of height H. Neglecting air resistance, show that the maximum range inside the tunnel is

$$\sqrt{8H\left(\frac{u^2}{g} - 2H\right)},$$

where u is the initial speed of the particle and g is the acceleration due to gravity. [6]

2. Consider a non-relativistic collision between two particles of masses M_1 and M_2. Show that the total kinetic energy in the laboratory frame can be written as

$$\frac{1}{2}(M_1 + M_2)V_{\mathrm{CM}}^2 + \frac{1}{2}\mu v^2\,,$$

where V_{CM} is the speed of the centre of momentum frame, v is the relative speed between M_1 and M_2, and μ is the reduced mass. State the physical meaning of the two terms in the expression, and deduce the loss of kinetic energy for a perfectly inelastic collision. [7]

3. (a) Show that the moment of inertia of a thin uniform rod of mass M and length L, about an axis passing through the rod's centre of mass and inclined at an angle α to its long axis, is given by

$$\frac{1}{12}ML^2\sin^2\alpha\,.$$

(b) Making use of the parallel axis theorem, or otherwise, find the moment of inertia of the rod about an axis in the plane of the rod, parallel to its long axis and a perpendicular distance D away. [5]

4. A particle of mass M is travelling at constant speed towards the rod from the previous question, in a direction perpendicular to the rod's long axis. The rod has the same mass M as the particle, and is initially at rest. Assuming a perfectly elastic collision, at what distance from the centre of the rod should the collision occur, so that the particle and centre of the rod move with equal speeds afterwards? Show that the relative speed of the particle and the point of contact on the rod is the same before and immediately after the collision. [7]

5. Write down an expression for the invariant interval in space-time. Using the form of this invariant, or otherwise, derive an expression for the time dilation between observers moving at constant relative speed.

A clock is placed in a jet aircraft. The clock measures a time interval of 1000 s when the aircraft flies directly from one location to another at a constant speed of $300\,\mathrm{m\,s^{-1}}$. According to special relativity, how much longer or shorter is the time interval for the aircraft's journey, as measured by clocks at rest on the ground? [7]

6. Derive the formula for the longitudinal relativistic Doppler effect. At what speed and in which direction would you have to move a laser source emitting light with wavelength $\lambda = 633\,\text{nm}$ in order to trigger an infrared sensor moving directly towards you at $0.2c$, whose response in its own rest frame is at $1000\,\text{nm}$? (Assume that all motion occurs along the same axis.) [8]

Section B

7. (a) A rocket of total instantaneous mass $M \equiv M(t)$ (including fuel) is launched vertically upwards from the surface of the Earth, and emits fuel backwards at a constant speed u relative to the rocket. Starting from consideration of the change in momentum of the rocket+fuel, derive the rocket equation

$$F = M\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}M}{\mathrm{d}t}\,,$$

where F is the net external force, and $v \equiv v(t)$ is the rocket's instantaneous velocity. [5]

The rocket moves under the influence of the Earth's gravity g, which may be assumed to be constant, and is also subject to a linear air resistance kMv. Given that fuel is burned at a rate such that $M(t) = M_0\mathrm{e}^{-bt}$, where M_0 is the initial total mass of the rocket, and b is a positive constant, find an expression for the velocity of the rocket, $v(t)$, at a time t. Give the condition on b in order for the rocket to launch. [5]

(b) Now consider two buckets filled with sand that are hanging on either side of an ideal, massless, frictionless pulley and connected by a massless rope. Initially, both buckets contain the same amount of sand and each has a total mass of M_0 (including the sand). However, one of the buckets has a small hole in it and is leaking, resulting in a loss of mass at a constant rate α.

Given that $M(t)$ and $v(t)$ are, respectively, the instantaneous mass and upwards velocity of the leaky bucket, show that

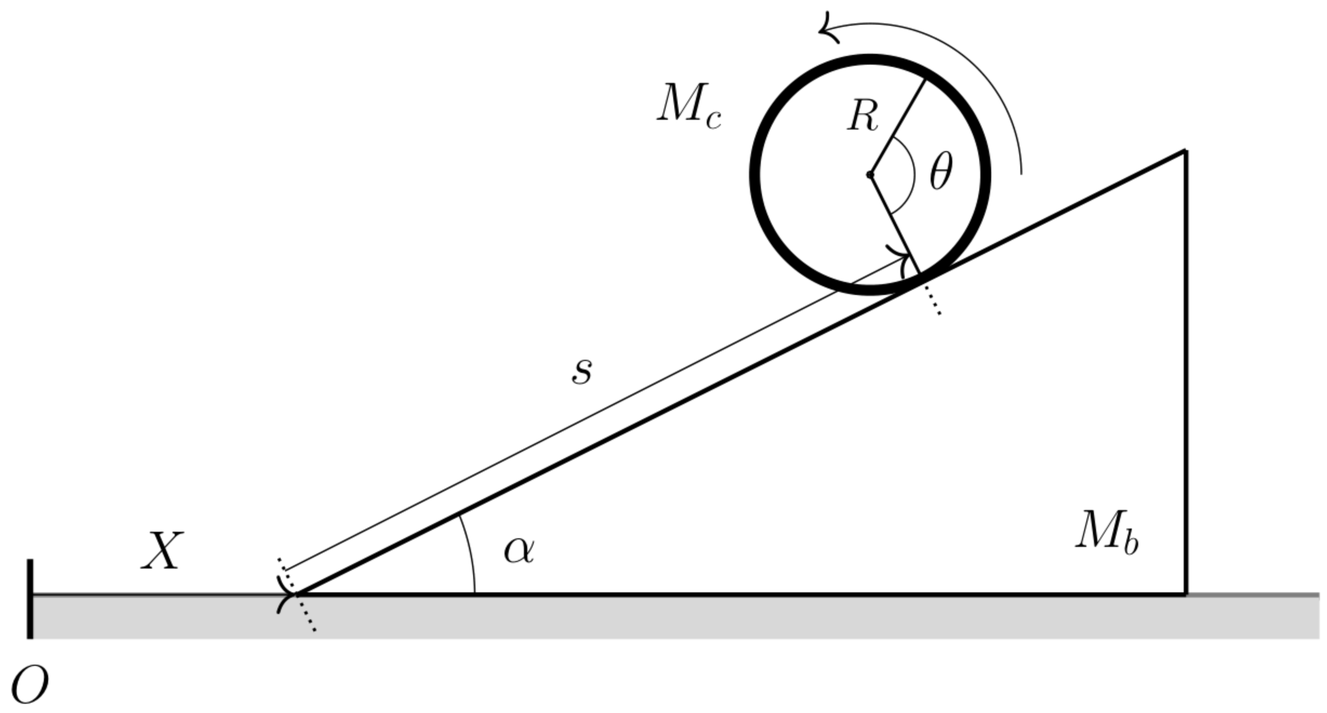
$$\frac{\mathrm{d}v}{\mathrm{d}M} = \frac{g}{\alpha}\left(\frac{M(t) - M_0}{M(t) + M_0}\right).$$

Given that ϵM_0 $(0 \leq \epsilon < 1)$ is the initial mass of the sand in each bucket, and the buckets are initially at rest, determine the speed of the leaky bucket at the instant it becomes empty. (You may assume that the bucket will empty before it hits the pulley.) Evaluate your result for the cases $\epsilon = 0$ and $\epsilon \to 1$. [10]

9. (a) Write down the Euler-Lagrange equations for a system described by generalised coordinates q_i and generalised velocities $\dot{q}_i$.

