Useful formulas - Radiative Processes

David Alonso - david.alonso@physics.ox.ac.uk
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These notes are not meant to be an exhaustive account of the materials we went over during the live lectures. Although I have tried to provide some semblance of a narrative to the discussion below, this is scant. The main aim instead is to provide a compendium of all the mathematical formulas that we used in the lectures, in order to amend any typos that an inattentive lecturer (or note taker!) may have incurred.

1 Basics of radiative transfer

1.1 Quantifying radiation

Intensity is the amount of energy per unit time, frequency, solid angle, and area perpendicular to the lightray:

$$I_\nu \equiv \frac{dE}{dt\, d\nu\, d\Omega\, dA_\perp} \tag{1}$$

Relation to photon distribution function and occupation number:

$$f_\gamma(\nu) = \frac{c^2}{h^4\nu^3} I_\nu = \frac{n_\nu}{h^3}. \tag{2}$$

Flux is energy per unit frequency, time, and arbitrarily-oriented area:

$$F_\nu \equiv \frac{dE}{dt\, d\nu\, dA} = \int d\Omega\, \cos\theta\, I_\nu. \tag{3}$$

For an isotropic source, integrating over a half-sphere (i.e. at the surface of the emitting medium)

$$F_\nu^{\rm iso} = \pi I_\nu. \tag{4}$$

Radiation pressure can be calculated from the momentum change of photons colliding against a surface:

$$P_\nu = \frac{2}{c}\int d\Omega\, \cos^2\theta\, I_\nu, \tag{5}$$

Other related quantities are the radiation energy density u_ν and angle-averaged intensity:

$$J_\nu \equiv \int \frac{d\Omega}{4\pi}\, I_\nu, \quad u_\nu \equiv \int d\Omega\, \frac{I_\nu}{c} = \frac{4\pi\, J_\nu}{c}. \tag{6}$$

Finally, for isotropic radiation:

$$P_\nu = \frac{u_\nu}{3}. \tag{7}$$

1.2 Thermal radiation

Non-relativistic particles (e.g. low-energy electrons) in thermal equilibrium have **Maxwell-distributed** momenta:

$$\frac{dN}{d\mathbf{x}^3\, d\mathbf{p}^3} = \bar{n}\,(2\pi m k_B T)^{-3/2} \exp\left[-\frac{p^2}{2\,m\,k_B T}\right], \tag{8}$$

thus, the velocity distribution is

$$\frac{dp}{dv} = 4\pi v^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{m\,v^2}{2k_B T}\right]. \tag{9}$$

Radiation that is itself in thermal equilibrium has a **black-body spectrum**:

$$I_\nu = B_\nu(T) \equiv \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_B T}-1}. \tag{10}$$

The flux emitted by a black-body source (integrated over frequencies and angles)

$$F = \sigma T^4 = \frac{2\pi^5 k_B^4}{15c^2h^3}T^4, \tag{11}$$

where σ is the Stefan-Boltzmann constant.

At low- and high- frequencies we get the Rayleigh-Jeans and Wien limits respectively:

$$B_\nu(T) = \begin{cases} \frac{2\nu^2}{c^2}k_B T & h\nu \ll k_B T \\ \frac{2h\nu^3}{c^2}e^{-h\nu/k_B T} & h\nu \gg k_B T \end{cases}. \tag{12}$$

Non-blackbody spectra can be assigned effective temperatures based on:

1. Intensity at low frequencies ("antenna temperature" or "brightness temperature"):

$$T_b = \frac{c^2}{2k_B\nu^2} I_\nu. \tag{13}$$

2. Color temperature: fit the shape of the spectrum to a blackbody shape with a free amplitude.

3. Effective temperature: apply Stefan-Boltzmann law to a measured flux

$$T_{\rm eff} = (F/\sigma)^{1/4}. \tag{14}$$

1.3 Radiative transfer equation

The radiative transfer equation (RTE) describes the changes to the radiation intensity as photons propagate through an emitting and absorbing medium:

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu. \tag{15}$$

α_ν is the absorption coefficient, with units of length^{-1}. j_ν is the emission coefficient:

$$j_\nu = \frac{dE_{\rm emitted}}{dV\,dt\,d\nu\,d\Omega}. \tag{16}$$

Defining the optical depth

$$\tau_\nu \equiv \int ds\,\alpha_\nu, \tag{17}$$

and the source function

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}, \tag{18}$$

the RT equation reads

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu \tag{19}$$

With solution:

$$I_\nu = e^{-\tau_\nu} I_\nu^0 + \int_0^{\tau_\nu} d\tau'_\nu e^{\tau'_\nu - \tau_\nu} S_\nu(\tau'_\nu). \tag{20}$$

For a homogeneous medium, this is

$$I_\nu = S_\nu + e^{-\tau}(I_\nu^0 - S_\nu). \tag{21}$$

For thermal sources, the **Kirchoff law** must hold, guaranteeing that the solution to the RTE is a blackbody:

$$j_\nu = \alpha_\nu B_\nu \tag{22}$$

2 Discrete transitions

2.1 Processes

1. **Spontaneous emission.** Einstein coefficient A_{21} defined as transition probability per unit time for a single atom. Emission coefficient is then:

$$j_\nu = \frac{h\nu\, n_2\, A_{21}}{4\pi}\varphi(\nu), \tag{23}$$

   where $\varphi(\nu)$ is the line profile, and n_2 is the number density of atoms/ions in the excited state. The line flux and luminosity (assuming no absorption) are then given by

$$L_{21} = h\nu_{12}A_{21}\int dV\, n_2, \quad F_{21} = \frac{h\nu_{12}A_{21}}{4\pi d^2}\int dV\, n_2. \tag{24}$$

2. **Absorption.** Einstein coefficient B_{21} defined such that $B_{21}u_{\nu_{12}}$ is the transition probability per unit time for a given atom, where u_ν is the radiation energy density. The absorption coefficient is then

$$\alpha_\nu^{12} = \frac{h\nu\, n_1 B_{12}}{c}\varphi(\nu). \tag{25}$$

   It is also common to define the B_{12} coefficient in terms of the **oscillator strength** f_{12} as

$$B_{12} = \frac{q^2}{4\epsilon_0 m_e h\nu_{12}} f_{12}, \tag{26}$$

   where q is the electron charge.

3. **Stimulated emission.** Thermodynamic inverse of absorption. Needed to guarantee detailed balance. Einstein coefficient B_{21} defined the same way as B_{12}. Effective absorption coefficient:

$$\alpha_\nu^{21} = -\frac{h\nu\, n_2 B_{21}}{c}\varphi(\nu). \tag{27}$$

   Absorption and stimulated emission can be considered at the same time with a single effective absorption coefficient:

$$\alpha_\nu = \frac{h\nu}{c}\left[n_1 B_{12} - n_2 B_{21}\right]\varphi(\nu). \tag{28}$$

4. **Collisional de-excitation.** The number of de-excitations per unit time and volume, given the collisional cross-section $\sigma_{21}(v_e)$, is

$$\frac{dN_{21}}{dV\,dt} = n_2 n_e \int_0^\infty dv_e\, v_e\, \frac{dp}{dv_e}\sigma_{21}(v_e) = n_e n_2 C_{21}, \tag{29}$$

which defines the Einstein C_{21} coefficient.

5. **Collisional excitation.** The only difference in this case is that only electrons with energy above $h\nu_{12}$ are able to produce an excitation:

$$\frac{dN_{12}}{dV\,dt} = n_1 n_e \int_{v_{\min}}^\infty dv_e\, v_e\, \frac{dp}{dv_e}\sigma_{12}(v_e) \equiv n_1 n_e C_{12}. \tag{30}$$

with $v_{\min} = \sqrt{2h\nu_{12}/m_e}$.

2.2 Collisional processes

Definition of Ω_{21} and Ω_{12}:

$$\sigma_{ij} = \frac{h^2}{4\pi m_e^2 v_e^2}\frac{\Omega_{ij}(v_e)}{g_i}. \tag{31}$$

Assuming a Maxwellian velocity distribution, the relations between Ω_{ij} and C_{ij} are then:

$$\begin{aligned} C_{21} &= \left(\frac{2\pi\hbar^4}{m_e k_B T_e}\right)^{1/2} \frac{1}{g_2}\bar{\Omega}_{21} \\ &= (8.63\times10^{-12}\,\mathrm{m^3\,s^{-1}})\left(\frac{1\,\mathrm{K}}{T_e}\right)^{1/2} g_2^{-1}\bar{\Omega}_{21}, \end{aligned} \tag{32}$$

$$\begin{aligned} C_{12} &= \left(\frac{2\pi\hbar^4}{m_e k_B T_e}\right)^{1/2} \frac{1}{g_1} e^{-\frac{h\nu_{12}}{k_B T_e}}\bar{\Omega}_{12} \\ &= (8.63\times10^{-12}\,\mathrm{m^3\,s^{-1}})\left(\frac{1\,\mathrm{K}}{T_e}\right)^{1/2} g_1^{-1} e^{-\frac{h\nu_{12}}{k_B T_e}}\bar{\Omega}_{12}, \end{aligned} \tag{33}$$

where we have defined

$$\bar{\Omega}_{ij} \equiv \int_0^\infty d\theta\, e^{-\theta}\Omega_{ij}(\theta), \tag{34}$$

with $\theta = m_e v_e^2/(2k_B T_e)$.

The two collisional coefficients are therefore related via

$$C_{12} = \frac{g_2\bar{\Omega}_{12}}{g_1\bar{\Omega}_{21}} e^{-\frac{h\nu_{12}}{k_B T_e}} C_{21}. \tag{35}$$

2.3 Thermal equilibrium, Einstein relations

In thermal equilibrium, processes involving photons must balance out, and the resulting radiation must be blackbody. The collisional processes must also balance out, and the resulting level populations must be related by a Boltzmann distribution ($n_2/n_1 = \exp(-h\nu_{12}/k_B T) g_2/g_1$). This implies the following relation between the Einstein coefficients:

$$\bar{\Omega}_{21} = \bar{\Omega}_{12}, \quad A_{21} = \frac{2h\nu_{12}^3}{c^2}B_{21}, \quad g_1 B_{12} = g_2 B_{21}. \tag{36}$$

These relations don't depend on temperature, and thus hold in general. Therefore, it is sufficient to know e.g. A_{21} and $\bar{\Omega}_{21}$ to determine all other coefficients.

2.4 Statistical equilibrium, emission lines

Statistical equilibrium is less restrictive than thermal equilibrium, and simply assumes that the level populations are constant over time. We looked at two simplified scenarios: the two-level and the three-level atoms.

2.4.1 The two-level atom

Statistical equilibrium for the ground level reads:

$$\frac{dn_1}{dt} = n_2 A_{21} + u_\nu (n_2 B_{21} - n_1 B_{12}) + n_e (n_2 C_{21} - n_1 C_{12}) = 0, \tag{37}$$

or

$$n_2 (A_{21} + u_\nu B_{21} + n_e C_{21}) = n_1 (u_\nu B_{12} + n_e C_{12}). \tag{38}$$

When exploring line emission, we studied the **weak-radiation limit**, where all terms involving u_ν are ignored. In this case, line fluxes can be used to determine the mass of a given element, M_A:

$$F_{21} = \frac{h\nu_{12}}{4\pi d^2} A_{21} \int dV n_2 = A_{21} \frac{h\nu_{12}}{4\pi d^2} \frac{M_A}{m_A} \left(1 + \frac{n_1}{n_2}\right)^{-1}. \tag{39}$$

The population ratio can then be found to be:

$$\frac{n_1}{n_2} = \frac{A_{21} + n_e C_{21}}{n_e C_{12}}. \tag{40}$$

We defined the **critical electron density**:

$$n_e^* \equiv A_{21}/C_{21}. \tag{41}$$

The low-density limit ($n_e \ll n_e^*$) is also called the **coronal approximation**. It is valid in planetary nebulae, for instance. In that case, the ratio of level populations is simply given by

$$\frac{n_1}{n_2} = \frac{A_{21}}{n_e C_{12}}. \tag{42}$$

In the opposite, **high-density limit**, the population ratio is

$$\frac{n_1}{n_2} = \frac{C_{21}}{C_{12}} = \frac{g_1}{g_2} e^{\frac{h\nu_{12}}{k_B T_e}} \simeq \frac{g_1}{g_2}, \tag{43}$$

where in the last equality we have assumed a small energy gap ($h\nu_{12} \ll k_B T_e$).

2.4.2 The three-level atom

In this case, the equations of statistical equilibrium are a bit longer. In the weak radiation limit:

$$n_3 \left[A_{31} + A_{32} + n_e (C_{31} + C_{32})\right] = n_e \left[n_2 C_{23} + n_1 C_{13}\right] \tag{44}$$

$$n_2 \left[A_{21} + n_e (C_{21} + C_{23})\right] = n_3 A_{32} + n_e \left[n_1 C_{12} + n_3 C_{32}\right]. \tag{45}$$

Line flux ratios can then be used to estimate the electron temperature and density. These are given by:

$$\frac{F_{32}}{F_{21}} = \frac{n_3}{n_2} \frac{A_{32}\nu_{23}}{A_{21}\nu_{12}}, \qquad \frac{F_{31}}{F_{21}} = \frac{n_3}{n_2} \frac{A_{31}\nu_{13}}{A_{21}\nu_{12}}. \tag{46}$$

Without further approximations, the level population ratio is given by

$$\frac{n_3}{n_2} = \frac{A_{21}C_{13} + n_e[C_{23}C_{12} + C_{23}C_{13} + C_{21}C_{13}]}{A_{31}C_{12} + A_{32}(C_{12} + C_{13}) + n_e[C_{31}C_{12} + C_{32}C_{12} + C_{32}C_{13}]}. \tag{47}$$

Note that the temperature dependence is implicit in the C coefficients ($C_{xy} \propto g_x^{-1} T_e^{-1/2} \exp(-h\nu_{xy}/k_B T_e)$). Note also, that the population ratio (and hence the line flux ratios) tend to a constant in the high- and low-density limits.

This result simplifies if one assumes large energy gaps, in which case $n_3 \ll n_2 \ll n_1$. This yields

$$\frac{n_3}{n_2} = \frac{C_{13}[A_{21} + n_e(C_{23} + C_{21})]}{C_{12}[A_{31} + A_{32} + n_e(C_{31} + C_{32})]}. \tag{48}$$

In class, we looked at two limits:

1. **Coronal approximation** (small n_e). In this case, the line flux ratios are e.g.:

$$\frac{F_{32}}{F_{21}} = \frac{A_{32}\nu_{23}}{(A_{31} + A_{32})\nu_{21}} \frac{\bar{\Omega}_{13}}{\bar{\Omega}_{12}} e^{-\frac{h\nu_{23}}{k_B T_e}}. \tag{49}$$

   This depends explicitly and trivially on T_e.

2. **Density diagnostic.** In this case we assumed the 3-2 transition was heavily forbidden and collisionally difficult. We also assumed the corresponding energy gap was small (in which case $h\nu_{23} \ll k_B T_e$, and $\nu_{31}/\nu_{21} \simeq 1$). The line flux ratio in this case is

$$\frac{F_{31}}{F_{21}} = \frac{A_{31}C_{13}(A_{21} + n_e C_{21})}{A_{21}C_{12}(A_{31} + n_e C_{12})} \simeq \begin{cases} \frac{C_{13}}{C_{12}} = \frac{\bar{\Omega}_{13}}{\bar{\Omega}_{12}} & n_e \ll n_e^* \\ \frac{A_{31}C_{13}C_{21}}{A_{21}C_{12}C_{31}} = \frac{A_{31}}{A_{21}} \frac{g_3}{g_2} & n_e \gg n_e^* \end{cases}. \tag{50}$$

   The result, in these two limits, does not depend on temperature, and therefore this is a good density diagnostic.

2.5 Absorption lines

We considered the case of background radiation propagating through a cold absorbing medium. The solution to the RTE in this case is

$$I_\nu = e^{-\tau_\nu} I_\nu^0, \quad \tau_\nu = \int ds\, \alpha_\nu. \tag{51}$$

In the case of line absorption, the effective absorption coefficient, accounting for both absorption and stimulated emission, is

$$\alpha_\nu = \frac{h\nu}{c} B_{12} n_1 \left[1 - \frac{b_1}{b_2} e^{-\frac{h\nu}{k_B T}}\right] \varphi(\nu), \tag{52}$$

where we have defined the **departure coefficient** b_2/b_1, parametrizing departure from thermal equilibrium in the level populations

$$\frac{n_2}{n_1} \equiv \frac{b_2}{b_1} \frac{g_2}{g_1} e^{-\frac{h\nu_{12}}{k_B T}}. \tag{53}$$

Assuming that b_2/b_1 is constant across the medium, the optical depth is then given by

$$\tau_\nu = \frac{h\nu_{12} B_{12}}{c} \left[1 - \frac{b_2}{b_1} e^{-\frac{h\nu_{12}}{k_B T}}\right] N_1\, \varphi(\nu), \tag{54}$$

where $N_1 \equiv \int ds\, n_1$ is the **column density**.

2.5.1 Equivalent width

Line absorption is quantified in terms of the equivalent width parameter W_λ, defined as

$$W_\lambda \equiv \int d\lambda \frac{I^0_\lambda - I_\lambda}{I^0_\lambda}, \tag{55}$$

where I^0 is the un-absorbed background spectrum. Using the solution to the RTE and going back to intensities defined with respect to frequency (instead of wavelength), this is then:

$$W_\lambda = \int d\nu \frac{c}{\nu^2} \left[1 - e^{-\tau_\nu}\right]. \tag{56}$$

2.5.2 Optically thin lines

If τ_ν is small, $1 - e^{-\tau_\nu} \simeq \tau_\nu$, and equivalent width and column density become proportional

$$W_\lambda = \int d\nu \frac{c}{\nu^2}\tau_\nu = \frac{hB_{12}}{\nu_{12}} \left[1 - \frac{b_2}{b_1} e^{-\frac{h\nu_{12}}{k_B T}}\right] N_1. \tag{57}$$

For cold media, in the case of optical/UV lines (i.e. $h\nu_{12} \gg k_B T$), the exponential term can be neglected, and the relation between N_1 and W_λ becomes

$$W_\lambda = \frac{hB_{12}}{\nu_{12}} N_1, \quad N_1 = (1.13 \times 10^{17}\,\mathrm{cm}^{-2}) \frac{W_\lambda/\mathrm{m\AA}}{f_{12}(\lambda_{12}/\mathrm{\AA})}. \tag{58}$$

On IR/radio wavelengths, the exponential term is often not negligible. Moreover, one can have population inversion, with a large b_2/b_1, where stimulation emission dominates over absorption (the resulting W_λ is negative!). This is common in masers, often due to diatomic molecules.

2.5.3 Optically thick lines

For thick media, the absorption line saturates the spectrum, and the line's shape (its tails) must be taken into account when relating W_λ and N_1. Three types of line broadening are common:

- **Doppler broadening.** Caused by random motions in the medium. The resulting profile is **Gaussian**:

$$\varphi(\nu) \propto \exp\left[-\frac{(\nu - \nu_{12})^2}{2(\sigma_v \nu_{21}/c)^2}\right], \tag{59}$$

  where σ_v is the velocity dispersion. Gaussian profiles have thick cores and narrow wings

- **Natural broadening.** Due to the uncertainty principle. This leads to a **Lorentzian profile**:

$$\varphi(\nu) \propto \frac{\Delta^2}{(\nu - \nu_{12})^2 + \Delta^2}. \tag{60}$$

  Lorentzian profiles have narrow cores and broad wings.

- **Collisional broadening.** Also causes Lorentzian broadening.

The result is a convolution of both types of profiles (Gaussian⊗Lorentzian), called a **Voigt profile**.

The behaviour of W_λ as a function of N_1 is the so-called **curve of growth**:

- In the *optically thin* regime: $W_\lambda \propto N_1$.
- In the *Gaussian* regime: $W_\lambda \propto \sqrt{\log N_1}$.
- In the *Lorentzian* regime: $W_\lambda \propto \sqrt{N_1}$.

3 Ionization and recombination

In this case we need to balance out continuum electrons e^-, and two ions of the same atom with different levels of ionization (which for concreteness we will label HI and HII here). In addition to this, one must consider the inonizing photons, with specific energy density u_ν.

3.1 Recombination

An HII ion captures a free electron and becomes HI. Let $\sigma^{fb}(v_e)$ be the cross section for recombination of an electron with velocity v_e. The number of recombination taking place per unit time and volume is

$$\frac{dN_{\rm rec}}{dV\,dt} = n_{\rm HII} n_e \int dv_e\, v_e\, \frac{dp}{dv_e} \sigma^{fb}(v_e) \equiv n_{\rm HII} n_e \alpha_r, \tag{61}$$

which defines the recombination coefficient $\alpha_r \equiv \langle v_e \sigma^{fb} \rangle$.

The electron velocity and the frequency of the emitted photon are related via energy conservation:

$$h\nu = \chi + \frac{1}{2} m_e v_e^2, \tag{62}$$

where $-\chi$ is the ionization potential (i.e. the energy of the bound level). With this in mind, the emission coefficient can be found to be

$$j_\nu = \frac{h^2 \nu}{4\pi\, m_e} n_{\rm HII} n_e \frac{dp}{dv_e} \sigma^{fb}(v_e), \tag{63}$$

where v_e is implicitly a function of ν via Eq. 62.

3.2 Photoionization

In this case, a photon of energy $h\nu$ extracts an electron from an HI ion/atom. If σ_ν^{bf} is the cross section for such interaction, the number of photoionizations per unit volume and time is

$$\frac{dN_{\rm io}}{dV\,dt} = n_{\rm HI} \int_{\nu_*}^{\infty} d\nu \frac{4\pi J_\nu}{h\nu} \sigma_\nu^{bf}, \tag{64}$$

where $\nu_* \equiv \chi/h$. The absorption coefficient is simply

$$\alpha_\nu = n_{\rm HI} \sigma_\nu^{bf} \tag{65}$$

The ionization cross section can be derived from quantum mechanics and, as a rule of thumb, has an "edge-like" shape:

$$\sigma_\nu^{bf} = \begin{cases} \sigma_1 \left(\frac{\nu_*}{\nu}\right)^3 & \nu \geq \nu_* \\ 0 & \nu < \nu_* \end{cases}. \tag{66}$$

The peak cross section σ_1 is approximately

$$\sigma_1 \sim (7 \times 10^{-18}\,{\rm cm}^2) \frac{n}{Z^2} g(n, \ell, Z, \nu), \tag{67}$$

where the *Gaunt factor* g has only mild dependence on principal and azimuthal quantum numbers n, ℓ, atomic number Z, and frequency ν.

3.3 Thermal equilibrium

Thermal equilibrium between all the species involved can be expressed via the **Saha equation**:

$$\frac{n_e n_{\rm HII}}{n_{\rm HI}} = 2\frac{g_{\rm HII}}{g_{\rm HI}}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\frac{\chi}{k_B T}}. \tag{68}$$

This is the equivalent of $n_1/n_2 = g_1/g_2\, e^{-\Delta E/(k_B T)}$ in the case of discrete energy levels.

Using the Saha equation, requiring detailed balance between recombination, ionization, and *stimulated recombination*[1], and using the black-body spectrum of the resulting radiation, we can derive the **Milne relation**, the analog of the Einstein relations for bound-bound transitions:

$$\frac{\sigma^{bf}}{\sigma^{fb}} = \frac{m_e^2 c^2 v_e^2}{h^2\nu^2}\frac{g_{\rm HII}}{g_{\rm HI}}. \tag{69}$$

As in the case of the Einsten relations, this does not depend on temperature, and holds in general.

3.4 Stromgen sphere

A strong UV source in a neutral medium will ionize a sphere around it. The radius of this "Stromgen sphere" can be calculated as

$$R_S = \left(\frac{3\dot{N}_\gamma}{4\pi n_e^2 \alpha_r}\right)^{1/3}, \tag{70}$$

where $\dot{N}_\gamma$ is the number of ionizing photons emitted by the central source per unit time. Recall that these could be related to the number of Hα photons as $\dot{N}_\gamma \simeq 2.2\dot{N}_{\rm H\alpha}$.

The radius of this sphere can be tracked approximately as a function of time:

$$R(t) = R_s\left(1 - e^{-t/t_{\rm rec}}\right)^{1/3}, \tag{71}$$

where the recombination timescale is $t_{\rm rec} = 1/(n_{\rm HI}\alpha_r)$.

4 Stellar spectra

Stars are very optically thick. Spectrum is mostly a black body. Absorption lines from the photosphere, where it thins out and LTE is no longer valid.

