

University of Oxford

PHYSICS UNDERGRADUATE

Problem Sheet Answers

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1 CP1: Classical Mechanics and Special Relativity

1.1 Classical Mechanics: Sheet 0

1. Powered rocket

A rocket starts from rest at the Earth's surface and accelerates vertically at a constant rate. After 60 s it is 36 km high and its motor is switched off. Neglect air resistance and the variation of g.

(a) Find the maximum height.

(b) Find the total flight time.

(c) Draw the velocity–time graph for the whole flight.

Solution.

Set $h_b = 3.60 \times 10^4\,\mathrm{m}$, $t_b = 60\,\mathrm{s}$ and $g = 9.81\,\mathrm{m\,s^{-2}}$.

$$h_b = \frac{1}{2}at_b^2 \quad\Longrightarrow\quad a = 20\,\mathrm{m\,s^{-2}}, \qquad v_b = at_b = 1200\,\mathrm{m\,s^{-1}}.$$

(a) At the highest point,

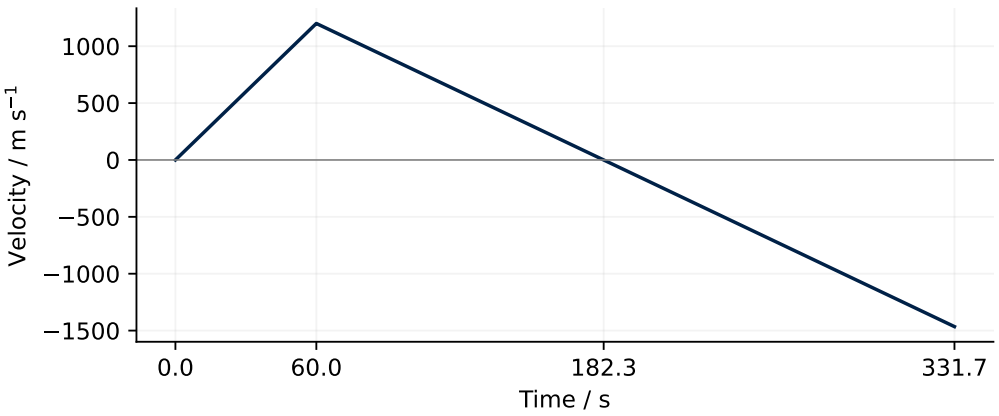
$$0 = v_b^2 - 2g(H - h_b) \quad\Longrightarrow\quad H = h_b + \frac{v_b^2}{2g}, \qquad \boxed{H = 109.4\,\mathrm{km}}.$$

(b) The ascent and descent times give

$$t_\uparrow = t_b + \frac{v_b}{g}, \qquad t_\downarrow = \sqrt{\frac{2H}{g}}, \qquad \boxed{t_f = t_\uparrow + t_\downarrow = 332\,\mathrm{s} = 5.53\,\mathrm{min}}.$$

(c) The velocity is continuous at cutoff:

$$v(t) = \begin{cases} 20t, & 0 \le t \le 60\,\mathrm{s}, \\ 1200 - g(t - 60\,\mathrm{s}), & 60\,\mathrm{s} < t \le t_f. \end{cases}$$



2. Constant acceleration

A particle moves on a line with constant acceleration a, initial position x_0 and initial velocity v_0. Derive

$$v = v_0 + at, \qquad x = x_0 + v_0 t + \tfrac{1}{2}at^2, \qquad a(x - x_0) = \tfrac{1}{2}(v^2 - v_0^2).$$

Find the minimum stopping distance of a car travelling at $100\,\mathrm{km\,h^{-1}}$ with maximum deceleration $10\,\mathrm{m\,s^{-2}}$.

Solution.

Integrating $\dot{v} = a$ and $\dot{x} = v$ gives $v = v_0 + at$ and $x = x_0 + v_0 t + at^2/2$. Alternatively $a = v\,dv/dx$ integrates to $a(x - x_0) = (v^2 - v_0^2)/2$. For stopping, $v_0 = 100/3.6\,\mathrm{m\,s^{-1}}$ and $a = -10\,\mathrm{m\,s^{-2}}$, so $\boxed{d = v_0^2/(2|a|) = 38.6\,\mathrm{m}}$.

3. Forces and stopping distances

(a) A 15-tonne boat drifts towards a dock at $0.3\,\mathrm{m\,s^{-1}}$. A woman can push it with a constant 400 N once it is 2 m away. Can she stop it in time?

(b) An 80 kg man drops 0.5 m onto a road. His motion is arrested over 2 cm. Find the average force on his bone structure.

(c) He now drops 1.5 m and bends his knees so that his centre of mass descends a further h. Find h if the average contact force is five times his weight.

Solution.

(a) The stopping distance is

$$Mv^2/(2F) = 15000(0.3)^2/(800) = \boxed{1.69\,\mathrm{m}} < 2\,\mathrm{m},$$

so she can stop the boat.

(b) Work during arrest gives $(N - mg)d = mgh$, so

$$\boxed{N = mg(1 + h/d) = 2.04 \times 10^4\,\mathrm{N}}.$$

The net decelerating force is $N - mg = 1.96 \times 10^4\,\mathrm{N}$.

(c) With $N = 5mg$, $(5mg - mg)h = mg(1.5)$, so

$$\boxed{h = 0.375\,\mathrm{m}}.$$

4. Banking and the wall of death

A car travels at speed v around a banked corner of radius r. Show that no friction is needed when the bank angle above horizontal satisfies $\tan\theta = v^2/(gr)$. Discuss a motorbike ridden around the inside of a vertical cylinder (the "Wall of Death"), with the bike and rider at a small angle to the horizontal. Explain the role of friction and the reason for the small angle.

Solution.

Without friction, $N\cos\theta = mg$ and $N\sin\theta = mv^2/r$, giving $\boxed{\tan\theta = v^2/(gr)}$. On a vertical cylinder the wall's normal reaction is horizontal, $N = mv^2/r$, and upward static friction supports weight: $mg \le \mu N$, hence $v^2 \ge gr/\mu$. To avoid a toppling torque, the rider-bike combination leans so that the contact resultant passes through its centre of mass; its angle α above horizontal satisfies $\tan\alpha = gr/v^2$. At large speed this angle is small.

5. Centripetal accelerations

Calculate the following centripetal accelerations in units of g and comment on their sizes.

(a) An object at latitude 45° on the rotating Earth, of radius 6378 km.

(b) The Moon, whose orbital radius is 384399 km and period is 27.3 days.

(c) An electron orbiting a proton at speed $0.007c$ and radius 0.05 nm.

Solution.

Use $a = \Omega^2 r_\perp$ or v^2/r. For Earth at 45°,

$$a/g = (2\pi/86400)^2 R\cos 45^\circ/g = \boxed{2.43 \times 10^{-3}}.$$

For the Moon, $(2\pi/P)^2 r/g = \boxed{2.78 \times 10^{-4}}$. For the stated electron orbit,

$$(0.007c)^2/(0.05\,\mathrm{nm}\,g) = \boxed{8.98 \times 10^{21}}.$$

Rotation weakly affects terrestrial apparent gravity; the small lunar acceleration reflects its large orbital radius; atomic accelerations can be enormous because the length scale is tiny.

6. Planet with zero equatorial weight

A planet has the Earth's mass and radius. An object stationary on its surface has zero apparent weight at the equator. Find the length of a day and the apparent gravitational acceleration at its poles. Include both gravitational and rotational effects.

Solution.

Zero apparent weight at the equator requires $\Omega^2 R = g$, so

$$\boxed{T = 2\pi\sqrt{R/g} \simeq 5.07 \times 10^3\,\mathrm{s} = 84.4\,\mathrm{min}.}$$

At the poles the distance from the rotation axis is zero, so there is no centrifugal term and the apparent gravity remains $\boxed{g}$ in the stated spherical model.

7. Repeated upward shots

A gun fires vertically upwards at $60\,\mathrm{m\,s^{-1}}$, once each second. Neglect air resistance. How many bullets are in the air at a time, and at what heights do they pass one another?

Solution.

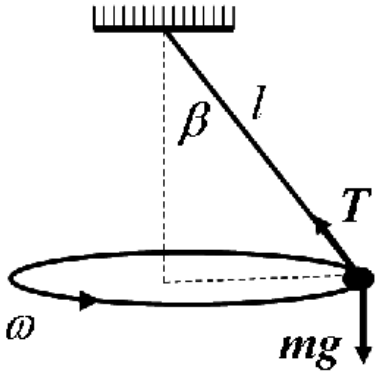
Each bullet remains airborne for $T = 2u/g = 12.23\,\mathrm{s}$. Once a steady firing sequence is established, the number aloft alternates between $\boxed{12 \text{ and } 13}$ according to the phase within each second. For shots separated by $\Delta = n\,\mathrm{s}$, equate $ut - gt^2/2 = u(t-\Delta) - g(t-\Delta)^2/2$. This gives $t = u/g + \Delta/2$ and

$$\boxed{h_n = \frac{u^2}{2g} - \frac{gn^2}{8}, \quad n = 1, \ldots, 12.}$$

Consecutive bullets pass at $h_1 \simeq 182.3\,\mathrm{m}$ for $g = 9.81\,\mathrm{m\,s^{-2}}$.

8. Conical pendulum

A mass m on a massless string of length l moves in a horizontal circle. The string makes a constant angle β with the vertical. Show that its angular speed is $\omega = \sqrt{g/(l\cos\beta)}$.

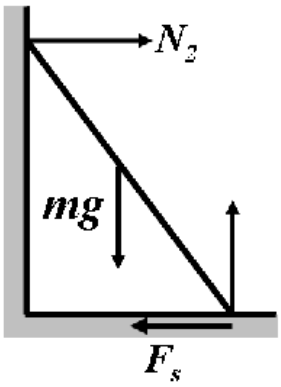


Solution.

The vertical and radial equations are $T\cos\beta = mg$ and $T\sin\beta = m\omega^2 l\sin\beta$. For a nonzero cone angle, eliminate T to obtain $\boxed{\omega^2 = g/(l\cos\beta)}$. The radius is $l\sin\beta$ and the tension is $m\omega^2 l$.

9. Ladder with friction

A uniform ladder rests against a smooth vertical wall and on a horizontal floor of friction coefficient μ. Show that its minimum angle to the floor without slipping satisfies $\tan\theta_{\min} = 1/(2\mu)$.



Solution.

For a uniform ladder of length L, moment balance about the foot gives $N_w L\sin\theta = mg(L/2)\cos\theta$. Horizontal balance requires floor friction $f = N_w$, while the floor normal is mg. The limiting condition $f \le \mu mg$ therefore gives

$$\boxed{\tan\theta \ge 1/(2\mu), \quad \theta_{\min} = \tan^{-1}(1/(2\mu))}.$$

1.2 Classical Mechanics: Sheet 1

1. Circular motion and planar vectors

At $t = 0$ a particle in the xy plane has position

$$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j}\,\text{cm}.$$

(a) Find a unit vector making an angle 20° with $\mathbf{r}$.

(b) Show geometrically that these two vectors span the plane.

(c) The particle begins circular motion about the origin at constant speed $15\,\text{cm}\,\text{s}^{-1}$. Explain why it accelerates, and illustrate why its velocity is tangential.

(d) Find its polar angle, position and velocity as functions of time.

Solution.

Initially $r_0 = 5(\cos\theta_0, \sin\theta_0)$ cm with $\theta_0 = \tan^{-1}(3/4) = 0.6435$. A unit vector at 20° to it is

$$\boxed{\widehat{u} = (\cos(\theta_0 \pm 20^\circ), \sin(\theta_0 \pm 20^\circ))},$$

numerically $(0.5465, 0.8375)$ or $(0.9570, 0.2902)$. Their determinant with r_0 is $\pm 5\sin 20^\circ \neq 0$, so $r_0, \widehat{u}$ span the plane. At constant counterclockwise speed $15\,\text{cm}\,\text{s}^{-1}$, $\omega = v/R = 3\,\text{s}^{-1}$. Hence

$$\boxed{\theta = 3t + \theta_0, \quad r = 5(\cos\theta, \sin\theta)\,\text{cm}, \quad v = 15(-\sin\theta, \cos\theta)\,\text{cm}\,\text{s}^{-1}.}$$

The derivative v is perpendicular to r, hence tangent; its changing direction gives inward acceleration $v^2/R = 45\,\text{cm}\,\text{s}^{-2}$.

2. Dimensional analysis

(a) Use dimensional analysis to show that the period of a pendulum of length l, bob mass m and angular amplitude θ_0 has the form $T = kf(\theta_0)\sqrt{l/g}$, with k dimensionless.

(b) For a satellite of mass $m \ll M$ in a circular orbit of radius R about mass M, show dimensionally that $T^2 = kR^3/(GM)$.

(c) Use Coulomb's law and the force between parallel current-carrying wires to find the dimensions of ϵ_0 and μ_0. Show that $(\mu_0\epsilon_0)^{-1/2}$ has dimensions of speed, evaluate it and comment.

Solution.

(a) Put $T \propto m^a l^b g^c$. Matching mass, length and time gives $a = 0$, $b = 1/2$, $c = -1/2$, with an arbitrary dimensionless amplitude dependence: $\boxed{T = kf(\theta_0)\sqrt{l/g}}$.

(b) Since $[GM] = L^3T^{-2}$, $\boxed{T = kR^{3/2}/\sqrt{GM}}$; Newtonian circular motion fixes $k = 2\pi$.

(c) With current dimension I, Coulomb's law gives $[\epsilon_0] = M^{-1}L^{-3}T^4I^2$ and the wire-force law gives $[\mu_0] = MLT^{-2}I^{-2}$. Thus $[1/\sqrt{\mu_0\epsilon_0}] = LT^{-1}$ and its value is

$$\boxed{2.998 \times 10^8\,\text{m}\,\text{s}^{-1}},$$

the speed of light.

3. Pendulum collision

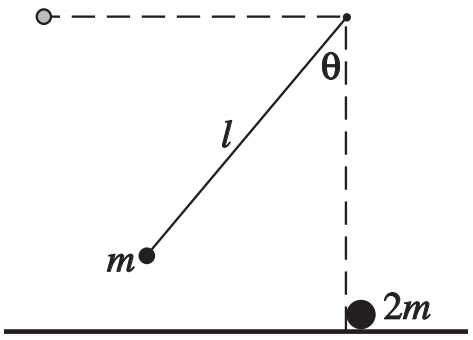
A bob of mass m is released from rest on a taut, horizontal string of length l. At the bottom it collides elastically with a stationary mass $2m$ on a smooth horizontal surface.

(a) Find its speed immediately before impact.

(b) Find the string tension then.

(c) Find the velocity imparted to $2m$.

(d) Find the bob's greatest subsequent angle to the vertical.



Solution.

Energy in the first swing gives $u = \sqrt{2gl}$. At the bottom, $T - mg = mu^2/l$, hence $\boxed{T = 3mg}$. Momentum and elastic energy for masses $m, 2m$ give $v_m = -u/3$ and $\boxed{v_{2m} = 2u/3}$. The bob's recoil energy raises it by $u^2/(18g) = l/9$, so $l(1 - \cos\theta_{\max}) = l/9$. Therefore

$$\boxed{\theta_{\max} = \cos^{-1}(8/9) = 27.27^\circ}.$$

4. Simple harmonic oscillator

A mass m moves in one dimension under $F = -kx$. Derive $\ddot{x} + \omega_0^2 x = 0$, where $\omega_0^2 = k/m$.

(a) Verify $x = A\cos\omega_0 t + B\sin\omega_0 t$ and write it as $x = x_{\max}\cos(\omega_0 t + \phi)$. Interpret ω_0.

(b) Find $U(x) - U(0)$ in terms of m and ω_0, and explain its sign.

(c) Set $U(0) = 0$. Differentiate $E = T + U$ to prove conservation, and express E in terms of $x_{\max}$ and $v_{\max}$.

Solution.

Newton's law gives $\ddot{x} + \omega_0^2 x = 0$, $\omega_0^2 = k/m$. Both sine and cosine satisfy it; ω_0 is the angular frequency. Set $A = x_{\max}\cos\phi$, $B = -x_{\max}\sin\phi$ to obtain $x = x_{\max}\cos(\omega_0 t + \phi)$. Since $F = -dU/dx$, $U(x) - U(0) = kx^2/2 \geq 0$: moving from equilibrium stores energy. Differentiating total energy gives $\dot{E} = \dot{x}(m\ddot{x} + kx) = 0$. Thus

$$\boxed{E = \frac{1}{2}m\omega_0^2 x_{\max}^2 = \frac{1}{2}mv_{\max}^2.}$$

5. A potential barrier

A particle of mass m approaches from $x = -\infty$ a potential

$$U(x) = \frac{U_0 a^2}{x^2 + a^2}, \qquad U_0 > 0.$$

(a) Find $F(x)$.

(b) Sketch F and U.

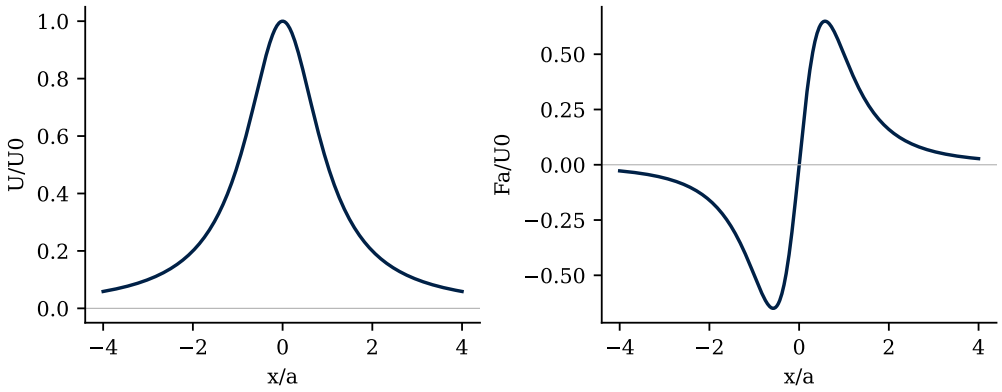
(c) Identify attractive and repulsive regions.

(d) Find the asymptotic speed after release arbitrarily close to the unstable equilibrium at rest.

(e) Find the threshold incident speed for passage from $-\infty$ to $+\infty$.

Solution.

Differentiation gives $\boxed{F(x) = 2U_0 a^2 x/(x^2 + a^2)^2}$. The even potential has maximum U_0 at zero and tends to zero; the odd force points away from the origin on both sides. It is therefore repulsive relative to the origin, vanishes at zero and has extrema at $x = \pm a/\sqrt{3}$. An arbitrarily small displacement from the unstable equilibrium releases energy approaching U_0, giving $\boxed{v_\infty = \sqrt{2U_0/m}}$. A particle arriving from infinity needs $E > U_0$ to cross: the infimum incident speed is this same value. At exact equality it approaches the barrier top only asymptotically.



6. Centre of mass and relative motion

Two masses m_1, m_2 move on a line, with attractive internal force of magnitude F_{int}, external forces F_1, F_2, and coordinates x_1, x_2.

(a) Write and add their equations of motion.

(b) Prove momentum conservation when $F_1 = F_2 = 0$.

(c) Define X_{cm} and show $P = (m_1 + m_2)\dot{X}_{\text{cm}}$.

(d) Derive the centre-of-mass equation and describe its force-free motion.

(e) In the isolated system, use $x_i' = x_i - X_{\text{cm}}$ to obtain the relative equation with reduced mass μ, where $\mu^{-1} = m_1^{-1} + m_2^{-1}$. Retain the direction of the internal force.

Solution.

Let f be the signed internal force on particle 1. Then $m_1\ddot{x}_1 = F_1 + f$ and $m_2\ddot{x}_2 = F_2 - f$. Adding gives $\dot{P} = F_1 + F_2$. With $X = (m_1 x_1 + m_2 x_2)/M$, $\boxed{P = M\dot{X}, \quad M\ddot{X} = F_1 + F_2}$. With no external forces, momentum is constant and the CM moves uniformly. Subtract the two acceleration equations in the isolated case: for $r = x_1 - x_2$,

$$\boxed{\mu\ddot{r} = f, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}.}$$

For an attractive force of magnitude F_{int}, $f = -F_{\text{int}} \operatorname{sgn} r$; its sign must be retained when the relative coordinate is signed.

7. Elastic equal masses

A frictionless truck A of mass m, initially travelling at u_0, collides elastically with an identical stationary truck B. Use momentum and energy conservation to find their final laboratory velocities. In the centre-of-mass frame: (a) find and explain the constancy of V_{cm}; (b) show that total momentum vanishes; (c) find and illustrate the initial and final velocities; (d) explain the advantage of this frame.

Solution.

Momentum gives $v_A + v_B = u_0$ and energy $v_A^2 + v_B^2 = u_0^2$. The separating postcollision solution is $\boxed{v_A = 0, v_B = u_0}$. The CM speed is $u_0/2$ and is unchanged because there is no external impulse. Subtracting it, initial CM velocities are $(u_0/2, -u_0/2)$ and final velocities are $(-u_0/2, u_0/2)$: both reverse and total CM momentum vanishes. The CM frame reduces an elastic head-on collision to reversal, after which one adds the unchanged CM velocity to recover the laboratory result.

8. Elastic unequal masses

Truck A of mass m_A and velocity u_A collides elastically with a stationary truck B of mass m_B.

(a) Derive $v_A = (m_A - m_B)u_A/(m_A + m_B)$ and $v_B = 2m_A u_A/(m_A + m_B)$. Give the limits $m_A \gg m_B$ and $m_A \ll m_B$.

(b) Find both initial and final velocities in the centre-of-mass frame.

Solution.

Momentum conservation and reversal of relative velocity give

$$\boxed{v_A = \frac{m_A - m_B}{m_A + m_B} u_A, \quad v_B = \frac{2m_A}{m_A + m_B} u_A.}$$

For $m_A \gg m_B$, $(v_A, v_B) \simeq (u_A, 2u_A)$; for $m_A \ll m_B$, $(v_A, v_B) \simeq (-u_A, 2m_A u_A/m_B)$. The CM velocity is $V = m_A u_A/(m_A + m_B)$. Initially $u'_A = m_B u_A/(m_A + m_B)$, $u'_B = -m_A u_A/(m_A + m_B)$; finally $v'_A = -u'_A$ and $v'_B = -u'_B$.

9. Inelastic equal masses

Repeat the equal-mass collision of Question 7 when half the initial kinetic energy is lost.

(a) Find both final laboratory velocities and account for the lost kinetic energy.

(b) Find $e = |v_B - v_A|/|u_B - u_A|$.

(c) Find both final centre-of-mass velocities and verify that e is unchanged.

Solution.

The final energy is half the initial, so $v_A + v_B = u_0$ and $v_A^2 + v_B^2 = u_0^2/2$. Their difference must vanish, giving $\boxed{v_A = v_B = u_0/2,\ e = 0}$. The missing translational kinetic energy appears as internal energy, deformation, heat and sound. In the CM frame both final velocities vanish; relative velocity is invariant under a Galilean boost, so $e' = e = 0$.

10. Infinite sequence of bounces

A ball falls from rest at height h and loses a fraction f of its incident kinetic energy at each bounce. Show that its height after bounce n is $h_n = (1-f)^n h$. Find the time from height h_n to h_{n+1}, sum the total time to rest, and evaluate it for $h = 5\,\mathrm{m}$, $f = 0.1$.

Solution.

With $\varepsilon = \sqrt{1-f}$, energy conservation between impacts gives $h_n = \varepsilon^{2n} h$. From apex h_n to apex h_{n+1}, the time is

$$\Delta t_n = \sqrt{2h/g}(\varepsilon^n + \varepsilon^{n+1}).$$

Summing the geometric series,

$$\boxed{t_{\mathrm{rest}} = \sqrt{\frac{2h}{g}}\frac{1+\varepsilon}{1-\varepsilon} = 38.3\,\mathrm{s} \quad (h = 5\,\mathrm{m}, f = 0.1).}$$

This idealized finite total time is the limit of infinitely many increasingly short flights.

11. Falling chains

(a) A uniform chain of length L and mass m lies on a smooth horizontal table, with length h hanging through a frictionless hole. Released from rest, how long does it take to fall off?

(b) The same chain is held straight vertically with its lower end just above a scale, then released. Find the scale reading when half the chain has landed.

Solution.

(a) Let x be the hanging length, $\lambda = m/L$. The entire chain moves at speed $\dot{x}$. Energy gives

$$\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}\lambda g x^2 = -\tfrac{1}{2}\lambda g h^2,$$

so $\ddot{x} = (g/L)x$ and $x = h\cosh\!\left(\sqrt{g/L}\,t\right)$. Therefore

$$\boxed{t = \sqrt{L/g}\,\operatorname{arcosh}(L/h)}.$$

(b) When a length x rests on the scale, its weight is $\lambda x g$; the falling material arriving at speed v transfers momentum at rate λv^2. With $v^2 = 2gx$, the reading is $3\lambda g x$, giving $\boxed{3mg/2}$ at $x = L/2$.

12. Sliding block and movable wedge

A block of mass m slides on a smooth wedge of mass M and angle θ, which can slide without friction on a horizontal table.

(a) Write vector equations of motion in the table frame.

(b) Find the block's acceleration along the wedge and the wedge's horizontal acceleration.

(c) Find the contact force.

(d) Prove conservation of total horizontal momentum.

Solution.

Choose X as wedge displacement and s as distance down a slope falling to the right. The block has $r = (X + s\cos\theta, -s\sin\theta)$. Put $D = M + m\sin^2\theta$. Horizontal momentum and the tangential equation give $(M+m)\ddot{X} + m\ddot{s}\cos\theta = 0$ and $\ddot{s} + \ddot{X}\cos\theta = g\sin\theta$. Thus

$$\boxed{\ddot{s} = \frac{(M+m)g\sin\theta}{D}, \quad \ddot{X} = -\frac{mg\sin\theta\cos\theta}{D}, \quad N = \frac{mMg\cos\theta}{D}.}$$

In vector form $m\ddot{r} = -mg\widehat{y} + N\widehat{n}$ and

$$M\ddot{X}\widehat{x} = -N\widehat{n} - Mg\widehat{y} + N_T\widehat{y},$$

where $\widehat{n} = (\sin\theta, \cos\theta)$. The conserved horizontal momentum is $(M+m)\dot{X} + m\dot{s}\cos\theta$.

1.3 Classical Mechanics: Sheet 2

1. Elastic collision in two dimensions

(a) Prove that each particle's speed in the centre-of-mass frame is unchanged by an elastic two-body collision.

(b) In the laboratory, a mass m with velocity $\mathbf{u}_1$ strikes an identical mass initially at rest. Show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$; hence either the incident particle stops or the outgoing trajectories are perpendicular. Establish its scattering-angle restriction from $\mathbf{u}_1 \cdot \mathbf{v}_1$.

(c) Obtain the same results by transformation from the centre-of-mass frame.

Solution.

(a) In the CM frame $m_1 u_1' = -m_2 u_2'$. Energy is $p'^2/(2\mu)$; its conservation fixes $|p'|$ and each speed.

(b) For equal masses with target at rest, $u_1 = v_1 + v_2$ and $u_1^2 = v_1^2 + v_2^2$. Squaring the momentum equation gives $\boxed{v_1 \cdot v_2 = 0}$. Also $u_1 \cdot v_1 = v_1^2 \geq 0$ unless $v_1 = 0$, so a moving projectile deflects by less than $90°$ when $v_1 \neq 0$.

(c) The CM moves at $u_1/2$. After scattering, CM velocities are $\pm u_1 \widehat{n}/2$. Adding the CM velocity gives $v_{1,2} = (u_1 \pm u_1 \widehat{n})/2$, whose dot product vanishes geometrically and algebraically.

2. Alpha particle maximum deflection

An alpha particle of mass $4m$ collides elastically and obliquely with a stationary proton of mass m, at nonrelativistic speed. Illustrate the outgoing velocities in the laboratory and centre-of-mass frames. Show that the alpha particle's laboratory deflection cannot exceed $14.5°$.

Solution.

The CM velocity is $V = 4u/5$ and the alpha particle's CM speed is $u/5$. Its possible laboratory velocities lie on a circle of radius $u/5$ centred a distance $4u/5$ from the velocity-space origin. The largest deflection is the tangent angle to this circle:

$$\boxed{\sin\theta_{\max} = \frac{u/5}{4u/5} = \frac{1}{4}, \quad \theta_{\max} = 14.48°.}$$

The proton's opposite CM momentum gives the corresponding recoil velocity after adding the same CM drift.

3. Inelastic recoil cone

A particle of mass $2m$ and laboratory speed u strikes a stationary mass m inelastically. The ratio of final to initial relative speed is α. Using the centre-of-mass frame, show that the lighter particle's laboratory speed at angle θ to the incident direction is

$$v_2 = \frac{2u}{3}\left(\cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}\right).$$

Solution.

The CM moves at $2u/3$. The lighter particle's initial CM speed is $2u/3$ and its final CM speed is $2\alpha u/3$. Hence its laboratory velocity satisfies $|v_2 - (2u/3)\widehat{x}|^2 = (2\alpha u/3)^2$. Resolving its direction at angle θ gives a quadratic in v_2, with solutions

$$\boxed{v_2 = \frac{2u}{3}\left(\cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}\right).}$$

Retain only real, nonnegative roots. For $0 \leq \alpha < 1$ the allowed cone satisfies $|\sin\theta| \leq \alpha$ and $\cos\theta \geq 0$.

4. Work-energy theorem and closest approach

(a) Derive $W_{ab} = T(b) - T(a)$ for a force $F(x)$ acting on a particle of mass m. Does this require a conservative force?

(b) A particle moves towards the origin under the repulsive force $F = k/x^2$ $(k > 0)$, with speed u at $x = x_0 > 0$. Find its distance of closest approach.

Solution.

Using $F = m\dot{v} = mv\,dv/dx$, integration gives

$$\boxed{\int_a^b F dx = \tfrac{1}{2}m[v(b)^2 - v(a)^2]}.$$

This holds for the total force, whether conservative or not. For $F = k/x^2$ on the positive axis, $U = k/x$. At turning, $k/x_{\min} = mu^2/2 + k/x_0$, so

$$\boxed{x_{\min} = \left(\frac{1}{x_0} + \frac{mu^2}{2k}\right)^{-1}}.$$

5. Two gravitational sources

Two fixed masses M lie at $(0, \pm a)$. A mass m moves along the x axis.

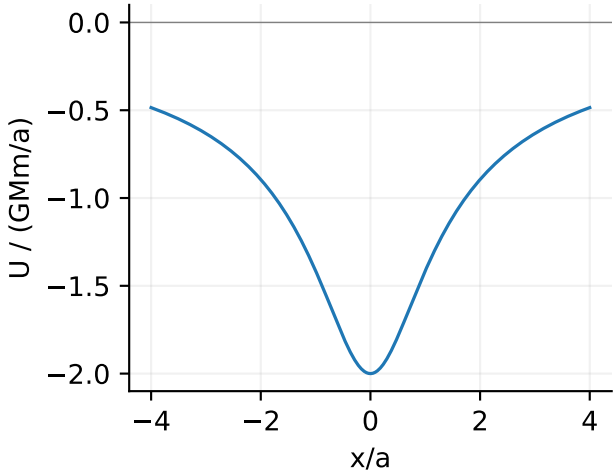
(a) Show $F_x = -2mMGx/(a^2 + x^2)^{3/2}$.

(b) Find its potential energy and discuss bound and unbound motion using its graph.

(c) Find the greatest speed after release from rest at $x = 3a/4$.

Solution.

The vertical forces cancel; the horizontal components sum to $\boxed{F_x = -2GMmx/(a^2 + x^2)^{3/2}}$. The potential is $\boxed{U = -2GMm/\sqrt{a^2 + x^2}}$. Its minimum is $-2GMm/a$; energies between that and zero give bound oscillation, while nonnegative energies permit escape. Release from $x = 3a/4$ gives initial $U = -8GMm/(5a)$. At zero the speed is maximal, and $mv_{\max}^2/2 = 2GMm/(5a)$, giving $\boxed{v_{\max} = \sqrt{4GM/(5a)}}$.



6. Lennard–Jones vibration

Two atoms, each of mass m, have interaction energy

$$U(r) = \epsilon\big[(r_0/r)^{12} - 2(r_0/r)^6\big].$$

(a) Sketch U and find its minimum and well depth.

(b) Expand to quadratic order about $r = r_0$.

(c) Derive the small-oscillation frequency.

(d) Estimate the effective spring constant for N_2 with vibrational frequency $3 \times 10^{13}\,\mathrm{Hz}$.

Solution.

Differentiating gives the minimum at $r = r_0$, $U(r_0) = -\epsilon$, and $U''(r_0) = 72\epsilon/r_0^2$. Thus

$$U = -\epsilon + 36\epsilon(r - r_0)^2/r_0^2 + O((r - r_0)^3).$$

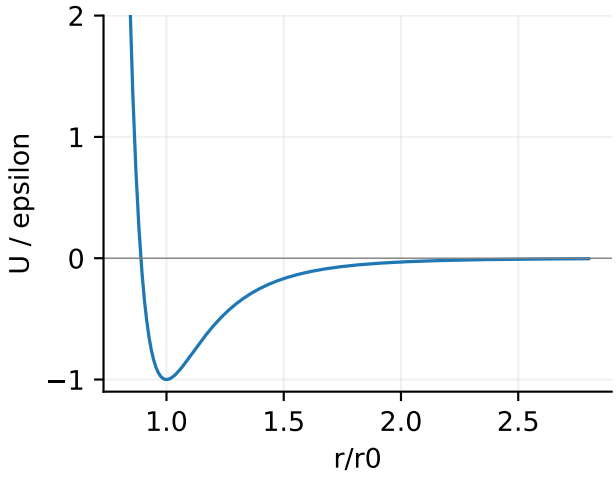
The relative coordinate has reduced mass $m/2$, so

$$\boxed{\omega = \sqrt{\frac{72\epsilon/r_0^2}{m/2}} = \frac{12}{r_0}\sqrt{\epsilon/m}.}$$

For N_2, $\mu \simeq 1.16 \times 10^{-26}\,\mathrm{kg}$ and

$$k = \mu(2\pi\nu)^2 \simeq \boxed{4.1 \times 10^2\,\mathrm{N\,m^{-1}}}$$

at $\nu = 3 \times 10^{13}\,\mathrm{Hz}$.



7. Projectile formulas and mortar data

A projectile is launched at speed V and angle α above level ground, without air resistance. Explain why its trajectory is planar. Find (a) flight time, (b) maximum height, (c) range, and (d) kinetic and potential energies at the top; verify energy conservation. Compare the predictions with the following mortar data, all at 45°, and explain discrepancies.

Muzzle speed ($\mathrm{m\,s^{-1}}$)	Range (m)	Time (s)
101.8	972	14.4
112.2	1160	15.7
121.9	1349	17.0
131.4	1539	18.2

Solution.

Uniform gravity and the initial velocity span a fixed plane. Taking horizontal x and vertical y, $x = V\cos\theta\, t$, $y = V\sin\theta\, t - gt^2/2$. Hence

$$\boxed{T = 2V\sin\theta/g, \quad H = V^2\sin^2\theta/(2g), \quad R = V^2\sin 2\theta/g.}$$

At the apex $K = mV^2\cos^2\theta/2$ and $U = mV^2\sin^2\theta/2$, summing to the initial kinetic energy. At $45°$, predicted (R, T) are approximately $(1056, 14.68), (1283, 16.17), (1515, 17.57), (1760, 18.94)$ in metres and seconds. The shorter observed ranges and times are consistent with air resistance, which reduces horizontal velocity and changes the vertical trajectory.

8. Average air resistance

A 1 kg body falls from rest through 100 m and reaches the ground at $40\,\mathrm{m\,s^{-1}}$. Find the average air resistance and specify the meaning of the average.

Solution.

Work-energy gives $mgh - \overline{F}_s h = mv_f^2/2$. Thus

$$\boxed{\overline{F}_s = mg - mv_f^2/(2h) = 1.81\,\mathrm{N}.}$$

This is a distance average $h^{-1}\int F_s\, ds$, rather than the generally different time average $t^{-1}\int F_s\, dt$.

9. Balloon drag

An air-filled balloon of diameter 30 cm has envelope mass 0.5 g and falls from rest in air.

(a) Using $F_1 = 6\pi a\eta v$, with
$$\eta = 1.7\times 10^{-5}\,\mathrm{kg\,m^{-1}\,s^{-1}},$$
find the terminal speed.

(b) Write $\dot{v} = g - \gamma v$, identify γ, solve the initial-value problem and find the time to reach 95% of terminal speed.

(c) Include the additional resistance $F_2 = 0.87(av)^2\,\mathrm{N}$ (a in metres, v in $\mathrm{m\,s^{-1}}$), and discuss which terminal-speed estimate is more realistic.

Solution.

The Stokes coefficient is

$$b = 6\pi a\eta = 4.81\times 10^{-5}\,\mathrm{kg\,s^{-1}}.$$

In the effective point-mass model, $m = 5.0\times 10^{-4}\,\mathrm{kg}$ gives $v_t = mg/b = \boxed{102\,\mathrm{m\,s^{-1}}}$ and $\lambda = b/m = 0.0961\,\mathrm{s^{-1}}$. Thus

$$\boxed{v = v_t(1 - e^{-\lambda t}),\ t_{95} = \ln 20/\lambda = 31.2\,\mathrm{s}}.$$

Including quadratic drag cv^2, $c = 0.87a^2 = 0.019575$, gives

$$\boxed{v_t = (-b + \sqrt{b^2 + 4cmg})/(2c) = 0.499\,\mathrm{m\,s^{-1}}}.$$

The much smaller value is more realistic because the Stokes-only prediction has very large Reynolds number. For a detailed transient model the inertia of the enclosed air and added air must also be included; its weight is approximately cancelled by buoyancy.

10. Vertical quadratic drag

A mass m is projected vertically upwards at speed v_0, subject to gravity and a resistance of magnitude αv^2 opposite its velocity.

(a) Write separate equations for ascent and descent. Show that the greatest height is $h = a\ln[1 + (v_0/v_l)^2]$, where $v_l = \sqrt{mg/\alpha}$ and $a = v_l^2/(2g)$.

(b) Show that its return speed satisfies $v_r^2 = v_l^2[1 - e^{-h/a}]$.

Solution.

Use upward velocity v. On ascent $m\dot{v} = -mg - \alpha v^2$, while on descent $m\dot{v} = -mg + \alpha v^2$. Put $v_l^2 = mg/\alpha$, $a = v_l^2/(2g)$. Using $\dot{v} = v\,dv/dy$, integrate from v_0 to zero:

$$\boxed{h = a\ln\bigl(1 + v_0^2/v_l^2\bigr).}$$

On descent let $w = -v \geq 0$ and downward distance s; $w\,dw/ds = g(1 - w^2/v_l^2)$ gives $w^2 = v_l^2(1 - e^{-s/a})$. At the launch height,

$$\boxed{v_r^2 = v_l^2(1 - e^{-h/a}) = \frac{v_0^2}{1 + v_0^2/v_l^2}.}$$

11. Earth's varying gravity

Let R_E be the Earth's radius and $g = GM_E/R_E^2$.

(a) Show that a fall from rest at infinity gives the same surface speed as a fall through R_E under uniform gravity g.

(b) For release from height $h \ll R_E$, show $v \simeq \sqrt{2gh}(1 - h/(2R_E))$.

(c) Show that a pendulum clock raised through h has fractional period change $\Delta T/T \simeq h/R_E$. Does it gain or lose time? Evaluate the error per week at $h = 100\,\mathrm{m}$.

Solution.

Energy for fall from $R + h$ to R gives $v^2 = 2GM(1/R - 1/(R + h))$. From infinity, $v = \sqrt{2GM/R} = \sqrt{2gR}$, equal to a constant-$g$ fall through height R. For $h \ll R$,

$$\boxed{v = \sqrt{2gh}[1 - h/(2R) + O(h^2/R^2)]}.$$

Since $g(R + h) \simeq g(1 - 2h/R)$ and $T \propto g^{-1/2}$, $\boxed{\Delta T/T = h/R}$. The elevated clock loses time, by about

$$604800(100/6.378 \times 10^6) = \boxed{9.48\,\mathrm{s}}$$

per week.

12. Projectile on an incline

A cannon on a plane inclined by α fires at speed V and angle β to the plane, along its line of greatest slope, uphill or downhill.

(a) Find the uphill range for $\alpha = \beta = 30°$.

(b) Show that the ratio of the greatest uphill and downhill ranges is $(1 - \sin\alpha)/(1 + \sin\alpha)$.

Solution.

Along the uphill plane, $\ddot{s} = -g\sin\alpha$; normal to it, $\ddot{n} = -g\cos\alpha$. With launch angle θ above the plane, $t_f = 2V\sin\theta/(g\cos\alpha)$. Substitution gives

$$R_\uparrow = \frac{2V^2\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}.$$

At $\theta = \alpha = 30°$, $\boxed{R_\uparrow = 2V^2/(3g)}$. Maximizing gives $\theta_* = \pi/4 - \alpha/2$ and $R_{\uparrow,\max} = V^2/[g(1+\sin\alpha)]$. Replacing α by $-\alpha$ gives

$$R_{\downarrow,\max} = V^2/[g(1-\sin\alpha)].$$

Their ratio is $\boxed{(1-\sin\alpha)/(1+\sin\alpha)}$.

13. Constant power with quadratic resistance

A mass M moves horizontally under constant supplied power P and resistance Mkv^2.

(a) Find the limiting speed v_m.

(b) Show that, from rest, the distance to $v_m/2$ is $\ln(8/7)/(3k)$.

(c) At that speed the power is cut off and a constant additional resisting force F is applied. Find the time to rest.

Solution.

Power balance gives $Mv\dot{v} = P - Mkv^3$, so $\boxed{v_m = (P/(Mk))^{1/3}}$. Using $\dot{v} = v\,dv/dx$, $dx = v^2dv/[k(v_m^3 - v^3)]$. Integrating from zero to $v_m/2$ yields $\boxed{x = \ln(8/7)/(3k)}$. Once power is removed, $M\dot{v} = -Mkv^2 - F$. The remaining stopping time is

$$\boxed{t = \sqrt{\frac{M}{Fk}}\tan^{-1}\left(\frac{v_m}{2}\sqrt{\frac{Mk}{F}}\right).}$$

14. Energy dissipated by quadratic drag

For the vertical quadratic-drag projectile in Question 10:

(a) Integrate the resisting force directly over the ascent to find the dissipated energy and verify the work–energy theorem.

(b) Use the return-speed formula to show that half the initial energy is lost over the round trip when $v_0 = v_l$.

Solution.

On ascent, $v^2(y) = (v_0^2 + v_l^2)e^{-y/a} - v_l^2$. Direct integration gives

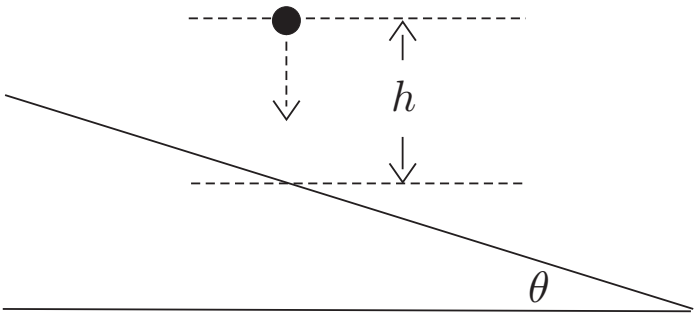
$$D_\uparrow = \int_0^h \alpha v^2 dy = \alpha a(v_0^2 + v_l^2)(1 - e^{-h/a}) - \alpha v_l^2 h = \boxed{\tfrac{1}{2}mv_0^2 - mgh}.$$

This equals the mechanical-energy decrease. For $v_0 = v_l$, $e^{-h/a} = 1/2$ and the return speed obeys $v_r^2 = v_l^2/2$. Thus final kinetic energy is half the initial, and $\boxed{D_{\text{round}} = K_i/2}$.

1.4 Classical Mechanics: Sheet 3

1. Elastic bounce from an incline

A sphere falls from rest through height h and rebounds elastically from a smooth plane inclined at $\alpha < 90°$ to the horizontal. Find the outgoing direction relative to the upward normal. Show that the distance down the plane to the next impact is $8h \sin\alpha$.



Solution.

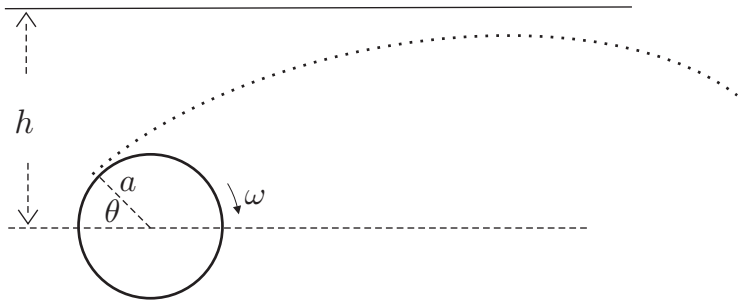
Let s point downhill and n normal out of the plane. Before impact the speed is $u = \sqrt{2gh}$ with components $(u\sin\theta, -u\cos\theta)$. A smooth elastic impact reverses only the normal component, so the outgoing velocity makes angle $\boxed{\theta}$ to the upward normal, towards downhill. The flight time back to the plane is $t = 2u/g$. The downhill displacement is

$$u\sin\theta\, t + g\sin\theta\, t^2/2 = \boxed{8h\sin\theta}.$$

2. Water drops from a wheel

Water drops leave tangentially from a wheel of radius a rotating in a vertical plane about a fixed horizontal axis at angular speed $\omega > \sqrt{g/a}$. Find the release point giving the greatest height above the axis. For a ceiling at height $h > a$, show that no drop reaches it if

$$(a\omega)^2 < gh + g\sqrt{h^2 - a^2}.$$



Solution.

Measure ϕ from the horizontal so that release height is $a\sin\phi$ and upward tangential velocity is $a\omega\cos\phi$. The maximum attained height is

$$H(\phi) = a\sin\phi + a^2\omega^2\cos^2\phi/(2g)$$

on the upward-moving half. Differentiation gives $\sin\phi_* = g/(a\omega^2)$ and

$$H_{\max} = \frac{a^2\omega^2}{2g} + \frac{g}{2\omega^2}.$$

Requiring $H_{\max} < h$ and using $a\omega^2 > g$ selects the upper-root bound

$$\boxed{(a\omega)^2 < gh + g\sqrt{h^2 - a^2}.}$$

3. Rocket equation

Masses m and $\Delta m \ll m$ initially move together at velocity v. The small mass is ejected backwards at speed w relative to m, whose velocity increases by Δv. Show $m\Delta v = w\Delta m$ to first order and discuss forward ejection. Derive the variable-mass rocket equation $m\dot{v} + w\dot{m} = 0$ with consistent signs, then include constant downward gravity.

Solution.

Momentum before separation is $(m+\delta m)v$; afterwards it is $m(v+\delta v)+\delta m(v+\delta v-w)$. To first order $m\delta v = w\delta m$. Forward ejection reverses the sign. For a rocket, ejected positive mass is $\delta m_{\text{ej}} = -dm$, so

$$\boxed{m\dot{v} = -w\dot{m}, \qquad m\dot{v} = -w\dot{m} - mg \text{ for upward flight.}}$$

Here $w > 0$ is backward exhaust speed relative to the rocket and $\dot{m} < 0$.

4. Vertically launched rocket

A rocket of initial mass M, half of which is fuel, launches vertically from rest. Fuel burns at positive mass rate α and exhaust speed V relative to the rocket. Gravity is constant.

(a) Derive $\dot{v} = -g + \alpha V/(M - \alpha t)$.

(b) Show that immediate lift-off requires $\alpha V > Mg$.

(c) Find burnout time.

(d) Find burnout velocity.

(e) Find burnout height and maximum height.

Solution.

With $m = M - \alpha t$, the rocket equation gives $\dot{v} = -g + \alpha V/(M - \alpha t)$, requiring $\alpha V > Mg$ for immediate liftoff. Fuel runs out at $t_b = M/(2\alpha)$. Integration gives

$$v(t) = V \ln \frac{M}{M - \alpha t} - gt, \quad v_b = V \ln 2 - \frac{gM}{2\alpha},$$

$$h_b = \frac{MV}{2\alpha}(1 - \ln 2) - \frac{gM^2}{8\alpha^2}.$$

Coasting adds $v_b^2/(2g)$, so

$$\boxed{H = \frac{MV}{2\alpha}(1 - 2\ln 2) + \frac{V^2 \ln^2 2}{2g}.}$$

5. One-stage and two-stage rockets

A rocket initially at rest has total mass Nm and payload mass m. Exhaust speed is u. The casing fraction of each stage's casing-plus-fuel mass is r. Neglect gravity and drag.

(i) For one stage, derive $v_1 = u\ln[N/(rN+1-r)]$.

(ii) For two stages, with initial second-stage mass nm and the first casing discarded at burnout, derive

$$v_2 = u\ln\frac{N}{rN+n(1-r)} + u\ln\frac{n}{rn+1-r}.$$

(iii) Maximise v_2 at fixed N, r, showing $n^2 = N$, and obtain $v_{2,\max}$.

(iv) With $u = 2.9\,\mathrm{km\,s^{-1}}$ and $r = 0.1$, can either design reach Earth escape speed?

Solution.

Integrating $dv = -u\,dm/m$ gives $u\ln(M_i/M_f)$. The one-stage final mass is $m[rN+(1-r)]$, yielding the stated v_1. For two stages the final masses before discarding stage 1 and after stage 2 are $m[rN+(1-r)n]$ and $m[rn+(1-r)]$, so

$$\boxed{v_2 = u\ln\frac{N}{rN+(1-r)n} + u\ln\frac{n}{rn+(1-r)}.}$$

Differentiation with respect to n gives a numerator proportional to $N-n^2$, so the maximum is at $n=\sqrt{N}$. Then $v_{2,\max} = 2u\ln[\sqrt{N}/(r\sqrt{N}+1-r)]$. As $N\to\infty$, $v_1 < u\ln(1/r) = 6.68\,\mathrm{km\,s^{-1}}$, below escape speed;

$$v_2 < 2u\ln(1/r) = 13.35\,\mathrm{km\,s^{-1}},$$

so two stages can in principle reach $11.2\,\mathrm{km\,s^{-1}}$ with sufficiently large mass ratio (about $N > 4.0\times10^2$ in this ideal model).

6. Path-dependent work

Find the work from $O=(0,0,0)$ to $P=(1,1,0)$ under $\mathbf{F} = \alpha(2xy, x^2, 0)$ along (a) $O\to(1,0,0)\to P$, (b) $O\to(0,1,0)\to P$, and (c) the straight segment OP. Repeat for $\mathbf{G} = \alpha(x^2, 2xy, 0)$ and determine which fields are conservative.

Solution.

For F,

$$F\cdot dr = \alpha(2xy\,dx + x^2dy) = d(\alpha x^2 y),$$

so all three paths give $\boxed{W_F = \alpha}$ and F is conservative. For G, direct integration gives

$$\boxed{W_G = 4\alpha/3,\ \alpha/3,\ \alpha}$$

along paths (a),(b),(c). In particular $(\nabla\times G)_z = 2\alpha y \neq 0$ generally, confirming nonconservativity.

7. Finite-amplitude pendulum

A pendulum of length l and angular amplitude θ_0 has period $T = kf(\theta_0)\sqrt{l/g}$.

(a) Use energy conservation to recover $kf \to 2\pi$ for small amplitudes and obtain $k = 2\sqrt{2}$,

$$f = \int_0^{\theta_0} (\cos\theta - \cos\theta_0)^{-1/2}\, d\theta.$$

(b) With $\sin(\theta/2) = \beta \sin\phi$, $\beta = \sin(\theta_0/2)$, derive

$$T = 4\sqrt{l/g} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \beta^2 \sin^2\phi}}.$$

(c) Expand through order θ_0^2 and estimate the small-angle error at $\theta_0 = \pi/3$.

Solution.

Energy gives

$$\dot\theta^2 = (2g/l)(\cos\theta - \cos\theta_0).$$

Four quarter-periods yield

$$T = 2\sqrt{2}\sqrt{l/g} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 4\sqrt{l/g} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2\phi}}, \quad \alpha = \sin(\theta_0/2).$$

Expand the last integrand: $1 + \alpha^2 \sin^2\phi/2 + \cdots$. Hence

$$\boxed{T = 2\pi\sqrt{l/g}\left[1 + \theta_0^2/16 + O(\theta_0^4)\right].}$$

At $\theta_0 = \pi/3$, the leading increase is 6.85% of the small-angle period; relative to this corrected period the small-angle estimate is 6.41% low. The exact elliptic-integral period gives about 6.82% underestimate.

8. Helix in a magnetic field

A particle of mass m and charge q starts at the origin with velocity $(0, v_0, w_0)$ in $\mathbf{B} = (0, 0, B)$. Starting from

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B},$$

integrate the component equations. Show $\ddot y + \omega^2 y = 0$, $\omega = qB/m$, and find x, y, z. Describe and sketch the path and explain the force balance.

Solution.

With $\omega = qB/m$, the components are $\ddot x = \omega\dot y$, $\ddot y = -\omega\dot x$, $\ddot z = 0$. Integrating the first gives $\dot x = \omega y$, and hence $\ddot y + \omega^2 y = 0$. Initial conditions yield

$$\boxed{x = \frac{v_0}{\omega}(1 - \cos\omega t), \quad y = \frac{v_0}{\omega}\sin\omega t, \quad z = w_0 t.}$$

The transverse path is a circle centred at $(v_0/\omega, 0)$, of radius $|v_0/\omega|$. The magnetic force supplies centripetal acceleration without changing speed; w_0 adds uniform motion along the field.

9. Crossed electric and magnetic fields

An electron of charge $-e$ starts at the origin with velocity $(v_0, 0, 0)$ in crossed fields $\mathbf{E} = (0, E_y, 0)$ and $\mathbf{B} = (0, 0, B_z)$. Write its component equations and show $z = 0$ and

$$m\dot{x} = -eB_z y + mv_0, \qquad m\dot{y} = -eE_y t + eB_z x.$$

Eliminate $\dot{x}$ to find $\ddot{y} + \omega^2 y = A$, identify ω, A, and obtain $y = a(1 - \cos\omega t)$ and $x = a\sin\omega t + bt$, with $b = E_y/B_z$. Sketch the cycloidal trajectory and indicate the fields.

Solution.

Put $\omega = eB_z/m$ and $b = E_y/B_z$. The equations are $\ddot{x} = -\omega\dot{y}$, $\ddot{y} = -eE_y/m + \omega\dot{x}$, $\ddot{z} = 0$. Thus $z = 0$ and $\ddot{y} + \omega^2 y = \omega(v_0 - b)$. With the given initial conditions,

$$\boxed{y = \frac{v_0 - b}{\omega}(1 - \cos\omega t), \qquad x = bt + \frac{v_0 - b}{\omega}\sin\omega t.}$$

This is circular gyromotion superposed on the $E \times B$ drift $b\widehat{x}$; depending on v_0 the planar curve is a trochoid, with a cycloid as a special case.

10. Accelerating lift

A passenger holds a parcel of mass m in a lift pulled upwards by constant force F. The lift plus passenger has mass $M \gg m$.

(a) Find the lift acceleration.

(b) Find the parcel's apparent weight.

(c) Find the time to fall through height h, using both an inertial frame and the lift frame.

(d) What force is needed for uniform lift velocity? Comment.

Solution.

The lift obeys $Ma = F - Mg$, so $\boxed{a = F/M - g}$. The parcel's apparent weight is $m(g + a) = \boxed{mF/M}$. In the lift frame a released parcel accelerates downward relative to the floor at $g + a = F/M$; thus $\boxed{t = \sqrt{2hM/F}}$ for $F > 0$. In an inertial frame subtract the floor trajectory from the parcel trajectory to obtain the same relative displacement $h - (g + a)t^2/2$. Uniform velocity requires $\boxed{F = Mg}$, restoring the ordinary weight and fall time.

11. Curvilinear acceleration

A particle follows a curve C with position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$.

(a) If $\dot{\mathbf{r}} = \mathbf{c} \times \mathbf{r}$ with constant $\mathbf{c}$, show that the motion is uniform circular motion.

(b) For unit tangent $\hat{\boldsymbol{\tau}}$, principal normal $\hat{\mathbf{n}}$ and arc length s, obtain

$$d\hat{\boldsymbol{\tau}}/ds = \hat{\mathbf{n}}/\rho,$$

where ρ is the radius of curvature. Discuss the derivatives of the unit vectors.

(c) Derive

$$\mathbf{a} = \dot{v}\hat{\boldsymbol{\tau}} + (v^2/\rho)\hat{\mathbf{n}}.$$

Solution.

(a) From $\dot{r} = c \times r$, both $d|r|^2/dt = 0$ and $d(c \cdot r)/dt = 0$. The trajectory is the intersection of a sphere and a plane: a circle about the c axis, with angular velocity $|c|$ (or a fixed point on that axis).

(b) In the osculating plane, a tangent turning through $d\psi$ over arc length ds gives $d\hat{t}/ds = \hat{n}/\rho$. For a general space curve, the derivative of the normal can also have a binormal component:

$$d\hat{n}/ds = -\hat{t}/\rho + \tau\hat{b}.$$

(c) Differentiate $v = v\hat{t}$ to obtain

$$\boxed{a = \dot{v}\hat{t} + (v^2/\rho)\hat{n}}.$$

For planar motion, the torsion is zero.

12. Projectile with linear drag

A projectile starts at the origin with horizontal and vertical velocities $u_0 = V\cos\alpha$, $v_0 = V\sin\alpha$ under gravity and drag $-mk\mathbf{v}$. Derive the equation for its horizontal range x on returning to the initial height:

$$\left(v_0 + \frac{g}{k}\right)\frac{x}{u_0} + \frac{g}{k^2}\ln\left(1 - \frac{kx}{u_0}\right) = 0.$$

For small k, show $x_{\max} \simeq 2v_0u_0/g - 8v_0^2u_0k/(3g^2)$.

Solution.

Solving $\dot{u} = -ku$ and $\dot{v} = -g - kv$ gives $x = u_0(1 - e^{-kt})/k$ and $y = (v_0 + g/k)(1 - e^{-kt})/k - gt/k$. Eliminating t and setting landing height to zero yields

$$\boxed{(v_0 + g/k)x/u_0 + (g/k^2)\ln(1 - kx/u_0) = 0.}$$

Expanding the logarithm, dividing by the nonzero root x, and writing $x = x_0 + kx_1 + \cdots$ gives $x_0 = 2u_0v_0/g$, $x_1 = -8u_0v_0^2/(3g^2)$. Thus

$$\boxed{x_{\text{range}} \simeq 2u_0v_0/g - 8ku_0v_0^2/(3g^2)}$$

for weak drag.

1.5 Classical Mechanics: Sheet 4

1. Polar acceleration on a ring

A mass m slides inside a smooth vertical semicircular track of radius r. Let θ be measured from the downward vertical.

(a) Derive polar velocity and acceleration, obtaining

$$\mathbf{a} = r\ddot{\theta}\hat{\boldsymbol{\theta}} - (v^2/r)\hat{\mathbf{r}}$$

on the track.

(b) Derive the small-oscillation period about the bottom.

(c) Find the normal reaction as a function of θ; for release from rest at the top of the semicircle, show that its magnitude at $\theta = 60°$ is $3mg/2$.

Solution.

Differentiating $\widehat{r}$ and $\widehat{\theta}$ gives

$$\dot{\widehat{r}} = \dot{\theta}\widehat{\theta},$$
$$\dot{\widehat{\theta}} = -\dot{\theta}\widehat{r}.$$

Thus

$$v = \dot{r}\widehat{r} + r\dot{\theta}\widehat{\theta}$$

and

$$a = (\ddot{r} - r\dot{\theta}^2)\widehat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\widehat{\theta}.$$

At fixed radius this reduces to the stated expression. Measure θ from downward vertical: $r\ddot{\theta} = -g\sin\theta$, giving $\boxed{T = 2\pi\sqrt{r/g}}$ for small oscillations. Release from the rim ($\theta = \pi/2$) gives $v^2 = 2gr\cos\theta$, and inward balance gives $N = mg\cos\theta + mv^2/r = 3mg\cos\theta$. Hence $\boxed{N(60°) = 3mg/2}$.

2. Angular momentum and area

A particle moves under a central attractive acceleration $-f(r)\hat{\mathbf{r}}$. Show that $\mathbf{r}\times\dot{\mathbf{r}}$ is constant and the orbit is planar. Define its angular momentum $\mathbf{J}$, prove constancy of $|\mathbf{J}|$ using the transverse polar equation, and derive the constant rate of swept area.

Solution.