Explain briefly what is meant by: (i) a cyclic coordinate, and its relation to a conservation law; and (ii) a holonomic constraint. [4]

(b) A uniform solid cylinder of mass M_c and radius R rolls without slipping down the inclined surface of a wedge-shaped block. The block has a mass M_b and an angle of inclination α, and is itself free to slide frictionlessly along a horizontal surface.



Justify that the system can be described in terms of two generalised coordinates, and find an expression for the Lagrangian in terms of $(X, s, \dot{X}, \dot{s})$, where X is the position of the lower left corner of the block and s is the distance along the inclined surface to the bottom of the cylinder.

Identify the cyclic coordinate and associated conserved quantity.

Use the Euler-Lagrange equations to determine the linear accelerations $\ddot{X}$ and $\ddot{s}$, and evaluate your expressions in the limit $M_b \gg M_c$. [9]

(c) Consider now the case where the block is fixed in position, and where the forces constraining the cylinder's motion (i.e. that it maintains contact with the block and rolls without slipping) are to be computed. Construct a Lagrangian for this problem, and use the Euler-Lagrange equations to determine the constraint forces, stating their physical origin. [Hint: introduce a new generalised coordinate, r, representing the perpendicular distance of the bottom of the cylinder from the inclined surface, and treat the system as having three generalised coordinates (r, s, θ), together with appropriate constraints.]

Given a coefficient of static friction μ_s, show that the cylinder rolls without slipping provided that $\mu_s \geq \frac{1}{3}\tan\alpha$. [7]

[The moment of inertia of the cylinder about its principal axis is $\frac{1}{2}M_cR^2$.]

Section A

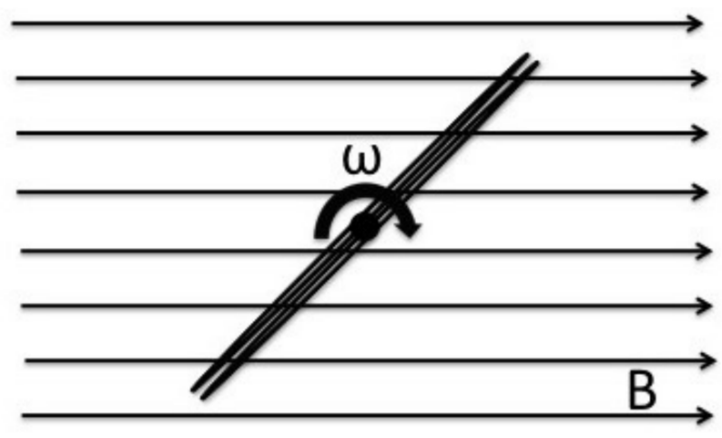
1. A long solenoid of length L and radius R is uniformly wound with N_1 closely spaced turns. It is tightly surrounded about the middle by a short coil closely wound with N_2 closely spaced turns on the same axis. What is the mutual inductance of the coils? [4]

2. Describe how the method of images can be used to evaluate the field configuration of a set of charges associated with conducting planes. Illustrate this by finding an expression for the force between the charge and the plane for a charge q a distance d above an infinite conducting plane. [5]

3. A hollow conducting sphere of radius R has an initial charge Q. By consideration of Gauss' law, find an expression for the electric field strength and for the potential inside and outside the sphere as a function of r, the distance from the centre of the sphere.

A second hollow conducting sphere of radius $2R$ and with no initial charge is brought into contact with the first sphere. Find expressions for the final charge on each sphere in this arrangement. [7]

4. A plane circular coil of radius r with N turns rotates at an angular velocity ω about an axis along a diameter and perpendicular to a uniform magnetic flux density B along the x-axis as shown in the figure.



Find an expression for the induced e.m.f. in the coil as a function of time t. What is the orientation of the coil with respect to the x-axis when the magnitude of the e.m.f. is a minimum? [6]

5. A transverse electromagnetic wave propagating in free space can be expressed as: $\mathbf{E}(x,t) = E_0 \cos(kx - \omega t)\,\hat{y}$. Explain all of the terms in this expression. Using the appropriate Maxwell's equation in differential form, find an expression for the **B** component and its direction of oscillation and hence find the speed of propagation of the wave. [5]

6. The most common forms of resistors used in electronic circuits are made out of carbon composite material. An alternative resistor composition uses resistive wire tightly wound around a small insulating substrate. Discuss the potential advantages and disadvantages of wire wound resistors compared to carbon composite devices. [3]

7. A thin equiconvex lens (a lens where the two convex surfaces have the same radius of curvature) is made of glass with refractive index n. Its spherical surfaces have radius of curvature R. Show that its focal length f is given by

$$f = \frac{R}{2(n-1)}.$$

[5]

An imaging system suffers from chromatic aberration so that an image in blue light is formed at a position 10 mm beyond an image in red light. A student wishes to correct for the aberration and form a corrected image on a screen 1 m from the red light image using a thin equiconvex lens placed between the aberrated images and the screen. Find expressions for the required focal lengths of the lens f_b and f_r at blue and red wavelengths. The lens has refractive indices $n_\mathrm{b} = 1.44$ and $n_\mathrm{r} = 1.40$ at blue and red wavelengths respectively. Calculate the radius of curvature of the lens surfaces and the focal lengths at blue and red wavelengths. [5]

Section B

8. A current I flows through a plane wire coil of radius a with N turns. Find an expression for the magnetic flux density at a distance x along the principal axis of the coil from the centre. [5]

A second identical coil is placed parallel to and coaxial with the first, and separated by a distance a. Identical currents I flow through the coils in the same direction. What is the magnetic flux density B on the axis midway between the pair of coils and at the centre of each coil? Sketch the total magnetic flux density and the contribution from each coil as a function of position on the axis between the coils. Describe briefly the properties of this coil arrangement. [9]

A stream of electrons is accelerated by a potential V and introduced into the region between the coils, perpendicular to the principal axis of the coils and close to the mid-point. An experimenter wants to deflect the electron stream so that it emerges from the region between the coils travelling parallel to but in the opposite direction to the input stream and separated by 2 cm. What is the value of the magnetic flux density between the coils required to achieve this, when the accelerating potential $V = 10$ kV? [6]

[You may neglect the effects of any B field external to the space between the two coils, and assume that the radius of the coils is $\gg 2$ cm.]

9. Compare and contrast the relative merits of the centre of momentum and laboratory reference frames when considering particle collisions. [3]

A non-relativistic particle of mass M_1 and initial speed u_1 collides *inelastically* and obliquely with a second particle of mass M_2, which is initially at rest. Write down expressions for the velocity of the centre of momentum frame, and the velocities of M_1 and M_2 in the centre of momentum frame before the collision. Show that the ratio of the final-to-initial speeds of *each* particle in the centre of momentum frame is equal to the coefficient of restitution, e. [5]

Following the collision, M_1 has a speed v_1 and a scattering angle θ_1 (measured with respect to the initial direction of M_1) in the laboratory frame. Show that the ratio of the final-to-initial speeds of M_1 in the laboratory, is given by

$$\frac{v_1}{u_1} = \frac{M_1}{M_1 + M_2}\left[\cos\theta_1 \pm \sqrt{\cos^2\theta_1 - 1 + e^2\left(\frac{M_2}{M_1}\right)^2}\right].$$

[5]

Making use of the above equation:

(i) calculate the possible values of v_1 for the case of a perfectly *elastic* and equal mass collision, and comment on the solutions;

