The Dirac Equation

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C4: particle physics

These notes consider the properties of the Dirac equation introducing gamma-matrices, the Dirac current, helicity and chirality.

1 Spinors

In the context of the quark model, we have used spin-$\frac{1}{2}$ particles as building-block, adding together, exploring the multiplets they must form. Now we examine the nature of the spin-$\frac{1}{2}$ particle to see them as quantum-mechanical objects called spinors rotating in SU(2) symmetry space. As a precursor, we remember that ordinary 3D rotations, $R(\boldsymbol{\theta})$, are described by a Lie group of SO(3) transformations,

$$R(\boldsymbol{\theta}) = \exp\left(i\boldsymbol{J}\cdot\boldsymbol{\theta}\right) . \tag{1}$$

where the three exponentiation generators, $\boldsymbol{J}$ are defined,

$$\boldsymbol{J} = -i\frac{\partial R(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=0} \tag{2}$$

and satisfy the commutation relation,

$$[J_i, J_j] = i\,\varepsilon_{ijk}J_k\,. \tag{3}$$

SU(2)

The SO(3) commutation relation, Eq. 3 can be realised using the 2×2, special ($\det U$), unitary ($U^\dagger = U^{-1}$) matrices of SU(2). To find these matrices remember that Lie operators are described by exponentiated generators, e.g. Eq.1. Here, the generator X has eigenvalues $X = Ix$ (x is a vector of eigenvalues) and U has eigenvalues u where $u = e^x$,

$$e^{\mathrm{Tr}\,X} = e^{\mathrm{Tr}(Ix)} = e^{\sum_i x_i} = \Pi_i e^{x_i} = \det(e^X) = \det(U)$$

as the determinant of a matrix is equal to the product of its eigenvalues. So for operator matrices with unit determinant, $\exp(\mathrm{Tr}\,X) = 1$, $\mathrm{Tr}\,X = 0$, their generators must be traceless.

Applying the unitary condition to a matrix, $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

gives $d = a^*$, $c = -b^*$. Writing out X with explicit real and imaginary parts,

$$X = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} \cancel{x_a} + iy_a & x_b + iy_b \\ -x_b + iy_b & \cancel{x_a} - iy_a \end{pmatrix}, \tag{4}$$

shows that the real part of a must be zero, otherwise X would have a non-zero trace. A natural basis is identified,

$$X = iy_b\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} + ix_b\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} + iy_a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}. \tag{5}$$

These are the three, traceless, Hermitian matrices known as the Pauli spin matrices. This is the expected dimensionality of SU(2). Complex $n\times n$ matrices have $2n^2$ real free parameters but the unitarity requirements imposes n^2 constraints. The "Special" condition, $\det(U) = 1$ imposes an extra constraint so,

$$\dim\left[SU(n)\right] = n^2 - 1\,, \tag{6}$$

which is indeed 3 in the case $n = 2$.

With 2×2 matrices, we expect two eigenvalues for each of the three Pauli matrices. Within quantum mechanics, these must be $m = -\frac{1}{2}, +\frac{1}{2}$, $j = \frac{1}{2}$. Thus the generators, X_i of rotations in SU(2)-space are defined,

$$s_i = \tfrac{1}{2}\,\sigma_i\,, \tag{7}$$

with commutation relation,

$$[s_i, s_j] = i\varepsilon_{ijk}s_k\,, \tag{8}$$

and for which a raising and lowering operator is

$$\begin{aligned} s_\pm &= s_x \pm is_y \\ s_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{9}$$

$$s_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Working backwards,

$$s_x = \frac{1}{2}(s_+ + s_-) \quad \text{and} \quad s_y = \frac{i}{2}(s_- - s_+)$$

and s_z is derived from the requirement:

$$s_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad s_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

confirms the SU(2) generators as the Pauli spin matrices:

$$s_x = \tfrac{1}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} \quad s_y = \tfrac{1}{2}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} \quad s_z = \tfrac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

The algebra of SU(2) transformation [almost] identical to SO(3) rotations but here, $s_i = \frac{1}{2}\,\sigma_i$ generates the rotation,

$$\begin{aligned} \overline{\xi} &= U\xi \\ U &= \exp\left[\tfrac{1}{2}\,i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}\right] . \end{aligned} \tag{10}$$

The equivalency of SU(2) (Eq. 10) and SO(3) (Eq. 1) rotations is known as an isomorphism. The correspondence of the SO(3) and SU(2) generators is written,

$$\boldsymbol{J} \longleftrightarrow \tfrac{1}{2}\,\boldsymbol{\sigma}\,. \tag{11}$$

An expansion of the exponential shows the importance of the factor, $\frac{1}{2}$. Using the y-component as an example,

$$(\sigma_y)^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{aligned} U_y =& 1 + \frac{(-i\,\frac{1}{2}\,\sigma_y\theta_y)}{1!} + \frac{(-i\,\frac{1}{2}\,\sigma_y\theta_y)^2}{2!} + \frac{(-i\,\frac{1}{2}\,\sigma_y\theta_y)^3}{3!} \\ &+ \frac{(-i\,\frac{1}{2}\,\sigma_y\theta_y)^4}{4!} + \frac{(-i\,\frac{1}{2}\,\sigma_y\theta_y)^5}{5!} + \frac{(-i\,\frac{1}{2}\,\sigma_y\theta_y)^6}{6!} + \ldots \\ =& \left[1 - \frac{\left(\frac{\theta_y}{2}\right)^2}{2!} + \frac{\left(\frac{\theta_y}{2}\right)^4}{4!} + \frac{\left(\frac{\theta_y}{2}\right)^6}{6!} + \ldots\right] \\ &- i\sigma_y\left[\frac{\left(\frac{\theta_y}{2}\right)}{1!} - \frac{\left(\frac{\theta_y}{2}\right)^3}{3!} + \frac{\left(\frac{\theta_y}{2}\right)^5}{5!} + \ldots\right] \\ U_y =& \cos\frac{\theta_y}{2} - i\sigma_y \sin\frac{\theta_y}{2}\,. \end{aligned} \tag{12}$$

The periodicity of the $\mathrm{SU}(2)$ rotations has a peculiar minus sign comes out of a 360° rotation, $U(2\pi)\,\xi = -\xi$ and it takes two rotations to achieved an null rotation, $U(4\pi)\,\xi = \xi$. The spin-$\frac{1}{2}$ objects, ξ, that are rotated by $\mathrm{SU}(2)$ rotations are not vectors (for which we expect 360° rotation to be null) but are called *spinors*. Though the rotation property of spinors seems initially strange, the Möbius strip, or Dirac belt trick can visualise circumstances where 720° is naturally required for a null rotation.

The Lorentz group of boosts and rotations

Now consider a Lorentz boost in x to velocity $+\beta$,

$$\Lambda_x(\eta) = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

where η is rapidity defined by $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$. Following Eq. 2, the generator of such transformations is defined,

$$\boldsymbol{K}(\boldsymbol{\eta}) = -i\frac{\partial \boldsymbol{\Lambda}}{\partial \boldsymbol{\eta}}\bigg|_{\eta=0} \tag{14}$$

which are written, together with the J matrices (in a 4×4 form),

$$K_x = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_x = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$K_y = -i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_y = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$K_z = -i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad J_z = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which can be used to demonstrate commutation relations,

$$\begin{aligned} \left[K_x, K_y\right] &= -iJ_z \\ \left[J_x, K_y\right] &= \;\; iK_z\,. \end{aligned} \tag{15}$$

These show Lorentz boost generators do not form a closed group because they are mixed up with rotations. However their combination,

$$L_{i+} = \tfrac{1}{2}\,(J_i + iK_i) \qquad L_{i-} = \tfrac{1}{2}\,(J_i - iK_i) \qquad \text{etc.} \tag{16}$$

do satisfy the commutation relations,

$$\begin{aligned} \left[L_{i+}, L_{j+}\right] &= i\,\varepsilon_{ijk}L_{k+} \\ \left[L_{i-}, L_{j-}\right] &= i\,\varepsilon_{ijk}L_{k-} \\ \left[L_{i+}, L_{j-}\right] &= 0\,. \end{aligned} \tag{17}$$

Importantly, these commutation relations show L_{n+} and L_{n-} each form a closed group; and by Eq. 17 they are distinct. The $\mathrm{SO}(3,1)$ Lorentz group of boosts and rotations is represented by two distinct types of transformations. This is significant because it allows two fundamentally different types of $\mathrm{SU}(2)$ spinors. This distinction is labelled *chirality*.
The two types of spinors are identified as chiral-right, ξ_R and chiral-left, ξ_L. Their $\mathrm{SU}(2)$ transformation, like Eq. 11, is still by the Pauli matrices, $\boldsymbol{L}_\pm \longleftrightarrow \frac{1}{2}\,\boldsymbol{\sigma}$, which requires the $\mathrm{SU}(2)$ isomorphic transformations to be $\boldsymbol{K} \longleftrightarrow \pm\,\frac{1}{2}\,i\boldsymbol{\sigma}$ such that,

$$\begin{aligned} \boldsymbol{K} \longleftrightarrow +\tfrac{1}{2}\,i\boldsymbol{\sigma} \;:\quad \boldsymbol{L}_+ &= \tfrac{1}{2}\left(\tfrac{1}{2}\,\boldsymbol{\sigma} + i\,\tfrac{1}{2}\,i\boldsymbol{\sigma}\right) = 0 \\ \boldsymbol{L}_- &= \tfrac{1}{2}\left(\tfrac{1}{2}\,\boldsymbol{\sigma} - i\,\tfrac{1}{2}\,i\boldsymbol{\sigma}\right) = \tfrac{1}{2}\,\boldsymbol{\sigma} \\ \boldsymbol{K} \longleftrightarrow -\tfrac{1}{2}\,i\boldsymbol{\sigma} \;:\quad \boldsymbol{L}_+ &= \tfrac{1}{2}\left(\tfrac{1}{2}\,\boldsymbol{\sigma} - i\,\tfrac{1}{2}\,i\boldsymbol{\sigma}\right) = \tfrac{1}{2}\,\boldsymbol{\sigma} \\ \boldsymbol{L}_- &= \tfrac{1}{2}\left(\tfrac{1}{2}\,\boldsymbol{\sigma} + i\,\tfrac{1}{2}\,i\boldsymbol{\sigma}\right) = 0\,. \end{aligned}$$

Together the transformation of the right-chiral and left-chiral spinors extend from the simple rotation of Eq. 10 to,

$$\overline{\phi}_{(R)} = \exp\left[\tfrac{1}{2}\,i\boldsymbol{\sigma}\cdot\boldsymbol{\theta} + \tfrac{1}{2}\,\boldsymbol{\sigma}\cdot\boldsymbol{\eta}\right]\phi_{(R)} \tag{18}$$

$$\overline{\chi}_{(L)} = \exp\left[\tfrac{1}{2}\,i\boldsymbol{\sigma}\cdot\boldsymbol{\theta} - \tfrac{1}{2}\,\boldsymbol{\sigma}\cdot\boldsymbol{\eta}\right]\chi_{(L)} \tag{19}$$

The distinct ϕ, χ spinors are the two-component Weyl spinors.

The unification of boosts and rotations is satisfying but there is an issue with two decoupled Weyl spinors: neither the transformation of ϕ (Eq. 18) nor that of χ (Eq. 19) conserve parity. Parity changes the sign of the boost $\beta \longrightarrow -\beta$, the sign of the rapidity $\eta \longrightarrow -\eta$ and so changes Eq. 18 in to Eq. 19. Were the right-chiral Weyl spinor of Eq. 18 to conserve parity, the parity operation would be part of the group described by Eq. 18. To ensure probability is preserved under an application of the parity operator, the two Weyl spinors must be treated together,

$$\psi_{(W)} = \begin{pmatrix} \chi_{(R)} \\ \phi_{(L)} \end{pmatrix}. \tag{20}$$

as the Weyl representation of the four-component Dirac spinor.

We now proceed to demonstrate the motivation for Dirac's equation, its algebra, its [spinor] solutions before returning to the Weyl representation at the end.

2 The Klein-Gordon problem

We remember the nonrelativistic energy-momentum relation for a free particle of mass, m, is

$$E = p^2/2m\,. \tag{21}$$

With quantum mechanical operators,

$$E \to i\hbar\frac{\partial}{\partial t}\,, \qquad \mathbf{p} \to -i\hbar\boldsymbol{\nabla}\,, \tag{22}$$

Eq. 21 becomes an operator equation acting on a complex wavefunction, $\psi(\mathbf{x},t)$,

$$i\frac{\partial\psi}{\partial t} + \frac{\hbar}{2m}\nabla^2\psi = 0\,, \tag{23}$$

which is the time-dependent Schrödinger equation in the absence of an external potential. We understand $\rho = |\psi|^2$ as the probability density and $|\psi|^2 d^3x$ as the probability of finding the particle in elemental volume d^3x.

