Radiative transfer Part I: Basic concepts

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Overview

Where we are going

Characterizing the radiation field

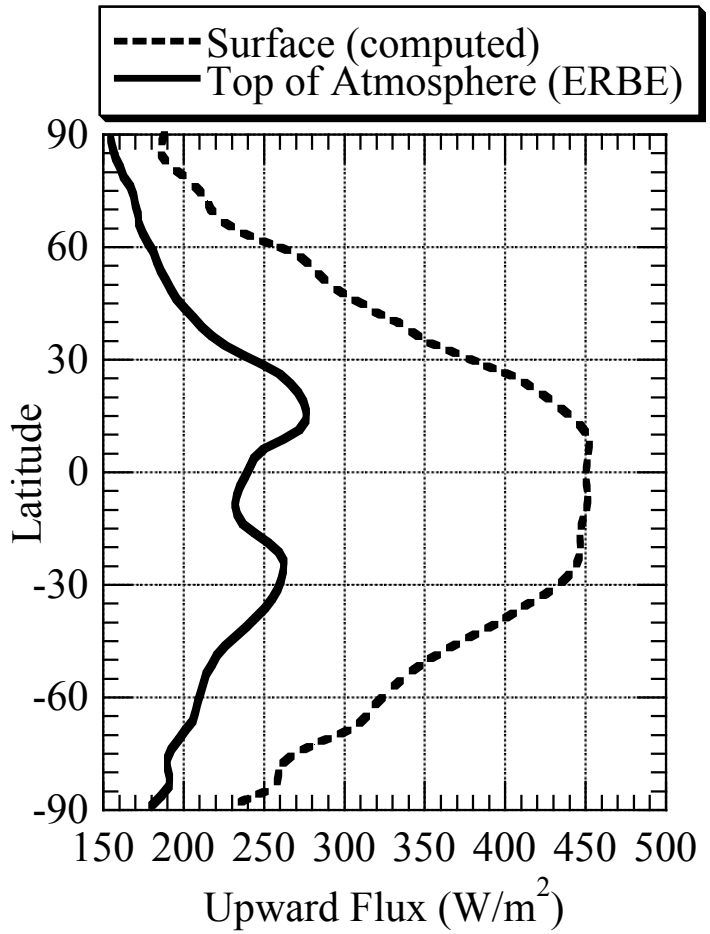
Blackbody Radiation

Absorption and emission basics

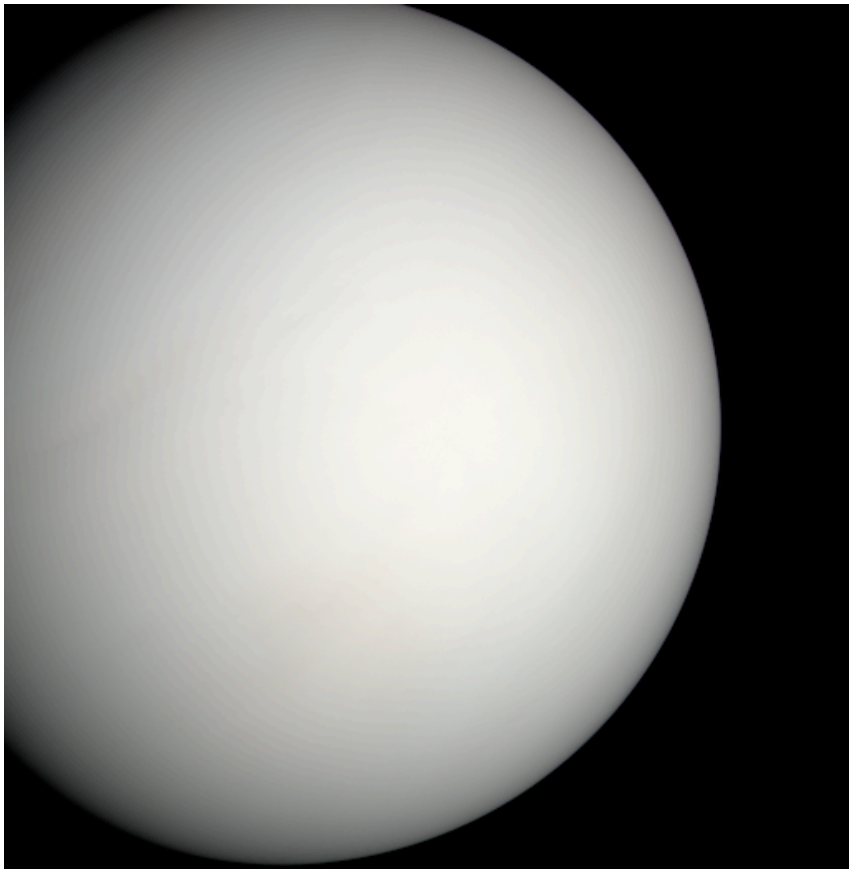
Molecular spectroscopy

Continuum absorption

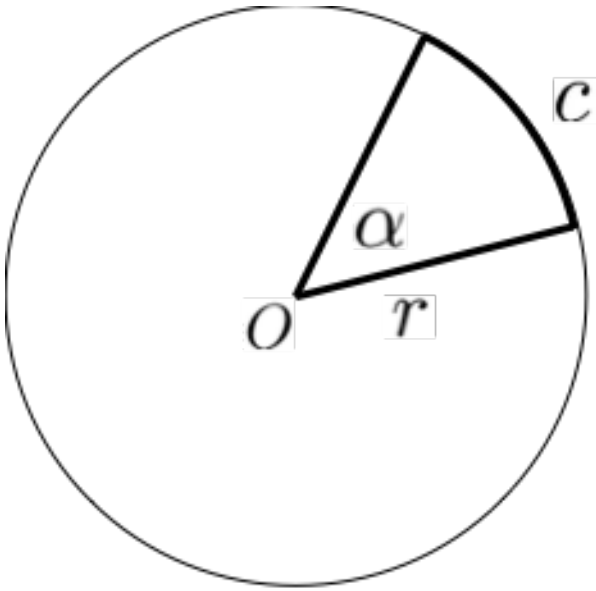
How does an atmosphere affect IR cooling to space?



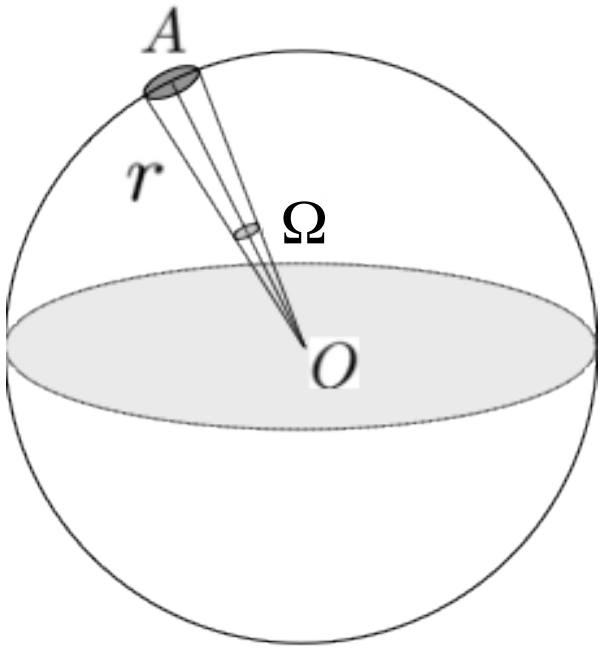
How does an atmosphere affect albedo?



Solid Angle



plane angle: $\alpha = \frac{c}{r}$



solid angle: $\Omega = \frac{A}{r^2}$

Plane angles are measured in radians
Solid angles are measured in steradians

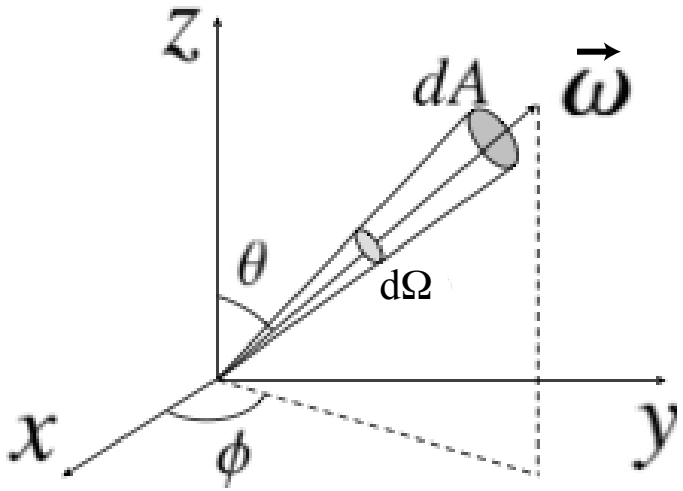
Rays vs.solid angle

$\vec{\omega}$ is a *ray*: described by a unit vector pointing in a given direction. It can be described by spherical coordinates θ and ϕ. It conveys information about direction, but not length/distance.

A solid angle Ω is a *measure* of a set of rays. E.g. $\Omega = 4\pi$ is the measure of the collection of rays pointing in every direction, $\Omega = 2\pi$ the measure of the collection of rays pointing in all upward directions.

When integrating, we integrate over a collection of rays.

Differential Solid Angle

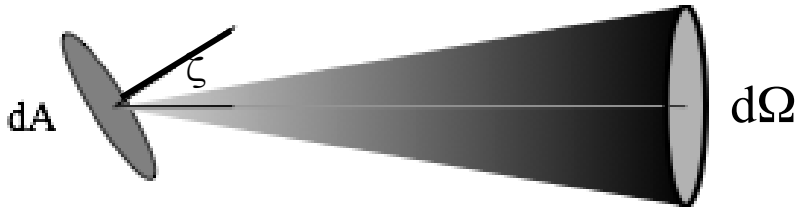


$$d\Omega = \sin\theta d\theta d\phi = -d\mu d\phi,\ \mu \equiv \cos\theta$$

Projected solid angle: $d\Omega_\perp = \cos\zeta d\Omega$

where ζ is the *zenith angle*, i.e. angle between direction of ray and normal to the surface where the ray intersects it. ($\zeta = \theta$ if coordinate system for rays is oriented with axis aligned with perpendicular to surface)

Radiance

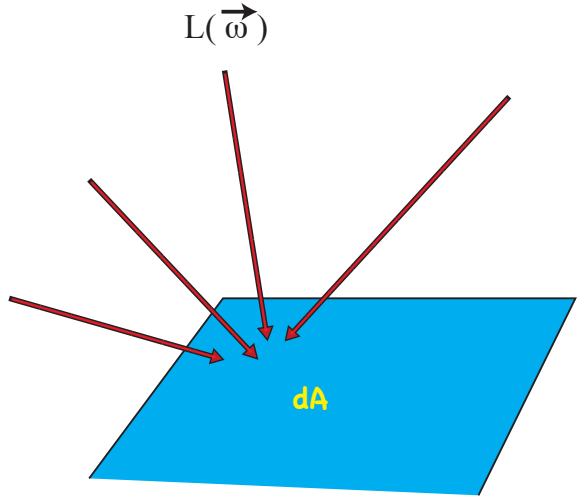


Consider a flat patch with area dA, whose normal makes an angle ζ relative to the direction $\vec{\omega}$. Some energy-carrying "stuff" is flowing through the patch. The energy passing through the patch per unit time (measured e.g. in Watts), carried by all the "stuff" moving in a solid angle $d\Omega$ about direction $\vec{\omega}$ is $d\Phi$. The *radiance* $L(\vec{\omega})$ is defined by

$$d\Phi = L\cos\zeta d\Omega dA = L d\Omega_{\perp} dA$$

Radiance is the energy per unit area per unit solid angle carried through a patch oriented normal to the direction of travel

Irradiance



$$E \equiv \int_{\Omega^-} L(\vec{\omega}) d\Omega_\perp$$

Ω^- is shorthand for integrating over the downward hemisphere

Irradiance is flux per unit area (e.g. $\mathrm{W/m^2}$. EdA is power (e.g. in W) incident on the patch (or coming out of it).

(When talking about radiation coming *out of* a patch, instead of radiation incident on a patch, this is called exitance instead, but it seems like a silly distinction to me.)

Radiance examples

- Radiance of parallel beam: $L = L_0 \delta(\vec{\omega} - \vec{\omega_0})$
  (Radiance from Sun reaching Earth is approximately parallel-beam)
- Radiance for isotropic flux: $L = L_0 = \text{const.}$

(Discuss corresponding irradiance of a patch of a planet's surface)

What is relation to electromagnetic field?

So far, we haven't said anything that restricts the description to the electromagnetic field. It could apply to flow of anything that carries energy. (Photons, neutrinos, water, watermelons ...)

- Electromagnetic energy density for plane wave is sum of energy density in electric field and magnetic field
- For unpolarized wave, $\mathcal{U} = \epsilon_0 E_0^2 = \frac{1}{\mu_0} H_0^2$
- Flux is energy density times speed, $= \mathcal{U} c$. That's the radiance.
- Can also be derived from Poynting vector
- Because quadratic, when superposing waves with different wavenumbers/frequencies, energies add

Spectral radiance (density)

Electromagnetic radiation is generally composed of a superposition of waves with many different frequencies. How do we describe the flux due to such a field?