(ii) determine the speed v_1 for a perfectly *elastic* collision when $M_2 \gg M_1$;

(iii) describe the motion of M_1 and M_2 for the case of a totally *inelastic* collision;

(iv) show that for a perfectly *elastic* collision with $M_1 > M_2$, the scattering angle θ_1 must take on a maximum value between $0 \leq \theta_1^{\max} < \frac{\pi}{2}$, depending on the relative values of M_1 and M_2. [7]

10. What is meant by an *invariant* quantity in the context of special relativity? Illustrate your answer by discussing the relationship between total energy, momentum and rest mass for a particle in special relativity. [5]

In the jet of a quasar, an ultra-relativistic electron collides head-on with a very low energy photon of the cosmic microwave background. No new particles are created. Show that the energy of the photon after the collision is given by

$$E = \frac{(E_1 + p_1 c)E_2}{(E_1 - p_1 c) + 2E_2}$$

where E_1 and p_1 are respectively the energy and the magnitude of the momentum of the incident electron, and E_2 is the energy of the initial photon. [10]

In the case where the initial wavelength of the photon is $\lambda = 2\,\text{mm}$ and the electron is travelling at $v = 0.99\,c$, find the photon wavelength after the collision. [5]

Section A

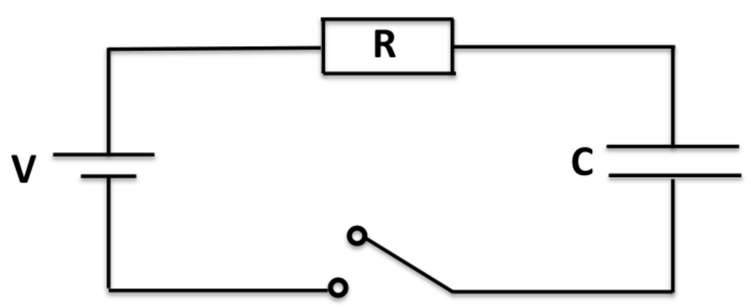
1. Derive an expression for the force per unit length between two parallel and infinitely long straight wires, each carrying a constant current I and separated by a distance r. Under what circumstances is the force repulsive? One of the long wires lies on the ground with the other one placed parallel to and directly above it. The current flowing through the wires produces a repulsive force, and the wires have a mass per unit length λ. Find an expression for the separation of the wires at which the gravitational and magnetic forces are balanced. [5]

2. A stationary charge $+Q$ is fixed at position (x, y) = (-d, 0) in a two dimensional space. A charge $-Q$ is brought from a large distance to a position $(x, y) = (+d, 0)$. What is the potential energy of this configuration? With these two charges in place, what is the **E**-field at positions $(x, y) = (\pm 2d, 0), (0, 0)$ and $(0, \pm 2d)$? [7]

3. Calculate the potential energy, that is the work required for its assembly against electro-static forces, of a solid sphere of radius R containing a uniform charge distribution with a total charge Q. [5]

4. A capacitor consists of two large parallel rectangular plates, separated by a distance x_o. The plates are charged to potentials of V_1 and V_2 respectively. The region between the plates is filled with a fixed, uniform space charge distribution of volume charge density ρ. Find and sketch the magnitude of the electric field and electric potential as a function of position x between the plates. [6]

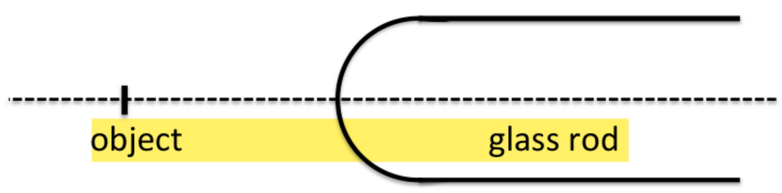
5. In the circuit below, a capacitor C is wired in series with a resistor R and a battery with potential difference V via a switch. Initially the switch is open and the capacitor is uncharged. The switch is closed at time $t = 0$. Find an expression for the voltage across the capacitor as a function of time. [4]



The resistor has resistance $100\,\mathrm{k}\Omega$ and the capacitor has capacitance $1\,\mu\mathrm{F}$. The battery is replaced by a square wave signal generator with frequency f and voltage range ± 12 V. Without detailed calculation, describe and sketch the voltage across the capacitor when $f = 100\,\mathrm{Hz}$, $10\,\mathrm{Hz}$ and $0.1\,\mathrm{Hz}$ after several periods have elapsed. [4]

6. An astronomer suspects that a star at a distance of 4×10^{17} m from the Earth is actually a binary system with an orbital separation of 1×10^{11} m. What diameter telescope is required to confirm this by imaging it at wavelength $\lambda = 410$ nm? [3]

7. A cylindrical glass rod has a hemispherical end with a radius of curvature $R = 20$ mm and a refractive index $n_g = 1.60$. A small object is placed 60 mm in front of the hemispherical end on the axis of the rod.



In the paraxial approximation, find an expression for the position of the image of the object. Find the magnification and position of the image when the rod is used in air with refractive index $n_0 = 1.00$, and in water with refractive index $n_w = 1.33$. [6]

Section B

8. State the laws of electromagnetic induction, clearly explaining the direction of the induced e.m.f. Starting from the expression for the Lorenz force on a moving charge, find an expression for the force resulting from a current flowing in a long wire placed within a magnetic field. [6]

Two long parallel conducting rods are inclined to the horizontal at an angle ϕ and separated by a distance s. A conducting bar of mass m rests perpendicular to and across the rods at the top. The rods are connected at the bottom by a large resistance R and a uniform, vertical magnetic field of flux density B fills the whole region. The bar is released and slides without friction down the rods, remaining perpendicular to them and maintaining contact with them.

Find:

(i) the current induced,

(ii) the equation of motion of the bar,

(iii) the terminal velocity of the bar,

(iv) the rate at which work is done by the bar sliding on the rods and compare it to the power dissipated by the resistor. [10]

In order to reverse the motion of the bar and send it up the slope, the system is modified by replacing the resistor with a wire and passing a large current up one rod, through the bar and down the other rod so that the bar moves with uniform speed. Find expressions for the forces acting on the bar and hence find an expression for the current required to maintain a uniform speed. In what direction must the current flow through the bar? [4]

9. State Maxwell's equations for electric and magnetic fields in vacuum in the differential form. Starting from these expressions, use the divergence theorem to derive Gauss' law and hence derive Coulomb's law for the force between two point electric charges. [5]

A capacitor consists of two thin concentric conducting spherical shells with radii a and b, with $b > a$, in vacuum, and a charge q on the inner shell. Find an expression for the potential between the shells, and hence show that the capacitance is [5]

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} \ .$$

The inner shell is raised to a potential V and then isolated and the outer shell removed. Find an expression for the residual electric field and hence determine the potential on the surface of the remaining sphere and at the position of the sphere that was removed.

An isolated, uncharged conducting spherical shell of radius $2a$ is connected to the remaining sphere with a long wire. What is the ratio of the electric fields on the surfaces of the two spherical shells? [10]

10. From consideration of Huygens' construction for a plane wavefront passing from a medium of refractive index n_1 to a medium of lower refractive index n_2, find an expression for the angle of refraction. What is the condition for total internal reflection? [4]

An optical fibre has a core with refractive index n_1 surrounded by a coaxial cladding layer with a lower refractive index n_2, and is used in air. Light enters the end of the fibre at an angle of incidence θ_i to its axis. Find an expression for the maximum angle of incidence for lossless light transmission along the fibre in terms of the refractive indices of the layers. Take the refractive index of air to be n =1.00. [4]

Signals consisting of very short pulses of light from two lasers operating at the same wavelength are incident on the axis of the fibre at the angle of incidence calculated above and at half that angle respectively. The refractive indices of the core and cladding are 1.5 and 1.45 respectively. Calculate the travel times of the signals along a fibre of length 2 km.