4.1 RTE in stars

RTE in stars usually written in terms of inwards-looking radial optical depth:

$$d\tilde{\tau}_\nu = -\alpha_\nu(R)\, dR, \tag{72}$$

where $\tilde{\tau}_\nu = 0$ at $R = R_s$. In terms of this, the RTE reads

$$\mu\frac{dI_\mu}{d\tilde{\tau}_\nu} - I_\nu = S_\nu, \tag{73}$$

where μ is the angle between the outgoing ray and the normal to the sphere's surface. The solution, for rays originating deep inside the sphere, is

$$I_\nu(0) = \int_0^\infty \frac{d\tilde{\tau}_\nu}{\mu} e^{-\tilde{\tau}_\nu/\mu} S_\nu(\tilde{\tau}_\nu). \tag{74}$$

Thus, the emergent spectrum is mostly due to the emission close to the surface, with $\tilde{\tau}_\nu \lesssim \mu$.

[1] Note that we did not derive stimulated recombination in the lectures! We also ignored collisional bound-free processes.

4.2 Eddington-Barbier relation

Expanding the source function around $\tilde{\tau}_\nu = 0$

$$S_\nu = a_\nu + \tilde{\tau}_\nu b_\nu, \tag{75}$$

the integral in Eq. 74 can be calculated explicitly:

$$I_\nu(0) = a_\nu + \mu b_\nu = S_\nu(\mu). \tag{76}$$

Thus we observe inside the star by an amount $\tilde{\tau} = \mu$, deeper close to the centre ($\mu = 1$), and less so at the edge $\mu = 0$.

In the **gray atmosphere** approximation (disregarding frequency dependence, assuming isotropic medium), the values of a_ν and b_ν can be related to the observed flux:

$$S(\tau) = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right), \quad \left(a = \frac{F}{2\pi}, \quad b = \frac{3F}{4\pi}\right). \tag{77}$$

From this we can quantify the phenomenon of **Limb darkening**:

$$\frac{I(\mu)}{I(1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right) = \frac{3}{5}\left(\sqrt{1 - \frac{r^2}{R^2}} + \frac{2}{3}\right), \tag{78}$$

where we have related μ to the projected distance from the centre of the star $r = R\sqrt{1-\mu^2}$. When taking the frequency dependence of the stellar spectrum into account, and the fact that temperature increases with optical depth, Limb darkening becomes more prominent at higher frequencies, above the peak of the BB spectrum.

5 Free-free (bremsstrahlung) emission

5.1 Radiation from accelerated charges

The EM scalar and vector potentials caused by a charge in motion are given by

$$\phi(t, \mathbf{x}) = \kappa\, q \int dt' \frac{1}{R(t')}\, \delta^D(t' - t + R(t')/c), \tag{79}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\kappa\, q}{c^2} \int dt' \frac{\mathbf{u}(t')}{R(t')}\, \delta^D(t' - t + R(t')/c), \tag{80}$$

where $\mathbf{r}(t)$ is the charge's trajectory, and

$$\kappa = \frac{1}{4\pi\epsilon_0}, \quad \mathbf{u} \equiv \dot{\mathbf{r}}, \quad \mathbf{R}(t) = \mathbf{x} - \mathbf{r}(t). \tag{81}$$

The electric and magnetic fields resulting from these potentials are

$$\mathbf{E}(t, \mathbf{x}) = \kappa q \left[\frac{(\hat{\mathbf{n}} - \boldsymbol{\beta})(1 - \beta^2)}{\xi^3 R^2}\right]_{\text{ret}} + \frac{\kappa q}{c}\left[\frac{\hat{\mathbf{n}}}{\xi^3 R} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]\right]_{\text{ret}}, \tag{82}$$

$$\mathbf{B}(t, \mathbf{x}) = [\hat{\mathbf{n}} \times \mathbf{E}]_{\text{ret}}, \tag{83}$$

where $\hat{\mathbf{n}} \equiv \mathbf{R}/R$ is a unit vector along the line of sight, $\boldsymbol{\beta} \equiv \mathbf{u}/c$, $\xi \equiv 1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta}$, and $[\cdots]_{\text{ret}}$ indicates that the quantity within brackets is evaluated at the retarded time ($c(t - t_{\text{ret}}) = R(t_{\text{ret}})$). Note that $\mathbf{B}$ is always perpendicular to $\mathbf{E}$ and $\hat{\mathbf{n}}$. Most importantly, the EM flux ($\propto \int dA\, |\mathbf{E}|^2$) of the first term in $\mathbf{E}$ drops to zero when evaluated at large distances, while that due to the second term remains constant. Thus, when studying

radiation from distant sources we discard the first term. Evaluating the second term in the non-relativistic limit ($\beta \ll 1$):

$$\mathbf{E} = \frac{\kappa q}{c^2}\left[\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\mathbf{u}})}{R}\right]_{\text{ret}}. \tag{84}$$

This expression can then be used to calculate the EM energy flux using the Poynting vector. The result is the **Larmor formula**, of central relevance for radiative processes:

$$\frac{dE}{dt\, d\Omega} = \frac{\kappa \dot{u}^2 q^2}{4\pi c^3} \sin^2 \alpha, \tag{85}$$

where α is the angle between the acceleration vector $\dot{\mathbf{u}}$ and the line of sight $\hat{\mathbf{n}}$. Integrating over all angles:

$$\frac{dE}{dt} = \frac{2\kappa q^2 \dot{u}^2}{3c^3}. \tag{86}$$

5.2 Bremsstrahlung radiation

Bremsstrahlung is caused by the acceleration/deceleration of an electron when passing close to an ion. The acceleration in the direction perpendicular to the original trajectory is:

$$\dot{u}_z = -\frac{\kappa Z q^2}{m_e b^2} \sin^3 \theta(t), \tag{87}$$

where b is the impact parameter, and $\theta(t)$ is the angle between the electron's velocity and the line connecting it with the proton.

Substituting this in Larmor's formula:

$$\frac{dE}{dt} = \frac{Z^2 q^2 c \sigma_T}{16\pi^2 \epsilon_0 b^4} \sin^6 \theta(t), \tag{88}$$

where σ_T is the Thomson scattering cross section (see Eq. 106). What follows is a set of approximations that will make the final result only accurate to order ~ 1.

The angle $\theta(t)$ can be calculated approximately assuming an unperturbed electron trajectory

$$\tan\theta = \frac{b}{v_e t} \quad \longrightarrow \quad \sin^6\theta = \left(1 + \frac{v_e^2 t^2}{b^2}\right)^{-3}. \tag{89}$$

The time dependence of the energy emitted in each electron-ion encounter is therefore a "blip" of width $\Delta t = 2b/v_e$. The corresponding spectrum should be approximately flat up to a maximum frequency $\nu_{\max} \sim 1/\Delta t$. The amplitude of this approximate "top-hat" spectrum can be calculated using Parseval's theorem, and the result is

$$\frac{dE}{d\nu} = \begin{cases} \frac{Z^2 q^2 c \sigma_T}{8\pi^2 \epsilon_0 b^2 v_e^2} & \nu < \nu_{\max} = v_e/(2b) \\ 0 & \nu \geq \nu_{\max} \end{cases}. \tag{90}$$

This is the power emitted by a single encounter with velocity v, and impact parameter b. The number of such encounters per interval of time, volume, impact parameter, and velocity, is

$$\frac{dN_{e-p}}{dt\, dv_e\, db\, dV} = n_i n_e 2\pi\, b\, v_e \frac{dp}{dv_e}. \tag{91}$$

The emission coefficient for bremsstrahlung can then be calculated by multiplying this by the power emitted in each pulse, and integrating over v_e and b. This required choosing some ad-hoc limits for b. The upper bound is given by the fact that only impact parameters $b < v_e/(2\nu)$ contribute at frequencies $< \nu$. The lower bound was chosen as the distance below which quantum effects become relevant due to the uncertainty principle,

$b_{\min} = \hbar/(m_e v_e)$. Finally, integrating over v_e assuming dp/dv_e is Maxwellian can be done approximately by substituting $m_e v_e^2/2 = 3k_B T_e/2$. The final approximate result is:

$$\begin{aligned} j_\nu &\sim \frac{Z^2 q^2 c \sigma_T}{16\pi^2 \epsilon_0} \sqrt{\frac{m_e}{3k_B T_e}} n_i n_e \log\left(\frac{3\pi k_B T_e}{h\nu}\right) \\ &\propto \frac{n_e^2}{T_e} \log\left(\frac{3\pi k_B T_e}{h\nu}\right). \end{aligned} \tag{92}$$

5.3 Cluster masses

Assuming that a massive cluster is a spherical cow, and that the gas within it is in hydrostatic equilibrium:

$$\frac{dp}{dr} = -\frac{GM(<r)\rho_{\rm gas}}{r^2}, \tag{93}$$

where p is pressure, $M(<r)$ is the mass within a sphere of radius r, and $\rho_{\rm gas}$ is the gas density. For an ideal gas, the pressure is $p = \rho k_B T/(\mu\, m_H)$, where m_H is the hydrogen atom mass, and $\mu \sim 0.6$ is the mean molecular mass of the gas. Then the mass profile can be calculated as

$$M(<r) = -\frac{k_B T r}{G\mu m_H}\left[\frac{d\log n_e}{d\log r} + \frac{d\log T_e}{d\log r}\right]. \tag{94}$$

Cluster masses can therefore be measured under these assumptions, estimating n_e and T_e from bremsstrahlung radiation.

6 Thomson and Compton scattering

6.1 Scattering generalities

Scattering can be understood as the absorption of a photon and its re-emission in a different direction. As such, it can be effectively included in the RTE as a standard absorption term and a non-standard emission term that depends on the radiation field. As a toy model, we considered isotropic scattering. In this case, the RTE reads

$$\frac{dI_\nu}{ds} = j_\nu + \alpha_\nu^s J_\nu - (\alpha_\nu + \alpha_\nu^s) I_\nu. \tag{95}$$

Here, α_ν^s is the absorption coefficient due to scattering, given in terms of the number density of scatterers and the scattering cross section as $\alpha_\nu^s = n_* \sigma_\nu^s$.

In this medium, a photon will have a mean free path

$$\ell_* = (\alpha_\nu + \alpha_\nu^s)^{-1}. \tag{96}$$

Each encounter with a medium particle will result in either a random scattering or absorption. The probabilities for each are:

$$p_a = \frac{\alpha_\nu}{\alpha_\nu + \alpha_\nu^s}, \quad p_s = 1 - p_a. \tag{97}$$

The probability for a photon to be absorbed after n scatterings is

$$p(n) = p_s^n p_a, \tag{98}$$

and therefore the mean number of scatterings before absorption is

$$\bar{n} = \frac{p_s}{p_a} \simeq \frac{1}{p_a}, \tag{99}$$

where in the last step we have assumed $p_s \gg p_a$. The effective mean path length for absorption is then

$$\ell_{\rm eff} = \ell_* \sqrt{\bar{n}} = [\alpha_\nu(\alpha_\nu + \alpha_\nu^s)]^{-1/2}, \tag{100}$$

and the effective optical depth is

$$\tau_{\rm eff} = \sqrt{\tau_a(\tau_s + \tau_a)}, \tag{101}$$

where τ_a and τ_s correspond to α_ν and α_ν^s respectively.

6.2 Thomson scattering

In Thomson scattering, a plane EM wave makes a charged particle oscillate, and in turn it will emit radiation in different directions at the same frequency. The acceleration due to a wave with electric field amplitude E_0 and angular frequency ω is

$$\dot{u} = \frac{qE_0}{m_e} \sin \omega t. \tag{102}$$

Substituting this in the Larmor formula, the power in scattered waves is

$$\frac{dE}{dt\, d\Omega} = \frac{\kappa q^4 E_0^2}{4\pi c^3 m_e^2} \sin^2 \alpha. \tag{103}$$

Relating this to the incoming mean flux $\langle S \rangle = 2c\epsilon_0 \langle |\mathbf{E}|^2 \rangle = c\epsilon_0 E_0^2$, we obtain the differential cross-section

$$\frac{dE}{dt\, d\Omega} = \langle S^2 \rangle \frac{d\sigma}{d\Omega}, \quad \longrightarrow \quad \left.\frac{d\sigma}{d\Omega}\right|_p = \left(\frac{\kappa q^2}{m_e c^2}\right)^2 \sin^2 \alpha. \tag{104}$$

This is for a wave polarized in a particular direction. The cross section for unpolarized light can be found by adding the cross section for a wave with an orthogonal polarization. In this case $\alpha = \pi/2$, and the result is the **Thomson differential cross section**:

$$\frac{d\sigma_T}{d\Omega} = \left(\frac{\kappa q^2}{m_e c^2}\right)^2 \frac{1 + \cos^2 \theta}{2} = r_e^2 \frac{1 + \cos^2 \theta}{2}, \tag{105}$$

where r_e is the classical electron radius, and θ is the angle between the incoming wave and the outgoing one. Integrating over angles we obtain the total cross-section:

$$\sigma_T = \frac{8\pi}{3} \left(\frac{\kappa q^2}{m_e c^2}\right)^2 = 6.65 \times 10^{-29}\, \mathrm{m}^2. \tag{106}$$

6.3 Compton scattering

Thomson scattering is the low-energy limit of Compton scattering, which accounts for QED effects, as well as the momentum gain of the electron, and the corresponding momentum loss of the outgoing photon. The difference between the wavelengths before and after the encounter is

$$\lambda' - \lambda = \lambda_c (1 - \cos \theta), \tag{107}$$

where $\lambda_c \equiv h/(m_e c)$ is the Compton wavelengths. These energy losses in the rest frame of the electron can therefore be ignored as long as $h\nu \ll m_e c^2$. This covers much of the spectrum, but becomes relevant at hard X-ray and γ-ray frequencies.

The exact cross section for Compton scattering can be found in QED:

$$\frac{d\sigma_c}{d\Omega} = r_e^2 \frac{\nu'}{\nu} \frac{1}{2} \left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - \sin^2 \theta\right). \tag{108}$$

It reduces to the Thomson result in the limit $\nu = \nu'$.

6.4 Inverse Compton scattering and the Sunyaev Zel'dovich effect

The previous results, where photons either maintain their energy or lose some, have been derived in the electron reference frame. In the frame of a cloud of free electrons with relatively large thermal velocities, photons actually gain energy via inverse Compton scattering. A photon background, such as the CMB, propagating through a hot gas cloud, such as a galaxy cluster, will therefore see its spectrum modified, by effectively shifting the photon energies. This leads to a depletion on small frequencies and an increase on high frequencies, with a characteristic spectral distortion. This is the so-called Sunyaev-Zel'dovich (SZ) effect.

To find the exact form of the SZ spectrum, one must unfortunately solve the RTE with scattering terms. The corresponding **Boltzmann equation** is

$$\frac{df_\gamma}{dt} = C[f_\gamma], \tag{109}$$

where the collision operator due to Compton scattering is

$$C[f_\gamma] = n_e \int d^3v \int d\Omega \, \frac{d\sigma_T}{d\Omega} \left\{ f_e(\mathbf{v}') f_\gamma(\mathbf{p}') \left[1 + f_\gamma(\mathbf{p})\right] - f_e(\mathbf{v}_e) f_\gamma(\mathbf{p}) \left[1 + f_\gamma(\mathbf{p}')\right] \right\}. \tag{110}$$

The two terms represent contributions from photons up-scattering from and to $\mathbf{p}'$ respectively. $f_e(v)$ is the distribution of electron velocities, and the $1 + f_\gamma$ terms account for stimulated scattering (which will be irrelevant anyway). Remember the connection between the photon phase-space distribution f_γ, intensity I_ν and occupation number n_ν in Eq. 2.

In the limit of:

- Non-relativistic, thermally distributed electrons,
- low-energy photons,
- ignoring changes to the electron velocity distribution,

the Boltzmann equation can be simplified into the **Kompaneets equation**:

$$\frac{\partial n_\nu}{\partial t} = c n_e \sigma_T \frac{k_B T_e}{m_e c^2} \frac{1}{x_e} \frac{\partial}{\partial x_e} \left[x_e^4 \left(\frac{\partial n_\nu}{\partial x_e} + n_\nu + n_\nu^2 \right) \right], \tag{111}$$

where we have defined $x_e \equiv h\nu/(k_B T_e)$. We will also use the shorthand

$$x \equiv \frac{h\nu}{k_B T_\gamma}, \tag{112}$$

where T_γ is the photon temperature.

In all cases of interest, photons are close to blackbody ($n_\nu \simeq (e^x - 1)^{-1}$), $T_e \gg T_\gamma$, and therefore

$$\frac{\partial n_\nu}{\partial x_e} = \frac{T_e}{T_\gamma} \frac{\partial n_\nu}{\partial x} \gg n, n^2. \tag{113}$$

In this case, the Kompaneets equation can be simplified to

$$\frac{\partial n_\nu}{\partial y} = x^{-2} \partial_x (x^4 \partial_x n), \tag{114}$$

where we have defined the **Compton-y parameter**

$$y = \int \frac{\sigma_T}{m_e c^2} n_e k_B T_e \, ds = \frac{\sigma_T}{m_e c^2} \int P_e \, ds, \tag{115}$$

where P_e is the electron pressure.

Since y is usually fairly small ($y \lesssim 10^{-5}$), the equation can be solved approximately by simply substituting $n_\nu = (e^x - 1)^{-1}$ on the right hand side. The result, in terms of intensity, is

$$\frac{\Delta I_\nu}{I_\nu} = y \, \frac{x e^x}{e^x - 1} \left[x \frac{e^x + 1}{e^x - 1} - 4 \right] \simeq -2y, \tag{116}$$

where the last equality holds in the Rayleigh-Jeans limit (small frequencies). The resulting spectral distortion therefore has a universal spectrum that does not depend on the properties of the medium, with an amplitude y that directly probes the gas pressure (or the product $n_e T_e$). The distortion is zero at $\nu \sim 217\,\mathrm{GHz}$, and reaches a minimum and a maximum at $\nu \simeq 130\,\mathrm{GHz}$ and $\nu \simeq 370\,\mathrm{GHz}$ respectively.

Modelling a cluster as a spherical cow of radius R_c, the signature at low frequencies is

$$\frac{\Delta I_\nu}{I_\nu} = -2y \simeq -4 R_c \sigma_T \frac{k_B T_e}{m_e c^2} n_e. \tag{117}$$

7 Dust

Dust extinction is the combined effect of absorption and scattering of optical/UV photons by cosmic dust particles. The combined cross section for extinction (see our discussion in Section 6.1) is

$$\sigma_\nu^{\mathrm{ext}} = \sigma_\nu^{\mathrm{abs}} + \sigma_\nu^{\mathrm{scat}}. \tag{118}$$

It is common to relate these cross section to the dust grain radius a via efficiency factors Q_ν^A:

$$\sigma_\nu^A = \pi a^2 \, Q_\nu^A. \tag{119}$$

An example is Rayleigh scattering (scattering of light by neutral spheres with a different effective refractive index n, in the limit $a \ll \lambda$). The differential cross section in this case is

$$\frac{d\sigma}{d\Omega} = \left(\frac{n^2 - 1}{n^2 + 2} \right)^2 a^6 \left(\frac{2\pi}{\lambda} \right)^4 \frac{1 + \cos^2\theta}{2}. \tag{120}$$

This is the dominant form of scattering on optical/NIR wavelengths, and has a steep frequency dependence $\propto \nu^4$.

The size distribution is approximately

$$N(a) \propto \begin{cases} a^{-3.5} \exp(a/a_0) & a \geq a_{\min} \\ 0 & a > a_{\min} \end{cases}, \tag{121}$$

with $a_0 \sim 0.25\,\mu\mathrm{m}$, and $a_{\min} \sim 1\,\mathrm{nm}$.

7.1 Quantifying extinction

The extinction parameter A_ν is defined in terms of magnitudes:

$$m_\nu = m_\nu^0 + A_\nu, \tag{122}$$

where m_ν and m_ν^0 are the observed and un-extincted magnitudes. Using the relation between magnitudes and fluxes

$$m_\nu \equiv -2.5 \log_{10} \frac{F_\nu}{F_\nu^{\mathrm{ref}}}, \tag{123}$$

and the solution of the RTE in the presence of absorption $F_\nu = F_\nu^0 e^{-\tau_\nu}$, we can see that

$$A_\nu = \tau_\nu \, 2.5 \log_{10} e = 1.086 \, \tau_\nu, \tag{124}$$

and thus, to a reasonable approximation, $A_\nu \simeq \tau_\nu$.

We described two ways of measuring A_ν:

1. **Star pairs.** Take a pair of stars with similar spectra (i.e. $m^0_{2,\nu} = m^0_{1,\nu} + C$). Assuming one of them (e.g. 1) has suffered negligible extinction (e.g. because it's close by, or far away from the Galactic plane), and that we can measure their spectra at a low frequency ν_L where extinction is negligible, A_ν can be estimated as
$$A_\nu = (m_{2,\nu} - m_{2,\nu_L}) - (m_{1,\nu} - m_{1,\nu_L}). \tag{125}$$

2. **Line ratios.** We discussed that the amplitudes of hydrogen recombination lines can be predicted on theoretical grounds, and have a fairly low dependence on electron temperature or density. Thus, if we measure hydrogen lines on a wide range of frequencies, A_ν can be found at the position of each line by rescaling the line to the predicted amplitude given a measurement of a low-frequency line where extinction is negligible.

7.2 Dust heating and emission

The energy absorbed by dust particles in the optical and UV heats those particles, and is emitted in the infrared. Empirically, the dust emission spectrum has a blackbody-like maximum, and is usually parametrized through a frequency-dependent quality factor $Q^{\rm em}_\nu$:
$$I_\nu = Q^{\rm em}_\nu B_\nu(T). \tag{126}$$

The dust temperature can be roughly estimated by balancing absorbed and emitted energy. The energy absorbed by a dust grain a distance d away from an optical/UV source is
$$E_{\rm abs} = \int d\nu\, L_\nu \frac{\sigma^{\rm abs}_\nu}{4\pi d^2} = \frac{\pi a^2}{4\pi d^2}\, L\, Q_{UV}, \tag{127}$$
where we have defined the effective UV efficiency Q_{UV} and total luminosity L
$$L \equiv \int d\nu\, L_\nu, \quad Q_{UV} \equiv \frac{1}{L}\int d\nu\, L_\nu\, Q^{\rm abs}_\nu. \tag{128}$$

The emitted energy, on the other hand, is
$$E_{\rm em} = 4\pi a^2 \int d\nu\, \pi\, I_\nu = 4\pi a^2\, Q_{IR}\, \sigma T^4, \tag{129}$$
where we have used Eq. 4 to relate intensity and flux, Eq. 11 to associate blackbody flux and temperature, and defined the IR emission efficiency Q_{IR}
$$Q_{IR} \equiv \frac{\int d\nu\, Q^{\rm em}_\nu B_\nu(T)}{\int d\nu\, B_\nu(T)}. \tag{130}$$

The dust temperature can be estimated equating both energies.
$$\frac{\pi a^2}{4\pi d^2} L\, Q_{UV} = 4\pi a^2\, Q_{IR} \sigma T^4. \tag{131}$$
In doing so, it is also important to take into account the dust size distribution (Eq. 121), and the intrinsic dependence of the efficiency factors on a.

Estimates of dust temperatures typically range around $T \sim (10 - 100)\, K$.

Radiative processes in astrophysics

MPhys C1

Radiative Processes

Me!

- david.alonso@physics.ox.ac.uk
- Cosmologist - https://www.physics.ox.ac.uk/our-people/alonso

The course:

- **Books:**
  Radiative Processes in Astrophysics - Rybicki & Lightman
  High-Energy Astrophysics - Longair
- **Other resources:**
  Adam Ingram's notes
  Last-year's prerecorded lectures on canvas.