For central acceleration $\ddot{r} = -f(r)\widehat{r}$, $d(r\times\dot{r})/dt = r\times\ddot{r} = 0$. Therefore the trajectory lies in the fixed plane perpendicular to this vector (or is radial if it vanishes). In that plane $\boxed{J = mr^2\dot{\theta}}$; the transverse equation $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ makes it constant. A sector swept in dt has area $dA = r^2 d\theta/2$, so $\boxed{\dot{A} = J/(2m)}$ is constant.

3. Masses linked through a table

Two equal masses m are joined by a light inextensible string of length l passing through a smooth hole in a horizontal table. One mass moves on the smooth table; the other hangs vertically.

(a) Derive $\dot{r}^2 = A - B/r^2 - gr$. Initially $r = l/2$ and the table mass has transverse speed v_0. Determine A, B.

(b) Show that circular motion requires $v_0^2/(gl) = 1/2$.

Solution.

With $h = r^2\dot{\theta}$ conserved per unit mass, the energy is $E = m\dot{r}^2 + mh^2/(2r^2) + mgr$+constant; both masses contribute radial kinetic energy. Thus

$$\boxed{\dot{r}^2 = A - B/r^2 - gr, \quad A = (gl + v_0^2)/2, \quad B = l^2 v_0^2/8.}$$

Here $r_0 = l/2$ and $h = r_0 v_0$. Constant radius also requires $h^2/r_0^3 = g$, equivalent to $\boxed{v_0^2/(gl) = 1/2}$.

4. Geostationary orbit and binary stars

(a) Find the altitude of a satellite that remains over one city on the Earth's equator. Explain why the city must be equatorial.

(b) In Plaskett's nearly circular binary, each component has speed $220\,\mathrm{km\,s^{-1}}$ and the orbital period is 14.4 days. Explain why their masses are approximately equal, and find the separation and component masses.

Solution.

(a) Matching the orbital angular velocity to Earth's rotation gives $r = (GMT^2/4\pi^2)^{1/3}$, so

$$\boxed{h = r - R_E \simeq 3.58 \times 10^4\,\mathrm{km}}.$$

A fixed point in the rotating equatorial plane is possible only for an equatorial, prograde orbit.

(b) Opposite, equal speeds and zero CM momentum imply equal masses. Each orbital radius is $r = vP/(2\pi) = 4.36 \times 10^{10}\,\mathrm{m}$, so separation

$$d = 2r = \boxed{8.71 \times 10^{10}\,\mathrm{m}}.$$

From $Mv^2/r = GM^2/d^2$, each mass is

$$\boxed{M = 4rv^2/G = 1.26 \times 10^{32}\,\mathrm{kg}}.$$

5. Galaxy rotation curve

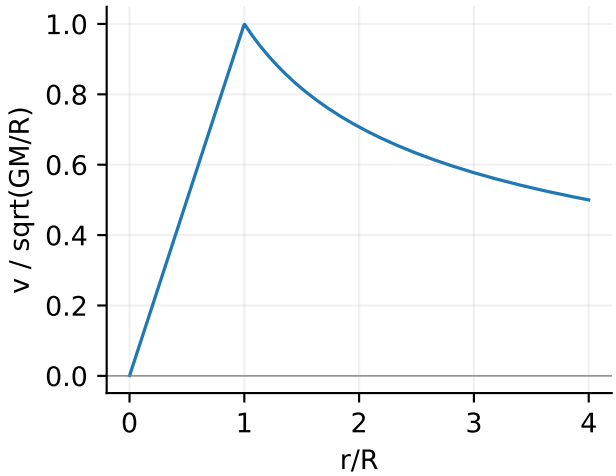
A galaxy is modelled as a sphere of radius R, total mass M and uniform density. Find and sketch the circular-orbit speed $v(r)$ both inside and outside it, using the gravitational shell theorem. Relate departures of measured rotation curves from luminous-matter predictions to the dark-matter problem.

Solution.

Spherical symmetry makes the interior gravitational force depend only on enclosed mass. For uniform density $M(r) = Mr^3/R^3$, so $v^2/r = GM(r)/r^2$. Therefore

$$\boxed{v(r) = \begin{cases} \sqrt{GM/R^3}\, r & r \le R, \\ \sqrt{GM/r} & r \ge R. \end{cases}}$$

The graph rises linearly to $\sqrt{GM/R}$ and then falls as $r^{-1/2}$; a measured outer curve that remains approximately flat requires a different mass distribution from this finite uniform model.



6. Launch to a specified apogee

A satellite launches from a nonrotating planet of mass M and radius R at 30° to the local vertical. Its apogee is $5R/2$. Neglect atmosphere and use energy and angular momentum to show $v_0^2 = 5GM/(4R)$.

Solution.

Specific angular momentum at launch is $h = Rv_0 \sin 30^\circ = Rv_0/2$. At apogee $r_a = 5R/2$, radial speed vanishes and $v_a = h/r_a = v_0/5$. Specific energy conservation gives $v_0^2/2 - GM/R = v_0^2/50 - 2GM/(5R)$. Rearrangement yields $\boxed{v_0^2 = 5GM/(4R)}$.

7. Hyperbolic comet

A comet approaches the Sun (mass M) from infinity with mass m, speed v and impact parameter b.

(a) Sketch its trajectory, impact parameter and deflection angle.

(b) Obtain $E = \frac{1}{2}m\dot{r}^2 + J^2/(2mr^2) - \alpha/r$ and identify J, α.

(c) Derive

$$r_{\min} = \frac{GM}{v^2}\left[\sqrt{1 + \left(\frac{bv^2}{GM}\right)^2} - 1\right].$$

Solution.

The incoming asymptote lies a perpendicular distance b from the Sun; attraction bends it towards the Sun into a hyperbola. With $J = mbv$, $\alpha = GMm$, energy is

$$E = m\dot{r}^2/2 + J^2/(2mr^2) - \alpha/r = mv^2/2.$$

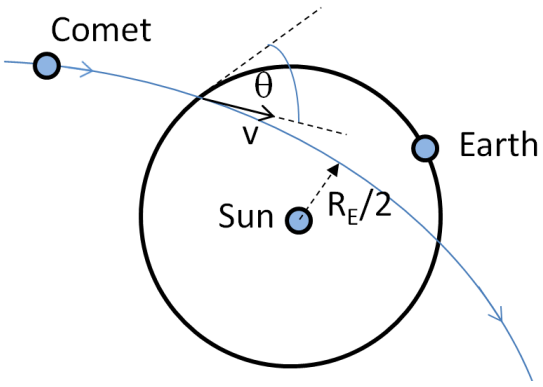
At perihelion $\dot{r} = 0$, giving $v^2r^2 + 2GMr - b^2v^2 = 0$. The positive root is

$$\boxed{r_{\min} = \frac{GM}{v^2}\left[\sqrt{1 + \left(\frac{bv^2}{GM}\right)^2} - 1\right].}$$

The angle between the incoming and outgoing asymptotes is the deflection angle; it obeys $\tan(\phi/2) = GM/(bv^2)$.

8. Comet crossing Earth's orbit

A comet's perihelion is half the Earth's circular orbital radius R_E. Its perihelion speed is $60\,\mathrm{km\,s^{-1}}$, twice the Earth's orbital speed. Neglect the Earth's attraction. Find its speed at $r = R_E$ and the angle its trajectory makes with the Earth's orbit there.



Solution.

Let Earth's circular speed be $v_E = 30\,\mathrm{km\,s^{-1}}$, so $GM = v_E^2 R_E$. At perihelion $r_p = R_E/2$, $v_p = 2v_E$, giving specific energy $v_p^2/2 - GM/r_p = 0$: the orbit is parabolic. At $r = R_E$,

$$\boxed{v = \sqrt{2}v_E = 42.4\,\mathrm{km\,s^{-1}}}.$$

Angular momentum is $h = r_p v_p = R_E v_E$, so the tangential speed there is v_E and the radial speed also has magnitude v_E. Thus the path makes $\boxed{45^\circ}$ with Earth's tangential orbit, inward or outward according to the crossing.

9. Impulsively changed satellite orbit

A satellite of mass m has circular-orbit speed v_0 and radius r_0 about the Earth (mass M).

(a) Show $r_0 = GM/v_0^2$.

(b) Obtain $J_0 = GMm/v_0$ and $E_0 = -mv_0^2/2$.

(c) An impulse changes J_0 to αJ_0 ($\alpha < 1$), keeping energy fixed. Describe the orbit and find its radial extremes for $\alpha = 12/13$.

(d) Instead, keep angular momentum fixed but increase kinetic energy by 1%. Find the difference of the extreme radii. Sketch both orbits relative to the original circle.

Solution.

Circular motion gives $r_0 = GM/v_0^2$, $J_0 = mr_0 v_0 = GMm/v_0$, $E_0 = -mv_0^2/2$. For fixed energy and $J = \alpha J_0$, the semimajor axis stays r_0 and $e = \sqrt{1-\alpha^2}$. Thus

$$\boxed{r_{\max,\min} = r_0(1 \pm \sqrt{1-\alpha^2})}; \quad \alpha = 12/13 \Rightarrow r_{\max} = 18r_0/13,\ r_{\min} = 8r_0/13.$$

For fixed $J = J_0$ and a 1% kinetic-energy increase, $E = -0.99mv_0^2/2$, $a = r_0/0.99$, $e = 0.1$. Therefore $\boxed{r_{\max} - r_{\min} = 2ae = 0.2020r_0}$. Both new orbits are ellipses with the Earth at a focus.

10. Attractive Coulomb scattering

A particle of mass m, charge q_1, incident speed v_0 and impact parameter $d > 0$ scatters from a fixed, oppositely charged nucleus of charge q_2. Sketch the attractive Rutherford orbit and derive the deflection formula, using the positive coupling $\kappa = |q_1 q_2|/(4\pi\epsilon_0)$. Adapt the result to a comet by setting $\kappa = GMm$.

Solution.

Let $\kappa = |q_1 q_2|/(4\pi\epsilon_0) > 0$. Specific orbit theory gives $e = \sqrt{1 + (mdv_0^2/\kappa)^2}$ for an attractive inverse-square force. The asymptote condition for the hyperbola gives $\sin(\alpha/2) = 1/e$, hence

$$\boxed{\cot(\alpha/2) = \frac{mdv_0^2}{\kappa} = \frac{4\pi\epsilon_0 mdv_0^2}{|q_1 q_2|}}.$$

The trajectory bends towards the nucleus. Replacing κ by GMm gives the comet result. Absolute charge product is needed if q_1, q_2 denote signed opposite charges.

11. Reduced-mass dynamics

For an isolated gravitational two-body system, define $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mu = m_1 m_2/(m_1 + m_2)$.

(a) Derive
$$\mu\ddot{\mathbf{r}} = -Gm_1 m_2 \hat{\mathbf{r}}/r^2.$$

(b) Show that the centre-of-mass kinetic energy is $\mu\dot{\mathbf{r}}^2/2$.

(c) Show that its total centre-of-mass angular momentum is $\mu\mathbf{r} \times \dot{\mathbf{r}}$. Explain the equivalent single-particle problem.

Solution.

Set $R = (m_1 r_1 + m_2 r_2)/M$, $r = r_1 - r_2$. Then $r_1 = R + (m_2/M)r$ and $r_2 = R - (m_1/M)r$. Subtracting Newton's equations gives
$$\boxed{\mu\ddot{r} = -Gm_1 m_2 \hat{r}/r^2}.$$
Substituting the coordinate decomposition into the kinetic energy and angular momentum cancels cross terms. In the CM frame,
$$\boxed{T = \mu\dot{r}^2/2, \quad J = \mu r \times \dot{r}},$$
exactly the single-particle formulas for mass $\mu = m_1 m_2/M$.

12. Two-dimensional Hooke orbit

A mass m experiences $\mathbf{F} = -k\mathbf{r}$ $(k > 0)$, with $\mathbf{r}(0) = (x_0, 0)$ and $\mathbf{v}(0) = (0, u_0)$.

(a) Solve the Cartesian equations, show that the orbit is an ellipse centred at the origin and find its axes.

(b) Derive the effective radial potential and show that the motion is bound.

Solution.

Let $\omega = \sqrt{k/m}$. Cartesian equations are $\ddot{x} + \omega^2 x = \ddot{y} + \omega^2 y = 0$. Initial conditions give
$$\boxed{x = x_0 \cos\omega t, \;\; y = (u_0/\omega)\sin\omega t},$$
an ellipse with semiaxes $|x_0|$ and $|u_0|/\omega$ (a degenerate line if either vanishes). With angular momentum J, $\boxed{U_{\text{eff}} = J^2/(2mr^2) + kr^2/2}$. It diverges at infinity, so every finite-energy orbit is bound; for $J \neq 0$ its inner centrifugal barrier also prevents reaching zero.

13. Mass loss in a binary

Stars of masses m_0 and $2m_0$ have circular relative orbit of separation r_0. The heavier star suddenly loses half its mass spherically symmetrically.

(a) Find the original centre-of-mass energy and period.

(b) Recover them with the reduced-mass description.

(c) Find the post-explosion energy in the new centre-of-mass frame and show that the binary stays bound.

(d) Sketch the new relative and individual orbits.

Solution.

Before the explosion, total mass is $3m_0$, reduced mass $2m_0/3$, and relative circular speed $v_r^2 = 3Gm_0/r_0$. Thus
$$\boxed{E_i = -Gm_0^2/r_0, \quad P_i = 2\pi\sqrt{r_0^3/(3Gm_0)}}.$$

The equivalent relative-coordinate problem gives exactly the same energy and period. Symmetric loss leaves each survivor's instantaneous velocity unchanged; afterwards $\mu' = m_0/2$ and the potential is $-Gm_0^2/r_0$. Hence

$$E_f = \mu' v_r^2/2 - Gm_0^2/r_0 = \boxed{-Gm_0^2/(4r_0)} < 0,$$

so the system stays bound. The new relative orbit has $a = 2r_0$, $e = 1/2$, with the explosion point at pericentre r_0 and apocentre $3r_0$. Each equal survivor follows a scaled ellipse about the new CM.

14. Transfer to the Moon

A spacecraft is boosted tangentially from a low circular Earth orbit (r_0, v_0) into an ellipse with perigee r_0 and lunar apogee r_a. Show $(v_p/v_0)^2 = 2r_a/(r_0 + r_a)$ and evaluate v_0, v_p for $r_0 \simeq R_E$, $r_a \simeq 60R_E$, $R_E = 6.38 \times 10^6\,\mathrm{m}$. Find the apogee error caused by a small fractional boost-speed error f, distinguishing $\delta r_a/r_a$ from $\delta r_a/r_0$. Show that a 0.1% speed error gives over 10% fractional apogee error. Discuss further requirements for rendezvous.

Solution.

At perigee and apogee, $r_0 v_p = r_a v_a$ and energy is conserved. Eliminating v_a and using $v_0^2 = GM/r_0$ gives $\boxed{(v_p/v_0)^2 = 2r_a/(r_0 + r_a)}$. For $r_a = 60r_0$ and $r_0 \simeq R_E$, $v_0 = 7.91\,\mathrm{km\,s^{-1}}$ and $v_p = 11.10\,\mathrm{km\,s^{-1}}$. Put $s = v_p/v_0$, $r_a/r_0 = s^2/(2 - s^2)$. Differentiation gives

$$\boxed{\delta r_a/r_a = 2(1 + r_a/r_0)f = 122f},$$

so $f = 0.001$ gives about 12.2% apogee error. Rendezvous also requires correct launch phase, orbital plane and arrival timing, followed by accounting for lunar gravity and insertion or landing requirements.

15. Ellipse and inverse-square gravity

An ellipse has focal distances r, r' with $r + r' = 2a$, focal separation $2a\epsilon$, and angle θ as in the focal polar construction.

(a) Derive
$$r^{-1} = (1 - \epsilon\cos\theta)/[a(1 - \epsilon^2)].$$

(b) With $r^2\dot\theta = C$, show
$$\ddot r = -\epsilon C\cos\theta\,\dot\theta/[a(1 - \epsilon^2)].$$

(c) Show that the radial acceleration is $-k/r^2$, find k, and discuss the implication for gravitation.

Solution.

Use the triangle between the two foci and the planet, together with $r + r' = 2a$. Squaring gives $r = a(1 - e^2)/(1 - e\cos\theta)$ with the stated angle convention. Constant areal speed gives $r^2\dot\theta = C$. Set $u = 1/r = (1 - e\cos\theta)/[a(1 - e^2)]$. Then
$$\dot r = -C\,du/d\theta = -Ce\sin\theta/[a(1 - e^2)],$$
and differentiating yields the stated $\ddot r$. The full radial acceleration is

$$\boxed{a_r = \ddot r - r\dot\theta^2 = -\frac{C^2}{a(1 - e^2)r^2}.}$$

Thus $k = C^2/[a(1 - e^2)]$, identifying the inverse-square force law. Here a is the semimajor axis.

1.6 Classical Mechanics: Sheet 5

1. Sticking disks

A uniform disc A of mass m and radius a spins freely at angular speed ω. Find its moment of inertia, angular momentum and kinetic energy. An identical, initially stationary disc B drops concentrically onto it and sticks. Find the final angular speed, angular momentum and energy.

Solution.

Integrate $I = \int r^2 dm$ over a uniform disk to obtain $I = ma^2/2$. Initially $L = I\omega$, $K = I\omega^2/2$. With negligible external axial torque, sticking doubles the inertia while conserving L, so

$$\boxed{\omega_f = \omega/2, \quad L_f = ma^2\omega/2, \quad K_f = ma^2\omega^2/8 = K_i/2.}$$

The lost kinetic energy becomes internal energy during slipping and sticking.

2. Frictional torques

A uniform cylinder of mass M and radius r slows from ω_0 to ω_1 in time t about a fixed axis. Determine the resisting torque for (a) constant torque and (b) torque proportional to the instantaneous angular speed.

Solution.

For $I = Mr^2/2$, constant opposing torque gives $I(\omega_1 - \omega_0) = -Nt$, so $\boxed{N = Mr^2(\omega_0 - \omega_1)/(2t)}$. If $N = b\omega$, $\omega = \omega_0 e^{-bt/I}$, so $b = I\ln(\omega_0/\omega_1)/t$ and the instantaneous magnitude is

$$\boxed{N(\omega) = Mr^2\omega\ln(\omega_0/\omega_1)/(2t)}.$$

3. Unwinding reel

A uniform cylindrical reel of mass m and radius r unwinds under gravity from a thread whose upper end is fixed. Find its initial downward acceleration and thread tension, using $I = mr^2/2$.

Solution.

For downward acceleration a, $mg - T = ma$. The rotational equation is $Tr = I\alpha$ and no slip gives $a = r\alpha$. With $I = mr^2/2$, $T = ma/2$. Hence $\boxed{a = 2g/3, \quad T = mg/3}$.

4. Rotating space station

A thin cylindrical space station of radius r and mass M rotates freely in gravity-free space, giving apparent surface gravity g.

(a) Find the minimum spin-up work.

(b) A point astronaut of mass m climbs a massless radial spoke to the axis. Find the fractional change in apparent surface gravity.

(c) The astronaut climbs halfway from the surface to the axis and lets go. Find the circumferential distance from the spoke's base to the impact point.

Solution.

(a) $\Omega^2 r = g$ and $I = Mr^2$, giving minimum work $\boxed{W = Mgr/2}$ for the cylinder alone.

(b) Including an astronaut initially at the rim, angular momentum conservation gives $Mr^2\Omega_f = (M+m)r^2\Omega_i$. Thus

$$\boxed{\Delta g_{\text{app}}/g_{\text{app}} = (1 + m/M)^2 - 1 \simeq 2m/M}.$$

(c) At release radius $r/2$, inertial velocity is tangential, $v = \Omega r/2$. The straight-line distance to the wall is $\sqrt{3}r/2$, taking time $\sqrt{3}/\Omega$. The collision radius advances by $\pi/3$ while the spoke advances by $\sqrt{3}$. The point of impact is therefore behind the spoke base by arc length $\boxed{r(\sqrt{3} - \pi/3)}$.

5. Torsion pendulum with an added ring

A uniform disc of mass m and radius a hangs from a torsion wire exerting torque $-c\theta$. Derive its SHM period. A wire ring of mass m and radius $a/2$ drops concentrically and sticks. Find changes in period, amplitude and energy when it lands (i) at a turning point and (ii) at the equilibrium crossing.

Solution.

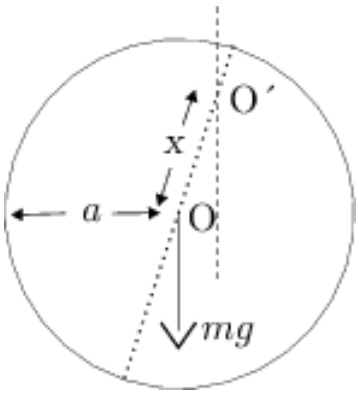
For $I = ma^2/2$, $I\ddot{\theta} = -c\theta$, so $T = 2\pi\sqrt{I/c}$. The ring adds $ma^2/4 = I/2$, making $I_f = 3I/2$ and $\boxed{T_f/T_i = \sqrt{3/2}}$. At a turning point the angular speed is zero: the amplitude and oscillation energy stay unchanged. At equilibrium, angular momentum gives $\dot{\theta}_f = (2/3)\dot{\theta}_i$. Hence $E_f/E_i = 2/3$ and $A_f/A_i = \boxed{\sqrt{2/3}}$. Only the latter case dissipates rotational energy during attachment.

6. Compound disk pendulum

A uniform disc of mass m and radius a swings about a horizontal axis through a hole a distance x from its centre.

(a) Derive its small-angle period using torque and angular momentum.

(b) Sketch $T^2(x)$ for $0 < x < a$, find its minimum and describe a method of measuring g from $T(x)$.

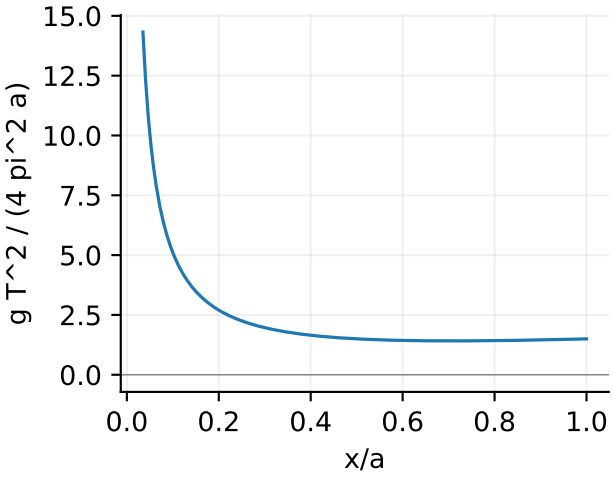


Solution.

The pivot inertia is $I_p = m(a^2/2 + x^2)$ and the restoring torque is $-mgx\sin\theta$. Linearizing gives

$$\boxed{T = 2\pi\sqrt{\frac{a^2/2 + x^2}{gx}}, \quad T^2 = \frac{4\pi^2}{g}\left(x + \frac{a^2}{2x}\right).}$$

The curve diverges as $x \to 0$ and has a unique minimum at $x = a/\sqrt{2}$. To measure g, plot xT^2 against x^2; the straight-line slope is $4\pi^2/g$.



7. Parallel axes and endpoint acceleration

State and prove the parallel-axis theorem. Integrate to find the moment of inertia of a uniform rod of mass m and length b about its centre and one end, and verify the theorem. The rod AB lies at rest on a smooth horizontal table. A horizontal force perpendicular to it is applied first at its centre, then in a separate experiment at B. Find the initial acceleration of A in each case and its direction. Where must the force act for A to have zero initial acceleration?

Solution.

Write each mass element's displacement from the new axis as $r' = r + d$, with $\int r\,dm = 0$ about the CM. Expanding and integrating gives $\boxed{I = I_{\mathrm{CM}} + md^2}$. A rod has $I_A = (m/b)\int_0^b x^2 dx = mb^2/3$ and

$$I_{\mathrm{CM}} = (m/b)\int_{-b/2}^{b/2} x^2 dx = mb^2/12,$$

satisfying the theorem. If force acts a distance s from A, CM acceleration is F/m and initial angular acceleration is $F(s - b/2)/I_{\mathrm{CM}}$. The acceleration of A along the force is

$$\boxed{a_A = \frac{F}{m}\left(4 - \frac{6s}{b}\right).}$$

Thus it is F/m for $s = b/2$, $-2F/m$ for $s = b$ (opposite the force), and zero for $s = 2b/3$.

8. Rolling spheres

Derive the central moments of inertia of a uniform thin spherical shell and a solid sphere, both of mass M and radius R. Each rolls without slipping from rest through distance L down a plane inclined at θ. Find their final speeds, identify the faster sphere, and show that their speed difference divided by their mean speed is about 8.7%.

Solution.

For a spherical shell, integrate $R^2\sin^2\theta$ over its uniform surface: $I = (2/3)MR^2$. Integrating such shells through a uniform solid sphere gives $I = (2/5)MR^2$. With drop $L\sin\theta$ and no slip, $MgL\sin\theta = Mv^2/2 + Iv^2/(2R^2)$. Hence

$$\boxed{v_{\mathrm{shell}} = \sqrt{6gL\sin\theta/5}, \quad v_{\mathrm{solid}} = \sqrt{10gL\sin\theta/7}.}$$

The solid sphere is faster. Their relative difference is $2(v_s - v_h)/(v_s + v_h) = \boxed{8.72\%}$.

9. Angular momentum decomposition

For masses m_1, m_2 at $\mathbf{r}_1, \mathbf{r}_2$, prove the decomposition of angular momentum about the origin:

$$\mathbf{L} = (m_1 + m_2)\mathbf{r}_{\text{cm}} \times \dot{\mathbf{r}}_{\text{cm}} + \mu \mathbf{r} \times \dot{\mathbf{r}}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mu^{-1} = m_1^{-1} + m_2^{-1}.$$

Solution.

Set $r_1 = R + (m_2/M)r$, $r_2 = R - (m_1/M)r$ and differentiate. Substitute into $L = \sum_i m_i r_i \times \dot{r}_i$. The mixed R, r terms cancel, leaving

$$\boxed{L = MR \times \dot{R} + \mu r \times \dot{r}, \quad \mu = m_1 m_2 / M.}$$

The two terms are CM orbital angular momentum and internal angular momentum.

10. Impulsively launched rod

A horizontal uniform rod of length l and mass M is released from rest and immediately receives an upward impulse at one end, acquiring angular speed ω_0 about its centre of mass.

(a) Describe the translational and rotational motion.

(b) Find the centre-of-mass maximum height and time to reach it.

(c) If the rod returns to its original position after time t, show $t^2 = 2\pi n l/(3g)$ for an integer n.

Solution.

An upward impulse J at one end satisfies $Jl/2 = I\omega_0$, $I = Ml^2/12$, so CM speed is $V = J/M = l\omega_0/6$. The CM follows $y = Vt - gt^2/2$, while the rod rotates uniformly at ω_0. Therefore

$$\boxed{h_{\max} = l^2\omega_0^2/(72g), \quad t_{\max} = l\omega_0/(6g).}$$

Return to the original CM level takes $t = 2V/g = l\omega_0/(3g)$. For the labelled rod to recover its orientation also requires $\omega_0 t = 2\pi n$, yielding $\boxed{t^2 = 2\pi n l/(3g)}$. If swapping indistinguishable ends is allowed, half-turn returns are possible as well.

11. Rotating rod collision

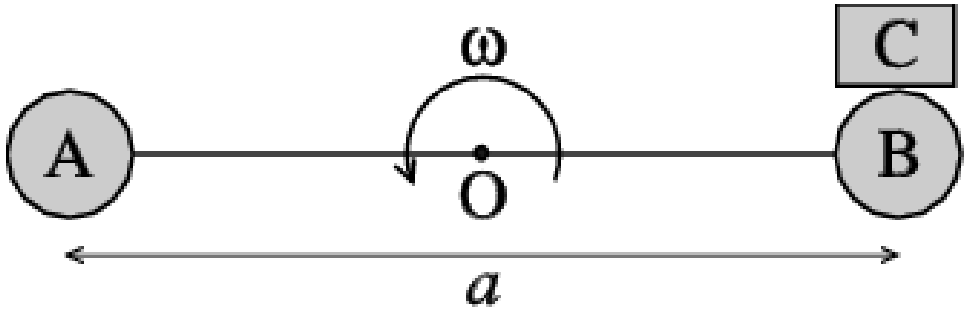
Point masses A, B, each M, are joined by a massless rod of length a and rotate at angular speed ω about their stationary midpoint O. Mass B strikes a stationary mass $C = M$ and sticks.

(a) Find the position and velocity of the three-body centre of mass O' immediately before impact.

(b) Calculate angular momentum about O' before and after impact, obtaining final angular speed $3\omega/4$.

(c) Show that the initial and final kinetic energies are $Ma^2\omega^2/4$ and $3Ma^2\omega^2/16$.

(d) Repeat for a perfectly elastic B–C collision, assuming no instantaneous impulse reaches A.



Solution.

Take A at $x = -a/2$, B and C at $x = a/2$, with $v_B = +a\omega\widehat{y}/2$. Before impact the three-body CM is $x_{O'} = a/6$, $V_{O'} = 0$. The total angular momentum about it is $L' = Ma^2\omega/2$, equal to its value about O because total linear momentum is zero. After sticking, distances from O' are $2a/3$ for A and $a/3$ for the double mass, giving $I_f = 2Ma^2/3$. Thus

$$\boxed{\omega_f = 3\omega/4, \quad K_i = Ma^2\omega^2/4, \quad K_f = 3Ma^2\omega^2/16.}$$

For the stipulated elastic collision, B stops and C takes B's initial velocity, while A's instantaneous velocity stays unchanged. The AB pair then has CM velocity $-a\omega\widehat{y}/4$ and angular velocity $\omega/2$; C moves at $+a\omega\widehat{y}/2$. The full system's kinetic energy is conserved.

1.7 Classical Mechanics: Sheet 6

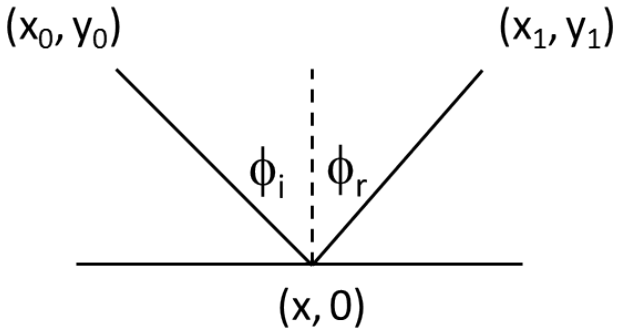
1. Fermat's principle

Light travels in the xy plane through refractive index $n(y)$.

(a) Use Fermat's principle to obtain the stationary travel-time functional

$$S[y] = c^{-1} \int_{x_0}^{x_1} n(y)\sqrt{1+y'^2}\, dx.$$

(b) For constant n, consider reflection from the plane mirror $y = 0$ between fixed endpoints. Show that the angles of incidence and reflection are equal.



Solution.

Travel time is

$$S = \int dt = c^{-1} \int n\, ds = c^{-1} \int n(y)\sqrt{1+y'^2}\, dx,$$

giving the specified functional. For a uniform medium reflected via point $(x, 0)$, differentiate the total path length $\sqrt{(x-x_0)^2+y_0^2} + \sqrt{(x_1-x)^2+y_1^2}$. Stationarity requires $(x-x_0)/r_0 = (x_1-x)/r_1$, i.e. $\sin\phi_i = \sin\phi_r$. Both physical angles lie in $[0, \pi/2]$, so $\boxed{\phi_i = \phi_r}$.

2. Polar Lagrangian and Hamiltonian

A particle of mass m moves in the plane under a position-dependent potential $V(\mathbf{r})$. Construct its Lagrangian and Hamiltonian in polar coordinates. Identify the conserved quantities, stating when angular momentum is conserved.

Solution.

For a central, time-independent potential,

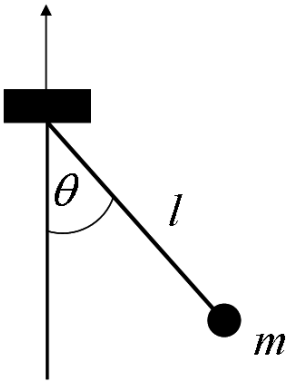
$$\boxed{L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \quad p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r).}$$

For a central potential, θ is cyclic and $p_\theta = J$ is conserved. For general $V(r,\theta)$, $\dot{p}_\theta = -\partial_\theta V$. In either case the time-independent Hamiltonian equals the conserved energy.

3. Pendulum with vertically accelerated support

(a) Use the Euler–Lagrange equation to obtain the small-angle period of a pendulum of length l and bob mass m.

(b) Find the period when its support accelerates vertically at signed upward acceleration a. For which a does the pendulum cease to oscillate about the downward position? Explain.

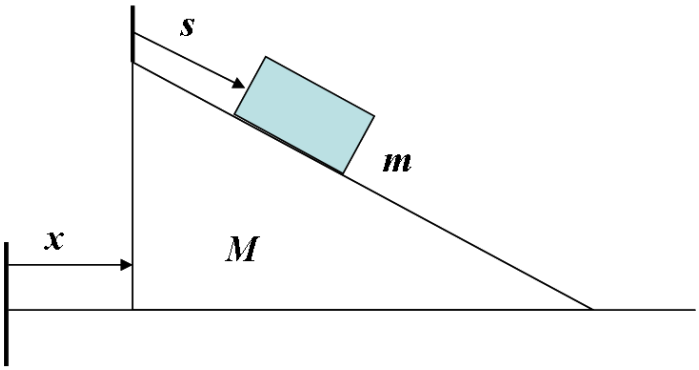


Solution.

For angle from downward vertical, $L = ml^2\dot{\theta}^2/2 + mgl\cos\theta$. Euler–Lagrange gives $\ddot{\theta} + (g/l)\sin\theta = 0$ and $\boxed{T = 2\pi\sqrt{l/g}}$. In a support frame accelerating upward at a, replace g by $g + a$, yielding $\boxed{T = 2\pi\sqrt{l/(g+a)}}$ for small oscillations about the downward position. At $a = -g$ the support is freely falling and the gravitational restoring term vanishes; for $a < -g$ the downward equilibrium is unstable.

4. Movable wedge by Euler–Lagrange

A block m slides on a smooth wedge M, itself on a smooth horizontal surface. Let x be the wedge displacement and s the block displacement relative to the wedge. Construct the Lagrangian, write both Euler–Lagrange equations and find the wedge acceleration.



Solution.

With s down the slope and x the wedge displacement,

$$L = \tfrac{1}{2}(M+m)\dot{x}^2 + m\dot{x}\dot{s}\cos\theta + \tfrac{1}{2}m\dot{s}^2 + mgs\sin\theta.$$

Euler–Lagrange gives $(M+m)\ddot{x} + m\ddot{s}\cos\theta = 0$ and $\ddot{s} + \ddot{x}\cos\theta = g\sin\theta$. Thus

$$\boxed{\ddot{x} = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}.}$$

This agrees with the Newtonian calculation in Mechanics Sheet 1, Q12.

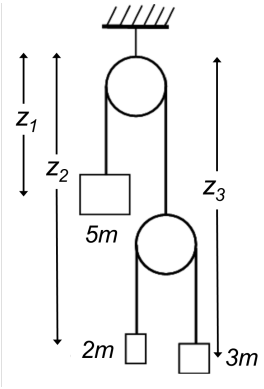
5. Compound Atwood machine

For the compound Atwood machine shown, with masses $5m, 2m, 3m$, the strings each have fixed length L, pulleys are massless and friction is negligible.

(a) Write the constraints in terms of z_1, z_2, z_3 and show that two generalised coordinates suffice.

(b) Use Euler–Lagrange equations to find all three accelerations.

(c) Find both string tensions and show that the upper is twice the lower. Explain why the upper pulley rotates although the masses on its two sides are equal.



Solution.

Take all z_i downward from the support. If the lower pulley is at z_p, string constraints are $z_1 + z_p = L$ and $z_2 + z_3 - 2z_p = L$, so $2z_1 + z_2 + z_3 = 3L$. Use z_1, z_2, with $z_3 = 3L - 2z_1 - z_2$. The Lagrangian is

$$L = \tfrac{1}{2}m(5\dot{z}_1^2 + 2\dot{z}_2^2 + 3\dot{z}_3^2) + mg(5z_1 + 2z_2 + 3z_3).$$

Its two Euler–Lagrange equations and the constraint give

$$\boxed{\ddot{z}_1 = g/49, \quad \ddot{z}_2 = -11g/49, \quad \ddot{z}_3 = 9g/49.}$$

From the individual force equations,

$$\boxed{T_{\text{lower}} = 120mg/49, \quad T_{\text{upper}} = 240mg/49}.$$

The massless lower pulley requires $T_{\text{upper}} = 2T_{\text{lower}}$. Equal total masses on the two upper sides do not imply balance, since the two lower masses also accelerate relative to their pulley.

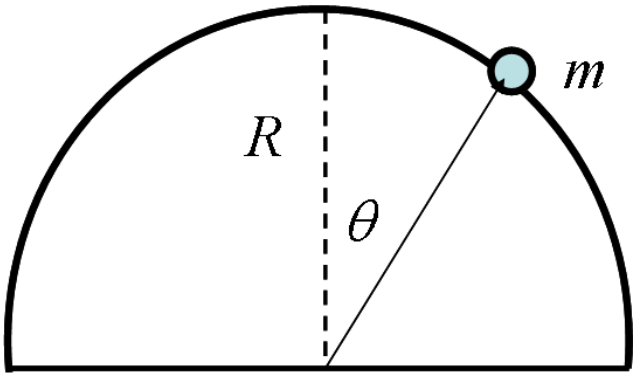
6. Sliding on a hemisphere

A mass m slides without friction on the outside of a hemisphere of radius R.

(a) Construct the constrained Lagrangian and equation of motion in polar coordinates.

(b) Release the radial constraint formally, adding contact potential $V(r)$, and derive the normal reaction $F_N = -\partial V/\partial r$ at $r = R$, $\dot{r} = \ddot{r} = 0$.

(c) For release from rest at the top, show that contact is lost at $\cos\phi_{\max} = 2/3$.



Solution.

With θ from upward vertical and contact $r = R$, $L = mR^2\dot{\theta}^2/2 - mgR\cos\theta$, yielding $\boxed{R\ddot{\theta} = g\sin\theta}$. To recover the reaction use

$$L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - mgr\cos\theta - V(r).$$

The radial equation at contact gives $\boxed{N = mg\cos\theta - mR\dot{\theta}^2}$. Release from the top gives $R\dot{\theta}^2 = 2g(1-\cos\theta)$, hence $N = mg(3\cos\theta - 2)$. Contact ends at $\boxed{\cos\theta = 2/3}$. An infinitesimal perturbation initiates motion from the exact unstable top.

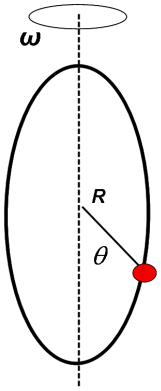
7. Bead on a rotating hoop

A bead m slides without friction on a vertical circular hoop of radius R, rotating about its vertical diameter at imposed angular speed ω. Use the bead angle ϕ from the vertical.

(a) Find the Lagrangian and equation of motion.

(b) Find the equilibria, their existence and stability, and small-oscillation frequencies about stable ones.

(c) Find total energy and Hamiltonian. Explain why the Hamiltonian is conserved while the bead's mechanical energy generally is not.



Solution.

Measure θ from the downward vertical. Then

$$L = \tfrac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + mgR\cos\theta, \quad \ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta.$$

The bottom is stable for $\omega^2 < g/R$, with frequency $\sqrt{g/R - \omega^2}$; the top is unstable. For $\omega^2 > g/R$, two additional stable equilibria satisfy $\cos\theta_* = g/(R\omega^2)$, with frequency $\omega|\sin\theta_*|$, while the bottom becomes unstable. At equality its linear frequency vanishes and the leading restoring term is cubic.

$$\boxed{H = \tfrac{1}{2}mR^2\dot{\theta}^2 - \tfrac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta = \text{constant}.}$$

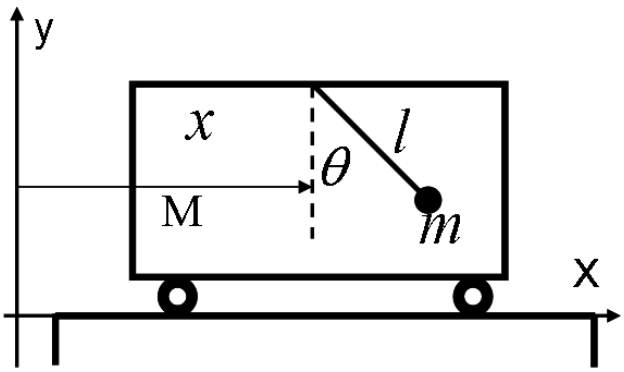
The mechanical energy instead contains the positive azimuthal kinetic term; the drive maintaining ω supplies or extracts work. Thus $H = E - \omega p_\phi$, rather than E.

8. Pendulum on a freely moving box

A box of mass M slides freely horizontally. A pendulum of length l and bob mass m hangs from its ceiling above its centre of mass. Its initial angular displacement is $\theta_0 \neq 0$.

(a) Find the Lagrangian and equations in x, the box position, and θ.

(b) Solve the small-angle motion and show that the box and bob oscillate about the common centre of mass at $\omega = \sqrt{(M+m)g/(Ml)}$. State any additional initial conditions used.



Solution.

The bob is at $(x + l\sin\theta, -l\cos\theta)$, so

$$L = \tfrac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

The equations are

$$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0$$

and

$$l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0.$$

Linearize and eliminate x to obtain

$$\boxed{\ddot{\theta} + \omega^2\theta = 0, \quad \omega^2 = \frac{M+m}{M}\frac{g}{l}.}$$

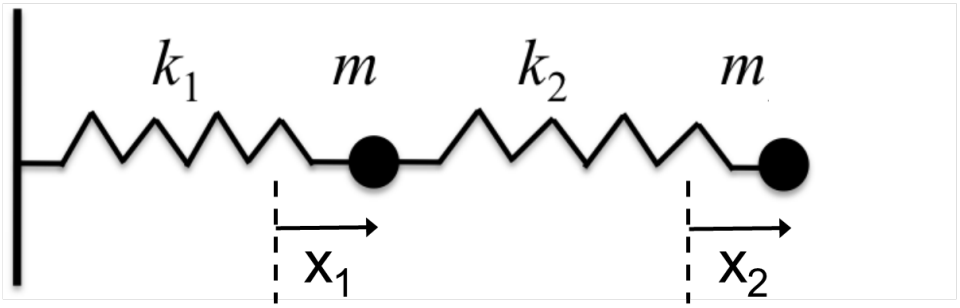
For release from rest, $\theta = \theta_0\cos\omega t$ and $x = x_0 + ml\theta_0(1-\cos\omega t)/(M+m)$. More general initial conditions add sine terms and uniform CM drift.

9. Two coupled springs

Two equal masses m move horizontally on a smooth table, connected to a wall by two springs as shown, with constants k_1, k_2.

(a) Write the Lagrangian in displacements x_1, x_2 and derive both equations.

(b) Find the normal-mode frequencies and amplitude ratios; give the frequency limits for $k_1 \ll k_2$ and $k_1 \gg k_2$.



Solution.

For the wall–k_1–mass–k_2–mass chain,

$$L = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}k_1 x_1^2 - \tfrac{1}{2}k_2(x_2 - x_1)^2.$$

Hence $m\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2$ and $m\ddot{x}_2 = k_2 x_1 - k_2 x_2$. The determinant of $K - m\omega^2 I$ gives

$$\boxed{\omega_\pm^2 = \frac{k_1 + 2k_2 \pm \sqrt{k_1^2 + 4k_2^2}}{2m}, \quad \frac{x_2}{x_1} = \frac{k_2}{k_2 - m\omega^2}.}$$

For $k_1 \ll k_2$, $\omega_-^2 \simeq k_1/(2m)$ and $\omega_+^2 \simeq 2k_2/m$; for $k_1 \gg k_2$, $\omega_-^2 \simeq k_2/m$ and $\omega_+^2 \simeq (k_1 + k_2)/m$.

10. Compound pendulum by variational dynamics

A uniform disc of mass m and radius a swings about a horizontal axis a distance x from its centre. Derive its small-oscillation period using the Euler–Lagrange equation.

Solution.

The pivot inertia is $I = m(a^2/2 + x^2)$, and $L = I\dot{\theta}^2/2 + mgx\cos\theta$. Euler–Lagrange gives $I\ddot{\theta} + mgx\sin\theta = 0$. For small angles,

$$\boxed{T = 2\pi\sqrt{(a^2/2 + x^2)/(gx)}.}$$

11. Cylinder rolling inside a cylinder

A uniform cylinder of radius a rolls without slipping inside a fixed cylinder of radius $b \gg a$. Use the Euler–Lagrange equation to show that its small-oscillation period equals that of a pendulum of length $3(b - a)/2$.

Solution.

The CM travels on radius $d = b - a$. No slip gives the cylinder's spin speed $|\dot{\phi}| = d|\dot{\theta}|/a$. Thus

$$T = md^2\dot{\theta}^2/2 + (ma^2/2)\dot{\phi}^2/2 = 3md^2\dot{\theta}^2/4,$$

and $U = mgd(1 - \cos\theta)$. Euler–Lagrange gives $\ddot{\theta} + 2g\sin\theta/(3d) = 0$, so

$$\boxed{P = 2\pi\sqrt{\frac{3(b-a)}{2g}}.}$$

This is the period of a simple pendulum of length $3(b - a)/2$; the derivation assumes $b > a$.

12. Unwinding reel by variational dynamics

A uniform cylindrical reel of radius r and mass m unwinds under gravity from a thread with fixed upper end. Using $I = mr^2/2$ and the Euler–Lagrange equation, find (a) its downward acceleration and (b) the string tension.

Solution.

Use downward displacement z and no slip $\phi = z/r$. The kinetic energy is

$$m\dot{z}^2/2 + I\dot{z}^2/(2r^2) = 3m\dot{z}^2/4.$$

Thus $L = 3m\dot{z}^2/4 + mgz$, and Euler–Lagrange gives $3m\ddot{z}/2 = mg$. Consequently

$$\boxed{\ddot{z} = 2g/3, \quad T = m(g - \ddot{z}) = mg/3}.$$

13. Sliding ladder

A uniform ladder of length l and mass m slides with its ends against a smooth wall and floor. Its angle to the vertical is θ; $I_{\mathrm{cm}} = ml^2/12$.

(a) Show that its centre of mass follows a circle about the wall–floor intersection.

(b) Find T, V.

(c) Form the one-coordinate Lagrangian and derive its equation of motion.

(d) Discuss conservation of energy, Hamiltonian and angular momentum while both contacts persist.

Solution.

With angle θ from the vertical, CM coordinates are $(l\sin\theta/2, l\cos\theta/2)$, a circle of radius $l/2$. Its translational and rotational energies sum to $T = ml^2\dot{\theta}^2/6$, and $U = mgl\cos\theta/2$. Thus

$$\boxed{L = ml^2\dot{\theta}^2/6 - mgl\cos\theta/2, \qquad \ddot{\theta} = \frac{3g}{2l}\sin\theta.}$$

While both contacts persist, the constraints are stationary and reactions do no work, so

$$\boxed{H = E = ml^2\dot{\theta}^2/6 + mgl\cos\theta/2}$$

is conserved. The angle is not cyclic, so its generalized momentum $ml^2\dot{\theta}/3$ is not conserved; gravity and contact reactions also prevent general angular-momentum conservation. The two-contact model ceases to apply once a required normal reaction becomes negative.

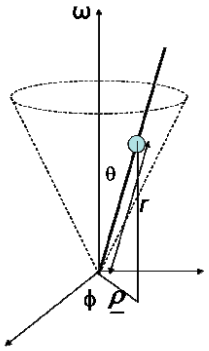
14. Bead on a rotating tilted wire

A bead m slides on a smooth straight wire inclined at fixed angle α to the upward vertical, rotating at imposed angular speed ω about that vertical. Its distance from the origin is r.

(a) Derive its equation of motion and check limiting inclinations.

(b) Solve with $r(0) = r_0$, $\dot{r}(0) = 0$. Show that the equilibrium $r_* = g\cos\alpha/(\omega^2\sin^2\alpha)$ gives circular motion, and discuss starts above or below it.

(c) Determine whether angular momentum, Hamiltonian and mechanical energy are conserved.



Solution.

For r increasing upward along the wire,

$$L = m(\dot{r}^2 + \omega^2 r^2 \sin^2\theta)/2 - mgr\cos\theta.$$

Thus

$$\boxed{\ddot{r} - \omega^2 \sin^2\theta\, r = -g\cos\theta.}$$

At $\theta = 0$ this reduces to vertical free fall; at $\theta = \pi/2$ it gives centrifugal outward acceleration. With $\Omega = \omega\sin\theta$ and $r_* = g\cos\theta/\Omega^2$, release from rest gives $\boxed{r = r_* + (r_0 - r_*)\cosh\Omega t}$. Equality $r_0 = r_*$ gives steady circular motion; larger initial radius moves outward and smaller moves inward. The autonomous Hamiltonian

$$H = m\dot{r}^2/2 - m\Omega^2 r^2/2 + mgr\cos\theta$$

is constant. The driven wire generally exchanges both mechanical energy and angular momentum with the bead.

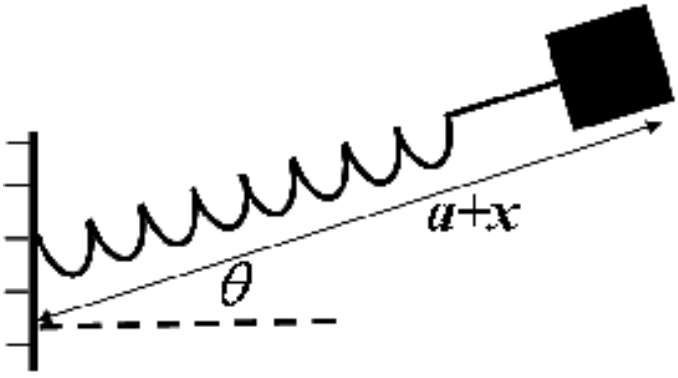
15. Two-dimensional spring

A mass m moves on a smooth horizontal plane, attached to a pivot by a massless Hooke-law spring of stiffness k and natural length a.

(a) Obtain the polar Euler–Lagrange equations, cyclic coordinate, conserved quantity and radial equation.

(b) Write the energy at fixed angular momentum J; use the effective potential to find the circular-orbit radius and compare with Newton's law.

(c) For $a \to 0$, find both semiaxes of the elliptical orbit in terms of E, J, m, k.



Solution.

The Lagrangian is

$$L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - k(r-a)^2/2.$$

Then $J = mr^2\dot{\theta}$ is conserved and

$$\boxed{m\ddot{r} = \frac{J^2}{mr^3} - k(r-a), \quad E = \tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{k}{2}(r-a)^2.}$$

Circular motion requires $k(r_c - a) = J^2/(mr_c^3)$, identical to centripetal force balance. For $a = 0$, turning radii obey $kr^4 - 2Er^2 + J^2/m = 0$, so the ellipse semiaxes are

$$\boxed{A, B = \sqrt{\frac{E}{k}\left(1 \pm \sqrt{1 - \frac{kJ^2}{mE^2}}\right)}.}$$

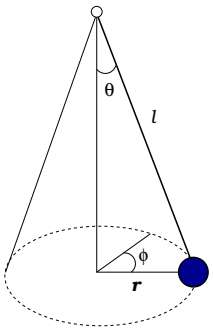
16. Spherical pendulum

A spherical pendulum consists of a bob m on a massless string of length l.

(a) Write its Lagrangian in polar angles θ, ϕ.

(b) Show that ϕ is cyclic, identify its conserved momentum and derive the θ equation. Show that steady circular motion at θ_0 has angular speed $\Omega^2 = g/(l\cos\theta_0)$.

(c) Release the length constraint formally to find the tension, and verify the circular-motion result from Newton's law.



Solution.

With θ from downward vertical,

$$L = \tfrac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mgl\cos\theta.$$

The cyclic coordinate ϕ gives $\boxed{J_z = ml^2\sin^2\theta\,\dot{\phi}}$. Euler–Lagrange in θ gives

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - (g/l)\sin\theta.$$

A conical motion at $\theta_0 \neq 0$ therefore has $\boxed{\Omega^2 = g/(l\cos\theta_0)}$. Relaxing the length constraint gives radial force balance and string tension

$$\boxed{T = mg\cos\theta + ml(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2).}$$

For conical motion this becomes $T = mg/\cos\theta_0 = ml\Omega^2$, as required by the Newtonian radial and vertical equations.

1.8 Special Relativity: Sheet 1

1. Simultaneity and time ordering

In frame S, event 1 has $(ct_1, x_1) = (x_0, x_0)$ and event 2 has $(ct_2, x_2) = (x_0/2, 2x_0)$, with $y = z = 0$.

(a) Find a frame sharing the spacetime origin in which the events are simultaneous, and their common time there.

(b) Prove that every spacelike pair is simultaneous in some inertial frame.

(c) Prove that its temporal order is reversed in some inertial frame.

Solution.

Use $ct' = \gamma(ct - \beta x)$. For the two events, simultaneous times require $1 - \beta = 1/2 - 2\beta$, giving $\boxed{\beta = -1/2,\ t' = \sqrt{3}x_0/c}$. More generally spacelike separation means $|c\Delta t| < |\Delta r|$. Boost along Δr with $v = c^2\Delta t/|\Delta r|$ to make $\Delta t' = 0$; the required speed is below c. Varying v slightly across this value reverses the sign of $\Delta t'$, so spacelike events have no invariant time order.

2. Superluminal light spot

A laser rotates on a turntable at $1200\,\mathrm{rev\,s^{-1}}$, with its beam at 30° above horizontal, illuminating clouds 50 km away. Find the spot speed and explain whether it conflicts with the speed-of-light limit. State how the quoted cloud distance is interpreted.

Solution.

The spot speed is $v_{\text{spot}} = \Omega R$, $\Omega = 2400\pi\,\mathrm{s^{-1}}$, where R is the horizontal radius of its sweep. If "50 km away" denotes the beam's line-of-sight distance, $R = 50\cos 30^\circ$ km and

$$\boxed{v_{\text{spot}} = 3.26 \times 10^8\,\mathrm{m\,s^{-1}} \simeq 1.09c}.$$

If it denotes cloud height, $R = 50/\tan 30^\circ$ km and the value is $2.18c$. Different cloud positions are illuminated by different photons; neither matter nor a signal travels sideways with the spot.

3. Delayed acknowledgement

Observers A, B have identical clocks. At their meeting, both read zero, and B moves away at $0.5c$. At time t on A's clock, A sends a light signal. After waiting $2t$ on her own clock from its reception, B returns a signal. At what time on A's clock does it arrive?

Solution.

Write T for A-frame time. B's worldline is $x = cT/2$. A's outgoing light ray, sent at $T = t$, is $x = c(T - t)$, meeting B at $T_1 = 2t$. B waits proper time $2t$, corresponding to $2\gamma t = 4t/\sqrt{3}$ in A's frame. The return is sent at $T_2 = 2t + 4t/\sqrt{3}$ from $x_2 = cT_2/2$. It reaches A at

$$\boxed{T_3 = T_2 + x_2/c = (3 + 2\sqrt{3})t.}$$

4. Cosmic-ray proper travel time

The Galaxy is 10^5 light-years across. A cosmic-ray proton has total energy $10^{19}\,\mathrm{eV}$ and rest energy $938\,\mathrm{MeV}$.

(a) Find its crossing time in the Galaxy frame and its own frame.

(b) Find the Galaxy's contracted width in the proton frame.

Solution.

The proton Lorentz factor is

$$\gamma = 10^{19}/(938 \times 10^6) = 1.066 \times 10^{10}$$

and $v \simeq c$. Across $D = 10^5$ light-years, the Galactic-frame time is $\boxed{t \simeq 10^5\,\mathrm{yr}}$. The proton's proper time is

$$\boxed{\tau = t/\gamma = 9.38 \times 10^{-6}\,\mathrm{yr} \simeq 296\,\mathrm{s}},$$

and the contracted Galactic width is

$$\boxed{D' = D/\gamma = 9.38 \times 10^{-6}\,\mathrm{ly} = 8.87 \times 10^{10}\,\mathrm{m}}.$$

5. Atmospheric muons

Muons produced in an atmospheric burst move towards the ground at $0.99c$. Their mean proper lifetime is $2.2 \times 10^{-6}\,\mathrm{s}$, and 1% survive. Estimate the burst altitude and its corresponding distance in the muon frame.

Solution.

The survival law is $N/N_0 = e^{-\tau/\tau_\mu}$. Thus $\tau = \tau_\mu \ln 100 = 10.13\,\mu\mathrm{s}$. At $\beta = 0.99$, $\gamma = 7.089$, so the lab flight time is $71.82\,\mu\mathrm{s}$ and

$$\boxed{h = \beta c\gamma\tau = 21.3\,\mathrm{km}, \qquad h' = h/\gamma = \beta c\tau = 3.01\,\mathrm{km}.}$$

6. Velocity boosts

(a) Compose collinear Lorentz boosts v_1, v_2 and derive $(v_1 + v_2)/(1 + v_1 v_2/c^2)$.

(b) An object moves at $(0.5c, 0, 0)$ in S. Find its velocity in S' moving at $(0.8c, 0, 0)$.

(c) Repeat for S' moving at $(0.8c/\sqrt{2}, 0.8c/\sqrt{2}, 0)$ with aligned spatial axes.

Solution.

(a) Compose $u' = (u - v_1)/(1 - uv_1/c^2)$ with a second boost to obtain combined speed

$$\boxed{v = (v_1 + v_2)/(1 + v_1 v_2/c^2)}.$$

(b)

$$\boxed{u'_x = (0.5 - 0.8)c/(1 - 0.4) = -0.5c}.$$

(c) Resolve u parallel and perpendicular to $V = 0.8c(1, 1)/\sqrt{2}$. With $u_\parallel = c/(2\sqrt{2})$ and $u_\perp = c/(2\sqrt{2})$, transform via

$$u'_\parallel = (u_\parallel - V)/(1 - Vu_\parallel/c^2)$$

and

$$u'_\perp = u_\perp/[\gamma_V(1 - Vu_\parallel/c^2)].$$

Returning to the aligned Cartesian axes gives

$$\boxed{u' = (-0.2310, -0.6493, 0)c}.$$

7. Radio signals and the twin journey

On 1 January 2030, astronaut A leaves Earth at $0.8c$ for a star 4 light-years away, turns around immediately and returns on 1 January 2040 (Earth time). Twin B stays on Earth. Each sends New Year greetings once per year of their own clock, through the return date.

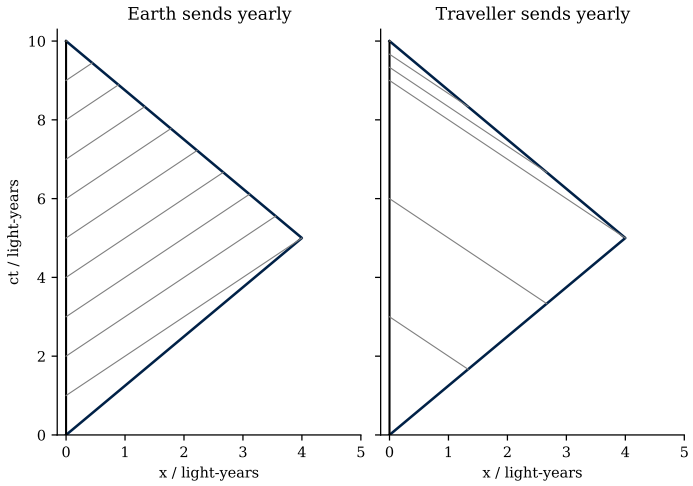
(a) Show that A sends six messages and B ten.

(b) Draw the Earth-frame spacetime diagram with all B's signals, verifying one reception by A before turnaround and nine afterwards.

(c) Draw all A's signals, verifying that B receives one every three Earth years for nine years, then three in the last year.

Solution.

In Earth coordinates (t, x) in years and light-years, the traveller's worldline is $x = 0.8t$ for $0 \le t \le 5$ and $x = 0.8(10 - t)$ afterwards. Proper time is t/γ on each leg, with $\gamma = 5/3$, so the traveller ages 6 years and Earth twin 10 years: they send 6 and 10 annual messages, respectively. An Earth message sent at integer year n has worldline $x = t - n$. On the outbound leg it arrives at $t = 5n$, so only $n = 1$ arrives by the turn. On return its intersection is $t = (8 + n)/1.8$, for $n = 2, \ldots, 10$. Traveller messages $k = 1, 2, 3$ are emitted at $t = 5k/3$, $x = 4k/3$ and arrive at Earth at $\boxed{t = 3, 6, 9}$. Messages $k = 4, 5, 6$ arrive at $\boxed{t = 8 + k/3 = 9\tfrac{1}{3}, 9\tfrac{2}{3}, 10}$. These line equations specify the complete spacetime diagrams, with all light lines at 45° when the axes are x, ct.