Describe the effect on the transmission of light through a fibre when it is bent into a tight circular loop. Calculate the radius of curvature R (the radius of curvature of the loop to the centre of the fibre) at which light entering a fibre of core radius $r = 80\,\mu$m at an angle of incidence of 20^o is not transmitted. [12]

6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Show that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues. [4]

7. Solve the equation

$$\frac{dy}{dx} = \frac{4x + 2y - 10}{2x - y - 3}$$

by first making the change of variables: $x = X + a$, $y = Y + b$. Explain why this method works, by referring to the lines $4x + 2y - 10 = 0$ and $2x - y - 3 = 0$. [6]

Section B

8. (a) Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$.

(i) Describe the vector space spanned by $\mathbf{a}$ and $\mathbf{b}$ and give its dimension depending on the properties of the two vectors.

(ii) Prove that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. [7]

(b) Show that the set of vectors in $\mathbb{R}^3$ orthogonal to a vector $\mathbf{v}$ form a vector subspace. Does this mean that every plane forms a vector subspace of $\mathbb{R}^3$? [4]

(c) Let $\mathcal{U}$ and $\mathcal{V}$ be planes which are vector subspaces of $\mathbb{R}^3$. State an equation relating $\dim(\mathcal{U} + \mathcal{V})$ and $\dim(\mathcal{U} \cap \mathcal{V})$ and determine their possible values. [4]

(d) Now suppose additionally that $\mathcal{U}$ passes through the points $(1, 1, 3)$ and $(1, 3, 1)$, and that $\mathcal{V}$ passes through $(1, 1, -2)$ and $(2, 1, 2)$. Find the shortest distance from the point $(1, 1, 1)$ to the intersection of the planes. [5]

9. (a) Let $\mathbf{M}$ be a Hermitian matrix. Prove that the matrix $e^{i\mathbf{M}}$ is unitary by writing it as a power series. [6]

(b) Let $\mathbf{A} = (a_1, a_2, \dots)$ be an infinite series of real numbers such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. The set of all such series forms a vector space V with addition and scalar multiplication defined by

$$\alpha\mathbf{A} + \mathbf{B} = (\alpha a_1 + b_1, \alpha a_2 + b_2, \dots).$$

Show that $\langle \mathbf{A}, \mathbf{B} \rangle = a_1 b_1 + a_2 b_2 + \dots$ defines a scalar product in V. Further, show that the series defined by this scalar product converges for any pair of vectors in V. [5]

(c) G_i is a series defined by $G_{i+2} = (G_i + G_{i+1})/2$, with $G_0 = 0$ and $G_1 = 1$. By expressing this definition in the form

$$\begin{pmatrix} G_{i+2} \\ G_{i+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} G_{i+1} \\ G_i \end{pmatrix},$$

where $\mathbf{A}$ is a 2×2 matrix, show that $G_i \to 2/3$ as $i \to \infty$. [9]

10. (a) Solve the differential equation

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0.$$

[Hint: one approach is to introduce the parameter $p = dy/dx$ and then show that $dp/dx = p/x$.]
How many integral curves pass through each point of the (x, y) plane (except $x = 0$)? Why is the solution at each point not unique? [11]

(b) The differential equation

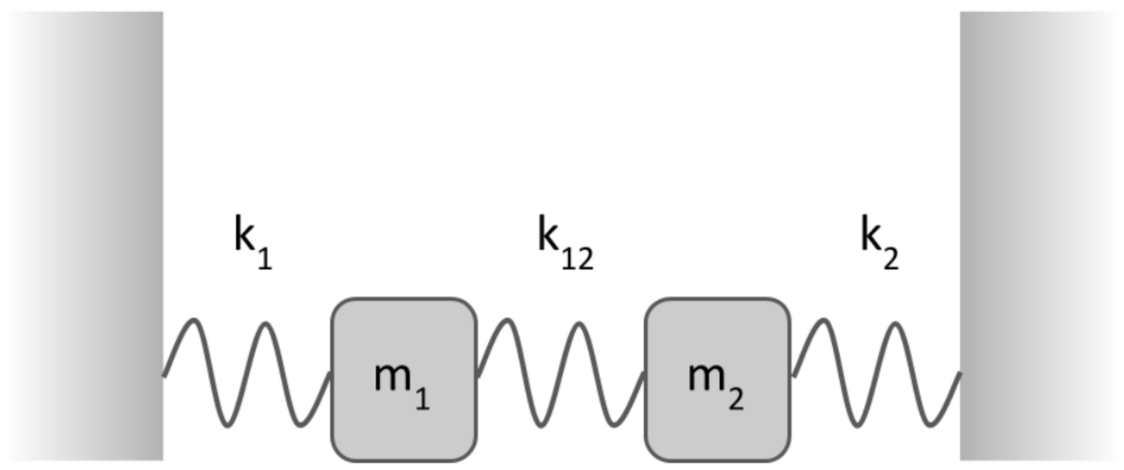
$$\left(\frac{dy}{dx}\right)^2 + f(x, y)\frac{dy}{dx} - 1 = 0$$

represents a set of curves such that two curves pass through any given point. Show that these curves intersect at right angles at the point. For $f = -2y/x$, verify this property for the point (3, 4). [9]

Section B

8. Describe briefly the properties of the *normal modes* of an oscillating system. [2]

A coupled oscillator is sketched in the figure below. It comprises two weights of equal masses $m_1 = m_2$, on a frictionless surface. Mass m_1 is connected to a fixed wall by a spring of spring constant k_1, and similarly mass m_2 is connected to a fixed wall by a spring of spring constant k_2. The two weights are coupled by a spring of spring constant k_{12}.



Write out the equations of motion for the system in the case that masses m_1 and m_2 undergo small horizontal displacements x_1, x_2 from their equilibrium positions. [2]

Calculate the two frequencies of normal modes of oscillation of the system, ω_- and ω_+, where $\omega_+ > \omega_-$. State any assumptions you make about the form of the general solutions to the equations of motion. Sketch how x_1 and x_2 will vary as a function of time in the event that both normal modes are excited and are *weakly coupled.* [6]

The oscillator is now *uncoupled* by holding m_2 fixed. What is the frequency, ω_1 of oscillation of m_1 in this situation if it is displaced from its equilibrium position by a small amount? Similarly, at what frequency, ω_2, does m_2 oscillate if now m_1 is held fixed and m_2 is displaced? [4]

If $k_1 > k_2$, place the four calculated frequencies $(\omega_+, \omega_-, \omega_1, \omega_2)$ in order of ascending frequency. [4]

9. (i) State Stokes' theorem, carefully describing all of the terms. For what kind of topology is Stokes' theorem valid? Show how Stokes' theorem applied in a two-dimensional plane becomes Green's theorem. [4]

(ii) Show that

$$\oint_C \mathbf{r} \cdot \mathbf{dr} = 0$$

where C is a closed loop and $\mathbf{r}$ is the position vector. [4]

(iii) Consider how a surface with cylindrical topology can still be used with Stokes' theorem by imagining a vertical (i.e. in the z direction in the cylindrical coordinate system) cut down one of the sides. Sketch such a geometry, with the cut, indicating on each line segment in which direction the line integral should be evaluated.