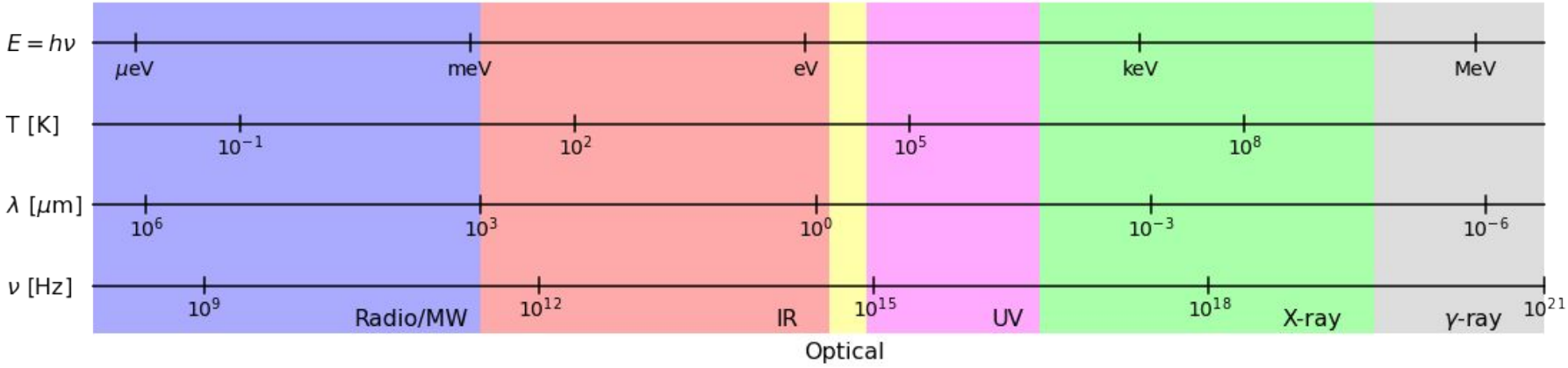
Radiative Processes

Syllabus

1. **Basics of radiation:** emission, absorption, scattering. The radiative transfer equation.
2. **Discrete transitions:** atomic transitions, Einstein coefficients,, collisional processes. Collisional processes.
3. **Line emission:** atomic spectra, the 2-level and the 3-level atom
4. **Ionization and recombination:** bound-free processes, Saha equation and Milne relations. Stromgen sphere, recombination lines.
5. **Absorption lines:** formation of absorption lines, equivalent width, line broadening, the Lyman-alpha forest.
6. **Stellar spectroscopy:** stellar types, emergent flux, Eddington-Barbier relation and gray atmospheres, Limb darkening, spectral features.
7. **Free-free emission:** radiation from moving charges, Larmor formula, Bremmstrahlung, cluster masses and HII regions.
8. **Thomson and Compton scattering:** scattering in general, Thomson/Compton scattering, the Sunyaev-Zel'dovich effect.
9. **Dust:** facts about dust. Extinction mechanisms. The extinction curve. Galactic and extragalactic dust.

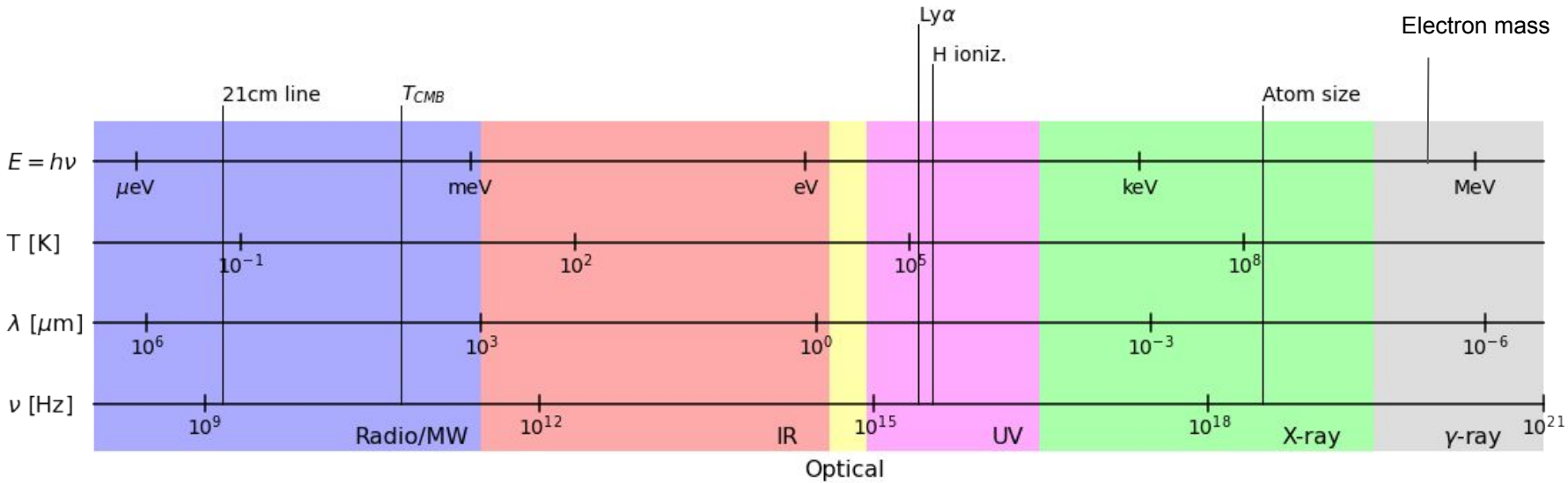
Lecture 1
Radiation basics and radiative transfer

The electromagnetic spectrum



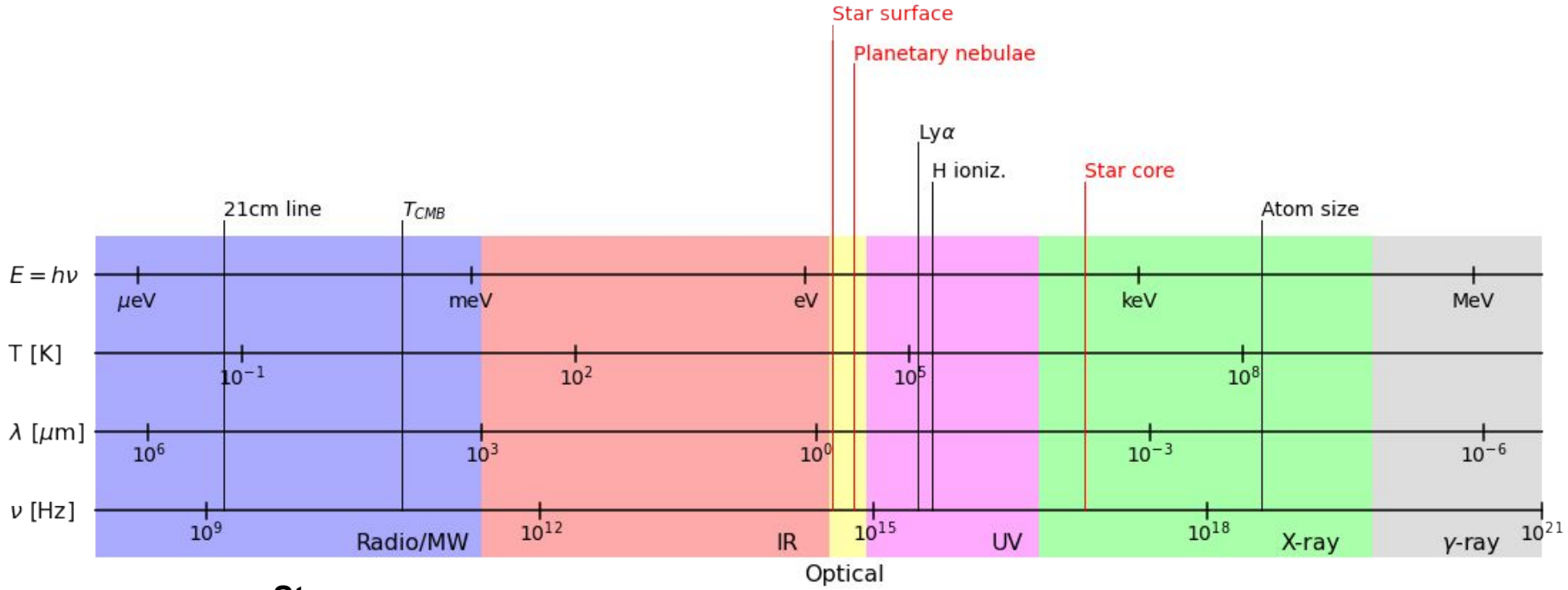
$$\nu = c/\lambda = E/h = k_B T/h$$

The electromagnetic spectrum



- CMB temperature: $T = T_0 (1+z)$, $T_0 = 2.7255^o$ K
- Lyman-limit (hydrogen ionization potential): 13.6 eV / 91.2 nm
- Lyman-α (n=1 - n=2 transition): 10.2 eV / 121 nm
- 21cm: hyperfine spin-flip transition. Radio tracer of neutral gas - epoch of reionization
- Atomic structure probed with X-rays. e- relativistic at gamma-ray energies.

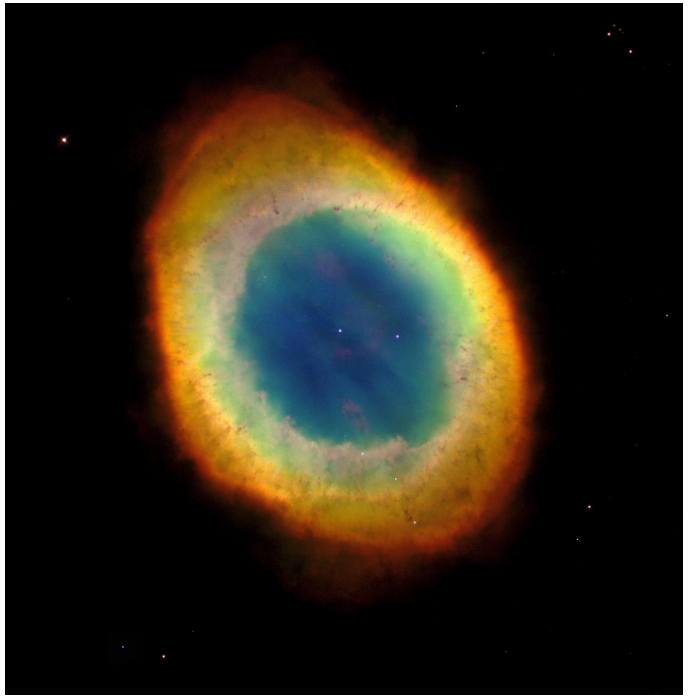
The electromagnetic spectrum



Stars:

- Core (nucleus): ~$10^{6\text{-}7}$ K (X-rays).
- Surface: ~10^{4} K (IR-vis-UV)
- Corona (surrounding hot plasma): ~10^{6} K
- Planetary nebulae (stellar remnants): gas ionized by remaining core, strong emission lines ($10^{4\text{-}5}$ K, optical)

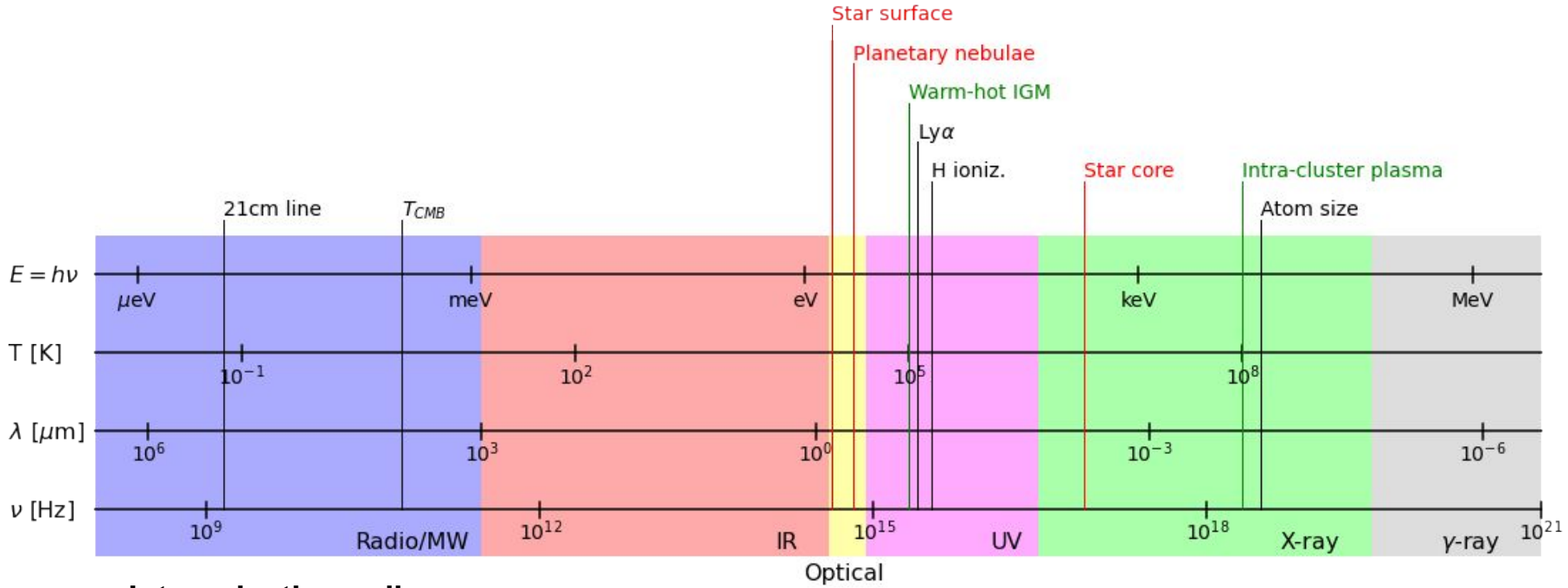
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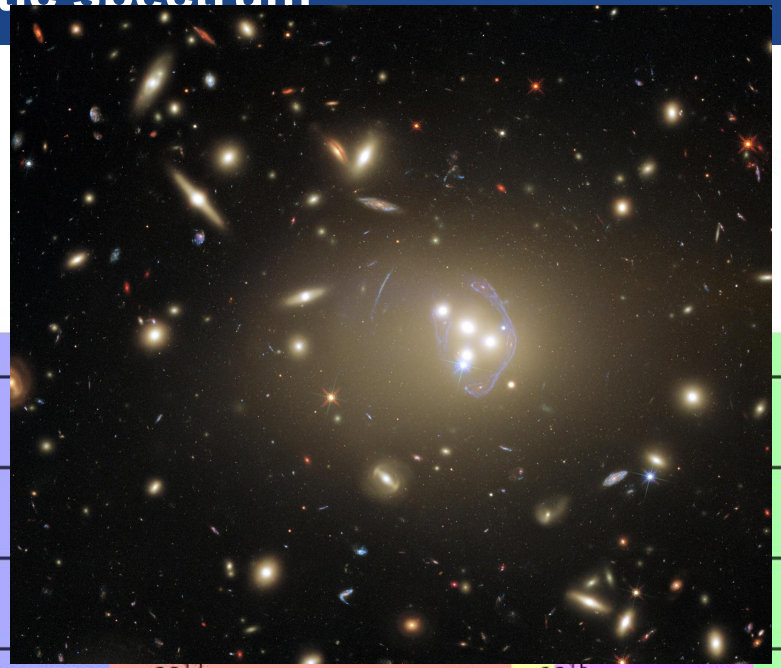
The electromagnetic spectrum



Intergalactic medium:

- Intra-cluster medium: Infalling gas shock-heated to ~10^8 K (X-rays). Low densities (10^{-3} cm^{-3}), free path (ℓ ~ O(Ly)).
- Warm-hot IGM (WHIM): gas in less dense parts of the Universe (e.g. filaments). ~$10^{5\text{-}7}$ K Main reservoir of cosmic baryons. Hard to observe.
- Active galactic nuclei: central $10^{6\text{-}10}M_{sun}$ black hole. Emission in wide range of freqs.

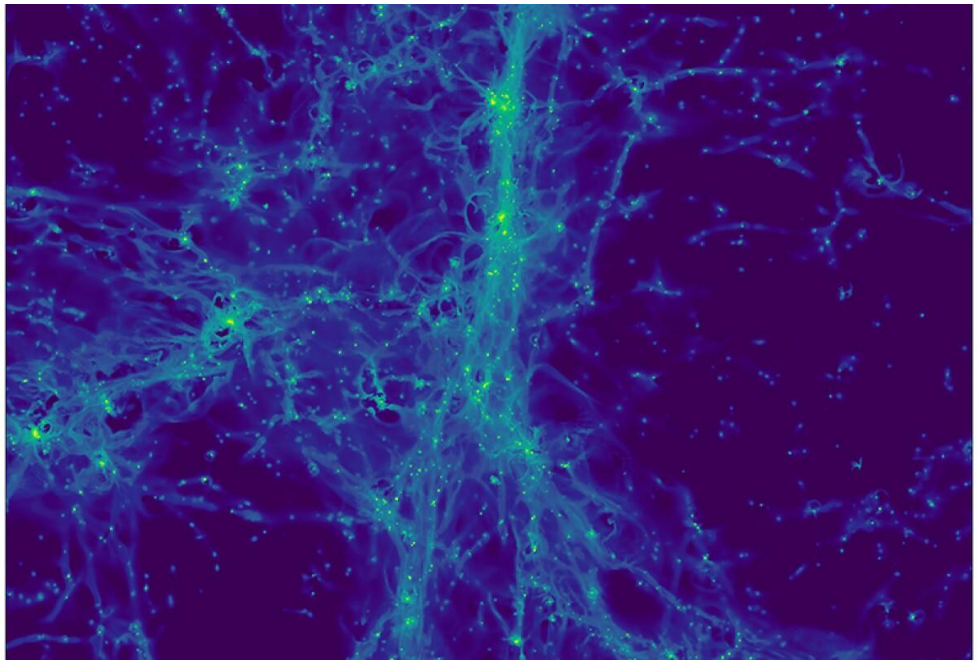
The electromagnetic spectrum

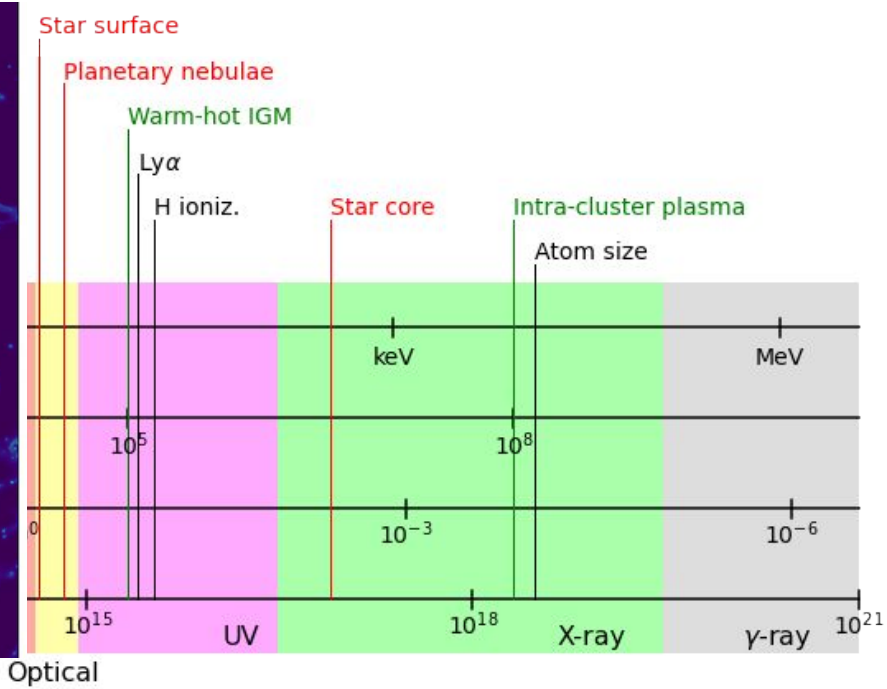


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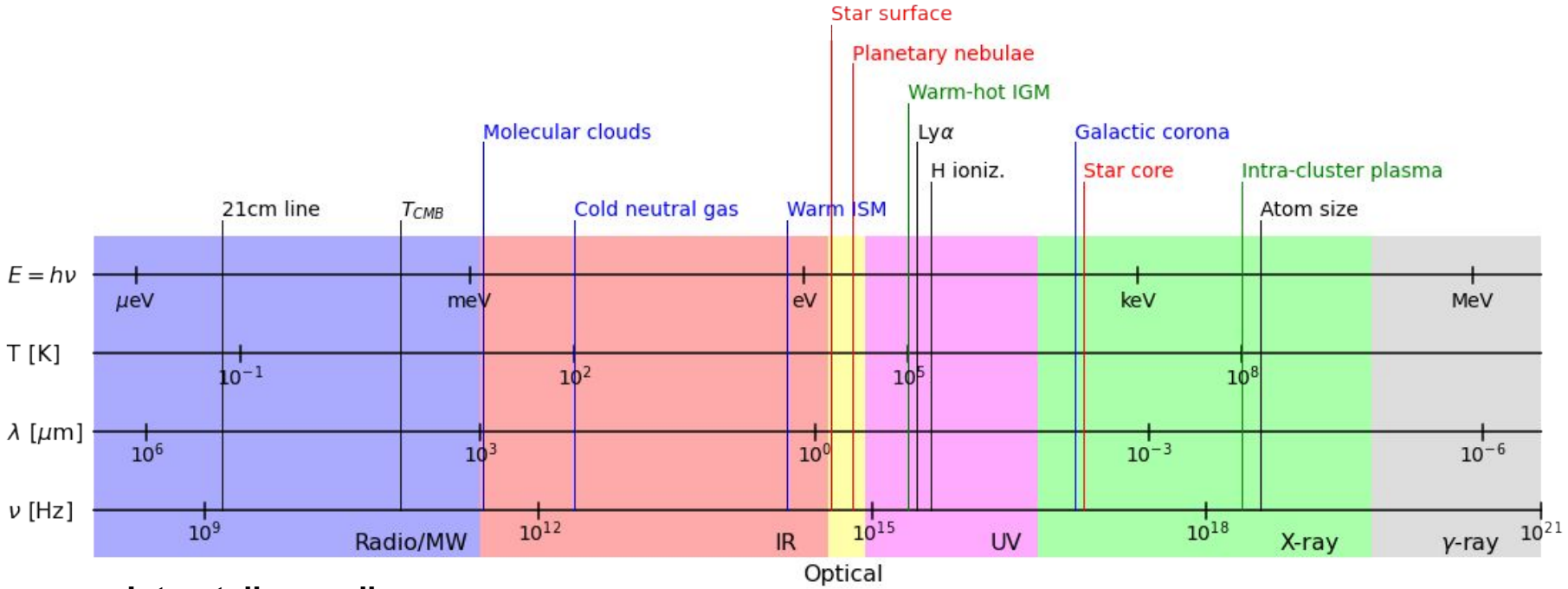




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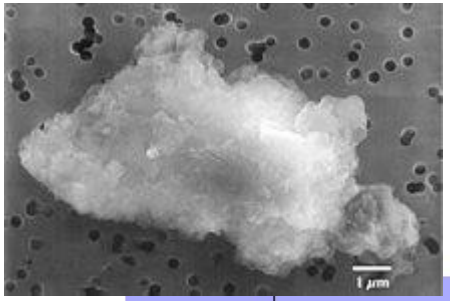
The electromagnetic spectrum



Interstellar medium:

- Gas between star systems. Composition/properties depend on temperature.
- Coldest regions: molecular clouds (observed in radio/IR lines).
- Cold neutral ISM: atomic hydrogen/helium. 21cm absorption.
- Warm ISM: neutral/ionized hydrogen. 21cm and Hα emission.
- Galactic corona: ejected hot gas/plasma (e.g. SNe). UV lines / X-ray continuum.