8. Rocket and station exchange

State the Lorentz transformation for a boost along x and derive time dilation and length contraction. A rocket passes Earth at $0.8c$ with both clocks reset to zero.

(a) After 30 rocket minutes it passes a station stationary relative to Earth. Find the Earth time.

(b) Find the Earth–station distance in both frames.

(c) The rocket sends a signal to Earth then; find its Earth reception time.

(d) Earth replies immediately; find the rocket reception time.

Solution.

The Lorentz transformation is $x' = \gamma(x - vt)$, $t' = \gamma(t - vx/c^2)$. Along one moving clock $\Delta x = v\Delta t$, giving $\Delta t = \gamma\Delta\tau$. Measuring a moving rod's endpoints at one lab time gives $L = L_0/\gamma$. For $v = 0.8c$, $\gamma = 5/3$. Thus

(a) $\boxed{t_E = 50\,\text{min}}$;

(b) station distance is

$$\boxed{\text{40 light-minutes in Earth coordinates, 24 light-minutes in rocket coordinates}}.$$

(c) The return signal reaches Earth at $50 + 40 = \boxed{90\,\text{min}}$.

(d) Earth's immediate reply obeys $x/c = t - 90$ and catches the rocket $x/c = 0.8t$ at $t = 450\,\text{min}$, or $\boxed{\tau_R = 270\,\text{min}}$.

E1. Wave equations and boosts

Consider $(c^{-2}\partial_t^2 - \partial_x^2)E_y = 0$.

(a) Verify $E_y = f(x - ct) + g(x + ct)$ for twice-differentiable f, g.

(b) Under $x' = x - vt$, $t' = t$ and $E'_y = E_y$, obtain

$$\left[c^{-2}\partial_{t'}^2 - (1 - v^2/c^2)\partial_{x'}^2 - 2vc^{-2}\partial_{x'}\partial_{t'}\right] E'_y = 0$$

and verify $f(x' - (c - v)t') + g(x' + (c + v)t')$.

(c) Show invariance under Lorentz boosts. Does the same hold for a wave speed below c?

(d) For

$$i\hbar\partial_t\psi = -\hbar^2\partial_x^2\psi/(2m),$$

verify

$$\psi = Ae^{ikx - i\omega t} + Be^{-ikx - i\omega t}$$

and find the dispersion relation. Show Galilean covariance using

$$\psi' = \psi e^{-imvx'/\hbar}e^{-imv^2t'/(2\hbar)}.$$

Solution.

(a) For $F = f(x - ct) + g(x + ct)$, $F_{tt} = c^2F_{xx}$ directly.

(b) The Galilean chain rule gives $\partial_t = \partial_{t'} - v\partial_{x'}$, $\partial_x = \partial_{x'}$. Substitution produces the stated mixed-derivative equation; its characteristics have speeds $c - v$ and $-(c + v)$, giving the stated transformed solution.

(c) Under a Lorentz boost,

$$\partial_t = \gamma(\partial_{t'} - v\partial_{x'})$$

and

$$\partial_x = \gamma(\partial_{x'} - v\partial_{t'}/c^2).$$

Cross terms cancel in $c^{-2}\partial_t^2 - \partial_x^2$, so this operator is invariant. A medium wave equation with speed below c is not invariant under the same vacuum Lorentz boosts without transforming its medium.

(d) Substitution of $e^{\pm ikx - i\omega t}$ gives $\boxed{\omega = \hbar k^2/(2m)}$. Insert

$$\psi' = e^{-imvx'/\hbar - imv^2t'/(2\hbar)}\psi(x' + vt', t');$$

differentiating its phase cancels the Galilean mixed first-derivative and constant terms, leaving the same Schrödinger equation.

E2. Reverse triangle inequality

For future-directed timelike four-vectors U, V with metric signature $(+---)$, prove that $U+V$ is future-directed and

$$\sqrt{(U+V)\cdot(U+V)} \geq \sqrt{U\cdot U} + \sqrt{V\cdot V}.$$

You may decompose $V = (U\cdot V)U/(U\cdot U) + W$ with $W\cdot U = 0$ and derive the reversed Schwarz inequality.

Solution.

Use metric $(+---)$. In the rest frame of future-timelike U, $U\cdot V = \sqrt{U^2}\,V^0 \geq \sqrt{U^2V^2}$. Hence

$$(U+V)^2 = U^2 + V^2 + 2U\cdot V \geq (\sqrt{U^2} + \sqrt{V^2})^2 > 0.$$

Its time component is positive, so $U+V$ is future-timelike. Taking positive square roots proves

$$\boxed{\sqrt{(U+V)^2} \geq \sqrt{U^2} + \sqrt{V^2}},$$

with equality only when U, V are positive multiples of each other.

E3. Twins in a periodic universe

In a spatially periodic universe, one twin remains on Earth while the other travels inertially around the universe and returns from the opposite direction without accelerating. Which twin is older on reunion? Explain how the global spacetime identification resolves the apparent symmetry.

Solution.

For a spatial identification $x \sim x + L$ at equal time in the Earth's global rest frame, Earth reunion time is $t = L/v$ while the winding twin ages $\tau = t/\gamma$. Thus $\boxed{\text{the traveller is younger}}$. A boost changes the identification to

$$(\Delta t', \Delta x') = (-\gamma v L/c^2, \gamma L),$$

so the spatially periodic topology singles out a global rest frame; the twins are not interchangeable globally even though neither accelerates.

E4. Universal limiting-speed derivation

Frame S' moves at velocity v along x relative to S, with parallel axes and coincident origins at $t = t' = 0$.

(a) Use homogeneity and isotropy to motivate a linear transformation $t' = at + bx$, $x' = ct + dx$, $y' = y$, $z' = z$.

(b) Show it can be written $x' = f(v^2)(x - vt)$, $t' = g(v^2)t - vh(v^2)x$, and write its inverse.

(c) Show inverse consistency implies $f = g$ and $f^2 - v^2 fh = 1$.

(d) Derive $u = (u' + v)/(1 + vu'h/f)$. Use a universal limiting speed C to show $h = f/C^2$ and recover the Lorentz transformation.

Solution.

Homogeneity makes coordinate differences transform independently of origin, forcing a linear map; isotropy removes preferred transverse directions. The moving origin $x = vt$ gives $x' = f(x - vt)$. Reflection $x \to -x, v \to -v$ makes f, g, h functions of v^2, with $t' = gt - vhx$. Reciprocity gives the inverse with $v \to -v$. Multiplying the two matrices yields $\boxed{f = g,\ f^2 - v^2 fh = 1}$. A velocity transforms as $u = (u' + v)/[1 + vu'(h/f)]$. Requiring $u' = C$ to imply $u = C$ gives $h/f = 1/C^2$. Thus

$$\boxed{x' = \frac{x - vt}{\sqrt{1 - v^2/C^2}}, \quad t' = \frac{t - vx/C^2}{\sqrt{1 - v^2/C^2}}.}$$

1.9 Special Relativity: Sheet 2

1. Two-body meson decay

A stationary K^0 meson of rest mass $498\,\mathrm{MeV}/c^2$ decays into $\pi^+\pi^-$, each of mass $140\,\mathrm{MeV}/c^2$. Show that each pion has speed approximately $0.83c$ in the kaon rest frame.

Solution.

In the kaon's rest frame the pion momenta are equal and opposite, and equal pion masses give equal energies $E_\pi = 498/2 = 249\,\mathrm{MeV}$. Hence $\gamma_\pi = 249/140$ and

$$\boxed{v/c = \sqrt{1-(140/249)^2} = 0.827 \simeq 0.83}.$$

2. Solar mass loss

Solar radiation at the Earth, $1.50\times10^{11}\,\mathrm{m}$ from the Sun, has flux $1400\,\mathrm{W\,m^{-2}}$ above the atmosphere. Find the Sun's mass-loss rate due to radiation.

Solution.

Assuming isotropic radiation,

$$P = 4\pi r^2 I = 4\pi(1.50\times10^{11})^2(1400) = 3.96\times10^{26}\,\mathrm{W}.$$

Mass-energy equivalence gives

$$\boxed{-\dot{M} = P/c^2 = 4.40\times10^{9}\,\mathrm{kg\,s^{-1}}}.$$

3. A 50 GeV electron

A SLAC electron has total energy 50 GeV. Find (a) its kinetic-energy fraction, (b) its three-momentum magnitude and (c) its speed.

Solution.

With $mc^2 = 0.510999\,\mathrm{MeV}$ and $E = 50000\,\mathrm{MeV}$,

$$\boxed{K/E = 1 - mc^2/E = 0.99998978, \quad |p|c = \sqrt{E^2 - m^2c^4} \simeq 50\,\mathrm{GeV}.}$$

The momentum vector points along the electron beam. Its speed is

$$\boxed{v/c = \sqrt{1-(mc^2/E)^2} \simeq 1 - 5.22\times10^{-11}}.$$

4. Reaction thresholds

(a) Find the incident-proton threshold energy for $p + p \to p + p + \pi^0$ with a stationary target, given rest masses $m_p c^2 = 938\,\mathrm{MeV}$ and $m_{\pi^0}c^2 = 135\,\mathrm{MeV}$. Distinguish total and kinetic threshold energy.

(b) Find the incident-electron threshold kinetic energy for $e^- + e^- \to e^- + e^- + e^+ + e^-$ with a stationary target electron.

Solution.

At threshold all final particles are at rest relative to each other in the CM frame, minimizing invariant mass.

(a) For a proton on a stationary proton,

$$s = 2m_p^2c^4 + 2m_pc^2E_{\mathrm{beam}} = (2m_p + m_\pi)^2c^4.$$

Thus

$$\boxed{K_{\mathrm{th}} = 2m_\pi c^2 + m_\pi^2 c^2/(2m_p) = 279.7\,\mathrm{MeV}, \quad E_{\mathrm{th}} = 1217.7\,\mathrm{MeV}.}$$

(b) For four final electron masses, $2m_e^2c^4 + 2m_ec^2E = 16m_e^2c^4$, so $E = 7m_ec^2$ and

$$\boxed{K_{\mathrm{th}} = 6m_ec^2 = 3.066\,\mathrm{MeV}}.$$

5. Relativistic sticking collision

A particle of mass m and speed $4c/5$ strikes an identical stationary particle and sticks to form one composite. Find the composite rest mass and speed.

Solution.

The incoming Lorentz factor is $5/3$. Initial energy and momentum are $E = 8mc^2/3$ and $p = 4mc/3$. Hence $\boxed{v_C = pc^2/E = c/2}$ and

$$\boxed{m_C = \sqrt{E^2 - p^2c^2}/c^2 = 4m/\sqrt{3}.}$$

The composite's increased rest mass contains the internal energy generated in the inelastic collision.

6. Single-photon emission

(a) Show that a free electron in vacuum cannot emit one photon.

(b) Explain why an excited hydrogen atom can emit one photon.

Solution.

Let $P = P' + K$ with $P^2 = P'^2 = m^2c^2$, $K^2 = 0$. Then $0 = 2P' \cdot K$, impossible for nonzero future-directed photon momentum and a future-timelike electron momentum. Thus a free electron cannot emit one photon while retaining its rest mass. An excited atom can emit because its initial invariant mass exceeds its ground-state invariant mass; the mass difference supplies both photon and recoil energies.

7. Atomic recoil

An atom has ground-state mass M and excited-state mass $M' = M + Q_0/c^2$.

(a) A stationary excited atom emits a photon and reaches its ground state. Find photon energy Q_1 and its first recoil correction for $Q_0 \ll Mc^2$.

(b) An excited atom moving towards a detector emits a forward photon and comes completely to rest. Find the detected energy Q_2.

(c) What photon energy can a stationary ground-state atom absorb?

(d) Explain enhanced resonant reabsorption at high temperature in gases and at low temperature in some solids (the Mössbauer effect).

Solution.

Write $M' = M + Q_0/c^2$.

(a) Four-momentum conservation for emission from rest gives

$$\boxed{Q_1 = \frac{(M'^2 - M^2)c^2}{2M'} = Q_0\left[1 - \frac{Q_0}{2Mc^2} + O\left(\frac{Q_0^2}{M^2c^4}\right)\right].}$$

(b) If the final atom is at rest, $Q_2 = \gamma M' vc = \gamma M'c^2 - Mc^2$. This requires the compatible speed $\beta = (M'^2 - M^2)/(M'^2 + M^2)$ and gives

$$\boxed{Q_2 = (M'^2 - M^2)c^2/(2M) = Q_0[1 + Q_0/(2Mc^2)]}.$$

(c) A stationary ground-state atom absorbs a photon with $\boxed{h\nu = Q_0[1 + Q_0/(2Mc^2)]}$, the recoil-shifted absorption energy.

(d) Thermal Doppler broadening in gases increases overlap of the separated emission and absorption lines. In a solid, recoil can be taken by the whole lattice with negligible recoil energy; lowering temperature increases the fraction of recoilless events, enhancing the Mössbauer resonance.

8. Relativistic Doppler shift

A source of proper frequency ν_0 recedes at speed βc. Show that the received frequency for a receding, longitudinal signal is $\nu = \nu_0\gamma(1-\beta)$. A Ca^+ line of rest wavelength 394 nm is seen at 475 nm in Hydra. Infer its radial motion relative to Earth.

Solution.

A receding source observed behind its motion has

$$\boxed{\nu/\nu_0 = \gamma(1-\beta) = \sqrt{(1-\beta)/(1+\beta)}}.$$

From $\lambda/\lambda_0 = 475/394$,

$$\boxed{\beta = [(475/394)^2 - 1]/[(475/394)^2 + 1] = 0.1848}:$$

Hydra is receding in the line-of-sight kinematic interpretation.

9. Two collinear photons

A neutral pion of rest mass $135\,\mathrm{MeV}/c^2$ and kinetic energy 1 GeV decays into two photons along its line of motion, one forward and one backward. Find their energies.

Solution.

The pion's total energy is $E = 1135\,\mathrm{MeV}$ and momentum satisfies $pc = \sqrt{E^2 - 135^2}\,\mathrm{MeV}$. The photons must travel in opposite directions along its line of motion. Conservation gives $E_+ + E_- = E$ and $E_+ - E_- = pc$, so

$$\boxed{E_\pm = \frac{1}{2}\left(1135 \pm \sqrt{1135^2 - 135^2}\right)\,\mathrm{MeV} \simeq 1130.97,\ 4.03\,\mathrm{MeV}.}$$

E1. Graviton mass estimate

A gravitational-wave signal from a merger 1 Gpc away has frequencies of order 100 Hz and duration about 1 s, with no large distortion from the predicted waveform. Using $E = h\nu$ for gravitons, estimate an upper bound on a hypothetical graviton rest mass from frequency-dependent propagation.

Solution.

For $mc^2 \ll E = hf$, $v/c \simeq 1 - m^2c^4/(2E^2)$. A propagation delay or dispersion over distance D is of order $\Delta t \sim (D/2c)(mc^2/E)^2$. Requiring it below the roughly one-second signal structure gives

$$\boxed{mc^2 \lesssim hf\sqrt{2c\Delta t/D} \simeq 1.8\times10^{-21}\,\mathrm{eV}, \quad m \lesssim 3.3\times10^{-57}\,\mathrm{kg}.}$$

This is the order-of-magnitude bound from the stated duration and frequency, rather than a fit to the detector waveform.

E2. Uniform acceleration and a horizon

An observer with constant proper acceleration a starting at (ct_0, x_0) follows

$$ct = ct_0 + \frac{c^2}{a}\sinh(a\tau/c), \qquad x = x_0 + \frac{c^2}{a}[\cosh(a\tau/c) - 1].$$

(a) Sketch the worldline and identify the region from which no signal can ever be received.

(b) Sketch two such observers starting at rest at $t = 0$, at $x_0 = c^2/a$ and $x_0 = L + c^2/a$, with equal a.

(c) A right-moving photon of initial-frame frequency ν passes first the trailing, then the leading observer. Use $U = c(\cosh\eta, \sinh\eta, 0, 0)$ and $x - ct$ to show that the trailing-minus-leading measured photon energy is $\Delta E = h\nu aL/c^2$.

Solution.

Put $a = a_{\text{inst}}$ and $\psi = a\tau/c$. The trajectory obeys $[x - x_0 + c^2/a]^2 - c^2(t - t_0)^2 = c^4/a^2$, a right-hand hyperbola asymptotic to null lines. Its future event horizon is $x - ct = x_0 - ct_0 - c^2/a$: events on or beyond the inaccessible side cannot send it a future light ray. The two specified worldlines are identical hyperbolae translated by L along x. For a photon travelling along $+x$, $x - ct$ is constant. At its two intersections,

$$\frac{c^2}{a}e^{-\psi_T} = L + \frac{c^2}{a}e^{-\psi_L}.$$

The measured photon energy is $E' = h\nu\gamma(1 - \beta) = h\nu e^{-\psi}$. Thus $\boxed{E'_T - E'_L = h\nu aL/c^2}$, whenever the ray reaches both observers.

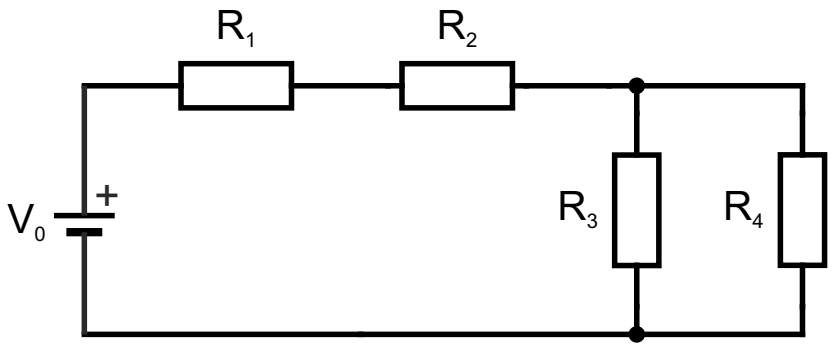
2 CP2: Electricity, Magnetism and Optics

2.1 Circuit Theory: Problems

Resistor circuits

1. Series and parallel resistors

For the circuit shown, find the current and voltage drop of every resistor, given $V_0 = 12\,\mathrm{V}$, $R_1 = 1\,\mathrm{k\Omega}$, $R_2 = R_3 = 3\,\mathrm{k\Omega}$ and $R_4 = 6\,\mathrm{k\Omega}$.



Solution.

$R_3 \parallel R_4 = 2\,\mathrm{k\Omega}$, giving total $R = 6\,\mathrm{k\Omega}$ and source current $2\,\mathrm{mA}$. Hence

	R_1	R_2	R_3	R_4
I (mA)	2	2	4/3	2/3
V (V)	2	6	4	4

All resistor currents run from the positive source terminal back towards its negative terminal; each voltage is the drop in that direction.

2. Opposite cube corners

A cube has resistance R on every edge. Find its equivalent resistance between opposite vertices.

Solution.

Symmetry makes the three vertices neighbouring each terminal equipotential. The cube reduces to three resistors in parallel, then six in parallel, then three in parallel, in series:

$$\boxed{R_{\mathrm{eq}} = R/3 + R/6 + R/3 = 5R/6.}$$

3. Other cube terminals

For the resistor cube of Question 2, find the resistance (a) between adjacent vertices and (b) between opposite vertices of one face.

Solution.

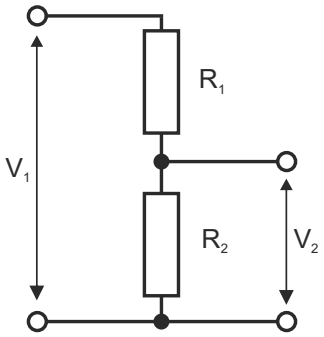
Ground one terminal, inject current I at the other, and apply $\sum_{j\sim i}(V_i - V_j)/R = I_i$ at each vertex. Solving these seven independent node equations gives

$$\boxed{R_{\text{adjacent}} = 7R/12, \qquad R_{\text{face diagonal}} = 3R/4.}$$

One reproducible evaluation is to label vertices by binary triples, connect pairs differing in one bit, and solve the cube graph Laplacian with unit edge conductance. The terminal voltage differences for unit injection are $7/12$ for $000 \to 001$ and $3/4$ for $000 \to 011$, multiplied by R.

4. Loaded voltage divider

A divider has $R_1 = 1\,\text{k}\Omega$, $R_2 = 4\,\text{k}\Omega$, input $V_1 = 5\,\text{V}$ and output across R_2. Find the unloaded output. A load R_L is connected in parallel with R_2; find the minimum R_L giving at most 5% output change.



Solution.

Unloaded, $V_2 = 5 \cdot 4/(1+4) = 4\,\text{V}$. Its Thevenin resistance is

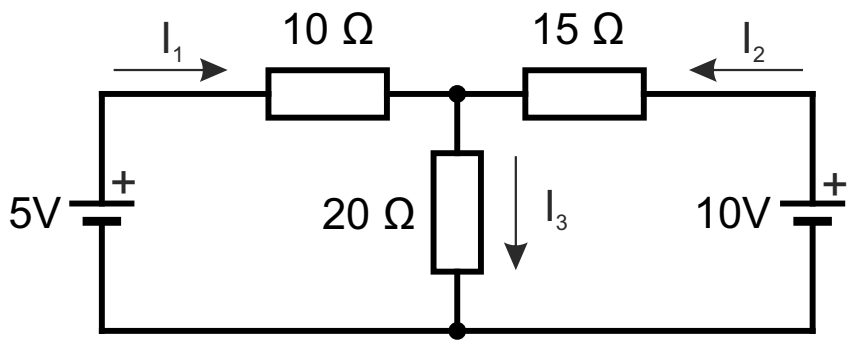
$$R_T = R_1 \parallel R_2 = 0.8\,\text{k}\Omega.$$

With load, $V_2' = 4R_L/(R_L + R_T)$. Requiring $V_2' \geq 0.95(4)$ gives

$$\boxed{R_L \geq 19R_T = 15.2\,\text{k}\Omega.}$$

5. Two sources and three resistors

Find the indicated currents I_1, I_2, I_3 in the circuit shown, using mesh currents or otherwise. Retain the source polarities and current directions indicated in the figure.



Solution.

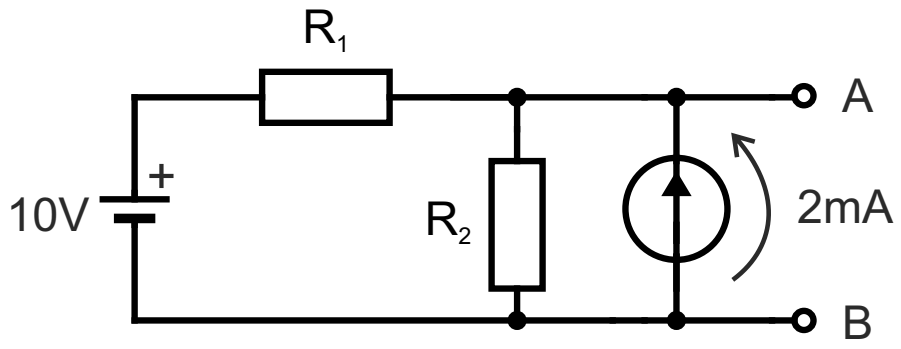
Take the lower node as zero and the central upper node as V. Kirchhoff's current law gives $(V-5)/10 + (V-10)/15 + V/20 = 0$, so $V = 70/13\,\mathrm{V}$. In the indicated arrow directions,

$$\boxed{I_1 = -1/26\,\mathrm{A}, \qquad I_2 = 4/13\,\mathrm{A}, \qquad I_3 = 7/26\,\mathrm{A}.}$$

The first negative sign means the actual current is opposite its arrow; $I_1 + I_2 = I_3$ checks the result.

6. Thevenin and Norton circuits

For the circuit shown with $R_1 = R_2 = 1\,\mathrm{k\Omega}$, find V_{AB} and the resistor currents. Find the Thévenin voltage and resistance at A, B, and the equivalent Norton current and resistance.



Solution.

Node A obeys

$$(10 - V_A)/(1\,\mathrm{k\Omega}) + 2\,\mathrm{mA} = V_A/(1\,\mathrm{k\Omega}),$$

giving

$$\boxed{V_{AB} = 6\,\mathrm{V},\ I_{R_1} = 4\,\mathrm{mA} \text{ to the right},\ I_{R_2} = 6\,\mathrm{mA} \text{ downward}}.$$

Killing independent sources means shorting the voltage source and opening the current source, so $R_T = R_1 \parallel R_2 = 500\,\Omega$. Thus

$$\boxed{V_T = 6\,\mathrm{V},\ R_T = 500\,\Omega}$$

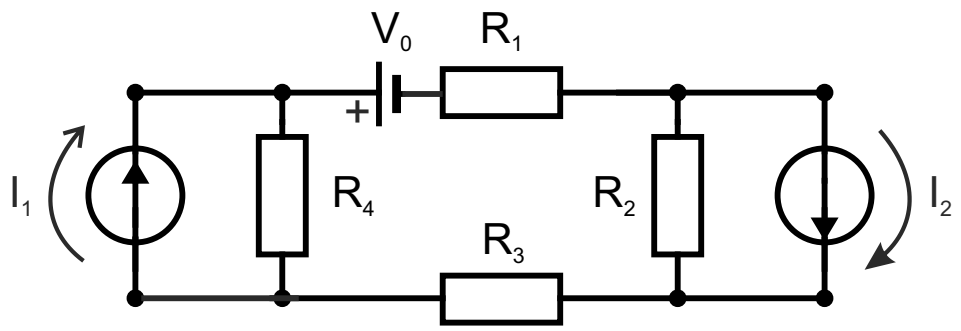
and the Norton equivalent is

$$\boxed{I_N = 12\,\mathrm{mA},\ R_N = 500\,\Omega},$$

with source current towards A.

7. Network with two current sources

Find the magnitudes and directions of the currents through R_1–R_4 in the circuit shown, with $V_0 = 2\,\mathrm{V}$, $I_1 = 1\,\mathrm{mA}$, $I_2 = 4\,\mathrm{mA}$, $R_1 = 1\,\mathrm{k\Omega}$, $R_2 = 1.5\,\mathrm{k\Omega}$, $R_3 = 0.5\,\mathrm{k\Omega}$ and $R_4 = 2\,\mathrm{k\Omega}$.



Solution.

Let the lower-left node be zero and the upper-left, upper-right, lower-right node potentials be A, B, C. The series voltage source makes the left end of R_1 equal to $A - 2\,\mathrm{V}$. Kirchhoff's equations are

$$(A-2-B)/1000 + A/2000 = 0.001,$$
$$(B-A+2)/1000 + (B-C)/1500 = -0.004,$$
$$C/500 + (C-B)/1500 = 0.004.$$

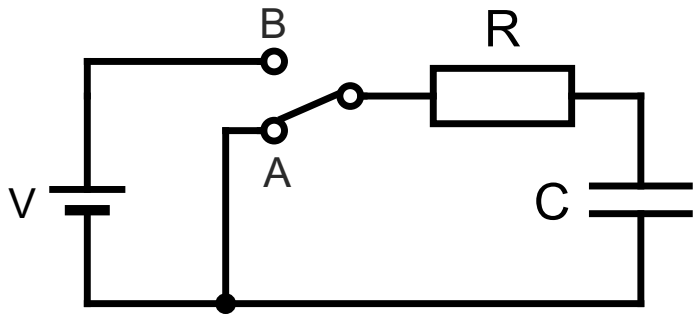
They give $(A, B, C) = (-0.4, -3.6, 0.6)\,\mathrm{V}$. Thus

$$\boxed{I_{R_1} = 1.2\,\mathrm{mA}\text{ right},\ I_{R_2} = 2.8\,\mathrm{mA}\text{ up},\ I_{R_3} = 1.2\,\mathrm{mA}\text{ left},\ I_{R_4} = 0.2\,\mathrm{mA}\text{ up}}.$$

Transients

8. RC charging and energy

The capacitor in the shown RC circuit is initially uncharged. At $t = 0$ the switch moves from A to B. Derive the resistor current. Integrate its dissipation to find the total heat, find the final capacitor energy and show that the battery supplies CV^2.



Solution.

Kirchhoff's voltage law gives $Ri + q/C = V$, $i = \dot{q}$, $q(0) = 0$. Therefore

$$\boxed{i = (V/R)e^{-t/(RC)},\ q = CV(1 - e^{-t/(RC)})}.$$

Direct integration gives $W_R = \int_0^\infty Ri^2 dt = CV^2/2$. Final capacitor energy is

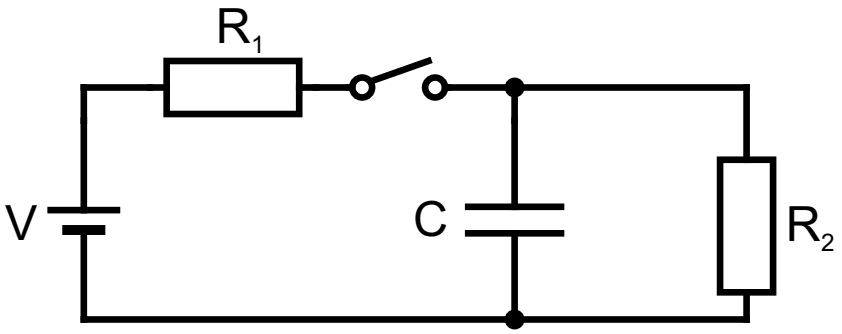
$$\boxed{U_C = CV^2/2}$$

and battery work is

$$\int V i\, dt = Vq(\infty) = \boxed{CV^2}.$$

9. Loaded RC charging

An initially uncharged capacitor is connected by closing the shown switch at $t = 0$. Derive $V_C = V_0(1-e^{-t/\tau})$. Find V_0, τ for $V = 6\,\mathrm{V}$, $R_1 = 10\,\mathrm{k\Omega}$, $R_2 = 22\,\mathrm{k\Omega}$, $C = 20\,\mu\mathrm{F}$. Find the steady resistor powers and capacitor energy.



Solution.

Seen by the capacitor, $V_T = V R_2/(R_1 + R_2)$ and $R_T = R_1 \parallel R_2$. Thus $\boxed{V_C = V_T(1 - e^{-t/(R_T C)})}$. Numerically

$$\boxed{V_T = 4.125\,\mathrm{V},\ \tau = 0.1375\,\mathrm{s}}.$$

At steady state

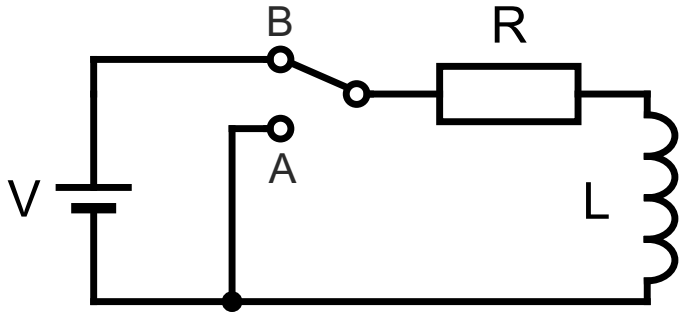
$$I = 6/(32\,\mathrm{k\Omega}) = 0.1875\,\mathrm{mA}.$$

Hence

$$\boxed{P_1 = 0.3516\,\mathrm{mW},\ P_2 = 0.7734\,\mathrm{mW},\ U_C = 0.1702\,\mathrm{mJ}}.$$

10. RL switch-on

The shown RL circuit is initially in its steady state with switch at A. At $t = 0$ the switch moves to B. Derive the current through R as a function of time.



Solution.

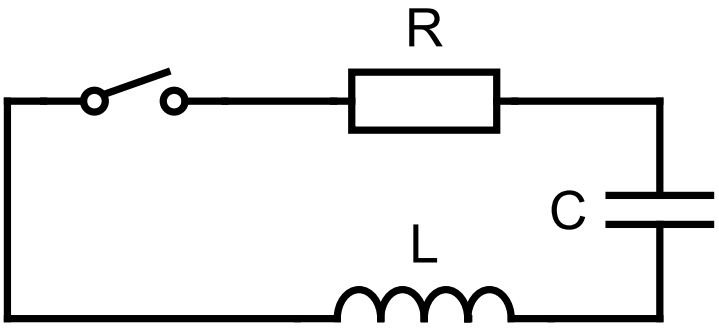
The inductor current starts at zero and obeys $L\dot{i} + Ri = V$. The solution is

$$\boxed{i(t) = \frac{V}{R}(1 - e^{-Rt/L}), \qquad \tau = L/R.}$$

The current and voltage polarities are chosen in the charging loop direction.

11. Damped series RLC discharge

In the shown series RLC circuit, the switch is initially open and the capacitor voltage is 1 V. At $t = 0$ the switch closes. Derive and sketch the damped oscillatory response quantitatively for $R = 150\,\Omega$, $L = 10\,\text{mH}$, $C = 10\,\text{nF}$.



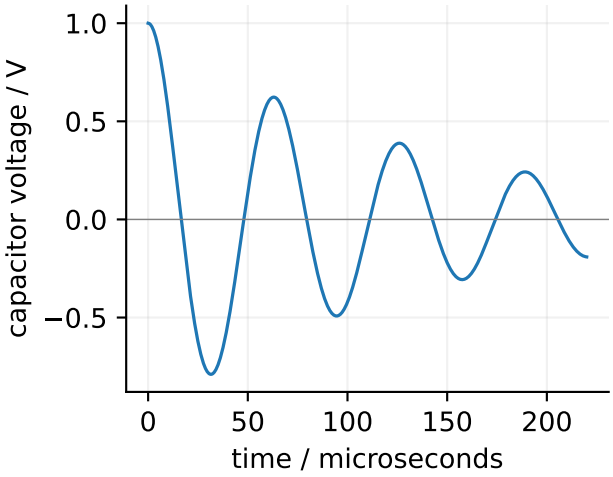
Solution.

The capacitor voltage obeys $LC\ddot{V}_C + RC\dot{V}_C + V_C = 0$. Initial voltage is 1 V and initial inductor current is zero, so $\dot{V}_C(0) = 0$. With $\alpha = R/(2L) = 7500\,\text{s}^{-1}$, $\omega_0 = 10^5\,\text{s}^{-1}$ and

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 9.972 \times 10^4\,\text{s}^{-1},$$

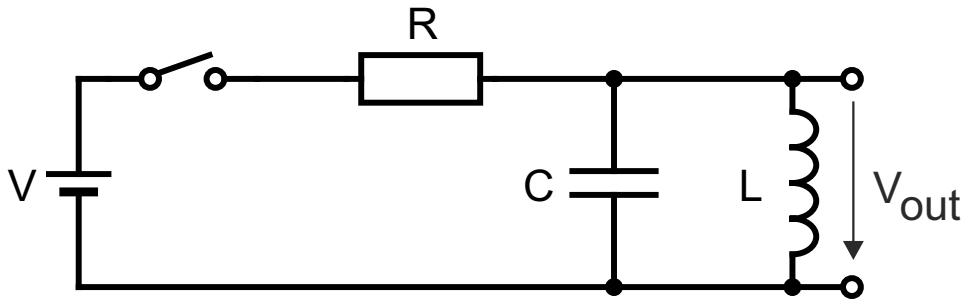
$$\boxed{V_C = e^{-7500t}\left[\cos(\omega_d t) + \frac{7500}{\omega_d}\sin(\omega_d t)\right]\ \text{V}.}$$

The envelope decays on 133.3 μs, with oscillation period 63.0 μs. The discharge current is $i = e^{-\alpha t}\sin\omega_d t/(L\omega_d)$, about 1.003 mA times the decaying sine.



12. Parallel LC with a feed resistor

The switch in the shown RLC network closes at $t = 0$. Solve for V_{out} and show that its oscillation envelope has decay time $2RC$. For $R = 100\,\text{k}\Omega$, $C = 10\,\text{nF}$, $L = 4\,\text{mH}$, verify $\omega_0 \simeq 1.6 \times 10^5\,\text{rad}\,\text{s}^{-1}$.



Solution.

Kirchhoff's law gives $(V - v)/R = C\dot{v} + i_L$, $L\dot{i}_L = v$. Differentiating gives $\ddot{v} + \dot{v}/(RC) + v/(LC) = 0$. For initially unenergized components, $v(0) = 0$ and $\dot{v}(0) = V/(RC)$. Thus

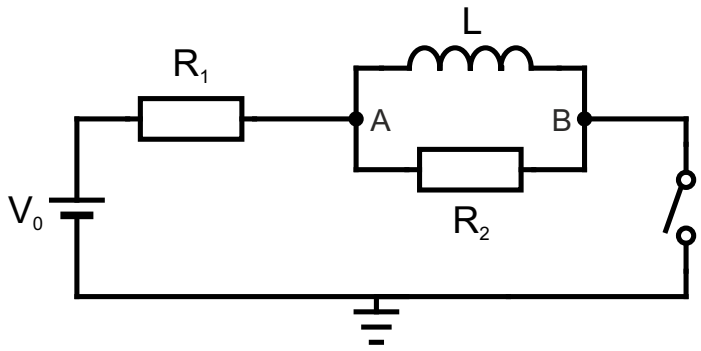
$$\boxed{v(t) = \frac{V}{RC\omega_d} e^{-t/(2RC)} \sin \omega_d t, \quad \omega_d^2 = 1/(LC) - 1/(4R^2C^2).}$$

The envelope time is $\boxed{2RC = 2.0\,\text{ms}}$, while

$$\boxed{\omega_0 = 1/\sqrt{LC} = 1.581 \times 10^5\,\text{rad}\,\text{s}^{-1} \simeq 1.6 \times 10^5}.$$

13. Inductor flyback

The shown switch has been closed for a long time, then opens. With $V_0 = 10\,\text{V}$, $R_1 = 100\,\Omega$, $R_2 = 10\,\text{k}\Omega$, $L = 100\,\text{mH}$, show that initially $V_A = V_B = 0$, then $V_A(0^+) = 10\,\text{V}$, $V_B(0^+) = 1010\,\text{V}$, and V_B decays towards $10\,\text{V}$ with time constant $10\,\mu\text{s}$. After equilibration, reclose the switch and find the rise time constant across R_1, including its approximately $1\,\text{ms}$ value.



Solution.

With the switch long closed the inductor shorts A to B, both at ground, and $i_L = V_0/R_1 = 0.1\,\text{A}$. On opening, the source branch carries no current, so $V_A = 10\,\text{V}$ while the unchanged inductor current returns through R_2. Thus

$$\boxed{V_B(0^+) = 10 + 0.1(10000) = 1010\,\text{V}}$$

and

$$\boxed{V_B(t) = 10 + 1000e^{-t/(10\,\mu\text{s})}\,\text{V}.}$$

After reclosing from the fully relaxed state,

$$\tau = L/(R_1 \parallel R_2) = \boxed{1.01\,\text{ms}}$$

and $V_{R_1} = 10 - [10R_2/(R_1+R_2)]e^{-t/\tau}$. For $R_2 \gg R_1$, $\tau \simeq L/R_1 = 1\,\text{ms}$.

Alternating currents

14. RMS voltages

Find RMS voltages for (a) $V(t) = V_C$; (b) an equal-duty square wave from 0 to V; (c) an equal-duty square wave from $-V/2$ to $V/2$, explaining its difference from (b); (d) the sawtooth $V(t) = V_0 t/T$ on $-T < t < T$, repeated with period $2T$.

Solution.

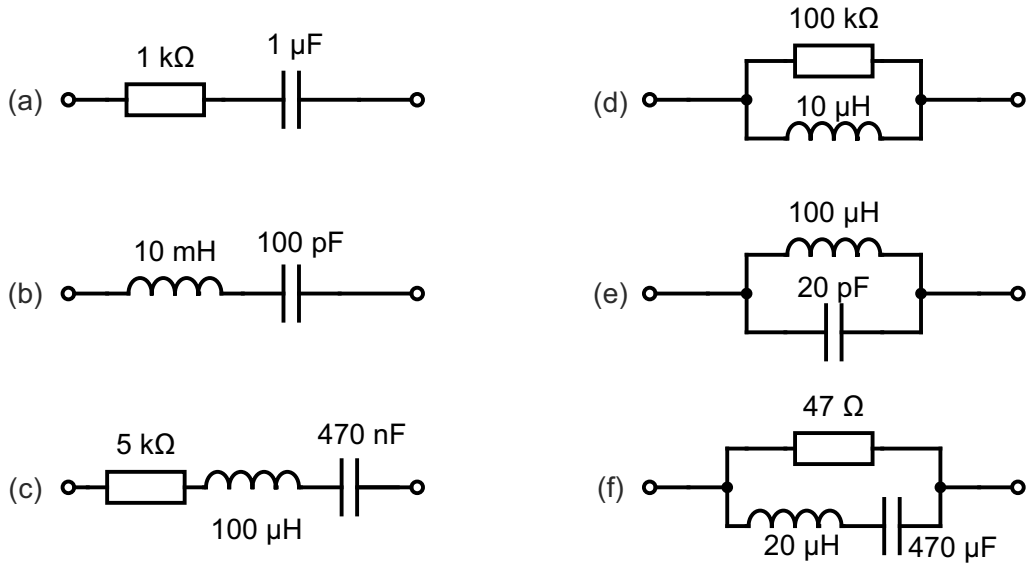
By $V_{\text{rms}} = \sqrt{\langle V^2 \rangle}$,

$$\boxed{(a)\ |V_C|, \quad (b)\ |V|/\sqrt{2}, \quad (c)\ |V|/2, \quad (d)\ |V_0|/\sqrt{3}.}$$

The 0-to-V square wave has a nonzero DC component; shifting it by $-V/2$ leaves its AC RMS equal to $V/2$ but removes that DC contribution. The sawtooth result follows by averaging $V_0^2 t^2/T^2$ over $[-T, T]$.

15. Complex impedance networks

For networks (a)–(f) in the figure, find (i) complex impedance, (ii) the current phase relative to applied $V_0 \sin \omega t$, stating lead or lag, and (iii) the peak resistor voltage in (a),(d), capacitor voltage in (b),(e), and inductor voltage in (c),(f), at $V_0 = 10\,\text{V}$, $\omega/(2\pi) = 10\,\text{kHz}$.



Solution.

Use phasors $e^{i\omega t}$, $Z_L = i\omega L$, $Z_C = 1/(i\omega C)$. The total impedances and their phase angles (voltage relative to current) are

	Z	$\arg Z$ at 10 kHz
a	$R + 1/(i\omega C)$	$-0.912°$
b	$i(\omega L - 1/(\omega C))$	$-90°$
c	$R + i(\omega L - 1/(\omega C))$	$-0.316°$
d	$(R^{-1} + (i\omega L)^{-1})^{-1}$	$89.9996°$
e	$[(i\omega L)^{-1} + i\omega C]^{-1}$	$90°$
f	$[R^{-1} + \{i(\omega L - 1/(\omega C))\}^{-1}]^{-1}$	$88.510°$

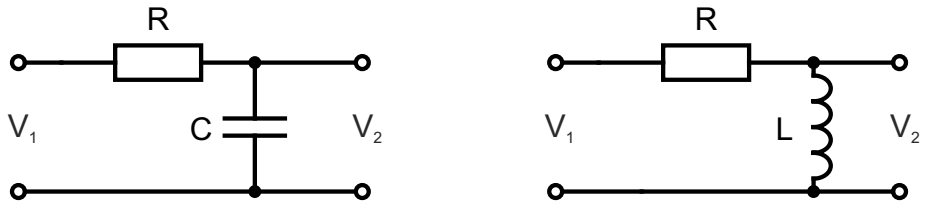
Thus current leads in (a,b,c) and lags in (d,e,f) for the stated values; at other frequencies the signs follow the displayed impedances. The requested peak element voltages are

$$\boxed{V_{R,a} = 9.9987\,\mathrm{V},\ V_{C,b} = 10.0396\,\mathrm{V},\ V_{L,c} = 0.01257\,\mathrm{V},\ V_{R,d} = 10\,\mathrm{V},\ V_{C,e} = 10\,\mathrm{V},\ V_{L,f} = 10.28\,\mathrm{V}.}$$

For a series element divide its impedance by total impedance; in (d,e) the requested element is directly across the source, and in (f) divide $i\omega L$ by the series LC branch impedance.

16. RC low-pass and RL high-pass

For each shown RC and RL network with input $V_1 = V_0 \sin \omega t$, draw the voltage phasors, find the output-to-input amplitude ratio and output phase, and sketch them against $x = \omega/\omega_0$, with $\omega_0 = 1/(RC)$ or R/L respectively.



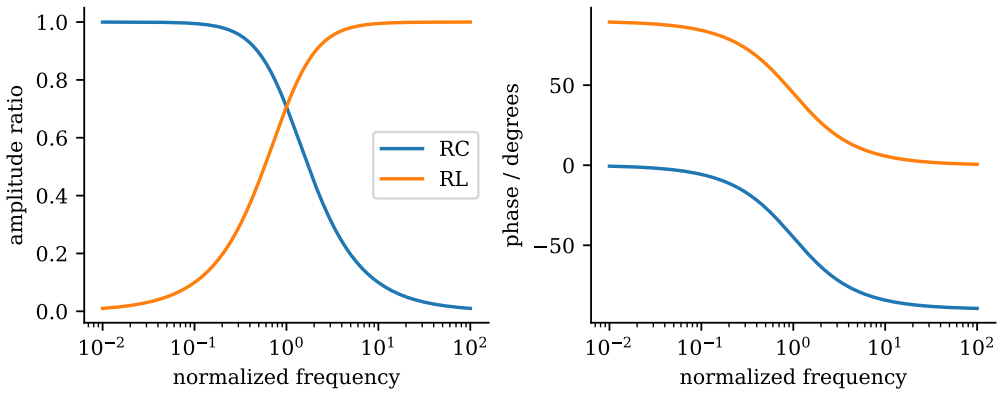
Solution.

Draw current on the real phasor axis: V_R is in phase, V_C lies 90° behind and V_L 90° ahead. Then $V_1 = V_R + V_C$ or $V_R + V_L$ respectively. With the stated normalized frequency,

$$\boxed{H_{RC} = \frac{1}{1+ix},\ |H_{RC}| = \frac{1}{\sqrt{1+x^2}},\ \phi_{RC} = -\tan^{-1} x;}$$

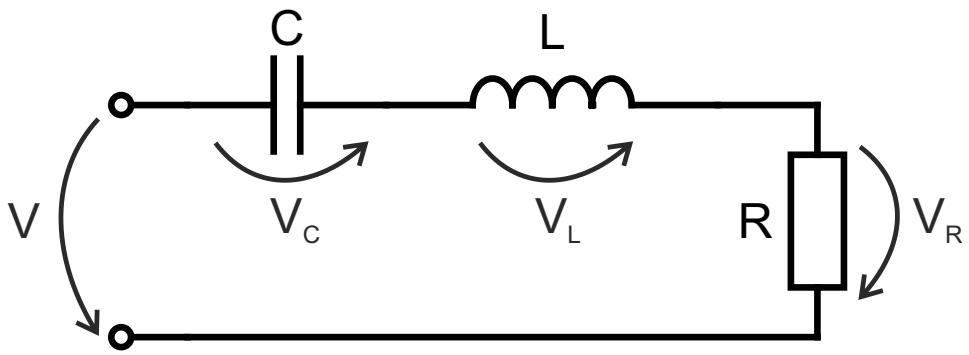
$$\boxed{H_{RL} = \frac{ix}{1+ix},\ |H_{RL}| = \frac{x}{\sqrt{1+x^2}},\ \phi_{RL} = \frac{\pi}{2} - \tan^{-1} x.}$$

The low-pass falls from unity to zero with phase 0 to $-\pi/2$; the high-pass rises from zero to unity with limiting phase $\pi/2$ to zero. Both have amplitude $1/\sqrt{2}$ at $x = 1$.



17. Series RLC with prescribed current

A series RLC circuit carries $I = I_0 \sin \omega t$, with $C = 10\,\text{nF}$, $L = 0.2\,\text{mH}$, $R = 100\,\Omega$, $I_0 = 0.1\,\text{A}$, $\omega = 10^6\,\text{rad}\,\text{s}^{-1}$. Find (a) all component and total voltage amplitudes and phases, (b) mean dissipated power, (c) $W_L(t), W_C(t)$, (d) their maxima and (e) sketches. (f) For fixed other parameters, what L makes $W_L + W_C$ constant?



Solution.

With $I = 0.1 \sin \omega t\,\text{A}$ and the given values,

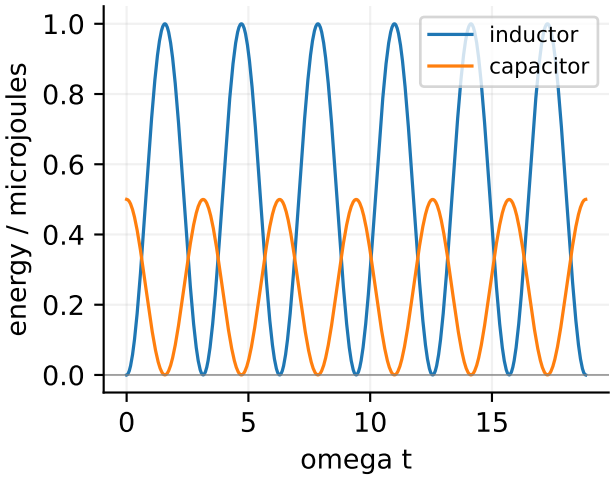
$$\boxed{V_R = 10 \sin \omega t, \quad V_L = 20 \cos \omega t, \quad V_C = -10 \cos \omega t, \quad V = 10\sqrt{2} \sin(\omega t + \pi/4)}$$

in volts. Only the resistor dissipates average power: $\boxed{P = I_0^2 R/2 = 0.5\,\text{W}}$. Stored energies are

$$\boxed{W_L = 1.0\,\mu\text{J} \sin^2 \omega t, \;\; W_C = 0.5\,\mu\text{J} \cos^2 \omega t}.$$

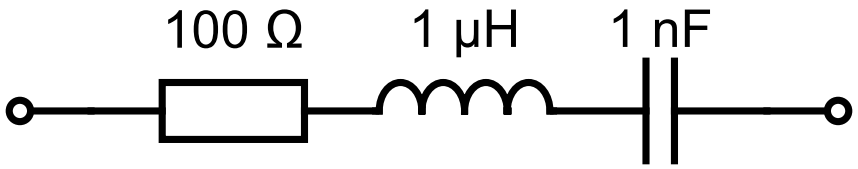
Their maxima are the prefactors; both vary at 2ω and reach maxima a quarter cycle apart. Their sum is constant when

$$\boxed{L = 1/(\omega^2 C) = 0.10\,\text{mH}}.$$



18. Series resonance

Find the frequency of minimum impedance for the shown network. For applied amplitude 10 V, show that the capacitor voltage amplitude there is 3.16 V.



Solution.

$|Z|^2 = R^2 + (\omega L - 1/(\omega C))^2$ is minimized at

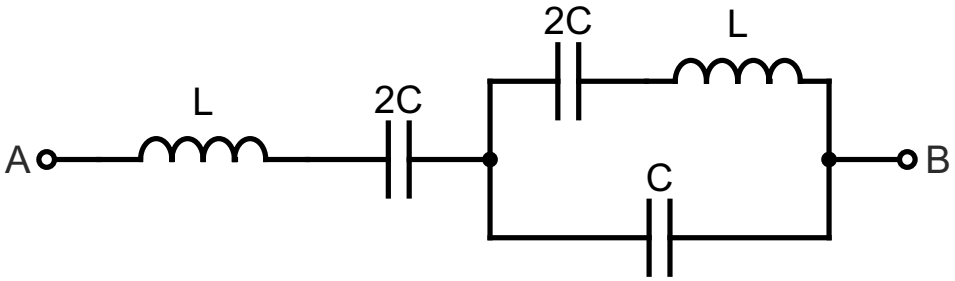
$$\boxed{\omega_0 = 1/\sqrt{LC} = 3.162 \times 10^7\,\mathrm{rad\,s^{-1}},\ f_0 = 5.033\,\mathrm{MHz}}.$$

At resonance $I_0 = V_0/R$, giving

$$\boxed{V_{C,0} = V_0/(R\omega_0 C) = 3.162\,\mathrm{V}}.$$

19. Two zero-impedance frequencies

For the shown LC network between A, B, show that its impedance vanishes at $\omega_1 = \sqrt{5/(2LC)}$ and $\omega_2 = \sqrt{1/(2LC)}$.



Solution.

Let $Z_s = i\omega L + 1/(2i\omega C)$ and $Z_c = 1/(i\omega C)$. The circuit is Z_s in series with $Z_s \parallel Z_c$, so $Z = Z_s(Z_s + 2Z_c)/(Z_s + Z_c)$. Its zeroes are $Z_s = 0$ and $Z_s + 2Z_c = 0$, with finite denominators, giving

$$\boxed{\omega_1 = \sqrt{5/(2LC)}, \qquad \omega_2 = \sqrt{1/(2LC)}.}$$

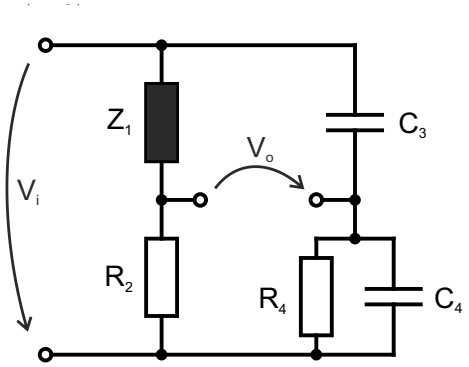
20. Balanced impedance bridge

In the shown bridge, R_2, C_3, R_4, C_4 are fixed and Z_1 is variable.

(a) Find the complex Z_1 for balance.

(b) If Z_1 is series R_1, C_1, find their values.

(c) Repeat for parallel R_1, C_1.



Solution.

Zero detector voltage means equal divider fractions, hence $Z_1/R_2 = Z_3/Z_4$. With $Z_3 = 1/(i\omega C_3)$ and $Z_4 = R_4/(1 + i\omega R_4 C_4)$,

$$\boxed{Z_1 = R_2 C_4/C_3 + R_2/(i\omega R_4 C_3).}$$

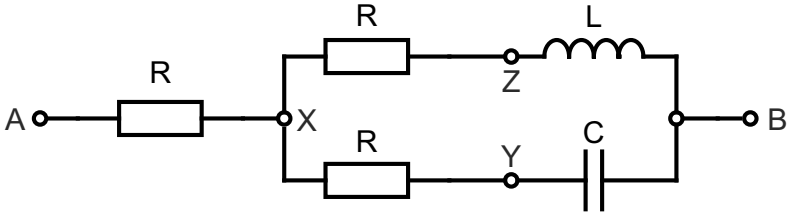
For a series RC pair, $\boxed{R_1 = R_2 C_4/C_3,\ C_1 = R_4 C_3/R_2}$. For a parallel pair equate real and imaginary parts of $1/Z_1 = 1/R_1 + i\omega C_1$. Writing $x = \omega R_4 C_4$ gives

$$\boxed{R_1 = \frac{R_2(1+x^2)}{\omega^2 R_4^2 C_3 C_4}, \qquad C_1 = \frac{R_4 C_3}{R_2(1+x^2)}.}$$

These parallel values balance at the selected frequency; the series values balance at every frequency in the ideal model.

21. Three-phase voltages

The shown network has $C = 1/(\omega_0 R\sqrt{3})$, $L = R\sqrt{3}/\omega_0$ and applied $V_{AB} = V_0 \cos\omega_0 t$. Show (a) $Z_{AB} = 3R$, (b) that V_{XA}, V_{XY}, V_{XZ} have equal amplitudes and (c) phases $0, \pi/3, -\pi/3$ relative to V_{AB}. (d) With a current source $I_0 \cos\omega t$ instead, plot the amplitude and phase of V_{AB} against ω.



Solution.

At ω_0, the two parallel branches have impedances $R(1 \pm i\sqrt{3})$, each of magnitude $2R$. Their parallel combination is $2R$, so total $\boxed{Z_{AB} = 3R}$. The input resistor drop is $V_0/3$ in phase with V_{AB}; the resistor drops V_{XY}, V_{XZ}

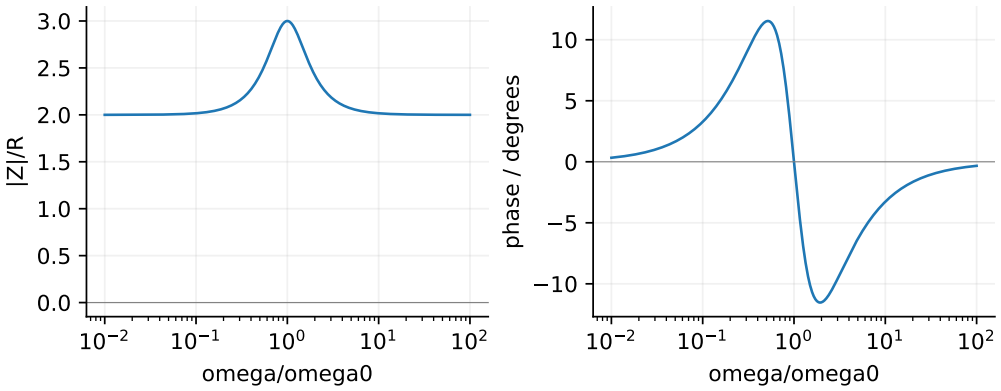
also have amplitude $V_0/3$ with phases $\boxed{+\pi/3, -\pi/3}$. Here the input-resistor voltage is taken as the drop from A to X; reversing terminal order adds π. For a current drive at arbitrary frequency put

$$u = \sqrt{3}(\omega/\omega_0 - \omega_0/\omega).$$

Then

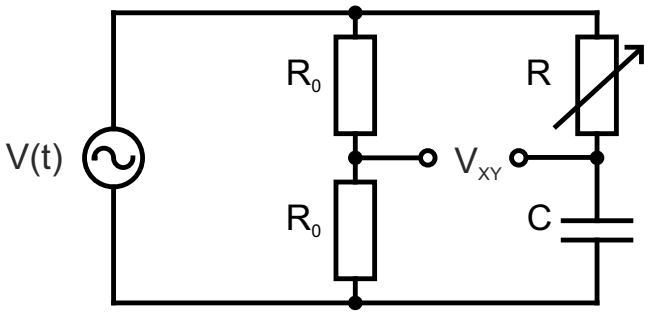
$$\boxed{Z/R = \frac{6+2iu}{2+iu}, \quad |V_{AB}| = 2I_0R\sqrt{\frac{9+u^2}{4+u^2}}, \quad \phi = \tan^{-1}(u/3) - \tan^{-1}(u/2).}$$

The magnitude tends to $2I_0R$ at both ends and peaks at $3I_0R$ at ω_0; phase is positive below, zero at and negative above resonance.



22. Constant-amplitude phase shifter

For the shown network, prove that the amplitude of V_{XY} is independent of R. Show that its phase relative to applied V is $\pi/2$ when $R = 1/(\omega C)$.



Solution.

The equal-resistor divider gives $V_X = V/2$ and the RC divider gives $V_Y = V/(1 + i\omega RC)$. Thus

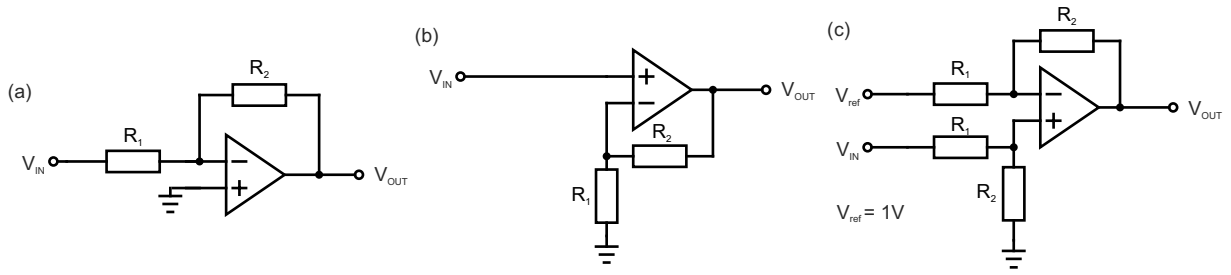
$$\boxed{\frac{V_{XY}}{V} = \frac{i\omega RC - 1}{2(1 + i\omega RC)}, \quad |V_{XY}| = |V|/2.}$$

Numerator and denominator have the same magnitude. At $\omega RC = 1$ the ratio is $i/2$, giving $\boxed{\phi = \pi/2}$.

Operational amplifiers

23. Ideal op-amp gains

Assuming ideal op-amps, find $V_{\text{out}}(V_{\text{in}}, R_1, R_2)$ for circuits (a)–(c), retaining the shown reference $V_{\text{ref}} = 1\,\text{V}$. For $R_2 = 4R_1$, find the input giving $V_{\text{out}} = 2\,\text{V}$ in each case.