Use Stokes' theorem to evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{dS}$$

where S is the surface defined by $x^2 + 4y^2 = 1$ and $-1 \leq z \leq 1$ and $\mathbf{F} = xz^2\mathbf{i} - yz^2\mathbf{j}$. [12]

10. (i) The divergence of a vector field $\mathbf{F}$ at a point P is defined as the limit of the ratio of the surface integral of $\mathbf{F}$ out of the surface of a closed volume V enclosing P to the volume V, as V shrinks to zero. Starting with this definition, and carefully defining all terms, derive the expression for the divergence. Show, stating any assumptions, how this leads to the expression for the divergence theorem for larger volumes. [4]

The surface S is defined by $x^2 + y^2 + z^2 = a^2$.

(ii) For the vector field $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}$, calculate both the vector surface integral of $\mathbf{F}$ through S, and the volume integral of the divergence of $\mathbf{F}$ for the volume enclosed by S. Are these consistent with the divergence theorem? [10]

(iii) Using the divergence theorem, determine the flux of the vector field $\mathbf{G} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through S. [6]

7. A vector addition operation $\oplus$ is defined for the set of n-dimensional vectors, $\mathbb{R}^n$, by

$$\mathbf{x} \oplus \mathbf{y} \equiv \mathbf{x} + \mathbf{y} - \mathbf{a}$$

where $\mathbf{a}$ is a fixed vector and the addition and subtraction operations on the right hand side are defined conventionally.

(i) Find the "null vector" $\mathbf{0}$ and $\mathbf{x}'$, the inverse of general vector $\mathbf{x}$, such that

$$\mathbf{x} \oplus \mathbf{0} = \mathbf{x},$$
$$\mathbf{x} \oplus \mathbf{x}' = \mathbf{0}.$$

(ii) Construct a corresponding scalar multiplication operation $\odot$ such that $\mathbb{R}^n$, together with operations $\oplus$ and $\odot$, forms a vector space over the field $\mathbb{R}$ and show that, with these definitions, scalar multiplication $\odot$ is distributive over vector addition $\oplus$. (It is not necessary to test all properties of the vector space.) [6]

Section B

8. (a) In each of the following cases describe the locus of points defined by the three-dimensional position vector $\mathbf{r}$ such that

(i) $|\mathbf{r} - \mathbf{a}| = c$

(ii) $\mathbf{r} = c\,\dfrac{\mathbf{R} \times \mathbf{a}}{|\mathbf{R}||\mathbf{a}|}$

where $\mathbf{R}$ is a variable vector (variable in both magnitude and direction), $\mathbf{a}$ is a constant vector and c is a constant scalar. [6]

(b) Lines L_1, L_2 are defined in vector form as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ respectively where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constant vectors and λ, μ are variable scalar parameters.

(i) Write down the vector joining general points on L_1 and L_2. [1]

(ii) Show that the shortest distance between these lines is given by $(\mathbf{a} - \mathbf{c}).\mathbf{n}$ where

$$\mathbf{n} = \frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}.$$ [2]

iii) Find λ as a function of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ at the point at which L_1 is closest to L_2. [6]

(c) Points on two lines satisfy the equations

$$(x - 3) = \frac{(4 - y)}{2} = \frac{(z - 1)}{3}$$

and

$$\frac{(x - 2)}{\alpha} = (y + 1) = \frac{(z - 5)}{2}$$

where α is an undetermined constant. What is the value of α for which the lines intersect? [5]

9. (a) Find, in the form $r\exp(i\theta)$, all values of complex number z which satisfy the following equations. Plot these solutions in the Argand diagram.

$$\text{(i)} \quad z^5 = 1 + \sqrt{3}i$$

$$\text{(ii)} \quad z = \left(\sqrt{2}(1+i)\right)^i$$

$$\text{(iii)} \quad \mathrm{Re}\left\{\frac{z-1}{z+1}\right\} = 0$$

[12]

(b) By considering the binomial expansion of $(1+e^{i\theta})^N$, or otherwise, show that

$$\sum_{n=0}^{N} \binom{N}{n} \sin(n\theta+\phi) = 2^N \cos^N\left(\frac{\theta}{2}\right) \sin\left(\frac{N\theta}{2}+\phi\right)$$

where $\binom{N}{n}$ represents the binomial coefficient $\left(\dfrac{N!}{(N-n)!n!}\right)$. [8]

10. (a) Matrices **U**, **V** are unitary. Show that their product **UV** is also unitary. [2]

(b) Matrix **M** and its adjoint $\mathbf{M}^\dagger$ have the same set of eigenvalues λ_i but different eigenvectors ($\mathbf{u}_i$, $\mathbf{v}_i$ respectively):

$$\begin{aligned} \mathbf{M}\mathbf{u}_i &= \lambda_i \mathbf{u}_i \,; \\ \mathbf{M}^\dagger \mathbf{v}_i &= \lambda_i \mathbf{v}_i \,. \end{aligned}$$

Show that $\mathbf{v}_i^\dagger \mathbf{u}_j = 0$ for $\lambda_i^* \neq \lambda_j$. [4]

(c) A matrix **A** can be transformed from one orthonormal basis to another by the unitary transformation $\mathbf{A}' = \mathbf{U}^\dagger \mathbf{A} \mathbf{U}$. State the corresponding transformation for column vector **x**. For the specific examples

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & i \\ 0 & 3 & 0 \\ -i & 0 & 0 \end{pmatrix} \,, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,,$$

construct a symmetric, unitary matrix **U** that can be used to transform **A** to a basis in which it is diagonal and find transformed matrix $\mathbf{A}'$ and vector $\mathbf{x}'$ in this basis. [14]

Section A

1. Find the limit as i) $x \to 0$ and ii) $x \to \infty$ of the function

$$f(x) = \frac{1}{x} + \frac{\ln x}{\sin x - x}.$$

[4]

2. Find the derivative of $f(x) = \left(x^2 + 2\right)^{\frac{1}{x}}$. [3]

3. Consider the functions

$$f_1(x,y) = \frac{\sin y}{x} + e^x \cos^2 y, \qquad f_2(x,y) = x \sin y + e^x \cos 2y, \qquad f_3(x,y) = \ln x \cos y - e^x \sin 2y.$$

Two of these functions can be combined to form an exact differential $\mathrm{d}f$ of a function $f(x,y)$ as follows:

$$\mathrm{d}f = f_i(x,y)\mathrm{d}x + f_j(x,y)\mathrm{d}y.$$

Identify these two functions and determine the corresponding function f, explaining your reasoning. [5]

4. Determine the first order Taylor expansion of the function $f(x,y) = e^{(x^2+\sqrt{2}y)}$ about the point $(x_0, y_0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$. Does the error in this approximation to f grow fastest along the x or the y axis? [5]

5. For $\mathbf{F} = (-x, y, z)$, evaluate $\iint \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over the surface of the plane $x + y + 2z = 4$ in the octant where x, y and z are all positive. [5]

6. Let $f_{X,Y}(x,y) = A \sin x \cos y$ define a joint probability density function normalised over the region bounded by the lines $y = \frac{\pi}{2} - x$, $y = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$. Find the value of the constant A and hence the expected value of the function $g(x) = \dfrac{1}{\sin x}$ over this region. [7]