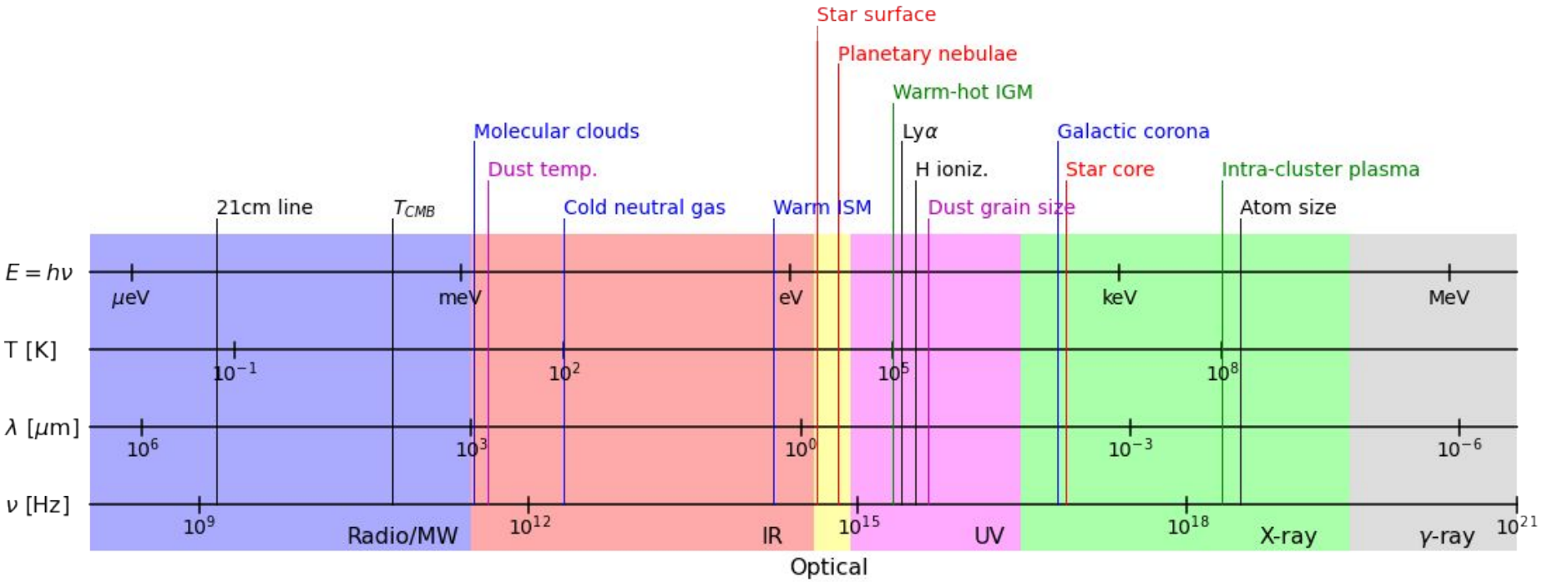
The electromagnetic spectrum



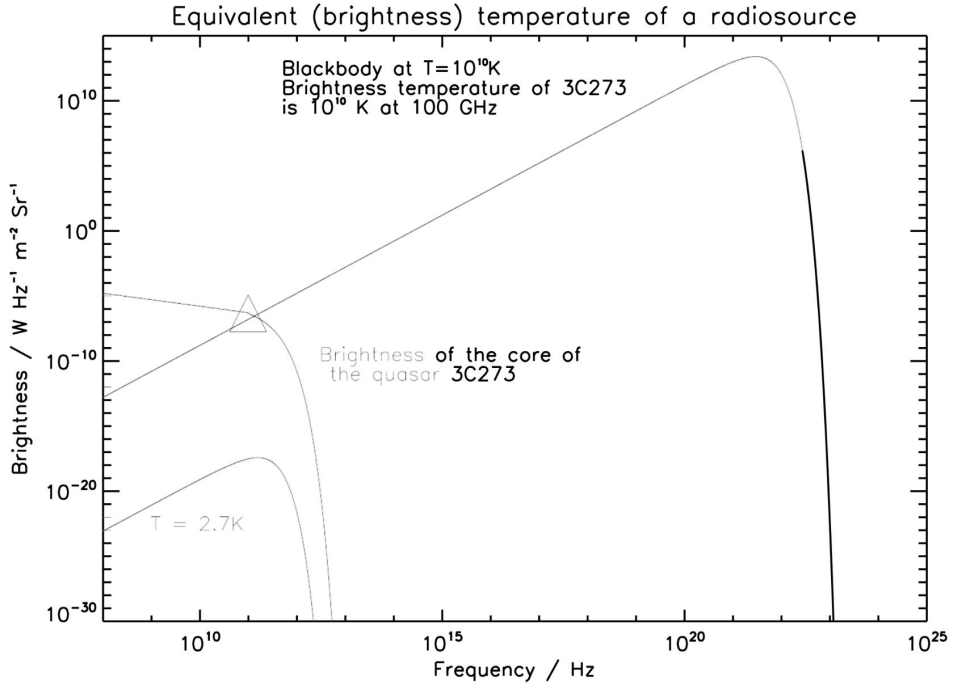
Dust:

- Large molecules / small grains from cooling gas of stellar outflows.
- Absorbs optical/UV light (grain size ~ wavelength). Main impediment for UV astronomy.
- Emits in infrared (vibrational and rotational modes), ~20 K.

The electromagnetic spectrum



Brightness temperature

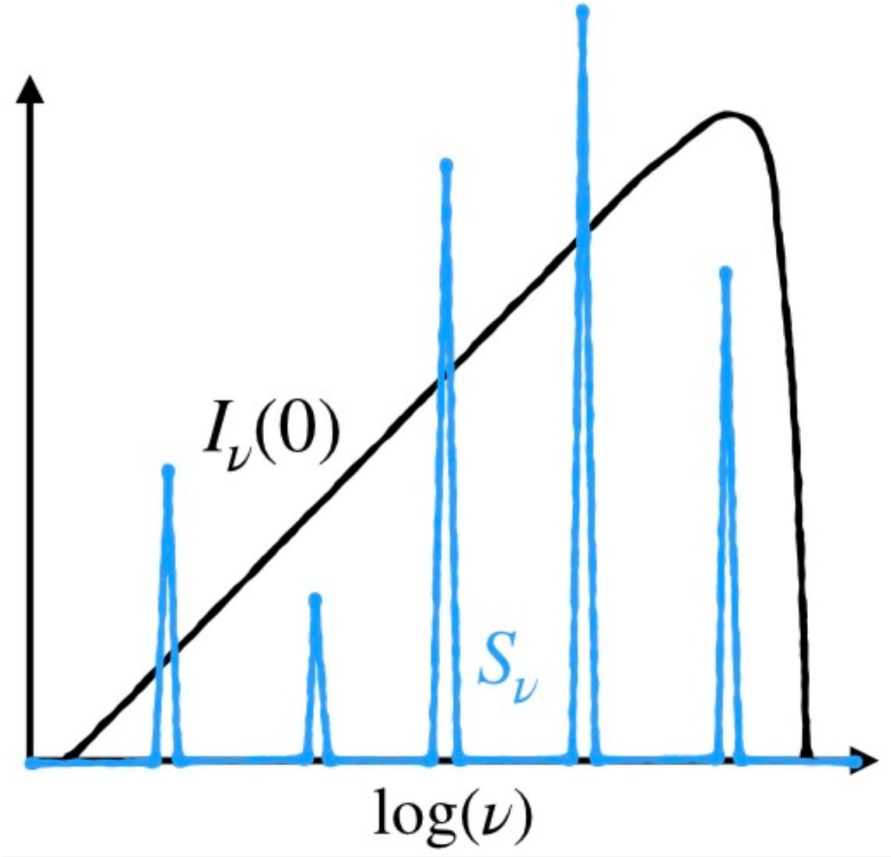


Brightness temperature only makes sense if spectrum is close to BB!

Lecture 2
Discrete transitions

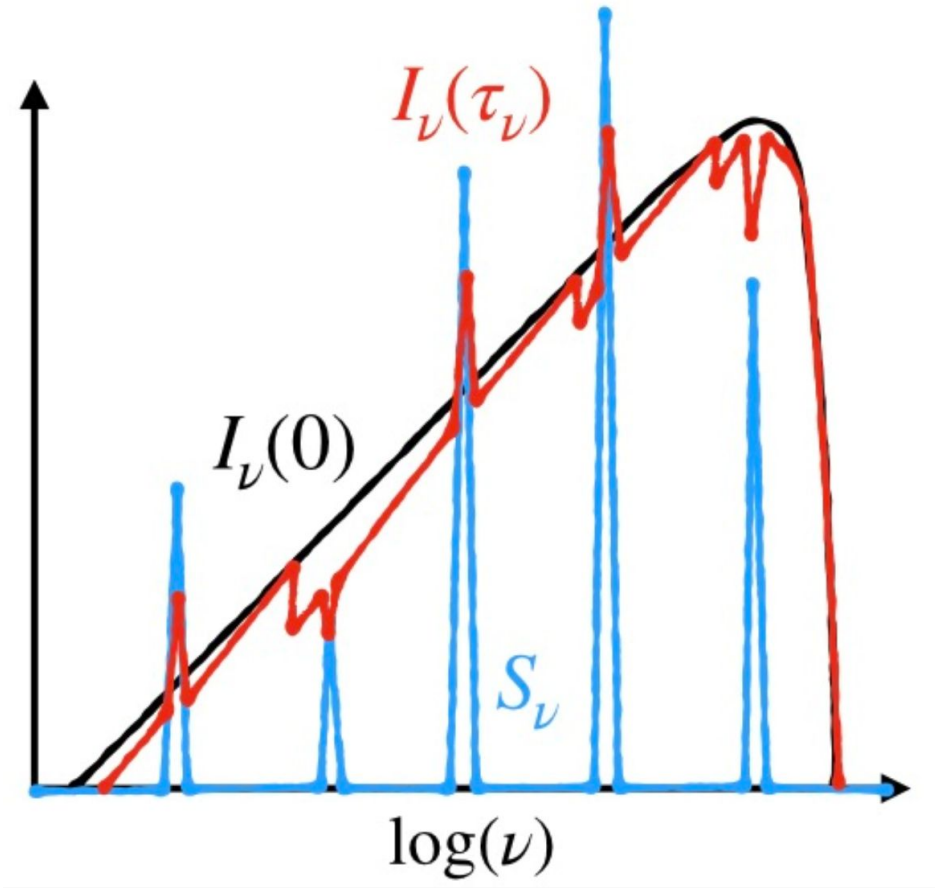
Emergent spectrum

$$I_\nu = I_\nu^0 + \tau_\nu (S_\nu - I_\nu^0)$$



Emergent spectrum

$$I_\nu = I_\nu^0 + \tau_\nu (S_\nu - I_\nu^0)$$



Emergent spectrum

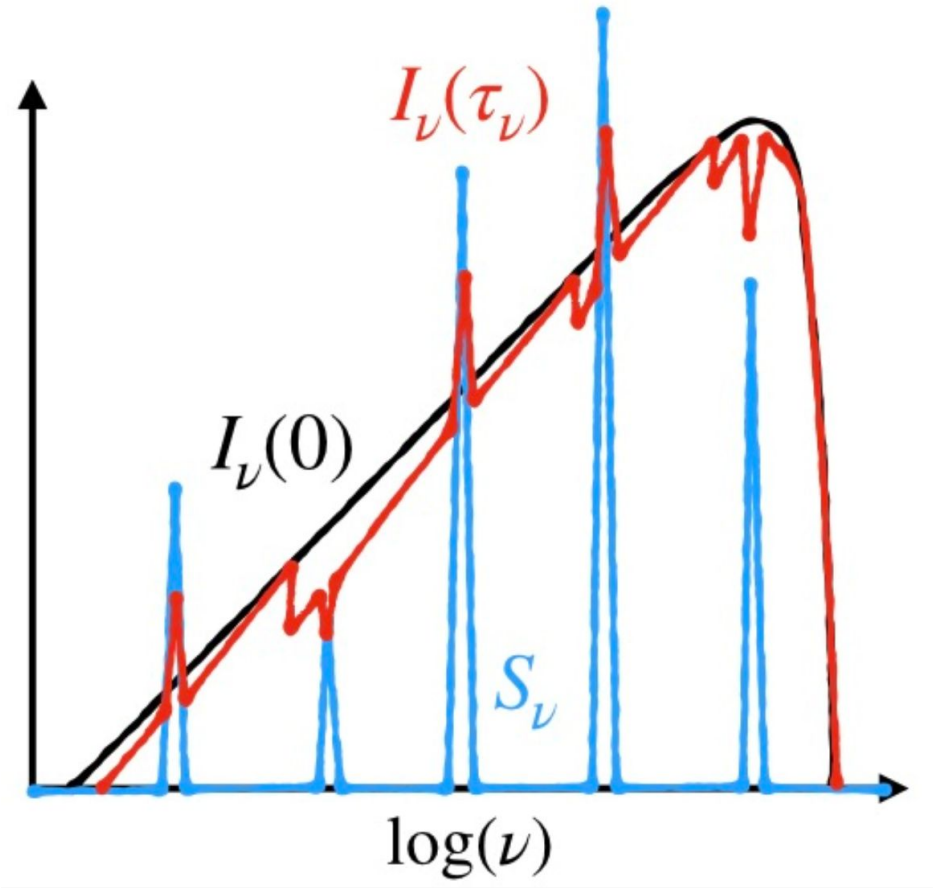
$$I_\nu = I_\nu^0 + \tau_\nu(S_\nu - I_\nu^0)$$

<u>Stellar interior is optically thick:</u>

- Optically thick
- Frequent collisions, LTE

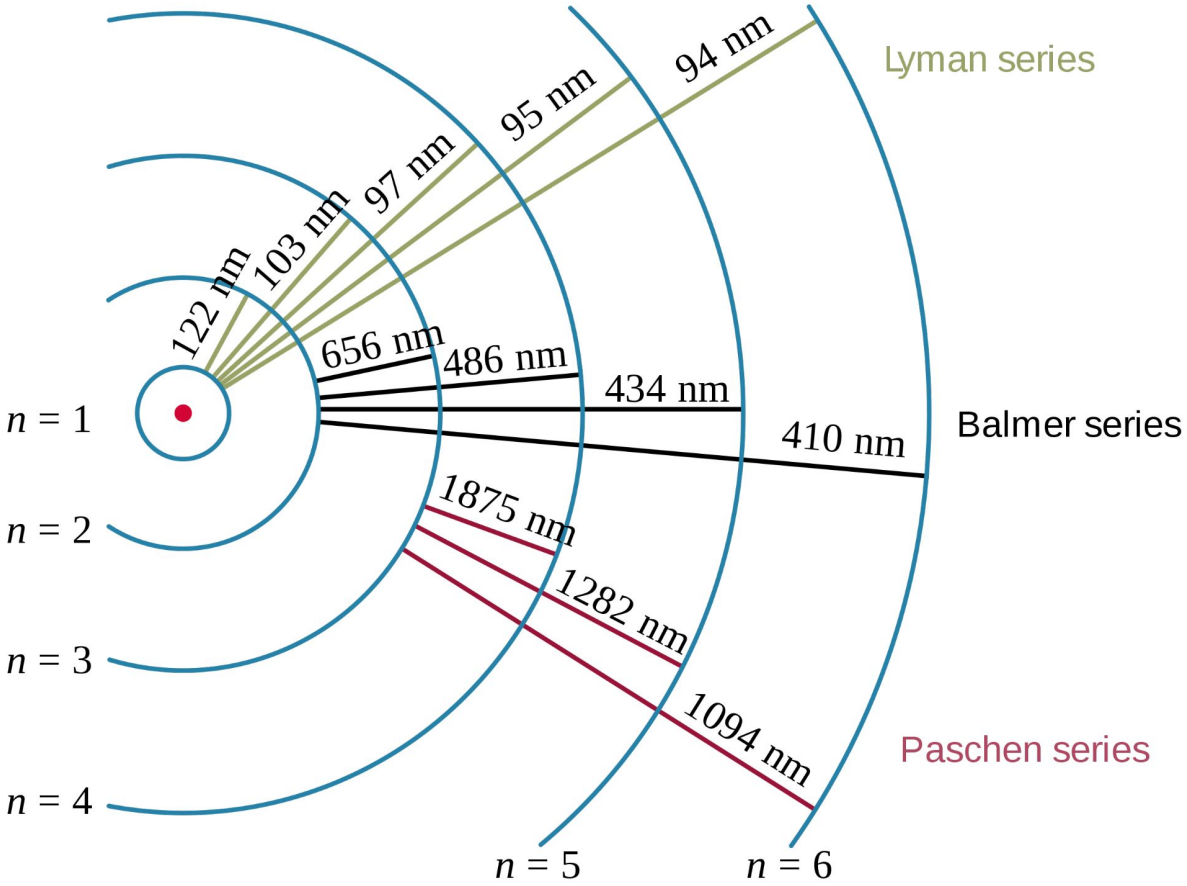
<u>Stellar atmospheres/planetary nebulae:</u>

- Optically thin, photons can escape.
- Strong spectral lines.

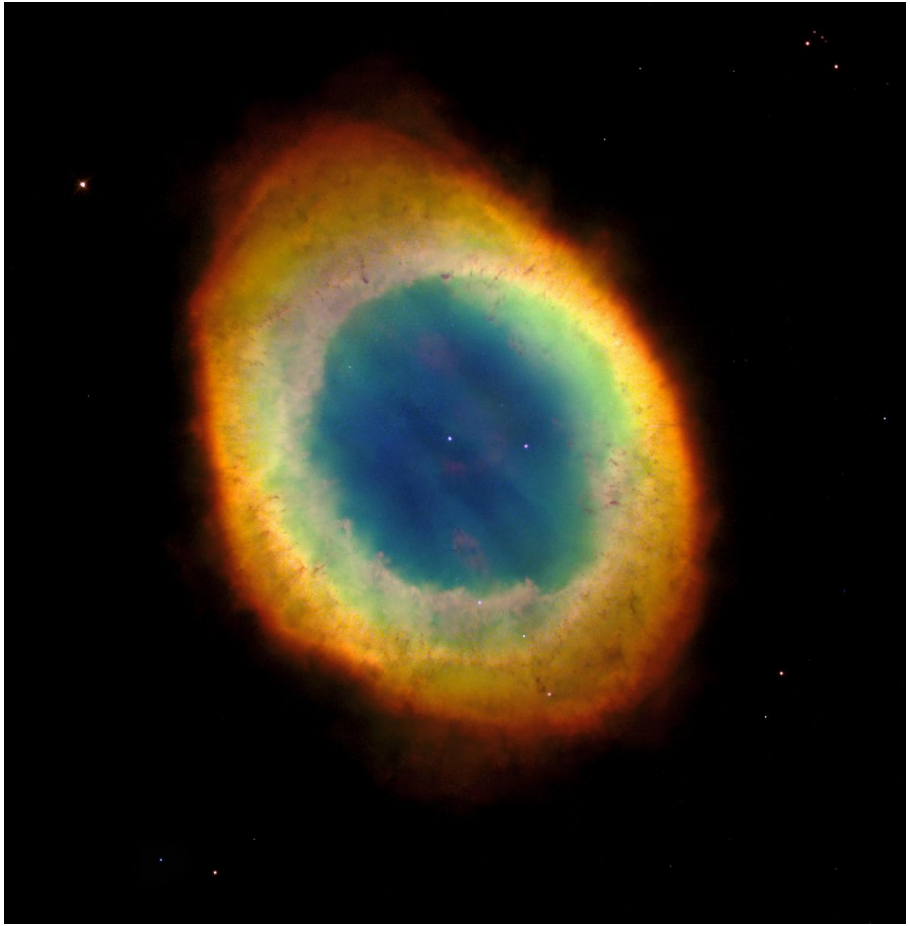


Lecture 3
Line emission

Atomic levels



Stratification



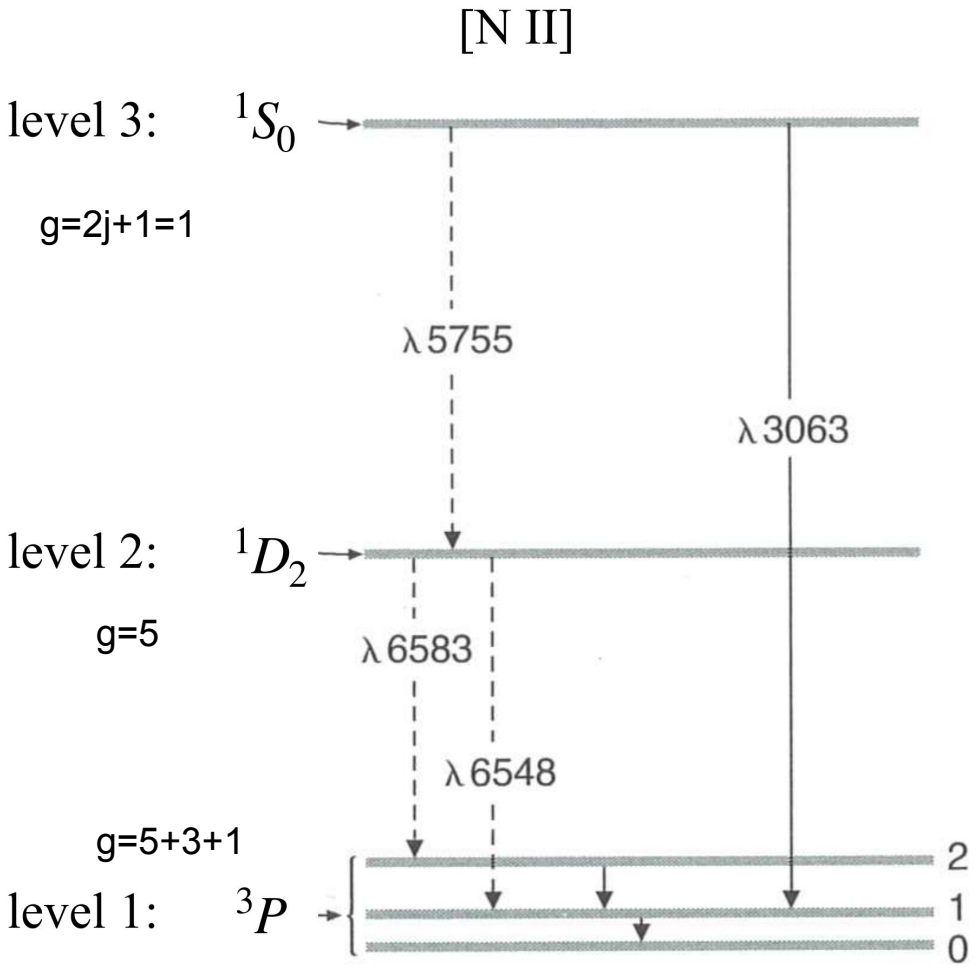
Temperature diagnostic

$A_{32} = A(^1S_0 - {}^1D_2)$

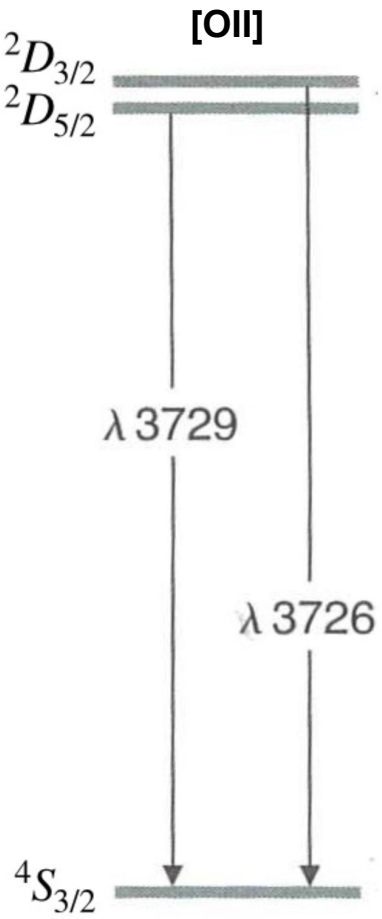
$A_{31} = A(^1S_0 - {}^3P_2) + A(^1S_0 - {}^3P_1) + A(^1S_0 - {}^3P_0)$

$A_{21} = A(^1D_2 - {}^3P_2) + A(^1D_2 - {}^3P_1) + A(^1D_2 - {}^3P_0)$

Also, for T~10^4 K, coronal approximation holds!
Good temperature diagnostic.