Solution.

Assume negative feedback, zero input currents and equal input voltages.

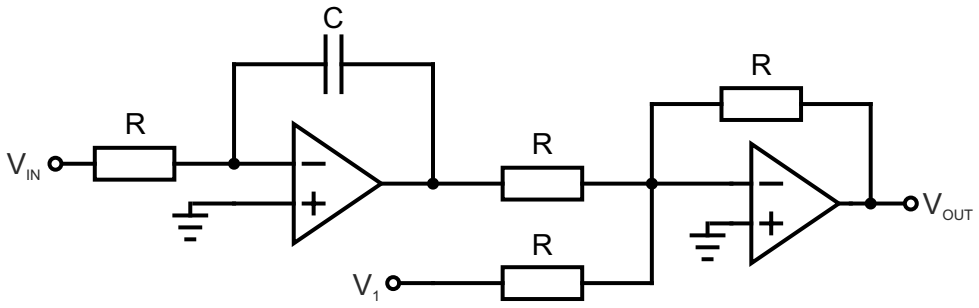
(a) Virtual ground and current balance give $\boxed{V_o = -(R_2/R_1)V_i}$.

(b) The feedback divider gives $\boxed{V_o = (1 + R_2/R_1)V_i}$.

(c) The noninverting input is $V_i R_2/(R_1 + R_2)$, so balance at the inverting input gives $\boxed{V_o = (R_2/R_1)(V_i - V_{\text{ref}})}$. At $R_2 = 4R_1$, $V_o = 2\,\text{V}$ needs inputs $\boxed{-0.5,\ 0.4,\ 1.5\,\text{V}}$ respectively, with $V_{\text{ref}} = 1\,\text{V}$.

24. Integrator and summing amplifier

For the shown op-amp circuit, find $V_{\text{out}}(t)$ when $V_{\text{in}} = 0$ for $t < 0$ and $1\,\text{V}$ for $t \geq 0$. Sketch it for $V_1 = 10\,\text{V}$, $R = 100\,\text{k}\Omega$, $C = 10\,\mu\text{F}$, and discuss the long-time output and saturation.

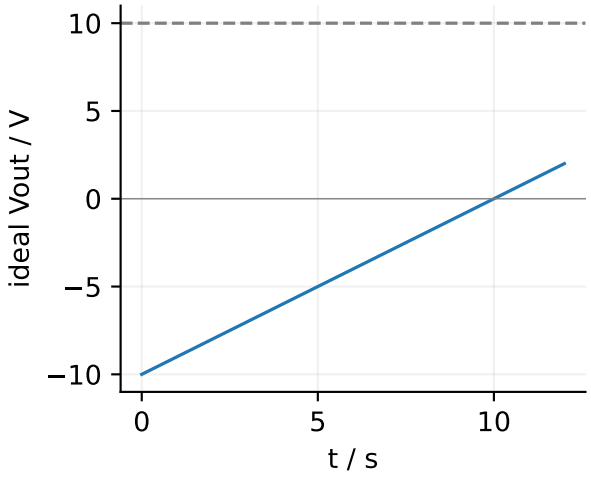


Solution.

For an initially uncharged feedback capacitor, the first stage is $u(t) = -(RC)^{-1}\int_0^t V_i(s)ds$. The second is an equal-resistor inverting summer, $V_o = -u - V_1$. Thus for the unit step,

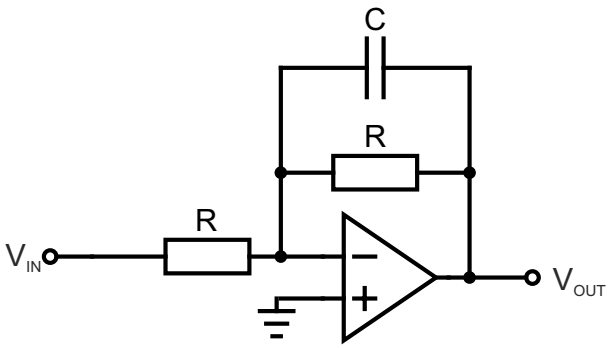
$$\boxed{V_o(t) = -V_1 + t/(RC) \quad (t \geq 0).}$$

With $RC = 1\,\text{s}$, $V_1 = 10\,\text{V}$, it begins at $-10\,\text{V}$ and rises at $1\,\text{V}\,\text{s}^{-1}$. The ideal model grows without bound; a real op-amp reaches its positive output rail. An initial capacitor voltage would add a constant offset.



25. Active low-pass filter

For the shown active filter, find $|V_{\text{out}}/V_{\text{in}}|$ in terms of R, C, ω and the frequency where it equals $1/\sqrt{2}$. Explain its advantages over a passive RC filter.

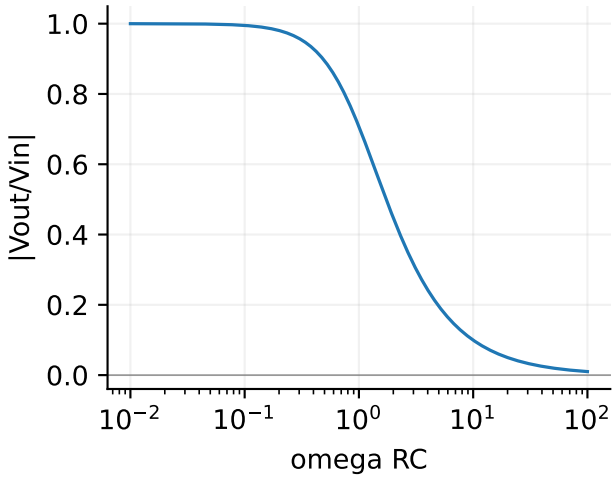


Solution.

The feedback impedance is $Z_f = R/(1 + i\omega RC)$ and input resistance is R. Therefore

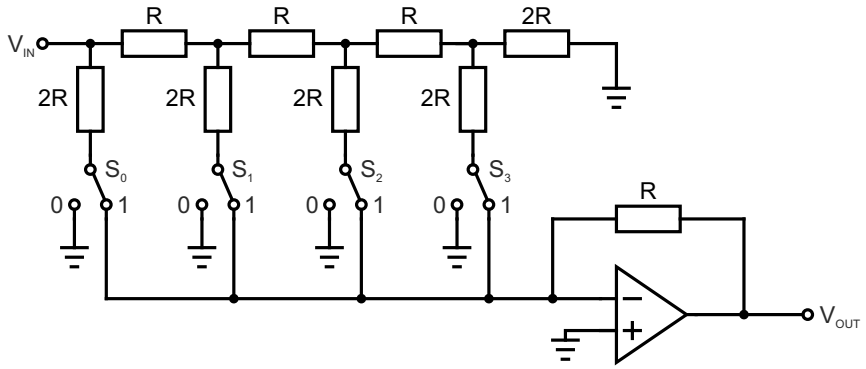
$$\boxed{\frac{V_o}{V_i} = -\frac{1}{1 + i\omega RC}, \quad \left|\frac{V_o}{V_i}\right| = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \quad \omega_{-3\text{dB}} = 1/(RC).}$$

The op-amp provides a low output impedance and suppresses load dependence; resistor ratios can also set passband gain. These benefits hold within its usable bandwidth and output range.



26. Four-bit R–2R DAC

For the shown R–$2R$ DAC, derive V_{out} as a function of V_{in} and switch bits $S_0, S_1, S_2, S_3 \in \{0, 1\}$. For $V_{\text{in}} = 10\,\text{V}$, find the output range and design a rescaling circuit spanning 0 to $+5\,\text{V}$.



Solution.

Since either switch state terminates its branch at zero potential (physical ground or virtual ground), the ladder node voltages are $V_i, V_i/2, V_i/4, V_i/8$. A branch switched to 1 sends current $S_k V_i/(2^{k+1}R)$ into the summing node. With feedback R,

$$\boxed{V_o = -V_i\left(\frac{S_0}{2} + \frac{S_1}{4} + \frac{S_2}{8} + \frac{S_3}{16}\right).}$$

For $V_i = 10\,\text{V}$, the 16 discrete levels range from 0 to $\boxed{-9.375\,\text{V}}$ in $0.625\,\text{V}$ steps. A further inverting stage of gain $-8/15$ with $R_{\text{in}} = 15\,\text{k}\Omega$ and $R_f = 8\,\text{k}\Omega$ maps the endpoints to $\boxed{0 \text{ and } +5\,\text{V}}$.

2.2 Electromagnetism: Problem Set 1

Section A

A.0. Field and potential

Define electric field and electrostatic potential, derive their relationship and explain superposition for charge distributions.

Solution.

For a positive test charge small enough not to change the source distribution, $E = \lim_{q_t \to 0} F/q_t$. Define $V(B) - V(A) = -\int_A^B E \cdot d\ell$; an infinitesimal displacement gives $\boxed{E = -\nabla V}$. Coulomb's law is linear in the source charge, so $V = \sum_i V_i$ and $E = \sum_i E_i$, or the corresponding integrals for continuous distributions.

A.1. Charges on a square

Charges $q, 2q, -5q, 2q$ occupy consecutive corners A, B, C, D of a square of side a.

(a) Find $\mathbf{E}, V$ at its centre and verify $\mathbf{E} = -\nabla V$.

(b) Find the electrostatic assembly energy from infinite separation.

Solution.

Let $k_e = (4\pi\epsilon_0)^{-1}$ and let $\widehat{c}$ point from the centre towards C. Every corner is at radius $a/\sqrt{2}$. The charges sum to zero, giving $V(0) = 0$. The B and D fields cancel; the A and C fields add along $\widehat{c}$, so $\boxed{E(0) = 12k_e q\widehat{c}/a^2}$. This also follows by differentiating $V(r) = k_e \sum_i q_i/|r - r_i|$ before setting $r = 0$. There are four adjacent pairs and two diagonal pairs. Their contributions are $-16k_e q^2/a$ and $-k_e q^2/(\sqrt{2}a)$, giving

$$\boxed{U = -\frac{q^2(32 + \sqrt{2})}{8\pi\epsilon_0 a}.}$$

A.2. Electric dipole

Charges $\pm q$ at $(0, 0, \pm d/2)$ have dipole moment $\mathbf{p} = qd\hat{\mathbf{z}}$.

(a) Derive

$$V = \mathbf{p} \cdot \mathbf{r}/(4\pi\epsilon_0 r^3)$$

at $r \gg d$.

(b) Find E_r, E_θ, E_ϕ.

(c) Find its energy in a uniform external field at angle θ to $\mathbf{p}$.

Solution.

Expanding $|r \mp (d/2)\widehat{z}|^{-1}$ to first order in d/r gives $V = k_e p\cos\theta/r^2 = k_e p \cdot r/r^3$. Differentiation in spherical coordinates yields

$$\boxed{E_r = 2k_e p\cos\theta/r^3, \quad E_\theta = k_e p\sin\theta/r^3, \quad E_\phi = 0.}$$

The external-field energy is

$$qV_{\text{ext}}(r_+) - qV_{\text{ext}}(r_-) = -p \cdot E_{\text{ext}}$$

to dipole order, hence $\boxed{W = -pE_{\text{ext}}\cos\alpha}$.

A.3. Monopole and quadrupole

A charge q_2 is at the origin and charges $-q_1$ are at $(0, 0, \pm a)$.

(a) Find the exact potential in spherical coordinates, with θ from z.

(b) Expand through order $(a/r)^2$ and identify monopole, dipole and quadrupole terms.

(c) For $q_2 = 2q_1$, give the far-field potential and its radial and angular electric fields.

Solution.

The exact potential is

$$V = k_e \left[\frac{q_2}{r} - \frac{q_1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{q_1}{\sqrt{r^2 + a^2 + 2ar\cos\theta}} \right].$$

Binomial expansion gives

$$V = k_e[(q_2 - 2q_1)/r - q_1 a^2(3\cos^2\theta - 1)/r^3] + O(a^4/r^5).$$

The first term is monopolar; symmetry kills the dipole; the next is quadrupolar. For $q_2 = 2q_1$,

$$\boxed{E_r = -3k_e q_1 a^2(3\cos^2\theta - 1)/r^4, \quad E_\theta = -6k_e q_1 a^2 \sin\theta\cos\theta/r^4.}$$

A.4. Finite charged rod

A thin rod of length $2l$ has uniform linear charge density λ. Integrate contributions to find the electric field (a) on its axis at distance z from the midpoint, beyond the ends, and (b) at perpendicular distance x from its midpoint.

Solution.

Take the rod along $z' \in [-l, l]$.

(a) For $z > l$,

$$E_z = k_e\lambda \int_{-l}^{l} dz'/(z - z')^2 = k_e\lambda[(z - l)^{-1} - (z + l)^{-1}].$$

For either side,

$$\boxed{E_z = \frac{2k_e\lambda l \ \operatorname{sgn} z}{z^2 - l^2} \quad (|z| > l).}$$

(b) The longitudinal components cancel and

$$E_x = k_e\lambda \int_{-l}^{l} x(x^2 + z'^2)^{-3/2}\, dz',$$

giving

$$\boxed{E_x = 2k_e\lambda l/[x\sqrt{x^2 + l^2}]}$$

for $x > 0$. The long-rod limit is $\lambda/(2\pi\epsilon_0 x)$.

A.5. Uniform disk

A thin disc of radius b in the xy plane has uniform surface charge density σ. Find the electric field on its axis by integrating charged rings. Give the limits $|z| \ll b$ and $|z| \gg b$.

Solution.

A ring has charge $2\pi\sigma r\,dr$. Its axial contribution integrates to

$$\boxed{E_z = \frac{\sigma}{2\epsilon_0}\left(\operatorname{sgn} z - \frac{z}{\sqrt{z^2+b^2}}\right), \quad z \neq 0.}$$

For $|z| \ll b$, $E_z \simeq \sigma \operatorname{sgn} z/(2\epsilon_0)$; for $|z| \gg b$, $E_z \simeq k_e(\pi b^2\sigma)\operatorname{sgn} z/z^2$. These are the infinite-sheet and point-charge limits. The field has its expected surface discontinuity at $z = 0$.

A.6. Charged ring

A thin ring of radius a has uniform charge q.

(a) Find the axial positions maximising $|E_z|$.

(b) Show that an electron near its centre on the axis executes small oscillations with ordinary frequency $\nu = \sqrt{eq/(16\pi^3\epsilon_0 a^3 m)}$ for $q > 0$.

Solution.

$E_z = k_e qz/(a^2+z^2)^{3/2}$. Its derivative is proportional to $a^2 - 2z^2$, so $|E_z|$ is maximal at $\boxed{z = \pm a/\sqrt{2}}$, with magnitude $2k_e q/(3\sqrt{3}a^2)$. For $q > 0$ and $|z| \ll a$, an electron obeys $m\ddot{z} = -ek_e qz/a^3$. Therefore

$$\boxed{\nu = \frac{1}{2\pi}\sqrt{\frac{eq}{4\pi\epsilon_0 m a^3}} = \sqrt{\frac{eq}{16\pi^3\epsilon_0 m a^3}}.}$$

This is stable for motion constrained to the axis.

A.7. Potential inside a shell

A spherical conducting shell of radius a carries uniform surface charge density σ. By direct integration over its surface, show that the potential at every interior point is $a\sigma/\epsilon_0$. Take the polar axis through the observation point.

Solution.

For $0 < r < a$, orient the polar axis through the observation point. Then

$$V = 2\pi k_e\sigma a^2 \int_{-1}^{1} \frac{du}{\sqrt{a^2+r^2-2aru}} = \frac{2\pi k_e\sigma a}{r}[(a+r)-(a-r)].$$

Thus

$$\boxed{V = 4\pi k_e\sigma a = a\sigma/\epsilon_0},$$

independent of r. At $r = 0$ the same result follows directly from the constant source distance a.

2.3 Electromagnetism: Problem Set 2

Section B

B.0. Conducting boundary

Explain what determines the tangential and normal electric-field components just outside a metal plate in electrostatic equilibrium.

Solution.

Electrostatic equilibrium requires zero field inside a conductor and constant potential on it, so the tangential exterior field is zero. A thin Gaussian pillbox gives $\boxed{E_{n,\text{out}} = \sigma/\epsilon_0}$ in vacuum. The normal field is determined by the induced surface charge; grounding fixes the surface potential rather than requiring that surface charge to vanish.

B.1. Image of a point charge

A point charge q lies a distance d normally above an infinite grounded conducting plane, with foot O.

(a) Use images to obtain the surface field at distance r from O.

(b) Sketch the field and find the force on q.

(c) Show that the total induced charge is $-q$.

(d) Find the work to move q to infinity. Estimate the escape energy in electron volts for an electron initially 0.1 nm above the plane.

Solution.

Put q at $z = d$ and its image $-q$ at $z = -d$. Their potential vanishes on $z = 0$. At cylindrical radius r,

$$E_z = -\frac{qd}{2\pi\epsilon_0(r^2 + d^2)^{3/2}}, \quad \sigma = \epsilon_0 E_z.$$

The charge is attracted with $\boxed{F_z = -q^2/(16\pi\epsilon_0 d^2)}$. Integrating $2\pi r\sigma\, dr$ from zero to infinity gives $Q_{\text{ind}} = -q$. The work against attraction to infinity is

$$\boxed{W = \frac{q^2}{16\pi\epsilon_0 d}.}$$

For an electron at $d = 0.1\,\text{nm}$ this model gives $W = 3.60\,\text{eV}$. Field lines end normally on the plate; in the image construction they run from q towards $-q$.

B.2. Two charges above a plane

Charges $+Q, -Q$ are separated horizontally by a, both at height b above a large grounded conducting plane. Find all components of the force on each charge.

Solution.

Place $+Q$ at $(0, b)$ and $-Q$ at (a, b) in the (x, z) plane. Include their opposite-sign images below the plane. Summing the three forces on $+Q$ gives

$$\boxed{\begin{aligned} F_x &= k_e Q^2 \left[\frac{1}{a^2} - \frac{a}{(a^2 + 4b^2)^{3/2}}\right], \\ F_z &= k_e Q^2 \left[-\frac{1}{4b^2} + \frac{2b}{(a^2 + 4b^2)^{3/2}}\right]. \end{aligned}}$$

For $-Q$, the horizontal component reverses while the vertical component is unchanged; the remaining component is zero.

B.3. Two perpendicular plates

Two grounded conducting half-planes meet at a right angle. A charge Q lies at distance a from each plane.

(a) Use three images to show that the field vanishes on the line of intersection.

(b) Find the field just outside either plane at its nearest point to Q.

(c) Sketch equipotentials near the charge and the planes.

(d) Find the induced surface charge density at both nearest points.

Solution.

For plates $x = 0, y = 0$, use images $-Q$ at $(-a, a)$ and $(a, -a)$, and $+Q$ at $(-a, -a)$. The potential vanishes on both planes; symmetry cancels every component of E on their join. At the point $(0, a)$ nearest the real charge,

$$\boxed{E_x = -\frac{Q}{2\pi\epsilon_0 a^2}\left(1 - \frac{1}{5\sqrt{5}}\right), \quad E_y = E_z = 0, \quad \sigma = \epsilon_0 E_x.}$$

At $(a, 0)$ interchange x, y. Near Q the equipotentials are approximately spherical; near either plate they approach locally parallel surfaces and meet the field at right angles.

B.4. Image of a line charge

An infinite line charge λ runs parallel to a grounded horizontal conductor at height d. Find the field at a point on the plane at perpendicular transverse distance x from the line's projection, using the infinite-rod limit of Question A.4.

Solution.

Use an image line $-\lambda$ at height $-d$. The horizontal components on the plate cancel, while the normal components add. With upward normal $\widehat{z}$,

$$\boxed{E(P) = -\frac{\lambda d}{\pi\epsilon_0(x^2 + d^2)}\widehat{z}.}$$

The finite-rod result tends to $\lambda/(2\pi\epsilon_0 s)$ per real or image line, at perpendicular distance $s = \sqrt{x^2 + d^2}$.

Section C

C.0. Gauss's law

State Gauss's law and explain how to determine the field from a spherically symmetric volume charge density $\rho(r)$.

Solution.

$\oint E \cdot dA = Q_{\text{enc}}/\epsilon_0$. A spherical charge density makes $E = E(r)\widehat{r}$, hence

$$\boxed{E(r) = \frac{1}{\epsilon_0 r^2}\int_0^r \rho(s)s^2\, ds.}$$

The flux law always holds; the symmetry is what allows the field to be taken outside the surface integral.

C.1. Solid sphere and shell

(a) Find $\mathbf{E}(r)$ and $V(r)$ inside and outside a uniformly volume-charged sphere of radius a and total charge q, with $V(\infty) = 0$.

(b) Repeat for a uniformly charged spherical shell.

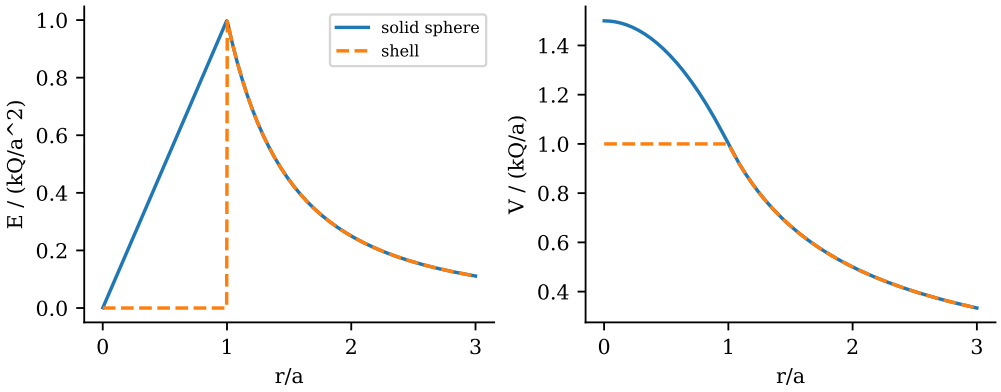
(c) Sketch both fields and potentials; explain continuity or discontinuity at $r = a$ and verify $E_r = -dV/dr$.

Solution.

For a uniform solid sphere, $Q_{\text{enc}} = q(r/a)^3$ inside and q outside. Integrate $E_r = -dV/dr$ from infinity to obtain

$$\boxed{E_r = \begin{cases} k_e q r/a^3 & r < a \\ k_e q/r^2 & r > a, \end{cases} \qquad V = \begin{cases} k_e q(3a^2 - r^2)/(2a^3) & r < a \\ k_e q/r & r \geq a. \end{cases}}$$

For the shell, $E_r = 0$ inside and $k_e q/r^2$ outside, while $V = k_e q/a$ inside and $k_e q/r$ outside. Solid-sphere E, V are continuous at a; shell V is continuous but E jumps by $q/(4\pi\epsilon_0 a^2)$.



C.2. Electrostatic self-energy

A nucleus has charge Ze uniformly distributed through a sphere of radius a. Derive its assembly energy $3(Ze)^2/(20\pi\epsilon_0 a)$. Repeat for the charge distributed on the spherical surface.

Solution.

Assemble concentric shells. The sphere already assembled at radius r has charge $Q(r) = Q(r/a)^3$ and the added charge is $dQ = 3Qr^2 dr/a^3$. Therefore $U = \int_0^a k_e Q(r) dQ/r = 3k_e Q^2/(5a)$. For a charged shell, $U = \int_0^Q k_e q\, dq/a = k_e Q^2/(2a)$. Thus

$$\boxed{U_{\text{volume}} = \frac{3(Ze)^2}{20\pi\epsilon_0 a}, \quad U_{\text{surface}} = \frac{(Ze)^2}{8\pi\epsilon_0 a}.}$$

C.3. Hydrogen charge cloud

The potential of an electron cloud is

$$V(r) = \frac{q}{4\pi\epsilon_0}\left[\frac{e^{-2r/a}-1}{r} + \frac{e^{-2r/a}}{a}\right], \qquad q > 0.$$

(a) Sketch it for $r \geq 0$.

(b) Find the electric field for $r \ll a$.

(c) A uniform external field displaces the still-spherical cloud relative to its equally charged nucleus. Find the induced dipole and show that its polarisability is $3\pi\epsilon_0 a^3$.

(d) Estimate the displacement for $a = 0.5 \times 10^{-10}\,\mathrm{m}$ and $E_{\mathrm{ext}} = 10^6\,\mathrm{V\,m^{-1}}$.

(e) Find the total cloud charge using Gauss's law.

Solution.

Expand the given potential:

$$V = k_e q\left[-\frac{1}{a} + \frac{2r^2}{3a^3} + O(r^3/a^4)\right], \qquad E_r = -\frac{4k_e q}{3a^3}r + O(r^2).$$

It rises monotonically from $-k_e q/a$ to zero, approaching $-k_e q/r$ at large r. For a nucleus displaced by s, force balance gives $qE_{\mathrm{ext}} = 4k_e q^2 s/(3a^3)$. Hence

$$\boxed{p = qs = 3\pi\epsilon_0 a^3 E_{\mathrm{ext}}, \quad \alpha = 3\pi\epsilon_0 a^3.}$$

At the given a, E_{ext}, $s = 6.5 \times 10^{-17}\,\mathrm{m}$. Finally $4\pi\epsilon_0 r^2 E_r \to -q$, so the cloud's total charge is $-q$.

2.4 Electromagnetism: Problem Set 3

Section D

D.0. Capacitance and energy

Define capacitance and derive the stored electrostatic energy of a capacitor and its field-energy interpretation.

Solution.

For fixed geometry in a linear dielectric, $Q = CV$. Charging reversibly gives $dU = V\,dQ = (Q/C)dQ$, so

$$\boxed{U = Q^2/(2C) = CV^2/2 = QV/2.}$$

Equivalently in vacuum $U = (\epsilon_0/2)\int E^2 d^3r$, derived by integrating the electrostatic energy density over space.

D.1. Spherical and cylindrical capacitors

Use Gauss's law to find the capacitance of (a) concentric spherical conductors of radii $R_1 < R_2$, and (b) long coaxial cylindrical conductors of radii $R_1 < R_2$ and length L. Show how both reduce to parallel-plate results for a small gap.

Solution.

Gauss's law gives $E = Q/(4\pi\epsilon_0 r^2)$ between spherical shells, and $E = Q/(2\pi\epsilon_0 Lr)$ between long cylinders. Integrating the potential drop yields

$$\boxed{C_s = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}, \qquad C_c = \frac{2\pi\epsilon_0 L}{\ln(R_2/R_1)}.}$$

For gap $d \ll R$, these reduce to $\epsilon_0(4\pi R^2)/d$ and $\epsilon_0(2\pi RL)/d$, respectively: $C = \epsilon_0 A/d$.

D.2. Immobile space charge

Two large conducting planes separated by d enclose a fixed uniform volume charge density ρ and have applied voltage V. Find the signed electric field at distance x from the positive plate, hence its magnitude, and discuss the incremental capacitance in the presence of fixed space charge.

Solution.

Take x from the positive towards the negative plate. The equations $dE_x/dx = \rho/\epsilon_0$ and $\int_0^d E_x dx = V$ give

$$\boxed{E_x = V/d - \rho(d - 2x)/(2\epsilon_0).}$$

The magnitude is $|E_x|$ if parameters allow a field reversal. The positive plate's inner-face charge is $Q = A(\epsilon_0 V/d - \rho d/2)$, so the incremental capacitance $dQ/dV = \epsilon_0 A/d$ is unchanged; the fixed charge produces an offset in Q, rather than a change in this derivative.

D.3. Breakdown-limited capacitor

An air-filled coaxial capacitor has outer radius $b = 10\,\mathrm{mm}$ and variable inner radius a. Air breaks down above $E_b = 3\,\mathrm{MV\,m^{-1}}$. Show that the greatest safe voltage occurs at $a = b/e$, while the greatest stored energy per unit length occurs at $a = b/\sqrt{e}$.

Solution.

The largest field occurs at $r = a$: $E(a) = V/[a\ln(b/a)] \le E_b$. Hence $V_{\max}(a) = E_b a\ln(b/a)$; differentiating gives $\boxed{a = b/e}$ and global maximum $E_b b/e = 11.0\,\mathrm{kV}$. The energy per unit length at the breakdown limit is $U' = \pi\epsilon_0 E_b^2 a^2\ln(b/a)$. Its derivative vanishes at $\boxed{a = b/\sqrt{e}}$, and both endpoint limits are zero, confirming a maximum.

D.4. Plate force and work

A parallel-plate capacitor has area $0.01\,\mathrm{m^2}$, gap $10\,\mathrm{mm}$ and voltage $600\,\mathrm{V}$.

(a) Derive the attractive force by energy accounting.

(b) Find the work to separate its plates to $20\,\mathrm{mm}$ (i) at fixed voltage and (ii) after charging and isolating it.

(c) In (i), explain why the stored field energy halves although positive mechanical work is supplied.

Solution.

The attractive pressure is $\epsilon_0 E^2/2$. Thus

$$F = \epsilon_0 A V^2/(2d^2) = \boxed{1.59\times10^{-4}\,\mathrm{N}}$$

initially. For $d \to 2d$,

$$\boxed{W_V = \int_d^{2d}\frac{\epsilon_0 A V^2}{2s^2}ds = \frac{\epsilon_0 A V^2}{4d} = 7.97\times10^{-7}\,\mathrm{J},\quad W_Q = \frac{Q^2 d}{2\epsilon_0 A} = 1.59\times10^{-6}\,\mathrm{J}.}$$

At fixed V, the stored energy halves: $\Delta U = -U_i/2$. Charge $Q_i/2$ returns to the battery, supplying it with U_i; the mechanical input $U_i/2$ plus the field-energy decrease $U_i/2$ accounts for that energy.

Section E

E.0. Magnetostatic laws

State the Biot–Savart and Ampère laws and explain how geometry and symmetry determine the appropriate method.

Solution.

Biot–Savart gives

$$dB = \mu_0 I\,d\ell\times\widehat{R}/(4\pi R^2)$$

and can be integrated over a known current distribution. Ampere's law is $\oint B\cdot d\ell = \mu_0 I_{\mathrm{enc}}$ in magnetostatics. Choose Ampere when symmetry fixes the direction and constancy of B on a convenient contour, such as an infinite wire or ideal long solenoid; otherwise integrate Biot–Savart. In vacuum $H = B/\mu_0$.

E.1. Wire segment and polygon

(a) A straight wire of length $2b$ carries current I. Find H at distance a along its perpendicular bisector.

(b) Find H at the centre of a regular N-sided current-carrying polygon circumscribing a circle of radius a, and recover the circular-loop limit as $N \to \infty$.

Solution.

(a) Integrate $dH = Ia\,dz/[4\pi(a^2+z^2)^{3/2}]$ from $-b$ to b:

$$\boxed{H = \frac{Ib}{2\pi a\sqrt{a^2+b^2}}.}$$

(b) Each side is at perpendicular distance a and half-length $b = a\tan(\pi/N)$; all centre fields are parallel. Thus $\boxed{H = NI\sin(\pi/N)/(2\pi a)}$. As $N \to \infty$, this tends to $I/(2a)$, the field strength at the centre of a circular loop.

E.2. Axial coil field and Helmholtz coils

(a) Show that a circular coil of N turns, radius a and current I produces axial field $H(x) = NIa^2/[2(a^2+x^2)^{3/2}]$.

(b) Two identical coaxial coils, separated by a, carry equal currents in the same sense. Show that both first and second axial field derivatives vanish at the midpoint. Explain the practical use.

Solution.

On the axis, transverse contributions cancel and Biot–Savart gives $H(x) = NIa^2/[2(a^2+x^2)^{3/2}]$. For separation a, add $H(x-a/2) + H(x+a/2)$. The sum is even, so its first and third derivatives vanish at zero. Since the second derivative of $(a^2+x^2)^{-3/2}$ is $3(4x^2-a^2)/(a^2+x^2)^{7/2}$, it also vanishes there. The central field is $\boxed{B_0 = \mu_0 NI(4/5)^{3/2}/a}$; its leading variation is quartic, providing a comparatively uniform calibration field.

E.3. Distant field in the coil plane

Find the magnetic flux density at an in-plane point a distance $r \gg a$ from the centre of a circular coil of radius a, N turns and current I. Verify it by replacing the coil with its magnetic dipole moment, retaining the field direction.

Solution.

The magnetic moment is $m = NI\pi a^2\hat{n}$. Expanding Biot–Savart for $r \gg a$ gives the dipole field

$$B = \mu_0[3(m\cdot\hat{r})\hat{r} - m]/(4\pi r^3).$$

In the coil plane $m\cdot\hat{r} = 0$, hence

$$\boxed{B = -\frac{\mu_0 NIa^2}{4r^3}\hat{n}.}$$

Its magnitude agrees with the stated result; its direction is opposite to the coil's centre field.

E.4. Finite solenoid

A solenoid of radius a and length l has n turns per axial length and current I. Its central axial flux density is $\sqrt{2}$ times the value at either end. Find l/a.

Solution.

Integrating circular-coil fields over the winding gives

$$B(z) = \frac{\mu_0 n I}{2}\left[\frac{z + l/2}{\sqrt{a^2 + (z + l/2)^2}} - \frac{z - l/2}{\sqrt{a^2 + (z - l/2)^2}}\right].$$

Thus

$$B(0)/B(l/2) = 2\sqrt{a^2 + l^2}/\sqrt{4a^2 + l^2}.$$

Set this equal to $\sqrt{2}$ and square: $4(a^2 + l^2) = 2(4a^2 + l^2)$, giving $\boxed{l/a = \sqrt{2}}$.

E.5. Cylindrical current annulus

Uniform axial current density $J\hat{\mathbf{z}}$ fills $a^2 < x^2 + y^2 < b^2$. Find and sketch the magnetic-field magnitude in $r < a$, $a < r < b$ and $r > b$.

Solution.

Apply $2\pi r B_\phi = \mu_0 I_{\text{enc}}$ with enclosed areas $0, \pi(r^2 - a^2), \pi(b^2 - a^2)$. Therefore

$$\boxed{B_\phi(r) = \begin{cases} 0 & r < a, \\ \mu_0 J(r^2 - a^2)/(2r) & a \le r \le b, \\ \mu_0 J(b^2 - a^2)/(2r) & r > b. \end{cases}}$$

The magnitude is continuous, rises monotonically through the annulus and falls as $1/r$ outside, reaching its maximum at $r = b$.

2.5 Electromagnetism: Problem Set 4

Section F

F.0. Induction

State Faraday's and Lenz's laws, and define self-inductance and mutual inductance.

Solution.

Faraday's law is $\mathcal{E} = -d\Phi_B/dt$ for a fixed loop, and its moving-loop form is

$$\oint (E + v \times B) \cdot d\ell = -d\Phi_B/dt.$$

Lenz's minus sign states that induced current opposes the change of flux. Flux linkage defines $N\Phi = LI$ for self-inductance and $N_2\Phi_{21} = MI_1$ for mutual inductance in a linear system.

F.1. Moving rectangular coil

A high-resistance rectangular loop lies in the plane of a long wire carrying current I. Its near side of length a is parallel to the wire, at distance D; its width away from the wire is b. It moves away at speed v.

(a) Derive the induced emf using motional emfs and flux change separately.

(b) Explain the high-resistance approximation.

(c) For resistance R, find the required force and show that mechanical power equals Joule heating.

Solution.

The external flux is $\Phi = \mu_0 I a \ln[(D+b)/D]/(2\pi)$. With $\dot{D} = v$,

$$\boxed{|\mathcal{E}| = \frac{\mu_0 I a b v}{2\pi D(D+b)}.}$$

The same result is $av[B(D) - B(D+b)]$ from the two parallel sides; the other two sides contribute zero. High resistance makes the induced current and its magnetic back-reaction small and permits neglect of self-inductance on the motion timescale. With $i = \mathcal{E}/R$, the opposing force is $ia[B(D) - B(D+b)] = \mathcal{E}^2/(Rv)$, so $\boxed{Fv = i^2 R}$.

F.2. Sliding rod

Parallel horizontal rails a distance L apart are joined by resistance R at $x = 0$. A conducting rod closes the circuit and slides at $x = vt$ in a uniform perpendicular field B. Neglect rail and rod resistance and self-inductance. Find (a) current, (b) required external force, (c) supplied power and (d) dissipated power, and compare them.

Solution.

The swept area grows at rate Lv, so $\boxed{|\mathcal{E}| = BLv, \quad |i| = BLv/R}$. Lenz's law fixes the direction to oppose increasing flux. The magnetic drag is BiL, requiring external force $\boxed{F = B^2L^2v/R}$. Therefore $\boxed{P_1 = Fv = B^2L^2v^2/R = i^2R = P_2}$: the mechanical input becomes Joule heat.

F.3. Homopolar generator

A conducting disc of radius $a = 3\,\mathrm{m}$, mass $m = 10^4\,\mathrm{kg}$ and rotation frequency $3000\,\mathrm{min}^{-1}$ is in axial field $B = 0.5\,\mathrm{T}$.

(a) Derive the axis-to-rim emf $\omega a^2 B/2$.

(b) A load $R = 10^{-3}\,\Omega$ is connected between axis and rim. Neglect other resistance and mechanical friction. Find initial current and the time to halve its speed.

Solution.

A radial element has motional emf $dV = B\omega r\,dr$, hence $V = B\omega a^2/2$. At $f = 50\,\mathrm{s}^{-1}$, $\omega_0 = 100\pi\,\mathrm{s}^{-1}$, so

$$\boxed{V_0 = 706.9\,\mathrm{V},\ I_0 = 7.07\times10^5\,\mathrm{A}}.$$

Write $K = Ba^2/2$ and $J = ma^2/2$. Electrical power is $K^2\omega^2/R = -J\omega\dot{\omega}$, so $\omega = \omega_0 e^{-K^2 t/(JR)}$. Thus

$$\boxed{t_{1/2} = \frac{2mR\ln 2}{B^2a^2} = 6.16\,\mathrm{s}.}$$

F.4. Coaxial self-inductance

A coaxial cable has a solid inner conductor of radius a and an outer conductor occupying $b < r < d$. Estimate its inductance per unit length for $(b-a) \gg a$ and $(b-a) \gg (d-b)$, explaining why these assumptions permit neglect of the internal magnetic-energy contributions.

Solution.

In the gap $B = \mu_0 I/(2\pi r)$. Its energy per unit length is

$$U' = \int_a^b B^2(2\pi r)dr/(2\mu_0) = \mu_0 I^2\ln(b/a)/(4\pi).$$

Thus

$$\boxed{L' \simeq \frac{\mu_0}{2\pi}\ln\frac{b}{a}.}$$

The approximation neglects flux and magnetic energy inside the conducting material; the stated wide-gap and thin-outer-wall assumptions motivate it. For uniform DC current a solid inner wire additionally contributes $\mu_0/(8\pi)$ per unit length, so the logarithmic approximation requires this to be small compared with the gap term.

F.5. Small coil and ring

A small coil of N turns and area A carries constant current I. It is coaxial with a ring of radius R at separation $d = d_0 + a\cos\omega t$. Use reciprocity to derive the induced voltage, with magnitude

$$|V| = \left|\frac{3\mu_0 NAI\omega aR^2 d}{2(R^2+d^2)^{5/2}}\sin\omega t\right|.$$

State the orientation convention for its sign.

Solution.

Reciprocity gives $M(d) = \mu_0 NAR^2/[2(R^2+d^2)^{3/2}]$, obtained from the large ring's on-axis field at the small coil. With constant I, $\mathcal{E} = -IM'(d)\dot{d}$, and $\dot{d} = -a\omega\sin\omega t$. Thus the emf magnitude, with polarity depending on the winding convention, is

$$\boxed{|\mathcal{E}| = \left|\frac{3}{2}\mu_0 NAI\omega\frac{aR^2 d}{(R^2+d^2)^{5/2}}\sin\omega t\right|.}$$

Reversing the voltage terminals reverses the sign of the induced voltage.

F.6. Coaxial solenoids

Two fully overlapping coaxial solenoids of length l, each with n turns per length, have radii $a, 2a$.

(a) Find each interior field far from its ends.

(b) Neglect end effects and calculate both self-inductances and mutual inductance.

(c) If the outer self-inductance is 40 mH and its current decreases at $2\,\mathrm{A\,s^{-1}}$, find the induced inner emf.

Solution.

Each ideal long solenoid has $B = \mu_0 n I$ inside and negligible field outside. Flux linkage gives

$$\boxed{L_1 = \mu_0 n^2 l \pi a^2, \quad L_2 = 4L_1, \quad M = \mu_0 n^2 l \pi a^2 = L_1.}$$

For $L_2 = 40\,\mathrm{mH}$, $M = 10\,\mathrm{mH}$. Therefore

$$|\mathcal{E}_1| = M|\dot{I}_2| = \boxed{20\,\mathrm{mV}};$$

its direction opposes the decreasing linked flux.

F.7. Magnetic energy

Derive the inductor energy $U = LI^2/2$ and hence the vacuum magnetic energy density $u_B = B^2/(2\mu_0)$.

Solution.

The source supplies power $P = I\mathcal{E} = LI\dot{I}$. Integration from zero current gives $\boxed{U = LI^2/2}$. For a long solenoid, $L = \mu_0 n^2 \mathcal{V}$ and $B = \mu_0 n I$, so $\boxed{u_B = U/\mathcal{V} = B^2/(2\mu_0)}$.

2.6 Electromagnetism: Problem Set 5

Section G

G.0. Lorentz force

State the force on a particle of charge q in electric and magnetic fields.

Solution.

In SI units $\boxed{F = q(E + v \times B)}$. The magnetic part is perpendicular to velocity and does no work: $dK/dt = qE \cdot v$. This distinguishes acceleration of the speed by E from bending by a magnetic field.

G.1. Bainbridge spectrometer

In a Bainbridge spectrometer, ions pass through a crossed-field velocity selector and then a magnetic-only analysing chamber.

(a) Sketch the apparatus and field directions.

(b) Find the selected speed for $E = 100\,\mathrm{V\,cm^{-1}}$ and $B = 0.2\,\mathrm{T}$.

(c) Can a 1 mm-wide beam separate $^3\mathrm{He}^+$ and $^4\mathrm{He}^+$ in a semicircular analyser?

Solution.

Choose the beam along $+x$, B along $+z$ and E along $+y$. Then $E + v \times B = 0$ selects

$$\boxed{v = E/B = 5.0 \times 10^4\,\mathrm{m\,s^{-1}}}.$$

In the magnetic-only chamber, $r = mv/(qB)$. For singly charged helium, $r_3 = 7.78\,\mathrm{mm}$ and $r_4 = 10.37\,\mathrm{mm}$. Their semicircular detector positions differ by

$$2(r_4 - r_3) = \boxed{5.18\,\mathrm{mm}},$$

exceeding the stated 1 mm beam width, so the model predicts separation.

G.2. Helical motion

(a) Show that generic motion of a charge in uniform constant $\mathbf{B}$ is helical.

(b) Particles of charge e, mass m and speed v leave a point source nearly parallel to $\mathbf{B}$. Show paraxial focusing at axial distances $2\pi mv/(Be)$ and its integer multiples.

(c) An electron has $v_\parallel = 0.01c$ in $B = 10^{-9}\,\mathrm{T}$. Find the number of turns over an axial light-year, using

$$e/m = 1.76 \times 10^{11}\,\mathrm{C\,kg^{-1}}.$$

Solution.

Take $B = B\hat{z}$ and $\omega_c = qB/m$. Then $\dot{v}_x = \omega_c v_y$, $\dot{v}_y = -\omega_c v_x$, $\dot{v}_z = 0$. Transverse velocity rotates at $|\omega_c|$ while longitudinal velocity is constant, producing a helix of radius $mv_\perp/(|q|B)$ and pitch $2\pi mv_\parallel/(|q|B)$. A small-angle beam returns to the axis after each cyclotron period, giving focus spacing $\boxed{2\pi mv/(eB)}$ to paraxial order. The stated electron makes

$$N = (eB/2\pi m)(1\,\mathrm{ly}/0.01c) = \boxed{8.84 \times 10^{10}}$$

revolutions.

G.3. Quadrupole lens

A particle (m, q) enters near the z axis at longitudinal speed v in a long quadrupole field $B_x = Ay$, $B_y = Ax$, $B_z = 0$.

(a) Derive paraxial transverse equations, neglecting end effects.

(b) Show focusing in one plane and defocusing in the other.

(c) For initially parallel rays, show that the first focal distance is $(\pi/2)\sqrt{mv/|qA|}$.

Solution.

With $v_z \simeq v$ constant, $m\ddot{x} = -qAvx$ and $m\ddot{y} = qAvy$. Thus one transverse direction oscillates while the other grows or decays exponentially; changing the sign of qA exchanges them. In the focusing direction, write $\ddot{x} + \Omega^2 x = 0$, $\Omega^2 = |qA|v/m$. A parallel incident ray has $\dot{x}(0) = 0$ and first reaches the axis at $t = \pi/(2\Omega)$, so

$$\boxed{z = \frac{\pi}{2}\sqrt{\frac{mv}{|qA|}}.}$$

Section H

H.0. Maxwell's equations

State Maxwell's equations with charge and current in vacuum and interpret each equation.

Solution.

In vacuum with sources,

$$\boxed{\nabla \cdot E = \rho/\epsilon_0, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\partial_t B, \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \partial_t E.}$$

These express electric flux from charge, absence of magnetic monopoles, induction, and the production of circulating magnetic field by conduction and displacement currents. Taking the divergence of the last equation recovers charge conservation.

H.1. Finite wire and displacement current

(a) Use Ampère's law for a long straight wire; explain why the same symmetry argument fails for a finite segment.

(b) For a segment of length $2b$, include the displacement current associated with endpoint charge accumulation to find B at perpendicular-bisector distance a. Compare with Biot–Savart, Question E.1(a).

Solution.

(a) For a long steady wire, cylindrical symmetry and Ampere's law give $B = \mu_0 I/(2\pi a)$. An isolated finite segment cannot carry steady current without charge building up at its ends; magnetostatic Ampere's law by itself is inadequate.

(b) Model charges accumulating at $z = \pm b$ at rates $\pm I$. Through a disk of radius a midway between the ends their changing electric flux gives $I_d = -I[1 - b/\sqrt{a^2 + b^2}]$. Therefore

$$\boxed{2\pi a B = \mu_0(I + I_d), \quad B = \frac{\mu_0 I b}{2\pi a\sqrt{a^2 + b^2}}.}$$

This agrees with Biot–Savart for the segment in the quasi-static model.

H.2. Vacuum waves

(a) Derive the vacuum electromagnetic wave equations.

(b) Give plane waves and their speed.

(c) Find $\mathbf{B}$ for $\mathbf{E} = (E_0 \sin(kz - \omega t), 0, 0)$.

(d) Find the free-space impedance $|E|/|H|$.

Solution.

With $\rho = J = 0$, take the curl of Faraday's law. Since

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E,$$

one obtains $\nabla^2 E = \mu_0 \epsilon_0 \partial_t^2 E$ and likewise for B. Plane waves require

$$\boxed{\omega = ck,\ c = (\mu_0 \epsilon_0)^{-1/2}}.$$

The stated field propagates along $+z$ with

$$\boxed{B = (0, E_0 \sin(kz - \omega t)/c, 0), \quad Z_0 = |E|/|H| = \sqrt{\mu_0/\epsilon_0} \simeq 377\,\Omega.}$$

H.3. Poynting flux

(a) Find the instantaneous and mean Poynting vectors of a vacuum plane wave.

(b) Solar luminosity is $3.83 \times 10^{26}\,\mathrm{W}$ and light takes 8.3 min to reach Earth. Find the mean flux at the Earth's surface after 30% atmospheric absorption.

Solution.

$S = E \times H = (E^2/Z_0)\widehat{k}$ for a plane wave. For sinusoidal peak field E_0, $\langle S \rangle = E_0^2/(2Z_0)$. The light-travel time gives

$$r = c(8.3 \times 60\,\mathrm{s}) = 1.493 \times 10^{11}\,\mathrm{m}.$$

Isotropic solar radiation after the stated atmospheric loss has

$$\boxed{\langle S \rangle = \frac{0.70(3.83 \times 10^{26})}{4\pi r^2} \simeq 9.57 \times 10^2\,\mathrm{W\,m^{-2}}.}$$

H.4. Energy entering a resistive rod

A long resistive rod of length l, radius r and resistance R carries uniform current I. Neglect end effects. Find the magnitude and direction of the surface Poynting vector and relate its inward flux to total dissipated power.

Solution.

Inside the rod, $E_z = IR/l$. At the cylindrical surface $H_\phi = I/(2\pi r)$. Hence

$$\boxed{S_r = -\frac{I^2 R}{2\pi r l}.}$$

The direction is radially inward. Multiplying by the lateral area gives inward power $2\pi r l |S_r| = I^2 R$, exactly the Joule dissipation. Electromagnetic energy enters through the surrounding field and is converted to heat in the conductor.

2.7 Electromagnetism: Revision

1.1. Coulomb's law

State Coulomb's law, define $\mathbf{E}(\mathbf{r})$ from it, and explain what it means for an electrostatic field to be conservative.

Solution.

For separation vector r from Q_1 to Q_2, $F_{12} = Q_1 Q_2 r/(4\pi\epsilon_0 |r|^3)$. Dividing the test-charge force by that charge defines E. Electrostatic conservativity means $\oint E \cdot d\ell = 0$, so work between endpoints is path independent and a scalar potential exists with $E = -\nabla V$.

1.2. Thundercloud and images

Model a thundercloud by charges $+40\,\mathrm{C}$ at $10\,\mathrm{km}$ and $-40\,\mathrm{C}$ at $6\,\mathrm{km}$ above an infinite conducting ground plane. Find the surface electric field directly below them, including its direction.

Solution.

Include opposite image charges below ground. Directly beneath a charge Q at height h, the vertical field is $-2k_eQ/h^2$. Thus, with upward positive,

$$\boxed{E_z = 2k_e\left[40/(6000)^2 - 40/(10000)^2\right] = 1.28\times 10^4\,\mathrm{V\,m^{-1}},}$$

directed upward. The closer negative charge dominates.

1.3. Assembly energy

For localised charges q_i with mutual potentials V_i excluding self-potentials, derive $W = \frac{1}{2}\sum_i q_i V_i$.

Solution.

Assembling charges one by one gives $W = \sum_{i<j} k_e q_i q_j / r_{ij}$. The sum

$$\sum_i q_i V_i = \sum_{i\neq j} k_e q_i q_j / r_{ij}$$

counts each pair twice. Therefore $\boxed{W = \frac{1}{2}\sum_i q_i V_i}$, with V_i excluding the charge's own singular potential.

1.4. Charged sphere energies

A sphere of radius a, far from other objects, carries charge q. Find its surface potential and assembly energy for charge distributed uniformly (a) over its surface and (b) throughout its volume, with potential zero at infinity.

Solution.

Gauss's law makes the surface potential $\boxed{V(a) = k_e q/a}$ in both cases. For a shell, $W = \int_0^Q k_e q\, dq/a = k_e Q^2/(2a)$. For a uniform solid sphere integrate $k_e Q(r) dQ/r$ with $Q(r) = Q(r/a)^3$, obtaining

$$\boxed{W_{\text{shell}} = q^2/(8\pi\epsilon_0 a),\ W_{\text{solid}} = 3q^2/(20\pi\epsilon_0 a)}.$$

1.5. Uniform density sphere

Find and sketch $E(r)$ and $V(r)$ for a sphere of radius R and uniform positive volume density ρ_0, with $V(\infty) = 0$.

Solution.

Set $Q = 4\pi\rho_0 R^3/3$. Gauss's law and $V(\infty) = 0$ give

$$\boxed{E_r = \begin{cases} \rho_0 r/(3\epsilon_0) & r \le R \\ \rho_0 R^3/(3\epsilon_0 r^2) & r \ge R, \end{cases} \qquad V = \begin{cases} \rho_0(3R^2 - r^2)/(6\epsilon_0) & r \le R \\ \rho_0 R^3/(3\epsilon_0 r) & r \ge R. \end{cases}}$$

The field rises linearly then decreases as r^{-2}; the potential decreases quadratically inside and as r^{-1} outside. Both are continuous at R.

1.6. Hydrogen density

A hydrogen electron cloud has $\rho(r) = -e\exp(-2r/a_0)/(\pi a_0^3)$, with $a_0 = 5.3 \times 10^{-11}\,\mathrm{m}$. Derive its radial field

$$E_r = \frac{e}{4\pi\epsilon_0}\left[\frac{e^{-2r/a_0} - 1}{r^2} + \frac{2e^{-2r/a_0}}{a_0 r} + \frac{2e^{-2r/a_0}}{a_0^2}\right].$$

Explain its response to a steady uniform external field E_0.

Solution.

Integrating $Q(r) = 4\pi\int_0^r \rho(s)s^2 ds$ gives

$$Q(r) = -e[1 - e^{-2r/a_0}(1 + 2r/a_0 + 2r^2/a_0^2)].$$

Divide by $4\pi\epsilon_0 r^2$ to obtain the stated electric field. Near zero $E_r = -er/(3\pi\epsilon_0 a_0^3) + O(r^2)$. A uniform external field displaces the negative cloud opposite to the field relative to the positive nucleus; the induced dipole points along it. In the rigid-cloud approximation $\boxed{p = 3\pi\epsilon_0 a_0^3 E_0}$.

1.7. Isolated inner sphere

Concentric conducting spheres of radii $a < b$ enclose air of relative permittivity one. Derive their capacitance. The inner sphere is charged to voltage V, isolated, and the grounded outer sphere removed. Show $V' = bV/(b-a)$. For $a = 0.9\,\mathrm{m}$, $b = 1.0\,\mathrm{m}$ and breakdown field $3000\,\mathrm{V\,mm^{-1}}$, find the greatest initial V.

Solution.

Integrating the radial field gives $\boxed{C = 4\pi\epsilon_0 ab/(b-a)}$. Isolation fixes $q = CV$. After removing the outer sphere, $V' = q/(4\pi\epsilon_0 a) = \boxed{bV/(b-a)}$. Initially $E(a) = Vb/[a(b-a)]$ is the maximum field. Therefore

$$V_{\max} = E_b a(b-a)/b = \boxed{2.70 \times 10^5\,\mathrm{V}}$$

for the stated values.

1.8. Dipole fields and transverse direction

A dipole consists of charges $\pm q$ separated by $2l$, with $\mathbf{p}$ directed from negative to positive. At $r \gg l$, derive (a) the potential, (b) radial and polar field components, and (c) its torque in a uniform field. Find the polar angle where its field is perpendicular to the dipole axis.

Solution.

Expand the two Coulomb potentials for $r \gg l$: $V = k_e p \cos\theta / r^2$, $E_r = 2k_e p \cos\theta / r^3$, $E_\theta = k_e p \sin\theta / r^3$. Summing torques on the two charges gives $\boxed{\tau = p \times E}$. The axial component of the dipole's own field is

$$E_z = E_r \cos\theta - E_\theta \sin\theta = k_e p (3\cos^2\theta - 1)/r^3.$$

It vanishes at $\boxed{\theta = \cos^{-1}(\pm 1/\sqrt{3})}$, i.e. 54.74° and 125.26°.

1.9. Dipole interaction energy

A second dipole of equal magnitude $p = 2ql$ is brought from infinity to distance r on the equatorial plane of the first, with angle θ between their axes. Show that the assembly work is $W = 4q^2 l^2 \cos\theta / (4\pi\epsilon_0 r^3)$.

Solution.

In the equatorial plane of dipole p_1, $E_1 = -k_e p_1 / r^3$. Hence the work in bringing dipole p_2 from infinity at fixed orientation is $W = -p_2 \cdot E_1 = k_e p^2 \cos\theta / r^3$. With $p = 2ql$,

$$\boxed{W = \frac{4q^2 l^2 \cos\theta}{4\pi\epsilon_0 r^3}.}$$

1.10. Orientation of a free dipole

A second dipole is free to rotate. Find its stable orientation relative to the first when placed (i) on its axis, $\theta = 0$, and (ii) in its equatorial plane, $\theta = \pi/2$.

Solution.

A free dipole minimizes $U = -p_2 \cdot E_1$, so it aligns with the local field. On the first dipole's positive axis the field is parallel to p_1, giving $\boxed{p_2 \parallel p_1}$. In its equatorial plane the field is antiparallel, giving $\boxed{p_2 \parallel -p_1}$.

2.1. Coil and Helmholtz pair

State Biot–Savart and derive the axial field of an n-turn circular coil of radius a carrying I. For two identical coaxial coils separated by d and carrying equal same-sense currents, write $B(x)$ about their midpoint. Show $B'(0) = 0$, find d for which $B''(0) = 0$, and verify that the change between $x = 0$ and $x = d/2$ is below 6%.

Solution.

Biot–Savart is

$$dB = \mu_0 I \, d\ell \times \widehat{R} / (4\pi R^2).$$

Axial integration gives $B(z) = \mu_0 n I a^2 / [2(a^2 + z^2)^{3/2}]$. The pair field is $B'(x) = B(x - d/2) + B(x + d/2)$; evenness gives $dB'/dx = 0$ at zero. Its second derivative is zero iff $\boxed{d = a}$. For this separation,

$$\frac{B'(a/2)}{B'(0)} = \frac{1 + 1/(2\sqrt{2})}{2(4/5)^{3/2}} = 0.9458,$$

a 5.42% reduction, below 6%.

2.2. Solenoid energy

For an ideal long solenoid of radius R and N turns per length, derive its uniform interior field and inductance per length. A superconducting solenoid has radius 0.5 m, length 7 m, 1000 total turns and current 5000 A. Estimate its field and stored energy using the long-solenoid approximation.

Solution.

An Ampere rectangle crossing a long winding gives $\boxed{B = \mu_0 N I}$ inside and approximately zero outside; cylindrical symmetry removes radial dependence in the ideal model. Flux linkage gives $L/l = \mu_0 N^2 \pi R^2$. For 1000 total turns over 7 m, $N = 1000/7\,\mathrm{m}^{-1}$, so

$$\boxed{B = 0.898\,\mathrm{T},\ L = 0.141\,\mathrm{H},\ U = LI^2/2 = 1.76\,\mathrm{MJ}}.$$

2.3. Coaxial cable

A long coax has thin-walled conductors of radii $a < b$, carrying equal opposite currents I. Find $B(r)$ in all three regions and derive $L = \mu_0 l \ln(b/a)/(2\pi)$. Sketch $B(r)$ if the inner conductor instead is solid and carries uniform current density, from $r = 0$ to $2b$.

Solution.

Ampere's law gives $\boxed{B_\phi = \mu_0 I/(2\pi r)}$ for $a < r < b$, and zero for $r < a$ or $r > b$ in the thin-shell model. Integrating $B^2/(2\mu_0)$ over the gap gives $\boxed{L = \mu_0 l \ln(b/a)/(2\pi)}$. For a solid inner wire with uniform current density, $B_\phi = \mu_0 I r/(2\pi a^2)$ inside: the graph rises linearly from zero, joins continuously to $1/r$ at a, and drops to zero across the outer sheet at b.

3.1. Annular generator

State the laws of induction. An annular metal disc has outer radius a, inner radius $a/2$, thickness d and resistivity ρ, with ideal ring brushes around both edges.

(a) Find its radial resistance.

(b) Find the brush voltage when it rotates at ω in axial field B.

(c) Find the load resistance for maximum delivered power.

Solution.

Faraday–Lenz: $\mathcal{E} = -d\Phi/dt$, with motional emf $\int (v \times B) \cdot d\ell$ where appropriate. A radial shell has resistance $dR = \rho\, dr/(2\pi r d)$. Hence

$$\boxed{R_D = \rho \ln 2/(2\pi d), \quad V = \int_{a/2}^{a} \omega B r\, dr = 3\omega B a^2/8.}$$

Load power is $V^2 R/(R + R_D)^2$; differentiating gives its maximum at $\boxed{R = R_D}$.

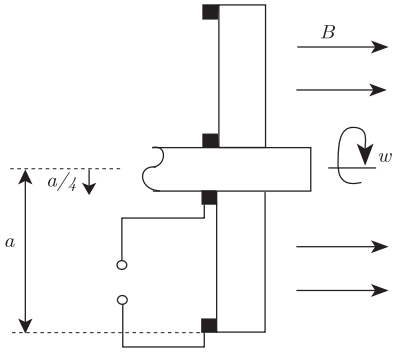
3.2. Annulus with a finite axle

A homopolar generator has disc radius a, axle radius $a/4$, thickness t, resistivity ρ and ideal ring brushes. It rotates at ω in axial field B.

(a) Find R_D.

(b) Derive emf $15\omega Ba^2/32$.

(c) With load R_L and drive removed, find the half-speed time, taking total inertia $ma^2/2$ and neglecting mechanical friction.