For the relativistic particle case, one might consider repeating the quantum mechanical operator substitution for the relativistic energy-momentum relation,

$$E^2 = |\mathbf{p}|^2c^2 + m^2c^4\,. \tag{24}$$

This gives the *Klein Gordon equation*,

$$\begin{aligned} -\hbar^2\frac{\partial^2\psi}{\partial t^2} &= -\hbar^2c^2\nabla^2\psi + m^2c^4\psi \\ \text{or}\quad \frac{\partial^2\psi}{\partial t^2} &= \left[\nabla^2 - m^2\right]\psi\,, \end{aligned} \tag{25}$$

in natural units, $\hbar = c = 1$. It is of interest to form the continuity equation by multiplying Eq. 25 by $-i\psi^*$ and its complex conjugate by $-i\psi$,

$$\begin{aligned} -i\psi^*\ddot{\psi} &= -i\psi^*\nabla^2\psi + im^2\psi^*\psi \\ -i\psi\ddot{\psi^*} &= -i\psi\nabla^2\psi^* + im^2\psi\psi^*\,, \end{aligned} \tag{26}$$

and subtracting one from another,

$$i\psi^*\ddot{\psi} - i\psi\ddot{\psi^*} = i\psi^*\nabla^2\psi - i\psi\nabla^2\psi^*\,. \tag{27}$$

Using the product rule $\frac{\partial}{\partial x}\left[a^*\dot{a} - a\dot{a^*}\right] = a^*\ddot{a} - a\ddot{a^*}$ (in reverse), on both time and space parts independently gives the continuity equation for the wavefunction probability,

$$\frac{\partial}{\partial t}\underbrace{\left[i\psi^*\dot{\psi} - i\psi\dot{\psi^*}\right]}_{\rho} + \boldsymbol{\nabla}\cdot\underbrace{\left[-i\psi^*\nabla\psi + i\psi\nabla\psi^*\right]}_{\mathbf{j}} = 0\,. \tag{28}$$

Thus for a free particle with a plane wavefunction,

$$\begin{aligned} \psi &= e^{\,i(\mathbf{p}\cdot\mathbf{x}-Et)} & \psi^* &= e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} \\ \dot{\psi} &= -iEe^{\,i(\mathbf{p}\cdot\mathbf{x}-Et)} & \dot{\psi^*} &= iEe^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} \end{aligned}$$

the probability density and probability current are deduced,

$$\rho = 2E\,, \qquad \mathbf{j} = 2\mathbf{p} \tag{29}$$

and together in the covariant from:

$$j^\mu = 2p^\mu\,, \qquad \text{where } \partial_\mu j^\mu = 0\,. \tag{30}$$

The problem is revealed when the plane wavefunction is substituted into Eq. 25 to give the energy eigenvalues,

$$E = \pm\sqrt{|\mathbf{p}|^2 + m^2}\,. \tag{31}$$

In addition to the expected positive energy solution are an equal number of negative energy solutions; and by Eq. 29 these negative energy solutions have negative probability.

3 The Dirac revelation (1928)

Paul Dirac sought to overcome the problem of negative probability in relativistic quantum mechanics by seeking an equation linear in both time and spatial derivatives like the Schrödinger equation,

$$E\psi \;=\; i\frac{\partial\psi}{\partial t} = \underbrace{(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)}_{\text{Hamiltonian}}\psi \qquad \mathbf{p} = -i\boldsymbol{\nabla}\,. \tag{32}$$

and require it satisfies special relativity (just $E^2 = |\mathbf{p}|^2 + m^2$) by applying the Hamiltonian twice,

$$-\frac{\partial^2\psi}{\partial t^2} = \left(-i\alpha_x\partial_x - i\alpha_y\partial_y - i\alpha_z\partial_z + \beta m\right)^2\psi \tag{33}$$

$$\begin{aligned} \frac{\partial^2\psi}{\partial t^2} = \big[\; & \alpha_x^2\partial_x^2 + \alpha_y^2\partial_y^2 + \alpha_z^2\partial_z^2 + \beta^2m^2 \\ & + (\alpha_x\alpha_y + \alpha_y\alpha_x)\partial_x\partial_y - m(\alpha_x\beta + \beta\alpha_x)\partial_x \\ & + (\alpha_y\alpha_z + \alpha_z\alpha_y)\partial_y\partial_z - m(\alpha_y\beta + \beta\alpha_y)\partial_y \\ & + (\alpha_z\alpha_x + \alpha_x\alpha_z)\partial_z\partial_x - m(\alpha_z\beta + \beta\alpha_z)\partial_z\;\big]\psi \end{aligned} \tag{34}$$

requiring of the terms: $\alpha_i^2 = \beta^2 = 1$; $(\alpha_i\alpha_j + \alpha_j\alpha_i) = 0$; and $(\alpha_i\beta + \beta\alpha_i) = 0$. Or more succinctly with $\beta \equiv \alpha_0$,

$$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu}\,. \tag{35}$$

This condition cannot be satisfied by numbers, α_μ must be matrices. The matrices are shown to be traceless by,

$$\begin{aligned} \mathrm{Tr}(\alpha_\mu) &= \mathrm{Tr}(\alpha_\nu^\dagger\alpha_\mu\alpha_\nu) \\ &= -\mathrm{Tr}(\alpha_\nu^\dagger\alpha_\nu\alpha_\mu) \;\text{ using Eq. 35} \\ &= -\mathrm{Tr}(\alpha_\mu) = 0\,, \end{aligned}$$

where the first equality is proved by,

$$\mathrm{Tr}(A_{ij}) = \mathrm{Tr}(\underbrace{B_{jk}B_{ki}^\dagger}_{=1\text{ if }B\text{ is hermitian.}}A_{ij}) = \mathrm{Tr}(B_{ki}^\dagger A_{ij}B_{jk})$$

This is appropriate as we require energy to be observable (real eigenvalues) so the Hamiltonian in Eq. 32, and thus α_i and β,

are hermitian. From the eigenvalue formula,

$$\begin{aligned} \alpha_\mu \boldsymbol{v} &= \lambda \boldsymbol{v} \\ \boldsymbol{v}^\dagger \underbrace{\alpha_\mu^\dagger \alpha_\mu}_{=1} \boldsymbol{v} &= \underbrace{\lambda^* \lambda}_{|\lambda|^2 = 1} \boldsymbol{v}^\dagger \boldsymbol{v} \quad \Rightarrow \lambda = \pm 1 \,. \end{aligned}$$

But the traceless condition, $\mathrm{Tr}(\alpha_\mu) = \sum_i^N \lambda_i = 0$, requires the dimension of α_μ to be even. For N=2 Pauli spin matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{36}$$

satisfy Eq. 35 but we need four such matrices $(\alpha_x, \alpha_y, \alpha_z, \beta)$ so $N = 4$ is required. Specifically the set of four hermitian, traceless 4×4 matrices define the *Dirac representation*,

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{37}$$

where σ_i are the Pauli spin matrices, satisfies Eq. 35. This is not unique, the *Weyl representation* is also valid, see Sec. 8. The 4×4 structure of the Dirac equation give four solutions, ψ. To check if this first-order equation indeed resolves the problems of Klein Gordon, the probability density and current is calculated in a similar manner as in Eq. 26. However, as this equation has matrices, the Hermitian conjugate is used rather than just the complex conjugate.

$$\begin{aligned} i\psi^\dagger \frac{\partial \psi}{\partial t} - (-i)\frac{\partial \psi^\dagger}{\partial t}\psi = \\ \psi^\dagger \left(-i\alpha_x \partial_x \psi - i\alpha_y \partial_y \psi - i\alpha_z \partial_z \psi + m\beta\psi \right) \\ - \left(i\partial_x \psi^\dagger \alpha_x + i\partial_y \psi^\dagger \alpha_y + i\partial_z \psi^\dagger \alpha_z + m\psi^\dagger \beta \right) \psi \,. \end{aligned}$$

then using the product rule on each term,

$$\begin{aligned} \underbrace{\psi^\dagger \frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t}\psi} + \underbrace{\sum_i \psi^\dagger \alpha_i \partial_i \psi + \partial_i \psi^\dagger \alpha_i \psi} = 0 \\ \frac{\partial}{\partial t}\left(\psi^\dagger \psi\right) \quad + \quad \boldsymbol{\nabla} \cdot \left(\psi^\dagger \boldsymbol{\alpha} \psi\right) \quad = 0 \end{aligned} \tag{38}$$

which is of the form $\dfrac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{j}$ with the Dirac probability current and probability density defined as,

$$\boldsymbol{j} = \psi^\dagger \boldsymbol{\alpha} \psi \qquad \rho = \psi^\dagger \psi \,. \tag{39}$$

The probability density, $\psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ is always positive, as required. Each four-vector component is generated from the multiplication of a spinor with its transpose conjugate, sometimes through a matrix. In this sense, a spinor is sometimes described as a “square-root" of a four-vector.

4 Intrinsic spin

A satisfying aspect of the Dirac equation is that it requires the particles it describes to have an intrinsic spin-$\frac{1}{2}$ angular momentum. Recall standard quantum mechanics commutations:

$$[p_i, p_j] = 0, \quad [x_i, x_j] = 0, \quad [p_i, x_j] = -i\hbar\delta_{ij}. \tag{40}$$

Also the commutation relation between Pauli spin matrices,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \,. \tag{41}$$

We require angular momentum to commute with the Hamiltonian because angular momentum quantum states are well established to have define states of definite energies. So,

$$[H, L] = [\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m, \boldsymbol{r} \times \boldsymbol{p}]. \tag{42}$$

The βm term commutes trivially, and for the rest, we exemplify the x component,

$$[\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_x] = \left[\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y\right] . \tag{43}$$

The only non-zero parts are terms of the form $[x_i, p_i] = -i\hbar$,

$$\begin{aligned} [H, L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\ &= -i\hbar(\alpha_y p_z - \alpha_z p_y) \\ &= -i\hbar(\boldsymbol{\alpha} \wedge \boldsymbol{p})_x \\ \therefore \quad [H, \boldsymbol{L}] &= -i\hbar\, \boldsymbol{\alpha} \wedge \boldsymbol{p} \quad \neq 0 \,! \end{aligned} \tag{44}$$

The Dirac equation appears not to commute with angular momentum, which cannot be a correct description of fermion states with definite angular momentum having definite energy.

This is solved by connecting to $SU(2)$, which described fundamental spin-$\frac{1}{2}$ objects (spinors), the rotations of which are generated by the Pauli spin matrices. Here we have 4×4 so try,

$$\boldsymbol{S} = \tfrac{1}{2}\hbar \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{45}$$

($\boldsymbol{\sigma}$ are Pauli spin matrices). Setting $\hbar = 1$, S_x, S_y, S_z are:

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}.$$

The commutation with the Hamiltonian proceeds as above: identify the non-trivial part and exemplify the x component.

$$\begin{aligned} [H, \boldsymbol{S}] &= [\boldsymbol{\alpha} \cdot \boldsymbol{p} + \underbrace{\beta m}, \boldsymbol{S}] \\ &\qquad \tfrac{1}{2}\begin{pmatrix} I & \\ & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}\begin{pmatrix} I & \\ & -I \end{pmatrix} = 0 \,. \\ &= [\boldsymbol{\alpha} \cdot \boldsymbol{p}, \boldsymbol{S}] \\ \text{e.g.} \Rightarrow [H, S_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, S_x] \\ &= p_x[\alpha_x, S_x] + p_y[\alpha_y, S_x] + p_z[\alpha_z, S_x] \,. \end{aligned}$$

Dealing with each of the three terms as well as employing the commutation relation for Pauli spin matrices,

$$\begin{aligned} [\alpha_x, S_x] &= \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_x^2 \\ \sigma_x^2 & 0 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_x^2 \\ \sigma_x^2 & 0 \end{pmatrix} = 0 \,. \end{aligned}$$

$$\begin{aligned} [\alpha_y, S_x] &= \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_y\sigma_x \\ \sigma_y\sigma_x & 0 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_x\sigma_y \\ \sigma_x\sigma_y & 0 \end{pmatrix} \end{aligned}$$

$$= \tfrac{1}{2}\begin{pmatrix} 0 & [\sigma_y\sigma_x] \\ [\sigma_y\sigma_x] & 0 \end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} = -i\alpha_z\,.$$

$$[\alpha_z, S_x] = \tfrac{1}{2}\begin{pmatrix} 0 & [\sigma_z\sigma_x] \\ [\sigma_z\sigma_x] & 0 \end{pmatrix} \quad \text{(by the same reasoning)}$$
$$= \tfrac{1}{2}\begin{pmatrix} 0 & +2i\sigma_y \\ +2i\sigma_y & 0 \end{pmatrix} = i\alpha_y\,.$$

Altogether, $[H, S_x] = i(p_z\alpha_y - p_y\alpha_z) = i\hbar(\boldsymbol{\alpha}\wedge\boldsymbol{p})_x$, hence,

$$[H, \boldsymbol{S}] = i\hbar\,\boldsymbol{\alpha}\wedge\boldsymbol{p} \tag{46}$$

The operator $\boldsymbol{S}$ that we have invented does not commute with the Hamiltonian either but $\boldsymbol{L}+\boldsymbol{S}$ does,

$$[H, \boldsymbol{L}+\boldsymbol{S}] = -i\hbar\,\boldsymbol{\alpha}\wedge\boldsymbol{p} + i\hbar\,\boldsymbol{\alpha}\wedge\boldsymbol{p} = 0\,. \tag{47}$$

The structure of the Dirac Hamiltonian (the α matrices) requires the particles whose motion it governs must have an intrinsic property that behave like, and can add to [orbital] angular momentum. Examining $\boldsymbol{S}$ further we note that,

$$S_z\psi = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}\psi = \pm\frac{\hbar}{2}\psi\,. \tag{48}$$

$$S^2\psi = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}\right.$$
$$\left.\begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}\right]$$
$$= \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\psi = \frac{3}{4}\hbar^2\psi = S(S+1)\hbar^2\psi \quad \text{for } S = \tfrac{1}{2}\,.$$

In summary, the properties of $\boldsymbol{S}$ are exactly that of quantum mechanical angular momentum with $S = \frac{1}{2}\hbar$,

$$\begin{aligned} S^2\psi &= S(S+1)\psi\,, \\ S_z\psi &= \tfrac{1}{2}\hbar\psi\,, \end{aligned} \tag{49}$$

and it is intrinsic to the Dirac Hamiltonian.