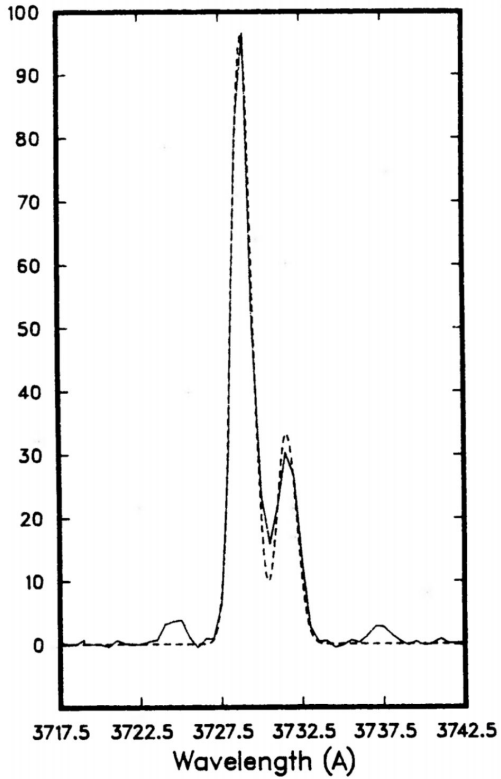


Density diagnostic



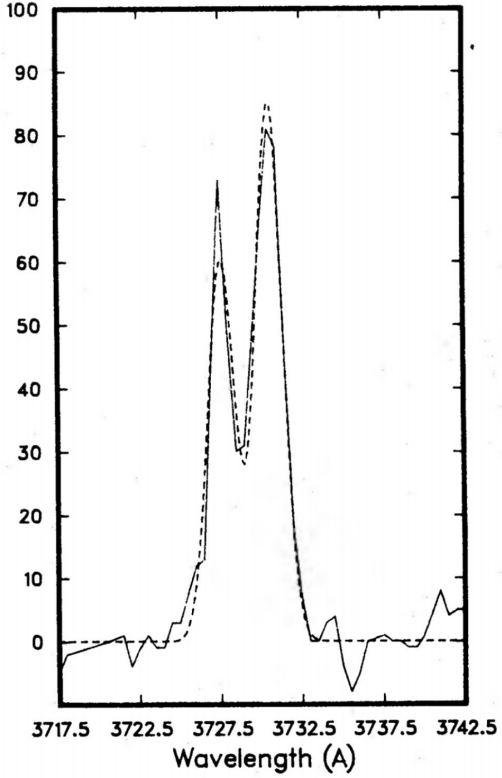
High-density limit

LMC N110



Low-density limit

SMC J2



Lecture 4
Ionization and recombination

Ionization front



Lecture 5
Absorption lines

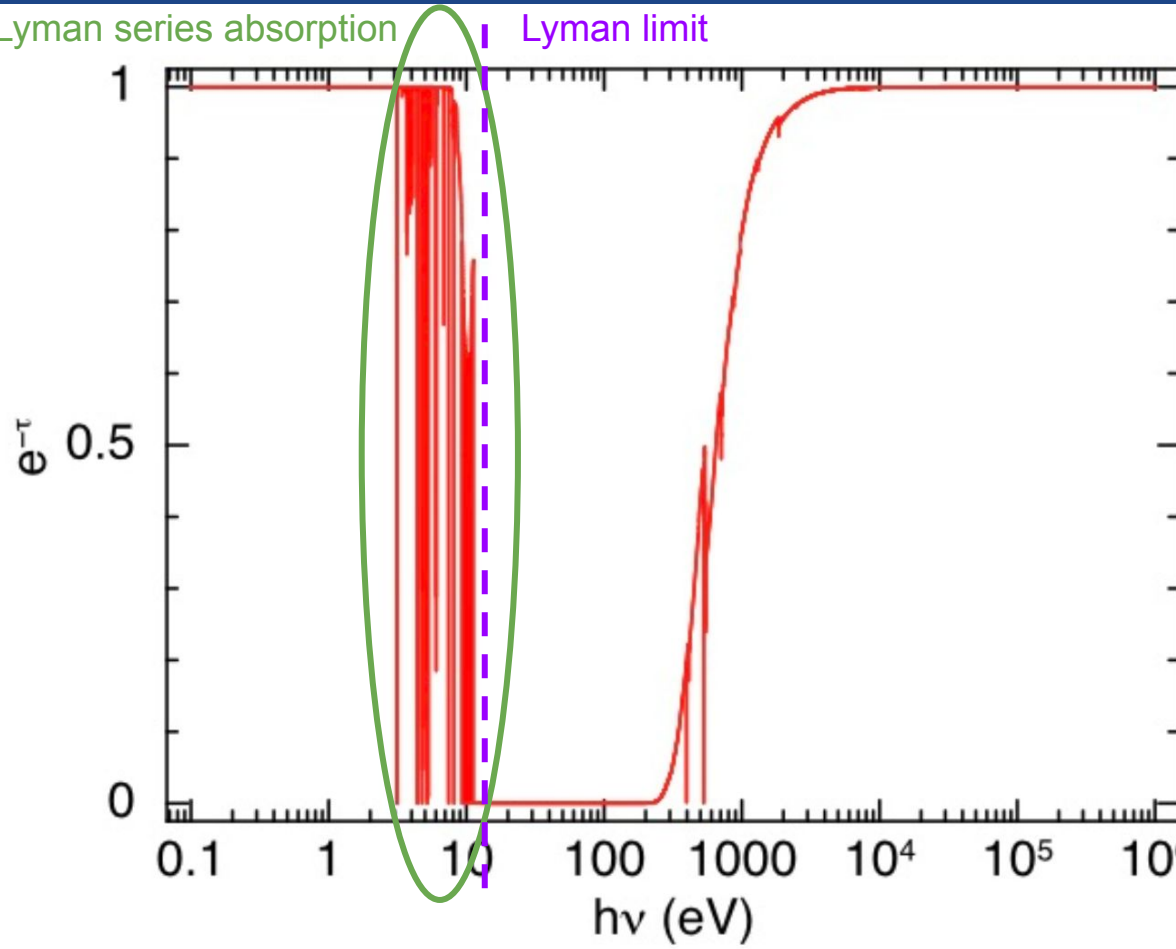
Absorption

$$S_\nu \ll I_\nu^0$$

$$I_\nu = e^{-\tau} I_\nu^0$$

- Low T_e: most HI in ground state
  Line absorption dominated by Lyman series (mostly Lyα)

- Above hν=13.6 eV (Lyman limit): ionization

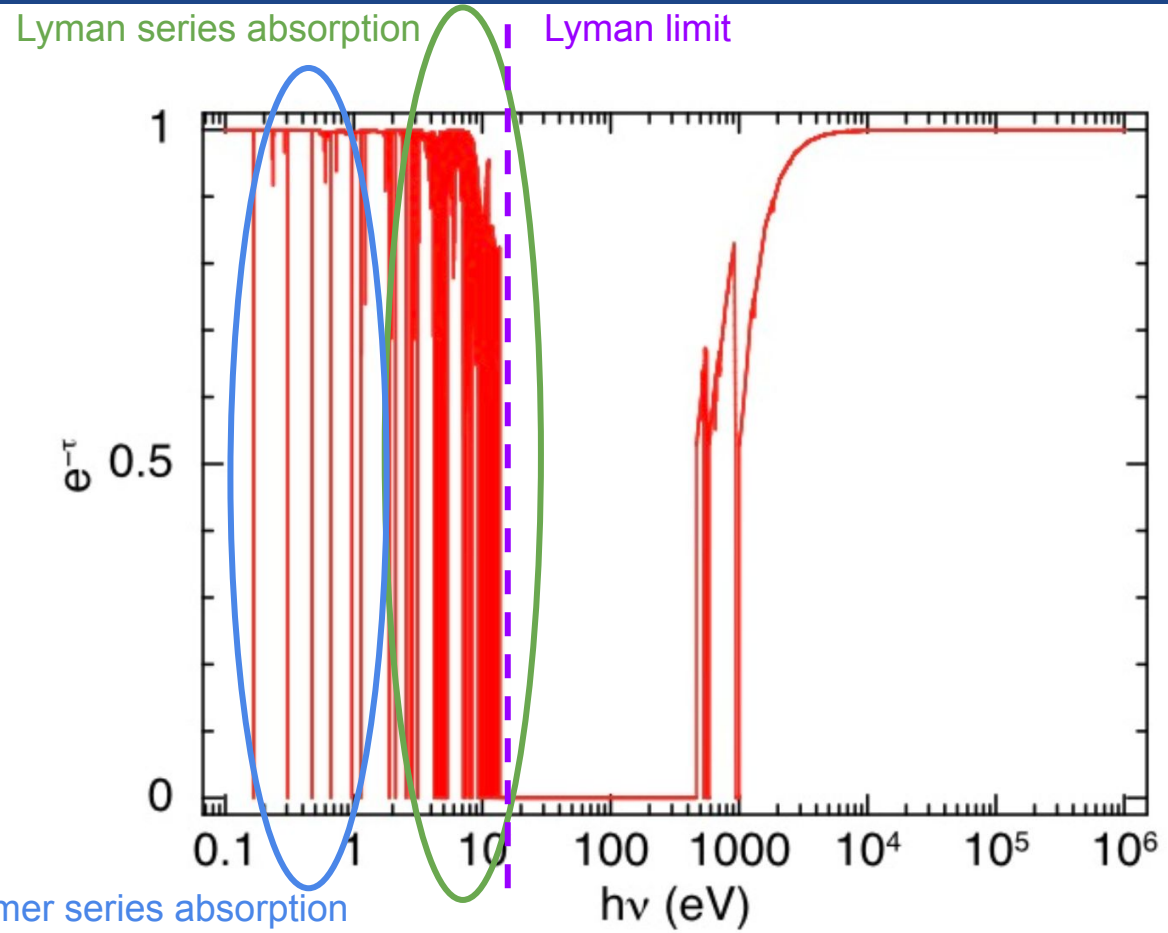


Absorption

$$S_\nu \ll I_\nu^0$$

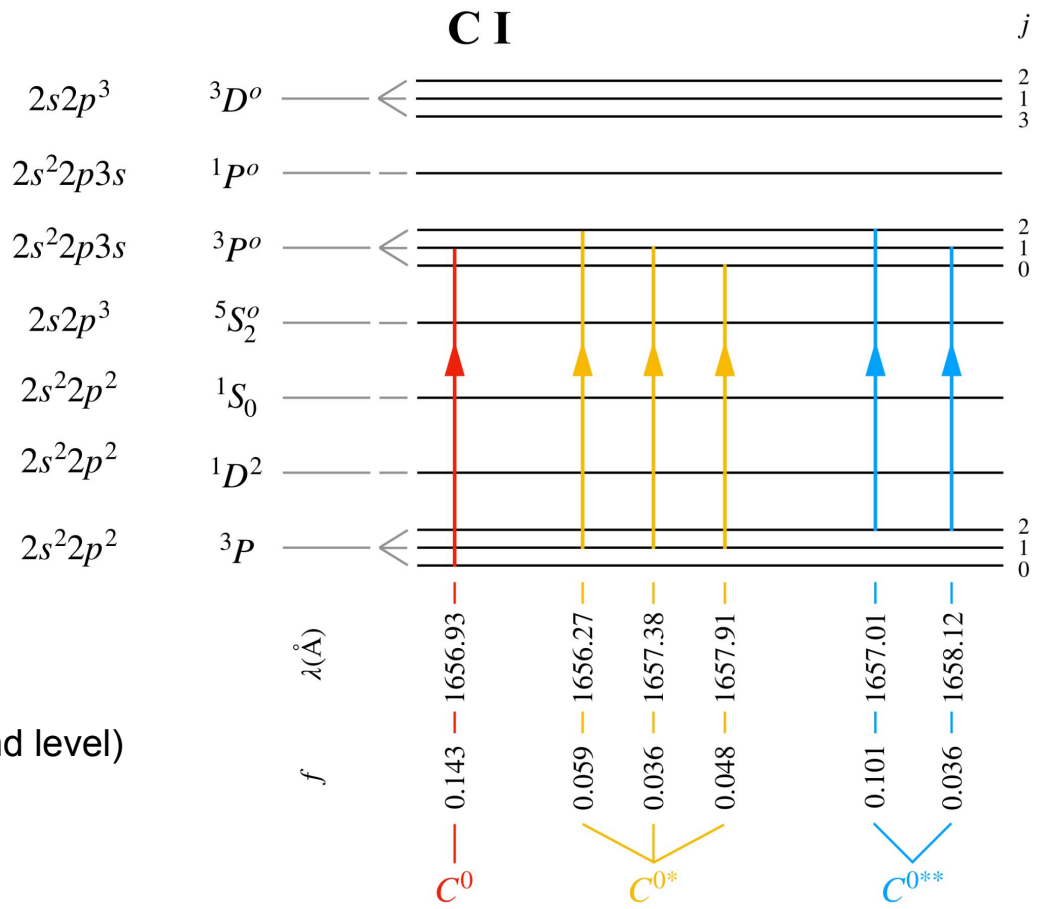
$$I_\nu = e^{-\tau} I_\nu^0$$

- Low T_e: most HI in ground state
  Line absorption dominated by Lyman series (mostly Lyα)
- Above hν=13.6 eV (Lyman limit): ionization
- Higher T_e: some excited HI
  Balmer lines

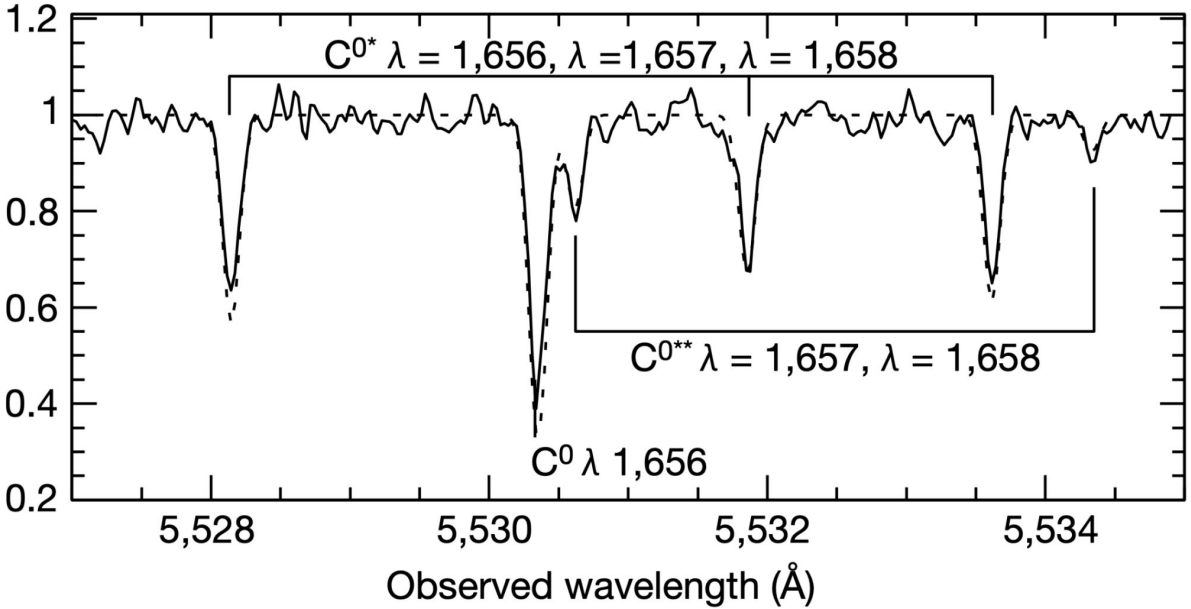


PKS1232+0815

- Very cold cloud. Heated by T_{CMB}.
- Most atoms in three split ground levels
- 3 main emission lines (from each ground level)

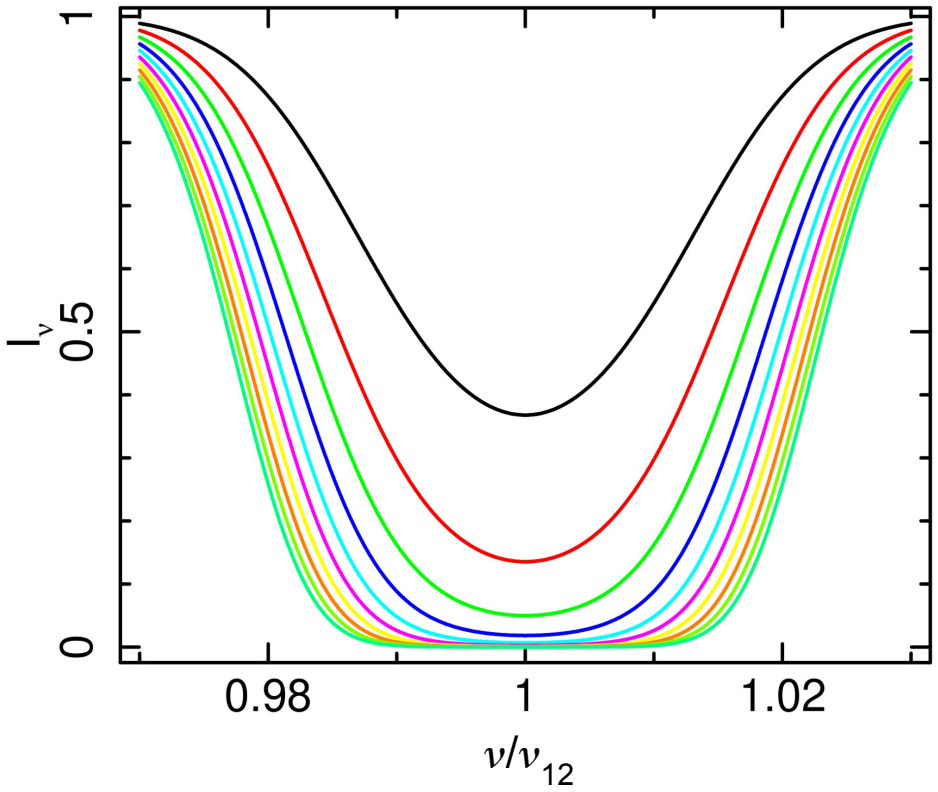


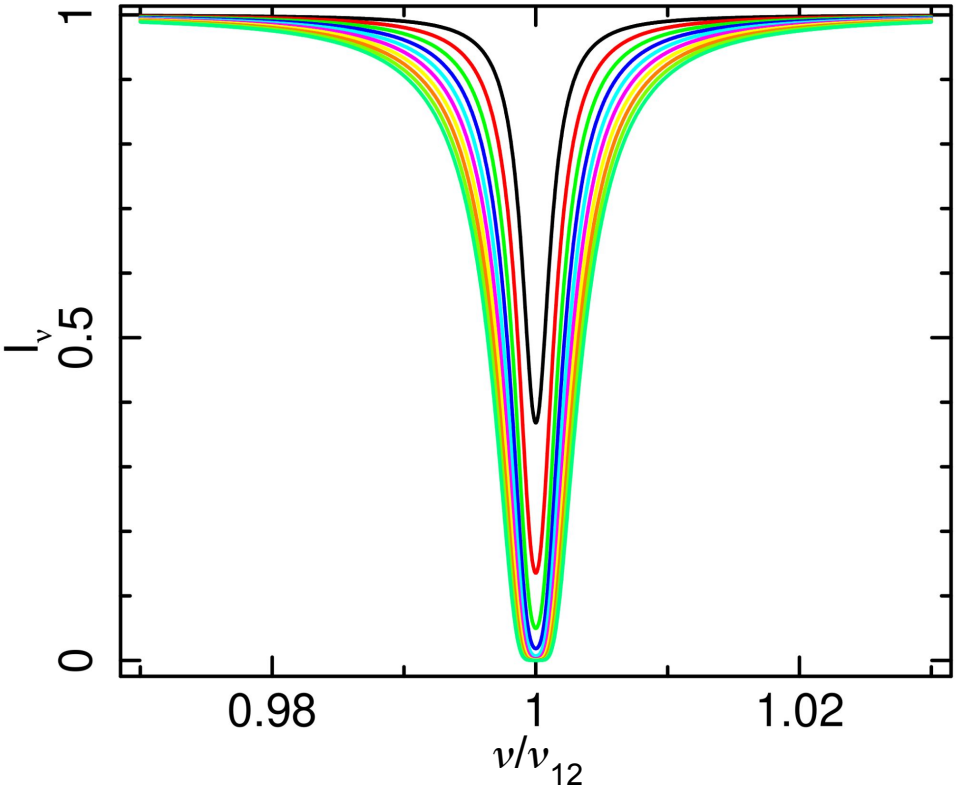
PKS1232+0815



- Very cold cloud. Heated by T_{CMB}.
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- 3 main emission lines (from each ground level)

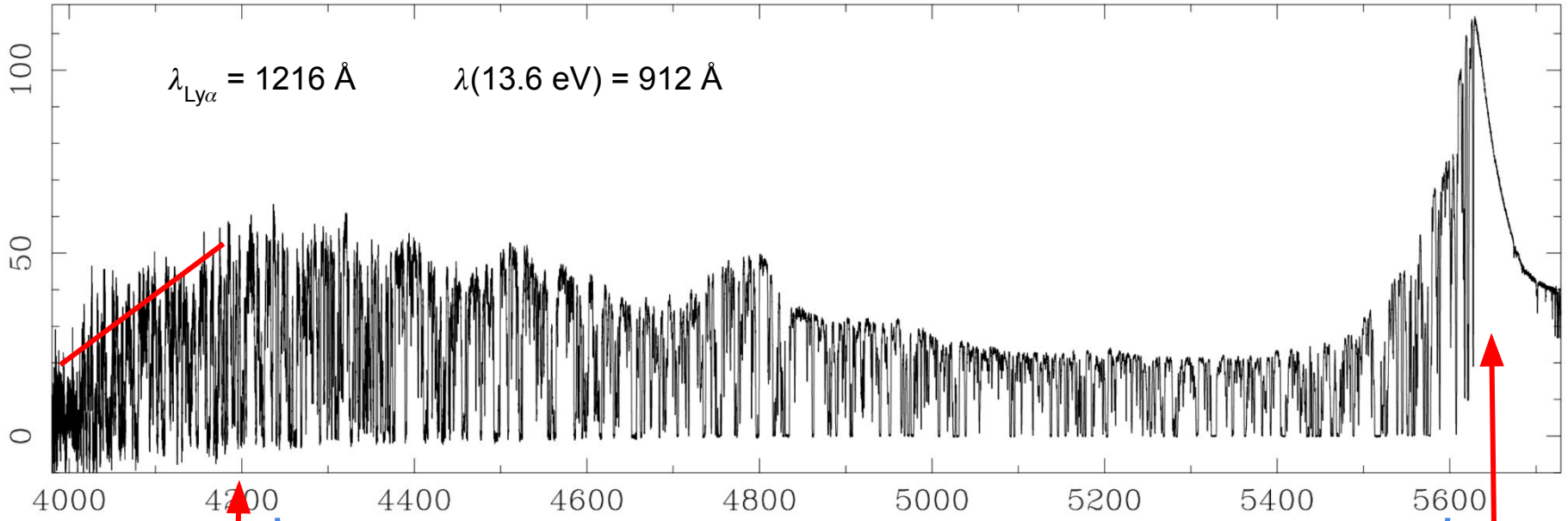
Line broadening





Linear increase in τ

Lyα forest



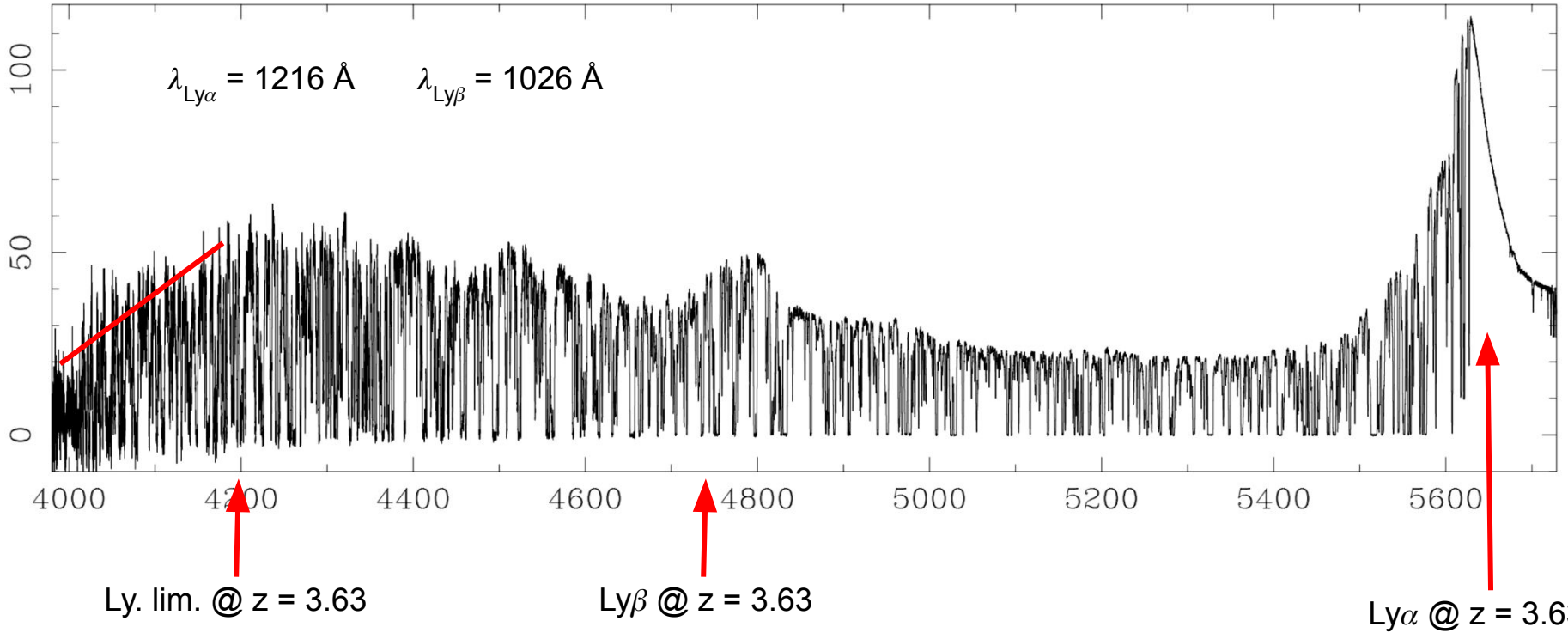
Lyman limit
@ z = 3.63

Lyα absorption from clouds of HI at z: $\lambda = \lambda_{Ly\alpha}(1+z)$

Most of it low-density stuff (filaments and sheets)