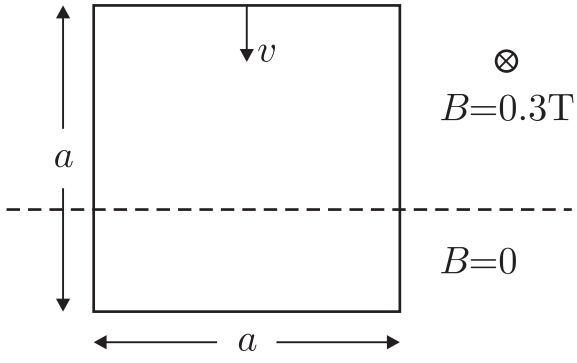
Solution.

Radial integration yields $\boxed{R_D = \rho \ln 4/(2\pi t)}$ and $V = K\omega$, $K = \int_{a/4}^{a} Br\,dr = 15Ba^2/32$. With $R_T = R_L + R_D$, electrical power is $K^2\omega^2/R_T$. Equating this to $-J\omega\dot{\omega}$, with $J = ma^2/2$, gives

$$\boxed{\tau_{1/2} = \frac{JR_T \ln 2}{K^2} = \left(\frac{32}{15}\right)^2 \frac{m(R_L + R_D)\ln 2}{2a^2B^2}.}$$

3.3. Falling loop

A vertical square wire loop of side a falls out of a horizontal field B, with speed v and resistance R. Show that the current while the upper side alone remains in the field is $I = Bav/R$, and describe all magnetic forces. Find the terminal falling speed for $a = 10\,\mathrm{cm}$, wire diameter $1\,\mathrm{mm}$, copper resistivity $1.7 \times 10^{-8}\,\Omega\,\mathrm{m}$, density $8960\,\mathrm{kg\,m^{-3}}$ and $B = 0.3\,\mathrm{T}$.



Solution.

As the loop leaves the field, $|\dot{\Phi}| = Bav$, so $\boxed{I = Bav/R}$. The upper horizontal side experiences an upward force IBa; opposite forces on the two vertical sides cancel horizontally. Terminal motion requires $mg = B^2a^2v/R$, hence $v_t = mgR/(B^2a^2)$. For cross-section $A_w = \pi(0.5\,\mathrm{mm})^2$, $R = 4a\rho/A_w = 8.66\,\mathrm{m\Omega}$ and $m = 4aA_w\rho_m = 2.81\,\mathrm{g}$. Thus $\boxed{v_t = 0.266\,\mathrm{m\,s^{-1}}}$ if reached before the upper side exits.

3.4. Inclined rails

Rails separated by d run down a plane inclined at θ and are joined at the bottom by resistance R. A horizontal conducting bar of mass m slides freely from rest down them in a uniform vertical field B. Neglect other resistance. Find the current, equation of motion and terminal speed.

Solution.

Only the field normal to the incline contributes to flux: $B_n = B\cos\theta$. Therefore $\boxed{I = Bdv\cos\theta/R}$, with circulation opposing the increasing flux. The force along the incline is $-IBd\cos\theta$, so

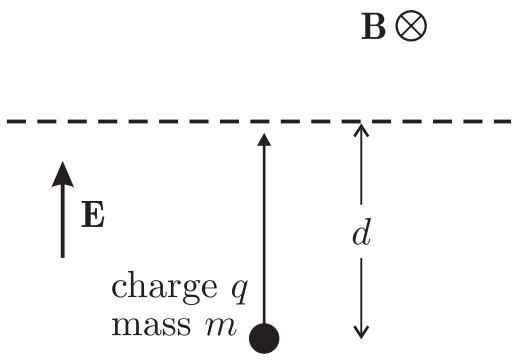
$$m\dot{v} = mg\sin\theta - \frac{B^2d^2\cos^2\theta}{R}v.$$

With release from rest, $v = v_t(1 - e^{-t/\tau})$, where

$$\boxed{v_t = mgR\sin\theta/(B^2d^2\cos^2\theta),\ \tau = mR/(B^2d^2\cos^2\theta)}.$$

3.5. Finite magnetic region

A positive charge q of mass m enters a magnetic slab $0 < x < b$ at speed $v\hat{\mathbf{x}}$, with field $B\hat{\mathbf{y}}$. Find the condition to reach $x > b$ and the exit angle. In a second experiment it is accelerated from rest through distance d in constant field E before entering a perpendicular magnetic field, as shown. Find $|B|$ for return to the electric-field boundary at separation d, sketch the magnetic arc and find its traversal time.



Solution.

The first trajectory is a circle of radius $r = mv/(q|B|)$ in the xz plane. It reaches $x > b$ iff $\boxed{r > b}$; at exit its deflection satisfies $\boxed{\sin\alpha = b/r = q|B|b/(mv)}$. Equality only grazes the boundary. In the second experiment, electric work gives $v = \sqrt{2qEd/m}$. To re-enter the electric region a distance d along its boundary, the magnetic semicircle needs diameter d, so

$$\boxed{|B| = \frac{2mv}{qd} = \sqrt{\frac{8mE}{qd}}, \qquad t_B = \frac{\pi m}{q|B|} = \frac{\pi d}{2v}.}$$

4.1. Displacement current and waves

Use a charging parallel-plate capacitor to explain displacement current $I_d = \epsilon_0 A\, dE/dt$. Derive differential Ampère–Maxwell law. State source-free vacuum Maxwell equations and deduce the wave equation and speed for $\mathbf{E}$.

Solution.

For a charging capacitor $Q = \epsilon_0 AE$, so $I_c = dQ/dt = \epsilon_0 A\dot{E}$. Surfaces through the wire or through the gap must give the same magnetic circulation; thus the gap carries displacement current $\boxed{I_d = \epsilon_0 A\dot{E}}$. Stokes's theorem gives

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \partial_t E.$$

In source-free vacuum, combine this with $\nabla \times E = -\partial_t B$ and zero divergences to obtain

$$\boxed{\nabla^2 E = \mu_0 \epsilon_0 \partial_t^2 E,\ c = 1/\sqrt{\mu_0 \epsilon_0}}.$$

4.2. Maxwell plane wave

State vacuum Maxwell equations with sources. Derive source-free wave equations for $\mathbf{E}, \mathbf{B}$ and their propagation speed. Construct a plane wave with only E_y, B_x nonzero, determine the propagation direction and amplitude relation, and sketch the field directions.

Solution.

The four source equations are $\nabla \cdot E = \rho/\epsilon_0$, $\nabla \cdot B = 0$, $\nabla \times E = -\partial_t B$,

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \partial_t E.$$

In vacuum without sources their curls give equal-speed wave equations for both fields. For example choose

$$\boxed{E_y = E_0 \cos(kz - \omega t), \quad B_x = -E_0 \cos(kz - \omega t)/c, \quad \omega = ck.}$$

Then $E \times B$ points along $+z$, and $|E_y| = c|B_x|$. Reversing the relative field sign reverses propagation.

2.8 Geometrical Optics

1. Huygens and refraction

State Huygens's principle and use it to derive Snell's law.

Solution.

Every point on an advancing wavefront acts as a source of secondary spherical wavelets; their envelope is the next wavefront. During the time one end of an oblique front travels a distance $v_1\Delta t$ to an interface, a wavelet in the second medium travels $v_2\Delta t$. The common interface geometry gives

$$\sin\theta_1/\sin\theta_2 = v_1/v_2 = n_2/n_1,$$

hence $\boxed{n_1\sin\theta_1 = n_2\sin\theta_2}$.

2. Critical angles

Derive the critical angle in terms of n_1, n_2. Calculate it for light entering air from ordinary glass ($n = 1.5$), lead crystal (1.8) and diamond (2.4). Discuss the relevance to jewellery.

Solution.

At critical incidence the refracted ray is tangential, so $\boxed{\sin\theta_c = n_2/n_1}$ for $n_1 > n_2$. For ordinary glass, lead crystal and diamond into air the angles are

$$\boxed{41.81^\circ, 33.75^\circ, 24.62^\circ}$$

respectively. Higher-index materials trap rays over a wider range of internal incidence; suitable facet angles exploit repeated total internal reflections to increase brilliance.

3. Fibre numerical aperture

A fibre has core index n_c and cladding index n_s, with flat end faces perpendicular to its axis. A ray enters on axis from index n_1 at incidence angle θ_i. Derive its numerical aperture, the largest allowed $n_1\sin\theta_i$ for guided meridional rays.

Solution.

If the internal ray angle to the fibre axis is θ, incidence at the side wall is $\pi/2 - \theta$. Total reflection requires $\cos\theta \geq n_s/n_c$. Entrance refraction gives $n_1\sin\theta_i = n_c\sin\theta$, hence

$$\boxed{(n_1\sin\theta_i)_{\max} = \min\{n_1, \sqrt{n_c^2 - n_s^2}\}.}$$

The usual numerical aperture $\sqrt{n_c^2 - n_s^2}$ applies when it does not exceed the outside medium's index; guiding requires $n_c > n_s$.

4. Apparent depth

A heron views a fish nearly vertically below the water surface, at actual depth 40 cm. Find the apparent depth for water index 4/3.

Solution.

For a paraxial ray emerging from water into air, $n_w\theta_w \simeq \theta_a$. The same lateral displacement satisfies $x = d\theta_w = d_{\text{app}}\theta_a$. Thus

$$\boxed{d_{\text{app}} = d/n_w = 40/(4/3) = 30\,\text{cm}}.$$

5. Thin-lens formula

For a thin biconvex lens of index n_2 in medium $n_1 < n_2$, derive $1/u+1/v = 1/f$ and $1/f = (n_2/n_1-1)(1/R_1+1/R_2)$, where R_1, R_2 are positive curvature magnitudes. State the paraxial and thin-lens assumptions.

Solution.

Paraxial refraction at a spherical surface gives $n_1/u + n_2/v = (n_2 - n_1)/R$ with a consistent signed-radius convention. Apply this to both surfaces, add, and neglect lens thickness; internal image distances cancel. Using positive magnitudes R_1, R_2 for a biconvex lens gives

$$\boxed{\frac{1}{u}+\frac{1}{v}=\frac{1}{f}, \qquad \frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right).}$$

The derivation assumes paraxial rays, a thin lens and homogeneous isotropic media.

6. Immersed lens

For lens index 1.60, find the factor by which its focal length changes on immersion from air into water of index 1.33.

Solution.

At fixed curvature $f^{-1} \propto n_g/n_m - 1$. Therefore

$$\boxed{\frac{f_{\text{water}}}{f_{\text{air}}} = \frac{1.60-1}{1.60/1.33-1} = 2.956.}$$

The lens is weaker in water because the refractive-index contrast is smaller.

7. Two lens positions

A source and screen are separated by fixed distance s. Show that a converging lens has two positions producing a focused image when $f < f_{\max}$, and find $f_{\max}$.

Solution.

Let object distance be u and image distance $s - u$. Then $u(s - u) = fs$, giving

$$\boxed{u_\pm = \frac{s \pm \sqrt{s^2 - 4fs}}{2}.}$$

There are two distinct positions when $s > 4f$, a repeated midpoint position at equality, and none when $s < 4f$. Thus $\boxed{f_{\max} = s/4}$.

8. Defocus spot

A converging lens has radius 20 mm and focal length 100 mm. A point source is 500 mm on one side and a screen 120 mm on the other. Neglect diffraction and find the illuminated spot diameter.

Solution.

The thin-lens equation gives $v = (1/100 - 1/500)^{-1} = 125\,\text{mm}$. Rays from the lens radius $a = 20\,\text{mm}$ converge linearly to zero radius there. At a screen distance 120 mm, $r_s = a|1 - 120/125| = 0.8\,\text{mm}$, so

$$\boxed{\text{spot diameter} = 1.6\,\text{mm}}.$$

9. Eye powers

Model the cornea and eye lens together as a thin lens 17 mm from the retina; the eyeball diameter is about 25 mm. Estimate the power range for focusing from a 25 cm near point to infinity. A myopic lecturer has a 50 cm far point. Find the type and power of correcting contact lens.

Solution.

The retinal image distance is 0.017 m. For an object at infinity, power is $1/f = 1/0.017 = 58.82\,\mathrm{D}$; at near distance 0.25 m, it is $58.82 + 4 = 62.82\,\mathrm{D}$. Thus

$$\boxed{P \simeq 58.8\text{–}62.8\,\mathrm{D}}.$$

A myopic far point at 0.50 m requires a diverging contact lens that images infinity virtually there: $\boxed{P_c = -2.0\,\mathrm{D}}$.

10. Magnifying glass

Explain the virtual-image use of a magnifying glass and derive its angular magnification, stating the viewing condition.

Solution.

Place the object inside the focal distance to form an upright virtual image. Compare its angular size with unaided viewing at near-point distance D. For a relaxed eye with the object at the focal plane, angular magnification is $\boxed{M = D/f}$. For a final virtual image at the near point and the lens close to the eye, the thin-lens equation gives $\boxed{M = 1 + D/f}$. The first convention minimizes accommodation; the second gives the larger nominal magnification.

11. Glass-sphere magnifier

Using the thin-lens approximation for a glass sphere of index 3/2, find how magnifying power depends on sphere radius and estimate it for radius 1 mm.

Solution.

Under the explicitly requested thin-lens approximation, $R_1 = R_2 = R$ and $n = 3/2$, so $1/f = 2(n-1)/R = 1/R$. Thus $\boxed{M \simeq D/R}$ for relaxed viewing, or $1 + D/R$ at the near point. A 1 mm sphere gives about $\boxed{250\times}$ with $D = 250\,\mathrm{mm}$. Thickness and spherical aberration make this only the stipulated approximation.

12. Spherical mirror

Derive the paraxial spherical-mirror equation $1/u + 1/v = 1/f = 2/r$.

Solution.

For a paraxial ray meeting the mirror at height h, small-angle geometry gives incidence/reflection symmetry $\alpha + \beta = 2\theta$, where $\alpha \simeq h/u$, $\beta \simeq h/v$, $\theta \simeq h/r$. Dividing by h gives

$$\boxed{1/u + 1/v = 2/r = 1/f.}$$

Signed object and image distances extend the relation to virtual images and convex mirrors.

13. Telescope magnification and pupil

A refracting telescope has objective diameter 80 mm and focal length 400 mm. For eyepiece focal lengths 9.7 mm and 26 mm, find magnification and exit-pupil diameter.

Solution.

For normal adjustment, angular magnification magnitude is f_o/f_e and exit pupil diameter is $D_o f_e/f_o$. The 9.7 mm eyepiece gives $\boxed{41.24\times,\ 1.94\,\text{mm}}$; the 26 mm eyepiece gives $\boxed{15.38\times,\ 5.20\,\text{mm}}$. Both images are inverted in the simple astronomical arrangement.

14. Compound microscope

A compound microscope has thin lenses of focal lengths f_o, f_e separated by L. Derive $|M| \simeq LD/(f_o f_e)$, stating the assumptions and near-point distance D. For $L = 200\,\text{mm}$, $f_o = 10\,\text{mm}$, $f_e = 20\,\text{mm}$, compare with the exact relaxed-eye magnification and find the object–objective distance.

Solution.

The objective's real-image magnification is approximately L/f_o when $L \gg f_o, f_e$; the relaxed-eye eyepiece magnifies angularly by D/f_e. Thus $\boxed{|M| \simeq LD/(f_o f_e)}$. Exactly for the final image at infinity, the intermediate image is f_e before the eyepiece, so $v_o = L - f_e = 180\,\text{mm}$ and

$$u_o = f_o v_o/(v_o - f_o) = \boxed{10.588\,\text{mm}}.$$

The objective magnification is 17, giving exact $\boxed{|M| = 212.5}$ for $D = 250\,\text{mm}$, versus the approximation 250.

E1. Right-angle prism

Show with a ray diagram how a symmetric right-angle glass prism in air retroreflects a ray entering normally through its hypotenuse by total internal reflection. Discuss a small incidence angle θ away from normal.

Solution.

A ray entering normally through the hypotenuse reaches the equal sides at 45°. For glass $n \simeq 1.5$, $45^\circ > \theta_c$, so two total internal reflections reverse its direction and it exits normally through the hypotenuse, laterally displaced. If the external incidence angle is a small θ, internal tilt is $\beta = \sin^{-1}(\sin\theta/n)$; the two incidence angles become $45^\circ \pm \beta$. Both reflections persist if $45^\circ - |\beta| > \theta_c$, and the emerging beam remains antiparallel to the incoming beam.

E2. Ray reflected inside a cylinder

A ray in air strikes a vertical glass cylinder of index μ at incidence angle θ, refracts, reflects once internally at the back surface and leaves the front at angle α. Show $\alpha = \theta$ and find the angle between incident and outgoing directions in terms of the refracted angle and θ.

Solution.

Let $\beta = \sin^{-1}(\sin\alpha/\mu)$ be the first internal refraction angle. Equal radii and equal chords make the internal incidence angles at each surface equal to β; exit refraction therefore gives $\boxed{\gamma = \alpha}$. Adding the two refraction deviations and one internal-reflection deviation gives

$$\boxed{\delta = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4\sin^{-1}(\sin\alpha/\mu)}$$

for the standard oriented deviation of the once-internally-reflected ray. If an unsigned smaller angle between beam directions is wanted, reduce this angle to $[0, \pi]$.

E3. Underwater vision

Explain why a diving mask restores clear underwater vision and how prescription swimming goggles can work in both air and water.

Solution.

The cornea provides much of the eye's focusing power through its index contrast with air. Water reduces this contrast, leaving the eye too weak to focus sharply on the retina. A diving mask restores an air layer before the cornea. Prescription goggles combine this air space with a corrective lens designed to create the appropriate object vergence for the eye; their correction can be designed for both external media, though their exact optical powers depend on the surface shapes and refractive indices.

E4. Parabolic thickness lens

Use Fermat's principle to show that a thin lens of thickness $w(y) = w_0 - \alpha y^2$ focuses paraxial rays. Find its focal length in air in terms of α, n and relate it to the biconvex lens-maker formula with radii r_1, r_2.

Solution.

For an axial object and image at distances u, v, the paraxial optical path through height y is a constant plus $y^2(1/u + 1/v)/2 + (n-1)(w_0 - \alpha y^2)$. Fermat stationarity across the aperture sets the y^2 coefficient to zero. Hence

$$\boxed{1/u + 1/v = 2\alpha(n-1), \quad f = [2\alpha(n-1)]^{-1}.}$$

A biconvex thin lens has $w = w_0 - y^2(1/R_1 + 1/R_2)/2 + \cdots$, recovering the lens-maker formula.

E5. Newton's lens formula

Derive Newton's lens formula $x_o x_i = f^2$ from the lens equation, with distances measured from the respective focal points. Find magnification in terms of f, x_o, and discuss signed distances for virtual images and diverging lenses.

Solution.

Put $u = f + x_o$, $v = f + x_i$ in $uv = f(u+v)$. Cancellation gives $\boxed{x_o x_i = f^2}$. The signed lateral magnification is $m = -v/u = -f/x_o = -x_i/f$. Both formulas apply to virtual images and diverging lenses with consistent signed distances, rather than interpreting every distance as a positive magnitude.

E6. Cassegrain geometry

A Cassegrain telescope has primary diameter 2.4 m, primary curvature radius 11.04 m, secondary curvature radius −1.36 m and mirror separation 4.91 m. For axial paraxial rays, find (a) both focal lengths, (b) the necessary secondary diameter, (c) the final focus location and (d) the effective focal length.

Solution.

The focal lengths are

$$\boxed{f_1 = 5.52\,\text{m},\ f_2 = -0.68\,\text{m}}.$$

The primary's converging cone has diameter at the secondary

$$D_2 = D_1(f_1 - d)/f_1 = \boxed{0.2652\,\text{m}},$$

the minimum for axial rays. For the secondary $u_2 = -(5.52 - 4.91) = -0.61$ m; $1/v_2 = 1/f_2 - 1/u_2$ gives $v_2 = \boxed{5.9257\,\text{m}}$. Thus the final focus lies $5.9257 - 4.91 = \boxed{1.0157\,\text{m}}$ behind the primary. The secondary magnifies by $|v_2/u_2| = 9.7143$, giving $\boxed{f_{\text{eff}} = 53.62\,\text{m}}$ for the stated mirror parameters.

2.9 Wave Optics

1. Young's fringes

Derive the screen intensity for Young's two narrow slits of separation d, illuminated normally by coherent wavelength λ, at screen distance $D \gg d$.

Solution.

The path difference is $d\sin\theta \simeq dx/D$, giving phase difference $\delta = 2\pi dx/(\lambda D)$. Adding two equal field amplitudes gives

$$\boxed{I(x) = 4I_1 \cos^2\left(\frac{\pi dx}{\lambda D}\right), \qquad \Delta x = \lambda D/d.}$$

Here I_1 is the intensity from one slit alone; coherent equal illumination gives zero ideal minima.

2. Slit separation

In Young's experiment, $\lambda = 550\,\text{nm}$, $D = 0.7\,\text{m}$ and the fifth bright fringe is 7 mm from the central maximum. Find the slit separation.

Solution.

The fifth bright fringe satisfies $x_5 = 5\lambda D/d$. Thus

$$\boxed{d = 5(550\,\text{nm})(0.7\,\text{m})/(7\,\text{mm}) = 0.275\,\text{mm}}.$$

3. Glass over one slit

A glass sheet of thickness 0.2 mm and index 1.66 is placed over one slit on the incident side in the preceding experiment. Find and describe the fringe displacement.

Solution.

The glass adds optical path

$$(n-1)t = 0.66(0.2\,\text{mm}) = 132\,\mu\text{m},$$

so the phase shifts by $2\pi(n-1)t/\lambda = 2\pi \times 240$. The fringe pattern translates towards the covered slit by

$$\boxed{\Delta x = D(n-1)t/d = 0.336\,\text{m} = 240 \text{ fringe spacings.}}$$

The spacing stays unchanged. At exactly the stated wavelength the displacement is an integer number of spacings, so an ideal infinite periodic pattern is indistinguishable modulo its fringe index; a finite diffraction envelope can reveal the shift.

4. Three spectral lines

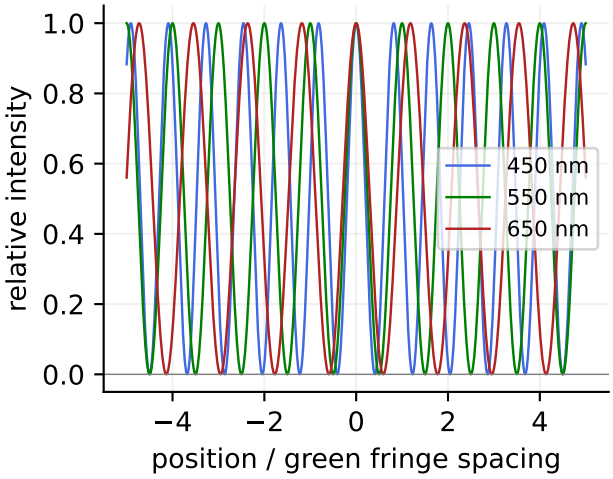
Remove the glass and replace the source by equal-intensity lines at 450, 550 and 650 nm. Sketch their separate Young patterns and describe their combined appearance.

Solution.

The three intensities add incoherently:

$$I(x) = 4I_1 \sum_{\lambda\in\{450,550,650\}\text{nm}} \cos^2(\pi dx/(\lambda D)).$$

Their spacings are $\boxed{1.145, 1.400, 1.655\,\text{mm}}$ with the earlier geometry. All have a coincident central maximum, producing a combined central bright stripe; farther out the blue, green and red fringes separate and overlap at different positions, producing a coloured composite pattern.



5. Grating maxima

For normally incident monochromatic light on a reflection grating of spacing d, derive the principal-maxima condition $n\lambda = d\sin\theta$. Compare the pattern with Young's two-slit fringes.

Solution.

At normal incidence the path difference between adjacent reflected rays is $d\sin\theta$. Constructive interference of all grooves requires $\boxed{d\sin\theta = n\lambda}$. Summing N groove amplitudes gives $I \propto [\sin(N\delta/2)/\sin(\delta/2)]^2$ with $\delta = 2\pi d\sin\theta/\lambda$. Compared with two-slit fringes, a many-groove grating produces much narrower principal maxima, weaker subsidiary structure and greater spectral resolving power.

6. Grating spectrometer

Describe a practical diffraction-grating spectrometer and how it measures wavelengths.

Solution.

Place a narrow entrance slit at a collimating lens's focal plane, direct the resulting parallel beam normally onto the grating, and use a telescope or camera lens to resolve outgoing directions. Calibrate the angular zero from the direct/zeroth-order beam, measure the same line on both sides, and use $\lambda = d\sin\theta/n$. The slit controls instrumental width; a collimated, well-aligned beam and known groove spacing permit wavelength measurement.

7. Numerical grating spectroscopy

A grating at normal incidence gives third-order hydrogen light at $\lambda = 656.3\,\text{nm}$ and $\theta = 36.2°$. Find its ruling density in lines per millimetre and first-order angle. Find the angular separation of the sodium lines 589.0, 589.6 nm in first and highest accessible orders.

Solution.

$$d = 3(656.3\,\text{nm})/\sin 36.2° \simeq 3.334\,\mu\text{m},$$

giving $\boxed{300\ \text{lines/mm}}$. The first-order hydrogen angle is $\boxed{11.35°}$. For sodium, $\Delta\theta \simeq n\Delta\lambda/[d\sqrt{1-(n\bar{\lambda}/d)^2}]$. The first-order separation is

$$\boxed{1.83\times10^{-4}\,\text{rad} = 0.0105°}.$$

The largest common order is $n = 5$, yielding

$$\boxed{1.92\times10^{-3}\,\text{rad} \simeq 0.110°}.$$

8. Single-slit diffraction

Derive the angular diffraction intensity of a uniformly illuminated slit of width w and wavelength λ, and the Rayleigh angular separation for one slit.

Solution.

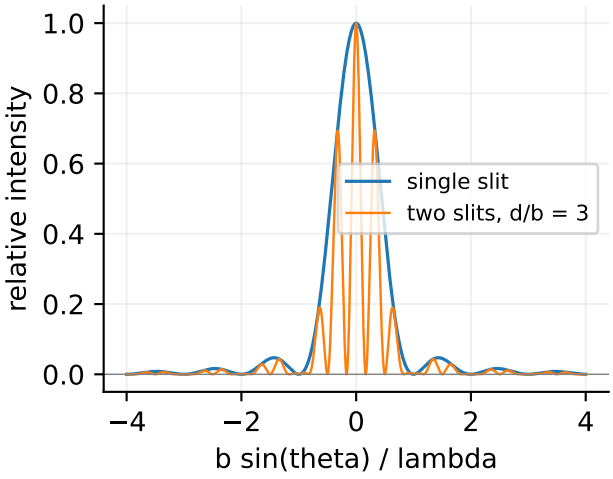
Integrate the field contributions across the aperture:

$$A(\theta) \propto \int_{-w/2}^{w/2} e^{ikx\sin\theta}\,dx = w\sin\beta/\beta,$$

where $\beta = \pi w\sin\theta/\lambda$. Therefore

$$\boxed{I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2, \quad \sin\theta_{\min} = \lambda/w.}$$

The first-zero Rayleigh angular separation for an ideal slit aperture is approximately λ/w.



9. Circular-aperture resolution

Explain why diffraction limits a lens image and state the Rayleigh resolution criterion for a circular aperture.

Solution.

A finite lens aperture diffracts a point source into an Airy pattern. For a circular pupil of diameter D, the first zero is at angle $\boxed{\theta_R \simeq 1.22\lambda/D}$. The Rayleigh convention calls two equal incoherent point sources just resolved when one image's central maximum lies at the other's first minimum; geometric focusing alone cannot shrink a point image below this diffraction scale.

E1. Defocus and diffraction

A lens has radius 20 mm and focal length 100 mm. A point source lies 500 mm before it and a screen 120 mm after it. Find the geometrical blur diameter, then the best-focus screen position and diffraction-limited spot size, stating the wavelength assumed.

Solution.

The geometric image is at $v = 125$ mm, as in Geometrical Optics Q8. At 120 mm the spot diameter is 1.6 mm. The best paraxial focus is $\boxed{125\,\text{mm}}$. At wavelength λ, the Airy first-zero diameter is $\boxed{d_A = 2.44\lambda v/D = 7.625\lambda}$ for $D = 40$ mm, e.g. $\boxed{4.19\,\mu\text{m}}$ at 550 nm. A numerical diffraction size requires a wavelength, which this subquestion does not specify.

E2. Eyes and Legolas

For a 5 mm eye pupil, compare the diffraction limit with reading 9 mm letters at 20 m (20/20 vision) and resolving 10 cm objects at 24 km (Legolas). State a representative visible wavelength.

Solution.

At visible $\lambda \simeq 550\,\text{nm}$ with a 5 mm pupil, $\theta_R \simeq 1.34 \times 10^{-4}\,\text{rad}$. A 9 mm letter at 20 m subtends $4.5 \times 10^{-4}\,\text{rad}$ and its five-stroke structure is of order $9 \times 10^{-5}\,\text{rad}$, consistent with the scale of normal acuity. Legolas's 0.10 m object at 24 km subtends $4.17 \times 10^{-6}\,\text{rad}$, demanding roughly $\boxed{D \simeq 0.16\,\text{m}}$ at the Rayleigh limit: far larger than a human pupil.

E3. Satellite dish alignment

Satellite television at 15 GHz uses a 0.7 m receiving dish. Estimate the necessary angular alignment accuracy.

Solution.

At 15 GHz, $\lambda = c/\nu \simeq 0.020\,\text{m}$. A 0.7 m aperture has diffraction angular scale

$$\boxed{1.22\lambda/D \simeq 0.0349\,\text{rad} = 2.0^\circ}.$$

Alignment should therefore be substantially better than a few degrees, typically within roughly a degree to stay near the main-lobe maximum.

E4. Apollo sites

A telescope with 2.4 m aperture observes the Moon at wavelength about 300 nm. Estimate its spatial resolution and explain whether non-detection of lunar modules is evidence against the landings.

Solution.

At $\lambda = 300\,\text{nm}$ and $D = 2.4\,\text{m}$,

$$\theta_R = 1.525 \times 10^{-7}\,\text{rad}.$$

At lunar distance $3.84 \times 10^8\,\text{m}$, this corresponds to $\boxed{59\,\text{m}}$ on the surface, much larger than a lunar module. The stated non-detection is therefore expected from the telescope's diffraction limit and provides no evidence against the landings.

E5. Two finite-width slits

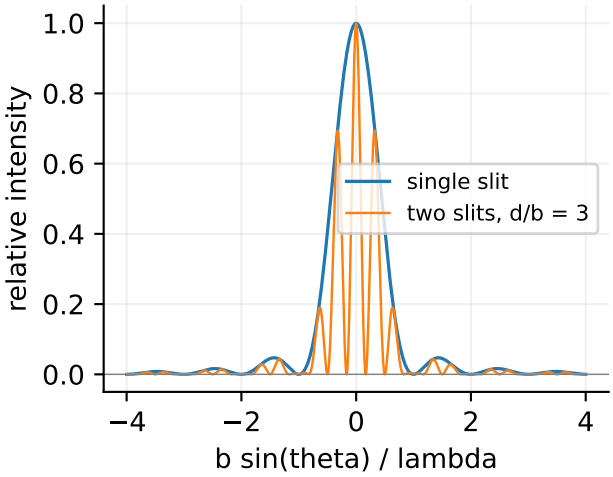
Two slits of width b are centred at $\pm d/2$, with $b < d$. Integrate the Fraunhofer field over both apertures to obtain amplitude and intensity versus diffraction angle, and interpret the diffraction envelope and interference factor.

Solution.

Sum integrals over $[-d/2 - b/2, -d/2 + b/2]$ and $[d/2 - b/2, d/2 + b/2]$. With transverse wavevector $q = k \sin\theta$,

$$\boxed{A(\theta) = 2A_s \frac{\sin(qb/2)}{qb/2} \cos(qd/2), \quad I(\theta) = 4I_s\,\text{sinc}^2\left(\frac{\pi b \sin\theta}{\lambda}\right) \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right).}$$

The narrow Young fringes are modulated by a single-slit diffraction envelope; interference maxima coincide with missing orders whenever the envelope vanishes.


1.0
0.8
0.6
0.4
0.2
0.0
relative intensity
single slit
two slits, d/b = 3
−4
−2
0
2
4
b sin(theta) / lambda

3 CP3: Mathematical Methods

3.1 Complex Numbers: Sheet 0

1. Polar forms

Convert to polar form: (i) $-i$, (ii) $(1-i\sqrt{3})/2$, (iii) $-3-4i$, (iv) $1+i$, (v) $1-i$, (vi) $(1+i)/(1-i)$.

Solution.

Using $z = |z|e^{i\arg z}$, the six answers are

$$\boxed{e^{-i\pi/2},\quad e^{-i\pi/3},\quad 5e^{i(-\pi+\tan^{-1}(4/3))},\quad \sqrt{2}e^{i\pi/4},\quad \sqrt{2}e^{-i\pi/4},\quad e^{i\pi/2}.}$$

Arguments are defined modulo 2π.

2. Complex arithmetic

For (a) $z_1 = 1+i$, $z_2 = -3+2i$, and (b) $z_1 = 2e^{i\pi/4}$, $z_2 = e^{-3i\pi/4}$, find z_1+z_2, z_1-z_2, z_1z_2, z_1/z_2, $|z_1|$ and z_1^*.

Solution.

(a) For $z_1 = 1+i, z_2 = -3+2i$, the requested answers are

$$\boxed{-2+3i,\ 4-i,\ -5-i,\ (-1-5i)/13,\ \sqrt{2},\ 1-i}.$$

(b) Let $w = e^{i\pi/4}$. Since $z_1 = 2w$, $z_2 = -w$, the answers are

$$\boxed{w,\ 3w,\ -2i,\ -2,\ 2,\ 2e^{-i\pi/4}}.$$

The last quantity is the complex conjugate z_1^*, not a reciprocal.

3. Real and imaginary parts

For $z = x+iy$, find real and imaginary parts of (i) z^2, (ii) $1/z$, (iii) i^{-5}, (iv) $(2+3i)/(1+6i)$, (v) $e^{i\pi/6} - e^{-i\pi/6}$.

Solution.

The results, written as real plus i times imaginary part, are

$$\boxed{\begin{aligned} &z^2 = (x^2-y^2)+2ixy, \quad z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}, \\ &i^{-5} = -i, \quad \frac{2+3i}{1+6i} = \frac{20}{37} - \frac{9}{37}i, \quad e^{i\pi/6} - e^{-i\pi/6} = i. \end{aligned}}$$

The reciprocal requires $z \neq 0$.

4. Complex-plane loci

Plot in the complex plane: (i) $3-2i$; (ii) $4e^{-i\pi/6}$; (iii) $|z-1|=1$; (iv) $\mathrm{Re}(z^2)=4$; (v) $\arg(z+3i)=\pi/4$; (vi) $|z+1|+|z-1|=8$; (vii) $z=te^{it}$ for real t; (viii) $\arg[(z-4)/(z-1)]=3\pi/2$, interpreted modulo 2π.

Solution.

(i) Point $(3,-2)$; (ii) point $(2\sqrt{3},-2)$; (iii) circle $(x-1)^2+y^2=1$; (iv) hyperbola $x^2-y^2=4$; (v) ray $y=x-3$ with $x>0$; (vi) ellipse $\boxed{x^2/16+y^2/15=1}$, foci $(\pm1,0)$. (vii) $x=t\cos t, y=t\sin t$ is a two-parameter-direction Archimedean spiral as t runs over positive and negative real values. (viii) For $w=(z-4)/(z-1)$, $\mathrm{Re}\,w=0$ gives $(x-4)(x-1)+y^2=0$ and $\mathrm{Im}\,w=3y/[(x-1)^2+y^2]$. Thus the locus with argument $3\pi/2$ (or $-\pi/2$) is the

$$\boxed{\text{lower semicircle with diameter } [1,4]},$$

excluding both endpoints. A principal-argument convention $(-\pi,\pi]$ should represent the specified angle as $-\pi/2$.

5. Fourth-angle identity and half angles

Use de Moivre's theorem to prove

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Deduce $\cos(\pi/8)=\sqrt{(2+\sqrt{2})/4}$ and find $\cos(3\pi/8)$.

Solution.

The real part of $(\cos\theta+i\sin\theta)^4$ is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Half-angle identities, with positive roots in the first quadrant, give

$$\boxed{\cos(\pi/8)=\frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \cos(3\pi/8)=\frac{\sqrt{2-\sqrt{2}}}{2}.}$$

6. Fourier expansion of a sixth power

Express $\sin^6\theta$ as a sum of terms in $\cos n\theta$ for integer n.

Solution.

Use $\sin^2\theta=(1-\cos 2\theta)/2$, cube it, and reduce $\cos^2(2\theta)$ and $\cos^3(2\theta)$ with double/triple-angle identities. The result is

$$\boxed{\sin^6\theta=\frac{10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta}{32}.}$$

7. Integer powers

Evaluate (i) $(1+i)^9$ and (ii) $(1-i)^9/(1+i)^9$.

Solution.

Since $1 \pm i = \sqrt{2}e^{\pm i\pi/4}$,

$$\boxed{(1+i)^9 = 16(1+i), \qquad \frac{(1-i)^9}{(1+i)^9} = e^{-9i\pi/2} = -i.}$$

8. All roots

Find all complex values of (i) $\sqrt[4]{(-1-i\sqrt{3})/2}$, (ii) $(-8i)^{2/3}$ and (iii) $\sqrt[8]{16}$.

Solution.

(i) The radicand is $e^{-2\pi i/3}$, so its fourth roots are $\boxed{e^{-i\pi/6+ik\pi/2},\ k=0,1,2,3}$. (ii) The three possible values of $(-8i)^{2/3}$ are $\boxed{4e^{-i\pi/3+4\pi ik/3},\ k=0,1,2}$, equivalently $4e^{i\pi/3}, -4, 4e^{-i\pi/3}$. (iii) The eight eighth roots of 16 are $\boxed{\sqrt{2}e^{ik\pi/4},\ k=0,\ldots,7}$.

9. Root sums

(i) Show that the sum of all nth roots of a complex number is zero for $n > 1$.

(ii) By considering $z^{2n+1}+1=0$, prove $\sum_{k=-n}^{n}\cos[(2k+1)\pi/(2n+1)] = 0$ for positive integer n.

Solution.

(i) For $n \geq 2$, the roots of a nonzero $z = re^{i\theta}$ are $r^{1/n}e^{i\theta/n}\omega^k$, with $\omega = e^{2\pi i/n}$. Their sum is proportional to $(1-\omega^n)/(1-\omega) = 0$. Zero also has zero root sum. (ii) The $2n+1$ roots of $z^{2n+1}+1$ are $e^{i(2k+1)\pi/(2n+1)}$, $k=-n,\ldots,n$. Taking the real part of their zero sum gives precisely the requested cosine identity.

10. Polynomial roots from a ratio

Find the roots of $(z-1)^n + (z+1)^n = 0$. Hence solve $x^3 + 15x^2 + 15x + 1 = 0$.

Solution.

Set $w = (z-1)/(z+1)$; then $w^n = -1$. For $\theta_k = (2k+1)\pi/n$, $w = e^{i\theta_k}$ and

$$\boxed{z = \frac{1+w}{1-w} = i\cot\frac{(2k+1)\pi}{2n}, \quad k=0,\ldots,n-1.}$$

For $n=6$, $(z-1)^6+(z+1)^6 = 2(z^6+15z^4+15z^2+1)$. Set $x=z^2$ to find

$$\boxed{x=-1, \quad x=-7+4\sqrt{3}, \quad x=-7-4\sqrt{3}.}$$

11. Geometric trigonometric series

Show that

$$\sum_{n=0}^{\infty} 2^{-n}\cos n\theta = (1-\tfrac{1}{2}\cos\theta)/(\tfrac{5}{4}-\cos\theta).$$

Solution.

The absolutely convergent sum is the real part of

$$\sum_{n\geq 0}(e^{i\theta}/2)^n = 1/(1-e^{i\theta}/2).$$

Multiply numerator and denominator by $1-e^{-i\theta}/2$ to obtain

$$\boxed{\sum_{n=0}^{\infty} 2^{-n}\cos n\theta = \frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}.}$$

12. Binomial sine identity

Prove

$$\sum_{r=1}^{n}\binom{n}{r}\sin(2r\theta) = 2^n\sin(n\theta)\cos^n\theta$$

using complex exponentials and the binomial theorem.

Solution.

The imaginary part of $(1+e^{2i\theta})^n$ is $\sum_{r=0}^{n}\binom{n}{r}\sin 2r\theta$. Since $1+e^{2i\theta} = 2\cos\theta e^{i\theta}$, the same imaginary part is $\boxed{2^n\cos^n\theta\sin n\theta}$. The $r=0$ summand is zero, giving the stated lower limit.

13. Elementary complex functions

Find real and imaginary parts, specifying branches where needed: (i) $e^{3\ln 2-i\pi}$; (ii) $\ln i$; (iii) $\ln(-e)$; (iv) $(1+i)^{iy}$; (v) $\sin i$; (vi) $\cos(\pi-2i\ln 3)$; (vii) $\tanh(x+iy)$; (viii) $\tan^{-1}(i\sqrt{3})$; (ix) $\sinh^{-1}(-1)$.

Solution.

Use principal values unless another branch is requested. (i) $e^{3\ln 2-i\pi} = -8$. (ii) $\log i = i\pi/2$; all branches add $2\pi ik$. (iii) $\log(-e) = 1+i\pi$; all branches have imaginary part $(2k+1)\pi$. (iv)

$$(1+i)^{iy} = e^{-\pi y/4}[\cos((y/2)\ln 2) + i\sin((y/2)\ln 2)]$$

for real y on the principal logarithm branch. (v) $\sin i = i\sinh 1$. (vi) $\cos(\pi-2i\ln 3) = -\cosh(2\ln 3) = -41/9$. (vii)

$$\boxed{\tanh(x+iy) = [\sinh 2x + i\sin 2y]/[\cosh 2x + \cos 2y]}$$

away from poles. (viii) With $\arctan z = [\log(1+iz) - \log(1-iz)]/(2i)$,

$$\boxed{\arctan\left(\sqrt{3}i\right) = \pi/2 + (i/2)\ln\left(2+\sqrt{3}\right)}$$

using the principal logarithms in this expression; the cut has a side-dependent boundary value and other inverse values differ by integer multiples of π. (ix) The real inverse hyperbolic sine is

$$\boxed{\operatorname{arsinh}(-1) = -\ln\left(1+\sqrt{2}\right).}$$

3.2 Ordinary Differential Equations: Sheet 1

1.1. Order and linearity

Determine the order and linearity of (i) $y'' + k^2 y = f(x)$, (ii) $y''' + 2yy' = \sin x$, (iii) $y' + y^2 = xy$.

Solution.

(i) Second order and linear, because y, y', y'' occur only to first power with coefficients depending on x. (ii) Third order and nonlinear, because of yy'. (iii) First order and nonlinear, because of y^2.

1.2. First-order solution methods

Solve by the indicated methods; implicit forms suffice.

$$\text{(a) Exact: } y' = \frac{(3x^2 + 2xy + y^2)\sin x - 2(3x + y)\cos x}{2(x + y)\cos x}.$$

$$\text{(b) Integrating factor: } y^2\,dx - (xy + x^3)\,dy = 0.$$

$$\text{(c) Separable: (i) } y' = \frac{xe^y}{1 + x^2}, \quad y(0) = 0; \qquad \text{(ii) } y' = \frac{2xy^2 + x}{x^2 y - y}.$$

$$\text{(d) Reducible to separable: } y' = 2(2x + y)^2.$$

$$\text{(e) Homogeneous: } 2y' = \frac{xy + y^2}{x^2}.$$

$$\text{(f) Reducible to homogeneous: } y' = \frac{x + y - 1}{x - y - 2}.$$

$$\text{(g) Linear: (i) } y' + y/x = 3, \quad y(1) = 3/2; \qquad \text{(ii) } y' + y\cos x = \sin 2x.$$

Solution.

(a) The numerator and denominator are the derivatives of $\Psi = (3x^2 + 2xy + y^2)\cos x$: $d\Psi/dx = 0$. Thus $\boxed{(3x^2 + 2xy + y^2)\cos x = C}$ on intervals where the stated slope exists.

(b) Treat x as a function of y. For $z = x^{-2}$, $z' + 2z/y = -2/y^2$, giving $\boxed{y^2/x^2 + 2y = C}$. (c)(i) $e^{-y}dy = x\,dx/(1 + x^2)$ integrates to $e^{-y} = 1 - \frac{1}{2}\ln\big(1 + x^2\big)$, so $\boxed{y = -\ln[1 - \frac{1}{2}\ln\big(1 + x^2\big)]}$ where the bracket is positive. (ii) Separation gives $\frac{1}{4}\ln\big(2y^2 + 1\big) = \frac{1}{2}\ln|x^2 - 1| + C$, or $\boxed{2y^2 + 1 = C(x^2 - 1)^2}$.

(d) Set $z = 2x + y$, so $z' = 2(1 + z^2)$, yielding $\boxed{y = \tan(2x + C) - 2x}$.

(e) Set $y = xv$: $2xv' = v(v - 1)$. Thus $\boxed{y = x/(1 - C\sqrt{|x|})}$, together with $y = 0$, on intervals excluding the singular coefficients.

(f) With $X = x - 3/2$, $Y = y + 1/2$, the equation is homogeneous. Put $Y = Xv$ and integrate $(1 - v)dv/(1 + v^2) = dX/X$, obtaining

$$\boxed{\arg(X + iY) - \tfrac{1}{2}\ln\big(X^2 + Y^2\big) = C}$$

locally with a continuous argument branch. (g)(i) Multiply $y' + y/x = 3$ by x: $(xy)' = 3x$. The initial value gives $\boxed{y = 3x/2}$ for $x > 0$. (ii) The integrating factor is $e^{\sin x}$. Substituting $u = \sin x$ in the right-hand integral gives $\boxed{y = 2(\sin x - 1) + Ce^{-\sin x}}$.

1.3. Further first-order equations

Solve the following; implicit solutions suffice.

(i) $y' = (x - y\cos x)/\sin x$; (ii) $(3x + x^2)y' = 5y - 8$;
(iii) $y' + 2x/y = 3$; (iv) $2y' = y/x + y^3/x^3$;
(v) $xyy' - y^2 = (x + y)^2 e^{-y/x}$; (vi) $x(x - 1)y' + y = x(x - 1)^2$;
(vii) $2xy' - y = x^2$; (viii) $y' = \cos(y + x),\ y(0) = \pi/2$;
(ix) $y' = (x - y)/(x - y + 1)$; (x) $y' = \cos 2x - y\cot x,\ y(\pi/2) = 1/2$.

Solution.

(i) $(y\sin x)' = x$, so $\boxed{y = (x^2/2 + C)/\sin x}$. (ii) Separate $dy/(5y - 8) = dx/[x(x + 3)]$ to obtain $\boxed{5y - 8 = C|x/(x + 3)|^{5/3}}$, including $y = 8/5$. (iii) Put $y = xv$ and integrate $-v\,dv/[(v - 1)(v - 2)] = dx/x$. This gives $\boxed{(y - 2x)^2 = C(y - x)}$, and the separately lost solution $y = x$; exclude points where the original denominator y vanishes. (iv) For $z = y^{-2}$, $z' + z/x = -x^{-3}$. Thus $\boxed{y^2 = x^2/(1 + Cx)}$, together with $y = 0$. (v) For $y = xv$, separation gives $ve^v\,dv/(1 + v)^2 = dx/x$. Therefore $\boxed{e^{y/x}/(1 + y/x) = \ln|x| + C}$, together with $y = -x$. (vi) The integrating factor is $(x - 1)/x$, giving $\boxed{(x - 1)y/x = x^2/2 - 2x + \ln|x| + C}$. (vii) The linear equation gives $\boxed{y = x^2/3 + C\sqrt{|x|}}$. (viii) Put $z = y + x$; then $z' = 1 + \cos z$. The initial value selects $\boxed{y = 2\tan^{-1}(x + 1) - x}$ with the continuous branch through $y(0) = \pi/2$. (ix) Put $z = x - y$, so $(z + 1)z' = 1$. Thus $\boxed{(x - y)^2/2 + x - y = x + C}$. (x) Multiply by $\sin x$ and integrate $\sin x\cos 2x$. The initial value gives

$$\boxed{y = [\cos x - (2/3)\cos^3 x + 1/2]/\sin x}.$$

1.4. Missing branches of a Riccati solution

With $z = 4y - x$, or otherwise, solve $y' - 16y^2 + 8xy = x^2$. Obtain the family $y = x/4 - \frac{1}{8}\tanh(2x + C)$ and determine the additional branches and exceptional solutions.

Solution.

For $z = 4y - x$, $z' = 4z^2 - 1$. Integration gives $(z - 1/2)/(z + 1/2) = Ce^{4x}$. The bounded branch is $z = -\frac{1}{2}\tanh(2x + C_1)$, which gives the bounded solution. Other real branches have $z = -\frac{1}{2}\coth(2x + C_1)$, on intervals avoiding their poles. The equilibrium values $z = \pm 1/2$ also solve it, so

$$\boxed{y = x/4 \pm 1/8}$$

must be included. The single tanh family is therefore not the full general real solution.

1.5. Integrating factors

Find integrating factors and solve (a) $2(x - y^4)\,dy = y\,dx$, looking for a factor depending on y, and (b) $y(2y\,dx - x\,dy) + x^2(y\,dx + 2x\,dy) = 0$.

Solution.

(a) Regard x as a function of y: $x' - 2x/y = -2y^3$. The integrating factor y^{-2} gives $\boxed{x = Cy^2 - y^4}$. The horizontal solution $y = 0$ also satisfies the original differential form.

(b) The monomial integrating factor $\mu = x^{-3}y^{-1}$ makes the equation exact:

$$d[-y/x^2 + \ln|x| + 2\ln|y|] = 0.$$

Thus $\boxed{-y/x^2 + \ln|x| + 2\ln|y| = C}$ on regions with $xy \neq 0$; $y = 0$ is an additional horizontal solution and $x = 0$ is a vertical integral curve of the differential form.

1.6. Power substitution

Solve $y\,dx + x(2xy+1)\,dy = 0$ by a substitution $y = z^n$ making it homogeneous. You may find x as a function of y.

Solution.

Set $y = z^n$. After substitution the two coefficient degrees match for $n = -1$, so $y = 1/z$ makes the equation homogeneous. More directly regard $x = x(y)$ and put $u = 1/x$. Then $u' - u/y = 2$, hence $(u/y)' = 2/y$. Therefore

$$\boxed{\frac{1}{xy} = 2\ln|y| + C.}$$

The coordinate axes are additional integral curves of the original differential form, excluded by this division.

1.7. Quasi-homogeneous equations

An ODE $y' = f(x,y)$ is quasi-homogeneous if

$$f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta-\alpha} f(x,y)$$

for nonzero α, β.

(a) Show that $y = x^{\beta/\alpha} z$ makes it separable.

(b) Solve in quadratures and include $y = z_0 x^{\beta/\alpha}$ when $f(1, z_0) = \beta z_0/\alpha$. When can composite solutions occur?

Solution.

Put $p = \beta/\alpha$ and $y = x^p z$ on $x > 0$. The scaling property gives $f(x, x^p z) = x^{p-1} f(1,z)$, so $xz' = f(1,z) - pz =: G(z)$. Thus

$$\boxed{\int^z \frac{ds}{f(1,s) - ps} = \ln x + C.}$$

Every zero z_0 of G gives $\boxed{y = z_0 x^p}$ separately. One can join such an equilibrium to a nonconstant branch after a finite waiting time only if that branch reaches z_0 in finite logarithmic time, i.e. the corresponding one-sided integral of $1/G$ converges, and the joined function satisfies the ODE. Local Lipschitz continuity of G at z_0 rules this out by uniqueness.

1.8. Orthogonal trajectories

(a) Differentiate the family $y = \tan[\ln(Cx)]$ and eliminate C to find its ODE.

(b) Show that orthogonal trajectories of $F(x,y,y') = 0$ satisfy $F(x,y,-1/y') = 0$.

(c) Obtain the orthogonal family $3x^2 + 2y^3 + 6y = C$.

Solution.

(a) Differentiate $y = \tan\ln(Cx)$: $\boxed{y' = (1+y^2)/x}$.

(b) Orthogonal finite slopes satisfy $m_1 m_2 = -1$, so the new derivative is $-1/y'$ in the original equation, giving the stated substitution; vertical and horizontal tangents are handled by continuity or differential forms.

(c) Therefore the orthogonal family obeys $y' = -x/(1+y^2)$. Integration gives $y + y^3/3 = -x^2/2 + C$, or $\boxed{3x^2 + 2y^3 + 6y = C}$.

1.9. Equations implicit in the derivative

Solve parametrically using $p = y'$ and resolving for y:

(i) $(y')^2 - 2xy' = x^2 - 4y$; (ii) $y' = e^{xy'/y}$; (iii) $y' = ye^{-xy'/y}$; (iv) $y' = y^2e^{-xy'/y}$.

Solution.

(i) Put $p = y'$ and solve $y = (x^2 + 2xp - p^2)/4$. Differentiation gives $(p - x)(p' + 1) = 0$. Thus

$$\boxed{y = -x^2/2 + Cx - C^2/4, \qquad y = x^2/2 \text{ (singular solution).}}$$

(ii) Since $p > 0$, $y = xp/\ln p$. Differentiation gives $dx/x = dp/(p \ln p)$, yielding the parametric family $\boxed{x = C \ln p,\ y = Cp}$, equivalently $y = Ce^{x/C}$ for $C \neq 0$. The singular straight line $\boxed{y = ex}$ also solves the equation on domains where its ratio is defined. (iii) Put $q = p/y > 0$. Then $x = -\ln q/q$, while $d \ln |y|/dx = q$. Hence

$$\boxed{x = -\ln q/q,\ y = Cq^{-1} \exp\big[(\ln q)^2/2\big]}$$

for $C \neq 0$. (iv) The same $q = p/y$ gives $y = qe^{xq}$. Differentiating shows $q'(1/q+x) = 0$. Thus $\boxed{y = Ce^{Cx}\ (C \neq 0)}$ or the singular solution $\boxed{y = -1/(ex)}$.

1.10. Folium parametrization

For $(y')^3 + x^3 = xy'$, put $y' = p = xu$, parametrise (x, p), then find $x(u), y(u)$. State valid parameter ranges and include branches lost by division.

Solution.

Set $p = y' = xu$. Apart from the isolated origin of the (x, p) curve, the cubic gives $x = u/(1 + u^3)$ and $p = u^2/(1 + u^3)$. Thus

$$\frac{dy}{du} = p\frac{dx}{du} = \frac{u^2(1 - 2u^3)}{(1 + u^3)^3}, \qquad \boxed{x = \frac{u}{1 + u^3}, \quad y = C + \frac{2}{3(1 + u^3)} - \frac{1}{2(1 + u^3)^2}.}$$

The parameter must avoid $u = -1$. A single-valued regular graph $y(x)$ is obtained on any interval where $dx/du \neq 0$, hence excluding or splitting at $u = 2^{-1/3}$. These monotonic intervals are $(-\infty, -1)$, $(-1, 2^{-1/3})$ and $(2^{-1/3}, \infty)$; endpoint limits can extend a branch where the derivative remains defined.

3.3 Ordinary Differential Equations: Sheet 2

2.1. Constant-coefficient homogeneous equations

Solve (i) $y'' + 2y' - 15y = 0$; (ii) $y'' - 6y' + 9y = 0$, $y(0) = 0, y'(0) = 1$; (iii) $y'' - 4y' + 13y = 0$, giving both complex-exponential and real forms; (iv) $y''' + 7y'' + 7y' - 15y = 0$.

Solution.

Substitute $y = e^{rx}$ and factor the characteristic polynomial. The resulting solutions are

$$\boxed{\begin{aligned}
&(i) \quad y = C_1 e^{3x} + C_2 e^{-5x}, \\
&(ii) \quad y = x e^{3x}, \\
&(iii) \quad y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) = D_1 e^{(2+3i)x} + D_2 e^{(2-3i)x}, \\
&(iv) \quad y = C_1 e^{x} + C_2 e^{-3x} + C_3 e^{-5x}.
\end{aligned}}$$

For a real solution in (iii), $D_2 = \overline{D_1}$. In (iv) the polynomial is $(r-1)(r+3)(r+5)$.

2.2. Damped harmonic oscillator

A damped oscillator is released from rest at displacement y_0. Derive $m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = 0$ and interpret its parameters.

(a) Find and sketch over-, critically and under-damped solutions, and give the undamped limit.

(b) For weak damping show $Q = \omega_0/\gamma$ from $Q = 2\pi\times$(stored energy)/(energy lost per period).

Solution.

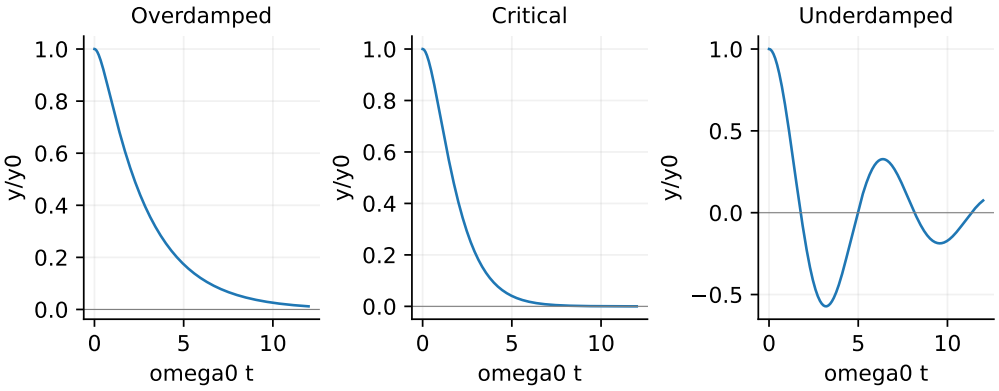
Newton's law with restoring force $-m\omega_0^2 y$ and drag $-m\gamma\dot{y}$ gives $\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0$. Here m is mass, γ damping rate and ω_0 undamped angular frequency. Put $a = \gamma/2$. For overdamping, $b = \sqrt{a^2 - \omega_0^2}$ and

$$\boxed{y = y_0 e^{-at}[\cosh bt + (a/b)\sinh bt]}.$$

At critical damping, $\boxed{y = y_0(1 + \omega_0 t)e^{-\omega_0 t}}$. For underdamping, $\Omega = \sqrt{\omega_0^2 - a^2}$ and

$$\boxed{y = y_0 e^{-at}[\cos \Omega t + (a/\Omega)\sin \Omega t]}.$$

With no damping, $y = y_0 \cos \omega_0 t$. The first two decay without oscillatory zero crossings; the third oscillates within an exponentially decreasing envelope. For weak damping, cycle-averaged energy decays as $e^{-\gamma t}$, so fractional loss per period is $2\pi\gamma/\omega_0 + O((\gamma/\omega_0)^2)$ and $\boxed{Q \simeq \omega_0/\gamma}$, the usual weak-damping definition.



2.3. Reduction of order

Verify that $y = x + 1$ solves $(x^2 - 1)y'' + (x + 1)y' - y = 0$ and use reduction of order to find its general solution. State intervals excluding singular points.

Solution.

Substitution verifies $y_1 = x + 1$. For $y = y_1 v$, the normalized equation gives $v'' + [2/(x + 1) + 1/(x - 1)]v' = 0$, hence $v' = C/[(x + 1)^2(x - 1)]$. Integrating and absorbing factors into constants gives

$$\boxed{y = C_1(x + 1) + C_2 \left[\frac{x + 1}{4} \ln\left|\frac{x - 1}{x + 1}\right| + \frac{1}{2}\right].}$$

This holds on regular intervals avoiding $x = \pm 1$. The two displayed functions have nonzero Wronskian there.

2.4. Lowering nonlinear order

Lower the order and solve (a) $1 + yy'' + (y')^2 = 0$ by a full derivative; (b) $xy'' = y' + (y')^3$ by introducing y'; (c) $(y - 1)y'' + (y - 1)^2 y' = (y')^2$ using $p(y) = y'$ and $y'' = p\, dp/dy$. Include exceptional solutions.

Solution.

(a) Since $(yy')' = -1$, $yy' = -x + C_1$, and $\boxed{y^2 = -x^2 + 2C_1 x + C_2}$ on intervals admitting a differentiable real square-root branch.

(b) Put $p = y'$: $xp' = p + p^3$. Separation gives $p = Cx/\sqrt{1 - C^2x^2}$ up to branch signs, leading to $\boxed{y = C_2 - \sqrt{1 - C^2x^2}/C}$ with branch sign and $C \neq 0$, together with constant solutions. Equivalently $\boxed{(y - C_2)^2 + x^2 = R^2}$ describes the nonconstant branches.