5 Gamma matrices

We continue by rearranging the Dirac equation of Eq. 32 and multiplying by the matrix β,

$$\beta\left[i\frac{\partial}{\partial t} + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \beta m\right]\psi = 0$$
$$\left[i\beta\partial_t + i\underbrace{\beta\alpha_x}_{\gamma^1}\partial_x + i\underbrace{\beta\alpha_y}_{\gamma^2}\partial_y + i\underbrace{\beta\alpha_z}_{\gamma^3}\partial_z - \underbrace{\beta^2}_{I}m\right]\psi =$$

to arrive at a manifestly covariant form of the Dirac equation,

$$\boxed{\left[i\gamma^\mu\partial_\mu - mI\right]\psi = 0} \tag{50}$$

where the identity matrix I is shown to make it explicit this is a matrix equation, acting on the four-component *Dirac spinor*.

It is important to understand that, though written with a μ superscript, γ^μ are <u>not</u> fourvectors. They are four matrices of constants, multipliers that do not change under any transformation. The four spacetime derivatives ∂_m are covariant fourvectors and m is a scalar. This is the covariant form of the Dirac equation. Similar to the α matrices that they are based on, the γ^μ matrices have an anticommutation relation,

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}} \tag{51}$$

where $g^{00} = 1, g^{aa} = -1, g^{ab} = 0$. Writing the gamma matrices explicitly confirms the γ^0 is hermitian and γ^a are antihermitian.

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix} \tag{52}$$

As $(\gamma^0)^2 = \beta^2 = I$, the continuity equation, Eq. 38 is written,

$$\partial_t\psi^\dagger\gamma^0\gamma^0\psi = \nabla\cdot\left(\psi^\dagger\gamma^0\gamma^0\alpha\psi\right)$$
$$\partial_t\underbrace{\overline{\psi}\gamma^0\psi}_{j^0} = \nabla\cdot\left(\underbrace{\overline{\psi}\gamma^a\psi}_{j^a}\right)$$
$$\partial_\mu j^\mu = 0 \tag{53}$$

with the *adjoint spinor* $\overline{\psi}$ defined, $\overline{\psi} = \psi^\dagger\gamma^0 = (\psi^*)^T\gamma^0$,

$$\overline{\psi} = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

And the Dirac probability current is re-identified,

$$j^\mu = \overline{\psi}\,\gamma^\mu\psi\,. \tag{54}$$

6 Solutions of the Dirac equation

We proceed by considering the four solutions to the Dirac equation for a free-moving particle (plane wave),

$$\begin{aligned} \psi(x^\mu) &= u(p^\mu)\ \ \exp[-ip_\mu x^\mu] \\ &= u(E,\boldsymbol{p})\exp[i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)] \end{aligned} \tag{55}$$

such that $i\partial_0 = i\frac{\partial}{\partial t} = E$ and $i\partial_a = -i\frac{\partial}{\partial x} = p_x$ etc.. This gives a version of the Dirac equation involving fourmomentum explicitly,

$$\boxed{\left[\gamma^\mu p_\mu - mI\right]\psi = 0} \tag{56}$$

Or in the most reduced form with the slash notation,

$$(\not{p}_\mu - m)\,\psi = 0 \tag{57}$$

The solutions are observed when this is expanded in full,

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix}\begin{pmatrix} u_a \\ u_b \\ u_c \\ u_d \end{pmatrix} e^{-ip_\mu x^\mu} = 0$$

First line gives $E^2 - p^2 - m^2 = 0$ if $u(p^\mu)$ is,

$$\underset{\downarrow}{(E-m)}\,\underset{\downarrow}{u_a} + \underset{\downarrow}{(-p_z)},\underset{\downarrow}{u_c} + \underset{\downarrow}{(-p_x+ip_y)}\,\underset{\downarrow}{u_d} = 0 \tag{58}$$
$$(E+m) \qquad (p_z) \qquad (p_x+ip_y)$$

$$\text{such that}\quad \psi_1 = u_1 e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)} = N\begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix} e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)} \tag{59}$$

The wavefunction normalisation factor, N, is found by requiring consistency with the Klein Gordon equation. Taking this or any Dirac spinor, the Dirac probability density from Eq. 53 is,

$$\begin{aligned} \rho = \psi^\dagger\psi &= (\,u^*(E,\boldsymbol{p})\,)^T\,u(E,\boldsymbol{p}) \\ &= N^2\begin{pmatrix} E+m & 0 & p_z & p_x-ip_y \end{pmatrix}\begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix} \\ &= N^2\left[E^2+m^2+2Em+p_z^2+p_x^2+p_y^2\right] \\ &= 2E\,N^2\,[E+m]\ . \\ \therefore \quad N &= \frac{1}{\sqrt{E+m}} \end{aligned} \tag{60}$$

gives the conventional probability density $\rho = 2E$, see Eq. 29.

Similarly, lines two,three,four give $E^2 - p^2 - m^2 = 0$ for $u(p^\mu)$,

$$\underset{\downarrow}{(E-m)}\,\underset{\downarrow}{u_b} + \underset{\downarrow}{(-p_x-ip_y)}\,\underset{\downarrow}{u_c} + \underset{\downarrow}{p_z}\,\underset{\downarrow}{u_d} = 0 \tag{61}$$
$$(E+m) \qquad (p_x-ip_y)(-p_z)$$

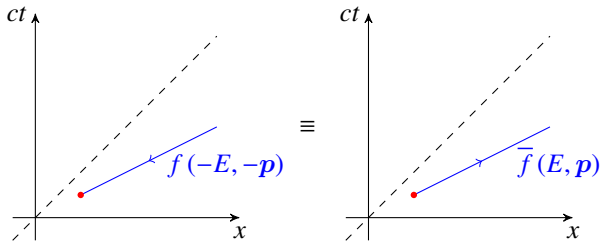
$$\underset{\downarrow}{(p_z)}\,\underset{\downarrow}{u_a} + \underset{\downarrow}{(p_x-ip_y)}\,\underset{\downarrow}{u_b} + \underset{\downarrow}{(-E-m)}\,\underset{\downarrow}{u_c} = 0 \tag{62}$$
$$(-p_z) \qquad (-p_x-ip_y) \qquad -E+m$$

$$\underset{\downarrow}{(p_x+ip_y)}\,\underset{\downarrow}{u_a} + \underset{\downarrow}{(-p_z)}\,\underset{\downarrow}{u_b} + \underset{\downarrow}{(-E-m)}\,\underset{\downarrow}{u_d} = 0 \tag{63}$$
$$(-p_x+ip_y) \qquad (p_z) \qquad -E+m\ .$$

The solutions of Eq. 62 and 63 are similar to Eq. 58 and 61 except $E \to -E$ and $\boldsymbol{p} \to -\boldsymbol{p}$. These negative energy solution is understood, within the *Feynman-Stückelberg Interpretation*, to be a particle, f traveling backwards in time, which we perceive as an antiparticle, $\overline{f}$ traveling forwards in time,

$$u(-E,-\boldsymbol{p})\,e^{i\,((-\boldsymbol{p})\cdot(-\boldsymbol{x})-(-E)\,(-t))} \quad\Rightarrow\quad v(E,\boldsymbol{p})\,e^{i\,(\boldsymbol{p}\cdot\boldsymbol{x}-E\,t)}$$

where the new label, $v(E,\boldsymbol{p})$ identifies antiparticle spinors.



Due to the sign-flip for antiparticles $i\frac{\partial}{\partial t} \to -i\frac{\partial}{\partial t}$ and $-i\frac{\partial}{\partial x} \to i\frac{\partial}{\partial x}$, it is usual to have a second version of Eq. 56 for antiparticles.

$$\boxed{\left[\gamma^\mu p_\mu + mI\right]\psi = 0} \tag{64}$$

or equivalently, $(\not{p}_\mu + m)\,\psi = 0$. Finally, the four solutions to the Dirac equation are shown below, where the antiparticle sign-flips with respect to Eq. 62 and 63 are highlighted for clarity.

$$\begin{aligned} u_1 &= \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \\ v_2 &= \sqrt{E+m}\begin{pmatrix} \frac{+p_x-ip_y}{+E+m} \\ \frac{-p_z}{+E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_1 = \sqrt{E+m}\begin{pmatrix} \frac{+p_z}{+E+m} \\ \frac{+p_x+ip_y}{+E+m} \\ 1 \\ 0 \end{pmatrix} \end{aligned} \tag{65}$$

This is sometimes written succinctly using $(s) \in 1,2$.

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \begin{pmatrix} p_z & p_x-ip_y \\ p_x+ip_y & -p_z \end{pmatrix} \qquad \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u^{(s)} = \sqrt{E+m}\begin{pmatrix} \chi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \qquad v^{(s)} = \sqrt{E+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{66}$$

In the rest frame, the four solutions simplify to,

$$\sqrt{2m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad \sqrt{2m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\quad \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\ . \tag{67}$$

The distinction between the first and second pair of spinor components is made; the difference between solution one and two (or three and four) is more enlightened using helicity.

7 Helicity

Returning to the discussion of intrinsic angular momentum, we remember that the spin-$\frac{1}{2}$ operator, $\boldsymbol{S}$, does not have eigenvalues that are constants of the motion ($[H,\boldsymbol{S}] \neq 0$). Instead, we introduce *helicity*, which is defined as the component of a particle's momentum along the z-axis of its spin.
It is manifestly observable because it is a scalar quantity: $\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{p}}$.

$$h = \frac{\boldsymbol{S}\cdot\boldsymbol{p}}{|\boldsymbol{S}||\boldsymbol{p}|} = \frac{2\boldsymbol{S}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\cdot\hat{\boldsymbol{p}} = \pm 1 \quad , \tag{68}$$

($\hat{\boldsymbol{p}}$: unit vector)

where the definition of $\boldsymbol{S}$ from Eq. 45 is used. Now consider the unit momentum vector at polar (θ) and azimuthal angle (ϕ) with respect to the $\hat{S}_z$,

$$\begin{aligned}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} &= \frac{1}{|\boldsymbol{p}|}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} &= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \end{aligned} \tag{69}$$

As helicity is observable, it is desirable to have Dirac solutions that are eigenstates of helicity (eigenvalues $h = \pm 1$).

$$\underset{4\times4\text{ matrix}}{\nearrow} \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm\begin{pmatrix} u_A \\ u_B \end{pmatrix} \underset{\text{Dirac Spinor}}{\nwarrow} \tag{70}$$

Noting that the top and bottom pairs have the same form:

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\begin{pmatrix} a \\ b \end{pmatrix}_{u_A} = \pm\begin{pmatrix} a \\ b \end{pmatrix}_{u_A} \qquad \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\begin{pmatrix} a \\ b \end{pmatrix}_{u_B} = \pm\begin{pmatrix} a \\ b \end{pmatrix}_{u_B} \tag{71}$$

and merging with Eq. 69, a relation is deduced,

$$\begin{aligned} a\cos\theta + b\sin\theta\, e^{-i\phi} &= \pm a \\ a\sin\theta\, e^{-i\phi} - b\cos\theta &= \pm b \end{aligned} \tag{72}$$

Rearranging either of these equations obtains the result (using half-angle trig. identities) for particle spinors,

$$\begin{aligned} h = +1: \quad & \frac{b}{a} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}e^{i\phi} \\ h = -1: \quad & \frac{b}{a} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}e^{i\phi}. \end{aligned} \tag{73}$$

For antiparticle spinors the result is the same but multiplied by -1. This due to the Feynman-Stückelberg Interpretation: our perception of the antiparticle momentum vector is opposite to that which the mathematics of the Dirac equation describes.

Using $\uparrow$ ($\downarrow$) to is label aligned (antialigned) momentum and spin vectors, the four Dirac solutions are written,

particle: $u_\uparrow \Longrightarrow$ $\Longleftarrow u_\downarrow$

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ \frac{|p|}{E+m}\cos\frac{\theta}{2} \\ \frac{|p|}{E+m}\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \qquad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \\ \frac{|p|}{E+m}\sin\frac{\theta}{2} \\ \frac{-|p|}{E+m}\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

antiparticle: $v_\uparrow \Longrightarrow$ $\Longleftarrow v_\downarrow$

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{|p|}{E+m}\sin\frac{\theta}{2} \\ \frac{-|p|}{E+m}\cos\frac{\theta}{2}e^{i\phi} \\ -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \qquad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{|p|}{E+m}\cos\frac{\theta}{2} \\ \frac{|p|}{E+m}\sin\frac{\theta}{2}e^{i\phi} \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

The problem with helicity is that it is not Lorentz invariant. It is possible to boost into a frame, faster than the particle, where the particle's momentum vector is reversed. Spin is an axial vector and does not change under the boost: thus helicity h changes sign. Helicity is Lorentz invariant only for massless particles travelling at the speed of light, in this case it is not possible to find a frame where the helicity flips.

In the *ultra-relativistic* case: $E \gg m$, $p_z \approx E$, the solutions are,

$$\begin{aligned} u_\uparrow &= \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_\downarrow = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \\ v_\uparrow &= \sqrt{E}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \qquad v_\downarrow = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned} \tag{74}$$

8 Chirality and the Weyl representation

Chirality is a fundamental property of a [Weyl] spinor, which are either left-chiral or right-chiral (see Eqs. 18&19). Chirality does not commute with the Hamiltonian so it is not a quantum mechanical observable. However, in the ultra-relativistic case, a particles chirality can be inferred from its helicity.