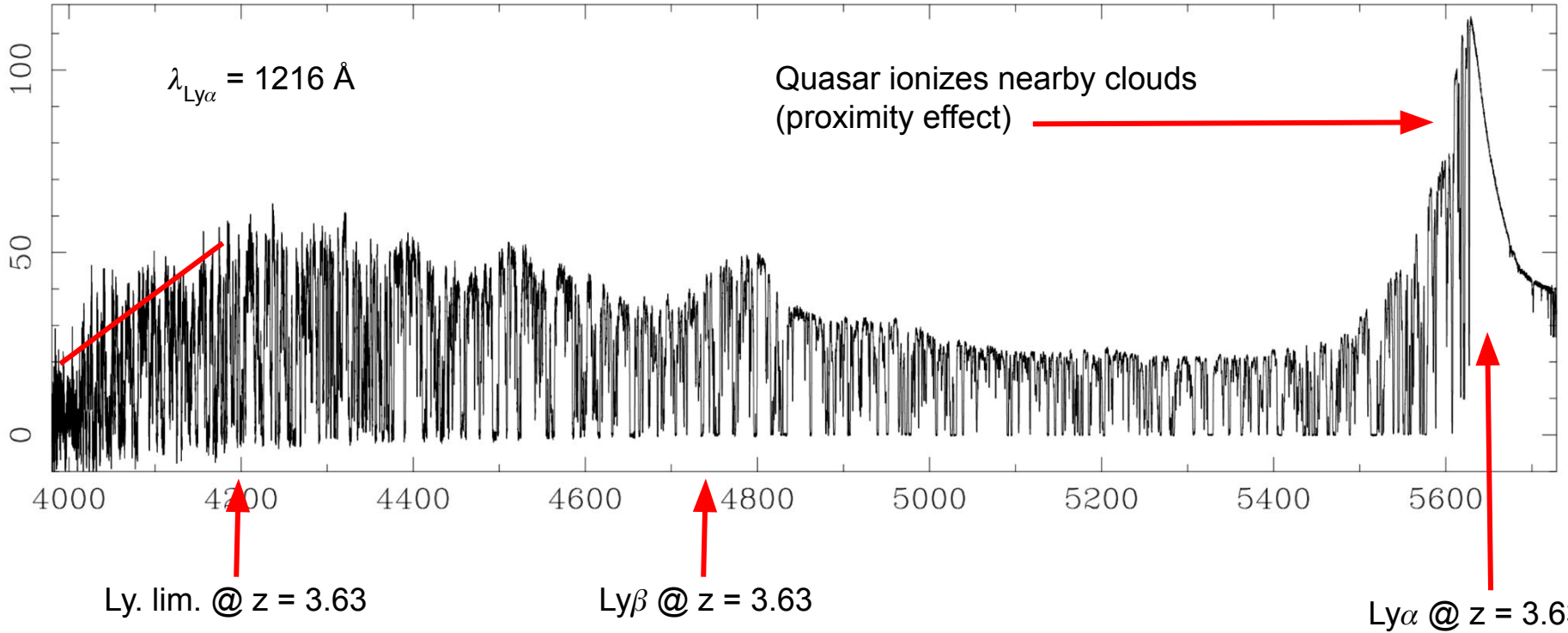
Quasar's Lyα line (z=3.63)

Lyα forest

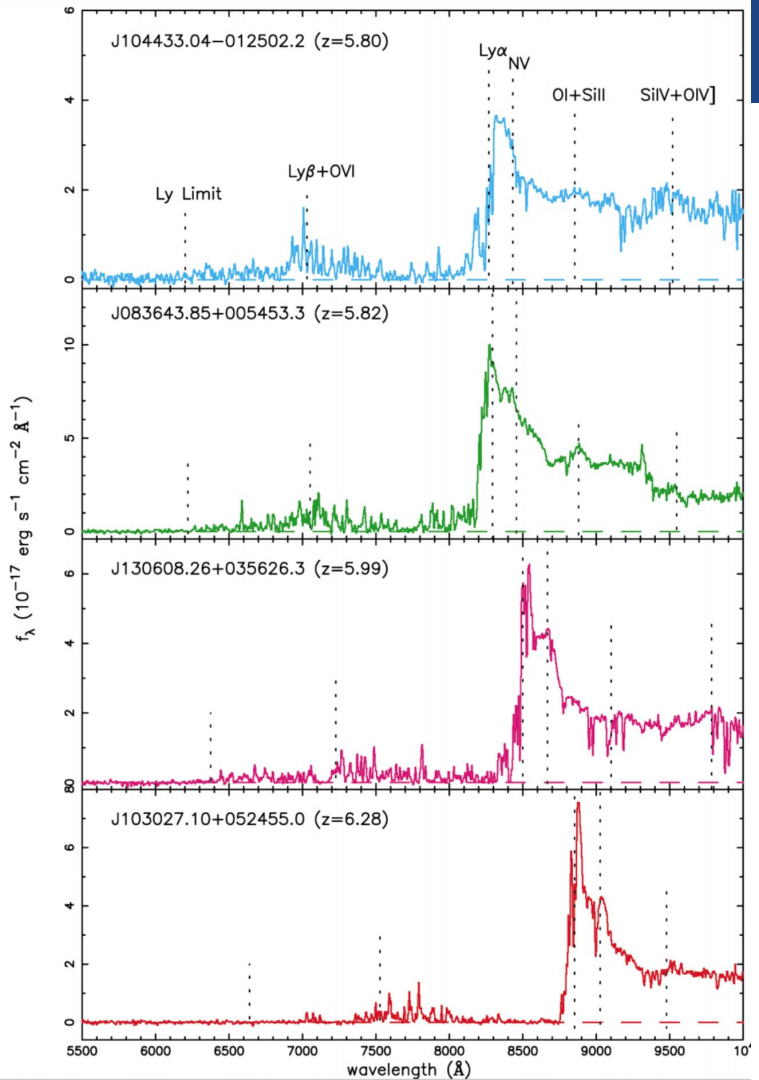


Lyβ appears at shorter wavelengths
(there aren't intrinsically more clouds).
In fact one woud expect $n_{HI} \sim (1+z)^3$

Lyα forest



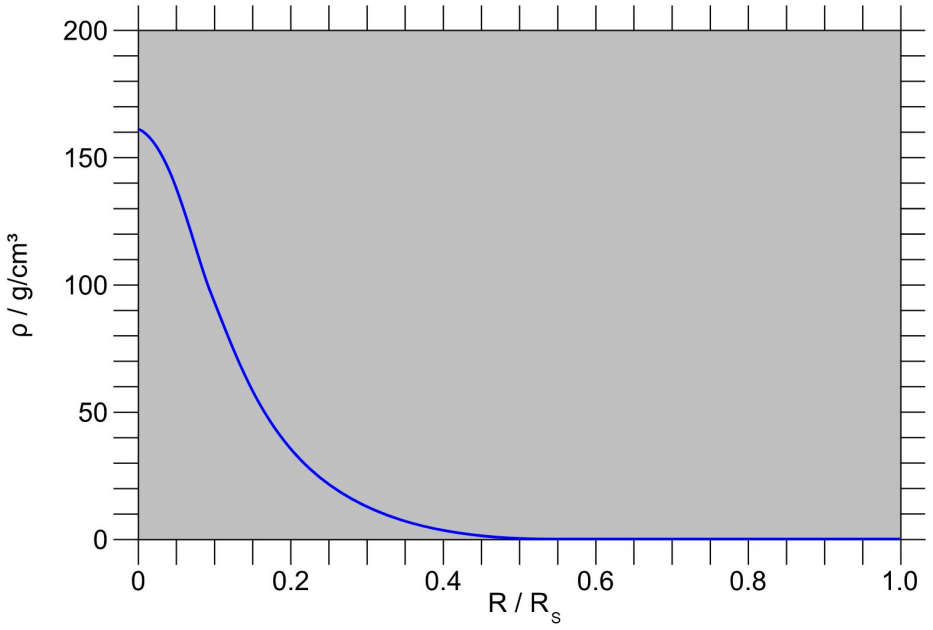
Gunn-Peterson trough

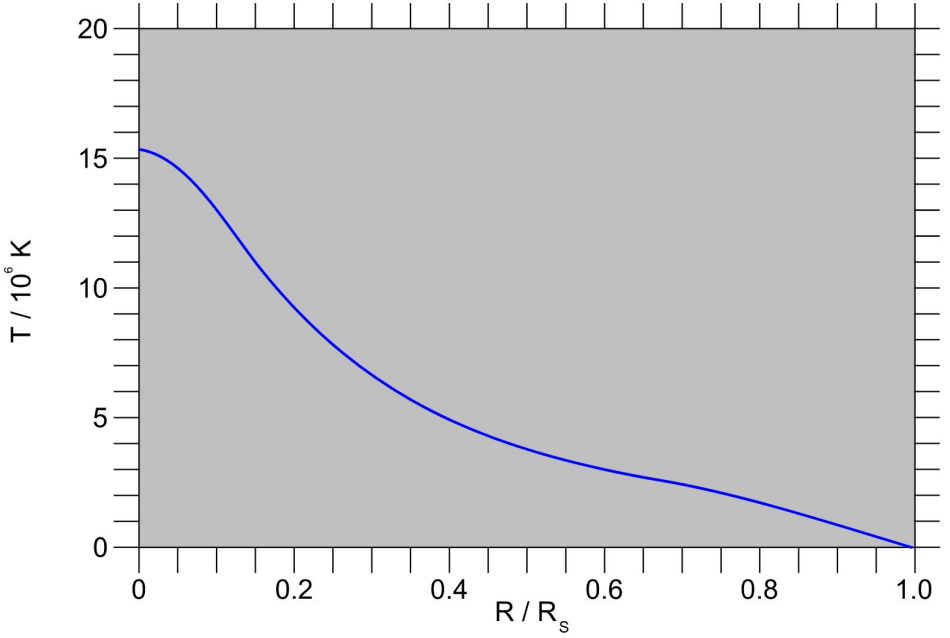


Becker et al. 2001

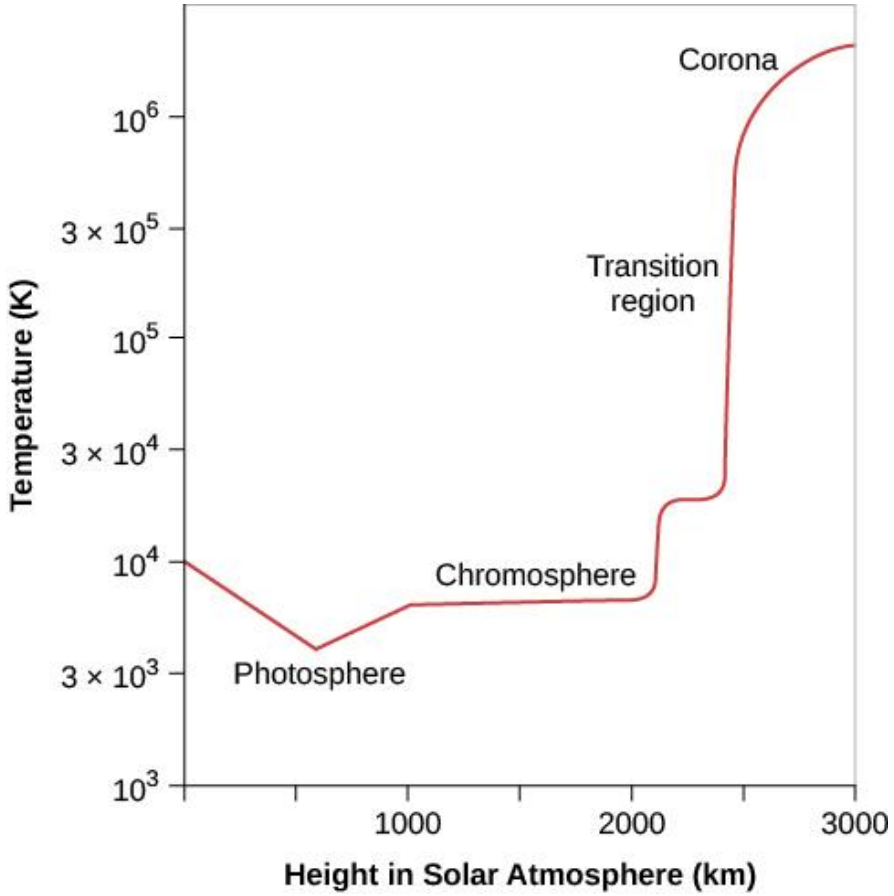
Lecture 6
Stellar spectroscopy

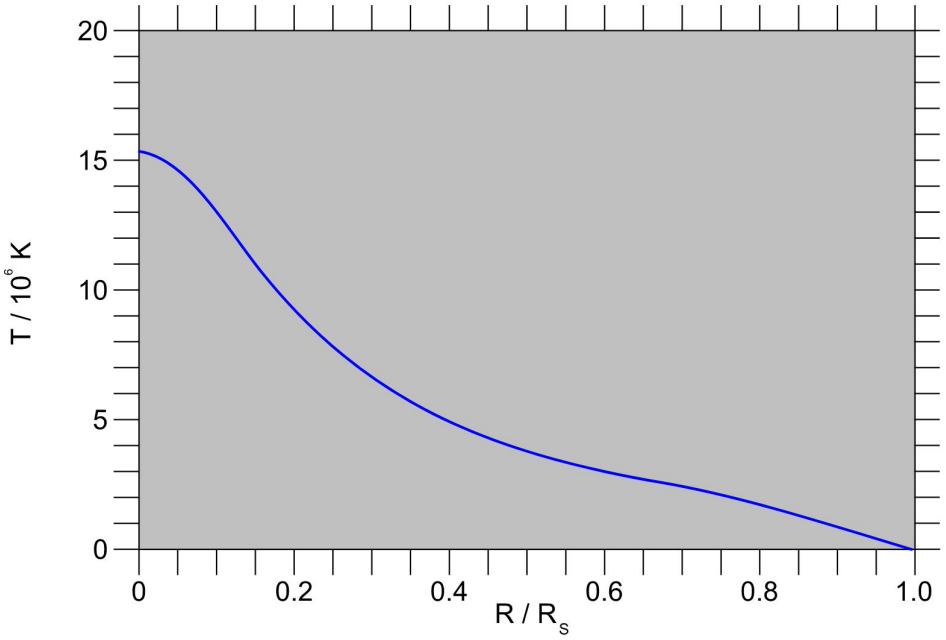
Stars


200
150
100
50
0
ρ / g/cm³
0
0.2
0.4
0.6
0.8
1.0
R / R_S

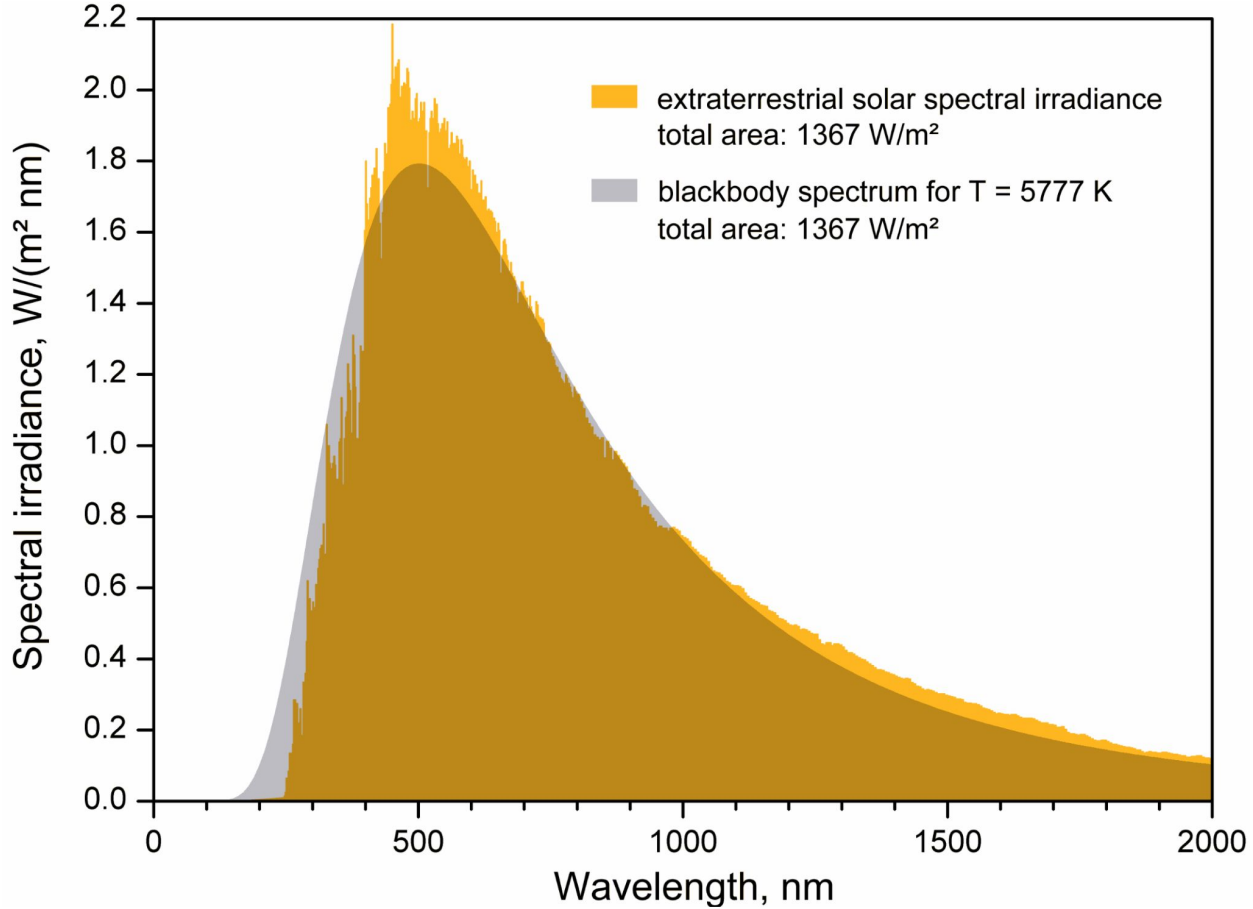

20
15
10
5
0
T / 10⁶ K
0
0.2
0.4
0.6
0.8
1.0
R / R_S

Stars

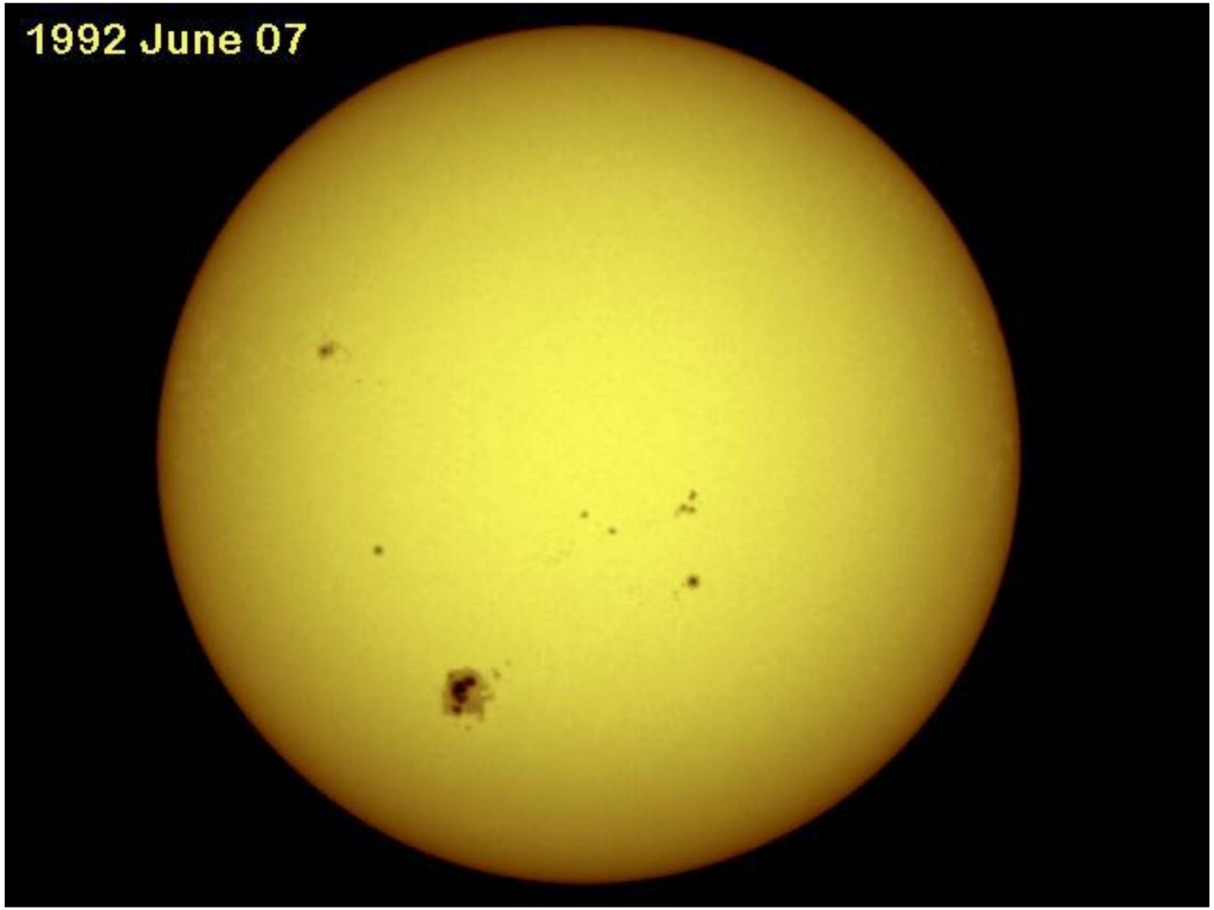

Corona
Transition region
Chromosphere
Photosphere
Temperature (K)
10^6
3×10^5
10^5
3×10^4
10^4
3×10^3
10^3
1000
2000
3000
Height in Solar Atmosphere (km)