(c) Put $p = p(y)$ and cancel p on nonconstant branches: $(y - 1)p' - p = -(y - 1)^2$. Thus $p = (y - 1)(A - y)$. The solution is given by

$$\boxed{\int \frac{dy}{(y - 1)(A - y)} = x + B.}$$

For $A \neq 1$, $y = 1 + (A - 1)/(1 + Ce^{-(A-1)x})$; for $A = 1$, $y = 1 + 1/(x + C)$. All constant functions are additional solutions of the original equation.

2.5. Homogeneous nonlinear equations

(a) Use $y' = yz$ to solve $xyy'' + x(y')^2 = 3yy'$ and recover $y^2 = C_1x^4 + C_2$.

(b) Solve the same equation by recognising full derivatives.

(c) For $x^2(y'' + 2yy') + 2xy^2 - 2y = 0$, $x > 0$, determine the scaling $x \mapsto \lambda x, y \mapsto \lambda^\alpha y$, then use $x = e^t$, $y = z(t)e^{\alpha t}$ to reduce and solve it.

Solution.

(a) For $y' = yz$, substitution into $xyy'' + x(y')^2 = 3yy'$ gives $xz' + 2xz^2 = 3z$. A simpler first integral follows from (b): if $u = y^2$, the equation is $xu'' = 3u'$. Thus $u' = Cx^3$ and

$$\boxed{y^2 = C_1x^4 + C_2.}$$

This also proves the same result by spotting the full derivative $yy'' + (y')^2 = (yy')'$.

(c) The scaling exponent is $\alpha = -1$. Put $t = \ln x$, $y = z(t)/x$. The equation reduces to $\ddot{z} + (2z - 3)\dot{z} = 0$. On nonconstant branches use $p = \dot{z}$ to get $dp/dz = 3 - 2z$, hence

$$\boxed{\int \frac{dz}{C_1 + 3z - z^2} = \ln x + C_2, \qquad y = z/x.}$$

Zeros of the denominator give constant z solutions as well.

2.6. Boundary-value eigenfunctions

(a) Solve $y'' + k^2 y = 0$, $k > 0$.

(b) On $[0, L]$ find the nonzero solutions satisfying $y(0) = y(L) = 0$ and allowable k. Interpret as an eigenvalue problem.

(c) Explain when a function vanishing at the endpoints can be expanded in these eigenfunctions and give its coefficients.

Solution.

(a) $y = A\cos kx + B\sin kx$.

(b) $y(0) = 0$ forces $A = 0$ and $y(L) = 0$ gives $B\sin kL = 0$. Nonzero solutions exist precisely at

$$\boxed{k_n = n\pi/L,\ y_n = B\sin(n\pi x/L),\ n = 1, 2, \dots}.$$

This is an eigenvalue problem for $-d^2/dx^2$ with Dirichlet boundary conditions.

(c) For $f \in L^2(0, L)$, its sine expansion is $f = \sum_{n\geq 1} b_n \sin(n\pi x/L)$ in L^2, where

$$\boxed{b_n = (2/L)\int_0^L f(x)\sin(n\pi x/L)dx}.$$

Additional regularity gives pointwise convergence at continuity points.

2.7. Fourth-order equations

Solve (i) $y^{(4)} - 6y''' + 8y'' + 6y' - 9y = 0$ and (ii) $y^{(4)} + 4y'' + 4y = 0$.

Solution.

The first polynomial factors as $(r-3)^2(r-1)(r+1)$, the second as $(r^2+2)^2$. Thus

$$\boxed{\begin{aligned} &(i) \quad y = (C_1 + C_2 x)e^{3x} + C_3 e^{x} + C_4 e^{-x}, \\ &(ii) \quad y = (C_1 + C_2 x)\cos\Big(\sqrt{2}x\Big) + (C_3 + C_4 x)\sin\Big(\sqrt{2}x\Big). \end{aligned}}$$

Repeated roots require the extra powers of x; omitting them loses independent solutions.

2.8. Series solutions

For $y'' + xy = 0$, seek $y = \sum_{n=0}^{\infty} c_n x^n$.

(a) Derive the coefficient recurrence.

(b) Find two independent series and the general solution.

(c) Apply the same method to $y'' + y = 0$ and identify the resulting sine and cosine series.

Discuss why general equations $x^2y'' + xp(x)y' + q(x)y = 0$ may require Frobenius series $x^\alpha \sum c_n x^n$, using Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ as an example.

Solution.

For $y = \sum c_n x^n$ in $y'' + xy = 0$, the constant coefficient gives $c_2 = 0$ and, for $n \geq 1$, $(n+2)(n+1)c_{n+2} = -c_{n-1}$. Equivalently

$$\boxed{c_{n+3} = -c_n/[(n+3)(n+2)], \quad c_0, c_1 \text{ free}, \quad c_2 = 0.}$$

The two solutions start $u = 1 - x^3/6 + x^6/180 - \cdots$ and $v = x - x^4/12 + x^7/504 - \cdots$. Their initial data are $(u, u') = (1, 0)$ and $(v, v') = (0, 1)$ at zero, so their Wronskian is one there; $y = C_1 u + C_2 v$. For $y'' + y = 0$, the recurrence is $c_{n+2} = -c_n/[(n+2)(n+1)]$, producing $\boxed{y = c_0\cos x + c_1 \sin x}$.

3.4 Ordinary Differential Equations: Sheet 3

3.1. Undetermined-coefficient ansatz

For $y'' - 3y' + 2y = f(x)$, give an appropriate particular-solution ansatz, and determine its coefficients, for (i) x^2, (ii) e^{4x}, (iii) e^x, (iv) $\sinh x$, (v) $\sin x$, (vi) $x \sin x$, (vii) $e^{2x} + \cos^2 x$.

Solution.

The complementary solutions are e^x, e^{2x}. Suitable particular-solution forms for (i)–(vii), respectively, are

$$\boxed{\begin{array}{c} Ax^2 + Bx + C; \quad Ae^{4x}; \quad Axe^x; \quad Axe^x + Be^{-x}; \\ A\cos x + B\sin x; \quad (Ax + B)\cos x + (Cx + D)\sin x; \\ Axe^{2x} + B + C\cos 2x + D\sin 2x. \end{array}}$$

Multiply a resonant forcing exponential by x; resolve $\sinh x = (e^x - e^{-x})/2$ and $\cos^2 x = (1 + \cos 2x)/2$ before selecting the ansatz. Substitution determines all coefficients without adding homogeneous terms.

3.2. Inhomogeneous linear equations

Solve the following inhomogeneous equations with the stated conditions.

(i) $5y'' + 2y' + y = 2x + 3$, $y(0) = -1,\ y'(0) = 0$;
(ii) $y'' - y' - 2y = e^{2x}$;
(iii) $4y'' - 4y' + y = 8e^{x/2}$, $y(0) = 0,\ y'(0) = 1$;
(iv) $y'' + 3y' + 2y = xe^{-x}$;
(v) $y'' - 4y' + 3y = 10\cos x$;
(vi) $y'' + 4y = x + \cos 2x$, $y(0) = 0$;
(vii) $y'' - 2y' + 2y = e^x(1 + \sin x)$, $y(0) = y(\pi/2) = 0$;
(viii) $y'' + 2y' + y = 2e^{-x} + x^3 + 2\cos x$;
(ix) $y'' - 2y' + y = 3e^x$, $y(0) = 3,\ y'(0) = 0$;
(x) $x^2y'' + xy' + y = x$.

Solution.

Adding complementary solutions to a particular solution and imposing the given data gives

$$\boxed{\begin{array}{rl} (i) & y = 2x - 1 - 5e^{-x/5}\sin(2x/5), \\ (ii) & y = C_1e^{2x} + C_2e^{-x} + \frac{1}{3}xe^{2x}, \\ (iii) & y = e^{x/2}(x^2 + x), \\ (iv) & y = e^{-x}(C_1 + x^2/2 - x) + C_2e^{-2x}, \\ (v) & y = C_1e^x + C_2e^{3x} + \cos x - 2\sin x, \\ (vi) & y = C\sin 2x + x/4 + x\sin 2x/4, \\ (vii) & y = e^x[1 - \cos x - \sin x - (x/2)\cos x], \\ (viii) & y = e^{-x}(C_1 + C_2x + x^2) + x^3 - 6x^2 + 18x - 24 + \sin x, \\ (ix) & y = e^x(3 - 3x + 3x^2/2), \\ (x) & y = C_1\cos\ln|x| + C_2\sin\ln|x| + x/2. \end{array}}$$

For (x), use $t = \ln|x|$ to turn the Euler operator $x^2y'' + xy'$ into d^2y/dt^2. In (vi) only one initial condition is specified, so one constant correctly remains free. All formulas apply on regular intervals of their original equations.

3.3. Undamped forced oscillator

A torsional pendulum of inertia I and natural angular frequency ω_0 receives torque $I\cos\omega t$ from rest at equilibrium at $t = 0$.

(a) Solve

$$\ddot{\theta} + \omega_0^2\theta = \cos\omega t$$

and show that its long-time RMS displacement is $1/|\omega_0^2 - \omega^2|$, with averaging time large compared with $1/|\omega_0 - \omega|$.

(b) Discuss the resonant limit $\omega \to \omega_0$.

Solution.

For $\omega \neq \omega_0$, initial rest at equilibrium gives

$$\boxed{\theta(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}.}$$

Over times long relative to the beat period and the oscillation periods, the squared cosines each average to $1/2$ and their cross term to zero, giving

$$\boxed{\theta_{\text{rms}} = 1/|\omega_0^2 - \omega^2|}$$

for distinct positive frequencies. As $\omega \to \omega_0$, the finite-time limit is

$$\boxed{\theta = t\sin\omega_0 t/(2\omega_0)}:$$

the amplitude grows linearly and no bounded steady resonant response exists.

3.4. Driven damped oscillator

For

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = F\cos\omega t:$$

(i) Define and find the stationary displacement and velocity.

(ii) Sketch their amplitudes and phases against ω.

(iii) Find displacement and velocity resonance frequencies.

(iv) Calculate the fractional full width at half maximum to leading order in γ/ω_0, specifying which response is plotted.

(v) Near weakly damped resonance, calculate stored energy and mean supplied power; recover $Q = \omega_0/\gamma$ and its bandwidth relation.

Solution.

After decaying transients, $y = A\cos(\omega t - \delta)$ with

$$\boxed{A = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta = \operatorname{atan2}(\gamma\omega, \omega_0^2 - \omega^2), \quad \dot{y} = -\omega A\sin(\omega t - \delta).}$$

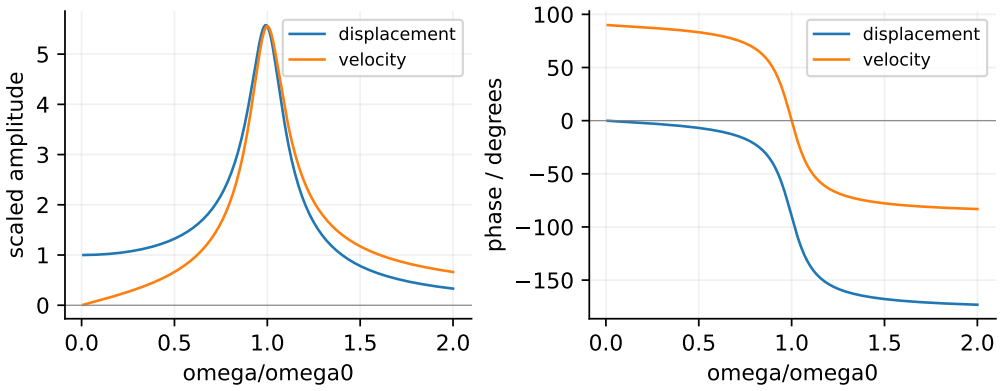
The displacement phase lag rises from zero through $\pi/2$ to π; velocity leads displacement by $\pi/2$. Displacement amplitude peaks at $\boxed{\sqrt{\omega_0^2 - \gamma^2/2}}$ if real (otherwise at zero), whereas velocity amplitude peaks at $\boxed{\omega_0}$. For the squared-amplitude or power peak,

$$\boxed{\Delta\omega \simeq \gamma, \quad \Delta\omega/\omega_0 \simeq Q^{-1}}.$$

Near resonance, $\langle E\rangle \simeq m\omega_0^2 A^2/2$ and $\langle P\rangle = m\gamma\omega^2 A^2/2$, yielding

$$Q = \omega_0\langle E\rangle/\langle P\rangle \simeq \omega_0/\gamma.$$

If "half maximum" refers to amplitude itself, the leading full width is instead $\sqrt{3}\gamma$.



3.5. Reduction of order with forcing

Solve $x(x+1)y'' + (2-x^2)y' - (2+x)y = (x+1)^2$.

(a) Use the known homogeneous solution $y_1 = 1/x$.

(b) Starting instead with the second homogeneous solution obtained in (a), recover y_1 and the general inhomogeneous solution.

Solution.

Substitution verifies both $y_1 = 1/x$ and $y_2 = e^x$. They have Wronskian $W = e^x(x+1)/x^2 \neq 0$ away from the singular points. A particular solution $y_p = ax + b$ gives coefficients $a = -1/2$, $b = -1$. Therefore

$$\boxed{y = C_1/x + C_2 e^x - x/2 - 1.}$$

Starting from either known homogeneous solution, the reduction-of-order formula $y_2 = y_1 \int e^{-\int P dx}/y_1^2 \, dx$ recovers the other, up to a homogeneous multiple, where $P = (2-x^2)/[x(x+1)]$.

3.6. Resonant parameter values

Solve $y'' - 2y' + (k^2+1)y = e^x \sin^2 x$ for real k. Identify and solve separately every exceptional value of k.

Solution.

Put $y = e^x z$ to obtain $z'' + k^2 z = (1 - \cos 2x)/2$. For $k \neq 0, \pm 2$,

$$\boxed{y = e^x \left[C_1 \cos kx + C_2 \sin kx + \frac{1}{2k^2} - \frac{\cos 2x}{2(k^2-4)} \right].}$$

At $k = 0$,

$$\boxed{y = e^x [C_1 + C_2 x + x^2/4 + \cos 2x/8].}$$

At $k = \pm 2$,

$$\boxed{y = e^x [C_1 \cos 2x + C_2 \sin 2x + 1/8 - x \sin 2x/8].}$$

The apparent poles in the generic formula signal resonant particular solutions, not nonexistence.

3.7. Piecewise forcing

Find a solution with continuous y, y' of

$$y'' + 2y' + 2y = \sin x + f(x), \qquad f(x) = \begin{cases} 0, & x \le 0, \\ x^2, & x > 0, \end{cases}$$

subject to $y(-\pi/2) = y(\pi) = 0$. Match the two branches at $x = 0$.

Solution.

A particular solution for $\sin x$ is $s(x) = (\sin x - 2\cos x)/5$. Let

$$g(x) = x^2/2 - x + 1/2 + \tfrac{1}{2}e^{-x}(-\cos x + \sin x),$$

so $g'' + 2g' + 2g = x^2$ and $g(0) = g'(0) = 0$. Then the required C^1 form is

$$\boxed{y = s(x) + e^{-x}(C\cos x + D\sin x) + \mathbf{1}_{x>0}g(x).}$$

The boundary data give

$$\boxed{D = -e^{-\pi/2}/5, \quad C = e^{\pi}(\pi^2/2 - \pi + 9/10) + 1/2}.$$

Because g, g' vanish at zero, both matching conditions hold without delta-function terms.

3.8. Amplitude-modulated forcing

A mass m on a spring of stiffness $m\omega_0^2$ has damping force $-m\gamma v$, with $\gamma \ll \omega_0$. It is driven by $mA\cos\Omega t \sin\omega t$, where $\Omega \ll \omega$ and $\omega = \omega_0$. Show that the long-time, near-resonant response is

$$y(t) \simeq -\frac{A\sin(\Omega t + \phi)\cos\omega t}{2\omega\sqrt{\Omega^2 + \gamma^2/4}}, \qquad \tan\phi = \frac{\gamma}{2\Omega}.$$

Solution.

Use

$$\cos\sigma t \sin\omega t = [\sin((\omega + \sigma)t) + \sin((\omega - \sigma)t)]/2.$$

Apply the oscillator susceptibility to each sideband. With $\omega = \omega_0$ and $\gamma, \sigma \ll \omega$, replace $\omega_0^2 - (\omega \pm \sigma)^2$ by $\mp 2\omega\sigma$ and the damping term by $\gamma\omega$. Recombining the two responses gives

$$\boxed{y \simeq -\frac{A\sin(\sigma t + \phi)\cos\omega t}{2\omega\sqrt{\sigma^2 + \gamma^2/4}}, \qquad \tan\phi = \frac{\gamma}{2\sigma}.}$$

This is the leading narrow-band stationary response; the exact response is the sum of the two susceptibility-weighted sidebands.

3.5 Ordinary Differential Equations: Sheet 4

4.1. Sum and difference modes

Solve $\dot{x} + ax - by = f$, $\dot{y} + ay - bx = 0$ for constant a, b, f, including exceptional parameter values.

Solution.

Let $u = x + y$, $v = x - y$. Adding and subtracting gives $\dot{u} + (a-b)u = f$, $\dot{v} + (a+b)v = f$. Define $F_\lambda(t) = (1 - e^{-\lambda t})/\lambda$ for $\lambda \neq 0$ and $F_0(t) = t$. Then

$$\boxed{u = C_1 e^{-(a-b)t} + fF_{a-b}(t), \quad v = C_2 e^{-(a+b)t} + fF_{a+b}(t), \quad x = (u+v)/2,\ y = (u-v)/2.}$$

This includes the degenerate cases $a = \pm b$ without division by zero.

4.2. Coupled affine system

Solve $y' + 2z' + 4y + 10z - 2 = 0$, $y' + z' + y - z + 3 = 0$, with $y(0) = 0, z(0) = -2$.

Solution.

Eliminating derivative combinations gives $y' = 2y + 12z - 8$, $z' = -3y - 11z + 5$. The equilibrium is $(-2, 1)$ and the matrix eigenvalues are $-2, -7$, with eigenvectors $(-3, 1)$ and $(-4, 3)$. The initial conditions give coefficients $6/5$ and $-7/5$. Thus

$$\boxed{y = -2 - \frac{18}{5}e^{-2x} + \frac{28}{5}e^{-7x}, \quad z = 1 + \frac{6}{5}e^{-2x} - \frac{21}{5}e^{-7x}.}$$

4.3. Real solutions of homogeneous systems

Find the general real solutions of $\dot{\mathbf{X}} = A\mathbf{X}$ for

$$\text{(i) } A = \begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \text{(ii) } A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Solution.

(i) Eigenvalues are $0, -1 \pm i$. A real eigenvector for zero is $v_0 = (0, 1, 1)$. For $-1+i$, use $a + ib = (-2, 0, 1) + i(0, 1, 0)$. Thus

$$\boxed{X = C_0 v_0 + e^{-t}\{C_1(a\cos t - b\sin t) + C_2(a\sin t + b\cos t)\}.}$$

(ii) The eigenvalues are $2, 3, 3$, with a complete eigenbasis. Hence

$$\boxed{X = C_1 e^{2t}(1,1,1) + e^{3t}[C_2(1,1,0) + C_3(1,0,1)].}$$

4.4. Inhomogeneous vector systems

Find the general real solutions of $\dot{\mathbf{X}} = A\mathbf{X} + \mathbf{f}(t)$ for

	A	$\mathbf{f}(t)$
(i)	$\begin{pmatrix} 4 & 3 & -3 \\ -3 & -2 & 3 \\ 3 & 3 & -2 \end{pmatrix}$	$(0, 0, 2e^{-t})^T$
(ii)	$\begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix}$	$(\cosh t, 2\sinh t + \cosh t, -2\cosh t)^T$

Solution.

(i) The homogeneous eigenvalues are $-2, 1, 1$, with eigenvectors $(1,-1,1)$, $(-1,1,0)$ and $(1,0,1)$. Substituting a particular solution ve^{-t} gives $v = (3,-3,2)$. Hence

$$\boxed{X = C_1e^{-2t}(1,-1,1) + e^t[C_2(-1,1,0) + C_3(1,0,1)] + e^{-t}(3,-3,2).}$$

(ii) The real eigenvector is $v_0 = (-1,-1,1)$ at -2; at $-1+i$ take $a+ib = (-1/2,0,1) + i(0,-1/2,0)$. Resolving the forcing into e^t, e^{-t} gives the particular solution $(1,1,-2)\sinh t$. Thus

$$\boxed{X = C_0e^{-2t}v_0 + e^{-t}[C_1(a\cos t - b\sin t) + C_2(a\sin t + b\cos t)] + (1,1,-2)\sinh t.}$$

Substitution checks both particular solutions, and the independent homogeneous vectors provide all three real constants in each case.

4.5. A differential-algebraic degeneracy

Solve $2y'' - 3y' + 2z' + 3y + z = e^{2x}$ and $y'' - 3y' + z' + 2y - z = 0$. Can $y(0) = z(0) = 0$ hold? Explain the degeneracy of the system.

Solution.

Subtract twice the second equation from the first: $3y' - y + 3z = e^{2x}$. Substitute $z = e^{2x}/3 - y' + y/3$ back into the second; second derivatives cancel and leave $y' - y = e^{2x}/5$. Therefore

$$\boxed{y = Ce^x + e^{2x}/5, \quad z = -2Ce^x/3.}$$

The data $y(0) = 0$ require $C = -1/5$, which makes $z(0) = 2/15 \neq 0$, so the proposed simultaneous zero values are impossible. The highest-derivative coefficient matrix is singular: the system has an algebraic compatibility constraint and only one free integration constant, rather than the naive order count.

4.6. Constant fields in Gaussian units

A charge q of mass m starts at rest in uniform constant electric and magnetic fields which are neither parallel nor perpendicular. In the unit convention

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c),$$

find $\mathbf{v}(t)$ and sketch its motion.

Solution.

Use Gaussian units. Put $\widehat{b} = B/|B|$, $\Omega = q|B|/(mc)$, $E_\parallel = (E \cdot \widehat{b})\widehat{b}$, and $u = cE \times B/B^2$. The perpendicular velocity equation is

$$\dot{v}_\perp = \Omega(v_\perp - u) \times \widehat{b}.$$

Initial rest gives

$$\boxed{v(t) = \frac{q}{m}E_\parallel t + u(1 - \cos\Omega t) - (u \times \widehat{b})\sin\Omega t.}$$

Integrating from initial position r_0 gives

$$r = r_0 + qE_\parallel t^2/(2m) + u[t - \sin\Omega t/\Omega] + (u \times \widehat{b})(\cos\Omega t - 1)/\Omega.$$

Thus the path combines transverse cycloidal drift with longitudinal acceleration. In SI units omit c from the Lorentz force and from the corresponding drift/cyclotron definitions.

4.7. Defective systems

Find general solutions of $\dot{\mathbf{X}} = A\mathbf{X}$ by triangularisation or Jordan chains:

$$\text{(i) } A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}; \quad \text{(ii) } A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}; \quad \text{(iii) } A = \begin{pmatrix} 2 & -5 & -8 \\ 7 & -11 & -17 \\ -3 & 4 & 6 \end{pmatrix}.$$

Solution.

(i) Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ and $N = A - 3I$. Since $N^2 = 0$,

$$\boxed{X = e^{3t}(I + tN)C.}$$

(ii) The matrix has eigenvalues $-1, 1, 1$. Put $v_- = (-2, -2, 1)$, $v = (-2, 0, 1)$ and $w = (-2, 1, 0)$; then $(A-I)w = v$. Thus

$$\boxed{X = C_1 e^{-t} v_- + e^{t}[C_2 v + C_3(w + tv)]}.$$

(iii) Put $N = A + I$ for the given 3×3 matrix. Direct multiplication gives $N^3 = 0$, $N^2 \neq 0$. Therefore

$$\boxed{X = e^{-t}(I + tN + t^2 N^2/2)C, \quad C \in \mathbb{R}^3.}$$

These polynomial factors are the missing solutions that pure exponential eigenvector ansatzes fail to produce.

3.6 Vectors and Matrices: Sheet 1

1. Linear independence

Determine linear independence in each set. For each dependent set, exhibit a maximal independent subset and justify it.

	$(\mathbf{v}_1)^T$	$(\mathbf{v}_2)^T$	$(\mathbf{v}_3)^T$
(a)	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
(b)	$(0,1,1)$	$(1,1,1)$	$(0,0,1)$
(c)	$(1,0,1)$	$(2,3,1)$	$(1,6,-1)$
(d)	$(1,2,0,-3)$	$(2,1,1,-4)$	$(-3,6,-4,1)$

Solution.

Put the vectors in the columns of A. A vanishing linear combination is exactly a vector in $\ker A$.

(a) $A = I_3$, so the three vectors are independent.

(b) $\det A = -1$, so all three are independent.

(c) $v_3 = -3v_1 + 2v_2$, while v_1, v_2 are not proportional.

(d) $-5v_1 + 4v_2 + v_3 = 0$, while the first two columns have a nonzero 2×2 minor. Thus

$$\boxed{\operatorname{rank} A = 3, 3, 2, 2 \quad \text{in (a)–(d), respectively.}}$$

In the last two cases $\{v_1, v_2\}$ is a maximal independent subset.

2. Subspaces

Determine whether each set is a vector subspace, giving reasons: (a) $(x, y, z) \in \mathbb{R}^3$ with $x = y = 2z$; (b) $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$; (c) the even real functions $f : \mathbb{R} \to \mathbb{R}$.

Solution.

(a) The set is $\{t(2,2,1) : t \in \mathbb{R}\}$, the span of one vector, hence a subspace.

(b) The unit circle omits 0 and is not closed under scaling, so is not a subspace.

(c) If f, g are even, $(af + bg)(-x) = af(x) + bg(x)$; the zero function is even. Hence the even functions form a subspace.

3. Symmetric and antisymmetric matrices

Find the dimension of the space of real 3×3 matrices. Show that its symmetric and antisymmetric matrices form subspaces, and give explicit bases and dimensions. Generalise to real $n \times n$ matrices.

Solution.

The ambient basis is E_{ij}, with a single unit entry, so its dimension is 9. Transpose is linear; therefore the equations $A^T = A$ and $A^T = -A$ are preserved by linear combinations.

$$\boxed{\begin{aligned} \operatorname{Sym}_n &= \operatorname{span}\{E_{ii}, E_{ij} + E_{ji} : i < j\}, \quad && \dim \operatorname{Sym}_n = \tfrac{1}{2}n(n+1), \\ \operatorname{Skew}_n &= \operatorname{span}\{E_{ij} - E_{ji} : i < j\}, && \dim \operatorname{Skew}_n = \tfrac{1}{2}n(n-1). \end{aligned}}$$

Independence follows by inspecting the individual upper-triangular entries. For $n = 3$ the dimensions are 6 and 3.

4. Coordinates in a nonstandard basis

Show that $(1,-1,0)^T$, $(0,1,-1)^T$, $(2,0,1)^T$ form a basis of $\mathbb{R}^3$. Expand $(x,y,z)^T$ in this basis and give its coordinates.

Solution.

For $v = c_1v_1 + c_2v_2 + c_3v_3$, equating components gives $c_1 + 2c_3 = x$, $-c_1 + c_2 = y$, $-c_2 + c_3 = z$. The basis matrix has determinant $3 \neq 0$, and

$$\boxed{[v]_{\mathcal{B}} = \frac{1}{3}\begin{pmatrix} x - 2y - 2z \\ x + y - 2z \\ x + y + z \end{pmatrix}.}$$

5. Sums and intersections

Let U, W be subspaces of a real vector space V.

(a) Prove that $U \cap W$ is a subspace.

(b) Prove that $U + W$ is a subspace.

(c) Verify the dimension formula for $V = \mathbb{R}^3$,

$$U = \operatorname{span}\{\mathbf{i} + 2\mathbf{j}, \mathbf{k}\},$$

$$W = \operatorname{span}\{\mathbf{j} + \mathbf{k}, -\mathbf{i} + 2\mathbf{j}\}.$$

(d) Prove $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ by extending a basis of the intersection.

Solution.

Both sets contain zero. If $u, v \in U \cap W$, every linear combination $au + bv$ belongs to each subspace, hence to their intersection. If $u_1 + w_1, u_2 + w_2 \in U + W$, their linear combination is $(au_1 + bu_2) + (aw_1 + bw_2) \in U + W$. In the example $U = \{(a, 2a, b)\}$ and $W = \{(-d, c + 2d, c)\}$, whence $U \cap W = \operatorname{span}\{(1, 2, 4)\}$. Thus $\dim U = \dim W = 2$, $\dim(U \cap W) = 1$, and $\dim(U + W) = 3$. For the general proof, extend a basis $e_1, \ldots, e_r$ of the intersection to bases $(e, u_1, \ldots, u_p)$ of U and $(e, w_1, \ldots, w_q)$ of W. Their union spans the sum. A dependence would put $\sum b_j w_j$ in $U \cap W$, so independence of (e, w) makes every b_j zero; independence of (e, u) then makes all other coefficients zero. Consequently

$$\boxed{\dim(U + W) = r + p + q = \dim U + \dim W - \dim(U \cap W).}$$

6. Vector identities

Prove the three-dimensional identities:

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$,
(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$,
(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.

Solution.

(a) Antisymmetry gives $b \times a = -a \times b$, and the dot product is symmetric.

(b) With repeated indices summed,

$$[a \times (b \times c)]_i = \epsilon_{ijk}\epsilon_{klm}a_j b_l c_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_j b_l c_m = (a \cdot c)b_i - (a \cdot b)c_i.$$

(c) Similarly

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

gives

$$\boxed{(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c).}$$

7. Consequences of vector identities

Prove the following identities and implications.

(a) $\mathbf{a} \times \mathbf{b} = \mathbf{a} - \mathbf{b}$ implies $\mathbf{a} = \mathbf{b}$.

(b) $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ implies $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$.

(c) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ implies $\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = \pm|\mathbf{c}|\,|\mathbf{a} - \mathbf{b}|$.

(d) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})]$.

Solution.

(a) Dot $a \times b = a - b$ with $a - b$: $0 = |a - b|^2$, hence $\boxed{a = b}$.

(b) $(a \times b) \cdot (\lambda a + \mu b) = 0$ because the cross product is perpendicular to both factors.

(c) Set $d = a - b$. Then $d \times c = 0$. If either vector is zero, the result is immediate; otherwise they are parallel, and $\boxed{c \cdot d = \pm|c||d|}$.

(d) The vector triple-product identity gives

$$(a \times b) \times (c \times b) = -b[(a \times b) \cdot c] = \boxed{b[b \cdot (a \times c)]}.$$

8. Triangle area

The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form the directed sides of a triangle. Prove

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}|.$$

Find the area of the triangle with vertices $(1, 3, 2)$, $(2, -1, 1)$ and $(-1, 2, 3)$.

Solution.

Orient the sides consecutively, so $a + b + c = 0$. Then $b \times c = b \times (-a - b) = a \times b$, and cyclically all three cross products agree. The area is half their common magnitude. In the stated example

$$(Q - P) \times (R - P) = (-5, 1, -9), \qquad \boxed{\mathcal{A} = \frac{\sqrt{107}}{2}.}$$

9. Coplanarity

(a) Prove that $(1, 1, 2)^T$, $(2, 3, -1)^T$, $(1, 2, -3)^T$ are coplanar.

(b) For $\mathbf{a} = (1, 1, 1)^T$ and $\mathbf{b} = (2, -1, 0)^T$, find a nonzero vector in their plane perpendicular to $\mathbf{a}$.

Solution.

(a) Here $c = b - a$, so the three vectors are dependent and coplanar.

(b) Remove from b its component along a: $b - (a \cdot b)a/|a|^2 = (5, -4, -1)/3$. Thus any nonzero multiple of $\boxed{(5, -4, -1)}$ is suitable.

10. Reciprocal basis

For independent $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, put

$$V = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$$

and define reciprocal vectors

$$\mathbf{v}_1' = (\mathbf{v}_2 \times \mathbf{v}_3)/V,$$

cyclically.

(a) Show $\mathbf{v}_i \cdot \mathbf{v}_j' = \delta_{ij}$ and that the coordinates of $\mathbf{w}$ are $\mathbf{w} \cdot \mathbf{v}_i'$.

(b) Show

$$V' = \mathbf{v}_1' \cdot (\mathbf{v}_2' \times \mathbf{v}_3') = 1/V.$$

(c) Show that taking reciprocals again recovers the original basis.

Solution.

The scalar triple product is alternating, so $v_i \cdot v_j' = \delta_{ij}$. Dotting $w = \sum c_i v_i$ with v_j' gives $\boxed{c_j = w \cdot v_j'}$. If $B = (v_1\ v_2\ v_3)$, the reciprocal basis matrix is B^{-T}. Hence $V' = \det\big(B^{-T}\big) = \boxed{1/V}$. Applying the construction again gives $(B^{-T})^{-T} = B$, which proves all three reciprocal-of-reciprocal formulas.

11. Distance to an axis

Find the shortest distance from $(2, 3, 4)$ to the x axis.

Solution.

The nearest point on the x axis is $(2, 0, 0)$. Therefore $\boxed{d = \sqrt{3^2 + 4^2} = 5}$.

12. Distance to a line

Write the vector equation of $(x-2)/4 = (y-1)/3 = (z-5)/2$ and find its minimum distance from the origin.

Solution.

The line is $r = (2, 1, 5) + t(4, 3, 2)$. Minimizing $|q + tm|^2$ gives $t_* = -q \cdot m/|m|^2 = -21/29$. Equivalently

$$\boxed{d = \frac{|q \times m|}{|m|} = \sqrt{\frac{429}{29}}.}$$

13. Distance between skew lines

Derive the shortest distance between

$$\mathbf{r}_i = \mathbf{q}_i + \lambda_i \mathbf{m}_i$$

$(i = 1, 2)$. Apply it to $(x-2)/2 = y - 3 = (z+1)/2$ and $x + 2 = (y+1)/2 = z - 1$.

Solution.

For nonparallel directions let $n = m_1 \times m_2$. The separation vector's projection on n is unaffected by sliding along either line, and the shortest connector is parallel to n. Thus

$$\boxed{d = \frac{|(q_2 - q_1) \cdot (m_1 \times m_2)|}{|m_1 \times m_2|}.}$$

Here $m_1 = (2, 1, 2)$, $m_2 = (1, 2, 1)$ and $q_2 - q_1 = (-4, -4, 2)$, giving $d = 18/\sqrt{18} = \boxed{3\sqrt{2}}$. For parallel directions instead use $d = |(q_2 - q_1) \times m_1|/|m_1|$.

14. Plane through three points

Find the Cartesian plane through $(2,-1,1)$, $(3,2,-1)$ and $(-1,3,2)$.

Solution.

A normal is $(P_2-P_1)\times(P_3-P_1)=(11,5,13)$. Therefore

$$\boxed{11x+5y+13z=30.}$$

15. Cartesian and vector planes

Give Cartesian and vector equations of the plane through $(3,2,1)$, $(-1,3,-2)$ and $(2,2,2)$.

Solution.

The displacement vectors $b-a=(-4,1,-3)$ and $c-a=(-1,0,1)$ have cross product $(1,7,1)$. Hence

$$\boxed{x+7y+z=18, \qquad r=(3,2,1)+s(-4,1,-3)+t(-1,0,1).}$$

16. Line-plane intersection

Find the intersection of the line through $0,(1,1,1)$ with the plane through $(-1,1,-2)$, $(1,5,-5)$ and $(0,2,-3)$.

Solution.

The plane normal is proportional to $(1,1,2)$ and its equation is $x+y+2z=-4$. The line is $r=t(1,1,1)$, so $4t=-4$. Thus the intersection is $\boxed{(-1,-1,-1)}$.

C1. Computational linear independence

Use Mathematica to check the linear independence results of Question 1(b)–(d).

Solution.

In Mathematica use `MatrixRank[Transpose[vs]]` and `NullSpace[Transpose[vs]]`, with the vectors entered as rows of `vs`. The ranks for Q1(b,c,d) are $3,2,2$; the last two null relations are $(-3,2,-1)$ and $(-5,4,1)$, up to nonzero scaling. These reproduce the explicit dependencies in Q1.

C2. Computational vector identities

Use symbolic computation to verify the vector identities in Question 6(a)–(c).

Solution.

Set `aa=Array[a,3]; bb=Array[b,3]; cc=Array[c,3]; dd=Array[d,3]`. Apply `Expand` to the difference between the two sides of each identity, with `Cross` for cross products and `Dot` for dot products. The scalar outputs for (a,c) are 0 and the vector output for (b) is $\{0,0,0\}$, with no assumptions on the components.

C3. Computational planes

Write code returning Cartesian and vector equations of the plane through three given points in $\mathbb{R}^3$. Check Questions 14 and 15.

Solution.

Use `n=Cross[p2-p1,p3-p1]` and return `n.Array[x,3]==n.p1`, together with `p1+s(p2-p1)+t(p3-p1)`. Reject `n=={0,0,0}` because the points then fail to determine a unique plane. Substitution of the Q14 and Q15 points gives $11x + 5y + 13z = 30$ and $x + 7y + z = 18$.

C4. Perceptron

Implement a perceptron in Mathematica, train it on points on both sides of a specified line in $\mathbb{R}^2$, and test its classification on additional points.

Solution.

Augment each input (x, y) to $X = (1, x, y)$ and label it $s = \operatorname{sign}(y - 2x - 1)$. Start $w = 0$ and, whenever $s(w \cdot X) \leq 0$, update $w \leftarrow w + sX$. Repeat over a finite training set until an epoch has no errors. This terminates for strictly linearly separable data. For example the four points $(0, 0), (0, 2), (1, 2), (1, 4)$ have labels $-1, +1, -1, +1$ and are all correctly classified by $w = (-1, -2, 1)$. Test on independent points; points on the separating line require an explicit convention.

3.7 Vectors and Matrices: Sheet 2

1. Matrix identities

For compatible matrices and scalar λ, prove associativity; $(A+B)^T = A^T + B^T$, $(\lambda A)^T = \lambda A^T$, $(AB)^T = B^T A^T$; the corresponding Hermitian-adjoint identities, with λ^*; and $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$ when invertible.

Solution.

(a)

$$(A(BC))_{ij} = \sum_{k,l} A_{ik}B_{kl}C_{lj} = ((AB)C)_{ij}.$$

(b) Transposition exchanges the two free indices:

$$(AB)^T_{ij} = \sum_k A_{jk}B_{ki} = (B^T A^T)_{ij};$$

addition and scalar multiplication follow entrywise.

(c) Taking complex conjugates of (b) gives $(AB)^\dagger = B^\dagger A^\dagger$, $(\alpha A)^\dagger = \alpha^* A^\dagger$ and additivity.

(d) $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$, proving the inverse formula. Transpose $AA^{-1} = I$ and $A^{-1}A = I$ to obtain $(A^T)^{-1} = (A^{-1})^T$.

2. Parameter-dependent rank

Use row operations to find the rank for all real a:

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 3 & 2 & 0 & -4 \\ 1 & -2 & a & 0 \end{pmatrix}.$$

Solution.

Eliminate column 1 using $R_3 \leftarrow R_3 - 3R_1$ and $R_4 \leftarrow R_4 - R_1$. Then $R_3 \leftarrow R_3 + R_2/2$ gives $(0, 0, -1, -1/2)$. Using this to eliminate the third entry of the final row leaves a final pivot proportional to $a - 8$. Thus

$$\boxed{\operatorname{rank} A = 4 \ (a \neq 8), \qquad \operatorname{rank} A = 3 \ (a = 8).}$$

The first three rows remain independent even at $a = 8$; $\det A = a - 8$ confirms the exceptional value.

3. Matrix inverse

Find by row operations, and verify, the inverse of

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & -3 & 0 \end{pmatrix}.$$

Solution.

Row-reducing $(A \mid I)$ gives $(I \mid A^{-1})$, where

$$\boxed{A^{-1} = \begin{pmatrix} -6 & 3 & 1 \\ -2 & 1 & 0 \\ -7 & 3 & 1 \end{pmatrix}.}$$

For a direct check the columns solve $Ax = e_1, e_2, e_3$, respectively, so $AA^{-1} = I$.

4. Change of basis

The linear map $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ acts on $\mathbb{R}^2$. Find its matrix A' in the basis $(1,1)^T, (0,1)^T$. Find P with $A' = PAP^{-1}$ and interpret P.

Solution.

The new basis matrix is $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so old and new coordinates satisfy $v = S[v]'$. Consequently

$$\boxed{A' = S^{-1}AS = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \qquad P = S^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.}$$

Thus P converts old coordinates into new coordinates, and $A' = PAP^{-1}$.

5. Reflection map

For a fixed unit vector $\mathbf{n}$, define

$$f(\mathbf{v}) = \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n}.$$

(a) Prove linearity and $f^2 = \mathrm{id}$.

(b) Find its standard-basis matrix and show $A^2 = I$.

(c) Find its matrix in a basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n})$ with independent $\mathbf{u}_a \perp \mathbf{n}$.

(d) Give its geometric interpretation.

Solution.

Linearity follows from that of $v \mapsto n \cdot v$. Write $v = v_\perp + (n \cdot v)n$, so $f(v) = v_\perp - (n \cdot v)n$ and $f^2(v) = v$. In the standard basis

$$\boxed{A = I - 2nn^T, \quad A^2 = I, \quad \widehat{A} = \mathrm{diag}(1, 1, -1).}$$

The map reflects through the plane perpendicular to n.

6. Trace

For square matrices, prove

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

and similarity invariance of trace. Explain the basis-change interpretation and find the trace of the reflection in Question 5.

Solution.

Expand

$$\mathrm{tr}(AB) = \sum_{i,j} A_{ij} B_{ji}$$

and exchange the dummy indices to get $\mathrm{tr}(BA)$. Hence

$$\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(AP^{-1}P) = \mathrm{tr}\, A :$$

trace belongs to the linear map independently of basis. The reflection has trace $1 + 1 - 1 = \boxed{1}$.

7. A polynomial differential operator

On $P_2(\mathbb{R})$ define $D(p) = xp'' + (1-x)p' + 2p$.

(a) Prove linearity.

(b) Find its matrix in $(1, x, x^2)$.

(c) Find its kernel and corresponding polynomials.

(d) Verify these polynomials directly.

Solution.

Differentiation and multiplication by a fixed function are linear. On the ordered basis $(1, x, x^2)$,

$$D1 = 2, \quad Dx = 1 + x, \quad Dx^2 = 4x, \qquad \boxed{A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.}$$

The kernel solves $2c + b = 0$, $b + 4a = 0$, so $p = a(x^2 - 4x + 2)$. Direct substitution gives $2ax + (1-x)(2ax - 4a) + 2a(x^2 - 4x + 2) = 0$.

8. A homogeneous system

For real λ, analyse $A\mathbf{x} = 0$ with

$$A = \begin{pmatrix} 2-\lambda & 1 & 2 \\ 1 & 4-\lambda & -1 \\ 2 & -1 & 2-\lambda \end{pmatrix}.$$

Find its rank, predict the solution-space dimension and calculate the solutions explicitly in every case.

Solution.

For

$$M = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

the system is $(M - \lambda I)v = 0$. The eigenvalues are $4, 2 \pm \sqrt{6}$, all simple. Thus the rank is 3 otherwise and 2 at these three values. The solutions are zero in the first case and arbitrary multiples of

$$\boxed{(1, 0, 1)\ (\lambda = 4), \qquad (-1, -\lambda, 1)\ (\lambda = 2 \pm \sqrt{6}).}$$

Substitution verifies each vector and the factorization

$$\det(M - \lambda I) = (4 - \lambda)[(2 - \lambda)^2 - 6].$$

9. Consistency with a parameter

For real α, consider $x + y + z = 1$, $x + 2y + 4z = \alpha$, $x + 4y + 10z = \alpha^2$. Show that the coefficient matrix has rank two. Explain the solvability conditions and solve every consistent case.

Solution.

The third coefficient row is $3R_2 - 2R_1$, while the first two are independent. Thus rank is 2 and consistency requires $\eta^2 = 3\eta - 2$, or $\eta = 1, 2$. Set $z = t$; subtracting the first two equations gives $y = \eta - 1 - 3t$, and then $x = 2 - \eta + 2t$. Therefore

$$\boxed{(x, y, z) = (2 - \eta + 2t, \eta - 1 - 3t, t), \quad \eta \in \{1, 2\}.}$$

There is no solution for any other η.

10. Two parameters in a linear system

By augmented-matrix row reduction, solve $3x + 2y - z = 10$, $5x - y - 4z = 17$, $x + 5y + \lambda z = \mu$ for all real λ, μ.

Solution.

The first two equations imply $x = (44+9z)/13$, $y = (-1-7z)/13$. The final equation reduces to $(\alpha-2)z = \beta-3$. Hence for $\alpha \neq 2$ set $z = (\beta-3)/(\alpha-2)$ in these formulas. For $\alpha = 2, \beta = 3$, z is arbitrary; for $\alpha = 2, \beta \neq 3$, there is no solution.

11. Semi-magic squares

A rational 3×3 semi-magic square has equal sums for all rows and columns. Prove that these matrices form a vector space and that the following form a basis:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Solution.

The equal-row-sum and equal-column-sum conditions are homogeneous linear constraints, so their solutions form a vector space over $\mathbb{Q}$. Given common sum s, subtract $(s/3)M_1$ to leave a matrix B with zero row and column sums. Such a matrix is determined freely by its upper-left 2×2 block, hence has four parameters. More explicitly if that block is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$B = aM_2 + (a+b)M_3 + (a+c)M_4 + (a+b+c+d)M_5.$$

These coefficients are unique, proving that the five supplied matrices form a basis and $\boxed{\dim = 5}$.

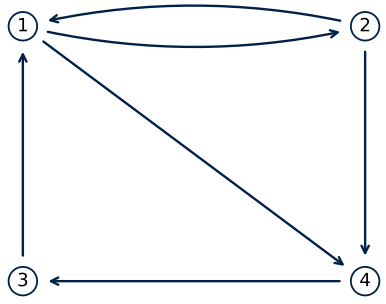
12. Page rank

In a directed internet with n sites, site j has $n_j > 0$ outgoing links. If L_i is the set linking to i, define page rank by $x_i = \sum_{j \in L_i} x_j / n_j$.

(a) Write $A\mathbf{x} = 0$ and find A_{ij}.

(b) Prove that a nonzero solution exists.

(c) Solve the four-site example with arrows $1 \to 2, 4$, $2 \to 1, 4$, $3 \to 1$, $4 \to 3$, normalising the ranks to sum to one. Which site ranks highest?



Solution.

Let $T_{ij} = 1/n_j$ if page j links to i, and zero otherwise. Then $A = I - T$ and $Ax = 0$. Each column of T sums to one, so $\mathbf{1}^T A = 0$, proving singularity and a nontrivial kernel (provided each $n_j > 0$). The displayed arrows give $1 \to 2, 4$, $2 \to 1, 4$, $3 \to 1$, $4 \to 3$. Thus

$$x_1 = x_2/2 + x_3, \quad x_2 = x_1/2, \quad x_3 = x_4, \quad x_4 = (x_1 + x_2)/2.$$

Normalizing $\sum x_i = 1$ gives $\boxed{x = (1/3, 1/6, 1/4, 1/4)}$; page 1 is ranked highest.

C1. Row reduction in code

Write code reducing a matrix to row-echelon form while displaying every row operation. Test it and verify Question 2.

Solution.

At each column, search at or below the current pivot row for a nonzero entry; swap it into the pivot position, divide that row by its pivot, and subtract multiples from lower rows. Print the matrix after each operation. Skip a column if its available entries vanish. This is Gaussian elimination and directly reproduces the final pivot $a - 8$ in Q2. In Mathematica, `RowReduce[A]` checks the resulting reduced form; treat $a = 8$ separately before reduction.

C2. Computational systems

Use computation to check the parameter-dependent linear systems in Questions 8–10.

Solution.

Apply `Solve[Thread[A.vars==rhs],vars]` to Q8–Q10, retaining the parameters. Alternatively compare `MatrixRank[A]` and `MatrixRank[Join[A,Transpose[{rhs}],2]]`. Equal ranks give consistency, and $n - \operatorname{rank} A$ free parameters. The exceptional parameter values are exactly those listed in the analytical solutions.

C3. Computational page ranking

Write code taking a directed graph and returning vertices ordered by page rank. Test the graph in Question 12 and other examples.

Solution.

Form the column-normalized adjacency matrix T, solve $(I - T)x = 0$ with $\sum x_i = 1$, and sort vertices by decreasing x_i. For Q12 the result is $(1/3, 1/6, 1/4, 1/4)$. A graph with a vertex of zero out-degree requires a rule for its column, e.g. a uniform distribution. For a disconnected graph the stationary distribution need not be unique; adding a positive teleportation term $T_d = dT + (1 - d)\mathbf{1}\mathbf{1}^T/n$, $0 < d < 1$, makes it unique.

3.8 Vectors and Matrices: Sheet 3

1. Determinants and matrix classes

Calculate determinants and classify the matrices below as real, diagonal, symmetric, antisymmetric, singular, orthogonal, Hermitian, anti-Hermitian and unitary:

$$A = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Solution.

$A^T = -A$ and $\overline{A} = -A$, so $A^\dagger = A$. Its odd dimension makes $\det A = 0$. Thus A is antisymmetric, singular and Hermitian; it is neither real, diagonal, symmetric, orthogonal, anti-Hermitian nor unitary. For B, the unscaled columns are pairwise orthogonal with squared length 8, giving $B^T B = I$. Their orientation is positive, so $\det B = 1$. Thus B is real, orthogonal and unitary; it is neither diagonal, symmetric, antisymmetric, singular, Hermitian nor anti-Hermitian.

2. Gram–Schmidt

Apply Gram–Schmidt to $(1,1,0)^T, (2,1,2)^T, (0,2,-1)^T$ to find an orthonormal basis of $\mathbb{R}^3$. Check all scalar products.

Solution.

First $e_1 = (1,1,0)/\sqrt{2}$. Subtract $(v_2 \cdot e_1)e_1$ from v_2 to obtain $(1,-1,4)/2$, then normalize. Removing the first two projections from v_3 gives a vector parallel to $(-2,2,1)$. Hence

$$\boxed{e_1 = \frac{(1,1,0)}{\sqrt{2}}, \quad e_2 = \frac{(1,-1,4)}{\sqrt{18}}, \quad e_3 = \frac{(-2,2,1)}{3}.}$$

Their pairwise dot products vanish and each squared norm is one.

3. Invertibility

Explain why invertibility is equivalent to nonzero determinant. Determine all (a,b) for which

$$A = \begin{pmatrix} a & 1 & a \\ 1 & b & -1 \\ 0 & -1 & a \end{pmatrix}$$

is singular.

Solution.

(a) If A is invertible, $\det A \det A^{-1} = 1$. Conversely $\det A \neq 0$ makes the adjugate formula $A^{-1} = \operatorname{adj} A / \det A$ valid.

(b) Expansion of the supplied determinant yields $\det A = a(ab-3)$. Thus

$$\boxed{A \text{ is singular exactly when } a = 0 \text{ or } ab = 3.}$$

4. Generalized cross product

For independent $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathbb{R}^n$, define

$$w_i = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{e}_i).$$

(a) Prove $\mathbf{w} \perp \mathbf{v}_a$.

(b) Prove

$$|\mathbf{w}| = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{w}/|\mathbf{w}|).$$

(c) Find $\mathbf{w}$ when $\mathbf{v}_a = \mathbf{e}_a$.

(d) Relate this construction to the cross product in three dimensions.

Solution.

For any vector u, multilinearity gives $u \cdot w = \det(v_1, \ldots, v_{n-1}, u)$. Setting $u = v_a$ produces a repeated column and hence zero. Setting $u = n = w/|w|$ gives the determinant $|w|$. If $v_a = e_a$, the only nonzero component is $w_n = 1$, so $w = e_n$. In three dimensions the defining scalar triple product gives $\boxed{w = v_1 \times v_2}$.

5. Three solution methods

Solve $x + 2y + 3z = 2$, $3x + 4y + 5z = 4$, $x + 3y + 4z = 6$ using (a) an inverse, (b) Cramer's rule, and (c) augmented row reduction. Which method would you use for a program handling large systems?

Solution.

For the coefficient matrix A and right-hand side b,

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ -7/2 & 1/2 & 2 \\ 5/2 & -1/2 & -1 \end{pmatrix}, \qquad \boxed{(x, y, z) = (-3, 7, -3).}$$

Here $\det A = 2$; replacing columns 1,2,3 by b gives determinants $-6, 14, -6$, so Cramer's rule gives the same answer. In row reduction, $R_2 - 3R_1$ gives $-2y - 4z = -2$ and $R_3 - R_1$ gives $y + z = 4$; hence $z = -3, y = 7, x = -3$. For numerical large systems choose Gaussian elimination with pivoting (usually LU factorization), avoiding explicit inversion and repeated determinants.

6. Gaussian-weighted polynomial inner product

On real polynomials, define

$$\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x)\, dx.$$

(a) Prove that this is an inner product.

(b) Show that $p_0 = 1/n_0$, $p_1 = 2x/n_1$, $p_2 = (4x^2 - 2)/n_2$ are orthogonal.

(c) Find positive n_a normalising them. You may use the Gaussian moments $\sqrt{\pi}$, $\sqrt{\pi}/2$, $3\sqrt{\pi}/4$ at orders $0, 2, 4$ and the vanishing odd moments.

Solution.

The integral is bilinear and symmetric; all polynomial moments converge. If $f \neq 0$, continuity makes $e^{-x^2} f(x)^2$ strictly positive on some interval, so its integral is positive. Odd integrands integrate to zero. The only even cross term is

$$\int e^{-x^2} (4x^2 - 2)\, dx = 4\sqrt{\pi}/2 - 2\sqrt{\pi} = 0.$$

The squared norms of $1, 2x, 4x^2 - 2$ are $\sqrt{\pi}, 2\sqrt{\pi}, 8\sqrt{\pi}$. Choose

$$\boxed{n_0 = \pi^{1/4}, \quad n_1 = \sqrt{2}\pi^{1/4}, \quad n_2 = 2\sqrt{2}\pi^{1/4}.}$$

7. Orthonormal coordinates

A complex inner-product space has orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$; take the inner product linear in its second entry.

(a) Prove that nonzero pairwise orthogonal vectors are independent.

(b) Show $v_i = \langle \mathbf{e}_i, \mathbf{v} \rangle$.

(c) Prove
$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_i \langle \mathbf{u}, \mathbf{e}_i \rangle \langle \mathbf{e}_i, \mathbf{v} \rangle.$$

(d) If $\mathbf{e}'_j = \sum_i U_{ij} \mathbf{e}_i$ is also orthonormal, show
$$U_{ij} = \langle \mathbf{e}_i, \mathbf{e}'_j \rangle$$
and $U^\dagger U = I$.

Solution.

Use the convention that $\langle \cdot, \cdot \rangle$ is antilinear in its first slot. If $\sum c_j w_j = 0$, taking the inner product with w_i gives $c_i \|w_i\|^2 = 0$, proving independence. Expanding $v = \sum v_i e_i$ gives $v_i = \langle e_i, v \rangle$ and
$$\langle u, v \rangle = \sum_i \langle u, e_i \rangle \langle e_i, v \rangle.$$

The same projection yields $U_{ij} = \langle e_i, e'_j \rangle$, and orthonormality of the primed basis gives $(U^\dagger U)_{jk} = \delta_{jk}$. Thus $\boxed{U^\dagger U = I}$.

8. Orthogonal transformations in two dimensions

(a) Show that preservation of the Euclidean inner product is equivalent to $R^T R = I$, and that $\det R = \pm 1$.

(b) Classify all 2×2 orthogonal matrices as rotations or reflections.

(c) Show $R(\phi_1) R(\phi_2) = R(\phi_1 + \phi_2)$ for rotations.

(d) If $\mathbf{x}' = R(\phi)\mathbf{x}$, relate $z' = x' + iy'$ and $z = x + iy$.

Solution.

The invariance condition is $v^T (R^T R - I) w = 0$ for every v, w, hence $R^T R = I$. Taking determinants gives $(\det R)^2 = 1$. A first unit column $(\cos\phi, \sin\phi)$ has second unit column either $(-\sin\phi, \cos\phi)$ or its negative. The determinant-$+1$ case is a rotation; the determinant--1 case is reflection through a line at angle $\phi/2$. Angle-addition identities give $R(\phi_1) R(\phi_2) = R(\phi_1 + \phi_2)$, and in complex notation $\boxed{z' = e^{i\phi} z}$ for a rotation.

9. Three-dimensional rotations

Construct right-handed axis rotations R_1, R_2, R_3.

(a) Multiply
$$R = R_1(\alpha_1) R_2(-\alpha_2) R_3(\alpha_3).$$

(b) Find its infinitesimal generators, retaining the minus sign in the second angle.

(c) Relate the infinitesimal displacement to a cross product with the corresponding signed angle vector.

Solution.

Write $c_i = \cos\alpha_i$, $s_i = \sin\alpha_i$. Direct multiplication, with the stated middle angle $-\alpha_2$, gives
$$R = \begin{pmatrix} c_2 c_3 & -c_2 s_3 & -s_2 \\ c_1 s_3 - s_1 s_2 c_3 & c_1 c_3 + s_1 s_2 s_3 & -s_1 c_2 \\ s_1 s_3 + c_1 s_2 c_3 & s_1 c_3 - c_1 s_2 s_3 & c_1 c_2 \end{pmatrix}.$$

To first order this is $I + [\widetilde{\alpha}]_\times$, where

$$\widetilde{\alpha} = (\alpha_1, -\alpha_2, \alpha_3)$$

and $[a]_\times v = a \times v$. Thus the three generators are $[e_1]_\times, -[e_2]_\times, [e_3]_\times$.

10. Lorentz transformations

On $\mathbb{R}^2$ use the bilinear form

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \eta \mathbf{w},$$

$\eta = \mathrm{diag}(-1, 1)$.

(a) Explain why it is not an inner product.

(b) Show that its preserving matrices satisfy $\Lambda^T \eta \Lambda = \eta$, with determinant ± 1.

(c) Parametrise the component with $\det \Lambda = 1$, $\Lambda_{11} > 0$ by

$$\Lambda(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

and then by $\beta = \tanh\theta$.

(d) Derive rapidity addition and the composition rule for β.

Solution.

The bilinear form is not positive definite: $\langle (1,0), (1,0) \rangle = -1$. Invariance is equivalent to $\Lambda^T \eta \Lambda = \eta$; determinants give $\det \Lambda = \pm 1$. In the determinant-+1, positive-time component, solving the column norm and orthogonality conditions gives

$$\boxed{\Lambda(\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}, \quad \beta = \tanh\xi.}$$

Hyperbolic addition yields

$$\Lambda(\xi_1)\Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2)$$

and

$$\boxed{\beta = (\beta_1 + \beta_2)/(1 + \beta_1\beta_2)}.$$

C1. Computational Gram–Schmidt

Implement Gram–Schmidt for a list of vectors and use it to check Question 2.

Solution.

Use `Orthogonalize[vs]` in Mathematica, or recursively set $u_j = v_j - \sum_{i<j} (e_i \cdot v_j) e_i$ and $e_j = u_j / \sqrt{u_j \cdot u_j}$. Reject a zero residual as a linear dependency. For Q2 the vectors are exactly those displayed in its solution; checking `Simplify[E.Transpose[E]]` gives the identity.

C2. Computational determinant

Check the determinant and singularity condition of Question 3(b) computationally.

Solution.

For `A={{a,1,a},{1,b,-1},{0,-1,a}}`, `Factor[Det[A]]` gives `a(-3+a b)`. Thus the symbolic calculation confirms both exceptional families $a = 0$ and $ab = 3$.

C3. Computational polynomial products

Check the Gaussian-weighted orthogonality and normalisations of Question 6(b),(c) computationally.

Solution.

Evaluate the nine integrals

```
Integrate[Exp[-x^2] hi hj,{x,-Infinity,Infinity}]
```

for $h = (1, 2x, 4x^2 - 2)$. The Gram matrix is $\sqrt{\pi}\,\mathrm{diag}(1, 2, 8)$. Dividing h_i by the positive square roots of these diagonal entries gives unit Gram matrix and precisely the n_i of Q6.

C4. Rotation generators and angular velocity

(a) Check the rotation products and infinitesimal limits of Question 9.

(b) Let the angles depend on time. Compute $W = \dot{R}R^T$, verify antisymmetry, and find the angular-velocity vector defined by $W_{ij} = \epsilon_{ikj}\omega_k$ and its magnitude.

Solution.

Differentiate $RR^T = I$ to obtain $W + W^T = 0$ for $W = \dot{R}R^T$. With the standard right-handed convention of Q9, the angular velocity in space coordinates is

$$\boxed{\omega = (\dot{\alpha}_1, 0, 0) - \dot{\alpha}_2(0, c_1, s_1) + \dot{\alpha}_3(-s_2, -s_1c_2, c_1c_2).}$$

This follows by differentiating the three factors separately. Hence

$$\boxed{|\omega|^2 = \dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \dot{\alpha}_3^2 - 2\dot{\alpha}_1\dot{\alpha}_3 \sin\alpha_2.}$$

Substitution into $W_{ij} = \epsilon_{ikj}\omega_k$ checks every entry.