The chirality operator, γ^5, is defined,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{75}$$

which respects the anticommutation relations,

$$\left(\gamma^5\right)^2 = I \quad \text{and} \quad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0\,. \tag{76}$$

The eigenstates of chirality are labelled by handedness with

eigenvalues, ± 1.

$$\begin{aligned} \gamma^5 u_R &= +u_R & \gamma^5 u_L &= -u_L \\ \gamma^5 v_R &= -v_R & \gamma^5 v_L &= +v_L \end{aligned} \tag{77}$$

Application of γ^5 on the helicity eigenstates in the ultra-relativistic case, Eq. 74, shows they return these eigenvalues. This means that in the ultra-relativistic case, the helicity eigenstates are effectively eigenstates of chirality. They are not he same, helicity is an observable of a particle, chirality is a property (that is not directly observable).

$$|p_z| = E: \qquad u_\uparrow = u_R \quad u_\downarrow = u_L \quad v_\uparrow = v_R \quad v_\downarrow = v_L$$

To understand the role in history of chirality, focus on the $E \gg m$ case where helicity an chirality have the same value. In the simplest case, reconsider the Schrödinger equation Eq. 32 for $m = 0$,

$$E\,\psi = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \cancel{\beta m})\,\psi\,. \tag{78}$$

In the deduction that lead to Eq. 37, it was concluded that four matrices are required. However, with m=0, β is not involved and the three 2×2 Pauli matrices, $\boldsymbol{\sigma}$ Eq. 36, are sufficient to satisfy commutation relation,

$$\text{Eq. 35:}\quad \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}\,.$$

Also, we notice that both $\alpha_i = \sigma_i$ and $\alpha_i = -\sigma_i$ satisfy this relation which leads to two decoupled equations,

$$\begin{aligned} E\chi = -\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi \quad &\Longrightarrow \quad \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\chi = -\chi \qquad = u_\downarrow\,(u_L) \\ E\phi = \boldsymbol{\sigma}\cdot\boldsymbol{p}\,\phi \quad &\Longrightarrow \quad \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\phi = +\phi \qquad = u_\uparrow\,(u_R) \end{aligned}$$

where the arrow means both sides are divided by $|E|$ to reveal the definition of the helicity operator. With helicity equal to chirality in the massless limit, we see the first equation describes a left-handed fermion and the second describes a right-handed fermion. Both equations is based on the relativistic energy-momentum relation, $E^2 = \boldsymbol{p}^2$ so the negative energy, (reversed momentum) solutions are equally described,

$$\begin{aligned} (-E)\chi = -\boldsymbol{\sigma}\cdot(-\boldsymbol{p})\,\chi \quad &\Longrightarrow \quad \boldsymbol{\sigma}\cdot(-\hat{\boldsymbol{p}})\,\chi = +\chi \qquad = v_\uparrow\,(v_R) \\ (-E)\phi = \boldsymbol{\sigma}\cdot(-\boldsymbol{p})\,\phi \quad &\Longrightarrow \quad \boldsymbol{\sigma}\cdot(-\hat{\boldsymbol{p}})\,\phi = -\phi \qquad = v_\downarrow\,(v_L) \end{aligned}$$

In summary the Weyl doublets are:

$$\chi = \begin{pmatrix} u_L \\ v_R \end{pmatrix} \qquad \phi = \begin{pmatrix} u_R \\ v_L \end{pmatrix} \tag{79}$$

This pair of decoupled equations were first found by Weyl (1929) but dismissed because parity is not conserved in either doublet. For example, χ describes u_L and v_R whereas $P\,u_L = u_R$, which is not in the χ doublet. .

Weyl vs. Dirac representation

Once the requirement of a 4×4 matrices was established, the two Weyl solutions come together in a parity-conserving quartet of the Dirac spinor. This forms the Weyl representation of the γ matrices,

$$\textbf{Weyl repr.}\quad \psi_{(\mathrm{W})} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$
$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma^a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{80}$$

which can be compared with the Dirac representation.

$$\textbf{Dirac repr.}\quad \psi_{(\mathrm{D})} = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$
$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{81}$$

Each representation is equally legitimate. Both are based on, and satisfy the anti-commutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,,$$

where $g^{00} = +1$, $g^{aa} = -1$ and $g^{ab} = 0$.
To switch between the two representations,

$$\begin{aligned} \gamma^\mu_{(\mathrm{D})} &= A\gamma^\mu_{(\mathrm{W})}A^{-1} \\ \psi_{(\mathrm{D})} &= A\psi_{(\mathrm{W})} \\ A = A^{-1} &= \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} \end{aligned} \tag{82}$$

Using this transformation, the Dirac spinor is shown to contain orthogonal combinations of the Weyl spinors.

$$\psi_{(\mathrm{D})} = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi + \phi \\ \chi - \phi \end{pmatrix}. \tag{83}$$

9 $(V \pm A)$ structure of the Dirac current

Define chiral projection operators,

$$\begin{aligned} P_R &= \tfrac{1}{2}(1+\gamma^5) & P_L &= \tfrac{1}{2}(1-\gamma^5) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} & &= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \end{aligned} \tag{84}$$

such that,

$$\begin{aligned} P_R u_R = u_R \quad P_R u_L = 0 \quad P_L u_R = 0 \quad P_L u_L = u_L \\ P_R v_R = 0 \quad P_R v_L = v_L \quad P_L v_R = v_R \quad P_L v_L = 0 \end{aligned}$$

Any spinor can be split into left-chiral and right-chiral parts,

$$\begin{aligned} P_R\psi = \psi_R \qquad P_L\psi = \psi_L \\ \psi = \tfrac{1}{2}\left(1+\gamma^5\right)\psi + \tfrac{1}{2}\left(1-\gamma^5\right)\psi\,. \end{aligned}$$

Applying the projection operators in the Weyl quartet shows,

$$P_{R(L)} = \tfrac{1}{2}\left(1 \pm \gamma^5\right) = \frac{1}{2}\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \pm \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\right]$$
$$P_L = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}_{\nwarrow 4\times 4\ \text{matrix}}$$

$$P_L\psi = \chi \qquad P_R\psi = \phi$$

confirming that projection operators identify chiral doublets. In the Dirac representation, the application of Eq. 84 to Eq. 83 is less trivial but yields the same result.

Parity

Next, we identify the parity operator $P = \gamma^0$ because it modifies the Dirac equation to the parity-transformed coordinates:

$$\begin{aligned} \psi(\boldsymbol{x},t) &= \gamma^0\psi'(-\boldsymbol{x},t) \\ \left[i\gamma^\mu\partial_\mu - m\right]\gamma^0\psi' &= 0 \end{aligned} \tag{85}$$

Expand and flip in the inverted axis, $\frac{\partial\psi'}{\partial x} = -\frac{\partial\psi'}{\partial x'}$ etc.

$$i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t} - i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x'} - i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y'} - i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z'} - m\gamma^0\psi' = 0$$

Use the anti-commutation relation, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$,

$$i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t} + i\gamma^0\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^0\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^0\gamma^3\frac{\partial\psi'}{\partial z'} - m\gamma^0\psi' = 0$$

Then pre-multiply by γ^0 and use $\gamma^0\gamma^0 = 1$,

$$\begin{aligned} i\gamma^0\frac{\partial\psi'}{\partial t} + i\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^3\frac{\partial\psi'}{\partial z'} - m\psi' &= 0 \\ \left[i\gamma^\mu\partial'_\mu - m\right]\psi' &= 0 \end{aligned}$$

recovering the Dirac equation in the parity-transformed coordinates and validating Eq. 85. Applying the parity operator, γ^0 to the Dirac spinor in their rest frame, we justify that Dirac particles have positive "intrinsic" (at rest) parity and the antiparticles have negative parity.

$$Pu_1 = \gamma^0 u_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix} = +u_1$$

$$Pv_1 = \gamma^0 v_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2m} \end{pmatrix} = -v_1\,.$$

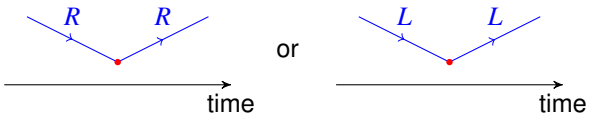
Now that we have the tool to project-out the orthogonal left and right parts of the Dirac spinor, we reconsider the Dirac probability current (Eq. 54),

$$j^\mu = \overline{\psi}\gamma^\mu\psi = \overline{\psi}_R\gamma^\mu\psi_R + \underbrace{\overline{\psi}_R\gamma^\mu\psi_L + + \overline{\psi}_L\gamma^\mu\psi_R}_{\text{Chirality not conserved.}} + \overline{\psi}_L\gamma^\mu\psi_L\,.$$

By expanding out the second or third term,

$$\begin{aligned} \overline{\psi}_R\gamma^\mu\psi_L &= \tfrac{1}{2}\psi^\dagger\left(1+\gamma^5\right)\gamma^0\gamma^\mu\,\tfrac{1}{2}\left(1-\gamma^5\right)\psi \\ &= \tfrac{1}{4}\psi^\dagger\gamma^0\gamma^\mu\underbrace{\left(1+\gamma^5\right)\left(1-\gamma^5\right)}_{1-\left(\gamma^5\right)^2 = 0}\psi \end{aligned}$$

the chirality non-conserving currents vanish. Chirality is a conserved quantity.



An expansion of the first and fourth terms,

$$\begin{aligned} \overline{\psi}_R\gamma^\mu\psi_R &= \overline{\psi}\left(\gamma^\mu + \gamma^\mu\gamma^5\right)\psi \\ \overline{\psi}_L\gamma^\mu\psi_L &= \overline{\psi}\underset{V}{\left(\gamma^\mu\right)} - \underset{A}{\left(\gamma^\mu\gamma^5\right)}\psi \end{aligned} \tag{86}$$

shows the Dirac current to have two constituents, labelled *V*ector and *A*xialvector. To justify these labels we consider the transformation of Dirac spinors under the parity transformation.

$$\psi'(x') = \gamma^0\psi(x) \tag{87}$$

and because $\gamma^{0\dagger} = \gamma^0$,

$$\overline{\psi}'(x') = \overline{\psi}\,\gamma^0\,.$$

Applying this parity transform to the Dirac current,

$$P(j^\mu) = \overline{\psi}'\,\gamma^\mu\psi' = \overline{\psi}\,\gamma^0\,\gamma^\mu\,\gamma^0\psi$$

and use the anticommutation relation to resolve the effect of the parity transformation on each part of Eq. 86 ,

$$\begin{aligned} &\text{for } V \quad \gamma^0\gamma^0\gamma^0 = +\gamma^0 \qquad \gamma^0\gamma^a\gamma^0 = -\gamma^a \\ &\text{for } A \quad \gamma^0\gamma^0\gamma^5\gamma^0 = -\gamma^0\gamma^5 \qquad \gamma^0\gamma^a\gamma^5\gamma^0 = +\gamma^a\gamma^5 \end{aligned}$$

we confirm that the $\overline{\psi}\,\gamma^\mu\psi$ term transforms as a *V*ector and the $\overline{\psi}\,\gamma^\mu\gamma^5\psi$ term transforms as an *A*xialvector. Parity is preserved as both left- and right-handed components are conveyed in the $(V+A+V-A)$ current.

Towards QED

Malcolm John
C4: particle physics

These notes apply the concept of the Dirac spinor to leptons in the context of Lagrangian mechanics. This requires inclusion of the photon field. This is not a comprehensive QFT treatment, rather they highlight the important features that allow rate calculations.

1 Gauge invariance

In classical mechanics, the Lagrangian is a scalar quantity,

$$L = \underset{\text{kinetic energy}}{T} - \underset{\text{potential energy}}{V}$$

from which equations of motion are derived using the Euler-Lagrange requirement,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{1}$$

where q_i is the general coordinates of an object and $\dot{q}_i = \frac{dq_i}{dt}$. With quantum mechanics, where objects are are not at discrete locations but are part of a continuous field, $\phi(x^\mu)$ we write,

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial\phi/\partial x^\mu)}\right) - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \tag{2}$$

where $\dot{q}$ has become $\partial_\mu\phi = \partial\phi/\partial x^\mu$ and $\mathscr{L}$ is a Lagrangian "density" such that $L = \int \mathscr{L} d^3x$. A simple example is,

$$\begin{aligned}
\mathscr{L} &= \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 \qquad (3)\\
\frac{\partial \mathscr{L}}{\partial \phi} &= -\,m^2\phi\\
\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu\phi\right)} &= \tfrac{1}{2}\left[(\partial^\mu\phi) + g_{\mu\mu}\left(\partial_\mu\phi\right)\right]\\
\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\right) &= \tfrac{1}{2}\left[\partial_\mu\partial^\mu\phi + \partial_\mu\partial^\mu\phi\right] = \partial_\mu\partial^\mu\phi\,.
\end{aligned}$$

Thus, Euler-Lagrange (Eq. 2) gives a Klein Gordon equation,

$$\begin{aligned}
&\partial_\mu\partial^\mu\phi + m^2\phi = 0\\
&\frac{\partial^2\phi}{\partial t} - \nabla^2\phi + m^2\phi = 0 \quad (-E^2 + \boldsymbol{p}^2 + m^2 = 0)\,.
\end{aligned}$$

The Lagrangian density for electromagnetism has a free field part and a part interaction with a charged current, j^μ.