20
15
10
5
0
$T / 10^6$ K
0
0.2
0.4
0.6
0.8
1.0
R / R_S

Stars

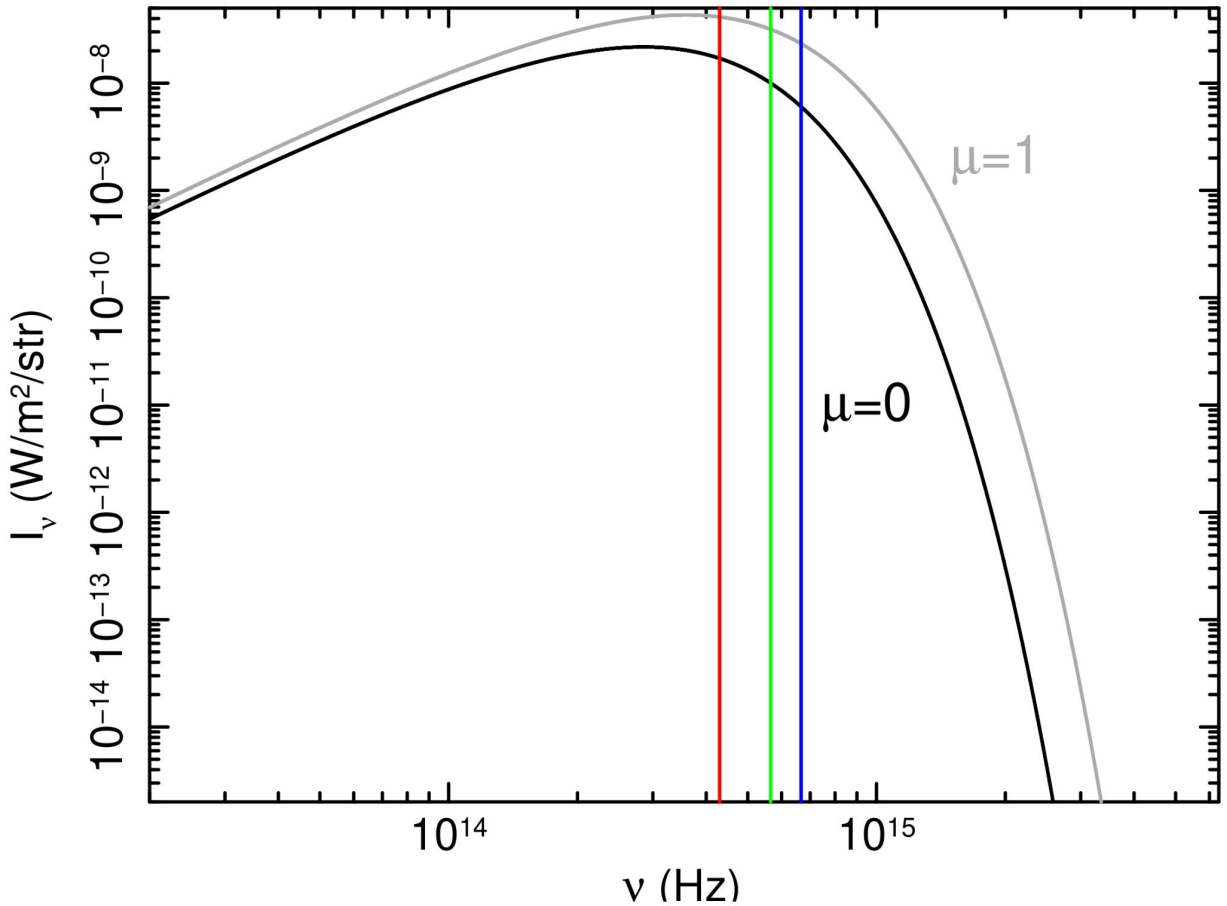

extraterrestrial solar spectral irradiance
total area: 1367 W/m²
blackbody spectrum for T = 5777 K
total area: 1367 W/m²
Spectral irradiance, W/(m² nm)
Wavelength, nm
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
0
500
1000
1500
2000

Limb darkening



Limb darkening

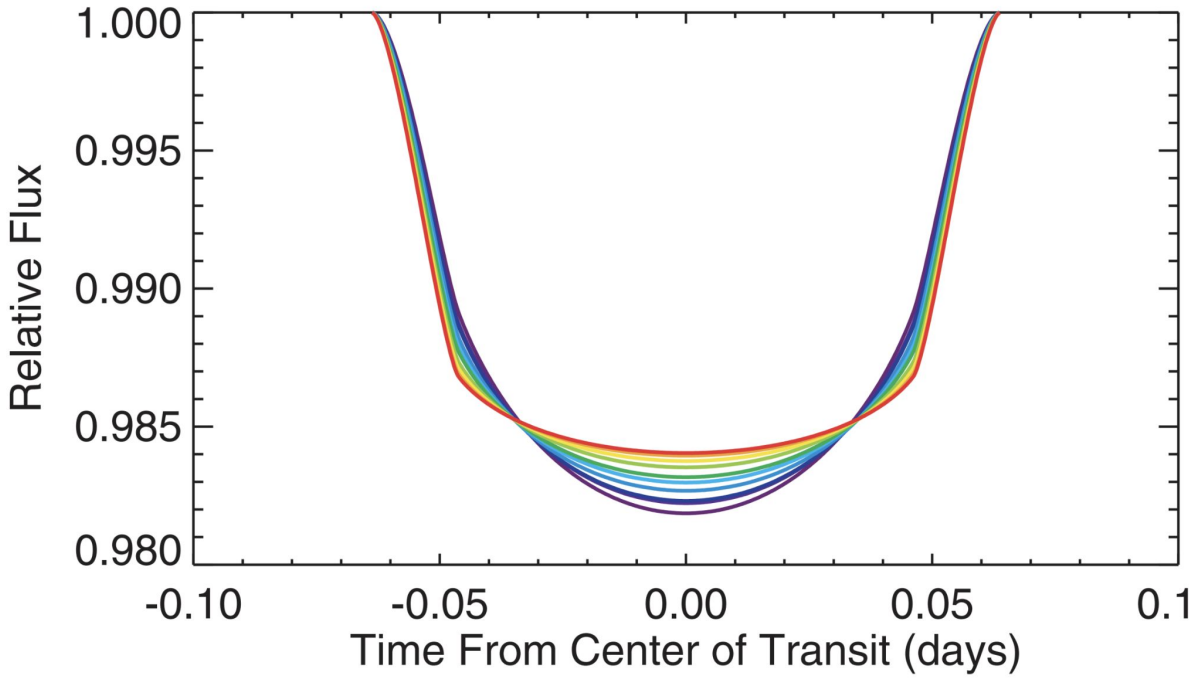
- Looking deeper (μ=1) means looking at a hotter BB spectrum.
- Limb-darkening is frequency-dependent



Limb darkening

Impact in exoplanet transits.

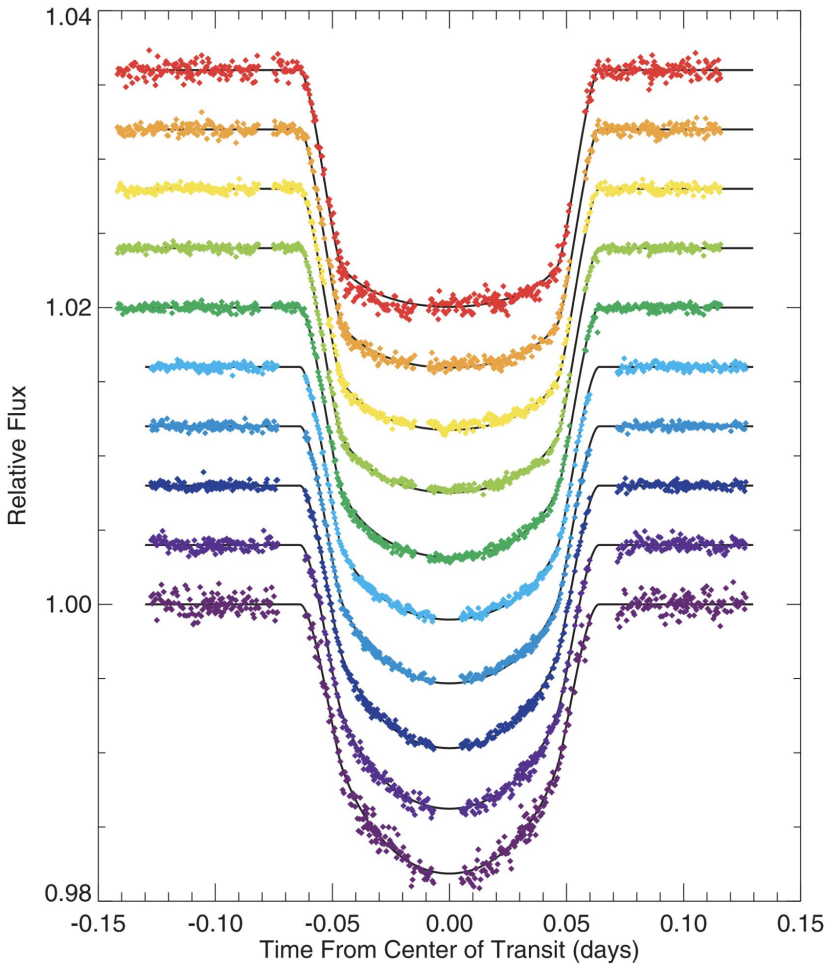
Duration/depth depends on frequency.



Limb darkening

Impact in exoplanet transits.

Duration/depth depends on frequency.

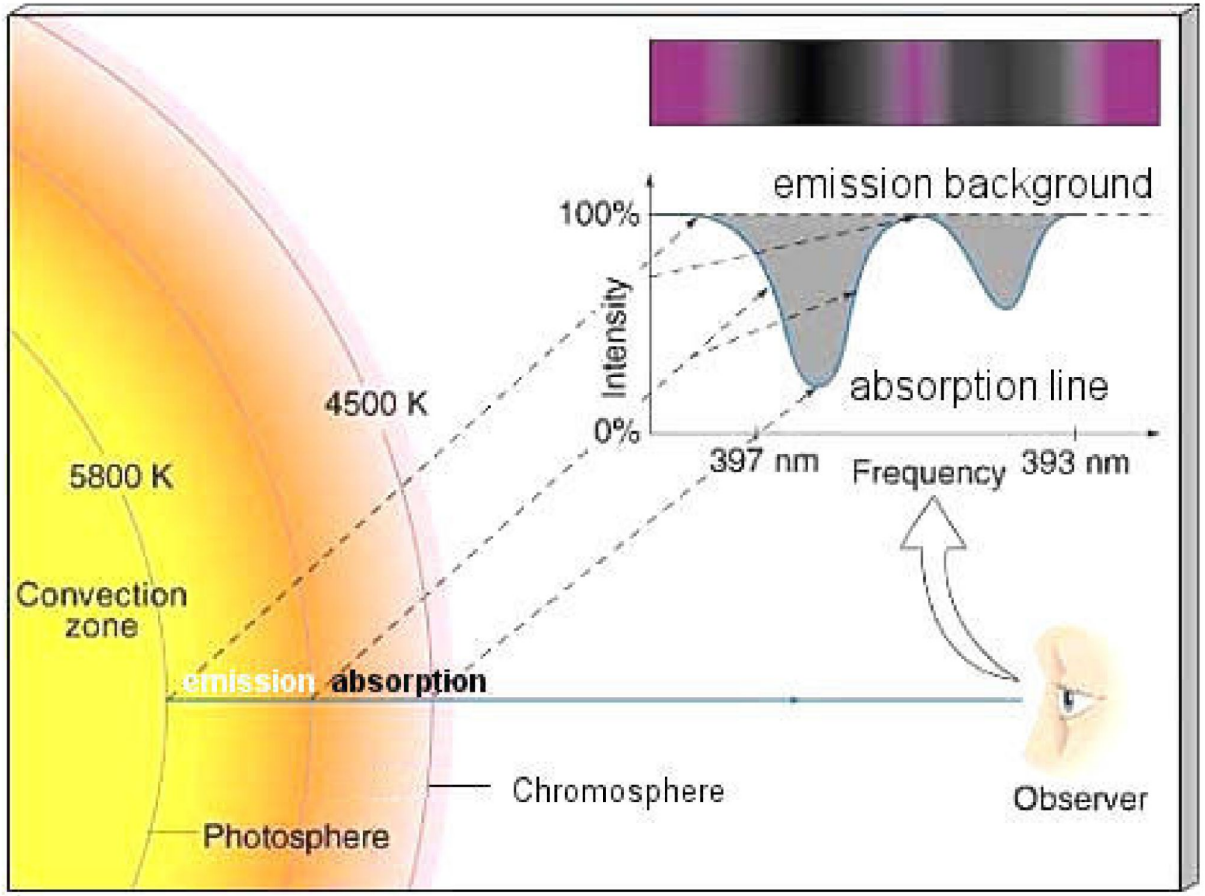


Emergent spectrum

We observe down to ~$2\tau/3$

But $\tau = \tau_\nu$!

So, if there's an absorption, line, at that ν you actually observe the BB at a cooler temperature.

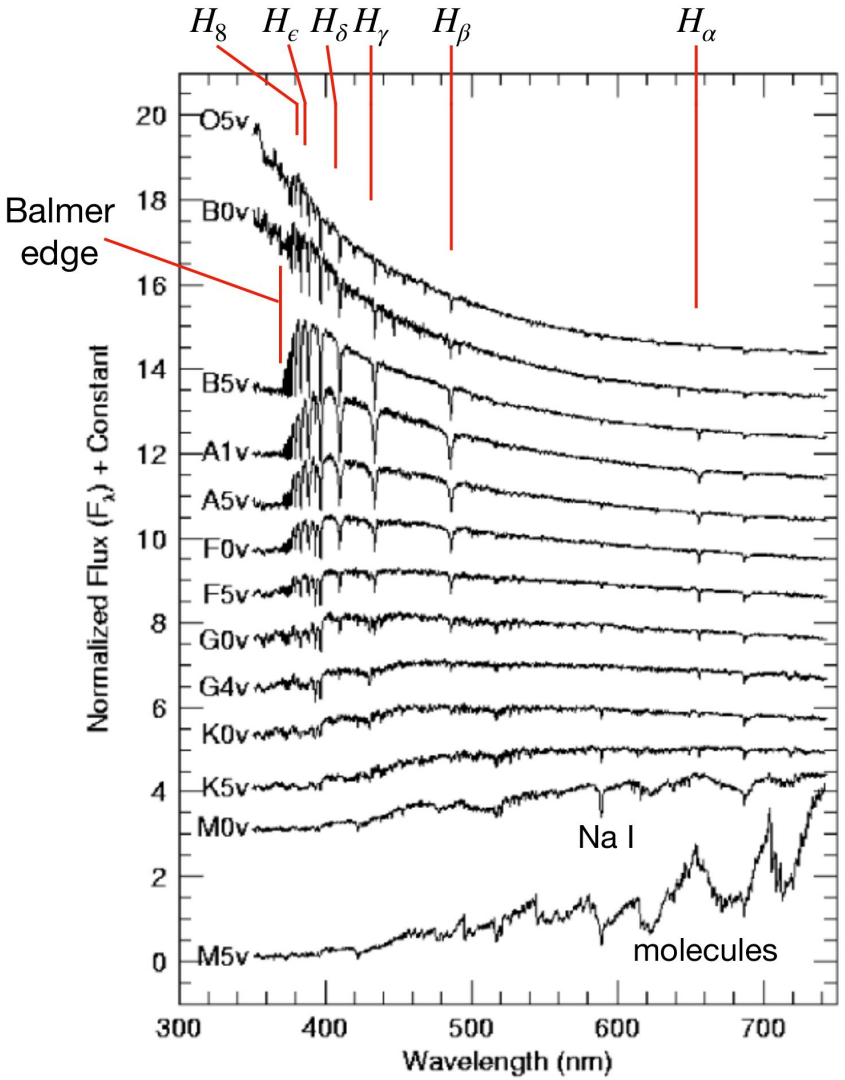


Emergent spectrum

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So, if there's an absorption, line, at that ν you actually observe the BB at a cooler temperature.



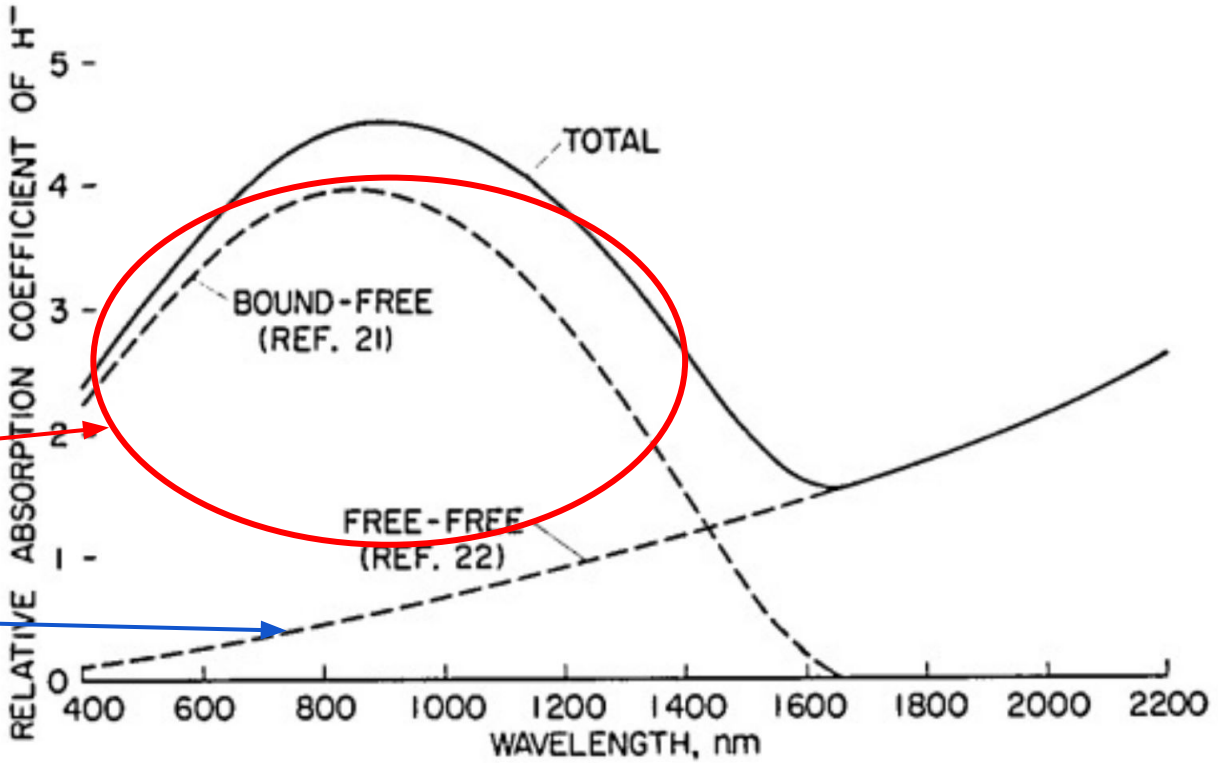
Cold stars

Spectrum suppressed in cold stars for low ν.

Reason: H- (= p + e + e !)

Ionization cross-section has an unusual frequency dependence (no edge).

Bremsstrahlung leads to absorption instead of emission.



Lecture 7
Free-free emission (Bremmstrahlung)

Galaxy clusters

M(DM) ~ 10x M(ICM)
M(ICM) ~ 10x M(stars)

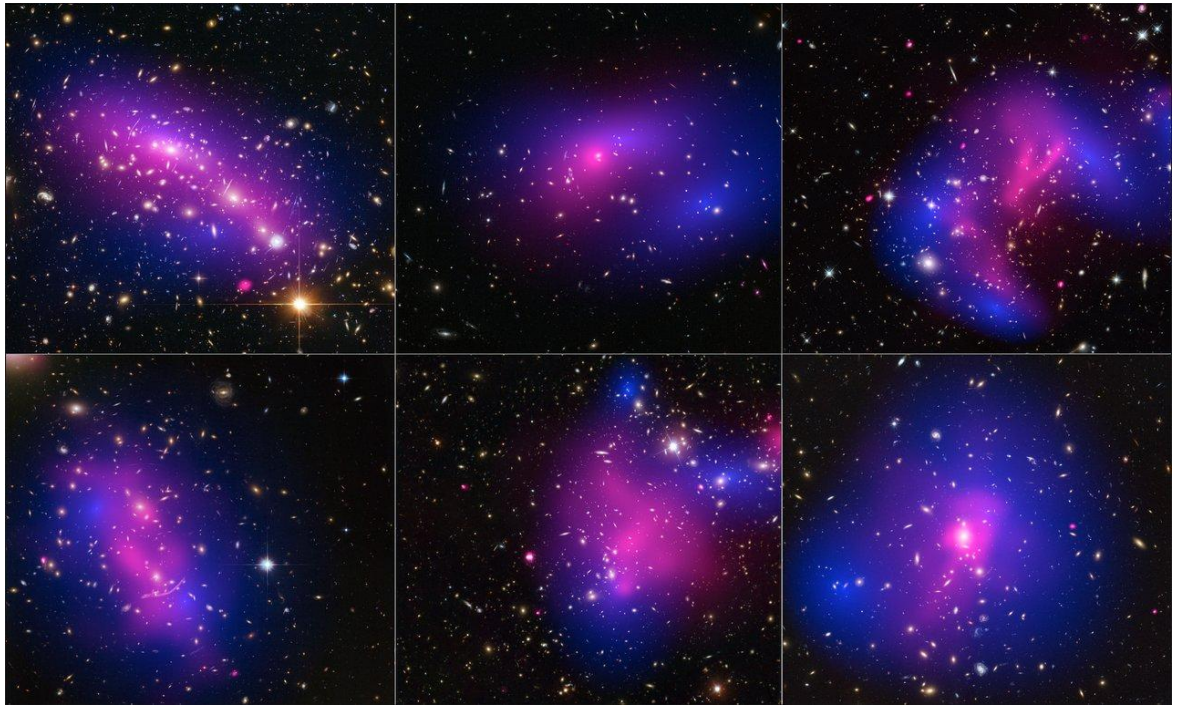
ICM:

- Plasma with large thermal motions.
- Shock-heated through gravitational infalling.

$$k_B T_e = k_B T_i \sim \frac{GMm_H}{R}$$

M ~ 10^{14} M_{sun}
R ~ 1 Mpc

=> $k_B T_e$ ~ 5 keV, T_e ~ 10^8 K

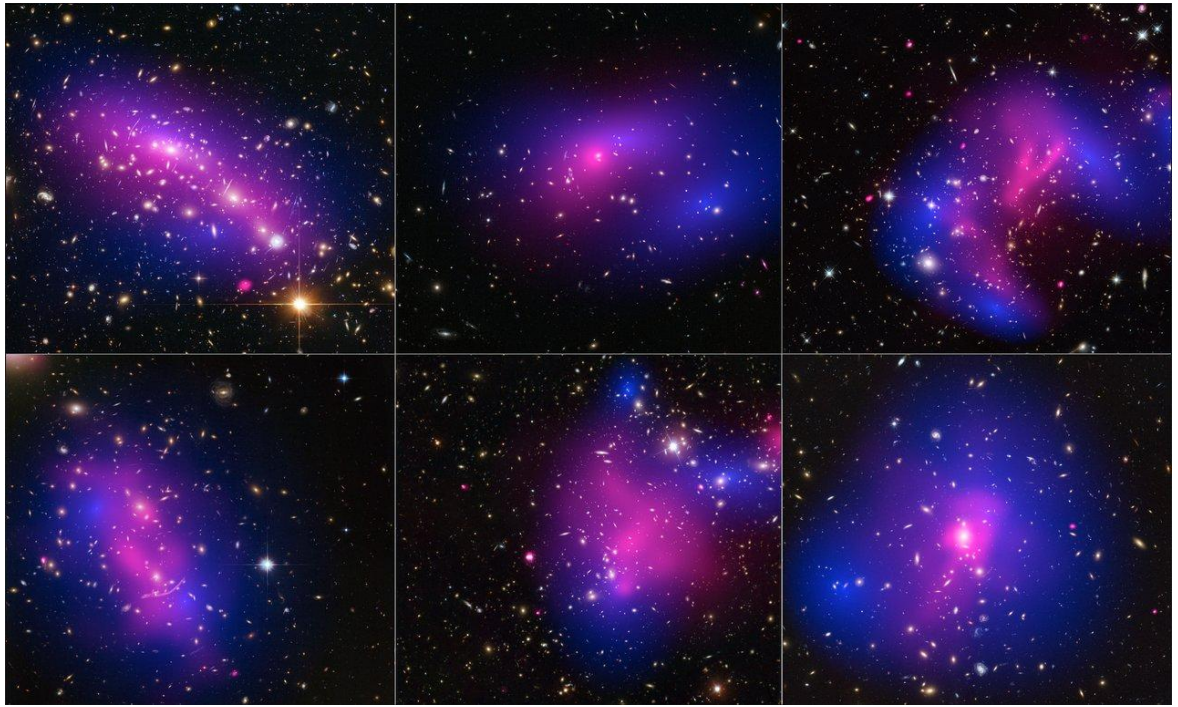


Galaxy clusters

M(DM) ~ 10x M(ICM)
M(ICM) ~ 10x M(stars)

ICM:

- Plasma with large thermal motions.
- Shock-heated through gravitational infalling.
- Not in thermal eq. with radiation!
- n_e too low to keep detailed balance.
- Main radiation mechanism: *Bremsstrahlung*

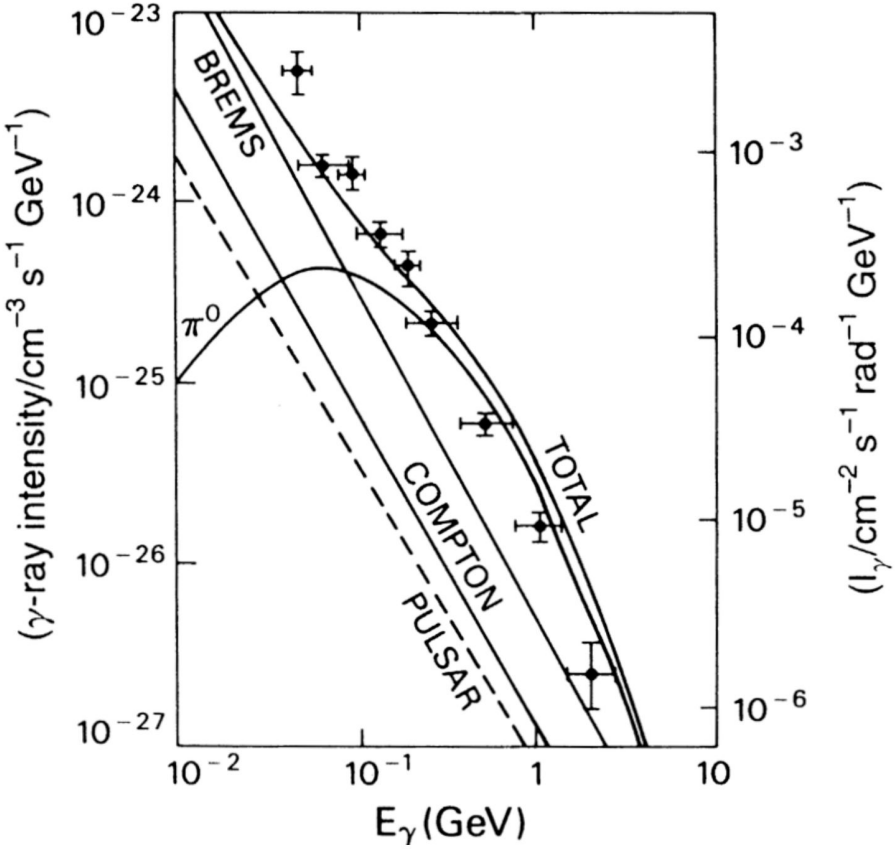


Relativistic free-free

m_e = 0.5 MeV -> γ-rays

γ-ray dominated by pion decay in CR collisions, but this drops off below ~10 MeV.