3.9 Vectors and Matrices: Sheet 4

1. Orthogonal diagonalization

Find eigenvalues and normalised eigenvectors of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Verify orthogonality and construct orthogonal R such that $R^T AR$ is diagonal.

Solution.

Solving $(A - \lambda I)v = 0$ gives eigenpairs

$$\boxed{(-1, (1, -1, 0)/\sqrt{2}), \quad (1, (1, 1, -2)/\sqrt{6}), \quad (4, (1, 1, 1)/\sqrt{3}).}$$

Their dot products vanish. With these unit vectors as columns of R, $R^T R = I$ and $R^T AR = \mathrm{diag}(-1, 1, 4)$.

2. Unitary diagonalization

Find eigenvalues and normalised eigenvectors of

$$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix},$$

using

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\dagger \mathbf{w}.$$

Construct unitary U diagonalising H.

Solution.

$$\det(H - \lambda I) = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11).$$

Corresponding orthonormal vectors are

$$\boxed{u_1 = (-i, 3)^T/\sqrt{10}, \qquad u_2 = (3i, 1)^T/\sqrt{10}.}$$

For $U = (u_1 \; u_2)$, $U^\dagger U = I$ and $U^\dagger H U = \mathrm{diag}(1, 11)$.

3. Principal axes of an ellipse

Write $x^2 + 3y^2 - 2xy = 1$ as $\mathbf{x}^T A \mathbf{x} = 1$. Diagonalise A, show that the curve is an ellipse and find its axis lengths and directions.

Solution.

Here $A = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$ has eigenvalues $\lambda_\pm = 2 \pm \sqrt{2} > 0$. In orthonormal principal-axis coordinates the equation is $\lambda_- X^2 + \lambda_+ Y^2 = 1$. Thus

$$\boxed{\text{full axis lengths} = \frac{2}{\sqrt{2 - \sqrt{2}}}, \quad \frac{2}{\sqrt{2 + \sqrt{2}}}.}$$

The major axis is along $(1, \sqrt{2} - 1)$; the minor axis is perpendicular to it.

4. Diagonalizable and defective matrices

(a) Diagonalise

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

over $\mathbb{C}$.

(b) Explain why

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

cannot be diagonalised.

Solution.

(a) The vectors $(1, -i)/\sqrt{2}$ and $(1, i)/\sqrt{2}$ have eigenvalues $e^{i\phi}$ and $e^{-i\phi}$ respectively, giving a unitary complex diagonalization.

(b) The triangular matrix has eigenvalues $1, 0, 0$; solving $Av = 0$ gives only $v \in \text{span}(1, -1, 0)$. Thus the repeated zero eigenvalue has only one independent eigenvector, so a basis of eigenvectors does not exist.

5. Definiteness

For a real symmetric matrix, define positive/negative definiteness using all nonzero vectors.

(a) Relate definiteness to eigenvalue signs.

(b) For $n = 2$ show $\operatorname{tr} M = \lambda_1 + \lambda_2$, $\det M = \lambda_1\lambda_2$.

(c) Give trace–determinant criteria, distinguishing semidefinite boundary cases.

(d) Classify $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$ and $\begin{pmatrix} -4 & 3 \\ 3 & -5 \end{pmatrix}$.

Solution.

In an orthonormal eigenbasis, $v^T M v = \sum_i \lambda_i c_i^2$, proving the stated eigenvalue criteria for nonzero v. For 2×2 matrices,

positive definite	$\det M > 0,\ \operatorname{tr} M > 0$
negative definite	$\det M > 0,\ \operatorname{tr} M < 0$
indefinite (both signs)	$\det M < 0$

If $\det M = 0$, distinguish positive/negative semidefinite from the zero matrix. The three examples have

$$(\operatorname{tr}, \det) = (3, 1), (0, -13), (-9, 11),$$

so they are positive definite, indefinite and negative definite.

6. Characteristic polynomials

Using $\chi_A(\lambda) = \det(A - \lambda I)$, derive (a) $\lambda^2 - \mathrm{tr}(A)\lambda + \det A$ for 2×2 matrices; (b) $-\lambda^3 + \mathrm{tr}(A)\lambda^2 - \frac{1}{2}[(\mathrm{tr}\,A)^2 - \mathrm{tr}(A^2)]\lambda + \det A$ for 3×3 matrices; (c) the coefficients of λ^n, λ^{n-1} and the constant term for general n.

Solution.

With the convention $\chi_A(\lambda) = \det(A - \lambda I)$, expanding the determinant gives

$$\boxed{\begin{aligned}\chi_2(\lambda) &= \lambda^2 - \mathrm{tr}(A)\lambda + \det A,\\ \chi_3(\lambda) &= -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \tfrac{1}{2}[(\mathrm{tr}\,A)^2 - \mathrm{tr}(A^2)]\lambda + \det A.\end{aligned}}$$

For general n, the leading two coefficients are $(-1)^n$ and $(-1)^{n-1}\,\mathrm{tr}\,A$, and the constant term is $\det A$. The 3×3 linear coefficient is minus the sum of all 2×2 principal minors, equal to the stated trace expression.

7. Orthogonal projectors

A real symmetric matrix satisfies $P^2 = P$.

(a) Find its possible eigenvalues and diagonal form.

(b) Explain the projection interpretation and rank $\mathrm{tr}\,P$.

(c) For a unit vector $\mathbf{n}$, show that $Q_{ij} = n_i n_j$ is a rank-one projector.

(d) Show that $I - Q$ is a projector and find its rank.

Solution.

If $Pv = \lambda v$, then $\lambda^2 = \lambda$, so $\lambda \in \{0, 1\}$. Symmetry allows an orthonormal eigenbasis in which $P = \mathrm{diag}(I_r, 0)$. It leaves its image fixed and kills the perpendicular complement, giving

$$\boxed{\mathrm{rank}\,P = \mathrm{tr}\,P = r}.$$

For $Q = nn^T$, $Q^2 = n(n^T n)n^T = Q$ and $Qv = n(n \cdot v)$, so it projects onto the one-dimensional span of n. Then $I - Q$ projects onto $n^\perp$ of dimension $n - 1$.

8. Three-dimensional rotations

(a) Show that eigenvalues of unitary matrices have modulus one.

(b) Prove that every real 3×3 proper orthogonal matrix has eigenvalue one.

(c) Find its form in an oriented orthonormal basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n})$ containing the rotation axis.

(d) Show $\cos\phi = (\mathrm{tr}\,R - 1)/2$.

(e) For

$$R = \frac{1}{3\sqrt{2}}\begin{pmatrix} 3 & 0 & 3 \\ -1 & -4 & 1 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2}\end{pmatrix},$$

verify that it is a rotation, find its characteristic polynomial, axis and rotation angle.

Solution.

(a) Norm preservation gives $\|Uv\| = |\lambda|\|v\|$, hence $|\lambda| = 1$.

(b) Nonreal eigenvalues of a real matrix come in conjugate pairs, with product one for an orthogonal matrix. In odd dimension there is a real eigenvalue ± 1; determinant $+1$ forces at least one $+1$.

(c) In an oriented orthonormal basis (u_1, u_2, n) the matrix is $\mathrm{diag}(R_2(\phi), 1)$, so

(d)

$$\boxed{\cos\phi = (\operatorname{tr} R - 1)/2}.$$

(e) Direct dot products of the supplied rows give $RR^T = I$, and $\det R = 1$. Put $c = -(10+\sqrt{2})/12$. Then

$$\boxed{\chi_R(\lambda) = (1-\lambda)(\lambda^2 - 2c\lambda + 1), \quad \cos\phi = c, \quad n \parallel (1+\sqrt{2}, -3+2\sqrt{2}, 1).}$$

Divide the last vector by its Euclidean norm to obtain a unit axis.

9. Coupled second-order equations

For $\ddot{\mathbf{x}} = -M\mathbf{x}$ with real symmetric M, explain solution by diagonalisation and the relation between eigenvalue signs and qualitative behaviour. Apply this to $\ddot{x}_1 = -kx_1 - lx_2$, $\ddot{x}_2 = -lx_1 - kx_2$ for all real k, l.

Solution.

Take $M = R\Lambda R^T$ and $q = R^T x$. The equations separate into $\ddot{q}_i = -\lambda_i q_i$. For $\lambda_i > 0$,

$$q_i = A_i \cos\Big(\sqrt{\lambda_i}t\Big) + B_i \sin\Big(\sqrt{\lambda_i}t\Big);$$

for $\lambda_i = 0$, $q_i = A_i + B_i t$; for $\lambda_i < 0$,

$$q_i = A_i e^{\sqrt{-\lambda_i}t} + B_i e^{-\sqrt{-\lambda_i}t}.$$

In (c), $q_\pm = (x_1 \pm x_2)/\sqrt{2}$ have $\boxed{\lambda_\pm = k \pm l}$. Both modes oscillate if $k > |l|$; each negative eigenvalue produces an exponential instability, and equality $k = \pm l$ produces a free linear mode.

C1. Computational eigensystems

Check computationally Questions 1, 2 and 3, and plot the ellipse obtained in Question 3.

Solution.

In Mathematica use `Eigensystem[A]`, then normalize each vector with the Hermitian norm

`Sqrt[Conjugate[v].v]`.

The eigenvalues are $(-1, 1, 4)$, $(1, 11)$ and $(2-\sqrt{2}, 2+\sqrt{2})$ for Q1–Q3. Verify `ConjugateTranspose[U].A.U` is diagonal. Plot Q3 parametrically as

$$x(t) = u_- \cos t/\sqrt{\lambda_-} + u_+ \sin t/\sqrt{\lambda_+},$$

$0 \le t \le 2\pi$.

C2. Trace formulas through fourth order

Verify the 2×2 and 3×3 characteristic-polynomial formulas of Question 6, and derive a trace–determinant expression for 4×4 matrices.

Solution.

Let $p_j = \operatorname{tr}(A^j)$. Newton's identities give $e_1 = p_1$, $e_2 = (p_1^2 - p_2)/2$, $e_3 = (p_1^3 - 3p_1p_2 + 2p_3)/6$, and

$$e_4 = \frac{p_1^4 - 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 - 6p_4}{24}.$$

Thus

$$\boxed{\chi_4(\lambda) = \lambda^4 - p_1\lambda^3 + e_2\lambda^2 - e_3\lambda + e_4}.$$

To check Q6 symbolically, construct a matrix with independent symbolic entries, expand its determinant, and compare its coefficients with these expressions; every difference simplifies to zero.

C3. Initial-value solver by diagonalization

Write code solving the symmetric system of Question 9 by diagonalisation, optionally imposing $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$. Test examples and plot the solutions.

Solution.

Compute $M = R\Lambda R^T$, $a = R^T x_0$, $b = R^T v_0$. Define

$$q_i(t) = \begin{cases} a_i \cos(\omega_i t) + b_i \sin(\omega_i t)/\omega_i, & \lambda_i = \omega_i^2 > 0, \\ a_i + b_i t, & \lambda_i = 0, \\ a_i \cosh(\kappa_i t) + b_i \sinh(\kappa_i t)/\kappa_i, & \lambda_i = -\kappa_i^2 < 0. \end{cases}$$

Return $\boxed{x(t) = Rq(t)}$. For $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $x_0 = (1, 0)$ and $v_0 = 0$, this gives $x_1 = (\cos\sqrt{3}t + \cos t)/2$, $x_2 = (\cos\sqrt{3}t - \cos t)/2$, which can be plotted directly.

4 CP4: Calculus, Normal Modes and Waves

4.1 Calculus: Problems

The original lettered question numbers are retained throughout.

0. Hyperbolic functions

(a) Sketch $\sinh x, \cosh x, \tanh x$.

(b) Verify $\cosh^2 x - \sinh^2 x = 1$, $\cosh^2 x + \sinh^2 x = \cosh 2x$, $2\sinh x \cosh x = \sinh 2x$, $1 - \tanh^2 x = \operatorname{sech}^2 x$ and $\coth^2 x - 1 = \operatorname{csch}^2 x$.

(c) Compare with trigonometric identities.

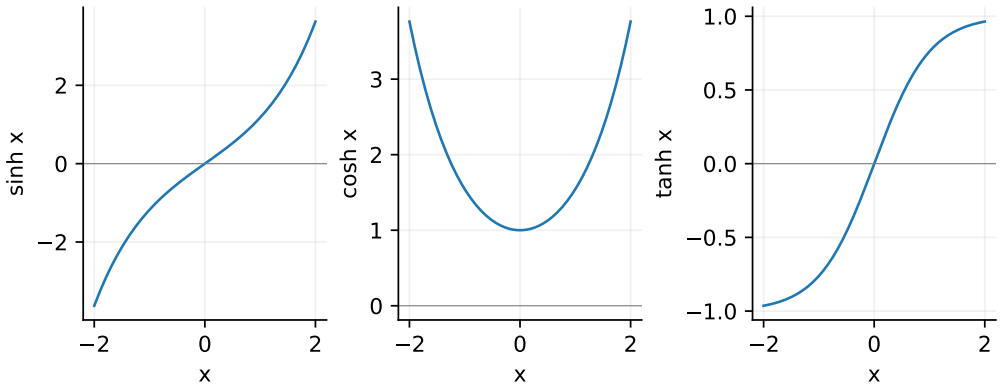
(d) Derive the derivatives of $\sinh x, \cosh x, \tanh x$.

Solution.

Define $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$. Their sum and difference of squares give $\cosh^2 x - \sinh^2 x = 1$, $\cosh^2 x + \sinh^2 x = \cosh 2x$ and $2\sinh x \cosh x = \sinh 2x$. Dividing the first identity by $\cosh^2 x$ or $\sinh^2 x$ gives the other two. The trigonometric analogues have corresponding sign changes, consistent with $\cos(ix) = \cosh x$ and $\sin(ix) = i\sinh x$. Differentiating the exponentials and taking a quotient yields

$$\boxed{\sinh' = \cosh,\ \cosh' = \sinh,\ \tanh' = \operatorname{sech}^2}.$$

The graphs are an increasing odd sinh, an even cosh with minimum 1 at zero, and an increasing odd tanh asymptotic to ± 1.



A. Differentiation

A1. Differentiation practice

Differentiate:

(a) $\sin x\, e^{x^3}$; $e^{x^3 \sin x}$; $\ln[\cosh(1/x)]$.

(b) $\arccos x$; $\operatorname{artanh}[x/(1+x)]$.

(c) $x^{\cos x}$; $\log_{10}(x^2)$.

(d) Implicitly differentiate $ye^y \ln x = x^2 + y^2$. If $t = ax^2 + bx + c$, show that acceleration is proportional to the cube of velocity.

(e) For $y = \sinh\theta, x = \cosh\theta$, find $dy/dx, d^2y/dx^2$. For $y = t^m + t^{-m}$, $x = t + t^{-1}$, prove $(x^2-4)(y')^2 = m^2(y^2-4)$ and $(x^2-4)y'' + xy' - m^2y = 0$.

Solution.

(a) The derivatives are $\boxed{e^{x^3}(\cos x + 3x^2 \sin x)}$,

$$\boxed{e^{x^3 \sin x}(3x^2 \sin x + x^3 \cos x)},$$

and $\boxed{-\tanh(1/x)/x^2}$.

(b) They are $\boxed{-1/\sqrt{1-x^2}}$ and $\boxed{1/(1+2x)}$ on the real inverse-function domains.

(c) Logarithmic differentiation gives

$$\boxed{(x^{\cos x})' = x^{\cos x}[-\sin x \ln x + \cos x/x]}$$

for $x > 0$;

$$\boxed{(\log_{10} x^2)' = 2/(x \ln 10)}$$

for $x \neq 0$. (d)(i) Implicit differentiation gives

$$\boxed{y' = (2x - ye^y/x)/[e^y(1+y)\ln x - 2y]}.$$

(ii) $v = dx/dt = 1/(2ax+b)$, hence

$$\boxed{\dot v = v\, dv/dx = -2av^3}.$$

(e)(i) $dy/dx = \coth\theta$ and

$$\boxed{d^2y/dx^2 = -1/\sinh^3\theta}$$

where $\sinh\theta \neq 0$. (ii) $dy/dx = m(t^{m-1} - t^{-m-1})/(1-t^{-2})$. Also $x^2 - 4 = (t - t^{-1})^2$ and $y^2 - 4 = (t^m - t^{-m})^2$, proving the first identity after squaring. Differentiate it with respect to x and cancel $2y'$ on regular points to obtain

$$\boxed{(x^2-4)y'' + xy' - m^2y = 0};$$

continuity extends it where the parametrization is regular.

A2. First principles

From the difference-quotient definition, derive the derivatives of x^2 and $\sin x$.

Solution.

$[(x+h)^2 - x^2]/h = 2x + h \to \boxed{2x}$. For sine, use

$$\sin(x+h) - \sin x = \sin x(\cos h - 1) + \cos x \sin h.$$

Since $\sin h/h \to 1$ and $(\cos h - 1)/h \to 0$, the derivative is $\boxed{\cos x}$.

A3. Fundamental theorem of calculus

For $I(x) = \int_a^x f(s)\,ds$, give an area-based graphical proof that $I'(x) = f(x)$.

Solution.

The increment $I(x+h) - I(x) = \int_x^{x+h} f(s)ds$ is the signed area of a narrow strip. If f is continuous at x, dividing by its width gives an average value approaching $f(x)$. Thus $\boxed{I'(x) = f(x)}$.

A4. Inverse-function derivatives

For a locally invertible differentiable function, prove

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}, \qquad \frac{d^2x}{dy^2} = -\frac{d^2y/dx^2}{(dy/dx)^3}.$$

Solution.

For a differentiable local inverse with $y' \neq 0$, differentiate $x(y(x)) = x$ to get $(dx/dy)(dy/dx) = 1$. Thus $\boxed{dx/dy = 1/y'}$. Differentiating with respect to y means applying $(1/y')d/dx$, so

$$\boxed{\frac{d^2x}{dy^2} = \frac{1}{y'}\frac{d}{dx}\left(\frac{1}{y'}\right) = -\frac{y''}{(y')^3}.}$$

A5. Changing differential-equation variables

Transform (a) $x^2y'' + (4x + 3x^2)y' + (2 + 6x + 2x^2)y = x$ by $z = x^2y$ into $z'' + 3z' + 2z = x$; (b) $4xy'' + 2(1 - \sqrt{x})y' - 6y = e^{3\sqrt{x}}$ by $t = \sqrt{x}$ into $y_{tt} - y_t - 6y = e^{3t}$.

Solution.

(a) With $z = x^2y$, $y = z/x^2$, $y' = z'/x^2 - 2z/x^3$, $y'' = z''/x^2 - 4z'/x^3 + 6z/x^4$. Substituting cancels all inverse powers and leaves $\boxed{z'' + 3z' + 2z = x}$. For example $z = C_1e^{-x} + C_2e^{-2x} + x/2 - 3/4$, and divide by x^2. (ii) With $t = \sqrt{x}$, $y_x = y_t/(2t)$ and $y_{xx} = y_{tt}/(4t^2) - y_t/(4t^3)$. Substitution gives $\boxed{y_{tt} - y_t - 6y = e^{3t}}$. Its general solution is $y = C_1e^{3t} + C_2e^{-2t} + (t/5)e^{3t}$ for $x > 0$.

A6. Eighth derivative by Leibniz's rule

Use Leibniz's theorem to find the eighth derivative of $x^2 \sin x$.

Solution.

Only derivatives of x^2 of orders zero through two survive:

$$D^8(x^2 \sin x) = x^2 D^8 \sin x + 8(2x)D^7 \sin x + 28(2)D^6 \sin x.$$

Using the four-cycle of sine derivatives gives $\boxed{(x^2 - 56)\sin x - 16x\cos x}$.

A7. Ranges and stationary points

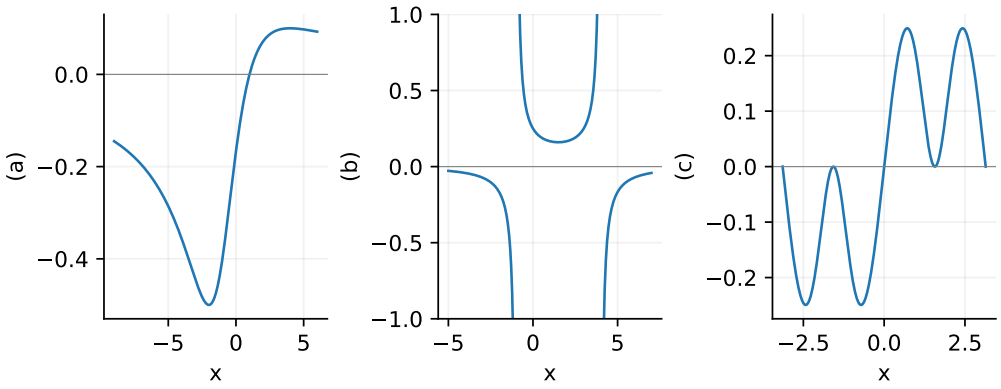
Find stationary points and ranges, and sketch (a) $(x-1)/(x^2+2x+6)$, (b) $1/(4+3x-x^2)$, (c) $8\sin x/(15+8\tan^2 x)$ on their domains.

Solution.

(a) The denominator $(x+1)^2+5$ is positive. Differentiating gives stationary points $x=-2,4$, with values $-1/2, 1/10$; the function tends to zero at infinity. Thus $\boxed{-1/2 \le y \le 1/10}$, with a global minimum and maximum at those points.

(b) Write the denominator as $25/4-(x-3/2)^2$. There are vertical asymptotes at $x=-1,4$; the positive branch has minimum $4/25$ at $x=3/2$, and the exterior branches are negative. Thus $\boxed{y\in(-\infty,0)\cup[4/25,\infty)}$.

(c) Put $u=\sin x$, so $y=8u(1-u^2)/(15-7u^2)$ where $\cos x\neq 0$. Its derivative in u has numerator $15-38u^2+7u^4$, giving $u^2=3/7$ in the domain. The periodic odd graph therefore has range $\boxed{[-8\sqrt{3/7}/21, 8\sqrt{3/7}/21]}$. Points $x=\pi/2+k\pi$ are excluded by the original tan expression, with limiting value zero.



B. Integration

B1. Integration practice

Evaluate the following integrals by the indicated methods.

(a) Inspection: $\int (x+a)(1+2ax+x^2)^{-3/2}\,dx$; $\int_0^{\pi/2}\cos x\, e^{\sin x}\,dx$; $\int_0^{\pi/2}\cos^3 x\,dx$; $\int_{-2}^{2}|x|\,dx$.

(b) Substitution: $\int (3+2x-x^2)^{-1/2}\,dx$; $\int_0^{\pi}(5+3\cos\theta)^{-1}\,d\theta$ using $t=\tan(\theta/2)$.

(c) Partial fractions: $\int dx/[x(1+x^2)]$.

(d) Parts: $\int x\sin x\,dx$; $\int \ln x\,dx$.

(e) Prove $I_n=\frac{1}{2}(n-1)I_{n-2}$ for $I_n=\int_0^{\infty}x^n e^{-x^2}\,dx$, $n\ge 2$, and find I_5.

(f) Trigonometric powers: $\int(\cos^5 x-\cos^3 x)\,dx$; $\int \sin^5 x\cos^4 x\,dx$; $\int \sin^2 x\cos^4 x\,dx$.

(g) Trigonometric substitutions: $\int \sqrt{x^2-9}\,dx/x$; $\int dx/[x^2\sqrt{16-x^2}]$.

Solution.

For every indefinite integral, add an independent constant of integration.

Part	Integral or definite value
$(a)(i)$	$-(1+2ax+x^2)^{-1/2}$
$(a)(ii),(iii),(iv)$	$e-1, \quad 2/3, \quad 4$
$(b)(i),(ii)$	$\sin^{-1}((x-1)/2), \quad \pi/4$
(c)	$\ln\lvert x\rvert - \frac{1}{2}\ln\bigl(1+x^2\bigr)$
$(d)(i),(ii)$	$-x\cos x+\sin x, \quad x\ln x - x$
(e)	$I_n = \frac{1}{2}(n-1)I_{n-2}, \quad I_1 = \frac{1}{2}, \quad I_5 = 1$
$(f)(i)$	$-\sin^3 x/3 + \sin^5 x/5$
$(f)(ii)$	$-\cos^5 x/5 + 2\cos^7 x/7 - \cos^9 x/9$
$(f)(iii)$	$x/16 + \sin 2x/64 - \sin 4x/64 - \sin 6x/192$
$(g)(i)$	$\sqrt{x^2-9} - 3\cos^{-1}(3/\lvert x\rvert), \quad \lvert x\rvert > 3$
$(g)(ii)$	$-\sqrt{16-x^2}/(16x), \quad 0 < \lvert x\rvert < 4.$

In (b)(i), complete the square. In (b)(ii), $u=\tan(\theta/2)$ gives $\int_0^\infty du/(4+u^2) = \pi/4$. For (c), use $1/[x(1+x^2)] = 1/x - x/(1+x^2)$. Integration by parts gives

(d) and the recurrence in (e); the boundary term in (e) vanishes. The sine substitution $u=\sin x$ gives (f)(i), the cosine substitution gives (f)(ii), and power-reduction identities give (f)(iii). Secant and sine substitutions give (g)(i) and (g)(ii), respectively; differentiating the displayed primitives verifies the answers on the stated domains.

B2. Integral symmetry and logarithms

(a) Which of $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$, $\int_{-\pi}^{\pi} x\sin x\,dx$, $\int_{-\pi}^{\pi} x^2\sin x\,dx$ vanish? Explain with integrand sketches.

(b) Prove that the symmetric integral of an integrable odd function vanishes.

(c) From $\ln x = \int_1^x dt/t$, prove $\ln x + \ln y = \ln(xy)$ for positive x, y.

Solution.

(a) Integrands (i) and (iii) are odd and integrate to $\boxed{0}$. In (ii), $x\sin x$ is even and its integral is $\boxed{2\pi}$ by integration by parts.

(b) Under $x=-u$,

$$\int_{-a}^{0} f(x)dx = \int_0^a f(-u)du = -\int_0^a f(u)du,$$

proving cancellation.

(c) For $x, y > 0$, split

$$\int_1^{xy} dt/t = \int_1^x dt/t + \int_x^{xy} dt/t.$$

Substitute $t=xu$ in the second term to obtain $\boxed{\ln(xy) = \ln x + \ln y}$.

B3. Lengths and surfaces of revolution

Find (a) the length of $y = \cosh x$ for $0 \le x \le 1$, (b) the length of $x = \cos t, y = \sin t$ for $0 < t < \pi/2$, and (c) the sphere's area and volume by revolving $y = \sqrt{R^2 - x^2}$ about x. Compare with known geometry.

Solution.

(a) $ds = \sqrt{1 + \sinh^2 x}\, dx = \cosh x\, dx$, so $\boxed{s = \sinh 1}$.

(b) $ds = \sqrt{\sin^2 t + \cos^2 t}\, dt = dt$, so the quarter-circle length is $\boxed{\pi/2}$.

(c) For $y = \sqrt{R^2 - x^2}$, $y\sqrt{1 + y'^2} = R$. Therefore surface area is $\int_{-R}^{R} 2\pi R dx = \boxed{4\pi R^2}$ and volume is

$$\int_{-R}^{R} \pi(R^2 - x^2)dx = \boxed{4\pi R^3/3},$$

the usual sphere formulas.

B4. Line integrals

Evaluate (a) $\int_C (x^2 + 2y)\, dx$ from $(0,1)$ to $(2,3)$ along $y = x + 1$; (b) $\int_C xy\, dx$ from $(0,4)$ to $(4,0)$ on the first-quadrant arc $x^2 + y^2 = 16$; (c) $\int_C (y^2\, dx + xy\, dy + zx\, dz)$ from $(0,0,0)$ to $(1,1,1)$ along the direct segment and along $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$.

Solution.

(a) Substitute $y = x + 1$, $0 \le x \le 2$: $\int_0^2 (x^2 + 2x + 2)dx = \boxed{32/3}$.

(b) On the short circular arc $y = \sqrt{16 - x^2}$, $\int_0^4 x\sqrt{16 - x^2}dx = \boxed{64/3}$. (c)(i) Along (t,t,t) the integrand is $3t^2 dt$, giving $\boxed{1}$. (ii) The first two segments contribute zero and the final segment has $y = z = 1$, giving $\int_0^1 dx = \boxed{1}$. Equality for these two paths alone does not establish path independence; the corresponding field has nonzero curl generally.

C. Series and limits

C1. Series notation

(a) Express $1/4 - 1/8 + 1/16 - \cdots$ as $\sum_{n=1}^{\infty} a_n$ and as $\sum_{n=0}^{\infty} b_n$, and sum it.

(b) Expand the first few terms of $\sum_{n=1}^{\infty} (-1)^n/n$.

(c) Write products and squares of series in summation notation, stating any required convergence conditions.

Solution.

(a) $a_n = (-1)^{n+1}/2^{n+1}$, $n \ge 1$, and $b_n = (-1)^n/2^{n+2}$, $n \ge 0$. The geometric sum is $\boxed{1/6}$.

(b) The first terms are $\boxed{-1 + 1/2 - 1/3 + 1/4 - \cdots}$.

(c) A product is

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j,$$

when the required convergence justifies the operation. Its Cauchy-product form is

$$\sum_{n=2}^{\infty}\sum_{i=1}^{n-1} a_i b_{n-i};$$

take $b = a$ for the square. Absolute convergence suffices to rearrange these terms.

C2. Taylor expansions

(a) Derive the Maclaurin series of e^x, $\sqrt{1+x}$ and $\arctan x$ through x^3.

(b) Use four Taylor terms of sine about $\pi/6$ to estimate $\sin 31^\circ$ and bound the error.

Solution.

(a)

$$\boxed{e^x = 1 + x + x^2/2 + x^3/6 + O(x^4)};$$

$$\boxed{\sqrt{1+x} = 1 + x/2 - x^2/8 + x^3/16 + O(x^4)};$$

$$\boxed{\tan^{-1} x = x - x^3/3 + O(x^5)}.$$

(b) Put $h = \pi/180$. The four terms about $\pi/6$ give

$$\boxed{\sin 31^\circ \simeq \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}h - \tfrac{1}{4}h^2 - \tfrac{\sqrt{3}}{12}h^3 = 0.515038073.}$$

Taylor's theorem bounds the omitted term by $h^4/24 < 3.9 \times 10^{-9}$, so at least the first seven decimal places are secure.

C3. Manipulating series

From sine and cosine series derive $\tan x = x + x^3/3 + 2x^5/15 + \cdots$. Compose the series of e^y and $\ln(1+x)$ to find the first four terms of $e^{\ln(1+x)}$ and comment.

Solution.

(a) Solve

$$(1 - x^2/2 + x^4/24 - \cdots)(x + ax^3 + bx^5) = x - x^3/6 + x^5/120 + \cdots .$$

Coefficient matching gives $a = 1/3$, $b = 2/15$, proving $\boxed{\tan x = x + x^3/3 + 2x^5/15 + O(x^7)}$.

(b) Substitute $u = x - x^2/2 + x^3/3 + \cdots$ into $e^u = 1 + u + u^2/2 + u^3/6 + \cdots$. The result through cubic order is $\boxed{1 + x + 0x^2 + 0x^3}$, as required by the exact inverse relation $e^{\ln(1+x)} = 1 + x$.

C4. Sine integral by a series

Expand $\sin x/x$ and integrate term by term to evaluate $\int_0^1 \sin x\,dx/x$ to four significant figures.

Solution.

$$\sin x/x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+1)!,$$

continuously extended to one at zero. Termwise integration on $[0,1]$ gives

$$\boxed{\int_0^1 \frac{\sin x}{x}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} = 0.946083\ldots \simeq 0.9461.}$$

The alternating remainder is bounded by the first omitted term.

C5. Continuity and differentiability

Sketch and test continuity and differentiability on $[-1,1]$ for (a) $f=0$ at $x\le 0$, $f=x$ at $x>0$; (b) $f=0$ at $x\le 0$, $f=x^2$ at $x>0$; (c) $f=0$ at $x\le 0$, $f=\cos x$ at $x>0$; (d) $f=|x|$.

Solution.

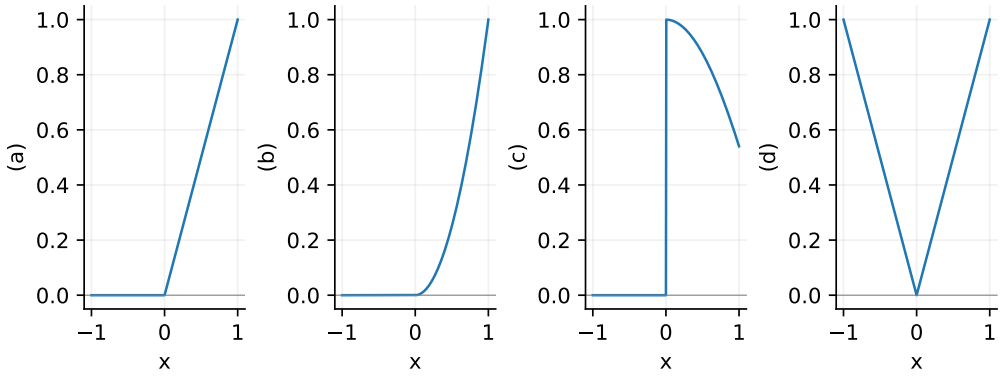
Only zero is potentially problematic.

(a) The two pieces meet at zero but have slopes $0,1$: continuous, not differentiable.

(b) They meet with slopes both zero: continuous and differentiable, with $f'(0)=0$ (but not twice differentiable).

(c) The right limit is one while $f(0)=0$: discontinuous and not differentiable.

(d) $|x|$ is continuous with slopes $-1,+1$, hence not differentiable at zero. The functions are smooth elsewhere in the given interval, with one-sided endpoint derivatives.



C6. Limits and local classification

Evaluate the limits at $x\to 0$ by both series and l'Hôpital's rule: (i) $\sin x/x$; (ii) $(1-\cos^2 x)/x^2$; (iii) $(\sin x - x)/(e^{-x}-1+x)$. Then find their limits as $x\to\infty$. Expand $[\ln(1+x)]^2$ through x^4. Classify the stationary point of $\cos 2x + [\ln(1+x)]^2$ at zero, and determine whether $[\ln(1+x)]^2/[x(1-\cos x)]$ has a finite limit there.

Solution.

At zero, $\sin x = x - x^3/6 + \cdots$, $1-\cos^2 x = x^2 + \cdots$ and $e^{-x}-1+x = x^2/2 + \cdots$. Thus the three limits are $\boxed{1,1,0}$. Applying L'Hopital once in (i), twice in (ii), and twice in (iii) yields the same results. As $x\to\infty$ they are $\boxed{0,0,-1}$, respectively. Also

$$\boxed{\ln^2(1+x) = x^2 - x^3 + 11x^4/12 + O(x^5)}.$$

Therefore

$$\cos 2x + \ln^2(1+x) = 1 - x^2 - x^3 + 19x^4/12 + \cdots$$

has a strict local maximum at zero. Finally $\ln^2(1+x)/[x(1-\cos x)] \sim 2/x$, so there is no finite or two-sided extended-real limit; the one-sided limits are $+\infty, -\infty$.

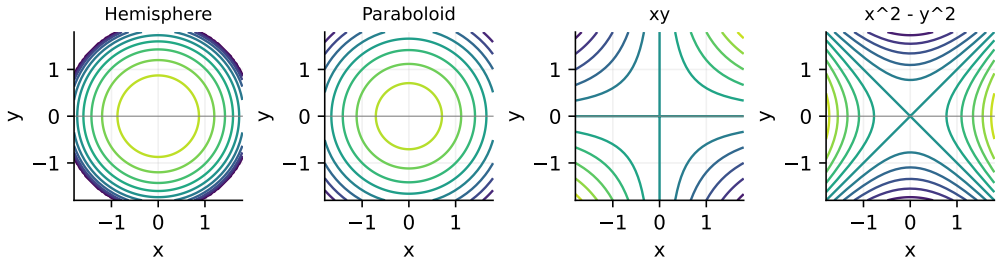
D. Partial differentiation

D1. Surfaces and contours

Sketch the surfaces and their contour maps: $z = \sqrt{4 - x^2 - y^2}$; $z = 1 - 2(x^2 + y^2)$; $z = xy$; $z = x^2 - y^2$.

Solution.

(i) $z = \sqrt{4 - x^2 - y^2}$ is the upper hemisphere of radius 2; a horizontal contour at height $c \in [0, 2]$ is $x^2 + y^2 = 4 - c^2$. (ii) $z = 1 - 2(x^2 + y^2)$ is a downward paraboloid with circular contours $x^2 + y^2 = (1 - c)/2$, $c \leq 1$. (iii) $z = xy$ is a saddle; nonzero contours are rectangular hyperbolae $xy = c$ and the zero contour is the two axes. (iv) $z = x^2 - y^2$ is a saddle rotated 45° relative to (iii); contours are hyperbolae $x^2 - y^2 = c$, degenerating into $y = \pm x$ at zero.



D2. Partial derivatives

(a) Find f_x for $\sqrt{x^2 + y^2}$, $\arctan(y/x)$ and y^x.

(b) Verify equality of mixed derivatives for $(x^2 + y^2)\sin(x + y)$ and $x^m y^n$.

(c) From $f_{xy} = 0$, derive the general form $f = F(x) + G(y)$ on a rectangular domain.

(d) Show that $V = f(x - ct) + g(x + ct)$ satisfies $V_{xx} - c^{-2} V_{tt} = 0$.

Solution.

(a)

$$\boxed{f_x = x/\sqrt{x^2 + y^2}, \quad -y/(x^2 + y^2), \quad y^x \ln y}$$

respectively, on their regular real domains.

(b) The two mixed derivatives both equal

$$\boxed{2(x + y)\cos(x + y) - (x^2 + y^2)\sin(x + y)}$$

in (i), and

$$\boxed{mnx^{m-1}y^{n-1}}$$

in (ii), where the power functions are smooth.

(c) If $f_{xy} = 0$ on a rectangular connected domain, f_x depends only on x, say $F'(x)$; integrating gives

$$\boxed{f = F(x) + G(y)},$$

and $f_y = G'(y)$.

(d) $V_{xx} = f''(x - ct) + g''(x + ct)$ and $V_{tt} = c^2[f'' + g'']$, giving the stated wave equation.

D3. Pendulum error propagation

Using $g = 4\pi^2 l/T^2$ and a linear error estimate, find the worst-case relative error in g when the length and period relative errors are 5% and 2%.

Solution.

Logarithmic differentiation gives $dg/g = dl/l - 2dT/T$. In the worst case the errors reinforce in absolute value, so

$$\boxed{|\Delta g|/g \lesssim 0.05 + 2(0.02) = 0.09 = 9\%}$$

to first order. Independent random errors would instead require a statistical combination, which is not the requested worst-case bound.

D4. Total derivatives

Differentiate in two ways: (a) $u = x^n y^n$, $x = \cos at$, $y = \sin at$, finding du/dt; (b) $u = x^2 y + y^{-1}$, $y = \ln x$, finding du/dx.

Solution.

(a) Direct substitution gives $u = 2^{-n}\sin^n(2at)$, hence

$$\boxed{\dot{u} = an2^{1-n}\sin^{n-1}(2at)\cos(2at)}.$$

Alternatively $\dot{u} = u_x\dot{x} + u_y\dot{y}$ gives the same expression.

(b) Substitute first to get $u = x^2\ln x + 1/\ln x$, so $\boxed{du/dx = 2x\ln x + x - 1/[x(\ln x)^2]}$. The chain rule $u_x + u_y y'$ with $y' = 1/x$ produces the same result, for $x > 0, x \neq 1$.

D5. Polar chain rule

For $w = e^{-x^2-y^2}$, $x = r\cos\theta, y = r\sin\theta$, find w_r, w_θ both directly and by the chain rule.

Solution.

Since $w = e^{-x^2-y^2} = e^{-r^2}$, direct differentiation gives $\boxed{w_r = -2re^{-r^2},\ w_\theta = 0}$. Alternatively use $w_x = -2xw$, $w_y = -2yw$, $x_r = \cos\theta$, $y_r = \sin\theta$, $x_\theta = -r\sin\theta$, $y_\theta = r\cos\theta$; the angular cross terms cancel.

D6. Cyclic partial-derivative identity

Using $PV = RT$ with constant R, verify the cyclic partial-derivative product

$$(\partial P/\partial V)_T(\partial V/\partial T)_P(\partial T/\partial P)_V = -1$$

and the reciprocal-derivative identity. Generalise both to a smooth relation $f(P, V, T) = 0$ where the relevant derivatives exist.

Solution.

For $PV = RT$, $(\partial P/\partial V)_T = -P/V$, $(\partial V/\partial T)_P = R/P$, $(\partial T/\partial P)_V = V/R$, whose product is $\boxed{-1}$. At fixed T, $(\partial V/\partial P)_T = -V/P$ is the reciprocal of the first derivative. For general smooth $f(P, V, T) = 0$, implicit differentiation gives $P_V|_T = -f_V/f_P$, $V_T|_P = -f_T/f_V$, $T_P|_V = -f_P/f_T$, proving the cyclic identity wherever these derivatives exist and denominators are nonzero. Interchanging numerator and denominator proves the reciprocal relation under the same held-variable condition.

D7. Change of variables

For $u = x^2 + y^2$, $v = 2xy$, express (a) z_x in terms of z_u, z_v, (b) z_u in terms of z_x, z_y, and (c) $z_u - z_v$ in terms of z_x, z_y, with the other coordinate held fixed appropriately. Verify (c) for $z = u + v$ and state invertibility conditions.

Solution.

The chain rule gives

$$\boxed{z_x = 2xz_u + 2yz_v,\ \ z_y = 2yz_u + 2xz_v}.$$

Invert this 2×2 system:

$$\boxed{z_u = \frac{xz_x - yz_y}{2(x^2 - y^2)}, \quad z_v = \frac{-yz_x + xz_y}{2(x^2 - y^2)}, \quad z_u - z_v = \frac{z_x - z_y}{2(x - y)}}.$$

The inverse formulas require $x^2 \neq y^2$. For $z = u + v = (x + y)^2$, $z_x = z_y = 2(x + y)$ and $z_u = z_v = 1$, verifying the difference is zero.

D8. Bivariate Taylor expansion

Expand $f(x, y) = e^{xy}$ about $(2, 3)$ through quadratic order (constant, linear and quadratic groups).

Solution.

Let $h = x - 2$, $k = y - 3$. Then $xy = 6 + 3h + 2k + hk$. Expand the exponential through total degree two:

$$\boxed{e^{xy} = e^6[1 + 3h + 2k + \tfrac{9}{2}h^2 + 7hk + 2k^2 + O((|h| + |k|)^3)]}.$$

These are the constant, linear and quadratic Taylor orders.

D9. Stationary points

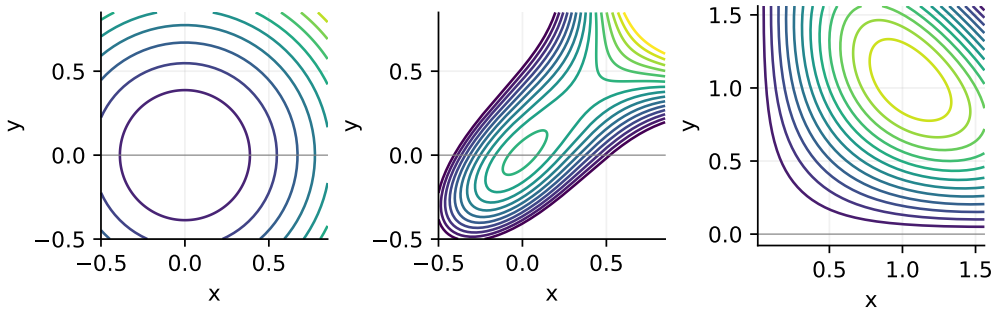
Find and classify stationary points and sketch contours for (i) $x^2 + y^2$; (ii) $x^3 + y^3 - 2(x^2 + y^2) + 3xy$; (iii) $\sin x \sin y \sin(x + y)$ in $0 < x, y < \pi/2$.

Solution.

(i) $\nabla f = (2x, 2y)$ vanishes only at $(0, 0)$; its positive Hessian $2I$ gives a global minimum, with concentric circular contours. (ii) Subtracting the two gradient equations gives $(x - y)[3(x + y) - 7] = 0$. The second branch yields no real stationary pair; the real solutions are $(0, 0)$ and $(1/3, 1/3)$. Their Hessian determinants are 7 and -5, and $f_{xx}(0, 0) = -4$. Thus

$$\boxed{(0, 0) \text{ is a local maximum; } (1/3, 1/3) \text{ is a saddle}}.$$

(iii) $f_x = \sin y \sin(2x + y)$, $f_y = \sin x \sin(x + 2y)$. Inside the specified square both vanish only at $\boxed{x = y = \pi/3}$. The Hessian has eigenvalues $-3\sqrt{3}/2, -\sqrt{3}/2$, so this is a strict maximum, of value $\boxed{3\sqrt{3}/8}$. Its nearby contours are nested ellipses; a saddle has crossing hyperbolic contour structure.



D10. Exact differentials

Decide which forms are exact and find primitives: $x\,dy + y\,dx$; $x\,dy - y\,dx$; $x\,dx + y\,dy + z\,dz$. Evaluate $\oint (x\,dy + y\,dx)$ around $x^4 + y^4 = 1$.

Solution.

(a)(i) $x\,dy + y\,dx = d(xy)$, so $\boxed{f = xy + C}$. (ii) $x\,dy - y\,dx$ is not exact, because

$$\partial(-y)/\partial y = -1 \neq 1 = \partial x/\partial x.$$

(iii) $x\,dx + y\,dy + z\,dz = d[(x^2 + y^2 + z^2)/2]$, so $\boxed{f = (x^2 + y^2 + z^2)/2 + C}$.

(b) The closed integral of $d(xy)$ around $x^4 + y^4 = 1$ is $\boxed{0}$.

E. Additional problems

E1. Relativistic binomial expansion

Expand $E = mc^2(1 - v^2/c^2)^{-1/2}$ through order v^4/c^4 and interpret the terms.

Solution.

With $\beta = v/c$,

$$(1 - \beta^2)^{-1/2} = 1 + \beta^2/2 + 3\beta^4/8 + O(\beta^6).$$

Thus

$$\boxed{E = mc^2 + \tfrac{1}{2}mv^2 + \tfrac{3}{8}mv^4/c^2 + O(mv^6/c^4).}$$

The terms are rest energy, Newtonian kinetic energy and the leading relativistic kinetic correction.

E2. Newton iteration

Derive Newton's iteration $x_{i+1} = x_i - f(x_i)/f'(x_i)$ from local linearisation and illustrate it geometrically.

Solution.

Near x_i, $f(x) \simeq f(x_i) + f'(x_i)(x - x_i)$. The zero of this tangent line is $\boxed{x_{i+1} = x_i - f(x_i)/f'(x_i)}$, provided $f'(x_i) \neq 0$. Graphically, follow the tangent at the current point to its intersection with the x axis. Near a simple root with sufficiently smooth f, the approximation improves quadratically once the iterates are close enough.

E3. Finite-difference derivatives

Use Taylor's theorem to derive the centred first- and second-derivative formulas and their leading errors. Evaluate them for $f(x) = \sin x$, $a = \pi/6$, $h = \pi/180$ and compare with exact derivatives. Construct a higher-order centred approximation for $f'(a)$.

Solution.

Subtract Taylor expansions at $a \pm h$ to get

$$[f(a + h) - f(a - h)]/(2h) = f'(a) + h^2 f'''(a)/6 + O(h^4).$$

Adding them and removing $2f(a)$ gives

$$[f(a + h) - 2f(a) + f(a - h)]/h^2 = f''(a) + h^2 f''''(a)/12 + O(h^4).$$

For the stated sine example these give

$$\boxed{f'_{\text{num}} = 0.8659814 \text{ (exact } 0.8660254), \quad f''_{\text{num}} = -0.4999873 \text{ (exact } -0.5)}.$$

To improve order, cancel the h^2 error by combining step sizes h and $2h$:

$$\boxed{f'(a) \simeq \frac{-f(a + 2h) + 8f(a + h) - 8f(a - h) + f(a - 2h)}{12h},}$$

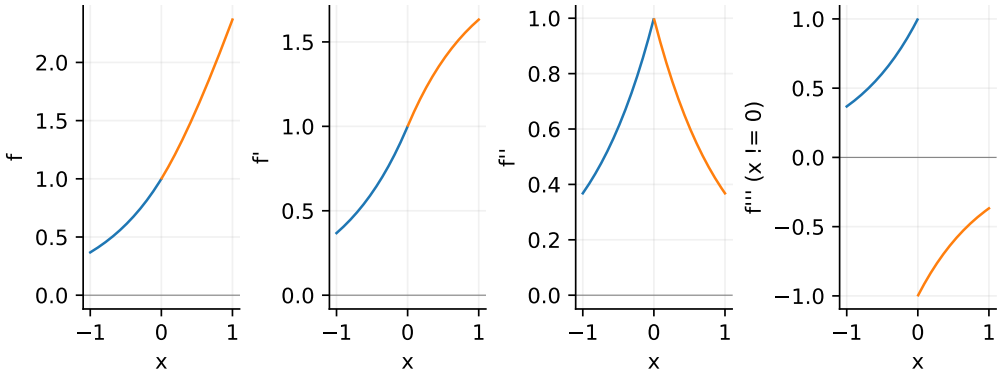
with truncation error $O(h^4)$; excessively small steps magnify floating-point subtraction error.

E4. Third-derivative discontinuity

Sketch $f(x) = e^{-x} + 2x$ for $x \geq 0$ and $f(x) = e^{x}$ for $x < 0$, together with its first three derivatives on either side. Determine derivative existence at zero and show where third-order smoothness fails.

Solution.

For $x < 0$, $f = f' = f'' = f''' = e^{x}$. For $x > 0$, $f = e^{-x} + 2x$, $f' = 2 - e^{-x}$, $f'' = e^{-x}$ and $f''' = -e^{-x}$. At zero the function and its first two derivatives match with values $1, 1, 1$, but the one-sided third derivatives are $\boxed{+1 \text{ and } -1}$. Thus f is C^2 and its third derivative does not exist at zero; the two smooth third-derivative branches exhibit a jump there. These formulas specify the graphs and their matching values.



E5. Additional limits

Find $\lim_{x\to -1} \sin(\pi x)/(1+x)$; $\lim_{x\to\infty} 2x\cos x/(1+x)$; $\lim_{x\to 0}(\sqrt{2+x}-\sqrt{2})/x$; $\lim_{x\to 0}(\sec x - \cos x)/\sin x$; $\lim_{x\to \pi/2}(\sin x)^{\tan x}$.

Solution.

(i) By one application of L'Hopital,

$$\boxed{-\pi}.$$

(ii) $2x/(1+x) \to 2$ but $\cos x$ oscillates, so

$$\boxed{\text{the limit does not exist}}.$$

(iii) Rationalize to get

$$1/[\sqrt{2+x}+\sqrt{2}] \to \boxed{1/(2\sqrt{2})}.$$

(iv)

$$(\sec x - \cos x)/\sin x = \tan x \to \boxed{0}.$$

(v) Write the logarithm as $\tan x \ln(\sin x)$. With $h = x - \pi/2$, this is $(-1/h + O(h))(-h^2/2 + O(h^4)) \to 0$, so the limit is

$$\boxed{1}.$$

E6. Chebyshev differential equation

For $y = \cos(a \arccos x)$, derive $(1-x^2)y'' - xy' + a^2 y = 0$. Differentiate n times to show $y^{(n+2)}(0) = (n^2 - a^2)y^{(n)}(0)$. Use it to obtain the terminating series for $a = 3$ and verify the polynomial in the ODE.

Solution.

Set $x = \cos\theta$, $y = \cos(a\theta)$. The chain rule applied twice gives $\boxed{(1-x^2)y'' - xy' + a^2 y = 0}$. Differentiate n times using Leibniz's rule and put $x = 0$: the terms yield $y^{(n+2)}(0) + [a^2 - n(n-1) - n]y^{(n)}(0) = 0$, hence $\boxed{y^{(n+2)}(0) = (n^2 - a^2)y^{(n)}(0)}$. For $a = 3$, $y(0) = 0, y'(0) = -3$, giving $y'''(0) = 24$ and every higher relevant derivative zero. Therefore $\boxed{y = 4x^3 - 3x}$; substitution in the differential equation verifies it directly.

E7. Radial derivatives

Use $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

(a) Find $\partial x/\partial r$ at fixed angles and $\partial r/\partial x$ at fixed y, z.

(b) For radial $f(r)$, derive $f_x = (x/r)f'$ and $f_{xx} = f'/r + (x^2/r)\, d(f'/r)/dr$, hence $\nabla^2 f = r^{-2} d(r^2 f')/dr$.

Solution.

(a)

$$\boxed{(\partial x/\partial r)_{\theta,\phi} = \sin\theta\cos\phi = x/r}$$

and

$$\boxed{(\partial r/\partial x)_{y,z} = x/r}.$$

They are not reciprocal partial derivatives because different variables are held fixed.

(b) For a radial f, $f_x = f'(r)x/r$. Differentiating again gives $\boxed{f_{xx} = f'/r + (x^2/r)d(f'/r)/dr}$. Summing cyclically and using $x^2 + y^2 + z^2 = r^2$ yields

$$\boxed{\nabla^2 f = f'' + 2f'/r = \frac{1}{r^2}\frac{d}{dr}(r^2 f').}$$

The formula is for $r > 0$; the origin is handled by a regular limit when f is smooth there.

4.2 Multiple Integrals and Vector Calculus: Set 1

1.1. Scalars and vectors

Which are best represented as vectors: temperature, magnetic field, acceleration, force, molecular weight, area, and angle of polarisation? Explain any geometric qualifications.

Solution.

Temperature and molecular weight are scalars. Magnetic field, acceleration and force require magnitude and direction and are vectors (magnetic field is specifically an axial vector). Area is a scalar magnitude unless an oriented surface is meant, when $\mathbf{A} = A\widehat{n}$ is a vector area. A polarization angle is an angular coordinate modulo π, not itself a three-dimensional vector; a polarization direction may be represented by a suitable direction vector or director.

1.2. Reversing double integrals

Sketch each region, evaluate the integral, reverse the order and evaluate again:

$$\text{(a)}\ \int_0^{\sqrt{2}} dy \int_{y^2}^{2} y\,dx; \quad \text{(b)}\ \int_0^{4} dx \int_0^{\sqrt{x}} y\sqrt{x}\,dy; \quad \text{(c)}\ \int_0^{1} dy \int_{-y}^{y^2} x\,dx.$$

Solution.

(a) The region is $0 \le y \le \sqrt{2}$, $y^2 \le x \le 2$. Direct integration gives $\int_0^{\sqrt{2}} y(2-y^2)dy = \boxed{1}$. Reversing gives $\int_0^2 dx \int_0^{\sqrt{x}} y\,dy = 1$.

(b) The region is $0 \le x \le 4$, $0 \le y \le \sqrt{x}$. Thus

$$\int_0^4 dx \int_0^{\sqrt{x}} y\sqrt{x}\,dy = \boxed{32/5}.$$

Reversal gives $\int_0^2 dy \int_{y^2}^4 y\sqrt{x}\,dx = 32/5$.

(c) Direct evaluation is $\int_0^1 [y^4 - y^2]dy/2 = \boxed{-1/15}$. Reversing requires a split at $x = 0$:

$$\int_{-1}^0 dx \int_{-x}^1 x\,dy + \int_0^1 dx \int_{\sqrt{x}}^1 x\,dy = -1/15.$$

These inequalities specify the respective parabolic regions for sketches.

1.3. Removing a sine-integral denominator

Reverse the integration order and evaluate $\int_0^\pi dy \int_y^\pi (\sin x/x)\,dx$.

Solution.

The region is $0 \le y \le x \le \pi$. Reversing gives

$$\boxed{\int_0^\pi dx \int_0^x \frac{\sin x}{x} dy = \int_0^\pi \sin x\,dx = 2.}$$

The integrand's limit at zero is finite, so there is no singularity obstruction.

1.4. Gaussian integral

Evaluate $\int_0^\infty e^{-\alpha x^2}\,dx$ for $\alpha > 0$ using a double integral or otherwise.

Solution.

For $\alpha > 0$, let $I = \int_0^\infty e^{-\alpha x^2}dx$. Squaring gives a first-quadrant double integral. Polar coordinates yield

$$I^2 = \int_0^{\pi/2} d\theta \int_0^\infty re^{-\alpha r^2}dr = \pi/(4\alpha).$$

Since $I > 0$,

$$\boxed{I = \frac{1}{2}\sqrt{\pi/\alpha}.}$$

1.5. Tangent plane

Find the tangent plane to $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

Solution.

For $F = 2xz^2 - 3xy - 4x - 7$, the normal is $\nabla F = (2z^2 - 3y - 4, -3x, 4xz)$. At the stated point it is $(7, -3, 8) \neq 0$. Thus

$$\boxed{7(x-1) - 3(y+1) + 8(z-2) = 0, \quad 7x - 3y + 8z = 26.}$$

1.6. Uniform semicircular lamina

A uniform semicircular lamina of radius a in the xy plane occupies $x \geq 0$ and has areal density k. Find its mass, centre of mass and moments of inertia about x, y, using polar coordinates.

Solution.

Use $x = r\cos\theta, y = r\sin\theta$, $0 \leq r \leq a$, $-\pi/2 \leq \theta \leq \pi/2$, with $dm = kr\,drd\theta$. Then

$$\boxed{M = \pi ka^2/2, \quad \bar{x} = 4a/(3\pi), \quad \bar{y} = 0, \quad I_x = I_y = \pi ka^4/8 = Ma^2/4.}$$

For example

$$\int x\,dm = k(a^3/3)\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = 2ka^3/3;$$

the squared-coordinate moments follow by integrating $\sin^2\theta$ and $\cos^2\theta$.

1.7. Half-Gaussian lamina

A planar sheet has areal density $k\exp\left[-(x^2+y^2)/a^2\right]$ for $x \geq 0$ and zero otherwise. Find its mass, centre of mass and moments about x, y, and compare with Question 6.

Solution.

The same angular interval applies but $r \in [0, \infty)$ and $dm = ke^{-r^2/a^2}r\,drd\theta$. Gaussian moments give

$$\boxed{M = \pi ka^2/2, \quad \bar{x} = a/\sqrt{\pi}, \quad \bar{y} = 0, \quad I_x = I_y = \pi ka^4/4 = Ma^2/2.}$$

The mass matches the semicircle's for the same k, a, but the Gaussian's unbounded tail moves the CM further from the straight edge and doubles both origin-axis moments.

1.8. Polar integral with an angle factor

Evaluate $\int_0^a dy \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\arctan(y/x)\,dx$.

Solution.

The region is the quarter-disk $0 \le r \le a$, $0 \le \theta \le \pi/2$, and $\arctan(y/x) = \theta$ there. Therefore

$$\boxed{\int_0^{\pi/2} \theta\, d\theta \int_0^a r^3 dr = \frac{\pi^2 a^4}{32}.}$$

4.3 Multiple Integrals and Vector Calculus: Set 2

2.1. Radial gradients

For $\mathbf{r} = (x, y, z)$ find $\nabla\phi$ when (a) $\phi = \ln|\mathbf{r}|$ and (b) $\phi = 1/|\mathbf{r}|$.

Solution.

Since $\nabla r = \mathbf{r}/r$ for $r > 0$,

$$\boxed{\nabla \ln r = \mathbf{r}/r^2, \qquad \nabla(1/r) = -\mathbf{r}/r^3.}$$

2.2. Gradient sum and product

For $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find $\nabla(F + G)$ and $\nabla(FG)$ at $(1, 0, -2)$.

Solution.

At $(1, 0, -2)$, $F = -1$, $G = 0$, $\nabla F = (-4, 1, 1)$ and $\nabla G = (0, 8, 0)$. Linearity and the product rule give

$$\boxed{\nabla(F + G) = (-4, 9, 1), \qquad \nabla(FG) = F\nabla G + G\nabla F = (0, -8, 0).}$$

2.3. Inverse Jacobians

Let $A = \partial(x, y)/\partial(u, v)$ and $B = \partial(u, v)/\partial(x, y)$ denote the corresponding derivative matrices.

(a) Use the chain rule to prove $AB = I$.

(b) Verify it for Cartesian and polar coordinates.

(c) Deduce the reciprocal relation of their Jacobian determinants, wherever the transformation is invertible.

Solution.