7. A tube with an elliptical cross-section, defined by $x^2 + 4z^2 = 4$, is intersected by the planes $y + z = 1$ and $z = y + 1$. Find the volume of the enclosed region, by transforming to coordinates (r, θ, y) given by $x = 2r\cos\theta$ and $z = r\sin\theta$. [8]

8. An isotropic dispersion relation $\omega(k)$ has the property that $k^n\omega^2$ is constant. For what value of n will the energy associated with a wave packet propagate in the same direction as the individual peaks and troughs but with twice the speed? [3]

Section B

9. (a) Show that the identity

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = (\mathbf{r} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{r} + \mathbf{a}(\nabla \cdot \mathbf{r}) - \mathbf{r}(\nabla \cdot \mathbf{a})$$

holds in the specific case that $\mathbf{a}$ is a constant vector and $\mathbf{r}$ is the position vector. Is $\mathbf{a} \times \mathbf{r}$ a conservative field? [6]

(b) i) State Stokes' Theorem, and show that the line integral from A to B given by $\int_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ is independent of path if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. [4]

ii) Now consider the functions

$$P = yf(xy) - \frac{y}{x^2}\,g\left(\frac{y}{x}\right), \qquad Q = xf(xy) + \frac{1}{x}\,g\left(\frac{y}{x}\right),$$

defined within the region $x > 0$. For general functions f and g, show that a function $\phi(x,y)$ can be uniquely defined by the line integral

$$\phi(x,y) = \int_{(1,1)}^{(x,y)} P(x',y')\,\mathrm{d}x' + Q(x',y')\,\mathrm{d}y'.$$

Show that ϕ can be separated into integrals of f and g as follows

$$\phi(x,y) = \int_1^{xy} f(s)\,\mathrm{d}s + \int_1^{y/x} g(t)\,\mathrm{d}t,$$

and hence determine the difference in $\phi(x,y)$ between the points (a,b) and (ab,b^2), given that $f(s) = 1/s^2$. [10]

10. Consider the function $z(x,y) = xy\left(x^2-4\right)\left(y^2-4\right)$.

(a) Determine the stationary points of z. Sketch a contour plot of z in the region of the (x,y) plane where $x, y \geq 0$. Indicate on your sketch any maxima, minima, saddle points and zeros, and also the direction of the gradient of z. [9]

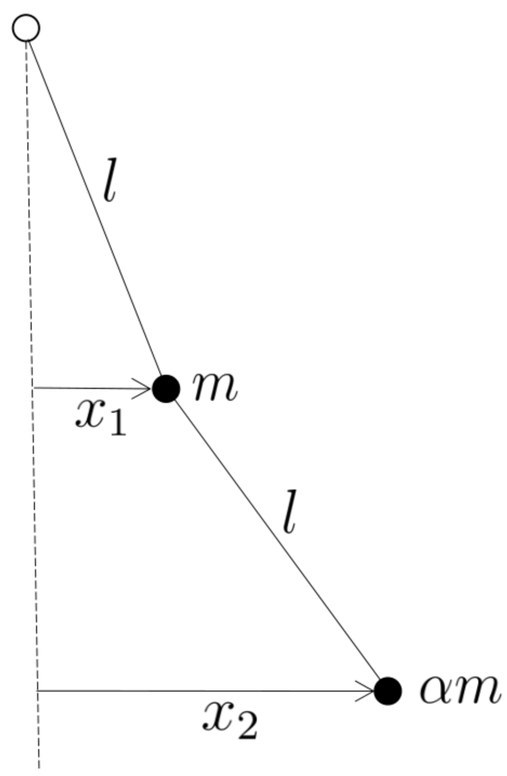
(b) For two-dimensional functions, the divergence theorem can be written as

$$\iint_A \nabla \cdot \mathbf{F}\,\mathrm{d}A = \oint_L \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}l = \oint_L M\,\mathrm{d}y - N\,\mathrm{d}x,$$

where L is a curve enclosing the area A, $\hat{\mathbf{n}}$ is the unit vector in the (x,y) plane perpendicular to L and $\mathbf{F} = (M,N)$. Verify this theorem by evaluating both the area and line integrals over the region where $0 \leq x \leq 2$ and $0 \leq y \leq 2$ in the case that $\mathbf{F} = \nabla z(x,y)$. [11]

11. Consider a double pendulum, consisting of two masses suspended by identical massless rods of length l. The upper and lower masses are m and αm respectively, where α is a constant. With x_1 and x_2 denoting the horizontal displacements of the upper and lower masses respectively, the equations of motion for small displacements can be written as

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -(1+\alpha)\frac{g}{l}x_1 + \alpha\frac{g}{l}(x_2 - x_1) \qquad \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -\alpha\frac{g}{l}(x_2 - x_1).$$



(a) Derive an expression for the frequencies of the normal modes of the system in terms of the constant α. [6]

(b) For the case where $\alpha = 1$ calculate the ratio of the amplitudes of the two displacements, and hence sketch the motion of the two masses. [6]

(c) Now assume that $\alpha \ll 1$ so that quadratic terms in α can be neglected. Determine the value that α should take for the lower frequency mode to have one quarter of the frequency of a single pendulum of length l and mass m. [4]

(d) Describe qualitatively what happens to the higher-frequency normal mode as $\alpha \to 0$. [4]

5. State the properties of an orthogonal matrix.

Matrix

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix} 3x & 2 & 2 \\ 2 & 1 & 3y \\ 2 & 3z & 1 \end{pmatrix}$$

is orthogonal. Find x, y and z. [5]

6. Show that W is a subspace of $V = \mathbb{R}^3$, where W consists of those vectors whose sum of components is zero, i.e. $W = \{(a, b, c) : a + b + c = 0\}$.

If U is the vector space of all $n \times n$ square matrices over a field F, show that Y is a subspace of U, where Y consists of all matrices which commute with a given matrix $\mathbf{T}$, i.e. $Y = \{\mathbf{A} \in U : \mathbf{AT} = \mathbf{TA}\}$. [5]

7. (i) Find the complete solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \cot x = 1,$$

subject to the initial condition $y = 2$ at $x = \pi/2$. [3]

(ii) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x + \sin(x).$$

[6]

Section A

1. Using the transformation $u = x + y$, $v = y/(x + y)$, evaluate the integral

$$I = \iint \frac{\exp(x + y)}{x + y}\,\mathrm{d}x\,\mathrm{d}y$$

over the area enclosed by the lines $x = 0$, $y = 0$ and $x + y = 1$. [6]

2. A particle moving in the xy plane experiences a force $\mathbf{F} = (ax^2y, by^2x)$ where a and b are constants. Calculate the work done by the force in moving the particle from $(0, 0)$ to $(1, 1)$ along the curve $y = x^2$. Does the work done depend on the path chosen? Justify your answer. [5]

3. Evaluate $\int_S \nabla f \cdot \mathrm{d}\mathbf{S}$, where $f = (x + 1)^2 + y^2 + z^2$ and S is the closed surface formed by the hemisphere $x^2 + y^2 + z^2 = a^2, z > 0$ and the xy plane:

(i) directly as a surface integral, and [5]

(ii) by using the divergence theorem. [2]

4. i) Find the MacLaurin series for the function $x/\sin(x)$ up to terms of order x^4. [3]

ii) Evaluate the following limit:

$$\lim_{x\to\pi/4} \frac{1}{\pi - 4x} \int_{\frac{\pi}{4}}^{x} \frac{\sin^2 t}{t^2}\,\mathrm{d}t\,.$$

[3]