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu\,, \tag{4}$$

where A_μ is a real vector potential and $F_{\mu\nu}$ is an antisymmetric field-strength tensor $F_{\mu\nu} = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$. Treating A_μ and their spacetime derivatives $\partial_\mu A_\nu$ as Lagrangian variables,

$$\begin{aligned}
-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\\
&= -\frac{1}{4}\Big[\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\nu A^\mu\\
&\qquad \underbrace{-\,\partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A^\mu}_{\text{relabel terms 3 and 4: } \mu\longleftrightarrow\nu}\Big]\\
&= -\frac{1}{2}\left(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu\right)\\
&= -\frac{1}{2}\left(g_{\mu\mu}g_{\nu\nu}\left(\partial_\mu A_\nu\partial_\mu A_\nu - \partial_\mu A_\nu\partial_\nu A_\mu\right)\right)\\
\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu A_\nu\right)} &= -\frac{1}{2}\left(2g_{\mu\mu}g_{\nu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right)\\
&= -\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\\
\therefore\quad \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu A_\nu\right)}\right) &= -\partial_\nu F^{\mu\nu} \qquad (5)\\
\text{and}\quad \frac{\partial \mathscr{L}}{\partial A_\mu} &= -j^\mu \qquad (6)
\end{aligned}$$

Employing Eq. 5, 6 in the Eular-Lagrange equation (Eq. 2) gives Maxwell's equations,

$$\partial_\nu F^{\mu\nu} = j^\mu\,, \tag{7}$$

and thus the continuity equation of electromagnetism,

$$\begin{aligned}
\partial_\mu j^\mu &= \partial_\mu\partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\\
&= 0\,,
\end{aligned} \tag{8}$$

due to the equivalence of the second derivative, $\partial_\mu\partial^\nu = \partial_\nu\partial^\mu$.

The Lagrangian, Eq. 4 has a symmetry group of unobservable transformations: the addition of the gradient of an arbitrary scalar function,

$$A_\mu \longrightarrow A_\mu + \partial_\mu\chi(x)\,, \tag{9}$$

leaves the field tensor unchanged. This is shown by,

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\\
&\longrightarrow \partial_\mu\left(A_\nu + \partial_\nu\chi(x)\right) - \partial_\nu\left(A_\mu + \partial_\mu\chi(x)\right)\\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \cancel{\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)}\chi(x)\\
&= F_{\mu\nu}\,.
\end{aligned} \tag{10}$$

This says the same electromagnetism is described by an infinite set of slightly-different four-vector potentials. We choose one by initially imagining a vector field $(A')^\mu$ whose differential is some spacetime function, $\partial_\mu(A')^\mu = f(x)$. It is always possible to have a $\chi(x)$ whose second differential equals $f(x)$.

$$\begin{aligned}
\partial_\mu\partial^\mu\chi(x) = f(x) &= \partial_\mu(A')^\mu\\
&= \partial_\mu\left(A^\mu + \partial^\mu\chi(x)\right)\\
&= \partial_\mu A^\mu + \partial_\mu\partial^\mu\chi(x)\\
\partial_\mu A^\mu &= 0
\end{aligned} \tag{11}$$

which is a Lorentz-invariant choice for algebraic convenience known as the Lorenz gauge, or Lorentz condition.

For the Dirac equation, a reasonable Lagrangian might be,

$$\mathscr{L}_{\mathrm{D}} = i\,\overline{\psi}\,\gamma^{\mu}\partial_{\mu}\psi - \overline{\psi}\,m\,\psi \tag{12}$$

And apply Euler-Lagrange, with respect to either ψ or $\overline{\psi}$,

$$\begin{aligned}
\frac{\partial\mathscr{L}}{\partial\psi} &= -\,\overline{\psi}\,m \\
\frac{\partial\mathscr{L}}{\partial\overline{\psi}} &= i\,\gamma^{\mu}\partial_{\mu}\psi - m\psi \\
\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\psi\right)} &= i\,\overline{\psi}\,\gamma^{\mu} \\
\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\overline{\psi}\right)} &= 0
\end{aligned}$$

yields the familiar Dirac equation, and its Hermitian conjugate.

$$\begin{aligned}
\text{E.L.}_{(\overline{\psi})} &: \quad i\gamma^{\mu}\partial_{\mu}\psi - m\,\psi = 0 \\
\text{E.L.}_{(\psi)} &: \quad i\partial_{\mu}\,\overline{\psi}\,\gamma^{\mu} + \overline{\psi}\,m = 0
\end{aligned} \tag{13}$$

As with electromagnetism, consider the the invariance of the Lagrangian variables with respect to an unobservable transformation. Specifically, we expect physics to be unchanged when a wavefunction is multiplied by a phase,

$$\begin{gathered}
\psi \to e^{i\alpha}\psi \\
\text{requiring} \quad |\psi|^2 \to |\psi|^2 .
\end{gathered}$$

This is called global "gauge" (phase) invariance and it describes the symmetry under the $U(1)$, Lie group operators, $U(\alpha) = e^{i\alpha}$. The symmetry of this unitary, Abelian group implies (by Noether) a conserved quantity and to understand this, we consider the Lagrangian under infinitesimal phase rotation, $\psi \to \psi + i\alpha\psi$ and require $\mathscr{L}\left(\psi, \partial_{\mu}\psi\right)$ to be unchanged,

$$\begin{aligned}
0 = \delta\mathscr{L} &= \left(\frac{\partial\mathscr{L}}{\partial\psi}\right)\underset{\uparrow}{\delta\psi} + \left(\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\psi\right)}\right)\delta\underset{\uparrow}{\left(\partial_{\mu}\psi\right)} \\
&\qquad\qquad i\alpha\,\psi \qquad\qquad\qquad i\alpha\,\partial_{\mu}\psi \\
&= i\alpha\left[\underbrace{\left(\frac{\partial\mathscr{L}}{\partial\psi} - \partial_{\mu}\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\psi\right)}\right)\right)}_{=\,0 \text{ by Eq. 2}}\psi + \partial_{\mu}\underbrace{\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\psi\right)}\psi\right)}_{i\,\overline{\psi}\,\gamma^{\mu}\psi}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i\partial_{\mu}j^{\mu}
\end{aligned}$$

where the Dirac probability j^{μ} current is reidentified. Thus, $\partial_{\mu}j^{\mu}$ must equal zero to maintain invariance under an unobservable global gauge transformation. This is the root of the Dirac continuity equation.

At this point, a coupling coefficient is introduced to set the scale of the Dirac probability current.

$$j^{\mu} = q\,\overline{\psi}\,\gamma^{\mu}\psi$$

As we shall see in a moment, this is electric charge. The spatial integral of the probability density multiplied by this elemental charge,

$$Q = \int j^{0} d^{3}x$$

is identified as total charge, which is a conserved quantity as a direct result of requiring global $U(1)$ gauge invariance.

The more general gauge condition is *local gauge invariance* where the change of phase is dependant on the coordinate,

$$\begin{gathered}
\psi \to e^{i\alpha(\boldsymbol{x})}\psi \\
\text{requiring} \quad |\psi|^2 \to |\psi|^2 .
\end{gathered}$$

An initial trial suggests that local gauge transformations fail to show Lagrangian invariance,

$$\begin{aligned}
\partial_{\mu}\psi \longrightarrow\ & \partial_{\mu}\left(e^{i\alpha(\boldsymbol{x})}\psi\right) \\
&= \underbrace{e^{i\alpha(\boldsymbol{x})}\partial_{\mu}\psi}_{\checkmark} + \underbrace{i\,\partial_{\mu}\alpha(\boldsymbol{x})}_{\neq 0}\,e^{i\alpha(\boldsymbol{x})}\psi
\end{aligned}$$

To have local gauge invariance we need,

$$D_{\mu}\left(e^{i\alpha(\boldsymbol{x})}\psi\right) = e^{i\alpha(\boldsymbol{x})}D_{\mu}\psi .$$

which is achieved with the *gauge-covariant Derivative*,

$$D_{\mu} = \partial_{\mu} + iqA_{\mu} . \tag{14}$$

This describes the vector field A_{μ} coupling to the Dirac spinor with coupling strength q. Simultaneously the local phase shift is compensated for with a simultaneous adjustment within the vector field,

$$A_{\mu} \longrightarrow A_{\mu} - \frac{1}{q}\partial_{\mu}\alpha(\boldsymbol{x}). \tag{15}$$

Exactly the electromagnetic gauge transformation of Eq. 9!

$$\begin{aligned}
D_{\mu}\left(e^{i\alpha(\boldsymbol{x})}\psi\right) &= \left(\partial_{\mu} + iqA_{\mu}\right)e^{i\alpha(\boldsymbol{x})}\psi \\
&= i\,\partial_{\mu}\alpha(\boldsymbol{x})\,e^{i\alpha(\boldsymbol{x})}\psi + e^{i\alpha(\boldsymbol{x})}\partial_{\mu}\psi \quad + iqe^{i\alpha(\boldsymbol{x})} \quad \overbrace{A_{\mu}\psi} \\
&= e^{i\alpha(\boldsymbol{x})}\left(\partial_{\mu}\psi + i\,\partial_{\mu}\alpha(\boldsymbol{x})\,\psi\right) + iqe^{i\alpha(\boldsymbol{x})}\overbrace{\left[A_{\mu} - \frac{1}{q}\partial_{\mu}\alpha(\boldsymbol{x})\right]}\psi \\
&= e^{i\alpha(\boldsymbol{x})}\left(\partial_{\mu} \cancel{+ i\partial_{\mu}\alpha(\boldsymbol{x})} + iqA_{\mu} \cancel{- i\partial_{\mu}\alpha(\boldsymbol{x})}\right)\psi \\
&= e^{i\alpha(\boldsymbol{x})}D_{\mu}\psi
\end{aligned}$$

Thus, the fundamental connection between electromagnetism and the dynamics of charged spin-$\frac{1}{2}$ particles is revealed.

$$\begin{aligned}
\mathscr{L}_{\mathrm{D}} &= i\,\overline{\psi}\,\gamma^{\mu}D_{\mu}\psi - m\,\overline{\psi}\,\psi \\
&= i\,\overline{\psi}\,\gamma^{\mu}\left(\partial_{\mu} + iqA_{\mu}\right)\psi - m\,\overline{\psi}\,\psi
\end{aligned} \tag{16}$$

The Dirac and electromagnetism Lagrangians merge into the QED Lagrangian,

$$\mathscr{L}_{\mathrm{QED}} = \underbrace{\overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi}_{\text{spinor}} - \underbrace{q\,\overline{\psi}\,\gamma^{\mu}\psi A_{\mu}}_{\text{interaction}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{free field}} \tag{17}$$

In the Klein Gordon Lagrangian, Eq. 3, there is a term $\frac{1}{2}m^2\phi^2$ that represent the mass of the boson associated with the scalar field, ϕ. It is tempting to introduce an equivalent term for the vector field of QED, i.e. $\frac{1}{2}m^2A_{\mu}A^{\mu}$. However, such a term is not

invariant under the required gauge transformation of Eq. 15,

$$\begin{aligned}\tfrac{1}{2}m^2A_\mu A^\mu &\longrightarrow \tfrac{1}{2}m^2\left(A_\mu-\frac{1}{q}\partial_\mu\alpha(\boldsymbol{x})\right)\left(A^\mu-\frac{1}{q}\partial^\mu\alpha(\boldsymbol{x})\right)\\ &\longrightarrow \underbrace{\tfrac{1}{2}m^2A_\mu A^\mu}_{\checkmark}-\underbrace{\tfrac{m^2}{2q}\left(\partial_\mu\alpha(\boldsymbol{x})A^\mu-A_\mu\partial^\mu\alpha(\boldsymbol{x})\right)+\ldots}_{\neq 0}\end{aligned}$$

Thus the vector boson of the A^μ vector field, the photon, is required to be massless by the requirement of gauge invariance.

Charge conjugation

With the Dirac current now understood as an electric current, the charge-parity of the Dirac spinors is considered. We start with "interacting" Dirac equation derived from applying Euler-Lagrange (Eq. 2) the covariant derivative Lagrangian (Eq. 16),

$$\begin{aligned}&\left[i\,\gamma^\mu\left(\partial_\mu+iqA_\mu\right)-\ m\right]\psi=0\\ \text{or}\qquad&\left[\gamma^\mu p_\mu-q\gamma^\mu A_\mu-m\right]\psi=0\end{aligned}\tag{18}$$

In terms of γ matrices the charge conjugation operator is

$$C=i\gamma^2=\begin{pmatrix}&&&+1\\&&-1&\\&-1&&\\+1&&&\end{pmatrix}.$$

Two applications of C is the identity, $CC=\left(i\gamma^2\right)^2=I$. It is useful to calculate the anticommutation relation for γ^2 and the complex conjugates of the γ matrices, which have real elements for $\mu\neq 2$, and all imaginary elements for $\mu=2$.

$$\begin{aligned}\gamma^2(\gamma^\mu)^*&=\ \gamma^2\gamma^\mu=-\gamma^\mu\gamma^2 &(\mu\neq 2)\\ &=-\gamma^2\gamma^2 &(\mu=2)\\ &=-\gamma^2\gamma^\mu\end{aligned}\tag{19}$$

Apply complex conjugation and premultiply by $C=i\gamma^2$,

$$\begin{aligned}&\left[-i\,(\gamma^\mu)^*\left(\partial_\mu-iqA^\mu\right)-\ m\right]\psi^*=0 &\text{(c.c.)}\\ &\left[\underbrace{\gamma^2(\gamma^\mu)^*}_{-\gamma^\mu\gamma^2}\left(\partial_\mu-iqA^\mu\right)-i\gamma^2m\right]\psi^*=0 &\text{(Eq. 19)}\\ &\left[i\,\gamma^\mu\left(\partial_\mu-iqA^\mu\right)-m\right]\underbrace{\left(i\gamma^2\psi^*\right)}_{\psi'=C\psi}=0\end{aligned}$$

(A^μ: opposite charge)

which is recasting Eq. 18 for a charge-conjugate particle ψ' with identical mass but opposite charge. Applying the charge conjugation to one of the Dirac spinor solutions, u_1 gives,

$$i\gamma^2u_1^*=\begin{pmatrix}&&&+1\\&&-1&\\&-1&&\\+1&&&\end{pmatrix}\begin{pmatrix}E+m\\0\\p_z\\p_x+ip_y\end{pmatrix}=\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E+m\end{pmatrix}=v_2.$$

Similarly $Cu_2=v_1$: the charge conjugate of a positive energy-electron is a positive-energy positron, and vice versa.