$L(\vec{\omega}, \nu)d\nu$ is radiance carried by waves with frequency in an infinitesimal interval $d\nu$ centered on ν

BLACKBOARD:

- How to recover monochromatic case from density (δ-functions again)
- Densities vs Functions; Example: Transformation to wavelength space

Digression: Measuring frequency as wavenumbers

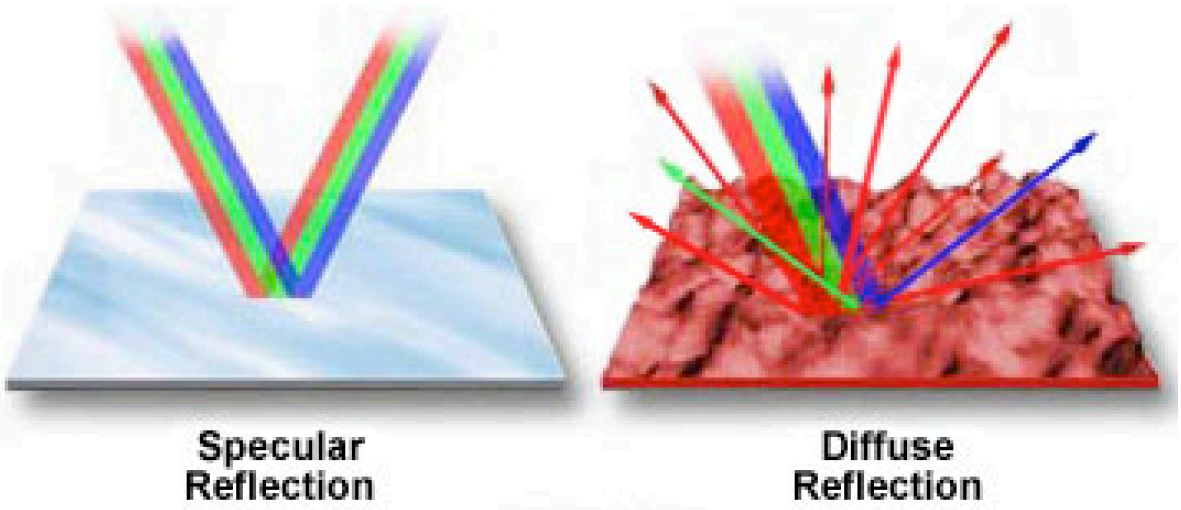
With ν measured in Hz (cycles/sec), λ in m and c in m/s,

$$\nu\lambda = c$$

so $\frac{1}{\lambda} = \frac{1}{c}\nu$

$\frac{1}{\lambda}$, called *wavenumber* is frequency measured in inverse-length units. For infrared, traditional to measure it in cm^{-1}

Diffuse vs specular reflection



Definition: *Bond albedo* of a surface is the ratio of reflected energy (summed over all directions) to incident energy. It is commonly a function of the direction of the incident light (which is often approx. parallel-beam in planetary applications, but could be composed of many directions itself, as in reflection by surface of diffuse radiation from the sky)

Radiance within a black body

(also applies to radiance from surface of a blackbody)

The Planck function (actually, density)

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

Properties

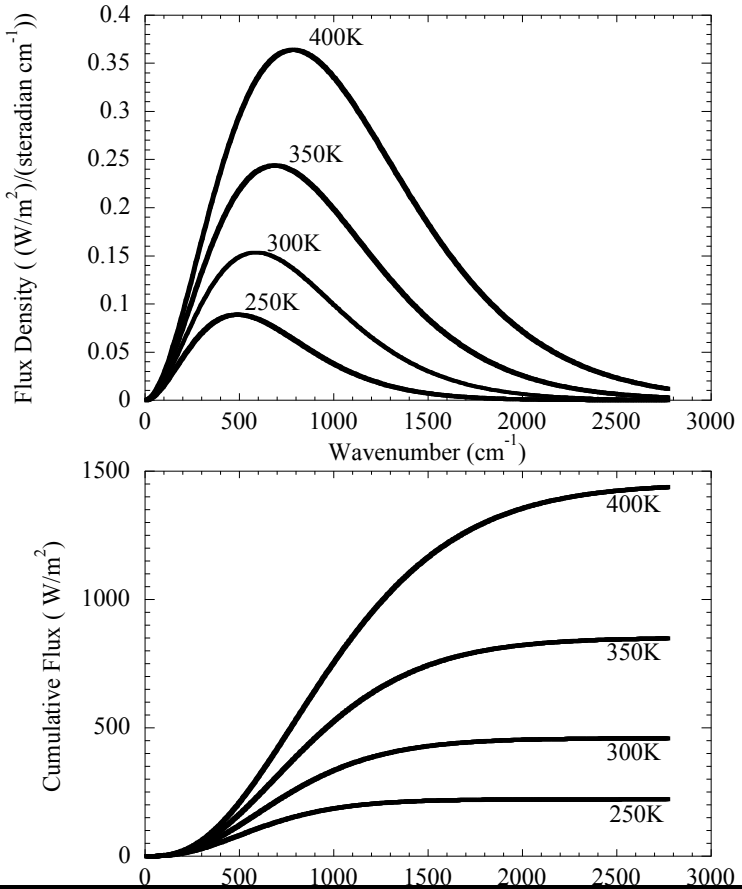
- Blackbody radiation is *isotropic*, ie. B independent of $\vec{\omega}$
- $h\nu$ is quantum of EM energy at frequency ν
- $\frac{1}{2}kT$ is energy per degree of freedom
- $\frac{h\nu}{kT} \ll 1$ is classical limit
  $B \approx \frac{2kT\nu^2}{c^2}$; h drops out. *Ultraviolet catastrophe!*

Irradiance through/from a patch of blackbody surface

i.e. add up energy over all rays passing through small patch with area dA

$$E = \int_{\Omega^+} B d\Omega_\perp = B \int_{\Omega^+} d\Omega_\perp = \pi B$$

Planck Density and cumulative Planck Density



Define $u \equiv \frac{h\nu}{kT}$, then

$$B = \frac{2(kT)^3}{h^2c^2}\frac{u^3}{e^u - 1}$$

Position of peak is proportional to T.

Wien's law

Also, $\frac{\partial B}{\partial T} > 0$ for fixed ν

Transforming to wavelength space

BLACKBOARD

- Planck density in wavelength space
- Implications for location of peak
- Cumulative emissions are coordinate-independant, so median emission wavenumber/wavelength is a more physially meaningful characterization

The Stefan-Boltzmann Law

Sum up blackbody irradiance over all frequencies

$$F(T) \equiv \int_0^\infty \pi B(\nu, T) d\nu = [\frac{2\pi k^4}{h^3 c^2} \int_0^\infty \frac{u^3}{e^u - 1} du] T^4 = \sigma T^4$$

Note: Depends on the two constants determining the nonclassical nature of Universe only via the combination $h^3 c^2$. Can have a classical universe with well-behaved blackbody radiation if $h \to 0$ and $c \to \infty$ with $h^3 c^2$ fixed!

Temperature of a planet vs. temperature of its star

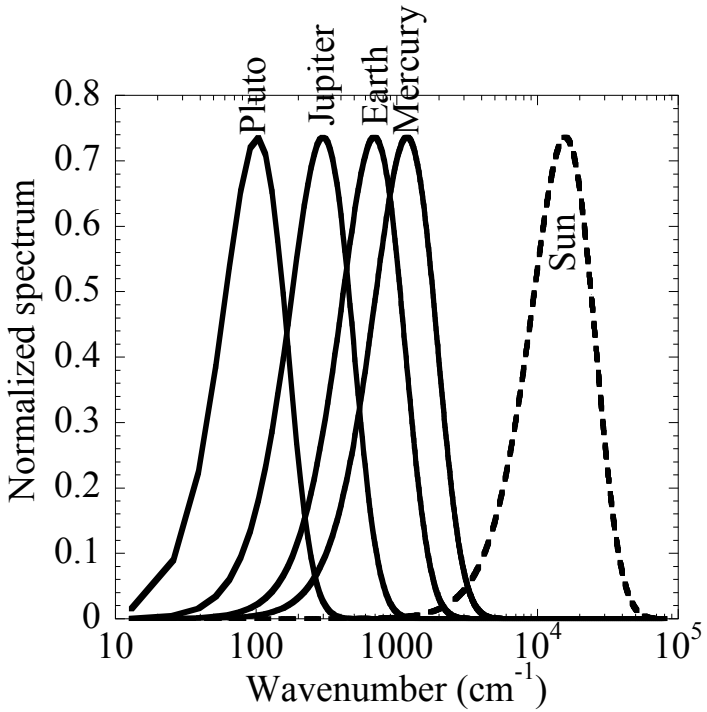
$$\sigma T^4 = \frac{1}{4}(1-\alpha)L_\circledast$$

$$T = \frac{1}{\sqrt{2}}(1-\alpha)^{1/4}\sqrt{\frac{r_\circledast}{r}}\,T_\circledast$$

where

- $L_\circledast$ = "instellation,"
  i.e. incident parallel-beam radiance at orbit
- $r_\circledast$ = radius of star, $T_\circledast$ = photospheric temperature of star
- r = orbital distance of planet

Most planets emit dominantly in the infrared



Spectral gap is what enables the greenhouse effect

But many exoplanets have a smaller spectral gap

e.g. 55Cancri-e (a.k.a. Vulcan)

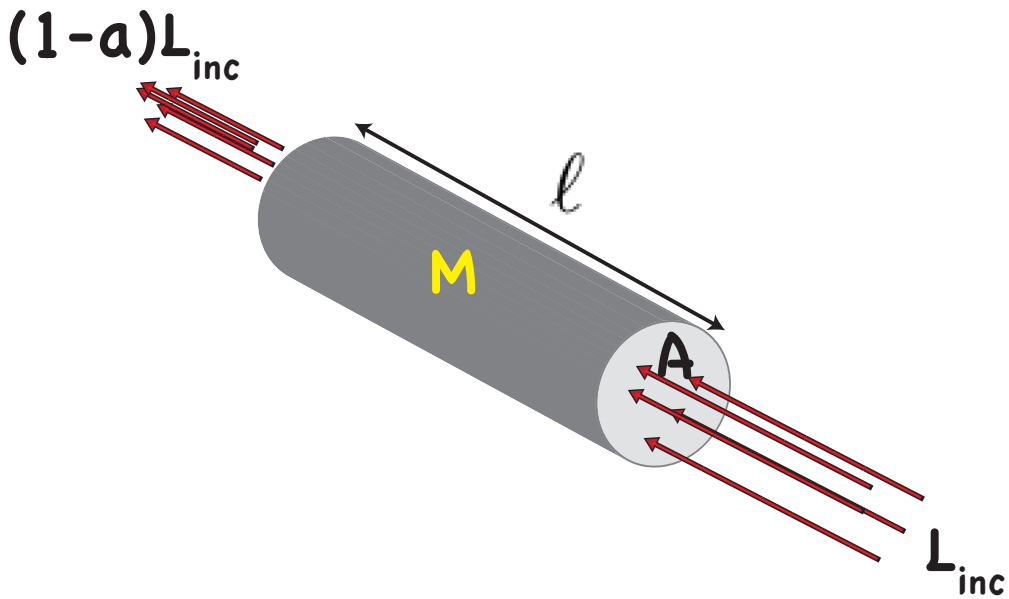
- .75 Earth day orbit around Sunlike G-star
- $T_\circledast = 5373K$, $r_\circledast = 1.152 r_\odot, M_\circledast = .95 M_\odot$

$$\frac{GM_\circledast}{r^2} = \frac{v^2}{r} = (\frac{2\pi r}{y})^2 \frac{1}{r}$$

$$r = (\frac{y}{2\pi})^{2/3} (GM_\circledast)^{1/3} \approx 3.64 r_\circledast$$

vs. 215. for Earth

Absorption and absorption cross-section



$\mathfrak{T} \equiv (1 - a)$ is the *transmission*; $\mu = \frac{M}{A} = \rho\ell$ is *path*

Monochromatic case: $\mathfrak{T} = e^{-k\ell} = e^{-\kappa\mu}$,

κ is *absorption cross-section*

More on absorption cross-section

- Units of κ are $\frac{\mathrm{m}^2}{\mathrm{kg}}$
- κ is effective area taken out of beam, per kg of absorber encountered
- Often measured in units of $\frac{\mathrm{m}^2}{\mathrm{molecule}}$ instead
- For well-mixed absorber in atmosphere with mass concentration q, mass path is $\frac{qp_s}{g}$
- *Optical thickness*, τ (for vertical path) is $\frac{\kappa q p_s}{g}$

Emission and emissivity

Let $L(\vec{\omega}, \nu)$ be the observed radiance from an isothermal experimental body with temperature T.

Then the *emissivity* is defined as

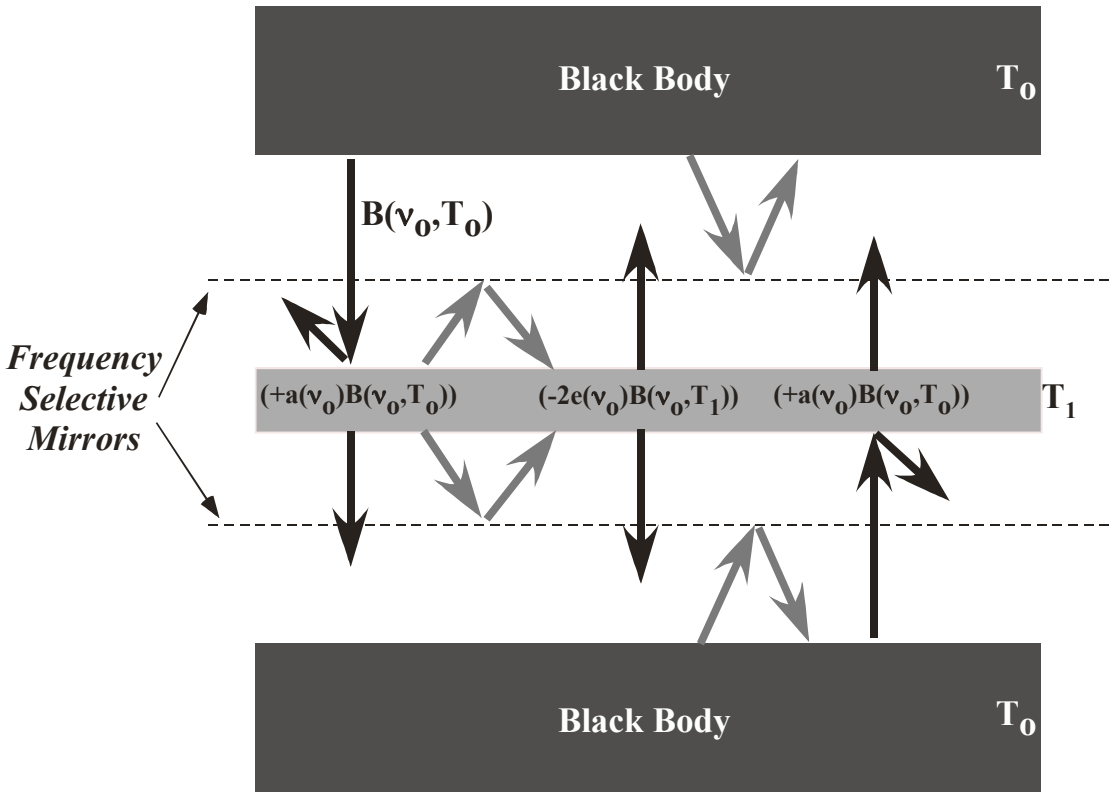
$$\epsilon(\vec{\omega}, \nu, T) = \frac{L}{B}$$

Kirchhoff's law: Empirical basis

Let $a(\vec{\omega}, \nu)$ be measured absorption of the body, $L(\vec{\omega}, \nu)$ the measured radiance

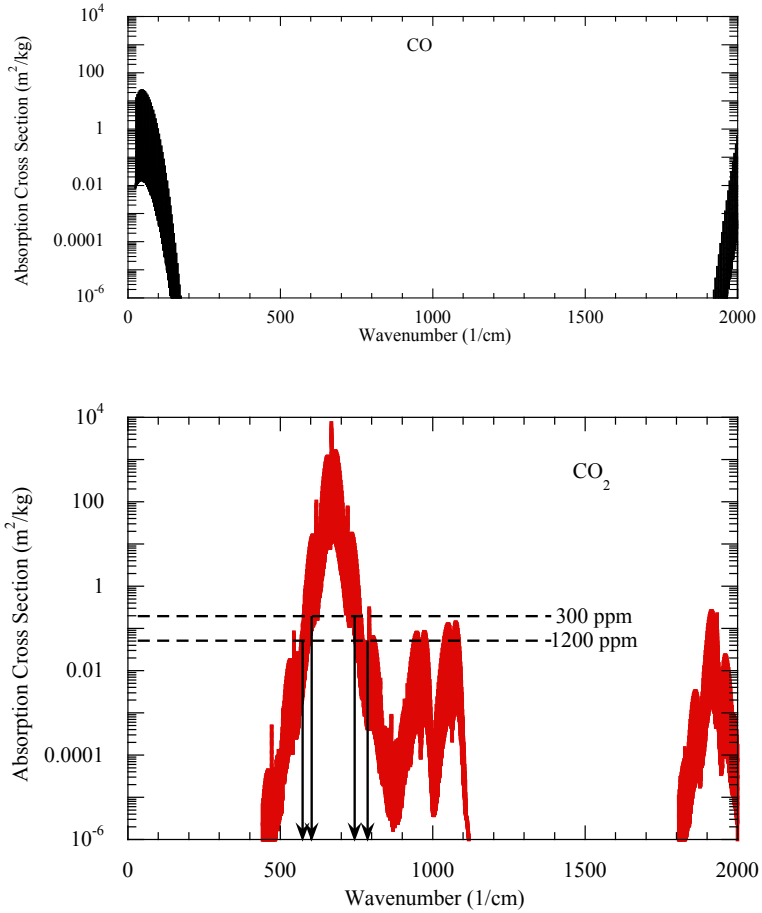
- Experiments show that $\frac{L}{a} = F(\nu, T)$ is a universal direction-independant function independent of what kind of matter the body is made of.
- F is the emission from
  an idealized perfectly absorbing (black) body.
  We know how to compute it (Planck) and now call it B
- Result can be stated as $\epsilon(\vec{\omega}, \nu, T) = a(\vec{\omega}, \nu, T)$