5. If $f \equiv f(x, y, z)$ and $g \equiv g(x, y, z)$, find the following partial derivatives in terms of partial derivatives of f and g with respect to x, y and z:

(i) $\left(\dfrac{\partial f}{\partial g}\right)_{x,y}$; [2]

(ii) $\left(\dfrac{\partial f}{\partial x}\right)_{z,g}$. [3]

6. The dispersion relation for gravity-driven waves on water is

$$\omega^2 = gk\tanh(kh)\ ,$$

where ω is the angular frequency, k the wavenumber, g the acceleration due to gravity and h the water depth. Derive expressions for the phase velocity c_p and group velocity c_g as functions of k in each of the limits i) $kh \ll 1$ (depth $\ll$ wavelength) and ii) $kh \gg 1$ (depth $\gg$ wavelength). On the same axes sketch $c_p/\sqrt{gh}$ and $c_g/\sqrt{gh}$ as functions of (kh) for a fixed depth, indicating at least one characteristic value on each axis for scale. [6]

7. A ball bounces perfectly elastically on a horizontal surface reaching an identical maximum height h on each bounce. The time spent by the ball within a range of heights is proportional to the probability of finding it in that range. Find the probability distribution $P(x)$ such that $P(x)\,\mathrm{d}x$ is the probability that its height, sampled over an integral number of bounces, lies between x and $x + \mathrm{d}x$. [5]

Section A

1. A tetrahedron has a single $1\,\Omega$ resistor between each of its four corners. Pick any two corners and find the resistance between them. [4]

2. Calculate the potential energy of a spherical, conducting shell of charge Q and radius a. [4]

3. A parallel plate capacitor, for which each plate has an area $A = 1\,\text{cm}^2$ and separation distance $d = 0.5\,\text{mm}$ between the plates, is connected in parallel with a $1.0\,\text{mH}$ inductor. The permittivity of the material between the parallel plates is ϵ_0. What is the resonant frequency f of this circuit? [3]

If a change of $1\,\text{Hz}$ in f can be measured, what is the fractional change in d that could be detected if this is the only parameter that changes in the circuit? Give your answer in parts per million (ppm). [3]

4. What is the Rayleigh criterion?

The pupil of a human eye has a diameter of about $5\,\text{mm}$. Estimate the angular resolution, in degrees, of the human eye.

A radio telescope has a circular dish having a diameter of $10\,\text{m}$ and it detects radio waves at a frequency of $10\,\text{GHz}$. Estimate its angular resolution in degrees. [6]

5. Four straight sections of wire, each of length $2a$, are connected in a square loop with corners ABCD. A current I flows around this loop. Find the magnetic flux density at the centre of the square. [6]

6. A particle of mass m and charge q moves in the z-direction with velocity $\mathbf{v}$ through a region with uniform electric field $\mathbf{E} = (E_x, 0, 0)$ and uniform magnetic flux density $\mathbf{B} = (0, B_y, 0)$. What $\mathbf{v}$ would lead to the particle continuing in a straight line? [4]

The power supply producing the electric field is suddenly switched off. Derive an expression for the subsequent trajectory of the particle, and hence describe its motion. [5]

7. Two coaxial solenoids with radii a and $3a$ are fully overlapping. Each solenoid has n turns per unit length and length l. Neglecting end effects, calculate the self-inductance of each coil and their mutual inductance. [5]

Section A

1. (a) Evaluate L_1, where

$$L_1 = \lim_{x \to \pi} \frac{\cos 2x - 1}{(x - \pi)^2}.$$ [2]

(b) Evaluate L_2, where

$$L_2 = \lim_{x \to 0} (\sin x)^{1/\ln x}.$$ [3]

2. Find all stationary points of the function $f(x, y) = x^2 + 2x - 2y^3 + 6y^2$, where $(x, y) \in \mathbb{R}^2$. Classify the stationary points as maxima, minima or saddle points. [4]

3. Calculate the area A_Ω of the surface Ω parameterised by

$$b\left(x^2 + y^2\right) - z = 0 \quad \text{and} \quad z \leq 4b,$$

where (x, y, z) are Cartesian coordinates and $b > 0$ is a real constant. [6]

4. The angular frequency ω and wave number k of certain waves in a magnetised plasma (called cyclotron waves) obey the following relation, known as a dispersion relation:

$$\omega^2 = \Omega^2 + k^2 c_s^2,$$

where c_s is the sound speed and Ω is the cyclotron frequency. Neither c_s nor Ω depend on k.

(a) Derive expressions for the phase velocity v_p and group velocity v_g of the cyclotron waves. [2]

(b) Sketch v_p and v_g versus k. What is the characteristic scale in k that sets the boundary between the 'long-wavelength' and 'short-wavelength' limits? [4]

(c) Two wave packets – one composed of long-wavelength plane waves and another composed of short-wavelength plane waves – are launched into a magnetised plasma. Discuss qualitatively how the shapes of the two wave packets will change with time as they travel through the plasma. [2]

5. Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for the vector field $\mathbf{F} = (-yz, x^2 z, xy^2)$, where the curve C starts at $(0, 0, 0)$, ends at $(1, 1, 1)$ and is defined by the intersection of the plane $x = y$ with the surface $z = x^2$. Here (x, y, z) are Cartesian coordinates. [5]

9. A string of uniform mass per unit length ρ and tension T is held fixed at $x = 0$ and $x = L$. The transverse displacement of the string $y(x, t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0,$$

where $c^2 = T/\rho$.

(a) Derive expressions for the allowed normal mode frequencies and wavenumbers. [6]

(b) At time $t = 0$, the string is released from rest with an initial transverse displacement

$$y_0(x) = \sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right).$$

Calculate the transverse displacement of the string at subsequent times. [5]

(c) Derive a general expression for the potential energy of the string due to its transverse displacement. [5]

(d) Given the initial conditions from (b), what is the ratio of energies stored in each normal mode at time t? [4]

10. (a) State Stokes' Theorem, precisely defining all terms. [4]

(b) Verify Stokes' Theorem for the vector field

$$\mathbf{F}(x, y, z) = \left(y^2 - y, 2xy, z\right)$$

defined on the surface S_1 parameterised by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad \text{and} \quad z = 0,$$

where $a > 0$ and $b > 0$ are real constants. [10]

(c) Evaluate

$$\int_{S_2} (\nabla \times \mathbf{G}) \cdot \mathrm{d}\mathbf{S}_2$$

for the vector field

$$\mathbf{G}(x, y, z) = \mathbf{F}(x, y, z) + (xz, 2yz, 0)$$

defined on the surface S_2 parameterised by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad z \geq 0,$$

where a, b and $\mathbf{F}$ are as given in part (b), and $c > 0$ is a real constant. Justify your result. [6]

11. Consider the function $\rho = \rho(t, x(t), y(t), z(t))$, where (x, y, z) are Cartesian coordinates and t is time.

(a) Write $\mathrm{d}\rho/\mathrm{d}t$ in terms of partial derivatives with respect to x, y, z and t. Be sure to state clearly which variables are held fixed for all partial derivatives. [4]

(b) Show that

$$\nabla \cdot (\phi \mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi \nabla \cdot \mathbf{A},$$

where ϕ is any scalar and $\mathbf{A}$ is any vector. [3]

(c) Use your result from (a) to express $\mathrm{d}\rho/\mathrm{d}t$ in terms of ρ and $\mathbf{u} = (\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t, \mathrm{d}z/\mathrm{d}t)$. Show that when $\mathrm{d}\rho/\mathrm{d}t = 0$ and $\nabla \cdot \mathbf{u} = 0$, the time derivative of ρ at fixed (x, y, z) can be written as the divergence of a vector. [4]