2 Electron dipole moment

Pre-multiply Eq. 18 by $\gamma^0=\beta$ and use $\gamma^a=\beta\alpha^a$ and $\beta=\begin{pmatrix}I&\\&-I\end{pmatrix}$.

$$\begin{aligned}\left[E-\ \alpha^ap^a-qA_0+q\alpha^aA^a-\beta\,m\right]\ \psi&=0\\ \left[E-qA_0-\beta\,m\right]\ \psi&=\left[\alpha^ap^a-q\alpha^aA^a\right]\psi\end{aligned}\tag{20}$$

Identifty A_0 as the scalar potential, $\frac{\phi}{c}$,

$$\begin{bmatrix}E-m-q\frac{\phi}{c}\\E+m-q\frac{\phi}{c}\end{bmatrix}\begin{pmatrix}u_A\\u_B\end{pmatrix}=\left[\begin{pmatrix}&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&\end{pmatrix}\cdot\mathbf{p}-q\begin{pmatrix}&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&\end{pmatrix}\cdot\mathbf{A}\right]\begin{pmatrix}u_A\\u_B\end{pmatrix}$$

Thus forming two simultaneous equations,

$$\begin{aligned}\left(E-m-q\tfrac{\phi}{c}\right)u_A&=(\boldsymbol{\sigma}\cdot\mathbf{p}-q\boldsymbol{\sigma}\cdot\mathbf{A})\,u_B\\ \left(E+m-q\tfrac{\phi}{c}\right)u_B&=(\boldsymbol{\sigma}\cdot\mathbf{p}-q\boldsymbol{\sigma}\cdot\mathbf{A})\,u_A\end{aligned}\tag{21}$$

Multiply the top line by $\left(E+m-q\frac{\phi}{c}\right)$, then sub. in the second,

$$\begin{aligned}\left(E-m-q\tfrac{\phi}{c}\right)\left(E+m-q\tfrac{\phi}{c}\right)u_A&=(\boldsymbol{\sigma}\cdot\mathbf{p}+q\boldsymbol{\sigma}\cdot\mathbf{A})\left(E+m-qA^0\right)u_B\\ &=(\boldsymbol{\sigma}\cdot\mathbf{p}-q\boldsymbol{\sigma}\cdot\mathbf{A})\,(\boldsymbol{\sigma}\cdot\mathbf{p}-q\boldsymbol{\sigma}\cdot\mathbf{A})\,u_A\end{aligned}$$

Expand the right-hand side and use these identities,

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\mathbf{a})^2&=|\mathbf{a}|^2\ ,\\ (\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b})&=\mathbf{a}\cdot\mathbf{b}+i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})\,,\\ (\nabla\times\boldsymbol{a})\psi&=\nabla\times(\boldsymbol{a}\psi)+\boldsymbol{a}\times(\nabla\psi)\,.\end{aligned}$$

$$\begin{aligned}&\left[(\boldsymbol{\sigma}\cdot\mathbf{p})^2-q^2\,(\boldsymbol{\sigma}\cdot\mathbf{A})^2+q\left[(\boldsymbol{\sigma}\cdot\mathbf{p})\,(\boldsymbol{\sigma}\cdot\mathbf{A})+(\boldsymbol{\sigma}\cdot\mathbf{A})\,(\boldsymbol{\sigma}\cdot\mathbf{p})\right]\right]u_A\\ &\left[\ |\mathbf{p}|^2\ +q^2\ \ |\mathbf{A}|^2\ -q\left[2\mathbf{p}\cdot\boldsymbol{A}+i\boldsymbol{\sigma}\cdot(\mathbf{p}\times\mathbf{A}+\mathbf{A}\times\mathbf{p})\right]\ \right]u_A\\ &(\mathbf{p}-q\mathbf{A})^2\,u_A\ -q\boldsymbol{\sigma}\cdot\underbrace{(\boldsymbol{\nabla}\times\mathbf{A}u_A+\mathbf{A}\times\boldsymbol{\nabla}u_A)}_{(\boldsymbol{\nabla}\times\mathbf{A})\,u_A\ \text{magnetic field}}\\ &\left[(\mathbf{p}-q\mathbf{A})^2-q\,\boldsymbol{\sigma}\cdot\ \boldsymbol{B}\right]u_A\end{aligned}$$

The Hamiltonian is derived for the non-relativistic case (kinetic energy, T) by multiplying-out, but neglecting the $1/c^2$ term.

$$\begin{aligned}&\left(E-m-q\tfrac{\phi}{c}\right)\left(E+m-q\tfrac{\phi}{c}\right)u_A\\ =\ &\left[\underbrace{E^2-m^2}_{p^2\approx 2mT}-2qE\tfrac{\phi}{c}+q^2\cancelto{\approx 0}{\tfrac{|\phi|^2}{c^2}}\ \right]u_A\\ \approx\ &2m\left(T-q\frac{\phi}{c}\right)u_A\end{aligned}$$

(E^2-m^2: $E\approx m$)

Together, this is the Schrödinger-Pauli Hamiltonian for a non-relativistic spin-$\frac{1}{2}$ particle of charge q in an electromagnetic field,

$$Tu_A=\left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2+q\frac{\phi}{c}-\frac{q}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]u_A\tag{22}$$

The first term is kinetic energy of the electron in terms of the canonical momentum. The second is the energy from any electric potential. The third term is the contribution from the electron spin interaction with any magnetic field.

As intrinsic spin is defined $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$, this term is better written,

$$\frac{q}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} \longrightarrow \mu_e \cdot \boldsymbol{B} \;,\; \mu_e = -g_e \underbrace{\frac{e}{2m_e}}_{\substack{\mu_\mathrm{B} \\ \text{Bohr magenton}}} \mathbf{S}\,, \quad g_e = 2.$$

for the electron of charge $-e$ and mass m_e; g_e is the gyromagnetic ratio. In S.I. units, the Bohr magneton, $|\mu_\mathrm{B}| = e\hbar/(2m_e) = (9.274 \times 10^{-24})$ joules/Tesla, or (5.8×10^{-5}) eV/Tesla.

Presumption based in atomic physics A magnetic moment is the strength of the dipole formed by a current, I in a loop of area $\mathbf{a}$,

$$\boldsymbol{\mu} = I\mathbf{a}$$

For a non-relativistic[1] description of the electron orbiting a hydrogen nucleus, the current is $I = -e/T$, where T is the period for one orbit $T = \dfrac{2\pi r_0}{v}$ and $\mathbf{a} = \pi r_0^2$. Inserting the electron mass and using the definition of orbital angular momentum, $\mathbf{L} = m\mathbf{r}\times\mathbf{v} = mvr_0$ in circular motion,

$$\boldsymbol{\mu}_H = \frac{-ev}{2\pi r_0}\pi r_0^2 \;= -\frac{e}{2m_e}m_e v r_0 \;=\; -\underbrace{\frac{e}{2m_e}}_{\substack{\mu_\mathrm{B} \\ \text{Bohr magneton}}} L\,.$$

Angular momentum is quantised with total orbital angular momentum $|\mathbf{L}|^2\psi = l(l+1)\psi$ and $L_z\psi = m\psi$, $m = -l, ..., +l$. So the hydrogen magnetic moment depends on the atomic 'shell'.

$$\begin{aligned} &S\text{-shell} : l = 0,\; L = \;\;0,\; m = 0, && \mu_H = \;0 \\ &P\text{-shell} : l = 1,\; L = \sqrt{2},\; m = -1, 0, +1, && \mu_H = -\mu_\mathrm{B},\; 0,\; +\mu_\mathrm{B} \end{aligned}$$

within this, the electron is $S = \frac{1}{2}$, $m = -\frac{1}{2}, +\frac{1}{2}$ so there is a further splitting: the anomalous Zeeman effect. The surprise is that the strength of the electron's magnetic moment is twice that expected from the analogy of orbital angular momentum.

Measurements of muon anomalous magnetic moment A charged particle moving in a magnet field will move in a circle by the Lorentz force, $F_\mathrm{L} = q\mathbf{v}\times\mathbf{B}$. The centripetal force is $F_C = m\omega^2/r$. Equating, $qvB = mv\omega$ (using $v = \omega r$) and the cyclotron frequency for an muon of charge e and mass m_μ is,

$$\omega_c = \frac{e}{m_\mu}B\,. \tag{23}$$

Then because of its magnetic moment, the muon feels a torque perpendicular to the moment vector and the magnetic field,

$$\begin{aligned} \text{torque} &= \boldsymbol{\mu}_\mu \times \mathbf{B} \\ &= -\frac{g_\mu e}{2m_\mu}|\mathbf{S}||\mathbf{B}|\sin\theta \end{aligned} \tag{24}$$

using the $\sin\phi$ definition of the cross product. The torque cause the angular momentum vector (specifically the component perpendicular to the field) to precess around the field lines with frequency,

$$\omega_s = \frac{\text{torque}}{|\mathbf{S}|\sin\theta} \tag{25}$$

$$= -\frac{g_\mu e}{2m_\mu}|\mathbf{B}| \tag{26}$$

Importantly, as $g_\mu = 2$, $\omega_s = \omega_c$ and the Larmor spin precession will rotate the lepton's magnetic moment exactly in phase with the orbital rotation by the Lorentz force. This means that, if the muon enters the magnetic field fully polarised with the spin aligned with the momentum vector, it will go around it a circle with the muon remaining fully polarised.

However, g_e and g_μ are not exactly 2. The first-order correction (involving an additional, internal photon line) was calculated by Schwinger in 1948 to be $\frac{\alpha}{2\pi} = 0.001162$. Higher order corrections lead to an anomalous magnetic moment.

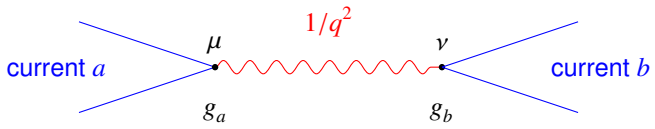
$$a_\mu = \omega_s - \omega_c = \frac{g_\mu - 2}{2}B\,.$$

Nowadays, these are some of the most precisely calculated and measured quantities in physics. For example the PDG records,

$$\frac{g_e - 2}{2} = 0.00115965218091 \pm 0.00000000000026\,.$$

3 The simplest QED matrix element

With the interaction between spinor and photon defined, the *matrix element* for QED can be attempted. We consider the transition from an initial to a final state via a propagator term proportional to $1/q^2$, consistent with perturbation theory,



where, from the Lagrangian, the QED interactions are of the form, $g_a = ej^\mu A_\mu$ and $g_b = ej^\nu A_\nu$ with the electric charge quantified as the elementary charge of the electron, $e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$ ($e^2 = 4\pi\alpha$ in natural units).

The connection between A_μ at one end of the propagator and A_ν at the other must be considered. The virtual photon (off mass shell) has four possible polarisations,

$$A^\mu = \varepsilon^\mu_{(k)} e^{iq\cdot x} \tag{27}$$

$$\varepsilon_{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \varepsilon_{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \varepsilon_{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \varepsilon_{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

scaler lefthanded righthanded longatudinal

The polarisation vector is Lorentz gauge invariant:

$$p_\mu \varepsilon^\mu = 0$$

[1] Using $\frac{1}{2}mv^2 \sim \frac{e^2}{4\pi\varepsilon_0 r}$ and the $r(mv) > \frac{\hbar}{2}$, $\langle\beta_e^H\rangle \sim 4\alpha \sim 0.03$ (non-rel.)

Virtual photons have $q^2 \neq 0$ and thus can have both longitudinal and transverse polarisations. The matrix element is the sum over all possible polarisation states (this is not obvious),

$$\sum_k \varepsilon^\mu_{(k)} e^{iq\cdot x} \left(\varepsilon^\mu_{(k)} e^{iq\cdot x}\right)^* = -g_{\mu\nu}\,. \tag{28}$$

Finally the QED matrix element using the Feynman rules is,

$$-i\mathscr{M}_{fi} \;=\; (ie)^2 \;\underbrace{j^\mu}_{\text{initial current}} \;\overbrace{\frac{-ig_{\mu\nu}}{q^2}}^{\text{propagator}} \;\underbrace{j^\nu}_{\text{final current}} \tag{29}$$

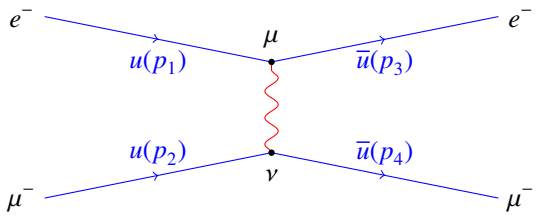
with $j^\mu = \overline{\psi}\,\gamma^\mu\psi$. Note: this is the scalar product of two fourvectors (fourcurrents) and is thus manifestly Lorentz invariant.