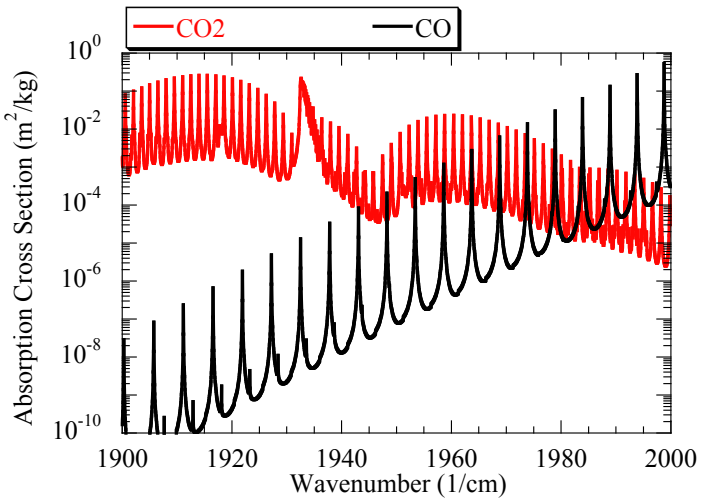
Kirchhoff's law: Basis in Second Law



(Finish derivation on BLACKBOARD

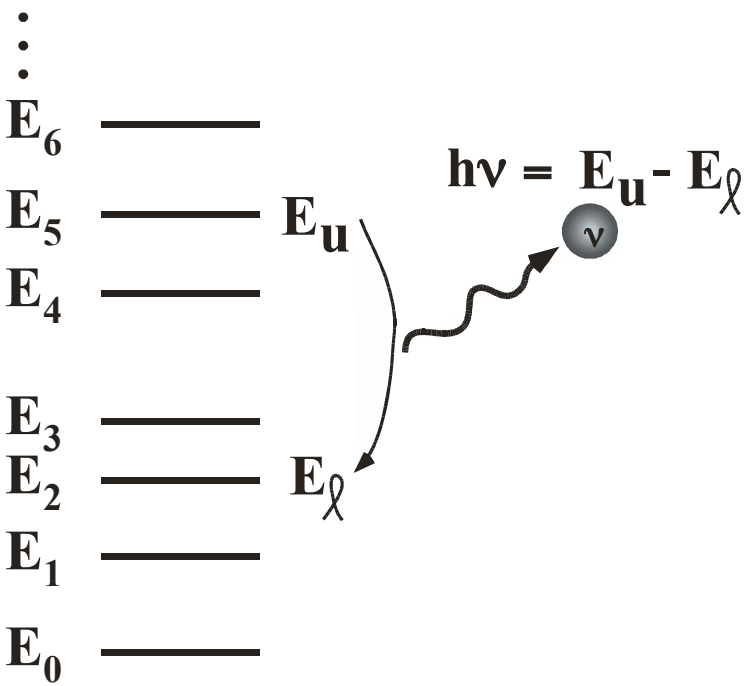
Absorption cross-section for CO and CO_2





(At 100mb pressure, 260K temperature)

Quantization and spectral lines



Types of transitions

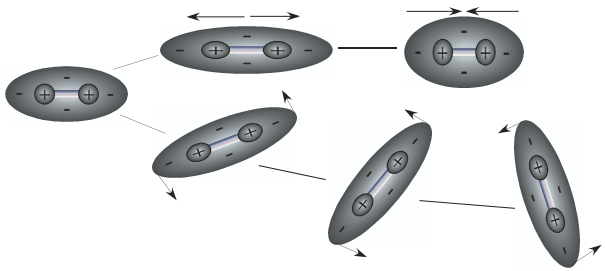
- Electron transitions (typically visible and UV)
- Vibrational transitions (includes "bending" modes) (typically IR)
- Rotational transitions (typically microwave)

Vibrational and rotational transitions can couple

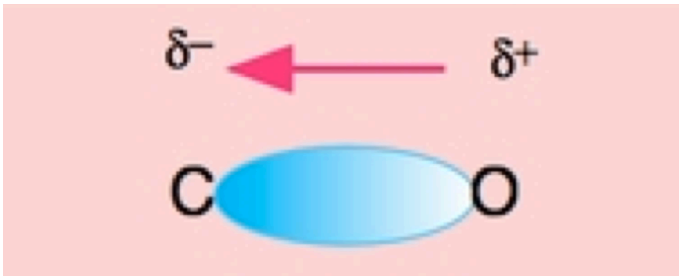
Requirement for coupling to electromagnetic field

In order to couple strongly to the electromagnetic field, the distortions of the molecule involved in rotational/vibrational transitions must change the molecule's dipole moment

Diatomic homoatomic molecules (e.g. N_2) don't have optically active transitions)

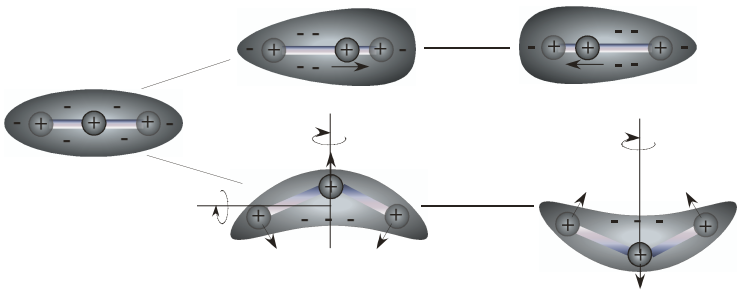


Asymmetric diatomic molecules (e.g. CO) do

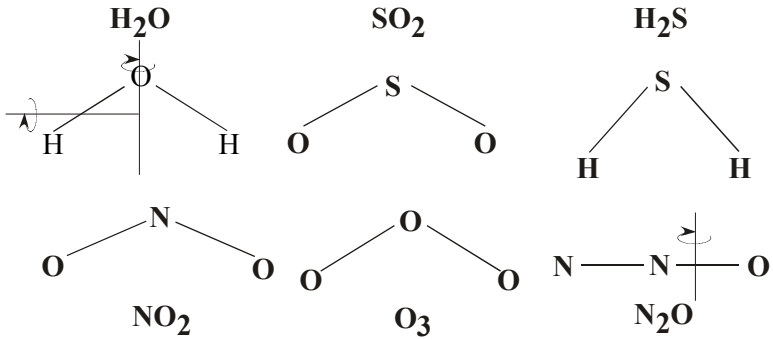


(Requirement not needed for electron transitions because those involve a moving charge)

Linear triatomic molecules (e.g. CO_2



Polar Triatomic Molecules



I walk the line

Spectral lines are characterized by

- Position ν_c
- Shape $F(x)$, $\int_{-\infty}^{\infty} F dx = 1$
- Width, δ (function of p and T)
- Strength, S (Mainly a function of T)

$$\kappa(\nu) = \frac{S}{\delta} F\left(\frac{\nu - \nu_c}{\delta}\right)$$

Intrinsic line width

$$\Delta E \Delta t = \frac{h}{4\pi}$$

Uncertainty Principle

$$\Delta E = h \Delta \nu$$

Doppler Broadening

Doppler-shifted frequency: $\nu = \nu_c \cdot (1 + \frac{v}{c})$
From Maxwell-Boltzmann:

$$F(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$\delta = \frac{\nu_c}{c} (\frac{2kT}{m})^{1/2}$$

where m is the mass of the molecule

Collisional (pressure) broadening

"Detune" the line by borrowing some energy from (unquantized) kinetic energy of motion of the molecules

$$F(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$$

Lorentz line shape. Valid near center, but not good in far tails.

$$\delta \approx \delta_0 \frac{p}{p_0} \left(\frac{T_0}{T}\right)^n$$

Often $n \approx \frac{1}{2}$, but value compiled in spectroscopic tables

What determines line strength?

$S(T)$ increases with T because high energy states are more populated

Actual form given by *total internal partition function* Q. For infrared transitions:

$$S(T) = S(T_0)\frac{Q(T_0)}{Q(T)}\left[\frac{1-\exp\left(-\frac{h\nu_c}{kT}\right)}{1-\exp\left(-\frac{h\nu_c}{kT_0}\right)}\right]\exp\left(-\frac{h\nu_\ell}{k}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right)$$

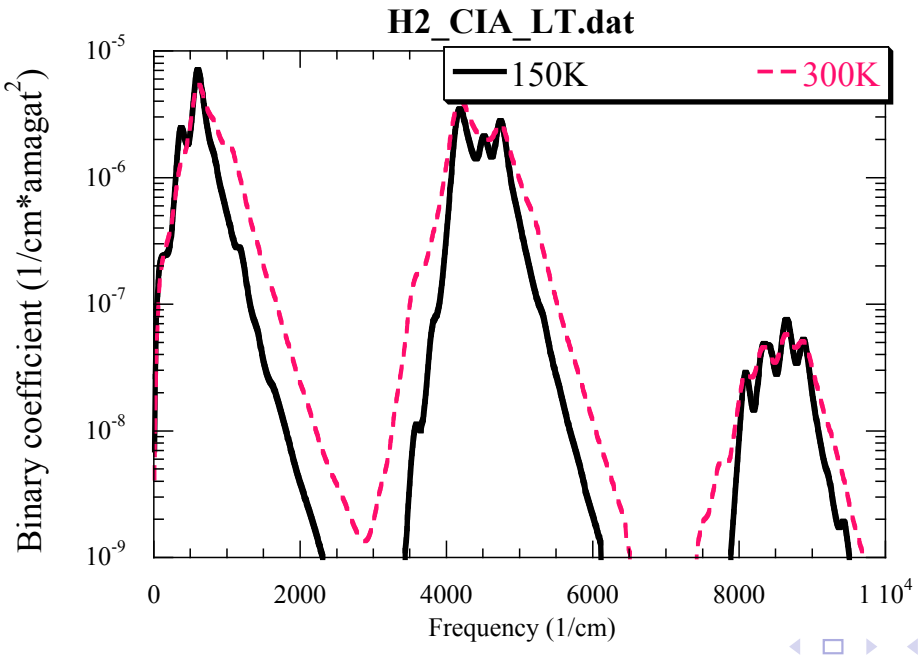
$Q \sim T$ for linear molecules, $Q \sim T^{1.5}$ for nonlinear molecules if T not too high. Precise form given in spectroscopic databases.

$h\nu_\ell$ is lower state energy of the transition. ν_c is line center.

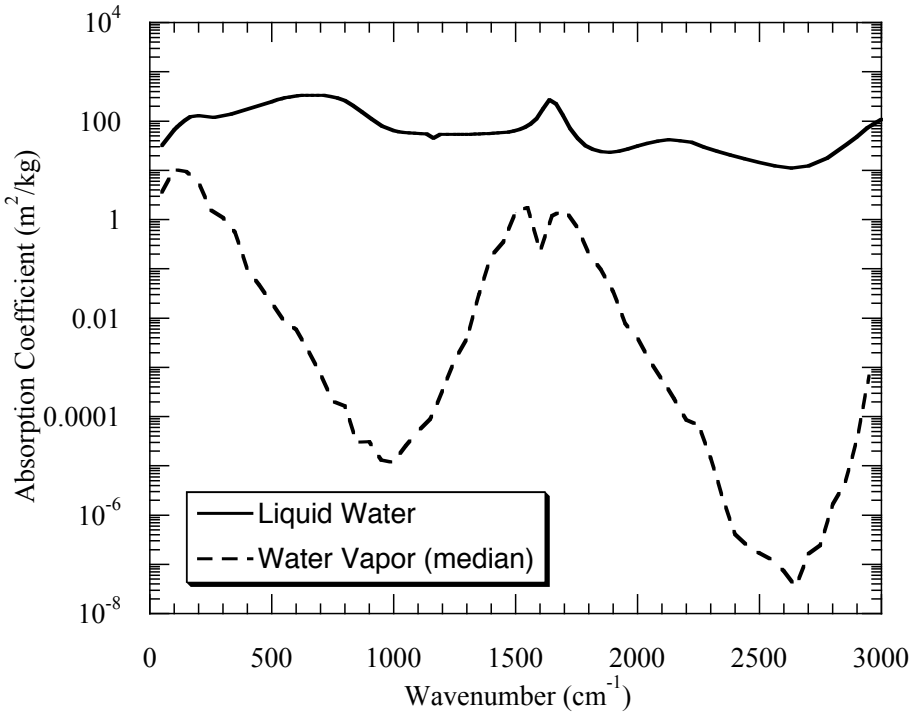
Collision-induced absorption

During collision, diatomic molecules which have no optically active transitions act as if they were a single larger molecule with a dipole moment

$\kappa_{AB} = b\rho_A\rho_B$, b is *binary coefficient*



Absorption by condensed substances:Water



But less absorption enhancement for CO_2 ice or liquid CH_4

Infrared (non-scattering) radiative transfer: Energy balance of planets, Greenhouse Effect and Radiative Forcing

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November 8, 2021

Overview

Reminder: Where we are going

Transmission Functions and optical thickness

The Equations of Radiative Transfer: Basic properties

Analytic solutions

Radiative forcing and climate response

Appendix: Radiative efficiency calculations
(Not covered 2019)