(d) At $t = t_0$, we have

$$\mathbf{u}(t_0, \mathbf{r}) = u_0 \frac{a^2}{r^2} \nabla r, \quad \rho(t_0, \mathbf{r}) = \rho_0 e^{-(r-r_0)^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the magnitude of the position vector $\mathbf{r}$, and u_0, ρ_0 and r_0 are real constants, with $\rho_0 > 0$ and $r_0 > 0$. Using the divergence theorem (or otherwise), calculate the time derivative of

$$N_V = \int_V \mathrm{d}^3\mathbf{r}\ \rho(t, \mathbf{r})$$

at $t = t_0$. The integration volume V is a sphere of radius a, centred at the origin. [7]

(e) Discuss the physical significance of the result in (d) for the case where ρ is the mass density of a fluid. [2]

Section A

1. Evaluate the following expressions:

$$\lim_{x\to 0} \frac{\cos(2x) - 1}{x^2}$$

and

$$\lim_{x\to 1} \frac{\ln(x) + 1 - x}{(e^x - e)^2}.$$

[4]

2. Two monochromatic plane waves, labelled by the subscripts + and −, respectively, have equal amplitudes and have frequencies $\omega_\pm = \omega_0 \pm \delta\omega$ and wave numbers $k_\pm = k_0 \pm \delta k$, where $\delta\omega \ll \omega_0$ and $\delta k \ll k_0$.

(a) By considering the superposition of these waves, derive an expression for the group velocity of the resulting wave packet and sketch a snapshot of the superposed waves. Be sure to label any special wavelengths on your sketch.

(b) Assuming the waves are propagating in a non-dispersive medium, express $\delta\omega$ in terms of ω_0, k_0 and δk. [6]

3. Suppose that $u = u(s,t)$ and $v = v(s,t)$.

(a) By calculating the total differentials of u and v, obtain expressions for $\mathrm{d}s$ and $\mathrm{d}t$ in terms of $\mathrm{d}u$, $\mathrm{d}v$ and partial derivatives of u and v with respect to s and t.

(b) By equating your expression for $\mathrm{d}s$ with the differential of $s(u,v)$, derive an expression for $(\partial s/\partial v)_u$ in terms of partial derivatives of u and v with respect to s and t. [5]

4. The joint probability density function $f_{XY}(x,y)$ over the continuous random variables X and Y is defined such that

$$f_{XY}(x,y) = 5\left(x^2 - y\right)$$

for $(-1 \le x \le 1,\ 0 \le y \le x^2)$ and zero otherwise. Calculate the following quantities:

(a) the marginal probability density function $f_X(x)$;

(b) the marginal probability density function $f_Y(y)$; and

(c) the expectation value for x^2. [7]

5. Evaluate the integral

$$\int_\Omega (3 - z)x^2 \mathrm{d}S,$$

where Ω is the part of the conical surface $3 - z = \sqrt{x^2 + y^2}$ that lies between $z = 1$ and $z = 2$, and dS is the element of area on Ω. [7]

6. The force acting on a particle in $\mathbb{R}^3$ is $\mathbf{F} = C_1\left(xyz\hat{\mathbf{x}} + x^2y\hat{\mathbf{y}} + x^2z\hat{\mathbf{z}}\right) - (C_2/r^3)\hat{\mathbf{r}}$, where (x,y,z) are Cartesian coordinates and $r = \sqrt{x^2 + y^2 + z^2}$ is the spherical radial coordinate. Calculate the work done by the force in moving the particle along the straight-line path between points A and B, with $A = (1,2,2)$ and $B = (2,4,4)$. [6]

7. In a simple model for one-dimensional gas dynamics, the density of particles ρ at any space-time location (x, t) is given by

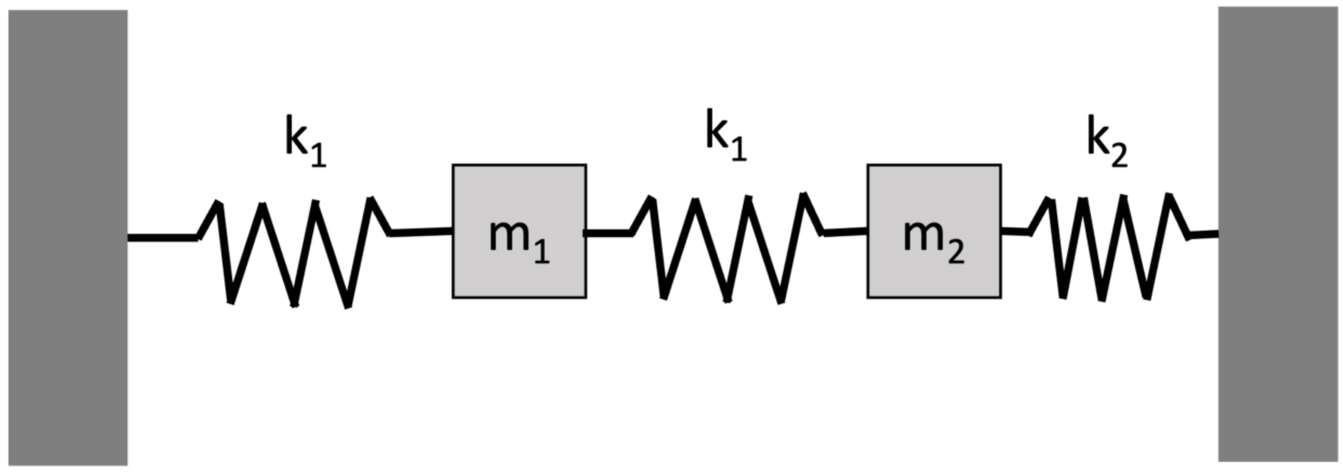
$$\rho(x,t) = \frac{\rho(x+\lambda, t-\tau)}{2} + \frac{\rho(x-\lambda, t-\tau)}{2},$$

with $\lambda > 0$ and $\tau > 0$ real constants. You may assume that ρ varies on space and time scales that are much larger than λ and τ, respectively.

(a) Taylor expand this expression about (x, t) to the lowest order in λ and τ required to obtain a partial differential equation relating the time derivative of ρ to its spatial derivatives.

(b) Now suppose that (x_0, t_0) is a critical point for ρ. Given the form of the differential equation from (a), what, if anything, can be determined about the nature of this critical point? [5]

10. Two masses m_1 and m_2 are connected to each other by a spring of spring constant k_1 and to adjacent walls by springs with spring constants k_1 and k_2, as shown in the diagram. Denote the displacements of the masses m_1 and m_2 from their equilibrium positions by x_1 and x_2, respectively.



(a) Write down the equations of motion for the masses. [4]

(b) Taking $m_2 = 2m_1$ and $k_2 = 3k_1$, calculate the frequencies associated with the normal modes of the system. [4]

(c) Calculate the ratio x_2/x_1 for each of the normal modes. [4]

(d) At $t = 0$ the masses are in their equilibrium positions with speeds

$$\frac{dx_1}{dt} = u, \quad \frac{dx_2}{dt} = 0.$$

Derive an expression for the position of the masses for $t > 0$. [4]

(e) Write down the normal coordinates for the system and express the kinetic energy in terms of the normal coordinates. What fraction of the total energy is associated with each normal mode? [4]