Feynman rules for QED

Incoming fermion	u	
Outgoing fermion	$\overline{u}$	
Incoming antifermion	$\overline{v}$	
Outgoing antifermion	v	
Incoming photon	ε_μ	
Outgoing photon	ε^*_μ	
Photon propagator	$\frac{-ig_{\mu\nu}}{q^2}$	
Fermion propagator	$\frac{i\left(\gamma^\mu q_\mu + m\right)}{q^2 - m^2}$	
Vertex factor	$ie\gamma^\mu$	
Matrix element	$-i\mathscr{M} =$	product of relevant factors

4 $e^-\mu^- \to e^-\mu^-$ scattering

What follows is a step-by-step calculation of the matrix element squared in this most-elementary QED scattering process. The Feynman diagram is,



for which the amplitude is,

$$-i\mathcal{M} = \left[\overline{u}(p_3)ie\gamma^\mu u(p_1)\right]\frac{-ig^{\mu\nu}}{q^2}\left[\overline{u}(p_4)ie\gamma^\nu u(p_2)\right]$$

$$\mathcal{M} = -\frac{e^2}{q^2}\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_4)\gamma_\mu u(p_2)\right]$$

which has: $2 \times 2 \times 2 \times 2 = 16$ spin configurations

The usual case is that fermions are unpolarised, meaning the spin is not known. So we calculate the spin-averaged matrix element, which is summed over all possible interactions and divided by the number of possible incoming states.

$$\overline{|\mathcal{M}|}^2 = \frac{1}{(2s_1+1)(2s_2+1)}\sum_{\text{all spins}}|\mathcal{M}|^2$$

$$= \frac{e^4}{q^4}L^{\mu\nu}_{(\text{electron})}L^{(\text{muon})}_{\mu\nu} \tag{30}$$

$$L^{\mu\nu}_{(\text{electron})} = \tfrac{1}{2}\sum_{\text{electron spins}}\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]^* \tag{31}$$

$$L^{(\text{muon})}_{\mu\nu} = \tfrac{1}{2}\sum_{\text{muon spins}}\left[\overline{u}(p_4)\gamma_\mu u(p_2)\right]\left[\overline{u}(p_4)\gamma_\mu u(p_2)\right]^* \tag{32}$$

Expanding the adjoint spinor in the second term of Eq. 31,

$$\left(u^\dagger(p_3)\gamma^0\gamma^\mu u(p_1)\right)^\dagger \qquad \text{its just a number so}\dagger = *$$
$$\left(u^\dagger(p_1)\gamma^{\mu\dagger}\gamma^0 u(p_3)\right) \qquad \gamma^0 \text{ is Hermitian}$$
$$\left(u^\dagger(p_1)\gamma^0\gamma^\mu u(p_3)\right) \qquad \text{use: } (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$
$$\left(\overline{u}(p_1)\gamma^\mu u(p_3)\right)$$

Thus the complex conjugate serves only to reverse the order of the matrix calculation. Let's examine the multiplication of each specific set of elements (specified by the subscripts),

$$L^{\mu\nu}_{(\text{electron})} = \tfrac{1}{2}\sum_{s_1}\sum_{s_3}\overline{u}_\alpha(p_3)\gamma^\mu_{\alpha\beta}u_\beta(p_1)\overline{u}_\gamma(p_1)\gamma^\nu_{\gamma\delta}u_\delta(p_3)$$

With the labels, this is a product of elements, which are associative so can move around. The summations can be moved because $AC + AD + BC + BD = A(C+D) + B(C+D)$. Thus,

$$= \tfrac{1}{2}\underbrace{\sum_{s_1}u_\beta(p_1)\overline{u}_\gamma(p_1)\,\gamma^\nu_{\gamma\delta}}_{(\not{p}_1+m)_{\beta\gamma}}\underbrace{\sum_{s_3}u_\delta(p_3)\overline{u}_\alpha(p_3)\,\gamma^\mu_{\alpha\beta}}_{(\not{p}_3+m)_{\delta\alpha}}$$

using the completness relations.

Completeness relations

$$u^{(s)} = \sqrt{E+m}\begin{pmatrix}\chi^{(s)}\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi^{(s)}\end{pmatrix} \qquad \chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix},\quad \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\overline{u}^{(s)} = \sqrt{E+m}\begin{pmatrix}\chi^{(s)\dagger} & -\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi^{(s)\dagger}\end{pmatrix}$$

$$\sum_{s=1,2}u^{(s)}(p)\overline{u}^{(s)}(p) = (E+m)\begin{pmatrix}I & -\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} & -\left(\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)^2\end{pmatrix}$$
$$= \begin{pmatrix}E+m & -\boldsymbol{\sigma}\cdot\mathbf{p}\\ \boldsymbol{\sigma}\cdot\mathbf{p} & -E+M\end{pmatrix}$$
$$= \gamma^\mu p_\mu + m$$
$$= \not{p} + m\,.$$

We now have the product of four 4×4 matrices, which gives a 4×4 matrix. The first and last index is the same, β so the resultant 4×4 matrix will be diagonal. So we write,

$$L^{\mu\nu}_{(\text{electron})} = \tfrac{1}{2}\,\text{Tr}\left[(\not{p}_1+m)\,\gamma^\nu\,(\not{p}_3+m)\,\gamma^\mu\right]\,. \tag{33}$$

Trace theorems

$$\mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b$$
$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$$
$$\mathrm{Tr}(\gamma^5) = 0$$

Trace of odd number of γ matrices =0.

Next we use the Trace theorems on Eq. 33,

$$L^{\mu\nu}_{\text{(electron)}} = \tfrac{1}{2}\,\mathrm{Tr}\Big[\underbrace{\not{p}_1\gamma^\nu\not{p}_3\gamma^\mu}_{\text{four }\gamma s} + \underbrace{m\not{p}_1\gamma^\nu\gamma^\mu}_{\text{three }\gamma s=0} + \underbrace{m\gamma^\nu\not{p}_3\gamma^\mu}_{\text{three }\gamma s=0} + \underbrace{m^2\gamma^\nu\gamma^\mu}_{\text{two }\gamma s}\Big]$$
$$= \tfrac{1}{2}\,\mathrm{Tr}\left(\not{p}_1\gamma^\nu\not{p}_3\gamma^\mu\right) + \tfrac{1}{2}\,m^2\,\mathrm{Tr}\left(\gamma^\nu\gamma^\mu\right)$$
$$= \tfrac{1}{2}\left[4\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - p_1\cdot p_3 g^{\mu\nu}\right] + 4m^2 g^{\mu\nu}\right]$$

The same logic is used for the muon (Eq. 32) such that,

$$L^{\mu\nu}_{\text{(electron)}} = 2\Big(\underset{1}{p_1^\mu p_3^\nu} + \underset{2}{p_1^\nu p_3^\mu} - \underset{3}{(p_1\cdot p_3)g^{\mu\nu}} + \underset{4}{m^2 g^{\mu\nu}}\Big)$$
$$L^{\text{(muon)}}_{\mu\nu} = 2\Big(\underset{1}{p_{2\mu}p_{4\nu}} + \underset{2}{p_{2\nu}p_{4\mu}} - \underset{3}{(p_2\cdot p_4)g_{\mu\nu}} + \underset{4}{M^2 g_{\mu\nu}}\Big) \qquad (34)$$

which are multiplied together in Eq. 30, to get,

$$\overline{|\mathcal{M}|}^2 = \frac{8e^4}{q^4}\Big[\underset{A}{(p_3\cdot p_4)(p_1\cdot p_2)} + \underset{B}{(p_3\cdot p_2)(p_1\cdot p_4)} - \underset{C}{m^2 p_4\cdot p_2} - \underset{D}{M^2 p_3\cdot p_1} + \underset{E}{2m^2M^2}\Big] \qquad (35)$$

The details of this multiplication are summarised as:

	1	2	3	4
1	$+A$	$+B$	$-X$	$+D$
2	$+B$	$+A$	$-X$	$+D$
3	$-X$	$-X$	$+4X$	$-4D$
4	$+C$	$+C$	$-4C$	$+4E$

Note the terms $X = (p_3\cdot p_4)(p_1\cdot p_2)$ cancel and the factors of 4 come from $g^{\mu\nu}g_{\mu\nu} = 4$.

This result of of limited use though it will be used later as a starting point for $e^- p \to e - p$ scattering.

5 Ultrarelativistic $e^+e^- \to \mu^+\mu^-$ annihilation

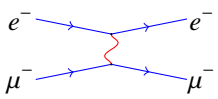
In the relativistic limit, the fermion masses can be neglected,

$$\overline{|\mathcal{M}|}^2 = \frac{8e^4}{q^4}\Big[(p_3\cdot p_4)(p_1\cdot p_2) + (p_3\cdot p_2)(p_1\cdot p_4)\Big] \qquad (36)$$

and Mandelstam variables used,

$$s = (p_1+p_2)^2 = \cancel{p_1^2}^{\,0} + \cancel{p_2^2}^{\,0} + 2p_1\cdot p_2 \simeq 2p_3\cdot p_4$$
$$t = (p_1-p_3)^2 = q^2$$
$$u = (p_1-p_4)^2 = \cancel{p_1^2}^{\,0} + \cancel{p_4^2}^{\,0} + 2p_1\cdot p_4 \simeq -2p_2\cdot p_3$$

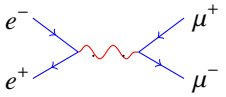
$$\overline{|\mathcal{M}|}^2 = 2e^4\frac{s^2+u^2}{t^2}$$

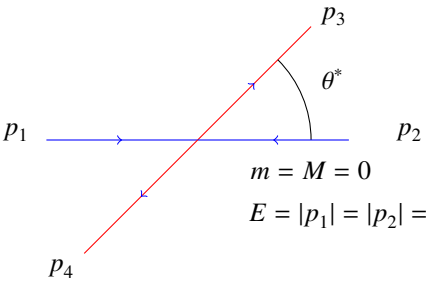


At this point we invoke “crossing symmetry". This relates scattering and annihilation processes that involve the same Feynman elements, just with interchanges of kinematics.

$$s \longleftrightarrow t \qquad (-p_3 \longleftrightarrow p_2)$$

$$\overline{|\mathcal{M}|}^2 = 2e^4\frac{t^2+u^2}{s^2}$$



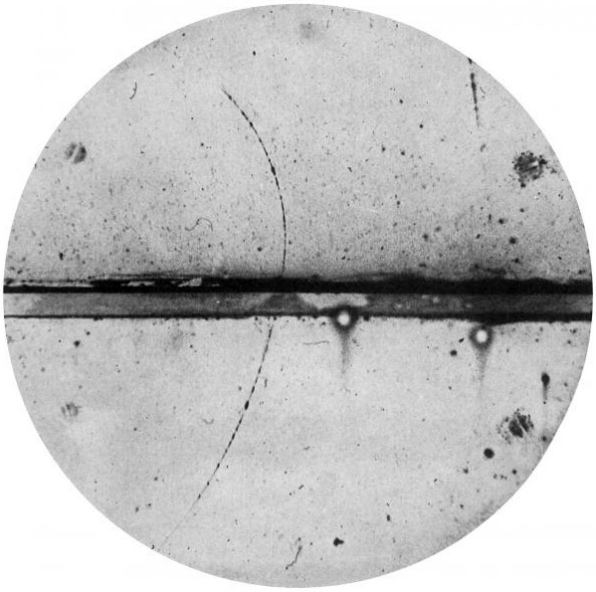


$$m = M = 0$$
$$E = |p_1| = |p_2| = |p_3| = |p_4|$$

$$s = (p_1+p_2)^2 \sim p_1\cdot p_2 = 4E^2$$
$$t = (p_1-p_3)^2 \sim -p_1\cdot p_3 = -2E^2(1-\cos\theta)$$
$$u = (p_1-p_4)^2 \sim -p_1\cdot p_4 = -2E^2(1+\cos\theta)$$

$$\overline{|\mathcal{M}|}^2 = 2e^4\frac{t^2+u^2}{s^2} = 2e^4\frac{\left(-2E^2(1-\cos\theta)\right)^2 + \left(-2E^2(1+\cos\theta)\right)^2}{\left(4E^2\right)^2} = e^4\left[1+\cos^2\theta\right]$$

which is the form of the matrix element found in the relativistic kinematics section when just considering conservation of angular momentum.



The Oxford MPhys Course

C4: Particle Physics

Problem Set Two

1. In natural units, $\hbar = c = 1$, and with $2m = 1$, the Schrödinger equation can be written as

$$-\nabla^2\psi + V\psi - i\frac{\partial\psi}{\partial t} = 0,$$

where V is a real potential.

(a) Multiply this equation by ψ^* from the left and multiply the complex conjugate equation by ψ from the right. Subtract the two resulting equations and show that the answer can be written in the form of a continuity equation,

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0,$$

where $\mathbf{j} = i^{-1}\left[\psi^*\nabla\psi - (\nabla\psi^*)\,\psi\right]$ and $\rho = \psi^*\psi$.