Effect of atmosphere on energy balance

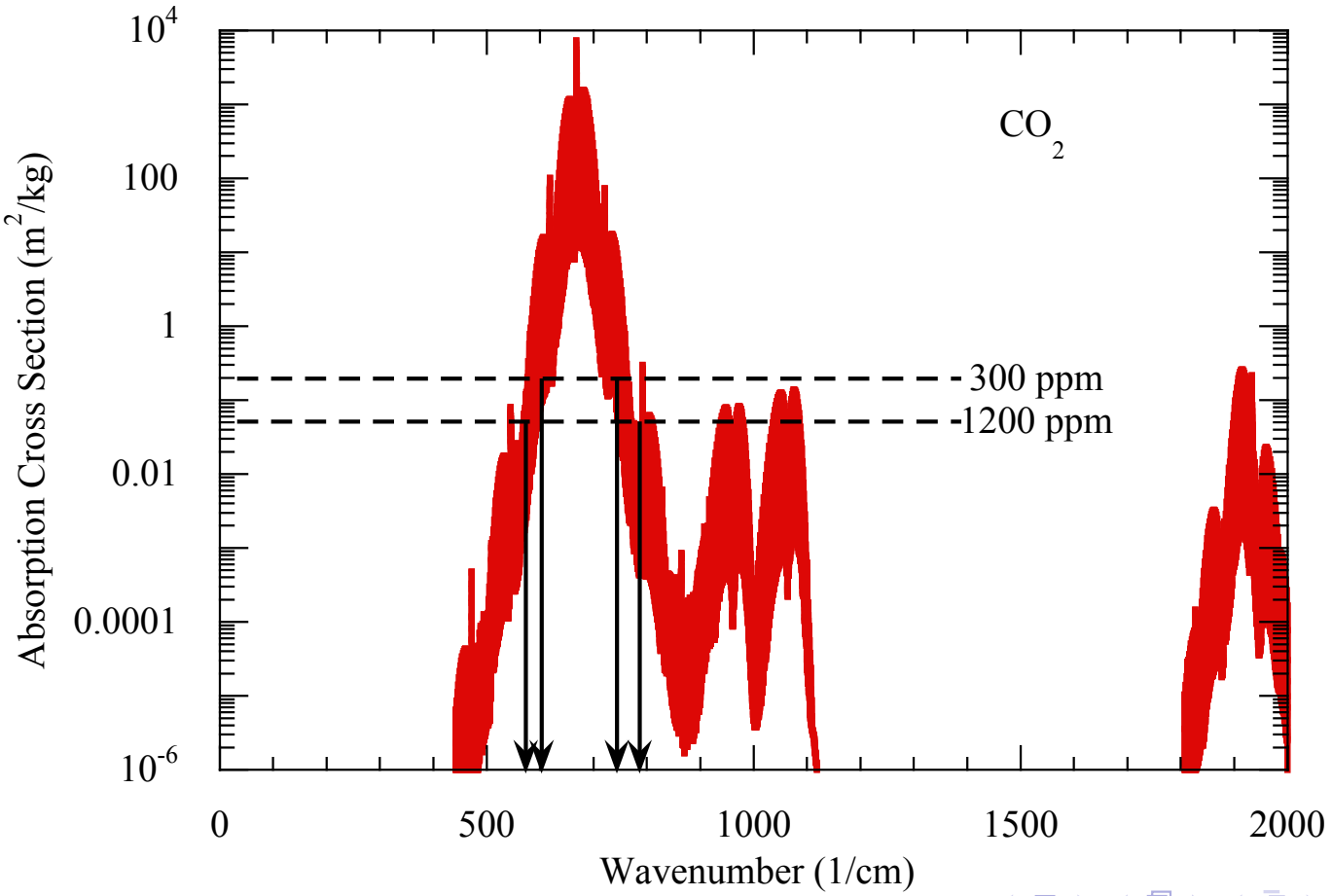
$$(1-\alpha)\frac{L_\circledast}{4} = OLR$$

(horizontally isothermal case)

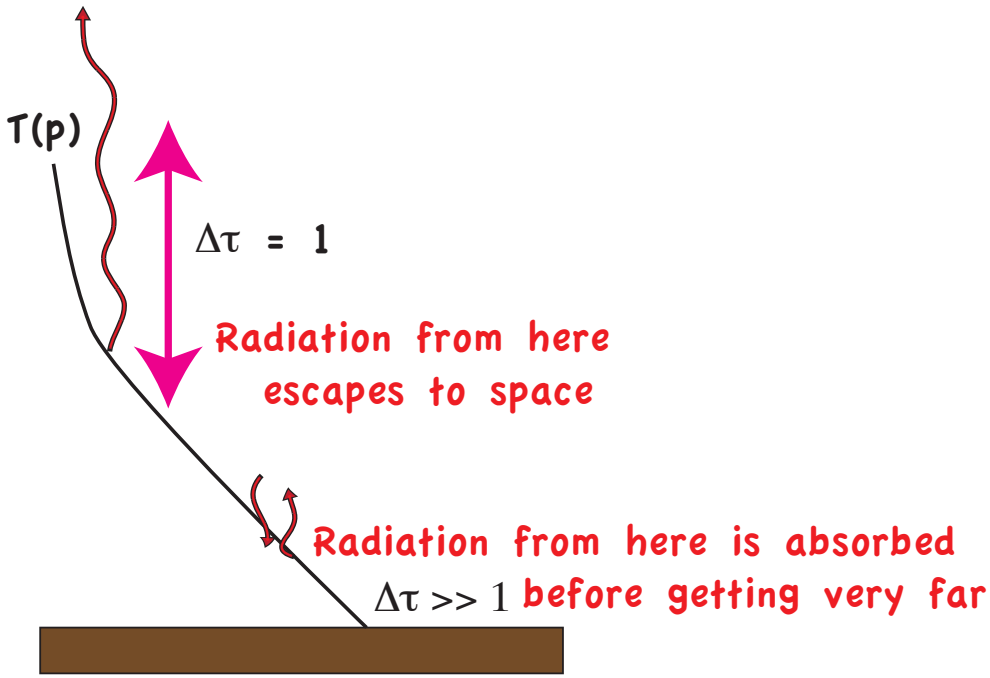
- $OLR \equiv$ "Outgoing Longwave Radiation," i.e. IR cooling to space
- $OLR = \sigma T_s^4$ for IR-transparent atmosphere
- For general atmosphere OLR is depend:
  - T_s, $T(p)$
  - $q(p)$ for each IR absorber/emitter in the atmosphere
  - (but for well-mixed gases, like CO_2 on Earth, $q(p) = \mathrm{const}$, independent of height.

We need to learn how to compute OLR

Reprise: CO_2 absorption spectrum



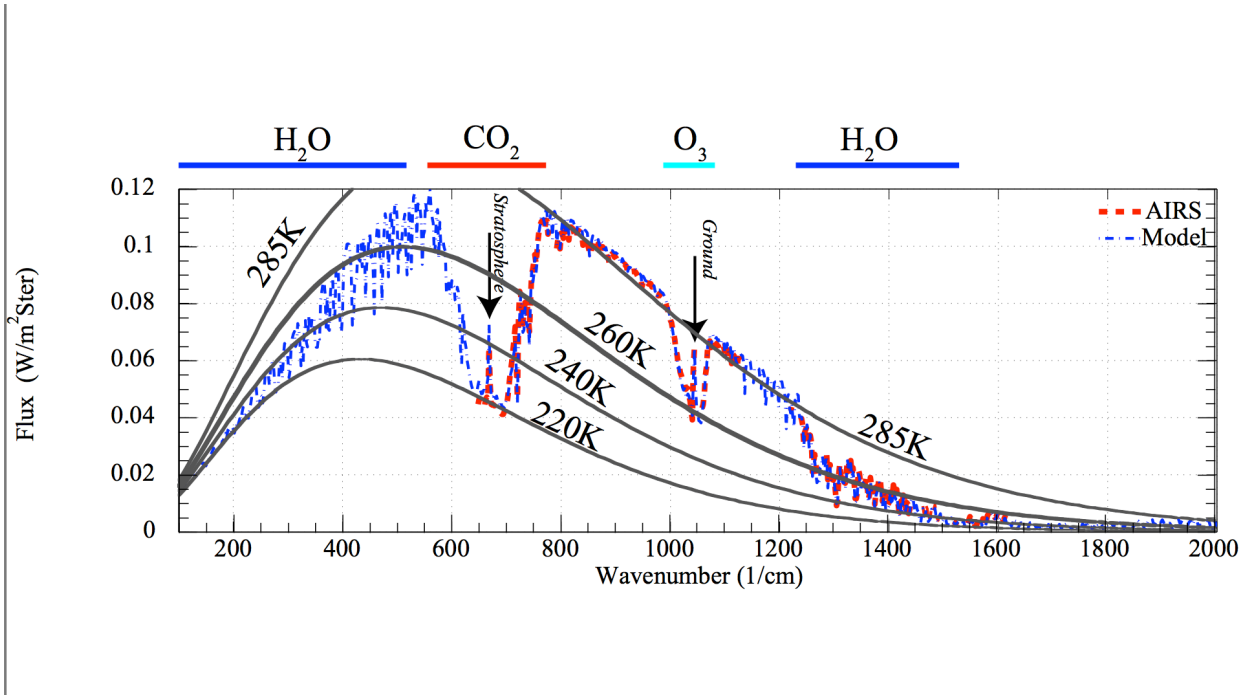
Radiating level

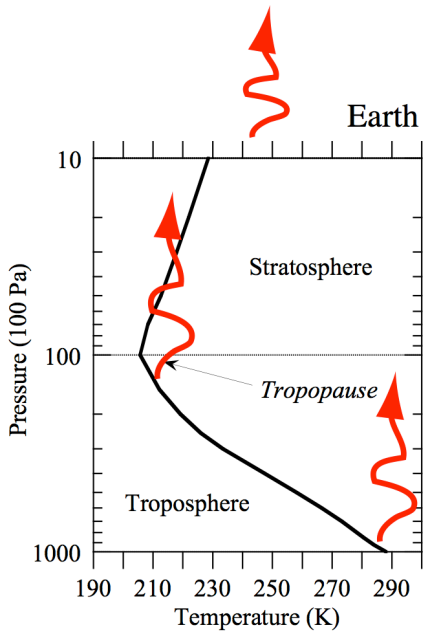


$$\Delta\tau_\nu(p) \approx \kappa_\nu \frac{p}{g}$$

Approximate because κ depends on T and p along path.

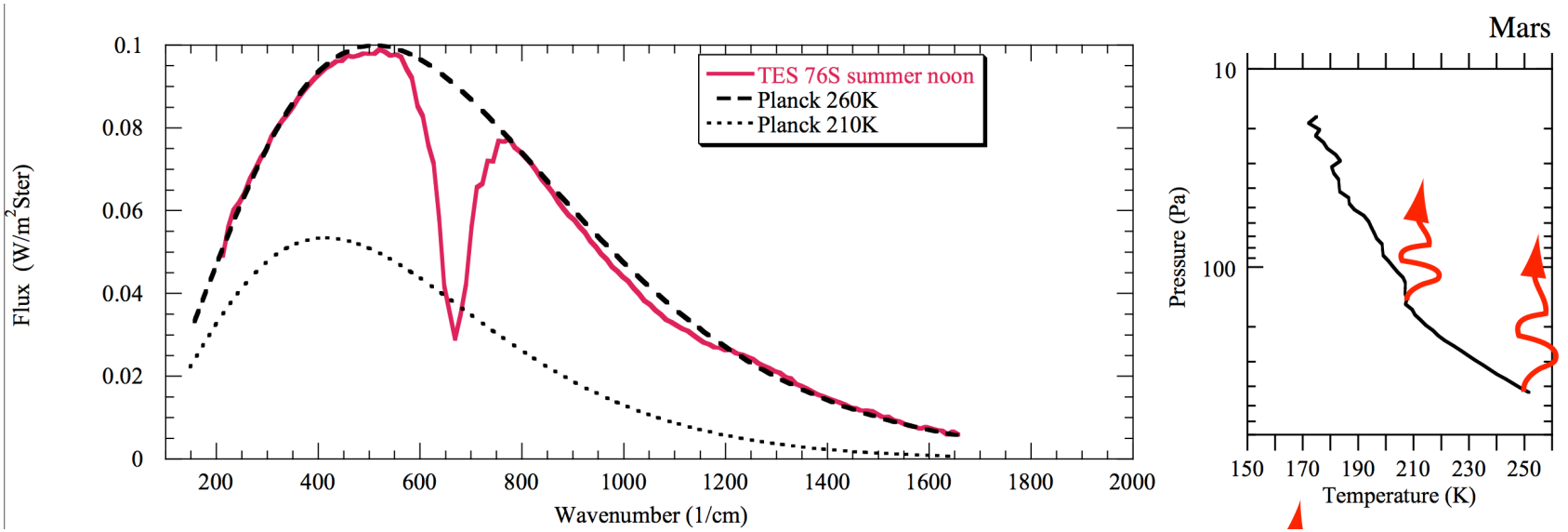
Observed *OLR* spectrum for Earth





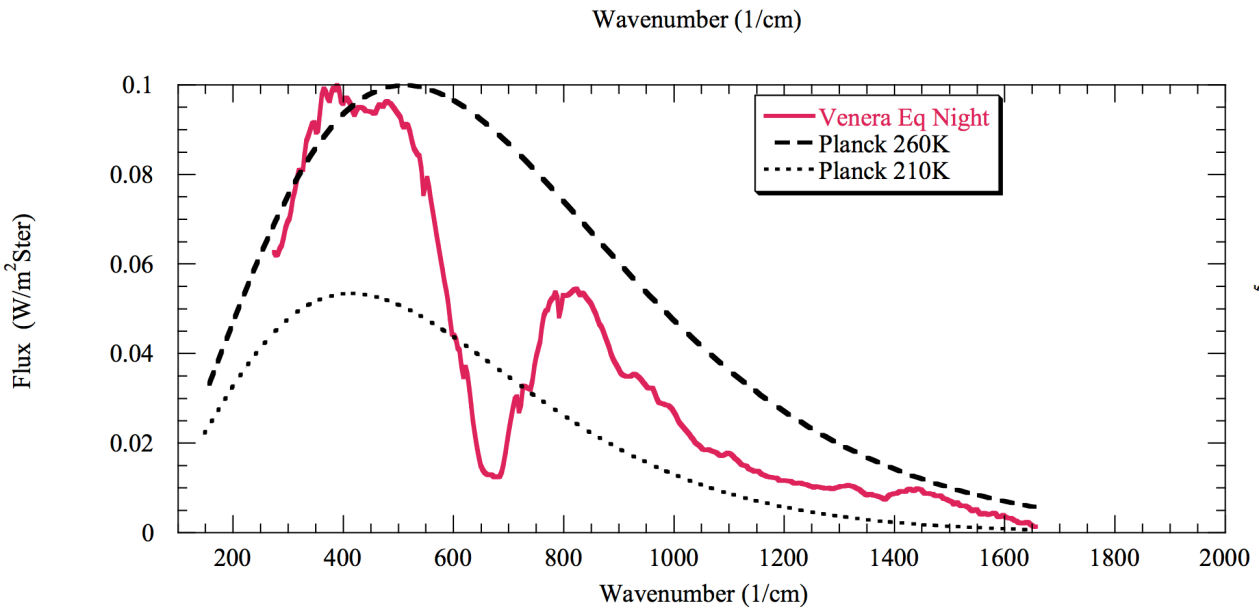
Clear sky

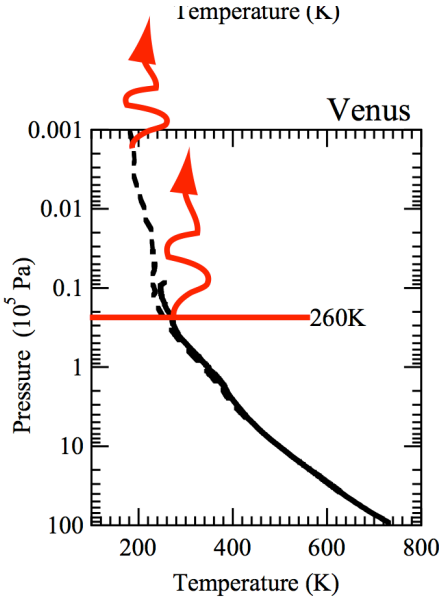
Observed *OLR* spectrum for Mars



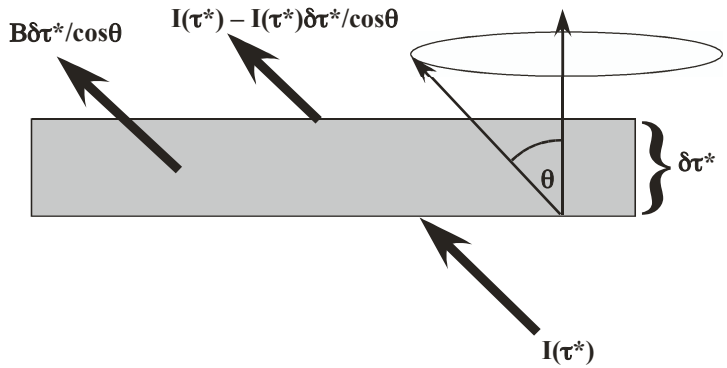
(Afternoon, non-dusty day)