The chain rule gives

$$(AB)_{ij} = \partial x_i/\partial x_j = \delta_{ij},$$

so $B = A^{-1}$ wherever the local coordinate transformation is invertible. For polar coordinates,

$$A = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix}.$$

Multiplication gives I and their determinants are r and $1/r$. In general

$$\boxed{\partial(u, v)/\partial(x, y) = [\partial(x, y)/\partial(u, v)]^{-1}},$$

with the right side the reciprocal determinant, not entrywise reciprocal matrix.

2.4. Non-Cartesian substitutions

(a) With $u = x + y$, $v = x - y$, evaluate $\iint (x^2 + y^2)\,dx\,dy$ over the region bounded by $y = x$, $y = x + 2$, $y = -x$, $y = -x + 2$.

(b) With $u = xy$, $v = y/x$, show that $\partial(u, v)/\partial(x, y) = 2y/x$. Evaluate $\iint e^{-xy}\,dx\,dy$ over $x, y > 0$, $xy < 1$, $1/2 < y/x < 2$.

Solution.

(a) $x = (u + v)/2$, $y = (u - v)/2$, $|\partial(x, y)/\partial(u, v)| = 1/2$, $x^2 + y^2 = (u^2 + v^2)/2$. The domain becomes $0 \le u \le 2, -2 \le v \le 0$. Thus the integral is

$$\int_0^2 \int_{-2}^0 (u^2 + v^2)\,dv du/4 = \boxed{8/3}.$$

(b) The forward determinant is $(y)(1/x)-(x)(-y/x^2) = \boxed{2y/x = 2v}$. The region becomes $0 < u < 1, 1/2 < v < 2$. Thus

$$\boxed{\int_0^1 e^{-u}du \int_{1/2}^2 \frac{dv}{2v} = (1 - e^{-1}) \ln 2.}$$

2.5. Conservative line integral

For $\mathbf{A} = (4x - y^4, -4xy^3 - 3y^2, 4)$, evaluate $\int \mathbf{A} \cdot d\mathbf{l}$ from $(0,0,0)$ to $(1,2,0)$ along (a) the direct segment and (b) the successive segments through $(0,0,1), (1,0,1), (1,2,1)$. Prove that $\mathbf{A}$ is conservative and find V with $\mathbf{A} = \nabla V$.

Solution.

Integrating the first component suggests $V = 2x^2 - xy^4 + h(y,z)$. Matching the other components yields $h = -y^3 + 4z$, so

$$\boxed{V = 2x^2 - xy^4 - y^3 + 4z + C, \quad A = \nabla V.}$$

Both paths therefore give $V(1,2,0) - V(0,0,0) = \boxed{-22}$. Directly, the broken path contributes $4+2-24-4 = -22$; along the straight line $(t, 2t, 0)$ the integrand is $4t - 24t^2 - 80t^4$, also integrating to -22.

2.6. Nonconservative line integrals

For $\mathbf{A} = (3x^2 + 6y, -14yz, 20xz^2)$, integrate from the origin to $(1,1,1)$ along (a) the direct segment, (b) successive segments through $(1,0,0), (1,1,0)$, and (c) $\mathbf{r}(t) = (t, t^2, t^3)$, $0 \le t \le 1$. Is the field conservative?

Solution.

(a) Along (t,t,t), $A \cdot dr = (6t - 11t^2 + 20t^3)dt$, giving $\boxed{13/3}$.

(b) The three coordinate segments contribute $1, 0, 20/3$, giving $\boxed{23/3}$.

(c) Along (t, t^2, t^3) the integrand is $9t^2 - 28t^6 + 60t^9$, giving $\boxed{5}$. The distinct results demonstrate nonconservativity; explicitly $(\nabla \times A)_z = -6 \neq 0$.

2.7. Hemisphere area

Find the curved area of a hemisphere of radius R (a) by integrating $R^2 \sin\theta \, d\theta \, d\phi$ and (b) by projection onto the xy plane.

Solution.

(a)

$$A = \int_0^{2\pi} d\phi \int_0^{\pi/2} R^2 \sin\theta \, d\theta = \boxed{2\pi R^2}.$$

(b) For $z = \sqrt{R^2 - x^2 - y^2}$, $dS = R\,dxdy/\sqrt{R^2 - x^2 - y^2}$. Projecting onto the disk and using polar coordinates gives

$$2\pi R \int_0^R r\,dr/\sqrt{R^2 - r^2} = \boxed{2\pi R^2}.$$

The rim singularity is integrable.

2.8. Area of a tilted disk

Find the area of the part of $x - 2y + 5z = 13$ inside the cylinder $x^2 + y^2 = 9$.

Solution.

Solve $z = (13 - x + 2y)/5$. Then

$$dS = \sqrt{1 + z_x^2 + z_y^2}\, dxdy = (\sqrt{30}/5)dxdy.$$

The projected region is $x^2 + y^2 \leq 9$, giving

$$\boxed{A = 9\pi\sqrt{30}/5.}$$

2.9. First-octant triangular lamina

Find by integration the area and centre of mass of the uniform first-octant triangular lamina $x + y + z = 1$. Check its area geometrically.

Solution.

Write $z = 1 - x - y$ over $x \geq 0, y \geq 0, x + y \leq 1$, with $dS = \sqrt{3}\, dxdy$. Then $A = \sqrt{3}\int_0^1 (1 - x)dx = \boxed{\sqrt{3}/2}$. Integrating x, y, z with this area measure, or using permutation symmetry, gives

$$\boxed{(\bar{x}, \bar{y}, \bar{z}) = (1/3, 1/3, 1/3)}.$$

The vertices lie on the three unit coordinate axes; the triangle is equilateral with side $\sqrt{2}$, confirming the area geometrically.

4.4 Multiple Integrals and Vector Calculus: Set 3

3.1. Volume Jacobians

Derive the coordinate Jacobians

$$\partial(x,y,z)/\partial(r,\theta,\phi) = r^2\sin\theta$$

for spherical coordinates and $\partial(x,y,z)/\partial(r,\phi,z) = r$ for cylindrical coordinates.

Solution.

Insert $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ in the determinant of their derivative matrix. Factoring the spherical scale factors gives

$$\boxed{\partial(x,y,z)/\partial(r,\theta,\phi) = r^2\sin\theta}.$$

For cylindrical $x = r\cos\theta, y = r\sin\theta, z = z$, the determinant is $\boxed{r}$. Thus the standard positively oriented volume elements are $r^2\sin\theta\,dr d\theta d\phi$ and $r\,dr d\theta dz$.

3.2. Cone solid angle

Find the solid angle of a cone of half-angle θ.

Solution.

On the unit sphere, integrate the area element over the cone cap:

$$\boxed{\Omega = \int_0^{2\pi} d\phi \int_0^{\alpha} \sin\theta\,d\theta = 2\pi(1-\cos\alpha).}$$

3.3. Solid hemisphere moments

A solid hemisphere of uniform density k occupies $x^2+y^2+z^2 \le a^2$, $z \ge 0$. Find its mass, centre of mass, three axial moments and three products of inertia, using symmetry.

Solution.

Uniform density gives $\boxed{M = 2\pi k a^3/3}$. Symmetry sets $\bar{x} = \bar{y} = 0$, and $\int z\,dm = \pi k a^4/4$, so $\boxed{\bar{z} = 3a/8}$. Each coordinate-squared integral is half its full-sphere value, giving $\boxed{I_{xx} = I_{yy} = I_{zz} = 2Ma^2/5}$ about the origin axes. All three products of inertia vanish by reflection symmetry in x or y: $\boxed{I_{xy} = I_{yz} = I_{zx} = 0}$.

3.4. Flux through a nonplanar loop

Air flows at $0.4\,\mathrm{m\,s^{-1}}$ in direction $(-1,-1,1)$. Find the volume flow per second through the loop joining $(1,1,0),(1,0,0),(0,0,0),(0,1,1),(1,1,1),(1,1,0)$, using the oriented vector area.

Solution.

For uniform flow, the oriented flux is $v\cdot S$ where $S = \frac{1}{2}\sum r_i \times r_{i+1}$ is the vector area of any spanning surface. The listed vertices give $S = (-1/2, 1, -1)$ in square metres. With $v = 0.4(-1,-1,1)/\sqrt{3}$,

$$\boxed{Q_{\text{signed}} = -\frac{0.6}{\sqrt{3}}\,\mathrm{m^3\,s^{-1}}, \qquad |Q| = 0.3464\,\mathrm{m^3\,s^{-1}}.}$$

The negative sign is relative to the orientation specified by the vertex order.

3.5. Flux of the position vector

Evaluate the outward flux $\iint_S \mathbf{r} \cdot \hat{\mathbf{n}}\, dS$ for (a) the unit cube in the first octant and (b) the sphere $r = a$.

Solution.

$\nabla \cdot r = 3$, so the divergence theorem gives $\oint r \cdot \widehat{n}\, dS = 3\mathcal{V}$. For the unit cube $\mathcal{V} = 1$, giving $\boxed{3}$. For a sphere of radius a, it gives $\boxed{4\pi a^3}$, also immediate from $r \cdot \widehat{n} = a$ and area $4\pi a^2$.

3.6. Three surface fluxes

Find the surface flux in each case, stating the normal orientation:

(a) $\mathbf{A} = (y, 2x, -z)$ through the first-octant part of $2x + y = 6$, $0 \le z \le 4$.

(b) $\mathbf{A} = (x + y^2, -2x, 2yz)$ through the first-octant triangle $2x + y + 2z = 6$.

(c) $\mathbf{A} = (6z, 2x + y, -x)$ out of the region inside $x^2 + z^2 = 9$, with $x, z \ge 0$ and $0 \le y \le 8$.

Solution.

Take normals outward from the corresponding origin-side regions.

(a) Parameterize the plane by $(x, 6 - 2x, z)$, $0 \le x \le 3$, $0 \le z \le 4$, with vector area $(2, 1, 0)dxdz$. The integrand is $12 - 2x$, giving $\boxed{108}$.

(b) Write $z = 3 - x - y/2$, with vector area $(1, 1/2, 1)dxdy$ and $0 \le y \le 6 - 2x$, $0 \le x \le 3$. The integrand is $2y(3 - x)$, giving $\boxed{81}$.

(c) The divergence is one, and the volume is the first-quadrant quarter-cylinder of radius 3 and length 8, so the closed outward flux is $\boxed{18\pi}$. Reversing a chosen open-surface normal reverses its flux sign.

3.7. Triangular prism flux

For $\mathbf{A} = (xy^2, x^2, yz)$, find the outward flux through every face of the prism bounded by $x = 0, y = 0, z = 0, x + y = 1, z = 1$. Sum them and verify the divergence theorem by a volume integral.

Solution.

The outward fluxes through the faces $x = 0, y = 0, z = 0, z = 1, x + y = 1$, respectively, are

$$\boxed{0,\ -1/3,\ 0,\ 1/6,\ 5/12.}$$

For the last face use $(x, 1 - x, z)$ with vector area $(1, 1, 0)dxdz$; its integrand is $x(1 - x)^2 + x^2$. The total is $\boxed{1/4}$. Independently $\nabla \cdot A = y^2 + y$, and

$$\int_0^1 dz \int_0^1 dx \int_0^{1-x} (y^2 + y)dy = \boxed{1/4},$$

verifying Gauss's theorem.

3.8. Product and divergence rules

For $\mathbf{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, find $\nabla \cdot \mathbf{A}$, $\mathbf{A} \cdot \nabla\phi$, $\nabla \cdot (\phi\mathbf{A})$ and $\nabla \cdot \nabla\phi$.

Solution.

Direct differentiation gives

$$\boxed{D := \nabla \cdot A = 3yz^2 + 6xy^2 - x^2y, \quad A \cdot \nabla\phi = 18x^2yz^2 - 2xy^3z + x^2y^2z.}$$

Since $\nabla\phi = (6x, -z, -y)$, the product rule gives

$$\boxed{\nabla \cdot (\phi A) = (3x^2 - yz)D + 18x^2yz^2 - 2xy^3z + x^2y^2z}.$$

Finally $\boxed{\nabla \cdot (\nabla\phi) = 6}$.

3.9. Solenoidal wire field

The field of a straight wire is azimuthal with magnitude $B = \mu_0 I/(2\pi r)$. Evaluate $\nabla \cdot \mathbf{B}$, addressing the axis appropriately.

Solution.

With wire along z, $B = \mu_0 I(-y, x, 0)/[2\pi(x^2 + y^2)]$. Thus $\partial_x B_x = \mu_0 Ixy/[\pi(x^2 + y^2)^2]$ and $\partial_y B_y$ is its negative. Hence $\boxed{\nabla \cdot B = 0}$ off the wire; there is no magnetic flux source on the wire either. Its singularity concerns curl and current, rather than divergence.

4.5 Multiple Integrals and Vector Calculus: Set 4

4.1. Vector areas and projections

Let $O = 0$, $A = (1,0,0)$, $B = (1,1,1)$ and $C = (0,2,0)$. Find the vector area of $OABCO$ (a) from coordinate-plane projections and (b) by triangulation. Find its maximum projected area and its projection along $(0,-1,1)$. Repeat for $OACBO$.

Solution.

For OABCO, triangulate into OAB and OBC:

$$\boxed{S = (a \times b + b \times c)/2 = (-1, -1/2, 3/2).}$$

The same components are the signed projected areas on the yz, zx, xy planes. The maximum projected area is $|S| = \boxed{\sqrt{14}/2}$, viewed along S; along $n = (0,-1,1)/\sqrt{2}$ it is $|n \cdot S| = \boxed{\sqrt{2}}$. For OACBO,

$$S' = (a \times c + c \times b)/2 = \boxed{(1,0,0)},$$

giving maximum area $\boxed{1}$ and the specified-direction projection $\boxed{0}$.

4.2. Sources and circulation

Sketch $\mathbf{A} = (x, y, 0)$ and $\mathbf{B} = (y, -x, 0)$. Calculate and interpret their divergences and curls.

Solution.

For $A = (x, y, 0)$,

$$\boxed{\nabla \cdot A = 2,\ \nabla \times A = 0}:$$

the planar field points radially outwards with a source-like divergence and no local circulation. For $B = (y, -x, 0)$,

$$\boxed{\nabla \cdot B = 0,\ \nabla \times B = (0,0,-2)}:$$

its arrows circle clockwise, with no local sources but a uniform negative-z vorticity.

4.3. Stokes on a hemisphere

Verify Stokes's theorem for $\mathbf{A} = (y, -x, z)$ on the outward-oriented hemisphere $r = 1$, $z \geq 0$, using the induced boundary orientation.

Solution.

$\nabla \times A = (0,0,-2)$. With outward hemisphere normal, $\int_S (\nabla \times A) \cdot dS = -2$ times the unit-disk projected area, giving $\boxed{-2\pi}$. The positively oriented rim is $(\cos t, \sin t, 0)$, $0 \leq t \leq 2\pi$, and $A \cdot dr = -dt$. Its circulation is also $\boxed{-2\pi}$.

4.4. Loops on an offset cylinder

For $\mathbf{A} = (y, -x, 0)$, find $\oint \mathbf{A} \cdot d\mathbf{l}$ for a general closed loop on $(x-3)^2 + y^2 = 2$. State its dependence on winding and orientation.

Solution.

The integrand $y\,dx - x\,dy$ depends only on the loop's projection onto the xy circle, of radius $\sqrt{2}$ centred at $(3,0)$. For a single positive winding, parameterize $x = 3 + \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$; the integral is $\int_0^{2\pi} (-2 - 3\sqrt{2}\cos t)dt = -4\pi$. In general

$$\boxed{\oint A \cdot d\ell = -4\pi N,}$$

where N is the signed winding number around the cylinder. A contractible loop has $N = 0$; merely saying "any closed loop" does not determine one common numerical value.

4.5. Rigidly rotating water

Water in a rotating bucket moves as a rigid body at angular velocity $\boldsymbol{\omega}$ about the vertical. Find and interpret $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$. Can $\mathbf{v}$ be a gradient field?

Solution.

$$v = \omega \widehat{z} \times r = (-\omega y, \omega x, 0).$$

Thus $\boxed{\nabla \cdot v = 0}$, consistent with incompressible steady rotation, and $\boxed{\nabla \times v = 2\omega \widehat{z}}$. A nonzero curl prevents representation as the gradient of a smooth scalar velocity potential on an open region of the fluid.

4.6. Component proofs

By writing out components, without index notation, prove $\nabla \times \nabla\phi = 0$ and $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for sufficiently smooth fields.

Solution.

For smooth ϕ, the x component of $\nabla \times \nabla\phi$ is

$$\partial_y \partial_z \phi - \partial_z \partial_y \phi = 0,$$

and the other two cancel similarly. For A, expand

$$\nabla \cdot (\nabla \times A) = \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) + \partial_z(\partial_x A_y - \partial_y A_x) = \boxed{0}.$$

Each mixed derivative appears once with each sign, proving both identities when the mixed derivatives commute.

4.7. Vector line integrals

On $\mathbf{r}(t) = (t^2, 2t, t^3)$, $0 \le t \le 1$, evaluate (a) $\int \nabla\phi \cdot d\mathbf{r}$ and (b) $\int (\nabla \times \mathbf{F}) \cdot d\mathbf{r}$, for $\phi = 2xyz^2$ and $\mathbf{F} = (xy, -z, x^2)$.

Solution.

Along $r = (t^2, 2t, t^3)$, $dr = (2t, 2, 3t^2)dt$, $F = (2t^3, -t^3, t^4)$ and $\phi = 4t^9$. Thus

$$\boxed{\int_\gamma F \times dr = (-9/10, -2/3, 7/5), \qquad \int_\gamma \phi \, dr = (8/11, 4/5, 1).}$$

For example $F \times r' = (-3t^5 - 2t^4, -4t^5, 4t^3 + 2t^4)$; integrate each component from zero to one.

4.8. Index notation

Use summation notation, with constant vectors $\mathbf{a}, \mathbf{b}$.

(a) Interpret $a_i b_j c_i$, $a_i b_j c_j d_i$, $\delta_{ij} a_i a_j$, $\delta_{ij}\delta_{ij}$, $\epsilon_{ijk} a_i b_k$, $\epsilon_{ijk}\delta_{ij}$.

(b) Find gradients of r^n and $\mathbf{a} \cdot \mathbf{r}$.

(c) Find divergences and curls of $\mathbf{r}$, $r^n\mathbf{r}$, $(\mathbf{a} \cdot \mathbf{r})\mathbf{b}$, $\mathbf{a} \times \mathbf{r}$.

(d) Prove the two identities of Question 6 by index notation.

Solution.

(a) The expressions are $(a \cdot c)b_j$, $(a \cdot d)(b \cdot c)$, $|a|^2$, 3, $(b \times a)_j$ and the zero vector (free index k), respectively.

(b)

$$\boxed{\nabla r^n = n r^{n-2} r, \quad \nabla(a \cdot r) = a}.$$

(c) The divergences and curls are respectively

F	$\nabla \cdot F$	$\nabla \times F$
r	3	0
$r^n r$	$(n+3)r^n$	0
$(a \cdot r)b$	$a \cdot b$	$a \times b$
$a \times r$	0	$2a$

where radial powers are evaluated for $r > 0$ unless regularly extendible.

(d)

$$[\nabla \times \nabla\phi]_i = \epsilon_{ijk}\partial_j\partial_k\phi = 0$$

because a symmetric derivative tensor contracts with an antisymmetric tensor. Similarly $\partial_i\epsilon_{ijk}\partial_j A_k = 0$.

4.9. Exponential distribution

A random variable has density $f(x) = ce^{-x}$ for $x \geq 0$, zero otherwise. Find (a) c, (b) its CDF, (c) its mean and median, and (d) $P(1 < X < 3)$.

Solution.

Normalization gives $c\int_0^\infty e^{-x}dx = c = 1$. Hence

$$\boxed{F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0, \end{cases} \quad E[X] = 1, \quad \text{median } X = \ln 2, \quad P(1 < X < 3) = e^{-1} - e^{-3}.}$$

The mean follows by integration by parts and the median by solving $F(m) = 1/2$.

4.10. Triangular joint density

The joint density is $f(x, y) = cx^2y$ on $0 \leq y \leq x \leq 1$, zero elsewhere. Sketch its support, normalise it, find both marginal densities and calculate $P(Y \leq X/2)$.

Solution.

The support is the triangle with vertices $(0,0), (1,0), (1,1)$. Its normalization is $c\int_0^1 x^2 \int_0^x y\,dy\,dx = c/10 = 1$, so $\boxed{c = 10}$. Integrating over the other variable gives

$$\boxed{f_X(x) = 5x^4 \ (0 \leq x \leq 1), \quad f_Y(y) = \frac{10}{3}y(1 - y^3) \ (0 \leq y \leq 1),}$$

and zero outside these ranges. Finally

$$\boxed{P(Y \leq X/2) = \int_0^1 \int_0^{x/2} 10x^2y\,dydx = 1/4}.$$

4.6 Normal Modes

1.1. Coupled pendula

Two pendula of equal length l and bob mass m oscillate in the same plane, joined at their bobs by a weak massless spring of stiffness k.

(a) Derive $m\ddot{x}_1 = -(mg/l)x_1 + k(x_2 - x_1)$ and its counterpart for bob 2.

(b) Write $m\ddot{\mathbf{x}} = -K\mathbf{x}$ and find K.

(c) Show that $\mathbf{x} = \mathbf{a}f(t)$ requires an eigenvector of K, and solve for f.

(d) Find all eigenvectors and the general motion.

(e) Describe release from rest at $x_1 = x_2 = A$.

Solution.

Linear restoring forces give $m\ddot{x} = -Kx$ with

$$K = \begin{pmatrix} mg/l + k & -k \\ -k & mg/l + k \end{pmatrix}.$$

A normal solution $x = af$ requires $Ka = \lambda a$ and $\ddot{f} + (\lambda/m)f = 0$. The independent modes are

$$\boxed{a_1 = (1,1),\ \omega_1^2 = g/l; \quad a_2 = (1,-1),\ \omega_2^2 = g/l + 2k/m}.$$

Independent sine and cosine amplitudes along each eigenvector give the general solution. Initial rest at (A, A) excites only mode 1: $\boxed{x_1 = x_2 = A\cos\omega_1 t}$.

1.2. Beats and initial data

For the coupled pendula of Question 1.1:

(a) Release from rest at $(x_1, x_2) = (A, 0)$. For $k/m = 0.105g/l$, show $x_1 = A\cos\bar{\omega}t\cos\Omega t$, $x_2 = A\sin\bar{\omega}t\sin\Omega t$, with $\bar{\omega} = 1.05\sqrt{g/l}$ and $\Omega = 0.05\sqrt{g/l}$. Sketch the beats and estimate oscillations per complete energy-transfer cycle.

(b) Give initial conditions exciting one normal mode.

(c) For $\mathbf{x}(0) = 0$, $\dot{\mathbf{x}}(0) = (0, v_0)^T$, derive the two sine-mode contributions.

(d) Sketch positions and velocities for the parameters in (a).

Solution.

(a) Projection gives $x_1 = A(\cos\omega_1 t + \cos\omega_2 t)/2$, $x_2 = A(\cos\omega_1 t - \cos\omega_2 t)/2$. Thus

$$\boxed{x_1 = A\cos\bar{\omega}t\cos\Delta t,\ x_2 = A\sin\bar{\omega}t\sin\Delta t},$$

where

$$\bar{\omega} = (\omega_1 + \omega_2)/2 = 1.05\sqrt{g/l}$$

and

$$\Delta = (\omega_2 - \omega_1)/2 = 0.05\sqrt{g/l}$$

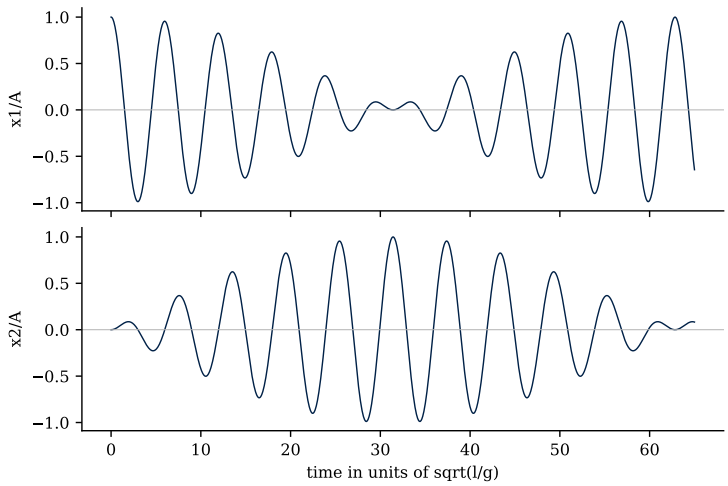
for the stated coupling. The first envelope returns at π/Δ, about $\boxed{10.5}$ carrier oscillations later.

(b) Initial rest at $(A, -A)$ excites only mode 2.

(c) For initial velocity $(0, v_0)$,

$$\boxed{x_1 = \frac{v_0}{2}\left(\frac{\sin\omega_1 t}{\omega_1} - \frac{\sin\omega_2 t}{\omega_2}\right), \quad x_2 = \frac{v_0}{2}\left(\frac{\sin\omega_1 t}{\omega_1} + \frac{\sin\omega_2 t}{\omega_2}\right)}.$$

(d) Differentiating gives velocities $v_0(\cos\omega_1 t - \cos\omega_2 t)/2$ and $v_0(\cos\omega_1 t + \cos\omega_2 t)/2$.



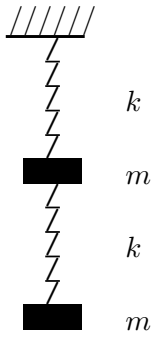
1.3. Vertical spring modes

Two masses m hang in a vertical series from a fixed support on identical springs k, as shown.

(a) Derive the equations in displacements from static equilibrium and show $\omega_{1,2}^2 = (k/m)(3 \mp \sqrt{5})/2$.

(b) Find eigenvectors.

(c) Explain why gravity does not enter these frequencies.



Solution.

Displacements from static equilibrium remove gravity. The equations are $m\ddot{x}_1 = -2kx_1 + kx_2$, $m\ddot{x}_2 = kx_1 - kx_2$. Their dimensionless characteristic equation $q^2 - 3q + 1 = 0$ gives

$$\boxed{\omega_{1,2}^2 = (k/m)(3 \mp \sqrt{5})/2, \quad c_{1,2} = (1, 2 - q_{1,2})}.$$

Gravity sets equilibrium extensions, but not the linear stiffness around them.

1.4. Zero-frequency mode

Two equal masses m joined by a spring k move freely along a smooth horizontal track. Find their normal frequencies and eigenvectors, explain the zero-frequency mode and give the full solution including uniform translation.

Solution.

The stiffness matrix is $k\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$. Hence

$$\boxed{x = (1,1)(A+Bt) + (1,-1)[C\cos\Big(\sqrt{2k/m}\,t\Big) + D\sin\Big(\sqrt{2k/m}\,t\Big)].}$$

The modes have $\omega = 0, \sqrt{2k/m}$. The zero mode translates the CM without extending the spring and includes uniform drift.

1.5. Mode energies

For the vertical spring system of Question 1.3, pull the lower mass down by $2a$, hold until static equilibrium with that imposed displacement, then release. Show $\mathbf{x}(0) = (a, 2a)^T$ and total oscillation energy ka^2. For a stiffness eigenvector $\mathbf{c}_i$ of eigenvalue λ_i, derive

$$E_i = \lambda_i[\mathbf{c}_i^T\mathbf{x}(0)]^2/(2\mathbf{c}_i^T\mathbf{c}_i)$$

and calculate each mode's energy.

Solution.

Holding the lower mass at $2a$ while the upper equilibrates requires $2kx_1 - 2ka = 0$, so $x(0) = (a, 2a)$. Initial energy is $x^TKx/2 = \boxed{ka^2}$. The eigenvector projection gives $E_i = \lambda_i[c_i^Tx(0)]^2/(2c_i^Tc_i)$. Substitution gives $\boxed{E_{1,2} = ka^2(1/2 \pm 1/\sqrt{5})}$, summing to ka^2.

1.6. Unequal masses

Replace the two equal pendulum masses in Question 1.1 by m_1, m_2, keeping length l and coupling stiffness k. Form $M\ddot{\mathbf{x}} = -K\mathbf{x}$ with the appropriate mass-dependent gravitational stiffnesses. Solve the generalised eigenvalue problem $K\mathbf{c} = \omega^2M\mathbf{c}$ and give the motion.

Solution.

The physical matrices are $M = \mathrm{diag}(m_1, m_2)$ and

$$K = \begin{pmatrix}m_1g/l + k & -k\\ -k & m_2g/l + k\end{pmatrix},$$

with $M\ddot{x} = -Kx$. The gravitational diagonal terms must change with the masses. Solving $Kc = \omega^2Mc$ gives

$$\boxed{\omega_1^2 = g/l,\ c_1 = (1,1); \quad \omega_2^2 = g/l + k(1/m_1 + 1/m_2),\ c_2 = (m_2, -m_1)}.$$

The modes satisfy $c_1^TMc_2 = 0$. A sine and cosine in each mode give the general motion.

1.7. Discrete string and continuum limit

Two masses m divide a horizontal distance $3l$ into equal segments and are joined to fixed walls by three identical springs of natural length $l_0 \ll l$ and equilibrium tension T_0.

(a) Show that small transverse extensions are quadratic, so the leading transverse restoring force uses T_0.

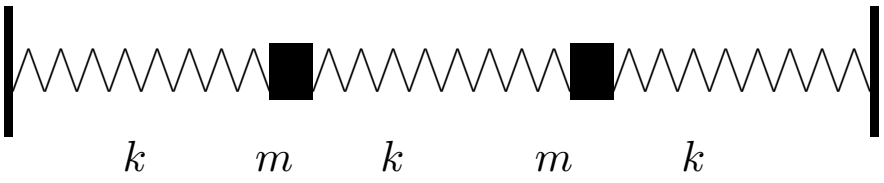
(b) Derive $m\ddot{\mathbf{y}} = -K\mathbf{y}$ with

$$K = (T_0/l)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(c) Generalise K to N equally spaced masses and tension T.

(d) Solve its eigenvalue recurrence using $c_i = \sin(i\theta)$; derive $\theta = n\pi/(N+1)$ and $\lambda_n = (2T/l)(1-\cos\theta)$.

(e) For $L = (N+1)l$, $k_n = n\pi/L$, $n \ll N$, show $\omega_n^2/k_n^2 \simeq Tl/m$ and relate it to the string wave equation.



Solution.

A spring length is $l + (y_{i+1} - y_i)^2/(2l) + O(y^4)$, so linear restoring forces use the unchanged equilibrium tension. Thus $m\ddot{y}_i = (T/l)(y_{i+1} - 2y_i + y_{i-1})$, with fixed endpoints. The stiffness matrix has diagonal $2T/l$ and off-diagonal neighbours $-T/l$. Substituting $c_i = \sin(i\phi)$ gives

$$\boxed{\phi_n = \frac{n\pi}{N+1}, \quad \lambda_n = \frac{2T}{l}(1-\cos\phi_n), \quad \omega_n^2 = \lambda_n/m.}$$

The first and last rows impose $c_0 = c_{N+1} = 0$. For low n, $\boxed{\omega_n^2 \simeq (Tl/m)k_n^2}$, $k_n = n\pi/[(N+1)l]$. This is $c^2 = T/\mu$ with $\mu = m/l$.

4.7 Waves I

2.1. Wave definitions

Explain standing and travelling waves. For $y = A\sin(kx - \omega t)$, identify direction, speed, amplitude, wavelength, wavenumber, wavevector, period, ordinary and angular frequencies. Show that it solves $y_{xx} = c^{-2}y_{tt}$ when $\omega^2 = c^2k^2$.

Solution.

A travelling wave translates its profile; a standing wave has fixed nodes. For $A\sin(kx - \omega t)$ with positive k, ω, phase moves in $+x$ at $\boxed{\omega/k}$. Amplitude is $|A|$, wavelength $2\pi/k$, spectroscopic wavenumber $k/(2\pi)$, wavevector $k\hat{x}$, period $2\pi/\omega$, frequency $\omega/(2\pi)$ and angular frequency ω. Substitution gives $\boxed{\omega^2 = c^2k^2}$.

2.2. Counter-propagating waves

Write an equal-amplitude wave travelling oppositely to $A\sin(kx - \omega t)$ and show that their sum is separable as a standing wave. Conversely express $A\sin kx\sin\omega t$ as two counterpropagating waves.

Solution.

Choose $y_2 = A\sin(kx + \omega t)$. Then $\boxed{y_1 + y_2 = 2A\sin kx\cos\omega t}$, with fixed nodes $kx = n\pi$. Conversely

$$\boxed{A\sin kx\sin\omega t = (A/2)[\cos(kx - \omega t) - \cos(kx + \omega t)]}.$$

2.3. Co-propagating envelopes

(a) Factor $A\cos(\omega_1 t + k_1 x) + A\cos(\omega_2 t + k_2 x)$, with $\omega_2 > \omega_1$, into carrier and envelope. Find both signed velocities and sketch at $t = 0$ and $t = \pi/(\omega_2 - \omega_1)$.

(b) For $y = \sum_i a_i e^{i(\omega_i t - k_i x)}$ near $(\bar{k}, \bar{\omega})$, linearise $\omega_i = \bar{\omega} + v_g(k_i - \bar{k})$, factor the carrier and envelope, and identify their speeds.

Solution.

(a) Sum-to-product gives

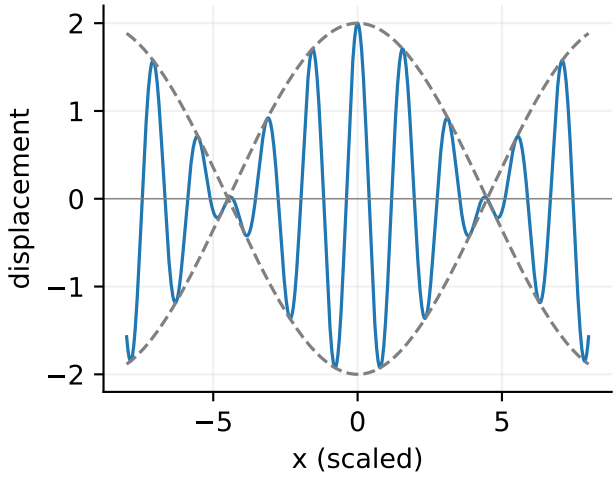
$$\boxed{y_1 + y_2 = 2A\cos[(\Delta\omega t + \Delta k x)/2]\cos[(\Sigma\omega t + \Sigma k x)/2]}.$$

With this plus-sign convention the envelope and phase velocities are $-\Delta\omega/\Delta k$ and $-\Sigma\omega/\Sigma k$. At $t = 0$ the envelope is $2A\cos(\Delta k x/2)$; at $t = \pi/\Delta\omega$ it is $-2A\sin(\Delta k x/2)$.

(b) Linearizing the dispersion gives

$$\boxed{y = e^{i(\bar{\omega}t - \bar{k}x)}\sum_i a_i e^{-i(k_i - \bar{k})(x - v_g t)}}.$$

Carrier phase speed is $\bar{\omega}/\bar{k}$ and envelope speed is v_g.



2.4. Dispersion

Define dispersion relation, phase velocity and group velocity; derive $v_g = v_p + k\,dv_p/dk$. Find both speeds for $\omega^2 = gk$, $\omega^2 = \sigma k^3/\rho$, $\omega^2 = gk + \sigma k^3/\rho$ and $\omega^2 = \omega_0^2 + c^2k^2$. What dispersion law gives $v_g = \alpha v_p$ with constant α? In the waveguide case, show $v_p v_g = c^2$ and explain the signal-speed implication.

Solution.

Dispersion means $v_p = \omega/k$ depends on k. Envelope speed is $v_g = d\omega/dk = v_p + k\,dv_p/dk$. For the four cases:

ω	v_p	v_g
$\sqrt{gk}$	$\sqrt{g/k}$	$v_p/2$
$\sqrt{\sigma k^3/\rho}$	$\sqrt{\sigma k/\rho}$	$3v_p/2$
$\sqrt{gk + \sigma k^3/\rho}$	$\sqrt{g/k + \sigma k/\rho}$	$(g + 3\sigma k^2/\rho)/(2\omega)$
$\sqrt{\omega_0^2 + c^2k^2}$	ω/k	c^2k/ω

A power law $\omega = Ck^\alpha$ gives $v_g = \alpha v_p$. In the waveguide $v_p > c$, $v_g < c$ and $v_p v_g = c^2$; a harmonic phase is not an independently controllable information signal.

2.5. Velocity minima

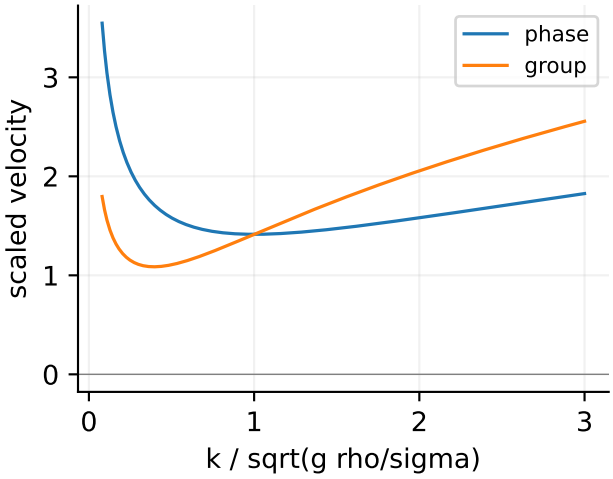
For gravity–capillary waves $\omega^2 = gk + \sigma k^3/\rho$:

(a) Show that the minimum of v_p is at $k^2 = g\rho/\sigma$, where $v_p = v_g$. Describe a packet centred there; the course demonstration uses $k \simeq 364\,\mathrm{m}^{-1}$ and packet range 340–390 m^{-1}.

(b) Show that v_g is minimal at $k^2 = [(2\sqrt{3} - 3)/3]g\rho/\sigma$, and describe propagation near this point. Verify numerically or with the course wave-packet plotter.

Solution.

Put $a = \sigma/\rho$. Differentiating $v_p^2 = g/k + ak$ gives $\boxed{k_p^2 = g\rho/\sigma}$, where $v_g = v_p$. Carrier and envelope then advance together to narrow-band order. Differentiating v_g gives $3a^2k^4 + 6agk^2 - g^2 = 0$, hence $\boxed{k_g^2 = (2\sqrt{3} - 3)g\rho/(3\sigma)}$. For $k_p = 364\,\mathrm{m}^{-1}$, $k_g \simeq 143\,\mathrm{m}^{-1}$. Here $d^2\omega/dk^2 = 0$, suppressing leading packet spreading.



2.6. String wave equation

Derive the transverse string wave equation with $c^2 = T/\mu$ for tension T and uniform linear density μ. Show that its differential operator is linear and homogeneous and explain the resulting superposition property.

Solution.

A segment has transverse force $T[y_x(x+dx) - y_x(x)] \simeq T y_{xx} dx$ and mass μdx. Thus

$$\boxed{\mu y_{tt} = T y_{xx},\ c = \sqrt{T/\mu}}.$$

The operator $\partial_x^2 - c^{-2}\partial_t^2$ is linear and the equation homogeneous, so linear combinations of solutions remain solutions, with correspondingly combined initial and boundary data.

2.7. Characteristics and fixed ends

Outline the derivation of $y = f(x-ct) + g(x+ct)$. For a string fixed at $x = 0$, find g from f. If it is also fixed at $x = L$, find the spatial and temporal periodicity. For release from rest, determine the relation between f, g, and establish the odd extension required by both fixed endpoints.

Solution.

In coordinates $u = x - ct, v = x + ct$, the equation becomes $y_{uv} = 0$, giving $\boxed{y = f(x-ct) + g(x+ct)}$. At $x = 0$, $g(s) = -f(-s)$; at L, $f(u + 2L) = f(u)$. Thus the spatial period is $2L$ and a common time period $2L/c$. Initial rest gives $g' = f'$; choosing the irrelevant split constant makes $g = f$. Fixed-end data then require $\boxed{f(-x) = -f(x)}$, with periodic extension.

2.8. Quarter-length pluck

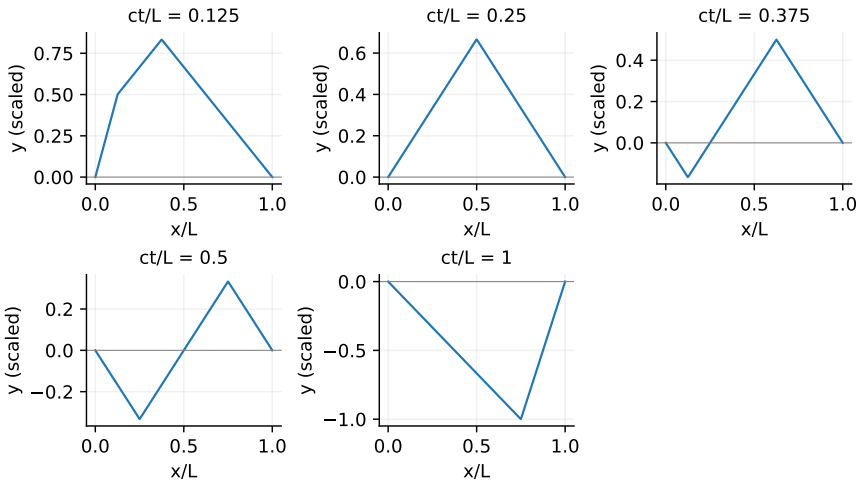
A string fixed at $0, L$ is plucked to height h at $L/4$ and released from rest. Find the odd periodic function f in its d'Alembert solution and sketch it on $[-L, 2L]$. Sketch the physical string at $t = L/(8c), L/(4c), 3L/(8c), L/(2c), L/c$ by superposing translated profiles.

Solution.

Let $Y = 4hx/L$ up to $L/4$ and $Y = 4h(L-x)/(3L)$ thereafter up to L. Extend it oddly and $2L$-periodically. Release from rest gives

$$\boxed{y(x,t) = [Y_{\text{ext}}(x-ct) + Y_{\text{ext}}(x+ct)]/2}.$$

The two half-height triangular profiles travel oppositely and invert at fixed-end reflections. This produces every requested snapshot; in particular $\boxed{y(x, L/c) = -Y(L-x)}$. The plot shows all five specified times.


ct/L = 0.125
ct/L = 0.25
ct/L = 0.375
ct/L = 0.5
ct/L = 1
y (scaled)
x/L
0.0
0.5
1.0
0.00
0.25
0.50
0.75
0.2
0.4
0.6
−0.2
−0.5
−1.0

4.8 Waves II

3.1. Separation of variables

For a string fixed at $x = 0, L$:

(a) Separate variables, explaining the allowed sign of the separation constant.

(b) Find the sine and cosine time modes and the general superposition.

(c) Find its common temporal period.

(d) For release from rest with midpoint displacement a and straight segments to the endpoints, derive the cosine-time Fourier form and find its sine-series coefficients.

(e) Instead take $y(x,0) = 0$, $y_t(x,0) = V(x)$ and give the solution and coefficient formula.

Solution.

For $y = X(x)T(t)$, $X''/X = T''/(c^2T) = -k^2$. Nonnegative separation constants give no nonzero solution with both fixed ends. Negative constants give $k_n = n\pi/L$, so

$$\boxed{y = \sum_{n\geq 1} \sin(n\pi x/L)[A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)].}$$

A common period is $2L/c$. A midpoint triangular pluck of height a released from rest has $B_n = 0$ and $\boxed{A_n = 8a\sin(n\pi/2)/(n^2\pi^2)}$, from $A_n = (2/L)\int_0^L Y \sin(n\pi x/L)dx$. For initial $y = 0, y_t = V(x)$,

$$\boxed{A_n = 0,\ B_n = 2(n\pi c)^{-1}\int_0^L V(x)\sin(n\pi x/L)dx}.$$

3.2. Two-mode motion

For the fixed-end string with $y(x,0) = \sin(\pi x/L) + 2\sin(2\pi x/L)$ and $y_t(x,0) = 0$, find $y(x,t)$ and sketch it at $t = 0, L/(4c), L/(2c), 3L/(4c), L/c$. Explain why the superposition is not a single standing or travelling wave.

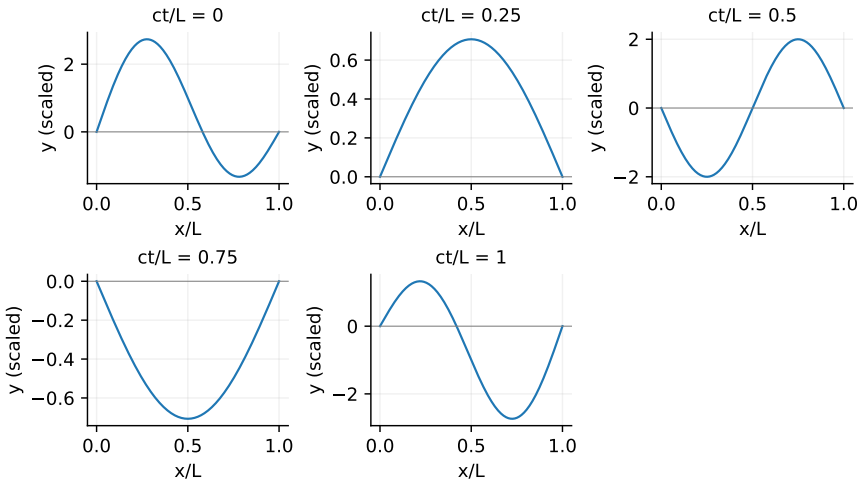
Solution.

The solution is

$$\boxed{y = \sin(\pi x/L)\cos(\pi ct/L) + 2\sin(2\pi x/L)\cos(2\pi ct/L)}.$$

Writing $u = \pi x/L$, the five requested profiles are

$$\boxed{\sin u + 2\sin 2u,\ \sin u/\sqrt{2},\ -2\sin 2u,\ -\sin u/\sqrt{2},\ -\sin u + 2\sin 2u}.$$



3.3. Energy density and flux

(a) Derive string energy densities $u_K = \mu y_t^2/2$, $u_P = T y_x^2/2$.

(b) Evaluate them for $y = A\sin(kx - \omega t)$ with $\omega = ck$.

(c) Show $u_K = u_P$.

(d) Derive the energy flux F.

(e) Calculate and interpret $F(x, 0)$.

(f) Prove local conservation $F_x + \partial_t(u_K + u_P) = 0$.

Solution.

The mass and quadratic length increase of a small segment give $\boxed{u_K = \mu y_t^2/2,\ u_P = T y_x^2/2}$. For the travelling sine with $\omega = ck$, both equal $\mu A^2\omega^2\cos^2(kx - \omega t)/2$ pointwise. The positive-x flux is

$$\boxed{F = -T y_x y_t = TA^2 k\omega\cos^2(kx - \omega t)}$$

and at zero time is this prefactor times $\cos^2 kx$. Differentiating and using $\mu y_{tt} = T y_{xx}$ gives

$$\boxed{\partial_t(u_K + u_P) + \partial_x F = 0},$$

local energy conservation.

3.4. Cancellation without loss of energy

A string carries $y = f(x - ct) + g(x + ct)$, where $f(u) = A\sin ku$ for $-2\pi \le ku \le -\pi$, zero otherwise, and $g(u) = A\sin ku$ for $\pi \le ku \le 2\pi$, zero otherwise.

(a) Sketch initial displacement and velocity.

(b) Find y at $t = 3\pi/(2kc)$ directly, and verify at $x = 0, 3\pi/k$ using the initial-data d'Alembert formula.

(c) Find the total energy initially and at that time.

Solution.

Initially $Y = f + g$ is a positive half-sine on $[-2\pi/k, -\pi/k]$ and negative on $[\pi/k, 2\pi/k]$, zero elsewhere; $V = -cf' + cg'$ supplies its cosine velocity pieces. At $t_* = 3\pi/(2kc)$ the two supports coincide on $[-\pi/(2k), \pi/(2k)]$, with

$$\boxed{f(x - ct_*) = A\cos kx,\ g(x + ct_*) = -A\cos kx,\ y(x, t_*) = 0}.$$

Both requested point values are zero, also on substituting the initial Y, V into d'Alembert's integral. At overlap $y_t = 2Akc\sin kx$ on this interval. Each initial pulse has energy $\pi TA^2k/2$; therefore $\boxed{E(0) = E(t_*) = \pi TA^2k}$. All energy at overlap is kinetic.

3.5. Reflection and transmission

Two semi-infinite strings of densities ρ_1, ρ_2 meet at $x = 0$ under common tension T. A unit-amplitude wave $\cos[\omega(t - x/c_1)]$ approaches from string 1. Find (a) reflected and transmitted displacement amplitudes r, t, (b) mean energy fluxes and verify conservation, and (c) coefficients r', t' for reverse incidence. Verify $r' = -r$ and $r^2 + tt' = 1$.

Solution.

Continuity gives $1 + r = t$ and $k_1(1 - r) = k_2 t$. With $Z_i = T/c_i = \sqrt{T\mu_i}$,

$$\boxed{r = \frac{Z_1 - Z_2}{Z_1 + Z_2},\quad t = \frac{2Z_1}{Z_1 + Z_2}.}$$

For unit incident displacement amplitude, incident/reflected/transmitted mean flux magnitudes are $Z_1\omega^2/2$, $Z_1\omega^2 r^2/2$, $Z_2\omega^2 t^2/2$. Hence

$$\boxed{\mathcal{R} = r^2,\ \mathcal{T} = (Z_2/Z_1)t^2,\ \mathcal{R} + \mathcal{T} = 1}.$$

Reverse incidence gives $r' = -r$, $t' = 2Z_2/(Z_1 + Z_2)$ and $r^2 + tt' = 1$.

3.6. Multiple pulse reflections

For the interface in Question 3.5:

(a) Give reflected and transmitted forms for general incident profile $f(t - x/c_1)$.

(b) For $\rho_2 = 4\rho_1$, find $c_2/c_1, r, r', t, t'$.

(c) Fix string 2 to a wall at $x = L$ and derive wall reflection coefficient -1.

(d) A short incident pulse has amplitude A and spatial width $W \ll L$, reaching the interface at $t = 0$. Sketch all pulses at $L/c_2, 3L/c_2, 5L/c_2, 9L/c_2$, including heights, widths and speeds.

(e) For a fixed pulse shape, derive $E \propto A^2/W$ and find the energy fractions in successive returned pulses.

Solution.

(a) The reflected and transmitted waves for arbitrary incident $f(t - x/c_1)$ are $rf(t + x/c_1)$ and $t_{12}f(t - x/c_2)$.

(b) $\mu_2 = 4\mu_1$ gives

$$\boxed{c_2/c_1 = 1/2,\ r = -1/3,\ r' = 1/3,\ t_{12} = 2/3,\ t_{21} = 4/3}.$$

(c) Fixed-end reflection has coefficient -1, because displacements must cancel.

(d) Put $\tau = L/c_2$ and let the incident spatial width in string 1 be W. The first reflected pulse is $-A/3$ moving at $-c_1$; the initially transmitted pulse is $2A/3$, width $W/2$, moving at c_2. At wall arrivals $\tau, 3\tau, 5\tau, 9\tau$ the heights are $\boxed{2A/3, -2A/9, 2A/27, 2A/243}$, each then reversing sign and direction. Outgoing string-1 return pulses are

$$\boxed{a_n = -\frac{8A}{9}(-1/3)^{n-1} \text{ emitted at } 2n\tau, \quad x_n = -c_1(t - 2n\tau), \quad n \geq 1,}$$

of width W, in addition to the initial reflected pulse. Here $c = c_2$ in the specified observation times.

(e) For a fixed shape $y = Af(x/W)$, $E = T\int y_x^2 dx \propto A^2/W$. Reflected energies are

$$\boxed{E/9,\ (64/81)(1/9)^{n-1}E\ (n \geq 1)}.$$

The geometric sum is E, so all energy is accounted for.

3.7. Wave impedances

A lossless transmission line obeys $I_x = -CV_t$, $V_x = -LI_t$, with L, C per unit length.

(a) Derive wave equations, speed c and forward-wave impedance $Z = V/I$.

(b) Find the backward-wave voltage–current relation and form the general superposition.

(c) With $U = (LI^2 + CV^2)/2$, $F = IV$, show local conservation.

(d) Repeat for plane electromagnetic waves with $H_x = -\epsilon_0 E_t$, $E_x = -\mu_0 H_t$.

(e) Map a string to the same form using velocity y_t and force $-Ty_x$; identify its speed, impedance, energy density and flux.

Solution.

(a) Differentiation gives $V_{xx} = LCV_{tt}$ and $I_{xx} = LCI_{tt}$, hence

$$\boxed{c = 1/\sqrt{LC},\ Z = \sqrt{L/C},\ I_+ = V_+/Z}.$$

(b) A backward wave has $I_- = -V_-/Z$; superposition gives $V = V_+ + V_-$ and $I = (V_+ - V_-)/Z$.

(c) $U = (LI^2 + CV^2)/2$, $F = IV$ give $U_t = LII_t + CVV_t = -IV_x - VI_x = -F_x$.

(d) Set $L = \mu_0, C = \epsilon_0$, $V \leftrightarrow E_y, I \leftrightarrow H_z$:

$$\boxed{c = 1/\sqrt{\mu_0\epsilon_0},\ Z_0 = \sqrt{\mu_0/\epsilon_0},\ U = (\mu_0 H^2 + \epsilon_0 E^2)/2,\ F = EH}.$$

(e) For a string choose $V = -Ty_x, I = y_t$. Then $I_x = -V_t/T$, $V_x = -\mu I_t$, identifying effective $C = 1/T, L = \mu$. Thus

$$\boxed{c = \sqrt{T/\mu},\ Z = \sqrt{T\mu},\ u = (\mu y_t^2 + T y_x^2)/2,\ F = -Ty_x y_t}.$$

Computational Companion

These executable definitions and examples accompany the Vectors and Matrices computational questions. The accompanying file is `Mathematica_checks.wl`. It has not been run in Mathematica.

```
(* Computational companions for CP3. Run in Mathematica. *)
EchelonTrace[a_]:=Module[{m=a,r=1,p,k},
 Do[If[r>Length[m],Break[]];
 p=Select[Range[r,Length[m]],!TrueQ[PossibleZeroQ[m[[#,j]]]]&];
 If[p==={},Continue[]];k=First[p];If[k!=r,m[[{r,k}]]=m[[{k,r}]];Print[MatrixForm[m]]];
 m[[r]]=Simplify[m[[r]]/m[[r,j]]];Print[MatrixForm[m]];
 Do[m[[i]]=Simplify[m[[i]]-m[[i,j]]m[[r]]];Print[MatrixForm[m]],
 {i,r+1,Length[m]}];r++,{j,Length[First[m]]}];m];
PlaneThrough[p1_,p2_,p3_]:=Module[{n=Cross[p2-p1,p3-p1]},
 If[n=={0,0,0},Return[$Failed]];
 {n.{x,y,z}==n.p1,p1+s(p2-p1)+t(p3-p1)}];
GramSchmidt[vs_]:=Module[{es={},u},
 Do[u=Simplify[v-Total[((Conjugate[#].v)#)&/@es]];
 If[TrueQ[Simplify[Conjugate[u].u]==0],Return[$Failed]];
 AppendTo[es,Simplify[u/Sqrt[Conjugate[u].u]]],{v,vs}];es];
PerceptronTrain[pts_,ys_,limit_:1000]:=
 Module[{xx=Prepend[#,1]&/@pts,w={0,0,0},errors},
 Do[errors=0;Do[If[ys[[j]](w.xx[[j]])<=0,
 w+=ys[[j]]xx[[j]];errors++],{j,Length[xx]}];
 If[errors==0,Return[w]],{limit}];$Failed];
PageRanks[edges_,n_]:=Module[{a=ConstantArray[0,{n,n}],out,T,sol},
 Do[a[[e[[2]],e[[1]]]]=1,{e,edges}];out=Total[a];
 T=Table[If[out[[j]]==0,1/n,a[[i,j]]/out[[j]]],{i,n},{j,n}];
 sol=Solve[Join[Thread[(IdentityMatrix[n]-T).Array[p,n]==0],
 {Total[Array[p,n]]==1}],Array[p,n]];
 {sol,Reverse@SortBy[Table[{j,p[j]/.First[sol]},{j,n}],Last]}];
SecondOrderModes[m_,x0_,v0_,time_]:=Module[{ev,es,a,b,q},
 {ev,es}=Eigensystem[m];es=Normalize/@es;a=es.x0;b=es.v0;
 q=MapThread[Function[{lam,aa,bb},
 Which[lam>0,aa Cos[Sqrt[lam]time]+bb Sin[Sqrt[lam]time]/Sqrt[lam],
 lam==0,aa+bb time,True,
 aa Cosh[Sqrt[-lam]time]+bb Sinh[Sqrt[-lam]time]/Sqrt[-lam]]],
 {ev,a,b}];Transpose[es].q];
Print[MatrixRank[Transpose[#]]&/@{
 {{0,1,1},{1,1,1},{0,0,1}},{{1,0,1},{2,3,1},{1,6,-1}},
 {{1,2,0,-3},{2,1,1,-4},{-3,6,-4,1}}}];
Print[PlaneThrough[{2,-1,1},{3,2,-1},{-1,3,2}]];
Print[PerceptronTrain[{{0,0},{0,2},{1,2},{1,4}},{-1,1,-1,1}]];
aa={{1,1,0,-1},{0,2,-2,1},{3,2,0,-4},{1,-2,a,0}};
Print[Factor[Det[aa]]];Print[RowReduce[aa/.a->8]];
Print[PageRanks[{{1,2},{1,4},{2,1},{2,4},{3,1},{4,3}},4]];
es=GramSchmidt[{{1,1,0},{2,1,2},{0,2,-1}}];
Print[es];Print[Simplify[es.Transpose[es]]];
Print[Factor[Det[{{a,1,a},{1,b,-1},{0,-1,a}}]]];
h={1,2x,4x^2-2};
Print[Table[Integrate[Exp[-x^2]h[[i]]h[[j]],{x,-Infinity,Infinity}],
 {i,3},{j,3}]];
R1[u_]:={{1,0,0},{0,Cos[u],-Sin[u]},{0,Sin[u],Cos[u]}};
R2[u_]:={{Cos[u],0,Sin[u]},{0,1,0},{-Sin[u],0,Cos[u]}};
R3[u_]:={{Cos[u],-Sin[u],0},{Sin[u],Cos[u],0},{0,0,1}};
```

```
rot=R1[a1[t]].R2[-a2[t]].R3[a3[t]];
ww=Simplify[D[rot,t].Transpose[rot]];
Print[Simplify[{ww[[3,2]],ww[[1,3]],ww[[2,1]]}]];
Print[Simplify[ww+Transpose[ww]]];
Print[Eigensystem[{{1,2,1},{2,1,1},{1,1,2}}]];
Print[Eigensystem[{{10,3I},{-3I,2}}]];
Print[SecondOrderModes[{{2,1},{1,2}},{1,0},{0,0},t]];
Print[{1,-p1,(p1^2-p2)/2,-(p1^3-3p1 p2+2p3)/6,
 (p1^4-6p1^2 p2+3p2^2+8p1 p3-6p4)/24}];

(* Further symbolic checks and a plotting companion. *)
aa=Array[a,3];bb=Array[b,3];cc=Array[c,3];dd=Array[d,3];
Print[Expand[Cross[aa,bb].Cross[cc,dd]
 -(aa.cc)(bb.dd)+(aa.dd)(bb.cc)]];
Print[Expand[Cross[aa,Cross[bb,cc]]-bb(aa.cc)+cc(aa.bb)]];
Print[Expand[aa.Cross[bb,cc]-bb.Cross[cc,aa]]];
SolveWithParameters[mat_,rhs_,vars_]:=Reduce[Thread[mat.vars==rhs],vars];
EllipsePlot[mat_]:=Module[{ev,es,u},
 {ev,es}=Eigensystem[mat];es=Normalize/@es;
 ParametricPlot[Evaluate[Transpose[es].({Cos[u],Sin[u]}/Sqrt[ev])],
 {u,0,2Pi},AspectRatio->Automatic]];
Print[EllipsePlot[{{3,1},{1,1}}]];
TracePolynomial4[mat_,lam_]:=Module[{p},
 p=Table[Tr[MatrixPower[mat,j]],{j,4}];
 lam^4-p[[1]]lam^3+(p[[1]]^2-p[[2]])lam^2/2
 -(p[[1]]^3-3p[[1]]p[[2]]+2p[[3]])lam/6
 +(p[[1]]^4-6p[[1]]^2p[[2]]+3p[[2]]^2
 +8p[[1]]p[[3]]-6p[[4]])/24];
mat4=Array[bmat,{4,4}];
Print[Simplify[TracePolynomial4[mat4,lam]-Det[lam IdentityMatrix[4]-mat4]]];
```

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