(b) Perform the same operations for the Klein-Gordon equation

$$-\nabla^2\psi + \frac{1}{4}\psi + \frac{\partial^2\psi}{\partial t^2} = 0$$

and derive the corresponding 'probability' density and current.

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(c) Show that for the free particle solution $\psi = Ne^{-ip^\mu x_\mu}$, where N is the normalisation, $p^\mu = (E, \mathbf{p})$ and $x = (t, \mathbf{x})$, the probability current $j^\mu = (\rho, \mathbf{j})$ is proportional to p^μ. Why might this result be a problem for the probabilistic interpretation of quantum mechanics?

2. For the free particle solutions $\psi = ue^{-ip^\mu x_\mu}$ of the Dirac equation

$$i\frac{\partial\psi}{\partial t} = (-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi,$$

the 4-component spinor u may be written in terms of two 2-component spinors

$$u = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

(a) Using the standard representation of the Dirac matrices, show that ϕ and χ satisfy the coupled equations,

$$(E - m)\phi = \boldsymbol{\sigma}\cdot\mathbf{p}\chi, \quad (E + m)\chi = \boldsymbol{\sigma}\cdot\mathbf{p}\phi\,,$$

where $p^\mu = (E, \mathbf{p})$.

(b) Show that for positive energy states $E = +|E|$ a solution to the Dirac equation is

$$u^+_{\uparrow\downarrow} = N\begin{pmatrix} \chi_{\uparrow\downarrow} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+m}\chi_{\uparrow\downarrow} \end{pmatrix},$$

where N is a normalisation factor and $\chi_{\uparrow\downarrow}$ are the 2-dimensional spinors that identify a spin up or down state,

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) Find the unnormalised positive energy and negative energy solutions $u^+_\uparrow$, $u^+_\downarrow$, $u^-_\uparrow$ and $u^-_\downarrow$ in terms of p_x, p_y, p_z, $|E|$ and m. Furthermore, give the relationships between $u_{\uparrow\downarrow}$ and the spinors for an anti-particle, $v_{\uparrow\downarrow}$, and write the components of $v_{\uparrow\downarrow}$ in terms of p_x, p_y, p_z, $|E|$ and m.

3. This question considers the commutation relations of the γ matrices.

(a) Verify explicitly that the matrices $\boldsymbol{\alpha}$ and β satisfy the Dirac anti-commutation relations.

(b) Defining the four γ matrices $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, where $\gamma^0 = \beta$ and $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$, find the anti-commutation relations between the γ matrices.

(c) Show that the Dirac equation may be written in the covariant form,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

(d) Define the conjugate spinor $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ and find the equation satisfied by $\bar{\psi}$ in the γ notation.

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(e) Show that the Dirac probability current $j^\mu = \bar{\psi}\gamma^\mu\psi$ satisfies the conservation law $\partial_\mu j^\mu = 0$.

4. The Dirac equation for a free fermion given is

$$(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\Psi = i\frac{\partial\Psi}{\partial t} \qquad (\hbar = c = 1)\,.$$

(a) How many solutions does the Dirac equation have? What are the quantum numbers labelling them?

(b) The general free particle solution to the Dirac equation is

$$\Psi(\mathbf{r}, t) = N \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{E_p+m}\chi \end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} - E_p t)}$$

where momentum $\mathbf{k} = (0,\, 0,\, k)$, $E_p = +\sqrt{m^2 + k^2}$, χ is a two-component spinor, and N is a normalisation factor. Find the value of N corresponding to a normalisation $u^\dagger u = 1$, where u is the 4-component Dirac spinor. Why is this not satisfactory in a relativistic theory? Give an alternative covariant normalisation.

(c) Give the helicity operator and interpret its eigenvalues. Find the probability of measuring negative helicity for an electron described by $\Psi(\mathbf{r}, t)$ with $\chi = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) At high energy ($E_p \gg m$), electrons described by $\Psi(\mathbf{r}, t)$ and $\chi = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ interact with unpolarized protons with charged-current cross section σ_{pol}. For unpolarized electrons interacting with unpolarized protons, the corresponding cross section is σ_0. Calculate σ_{pol}/σ_0.

5. $P_L = \frac{1}{2}(1 - \gamma^5)$ and $P_R = \frac{1}{2}(1 + \gamma^5)$ are left and right-hand projection operators.

(a) To verify this statement, show that,

$$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_L + P_R = 1, \qquad P_R P_L = P_L P_R = 0,$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the chirality operator.

(b) Show that $\bar{\nu}\gamma^\mu P_L \nu = \bar{\nu}_L\gamma^\mu\nu_L$, where $\nu_L = P_L\nu$ is the left-handed chiral component of the spinor ν. What is the value of $\bar{\nu}_L\gamma^\mu\nu_R$ and what are the implications of these results for the coupling of spin-$\frac{1}{2}$ particle to the photon?

(c) Starting from the standard representation of the free particle 4-component spinor

$$u(p) = \sqrt{E+m}\begin{pmatrix} \xi_\pm \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\xi_\pm \end{pmatrix} \quad \text{where} \quad \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

for a particle of mass m, and 4-momentum p, find the left- and right-handed states $u_L = P_L u$ and $u_R = P_R u$.

(d) Show that in the high energy limit $E \gg m$, u_L and u_R are eigenstates of the helicity operator

$$\Sigma \cdot \hat{\mathbf{p}} \equiv \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix}$$

Take $\mathbf{p} = (0, 0, p)$. Optional extra: repeat the last part for anti-particle spinors.

6. The Dirac equation can be written as $(\vec{\alpha} \cdot \vec{\mathrm{p}} + \beta m)\psi = \mathrm{p}^0 \psi$, where $\mathrm{p}^\mu \equiv (\mathrm{p}^0, \vec{\mathrm{p}}) = \left(i\frac{\partial}{\partial t}, -i\nabla \right)$. Free particle plane wave solutions have the form $\psi = u(p) \exp(-ip_\mu x^\mu)$, where u is a four-component spinor and $p^\mu = (E, \vec{p})$).

(a) Writing u in terms of 2-spinors u_A and u_B and using the standard (Dirac-Pauli) representation of the $\vec{\alpha}$ and β matrices show that

$$\vec{\sigma} \cdot \vec{\mathrm{p}}\, \tilde{u}_B = \left(\mathrm{p}^0 - m\right) \tilde{u}_A, \quad (1)$$
$$\vec{\sigma} \cdot \vec{\mathrm{p}}\, \tilde{u}_A = \left(\mathrm{p}^0 + m\right) \tilde{u}_B, \quad (2)$$

where $\tilde{u}_{A,B} = u_{A,B} \exp(-ip_\mu x^\mu)$. Use these to show that for a non-relativistic electron, traveling with a velocity v, the norm of u_B is smaller than the norm of u_A by a factor of approximately $v/2c$.

(b) Introduce the electromagnetic interaction into equations (1) and (2) by the operator substitution $\mathrm{p}^\mu \to \mathrm{p}^\mu - e\mathrm{A}^\mu(\mathrm{x})$. Then act on the first equation with the operator $(\mathrm{p}^0 - e\mathrm{A}^0 + m)$. Next, eliminate $\tilde{u}_B$ from the result by using the second equation. Finally, assuming that the electric potential is varying slowly in time and space and that $e\mathrm{A}^0 \ll m$, show that the resulting equation for $\tilde{u}_A$ is

$$\vec{\sigma} \cdot (\vec{\mathrm{p}} - e\vec{\mathrm{A}}) \vec{\sigma} \cdot (\vec{\mathrm{p}} - e\vec{\mathrm{A}})\, \tilde{u}_A = 2m(E_{nr} - e\mathrm{A}^0)\tilde{u}_A, \quad (3)$$

where $E_{nr} = E - m$.

(c) The operators in equation (3) do not commute. You need the following relations (which you can try to prove as an optional extra):

$$(i) \qquad (\vec{\sigma} \cdot \vec{\mathrm{a}})(\vec{\sigma} \cdot \vec{\mathrm{b}}) = \vec{\mathrm{a}} \cdot \vec{\mathrm{b}} + i\vec{\sigma} \cdot (\vec{\mathrm{a}} \times \vec{\mathrm{b}}),$$

where $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$ are operators.

$$(ii) \qquad \vec{\mathrm{p}} \times \vec{\mathrm{A}} + \vec{\mathrm{A}} \times \vec{\mathrm{p}} = -i\vec{\nabla} \times \vec{\mathrm{A}},$$

where $\vec{\mathrm{p}}$ is the 3-momentum operator and $\vec{\mathrm{A}}$ is function of position.

Show that the wave function $\tilde{u}_A = u_A \exp(-ip_\mu x^\mu)$ satisfies the Schrödinger-Pauli equation,

$$\left(\frac{1}{2m}(\vec{\mathrm{p}} - e\vec{\mathrm{A}})^2 - \frac{e}{2m}\vec{\sigma} \cdot \vec{\mathrm{B}} + e\mathrm{A}^0 \right) \tilde{u}_A = E_{nr}\tilde{u}_A, \quad (4)$$

where $\vec{\sigma} \cdot \vec{\mathrm{B}}$ is the spin magnetic field coupling.

(d) From equation (4) show that the predicted gyromagnetic ratio of the electron, g, is 2. Experimentally g is slightly larger than 2. What is missing from the approach here and why was the accurate measurement of $(g-2)_e$ so important for the development of Quantum Electro-Dynamics (QED)?

7. The Dirac spinor for a particle of momentum $\mathbf{p}$, energy E and mass m is

$$u = N \begin{pmatrix} \phi \\ \mathbf{p}\cdot\sigma/(E+m)\phi \end{pmatrix} \tag{5}$$

The helicity matrix for a particle with momentum $\mathbf{p}$ is given by

$$h(\mathbf{p}) = \begin{pmatrix} \hat{\mathbf{p}}\cdot\sigma & 0 \\ 0 & \hat{\mathbf{p}}\cdot\sigma \end{pmatrix}$$

where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$.

(a) Show that the Dirac spinors are eigenvectors of the helicity operator and find the eigenvalues.

(b) The chirality operator is given by γ_5. Considering the case of massless fermions, show that the eigenvectors of the chirality operator are simultaneous eigenvectors of the Dirac Hamiltonian.

(c) We define the left and right-handed projection operators by

$$\begin{aligned} P_L &= \frac{1}{2}(1-\gamma_5) \\ P_R &= \frac{1}{2}(1+\gamma_5) \end{aligned} \tag{6}$$

For the case of massless fermions show that P_L (P_R) project out negative (positive) helicity eigenstates.

(d) Now consider the case of finite mass but with $E \gg m$ (e.g. electrons from W decays). First show that

$$1 - \frac{\sigma\cdot\mathbf{p}}{E+m} = \frac{1}{2}(1-\frac{|\mathbf{p}|}{E+m})(1+\sigma\cdot\hat{\mathbf{p}}) + \frac{1}{2}(1+\frac{|\mathbf{p}|}{E+m})(1-\sigma\cdot\hat{\mathbf{p}}) \tag{7}$$

(e) Define the left-handed chiral state as $u_L = P_L u$ and the right-handed chiral state as $u_R = P_R u$. Evaluate u_L and u_R in terms of positive and negative helicity eigenstates for $E \gg m$ to leading order in m/E.

(f) Use this result to explain which combinations of helicity states are possible for ultra-relativistic gauge interactions in the Standard Model.

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8. past paper question. The Dirac equation can be written

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\psi = \mathrm{i}\frac{\partial\psi}{\partial t},$$

where ψ is a 4-component free-particle wave function, $\mathbf{p}$ is the momentum operator and β is a matrix, as are the three components α_j of $\boldsymbol{\alpha}$.

(a) Derive anticommutation relations for the α_j and β matrices. [6]

(b) Two positive-energy solutions are

$$\psi^s(\mathbf{r},t) = N\begin{pmatrix} \boldsymbol{\chi}^s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\boldsymbol{\chi}^s \end{pmatrix}\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar},$$

$$\text{where}\quad s=1,2:\quad \boldsymbol{\chi}_1 = \begin{pmatrix}1\\0\end{pmatrix},\quad \text{and}\quad \boldsymbol{\chi}_2 = \begin{pmatrix}0\\1\end{pmatrix}.$$

Find a normalization factor N and explain why these wave functions are not normalized to one particle per unit volume. [6]

(c) Find eigenstates and eigenvalues of the parity operator γ^0. Show that the intrinsic parity of the electron is the opposite of the intrinsic parity of the positron. [6]

(d) Define the helicity operator and interpret its eigenvalues. Find a negative helicity eigenstate in a reference frame in which $\mathbf{p} = (p_x, 0, p_z)$. [7]

9. optional (not marked). The magnetic dipole moment of the neutron has been measured as $\mu = -1{\cdot}91\mu_N$ (where μ_N is the nuclear magneton). The precise value of the electric dipole moment (EDM) of the neutron is unknown, but it is less than $2{\cdot}9\times10^{-26}$ e.cm (90% CL).

(a) Using parity (P) and time-reversal (T) symmetry arguments, explain this. What would be the significance of a non-zero electric dipole moment?

(b) Outline the key features of an experiment to measure the neutron EDM. Explain how fluctuations in the magnetic field can mimic the signal from an EDM. How large a change in magnetic field is needed to produce an EDM signal equivalent to the limit given above? How is this problem overcome? [See P. Harris et al, Phys. Rev. Lett. **82** (1999) 904. The applied electric field strength was $\pm4{\cdot}5$ kV/cm. The magnetic moment of the neutron is $\mu = -1{\cdot}91\mu_N$, where μ_N is the nuclear magneton.]

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