Observed *OLR* spectrum for Venus





Optical thickness along a slant path



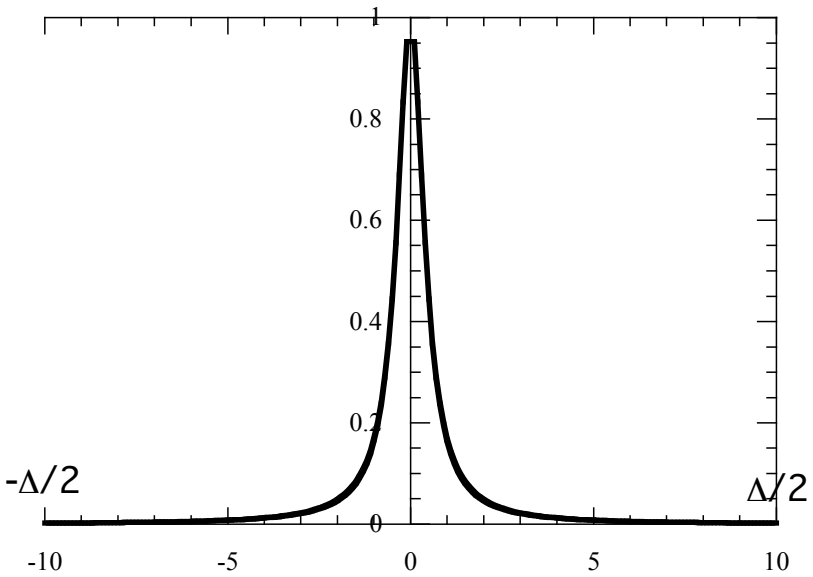
$$d\tau = -\frac{dp}{g\cos\theta} q(p)\kappa_\nu(p, T(p))$$

(with convention that τ increases *upward*

$$\tau(\nu, p1, p2) = -\frac{1}{g\cos\theta}\int_{p_1}^{p_2} \kappa_\nu(p, T(p))q(p)dp$$

BLACKBOARD: case of uniform path (constant κ).

Transmission function for isolated line



$$\mathfrak{T}(p1, p2) = \int_{\nu=\nu_0-\frac{\Delta}{2}}^{\nu_0+\frac{\Delta}{2}} e^{-\tau(\nu,p_1,p_2)} d\nu$$

Weak line limit

$\tau \ll 1$ along entire path

$$\mathfrak{T}(p1, p2) \approx \int_{\nu=\nu_0-\frac{\Delta}{2}}^{\nu_0+\frac{\Delta}{2}} (1 - \tau(\nu, p_1, p_2))d\nu = \Delta - \boxed{S\mu}$$

$$\mu \equiv \frac{1}{g \cos\theta} \int_{p_2}^{p_1} q(p)dp$$

(Note integral taken in direction of increasing pressure, so mass path is positive)

Result valid independent of line shape.

Boxed expression is *equivalent width*

Strong line limit

Transmission determined by shape of tail of line only.
For Lorentz line, approximate

$$\begin{aligned}\tau(\nu, p1, p2) &= -\frac{1}{g\cos\theta}\int_{p_1}^{p_2}\frac{S}{\pi}\frac{\delta}{(\nu-\nu_0)^2}q(p)dp \\ &= \frac{S}{\pi}\frac{\delta_0}{(\nu-\nu_0)^2}\mu_p\end{aligned} \tag{1}$$

where μ_p is the *pressure weighted path*

$$\mu_p \equiv \frac{1}{g\cos\theta}\int_{p_2}^{p_1}\frac{p}{p_0}q(p)dp$$

then

$$\mathfrak{T}(p1,p2) \approx \int_{\nu=\nu_0-\frac{\Delta}{2}}^{\nu_0+\frac{\Delta}{2}} \exp[-\frac{S}{\pi}\frac{\delta_0}{(\nu-\nu_0)^2}\mu_p]d\nu \approx \Delta - \boxed{2\sqrt{S\delta_0\mu_p}}$$

Boxed expression is *equivalent width*

Uses $\int_0^X \exp[-\frac{1}{\pi x^2}]dx \approx X-1$ for $X \gg 1$, which you derive by:

- Substitute $y = 1/x$
- Integrate by parts
- Use expression for Gaussian integral $\int_0^\infty \exp(-y^2/\pi)dy$

Lesson learned

Band-averaged transmission is not exponential in path!

(Like decay of a mix of radioactive elements with different decay rates)

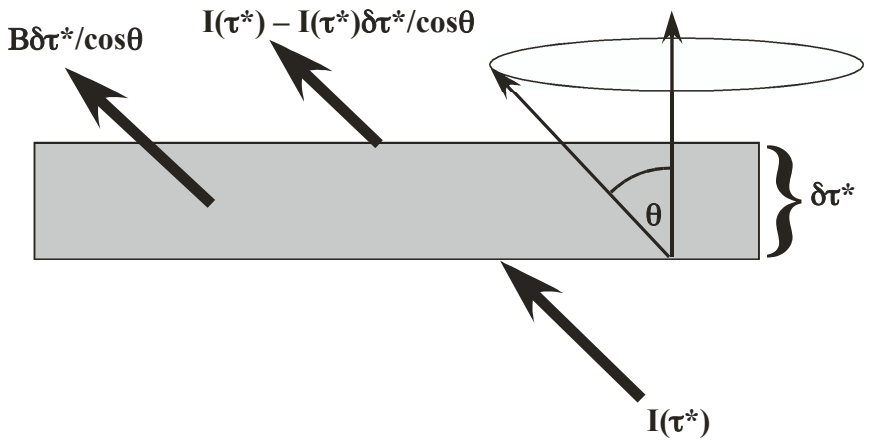
Superposition of lines

For a superposition of many (essentially) non-overlapping lines, you can just sum up the results for each line. An array of non-overlapping strong lines still give a square-root behavior in pressure-weighted path. Methane and $\mathrm{N_2O}$ behave rather like this for present-Earthlike concentrations. (but *not* $\mathrm{CO_2}$ and water vapor).

... but watch out for what happens if you try to "eat more than the whole pizza".

(cf. "random overlap" approximation)

Derivation



Let I be radiance in direction θ, at frequency ν
Let τ^* be normal-directed optical depth at frequency ν.

$$\frac{dI}{d\tau^*} = -\frac{1}{\cos\theta} I + \frac{1}{\cos\theta} B(\nu, T)$$

Sketch is shown for upward direction but just take $\theta > \frac{\pi}{2}$ to get downward direction

Recovery of blackbody flux:

If temperature is uniform, then $I = B$ is a solution, because

$$\frac{dI}{d\tau^*} = 0$$

Computing the net flux

- For observations, we need to know the radiance at a single angle
- For energy balance and climate, we want the net flux (irradiance) integrated over all upward/downward angles, and all frequencies
- For non-scattering radiative-transfer equations, it is not hard to do the calculation for several angles individually and sum the results (angular dependence is smooth so often 3 angles is enough)
- However, to make the algebra simpler, I will concentrate henceforth on the angle-averaged equations.
- I'll do angle-averaging first, and later we'll discuss how to do averaging over frequency (band-averaged transmission)

Net upward and downward monochromatic irradiance

$$I_+ \equiv \int_{\Omega^+} I(\vec{\omega}) \cos\theta d\Omega$$

$$I_- \equiv \int_{\Omega^-} I(\vec{\omega}) \cos\theta d\Omega$$

$$\frac{dI_+}{d\tau^*} = -\int_{\Omega^+} I(\vec{\omega}) d\Omega + 2\pi B$$

Approximate I isotropic, then $\int_{\Omega^+} I(\vec{\omega}) d\Omega = 2I_+$

$$\frac{1}{2}\frac{dI_+}{d\tau^*} = -I_+ + \pi B$$

$\tau \equiv 2\tau^*$, $\cos\tilde{\theta} = \frac{1}{2}$;(Could use other mean dispersion. 1.66 vs 2)

mutatis mutandum for I_- "Schwartzschild equations"

Integral solution

Recall $\tau = \tau_\nu$ depends on ν, so $\tau_\nu(p)$ coordinate mapping depends on ν.

$$I_+(\tau_\nu, \nu) = I_+(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} \pi B\left(\nu, T\left(\tau'_\nu\right)\right) e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu$$

$$I_-(\tau_\nu, \nu) = I_-(\tau_{\infty,\nu})e^{-(\tau_{\infty,\nu} - \tau_\nu)} + \int_{\tau_\nu}^{\tau_{\infty,\nu}} \pi B\left(\nu, T\left(\tau'_\nu\right)\right) e^{-(\tau'_\nu - \tau_\nu)} d\tau'_\nu \quad (2)$$

Recovering the radiating level from the solution

$$OLR_\nu \equiv I_+(\tau_{\infty,\nu})$$

$$OLR_\nu = I_+(0)e^{-\tau_{\infty,\nu}} + \int_0^{\tau_{\infty,\nu}} \pi B\left(\nu, T\left(\tau'_\nu\right)\right) e^{-(\tau_{\infty,\nu}-\tau'_\nu)} d\tau'_\nu$$

OLR_ν is emission from each level weighted by $e^{-\Delta\tau}$, where

$$\Delta\tau \equiv \tau_{\infty,\nu} - \tau'_\nu$$

(plus attenuated emission from ground; $I_+(0) = \pi B(\nu, T_g)$, usually)

"All Troposphere model"

- $T(p)$ is determined by convection over the depth of the whole atmosphere.
- (True for about 90% of mass of Earth's atmosphere, also many other planets).
- For dry convection, set $T(p) = T_s(\frac{p}{p_s})^{R/c_p}$
- Since you know $\tau(p)$, you can compute $T(\tau)$ and plug into integral formulae to get $OLR(T_s)$ or other fluxes.

Will not carry it out in detail, since in problem sheet you will do this for "gray gas."

Gray gas case

κ independent of ν, so $\tau(p)$ same for all ν.
In this case, can integrate RT equations and integral solution over frequency:

$$\frac{dI_+}{d\tau} = -I_+ + \sigma T^4$$

$$\frac{dI_-}{d\tau} = I_- - \sigma T^4$$

$$I_+(\tau) = \sigma T_g^4 e^{-\tau} + \int_0^\tau \sigma T(\tau)^4 e^{-(\tau-\tau')} d\tau'$$

... etc.

Pure radiative equilibrium

Def^n: Equilibrium when only heat transfer mechanism is radiation
Can generally integrate flux over IR alone, and deal with shortwave absorption separately.
Net IR heating is proportional to $-\frac{d}{d\tau}(I_+ - I_-)$
Assuming no shortwave (solar) absorption in atmosphere, then

$$\frac{d}{d\tau}(I_+ - I_-) = 0$$

$$(I_+ - I_-) = \text{const} = OLR$$

I_+ and I_- are *frequency integrated* fluxes

Pure radiative equilibrium temperature profile

To get further, analytically, make gray gas approximation.

$$0 = \frac{d}{d\tau}(I_+ - I_-) = -(I_+ + I_-) + 2\sigma T^4$$

$$\frac{d}{d\tau}(I_+ + I_-) = -(I_+ - I_-) = -OLR$$

so

$$(I_+ + I_-) = -OLR\tau + C$$

At top of atm, $I_+ + I_- = OLR$, so

$$C = (1+\tau)OLR; \sigma T^4 = \frac{1}{2}(1 + \tau_\infty - \tau)OLR$$

BLACKBOARD : Skin temperature. Optically thin limit. Temperature in general case decreases with height. Unstable temperature discontinuity at the ground.

Back to general non-Gray case

Define transmission function $\mathfrak{T}_\nu(p_1, p_2) \equiv e^{-(\tau_\nu(p_1)-\tau_\nu(p_2))}$
then integral solution to Schwartschild equations can be written:

$$
\begin{aligned}
I_+(p,\nu) &= I_{+,s}(\nu)\mathfrak{T}_\nu(p,p_s) - \int_{p'=p}^{p_s} \pi B(\nu, T(p'))d\mathfrak{T}_\nu(p,p') \\
I_-(p,\nu) &= I_{-,\infty}(\nu)\mathfrak{T}_\nu(0,p) + \int_{p'=0}^{p} \pi B(\nu, T(p'))d\mathfrak{T}_\nu(p,p').
\end{aligned}
\tag{3}
$$

$d\mathfrak{T}_\nu(p,p')$ taken with p fixed

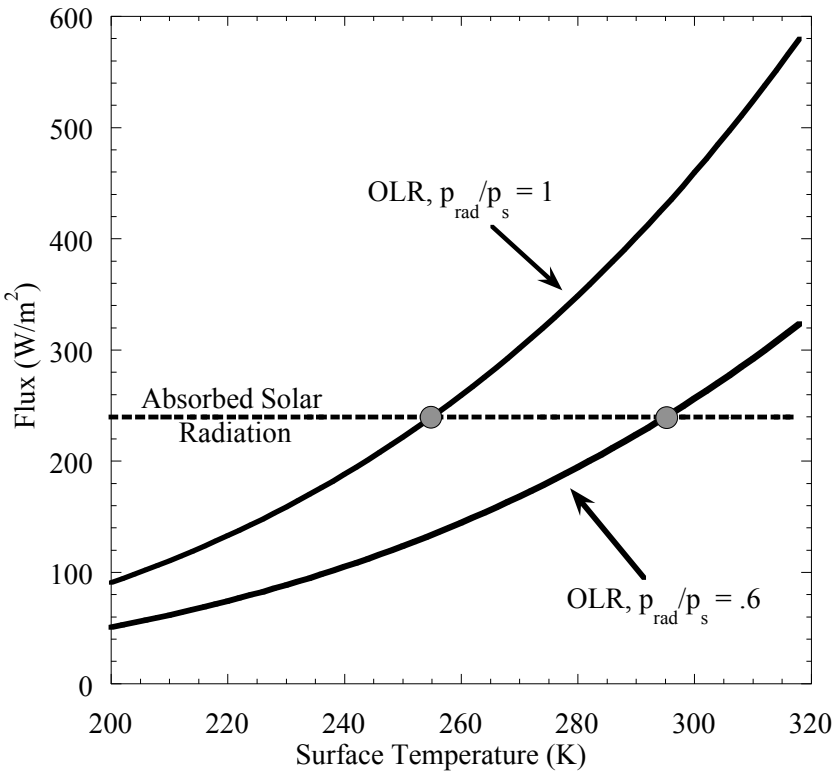
Equations are suitable for band-averaging over frequency intervals

Reprise: The *OLR* curve and climate equilibrium

Suppose we have a theory that relates $T(p)$ to the surface temperature T_s, e.g. dry adiabat, pure radiative equilibrium or mix of the two.

Then, from Schwartschild, we can compute $OLR(T_s)$. Increase of greenhouse gas concentration lowers the $OLR(T_s)$ curve, and shifts equilibria to warmer temperatures

Radiative forcing for greenhouse gases



Radiative forcing is the amount by which the *OLR* curve is shifted downward at a given T_s (the unperturbed equilibrium), when the greenhouse gas amount is increased with everything else held fixed.

Actually that's just the IR radiative forcing; in general radiative forcing could also come from the shortwave part of the energy balance, e.g. through change in solar luminosity or change in albedo from shortwave-reflecting or absorbing aerosols

In this lecture, I'll only discuss IR radiative forcing. The generalization to shortwave radiative forcing is straightforward.

Perturbed energy balance

Suppose OLR depends on a parameter λ (e.g. CO_2 concentration).
Suppose $T(p)$ can be computed uniquely from surface temp T_s.

$$0 = G \equiv OLR(T_s, \lambda) - (1-\alpha)\frac{L_\circledast}{4}$$

$$0 = \frac{d}{d\lambda} G = (\partial_{T_s} G)(\partial_\lambda T_s) + \partial_\lambda G$$

$$\partial_\lambda T_s = -\frac{\partial_\lambda G}{\partial_{T_s} G} = -\frac{\partial_\lambda OLR}{\partial_{T_s} OLR}$$

$-\partial_\lambda OLR$ is the *differential radiative forcing*

Computed with $T(p)$ and everything else held fixed

$\partial_{T_s} OLR$ is the *climate sensitivity parameter*

Computation from band-integrated integral solution

$$OLR_\Delta(q) = \pi B(\nu, T_s)\overline{\mathfrak{T}}_\nu(0, p_s) - \int_{p'=0}^{p_s} \pi B(\nu, T(p'))d\overline{\mathfrak{T}}_\nu(0, p'|q)$$

- For a single band; need to sum over all bands
- Overbar indicates band integral
- Assume band is narrow enough that $B(\nu, T)$ nearly constant over band width Δ

From some detailed calculations

- For CH_4, RF$\approx .036(\sqrt{M} - \sqrt{M_0})$
  M is concentration in ppbv; M_0 is baseline concentration.
- For CO_2, RF$\approx 3.6 \log_2 \frac{C}{C_0}$;
  C/C_0 is concentration relative to baseline.
  The logarithmic form arises from line overlap and the roughly exponential shape of the envelope of $\kappa(\nu)$ near the 15-micron absorption region.

But ... Feedbacks!

- Ice-albedo feedback
- Lapse rate feedback
- Water vapor feedback
- Cloud feedback

Computation of climate sensitivity parameter

(i.e. "Planck Feedback")

Assume:

- $\partial_{T_s} T(p) = 1$ (i.e. shifting T_s uniformly shifts temperatures at all levels)
- Temperature dependence of transmission is small

$$\partial_{T_s} OLR_\Delta = \pi \partial_{T_s} B(\nu, T_s) \overline{\mathfrak{T}}_\nu(0, p_s) - \int_{p'=0}^{p_s} \pi \partial_T B(\nu, T)|_{T(p')} d\overline{\mathfrak{T}}_\nu(0, p')$$

(note $d\overline{\mathfrak{T}}_\nu(0, p') < 0$)

Some general properties

- Transmission-weighted $\partial_T B$
- Sensitive primarily to temperature near radiating level
- For gray gas, sums up over all bands to approximately $4\sigma T_{rad}^3$

Computation of radiative forcing

Consider well-mixed gas, with uniform concentration q

$$\partial_q OLR_\Delta = \pi B(\nu, T_s)\partial_q \overline{\mathfrak{T}}_\nu(0, p_s) - \int_{p'=0}^{p_s} \pi B(\nu, T)|_{T(p')} d\partial_q \overline{\mathfrak{T}}_\nu(0, p')$$

q affects transmission through the mass path
Vanishes if T uniform in height and equal to T_s

Radiative forcing for a gas with all weak lines

Any gas is a weak-line gas for sufficiently small concentration

$$\begin{aligned}\partial_q OLR_\Delta &= -\pi B(\nu, T_s)\partial_q S\mu(0, p_s) + \int_{p'=0}^{p_s} \pi B(\nu, T) d\partial_q S\mu(0, p_s) \\ &= S \cdot [-\pi B(\nu, T_s)\frac{p_s}{g} + \int_{p'=0}^{p_s} \pi B(\nu, T(p'))\frac{1}{g} dp'] \end{aligned} \tag{4}$$

- Net radiative forcing for concentration q is proportional to q
- Radiative forcing is proportional to line strength
- Sensitive to vertical gradient of temperature.
  RF>0 when temperature decreases with height
- For multiple lines, just sum up S_j

Radiative forcing for a gas with separated strong lines

$$\begin{aligned}\partial_q OLR_\Delta &= -\pi B(\nu, T_s)\partial_q 2\sqrt{S\delta_0\mu_p(0, p_s)} \\ &+ \int_{p'=0}^{p_s} \pi B(\nu, T) d\partial_q 2\sqrt{S\delta_0\mu_p} \\ &= \sqrt{\frac{S\delta_0}{q}} \cdot [-\pi B(\nu, T_s)\frac{p_s}{\sqrt{2p_0 g}} \\ &+ \int_{p'=0}^{p_s} \pi B(\nu, T(p'))d\frac{p'}{\sqrt{2p_0 g}}]\end{aligned} \tag{5}$$

- Net radiative forcing for a given q increases like $\sqrt{q}$, i.e. slower than linear
- Proportional to $\sqrt{S\delta_0}$ (or summed over all lines in band)

The lesson learned

- Gases in weak line regime (i.e. with low concentration) tend to have high radiative forcing per molecule (e.g. SF_6, HFC's)
- As concentration increases and the gas enters the strong line regime, RF per molecule goes down (like $\frac{1}{\sqrt{q}}$)
- When lines start to overlap, functional form of decrease in "radiative efficiency" becomes different

An introduction to scattering

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November 16, 2020

Overview

Characterizing scattering
- Scattering by individual particles
- Characterizing scattering by a volume

Rayleigh scattering

Mie scattering

Appendix: Material not taught MT2019; Multiple scattering
- Formulation of the two-stream model
- Analytic solutions

Characterizing net scattering and absorption

Scattering and Absorption cross-section and efficiency

Let I_0 be incident parallel-beam irradiance. Flux taken out of parallel beam by *scattering* into other directions is $I_0 \chi_{sca}$. χ_{sca} is the effective area taken out of the beam.
Scattering efficiency is:

$$Q_{sca} \equiv \frac{\chi_{sca}}{\pi r^2}$$

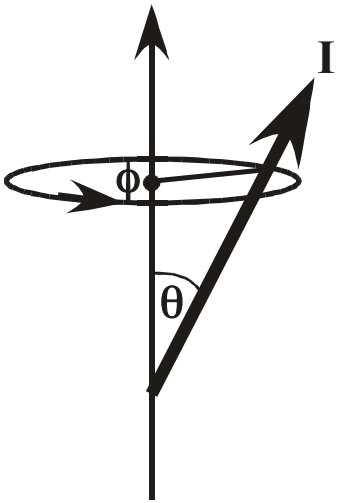
where r is particle radius. *Absorption cross section* and *Absorption efficiency* are defined analogously, Extinction $\chi_{ext} = \chi_{sca} + \chi_{abs}$,...

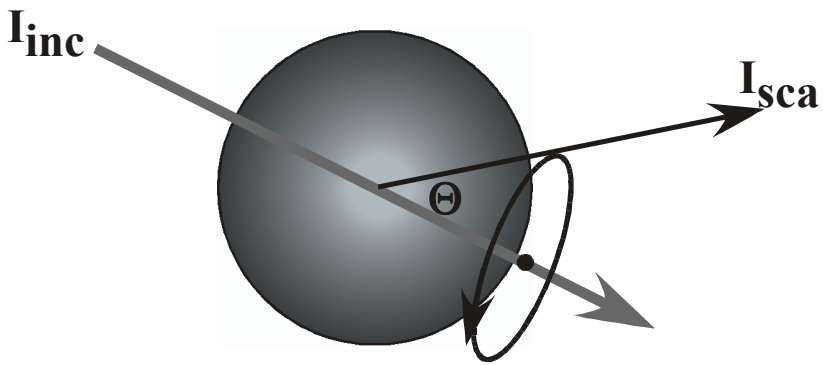
Single-scatter albedo

$$\tilde{\omega} = \frac{\chi_{sca}}{\chi_{sca} + \chi_{abs}} = \frac{Q_{sca}}{Q_{sca} + Q_{abs}}$$

Scattering angles

We use two different coordinate systems, one relative to an incident beam, and the other fixed with respect to the atmosphere





Angular distribution: The phase function

Let I_0 be irradiance of incident parallel beam radiation

Then, the phase function P for spherically symmetric scattering particles is defined by

$$I(\vec{\omega}) = \chi_{sca} I_0 \frac{1}{4\pi R^2} P(\Theta)$$

$$\int_{\text{sphere}} P d\Omega = 4\pi$$

where $I(\vec{\omega})$ is the radiance in direction $\vec{\omega}$, and R is distance from particle. Depends only on angle Θ relative to incident beam direction

For nonspherical particles, applies to average over randomly-oriented particles. "Equivalent spheres."

Asymmetry factor and model phase function

$$g \equiv \frac{1}{4\pi} \int_{\text{sphere}} P(\Theta) \cos\Theta d\Omega$$

$g = 0$ for symmetric scattering, $g = 1$ for all forward scattering, $g = -1$ for all back-scattering

Optical thickness of a medium

Consider a small volume containing a volume density of N scatterers with cross sections χ_{abs} and χ_{sca}, and a gaseous background with absorber mass concentration q and absorption (mass) cross section κ.

A parallel beam passing through the medium loses energy both to scattering into different directions and to absorption. The optical thickness including both processes is given by

$$d\tau = q\kappa\rho d\ell + N\chi_{ext} d\ell = (q\kappa\rho + N(\chi_{ext}))d\ell = (q\kappa + \frac{Nm}{\rho}\frac{\chi_{ext}}{m})d\mu$$

where m is particle mass. Nm/ρ is particulate mass concentration

Extinction coefficients

Attenuation of incident parallel beam is given by

$$I = I_0 e^{-\tau} = I_0 e^{-\kappa_{ext}\mu} = I_0 e^{-\beta_{ext}\ell}$$

where ℓ is the distance traveled.
β_{ext} is the *volume extinction coefficient*

(Note also:single-scatter albedo can be defined for a volume in the obvious way. Divide total scattering energy loss by total energy loss)

Planetary albedo for isotropic scatterers

(...neglecting multiple scattering; for simplicity assume ground has zero albedo and doesn't reflect any radiation back upward. Also assume no atmospheric absorption by particles or gases.)

- Scattering converts direct beam radiation into diffuse radiation.
- Energy lost from direct beam due to scattering is $I_0 - I_0 \exp(-\tau_{sca})$, where $\tau_{sca} = \frac{Nm}{\rho}\frac{\chi_{sca}}{m}\mu$ if scatterers have uniform concentration over a layer from p_1 to p_2. $\mu = (p_2 - p_1)/(g\cos\zeta)$, where ζ is the zenith angle. Phase function does not introduce a zenith-angle dependence of albedo, and the only zenith angle dependence comes from μ.

Planetary albedo for isotropic scatterers, cont'd

- For isotropic scatterers, half of energy scattered out of the direct beam goes upwards, and in the absence of multiple scattering, escapes to space. For an optically thin scattering layer, linearize the exponential and find that the albedo is $.5\tau_{sca}$.
- For an optically thick scattering layer, the albedo approaches .5, but in that case multiple scattering becomes important and can increase the albedo beyond .5.
- Note that introduction of absorption would reduce the albedo, both by reducing the proportion of direct beam available for scattering, and by absorbing the upward-scattered radiation.

Rayleigh Scattering: Definition and sketch of derivation

- Scattering by objects (particles or molecules) that are small compared to wavelength of incident electromagnetic wave. $\frac{r}{\lambda} \ll 1$
- Applied electromagnetic field uniform over extent of object. Induces oscillating dipole moment. Magnitude/orientation of dipole determined by *polarizability tensor*
- Then compute radiation by an oscillating dipole. (Can be obtained by superposing Maxwell Eqn. Green's functions for a charge undergoing small oscillations).
- For full derivation, see Goody and Yung

Rayleigh scattering cross-section

$$\chi_{sca} = \frac{6+3\delta}{6-7\delta}\frac{8\pi}{3}\left(\frac{2\pi}{\lambda}\right)^4 \alpha_p^2$$

- α_p is *polarizability*. Weakly dependent on λ
- Average over random orientations for non-spherical molecules, unpolarized incident beam
- For molecules there is no particular reason to introduce an effective radius
- δ is *depolarization factor*, representing effect of misalignment between induced dipole and electric field
  - $\rightarrow$ Zero for spherically symmetric (monatomic) case
  - $\rightarrow$ Generally pretty small (.054 for O_2, .0805 for CO_2

- Technically speaking, for polarizability in S.I. (mks) units, there should be some numerical factors multiplying the standard definition of α_p, but we have absorbed these into the definition, so units are m^3
- Can look up $\alpha_p(\lambda)$ in data compilations, but in practice works pretty well to look up χ_{sca} at a given λ and extrapolate from there.
- χ_{sca} per unit mass (of molecule) is a convenient statistic for summing up net scattering from a volume containing many particles. Same units, same meaning as (mass) absorption cross section κ we used to characterize gaseous absorption

	H_2	He	Air	N_2	O_2	CO_2	H_2O	NH_3	CH_4
χ_{sca}	1	0.0641	4.4459	4.6035	3.8634	10.5611	3.3690	7.3427	10.1509
χ_{sca}/m	1	0.0321	0.3066	0.3288	0.2415	0.4800	0.3743	0.8638	1.2689
τ_{ray}	0.40653	0.01305	0.12464	0.13367	0.09818	0.19513	0.15216	0.35116	0.51585

These results are based on observations of the index of refraction, and do not take into account variations in the polarization factors. The scattering cross-sections and cross-sections per unit mass for H_2 are $1.4 \cdot 10^{-38}\,\mathrm{m}^2$ ($4.215 \cdot 10^{-12}\,\mathrm{m}^2/\mathrm{kg}$) at a wavelength of 10 μm, $8.270 \cdot 10^{-33}\,\mathrm{m}^2$ ($2.490 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{kg}$) at a wavelength of 1 μm, and $3.704 \cdot 10^{-28}\,\mathrm{m}^2$ ($0.11\ \mathrm{m}^2/\mathrm{kg}$) at a wavelength of 0.1 μm. τ_{ray} is the optical depth due to Rayleigh scattering at $\frac{1}{2}$ μm for a 1 bar atmosphere of the indicated gas under Earth gravity.

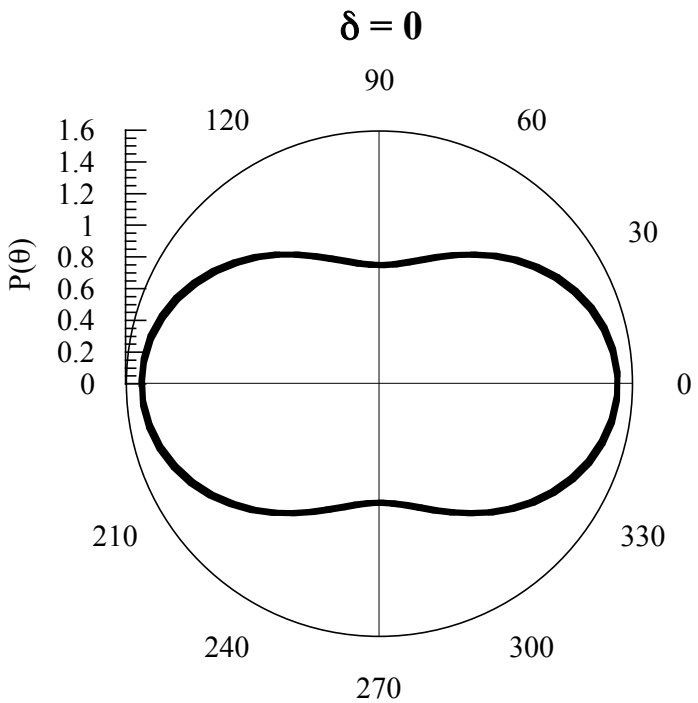
What about absorption?

Rayleigh scattering, even applied to atoms and molecules, is a fully classical theory. It is a conservative scattering theory, and there is no natural way to incorporate quantum effects on molecular absorption. (In practice, we generally just treat absorption separately)

Molecular scattering negligible for IR, so this is not an issue for IR absorption. For visible and UV, it is more problematic.

Rayleigh scattering phase function

$$P(\cos\Theta) = \frac{3}{2(2+\delta)}(1+\delta+(1-\delta)\cos^2\Theta)$$

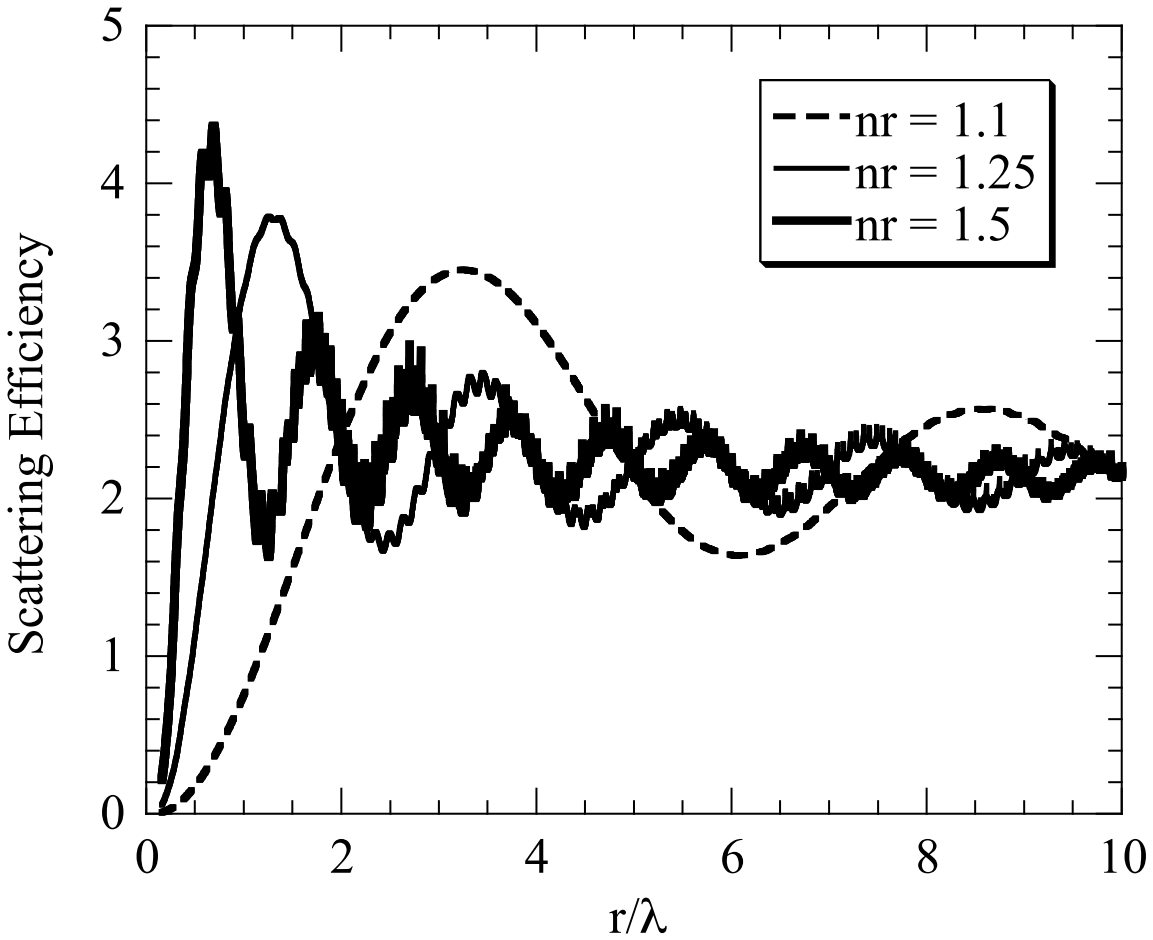


- Rayleigh scattering is symmetric ($g = 0$)
- Mildly anisotropic (weak side-scatter) for $\delta \approx 0$
- Becomes more isotropic as δ increases

Mie scattering: Definition and sketch of derivation

- Scattering by dielectric sphere with uniform index of refraction n embedded in medium with uniform index of refraction n_0 (often taken to be ≈ 1 as for air, but can be anything).
- Can apply to absorbing particles (use complex n), but here we will concentrate on conservative scattering case (real n)
- Have to solve Maxwell Eqns. inside and outside sphere (vector spherical harmonics!) apply continuity condition at interface, then decompose far-field into plane waves
- A magnificent feat of 19^{th} century special-function virtuosity, not applicable to any other problem you are likely to encounter. Generalizations (e.g. layered dielectrics) are done numerically (though still with vector spherical harmonics for angular part)

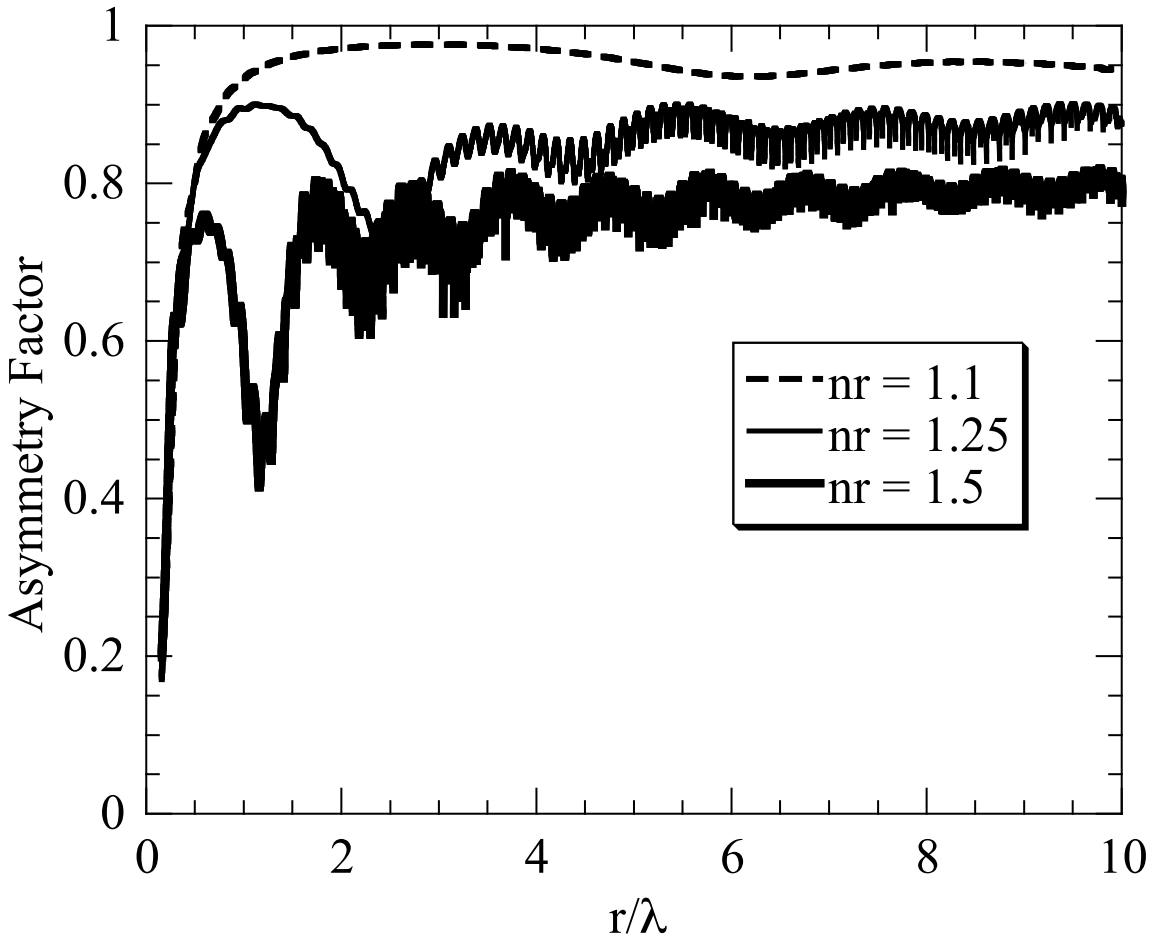
Numerical results: Scattering efficiency



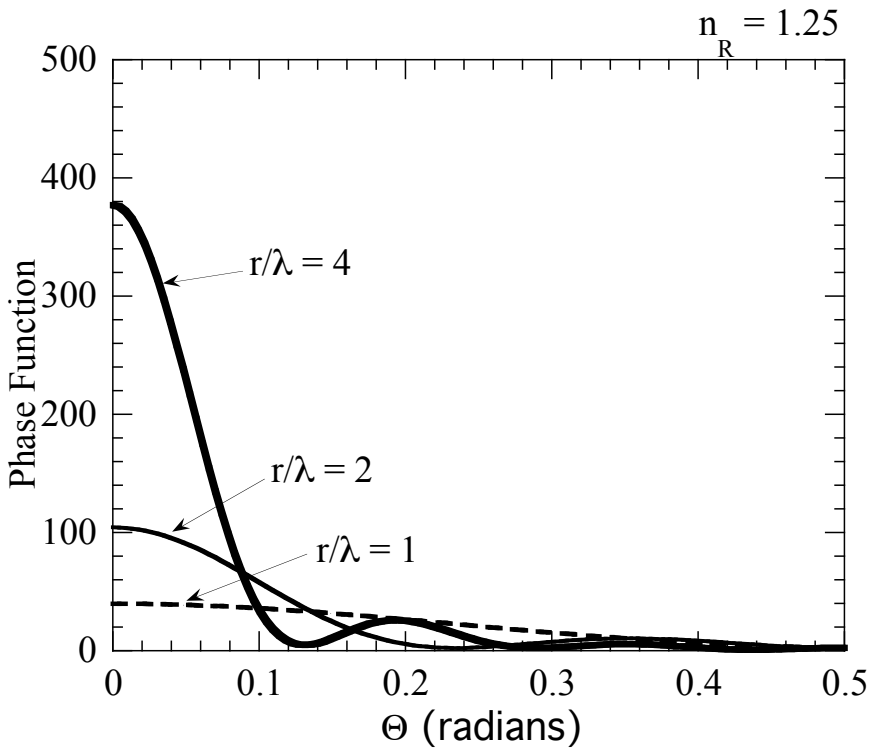
Limiting cases

- $x \ll 1$, Rayleigh limit
- $x \gg 1$, geometric optics (ray tracing)
  But note $Q \to 2$, not 1 as might be expected!

Numerical results: Asymmetry factor



For large particles, scattering is very forward-peaked



Very important for scattering by cloud droplets!

The basic problem:

- Define I_+ and I_- as before. These angle-averages are all we need for most climate calculations.
- We would like to derive closed equations for the fluxes in these two streams
- This cannot be done precisely, but we can get a useful approximate system
- Simplifications for these lectures:
  - Conservative scattering (no absorption or emission)
  - Symmetric scattering (no forward/backward asymmetry)

I_+ and I_- characterize the *diffuse radiation*. We need to also introduce a separate "stream" for the *direct beam* radiation which comes in at (essentially) a single direction from the star.

Two-stream scattering equations for diffuse radiation

$$\frac{dI_+}{d\tau^*} = -\gamma I_+ + \gamma I_-$$

$$\frac{dI_-}{d\tau^*} = \gamma I_- - \gamma I_+$$

- γ determined by assumption about angular distribution of the radiation
- $\gamma = 1$ for hemispherically isotropic case, $\frac{3}{4}$ for "Eddington" case
- "Eddington" more accurate for shortwave scattering without absorption/emission
- Easily show that energy conserved, $\frac{d}{d\tau^*}(I_+ - I_-) = 0$

The direct beam

Assume direct beam is purely downward (only diffuse reflection from surface)

$$I_{\circledast} = L_{\circledast} \exp(-(\tau^*_\infty - \tau^*)/\cos\zeta)$$

Concentrated in a single direction, with zenith angle ζ.

Coupling of diffuse radiation to direct beam

$$\frac{dI_+}{d\tau^*} = -\gamma I_+ + \gamma I_- + \gamma_+ I_\circledast$$

$$\frac{dI_-}{d\tau^*} = \gamma I_- - \gamma I_+ - \gamma_- I_\circledast$$

For isotropic scatterers, $\gamma_+ = \gamma_- = \frac{1}{2}$. Turns out to be true more generally for forward/backward symmetric scatterers.

Sketch of solution

First-order coupled linear inhomogeneous equation in two dependent variables, with constant coefficients. Solve by finding homogeneous and particular solution, then superposing to match boundary conditions.

- $I_- = 0$ at top of atmosphere
- $I_+ = 0$ at ground (because of zero ground albedo)

We primarily want to solve for $I_+(\tau^*_\infty)$, since that will give us the albedo.

Sketch of solution

For conservative scattering, problem simplifies because $I_+ - I_- - I_\circledast \cos\zeta$ is constant (call it F).

$$\frac{d}{d\tau^*}(I_+ + I_-) = -2\gamma(I_+ - I_-)$$

Reduces to quadrature since r.h.s. known in terms of F.
Integration introduces one additional constant of integration A.
Problem is solved by finding A and F that meet the two boundary conditions (results in a 2x2 linear system for A and F)

Albedo formula for perfectly absorbing ground

$$\beta_* \equiv 1 - \exp(-\tau^*_\infty / \cos\zeta)$$

$$\alpha = \frac{(\frac{1}{2} - \gamma\cos\zeta)\beta_* + \gamma\tau^*_\infty}{1 + \gamma\tau^*_\infty}$$

Infrared Radiative Transfer

Molecular Spectroscopy and radiation basics

1. **Spectral line width and position**

   It might appear that transitions between two energy levels in an atom would lead to an infinitely thin spectral line. What would the consequences for atmospheric absorption be if lines really were infinitely thin? Briefly discuss the three mechanisms that can lead to finite-width lines. Now consider the 2173 cm^{-1} line of carbon monoxide (CO), which has a full width of about .12 cm^{-1} at 1 bar and 290K. (That's actually just for broadening by collisions with air, but for the purposes of this problem assume that the same applies for collisions with other gases). For temperatures of 200K, 400K and 2000K determine the pressures at which each of the three broadening mechanisms become dominant. (The 2000K case is relevant to CO on hot exoplanets). For Pluto, which has atmospheric temperatures in the range of 50K to 100K and a surface pressure on the order of 11 microbars, what broadening mechanisms are relevant?

   To estimate the intrinsic (i.e. uncertainty) line width, you will need to know the typical lifetimes of excited vibrational states. These are given by the Einstein A-coefficients, whose units are usually $\sec^{-1}$, and which give the decay rate of populations of excited states by spontaneous emission. A coefficients are tabulated in major spectroscopic databases, but you can get a quick look which is good enough for this problem from these publications on CO:

   Chandra et al. A&A Suppl. Series 117 557-559 (1996)
   https://aas.aanda.org/articles/aas/pdf/1996/09/ds1033.pdf

   Thesis (U. of Lethbridge)
   https://opus.uleth.ca/bitstream/handle/10133/4823/ISLAM_NAZRUL_MSC_2016.pdf?sequence=1
   see Table 4.2 and 4.3

   [Question Total: 10]

2. **Secondary eclipse observation of an exoplanet** Consider an exoplanet in a system in which the planet transits its star. This means that the planet passes in front of the star (blocking some of the starlight) and also passes behind the star (blocking the thermal emission from the planet). The latter is called the *secondary eclipse*, and by looking at the drop in radiation as the planet goes into secondary eclipse, one can determine the dayside thermal emission from the planet when it is viewed face-on.

   Suppose that the star has radius $r_\circledast$, photospheric temperature $T_\circledast$, and radiates like an ideal blackbody. Suppose the planet orbits the star at a distance r, has radius r_p, and has uniform albedo α. The planet is an airless rocky body and its surface reflects as an ideal (isotropic) diffuse reflector. Being airless, each part of the surface can be considered to be in instantaneous equilibrium with the stellar radiation (per unit area) incident upon it. The planet is tide-locked to the star (as the Moon is tide-locked to Earth) so that the substellar point is at a fixed geographical position.

   (a) As a warm-up, to exercise some ideas for dealing with the angular distribution of radiation, let's consider the flux seen by a distant observer from an isothermal blackbody sphere of radius a. In this case, we can do a shortcut to get the flux, since the net power

put out is $4\pi a^2 \sigma T^4$, and for an observer at distance d the radiation is approximately parallel beam and spread out over a sphere with area $4\pi d^2$. Thus, the parallel beam flux seen by a distant observer looking directly at the sphere is just $\sigma T^4 (a/d)^2$, which is the inverse-square law. Now, as a warm-up for the rest of this problem, compute the same flux in a more complicated way, by integrating up the radiance *in the direction of the observer* coming out of each patch of the sphere. Note that radiance is a flux *per unit solid angle*, so that to get the flux seen by a distant observer – which is essentially the flux captured by a telescope mirror of area $1\mathrm{m}^2$ pointed directly at the object – you need to determine the appropriate differential of solid angle $d\Omega$ to multiply the radiance by in order to represent the collection of rays which intercepts the detector. Keep thinking about how to do this until you get a result that agrees with the $1/d^2$ result that comes from the shortcut. Would the flux intercepted change if the emitter were a blackbody flat disk with normal pointed toward the observer rather than a blackbody sphere?

(b) Compute the thermal emission flux from the planet as a proportion of the stellar flux just before it goes into secondary eclipse, expressed as parts per million. Note that this is the total flux emitted over all angles, which is greater than the flux seen by a distant observer, since that observer only captures the portion of emitted flux going in the direction of the observer. We'll correct the contrast ratio for that effect in a later part of this problem.

(c) Estimate (very roughly) the range of wavenumbers containing most of the thermal emission from the planet, and recompute the contrast ratio between planetary and stellar emission in just this wavenumber range.

(d) Compute the total reflected flux from the planet as a proportion of the stellar flux just before it goes into secondary eclipse, expressed as parts per million. Compare this to the contrast ratio for thermal emission. What does this say about the difficulty of determining the albedo of an exoplanet, as compared to the difficulty of determining its thermal emission?

(e) Assuming that the system is a distance d from a telescope in the Solar System, compute the flux from the planet and star (in W/m^2) incident upon the telescope. Whereas the previous parts of the problem can be computed just by energy balance, this problem requires integrating up the flux from each patch of the planet that is emitted *in the direction of the observer*. The star is an isothermal blackbody, so the simple computation of the flux seen by the observer applies. The planet is not isothermal though. The nightside is so cold it doesn't radiate at all. If we introduce a spherical polar coordinate system such that $\theta = 0$ corrresponds to the substellar point, then the energy balance for a ring of the planet at a position θ on the dayside is

$$\sigma T(\theta)^4 = (1-\alpha) L_\circledast \cos\theta$$

since the $\cos\theta$ factor gives the projected area of a segment of the dayside hemisphere, i.e. the effective area of the incident parallel beam from the star that is intercepted. You now need to compute the radiance emitted *in the direction of the distant observer*, integrated up over the surface of the dayside, and then allow for the solid angle of the radiation intercepted by the observer. The calculation is exactly the same as you did for the isothermal sphere, except that in this case the hemisphere is not isothermal. Use this calculation to

correct the contrast ratio seen by the observer. Assuming the planet to be a perfectly diffuse reflector (Lambert sphere) would the correction be any different for reflected light?

Assuming the mirror of the telescope to have an area A, estimate the number of photons per second from the planet's thermal emission intercepted by the mirror.

(f) Look up the parameters of the exoplanet Trappist 1b (you can find it in either the NASA Exoplanet Archive or Wikipedia), and evaluate the formulae derived above assuming Trappist 1b to be an airless rocky planet with uniform albedo .3.

[Question Total: 20]

3. **Dyson Spheres** The physicist Freeman Dyson has speculated that an advanced civilization might enclose its star within a spherical shell of material, and live on the inner surface. This gives them more living space, and avoids wasting all that starlight that would ordinarily escape to space. Consider a red dwarf star with photospheric temperature 4000 K and radius 200 000 km. What should the radius of the Dyson sphere be in order to have an inner surface temperature of 300 K? Assume that the sphere material is a perfect blackbody with zero albedo at all wavelengths, and that it is a good conductor of heat so that the inner surface and outer surface temperature are identical. You may also assume that the temperature of the photosphere remains unchanged after the Dyson sphere is built. If you were an astronomer looking for such a Dyson sphere, in what part of the electromagnetic spectrum should you look? What radius should the Dyson sphere have if it is instead made of a good thermal insulator, so that the outer surface is significantly colder than the inner surface? In the latter case, assume that the heat flux through the shell is proportional to the temperature difference between the inner and outer surface, and that this flux must match the radiation to space from the outer surface.

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[Question Total: 5]

Grey Gas Radiative Transfer

4. **Grey gas OLR** Compute $OLR(T_s)$, where T_s is surface temperature, for a gray-gas atmosphere for which $T(p)$ is on the dry adiabat all the way to $p = 0$, the absorption coefficient for the infrared absorber in the atmosphere is independent of pressure, and the surface temperature is equal to the temperature of the air immediately above the surface. (The full answer is an incomplete Gamma function, but you can get an analytic expression in the optically thick limit). Determine T_s as a function of the optical thickness of the atmosphere (which is proportional to the concentration of the absorber), and find an optical thickness that yields an Earthlike temperature of 280K when in equilibrium with an absorbed solar flux of 240 $\mathrm{W/m^2}$. Now suppose that the concentration of the absorbers is doubled (doubling the optical thickness). How much does the temperature increase. Based on your answer, discuss why it is a good thing for Earth's habitability that $\mathrm{CO_2}$ does not behave like a gray gas. (Note: This calculation does not take into account the importance of water vapor feedback, which adds to the opacity caused by doubling $\mathrm{CO_2}$)

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[Question Total: 15]

5. **Nuclear Winter** "Nuclear winter" refers to the climate following an event such as (perhaps) a global thermonuclear war or an asteroid impact, which lofts a high-altitude long-lived soot layer into the atmosphere. As an idealized form of this phenomenon, find the radiative-equilibrium temperature profile of a gray gas atmosphere below a high altitude soot cloud which is so thick that it completely absorbs the solar radiation incident upon it. You may assume that the soot cloud is isothermal, and acts like a blackbody in the infrared, and that the atmosphere above the soot cloud is thin enough to be essentially transparent to both shortwave and infrared radiation.

   Do you expect a troposphere to form in this situation? How does the surface temperature compare to what it would be without the soot cloud? What do you think would happen to the precipitation?

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[Question Total: 15]

Real gases

6. **Transmission function for Methane**

   The file `MethaneLinesC5.txt` contains a list of line properties for methane in the wavenumber range 1150 to 1400 cm^{-1}, which is the region most important for methane radiative forcing for the relatively small amounts of methane present in Earth's present atmosphere (on the order of 1 ppmv). This data was downloaded directly from the `HITRAN` database, which is a good source to become familiar with if you are going to be working with radiative transfer. It is one of the two main line databases in use, together with the more specialized `exomol` database. The `HITRAN` database is available at `hitran.org` . You need to register in order to download data, but registration is free and easy.

   The data file is stored in the C5 Canvas directory under the Radiative Transfer module. In order to make the data accessible, you will need to copy it to whichever directory you are running your Jupyter Notebook from. If running Python on your own computer you just download it and put it in your working directory which holds your Notebook files. If running Python on the Physics notebook server, after download you need to upload it to your Physics home directory that contains your notebook files. If using Google's CoLab, you need to upload the file to a place in your Google Drive where CoLab can access it (this is a bit tricky).

   A useful Python trick for reading in data from text files and putting them in `numpy` arrays for further processing is to use the `numpy.loadtxt` command, with the appropriate flags. Note that since the methane lines file contains a text header giving the column definitions, you will need to set the appropriate flag to skip the first line, and another flag to unpack the columns into separate `numpy` arrays. You can find complete documentation on `numpy.loadtxt` at:

   `https://numpy.org/doc/stable/reference/generated/numpy.loadtxt.html`

   A description of the data in the dowloaded line list can be found in the file `MethaneLinesC5-readme.txt` Read this carefully for the definition of the columns, since HITRAN uses a different notation for line width than we have been using in class (HITRAN uses γ, whereas we use δ, and HITRAN uses δ for the line *shift* coefficient, which we haven't discussed). Note that the units

of line strength are $\mathrm{cm}^{-1}/(\mathrm{molecule}/\mathrm{cm}^2)$ so in order to make use of the formulae we have derived in class, you will need to convert to units of $\mathrm{cm}^{-1}/(\mathrm{kg}/\mathrm{m}^2)$, making use of the mass of a methane molecule in kg. The funny units are because we measure frequency in units of cm^{-1}, even though we generally otherwise use S.I. units.

The line strength (intensity) value and line width are given for air pressure of 1 atmosphere (1.013 bar) and temperature 296K. In doing this problem, you may neglect the temperature dependence of line strength and line width. Assume general planetary parameters corresponding to the present Earth's atmosphere where you need them (e.g. mean molecular weight 29, 1 bar surface pressure, $9.8\mathrm{m}/\mathrm{s}^2$ gravity

(a) For a CH_4 concentration of 1ppmv, assumed uniform throughought the atmosphere, how many of the lines are strong lines for a mass path corresponding to the entire depth of the atmosphere? How many are weak lines? What are the numbers for concentrations of .1ppmv and 10 ppmv? In assessing whether the line is in a strong line or weak line limit, use the optical thickness at the line center for the layer of the atmophere between the ground (1 bar pressure) and the 0.1 bar level. What happens if you try to use the same formula all the way to the top of the atmosphere ($p = 0$)? What would you have to do to assess strong vs. weak lines for the layer between 0.1 bar and the top of the atmosphere? (You needn't actually carry out that modification; a discussion of the relevant issues will suffice).

(b) Compute the band-integrated transmission function between the ground and the 0.1 bar level in the wavenumber range covered by the file, as a function of concentration, for concentrations ranging from zero to 50 ppmv, assuming that lines do not overlap. At what concentration do the strong lines dominate the behaviour of the transmission? For what concentrations would you need to start taking overlapping of lines into account?

[Question Total: 25]

Scattering

7. **Radiative forcing from stratospheric aerosols** In this problem, we suppose that small stratospheric sulfate particles ("aerosols") are created when a volcano injects SO_2 gas into the stratosphere, resulting in an increase of the albedo. The object of this problem is to relate the increase in albedo due to stratospheric aerosols to the mass of SO_2 injected into the stratosphere. Stratospheric aerosols have a typical lifetime of a year or two before they fall out of the stratosphere and are washed out by rain, so the albedo change caused by the injected mass can be thought of as persisting for around a year.

   To do this problem accurately, one would need to use the Mie scattering phase function corresponding to the index of refraction for sulfuric acid, but to keep things simple, you may assume that the aerosols are isotropic scatterers with unit scattering efficiency.

   (a) Find the relationship between the mass of SO_2 injected and the scattering optical depth of the stratosphere associated with the aerosols. Assume that the aerosols form into droplets with a radius of $1\,\mu$m, and that each SO_2 molecule injected forms one $\mathrm{H}_2\mathrm{SO}_4$ molecule,

which forms a droplet which is composed of half water and half sulfuric acid by mass. Assume that the aerosols spread out into a horizontally uniform layer within the entire global stratosphere. The sulfate aerosols can be assumed to be nonabsorbing in the shortwave spectrum.

Based on what you know about Mie scattering, how good are the assumptions we have made about the scattering properties of the aerosols? How sensitive is the optical depth to the assumed size of the aerosol particles?

(b) Compute the albedo as a function of the optical depth you computed above, neglecting multiple scattering. . You may neglect the Rayleigh scattering from stratospheric air in this calculation. How does the albedo depend on the zenith angle?

Based on Earth's instellation $L_\odot = 1360\mathrm{W}/\mathrm{m}^2$, determine how much albedo increase would be needed to cause $4\mathrm{W}/\mathrm{m}^2$ of top of atmosphere radiative forcing (in the cooling direction), when averaged over the entire surface of the Earth. Use this to determine the mass of SO_2 that would need to be injected by the volcano to have a cooling effect comparable to the warming effect of doubling CO_2, which induces a radiative forcing of approximately $4\mathrm{W}/\mathrm{m}^2$ when averaged over the surface of the Earth.

[Question Total: 10]