Relativistic free-free could explain soft γ-ray spectrum.



Lecture 8
Thomson and Compton scattering

Boltzmann and Kompaneets equations

Boltzmann

$$\frac{df_\gamma}{dt} = C[f_\gamma]$$

$$f_\gamma(\nu) = \frac{c^2}{h^4\nu^3} I_\nu = \frac{1}{h^3} n_\nu$$

Photon distribution function

Intensity

Occupation number

Boltzmann and Kompaneets equations

Boltzmann

$$\frac{df_\gamma}{dt} = C[f_\gamma] \qquad f_\gamma(\nu) = \frac{c^2}{h^4\nu^3} I_\nu = \frac{1}{h^3} n_\nu$$

$$C[f_\gamma] = n_e \int d^3\mathbf{v}_e \int d\Omega \, \frac{d\sigma_T}{d\Omega} \left[f_e(\mathbf{v}') f_\gamma(\mathbf{p}') \left(1 + f_\gamma(\mathbf{p})\right) - f_e(\mathbf{v}) f_\gamma(\mathbf{p}) \left(1 + f_\gamma(\mathbf{p}')\right) \right]$$

Collision operator for Compton scattering

Boltzmann and Kompaneets equations

Boltzmann

$$\frac{df_\gamma}{dt} = C[f_\gamma] \qquad f_\gamma(\nu) = \frac{c^2}{h^4\nu^3} I_\nu = \frac{1}{h^3} n_\nu$$

$$C[f_\gamma] = n_e \int d^3\mathbf{v}_e \int d\Omega \, \frac{d\sigma_T}{d\Omega} \left[f_e(\mathbf{v}') f_\gamma(\mathbf{p}') \left(1 + f_\gamma(\mathbf{p})\right) - f_e(\mathbf{v}) f_\gamma(\mathbf{p}) \left(1 + f_\gamma(\mathbf{p}')\right) \right]$$

Photons upscattered from **p'**

Photons upscattered to **p'**

Stimulated scattering (will be irrelevant)

Boltzmann and Kompaneets equations

Boltzmann

$$\frac{df_\gamma}{dt} = C[f_\gamma] \qquad f_\gamma(\nu) = \frac{c^2}{h^4\nu^3} I_\nu = \frac{1}{h^3} n_\nu$$

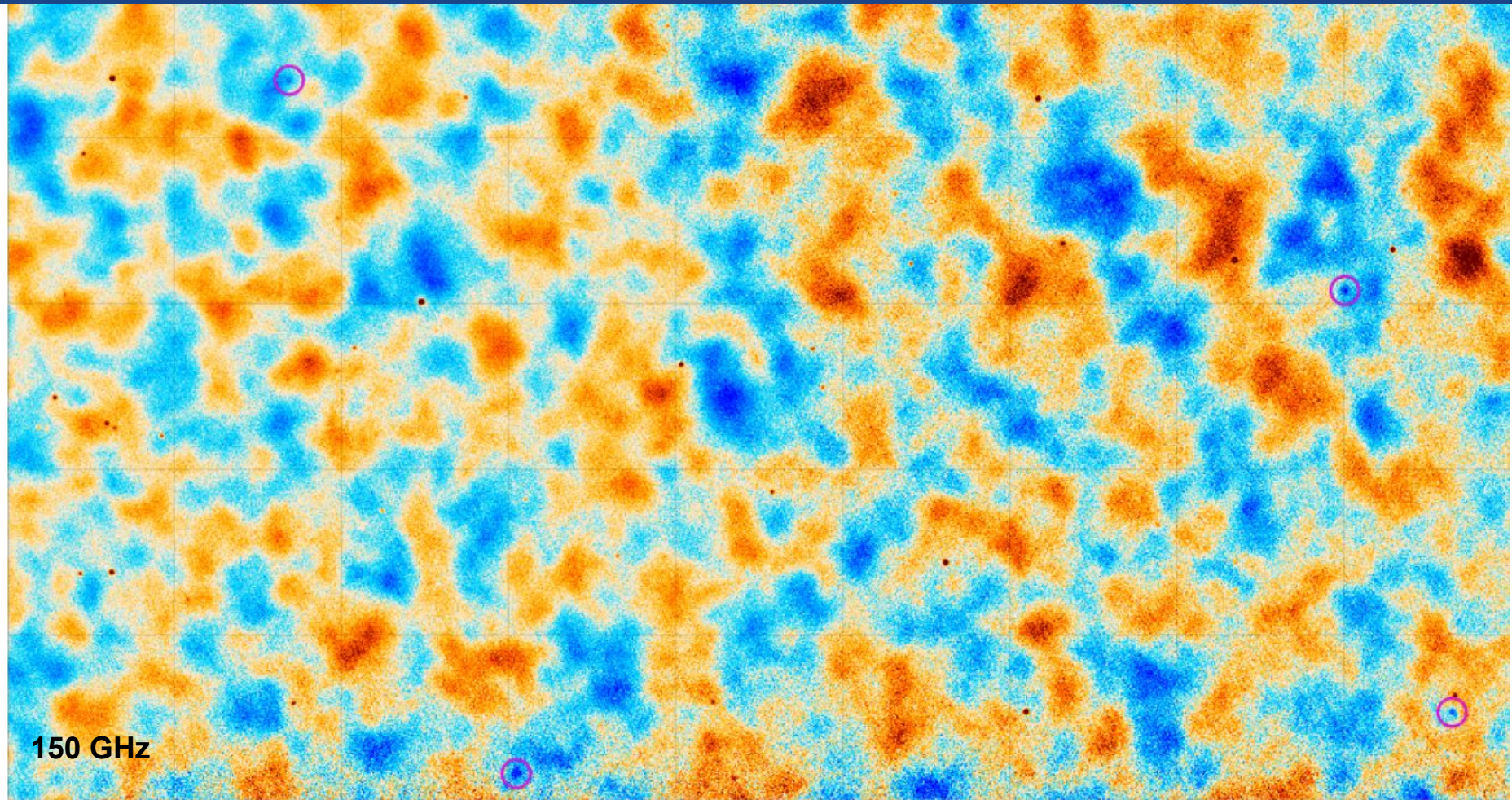
$$C[f_\gamma] = n_e \int d^3\mathbf{v}_e \int d\Omega \, \frac{d\sigma_T}{d\Omega} \left[f_e(\mathbf{v}') f_\gamma(\mathbf{p}') \left(1 + f_\gamma(\mathbf{p})\right) - f_e(\mathbf{v}) f_\gamma(\mathbf{p}) (1 + f_\gamma(\mathbf{p}')) \right]$$

Kompaneets

$$\frac{\partial n}{\partial t} = c n_e \sigma_T \frac{k_B T_e}{m_e c^2} \frac{1}{x_e^2} \frac{\partial}{\partial x_e} \left[x_e^4 \left(\frac{\partial n}{\partial x_e} + n + n^2 \right) \right]$$

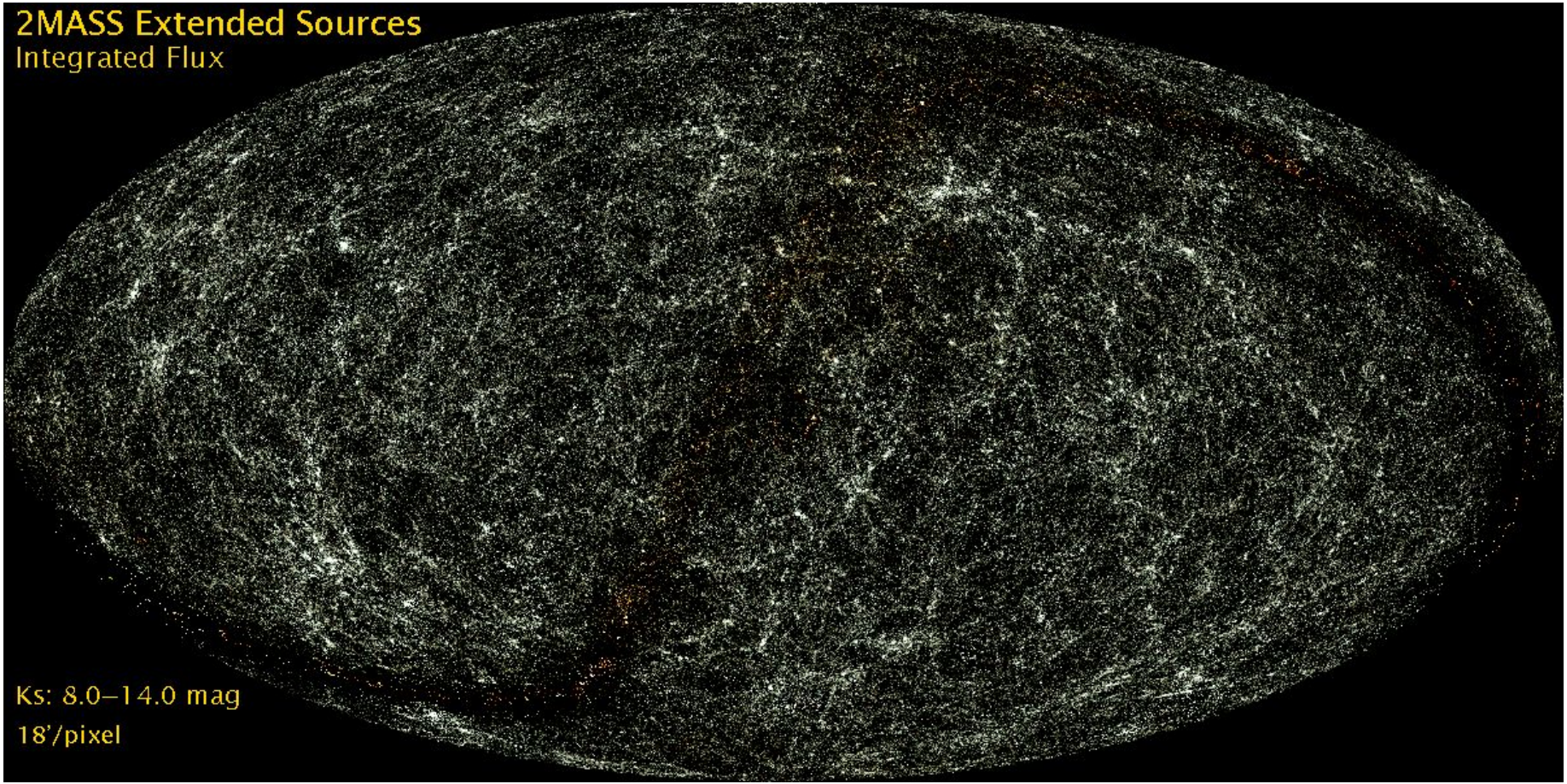
$$x \equiv \frac{h\nu}{k_B T_\gamma} \qquad x_e \equiv \frac{h\nu}{k_B T_e} = x \frac{T_\gamma}{T_e} \qquad n_{BB} = \frac{1}{e^x - 1}$$

SZ clusters

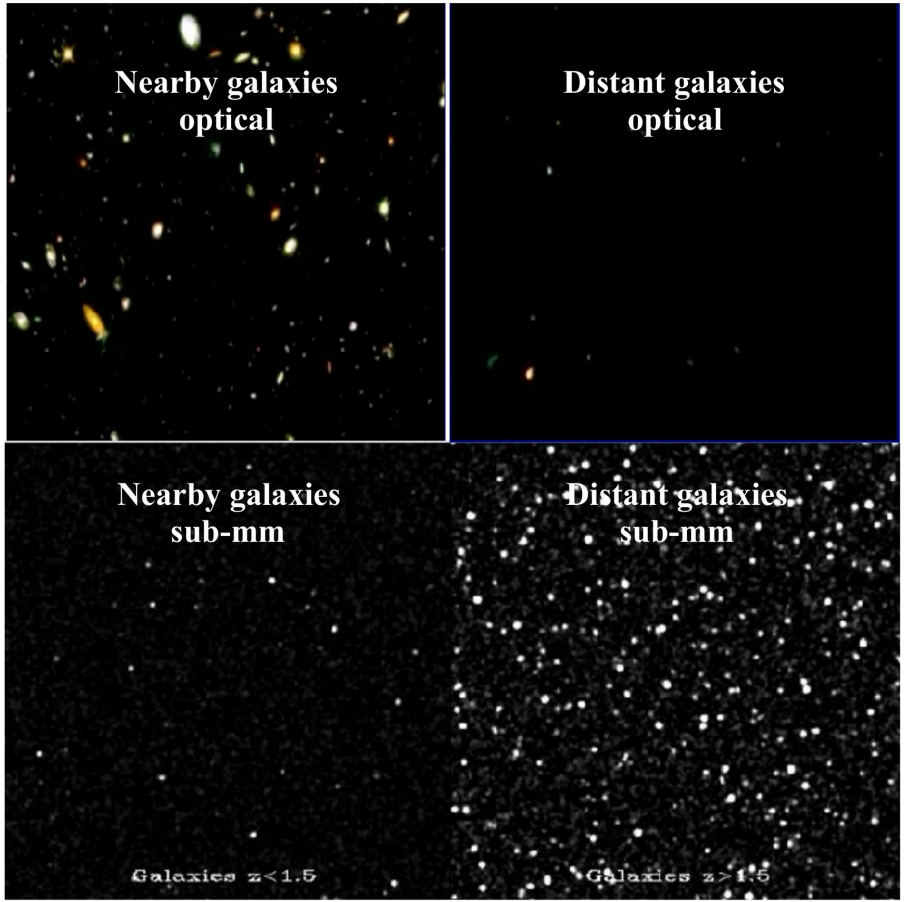


Lecture 9
Dust

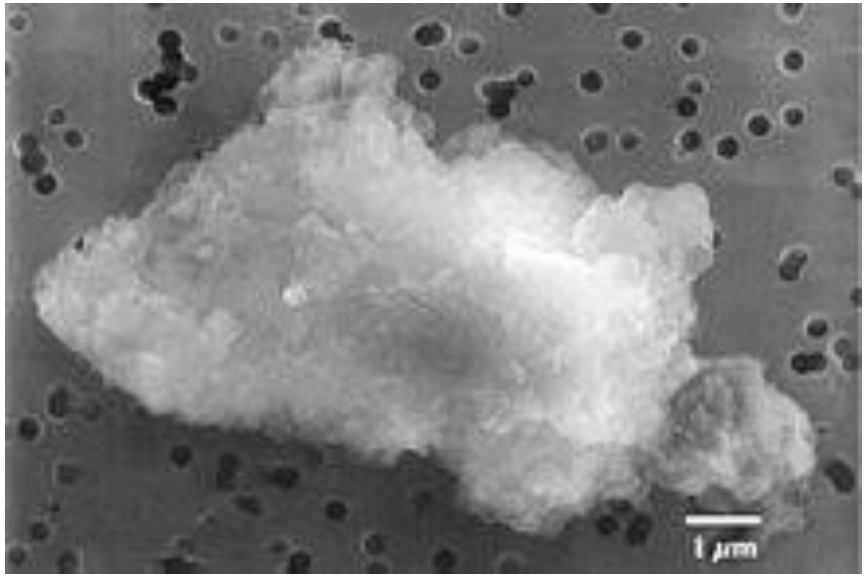
Cosmic dust


2MASS Extended Sources
Integrated Flux
Ks: 8.0–14.0 mag
18'/pixel

Cosmic dust


Nearby galaxies
optical
Distant galaxies
optical
Nearby galaxies
sub-mm
Distant galaxies
sub-mm
Galaxies z<1.5
Galaxies z>1.5

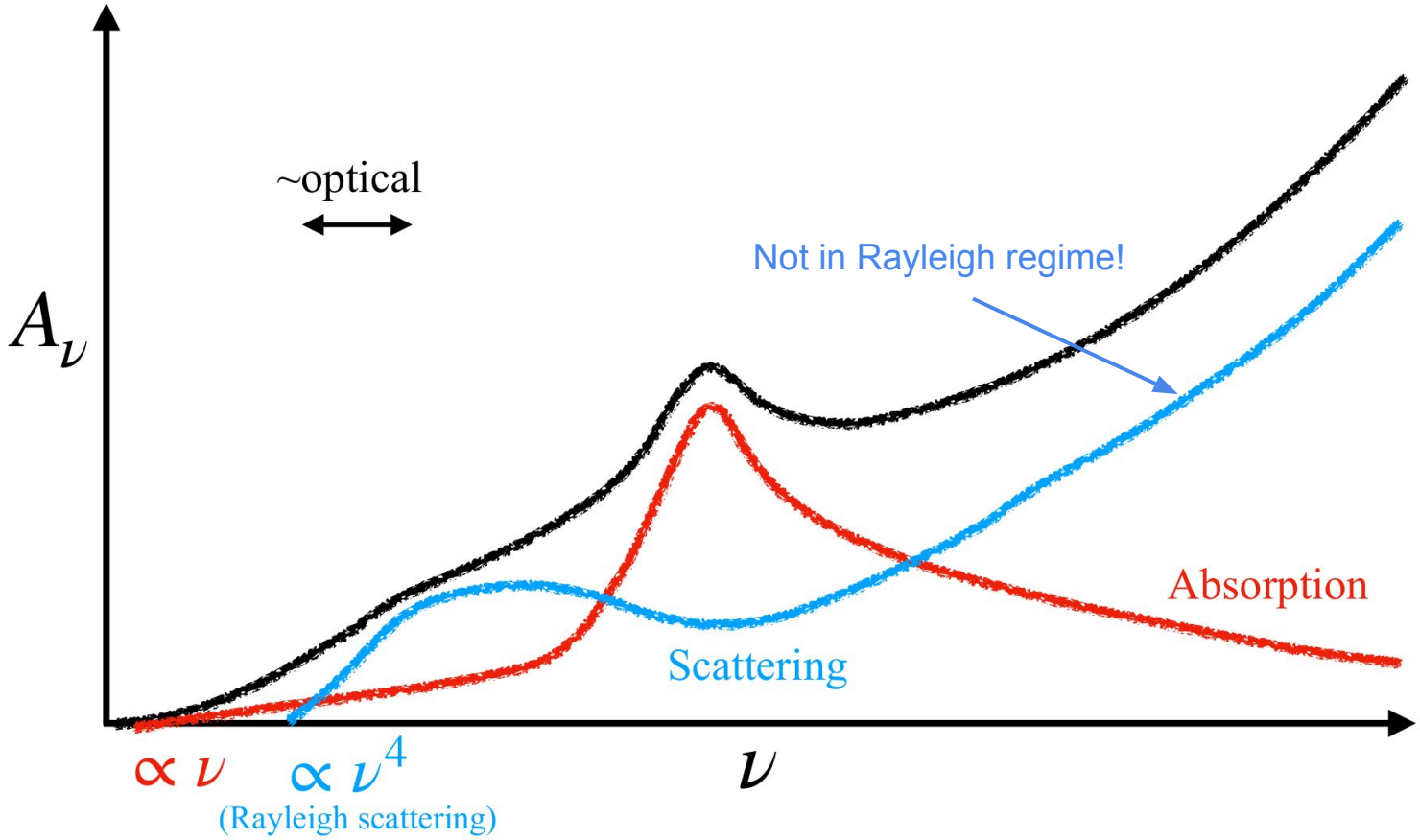
Dust particles



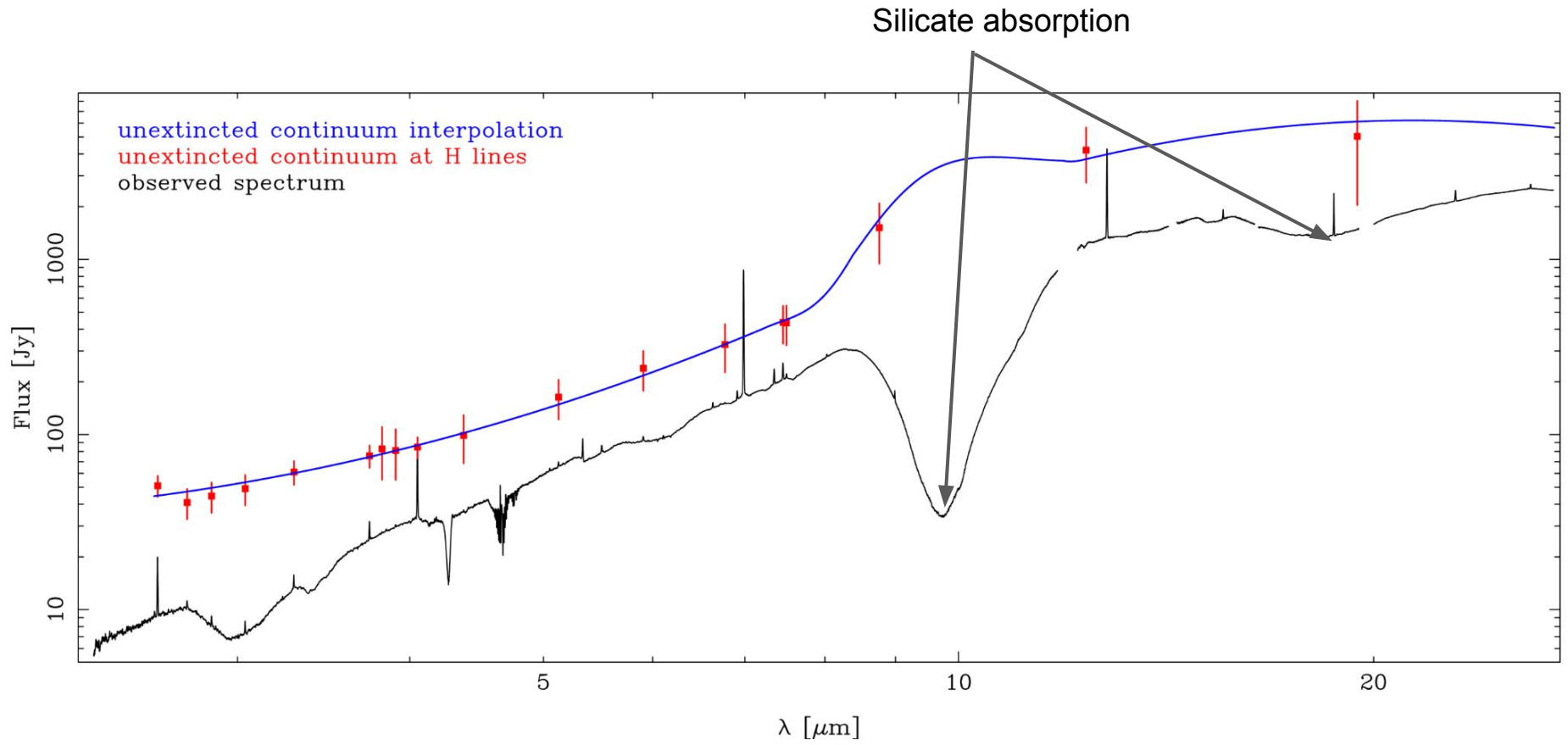
Rayleigh scattering



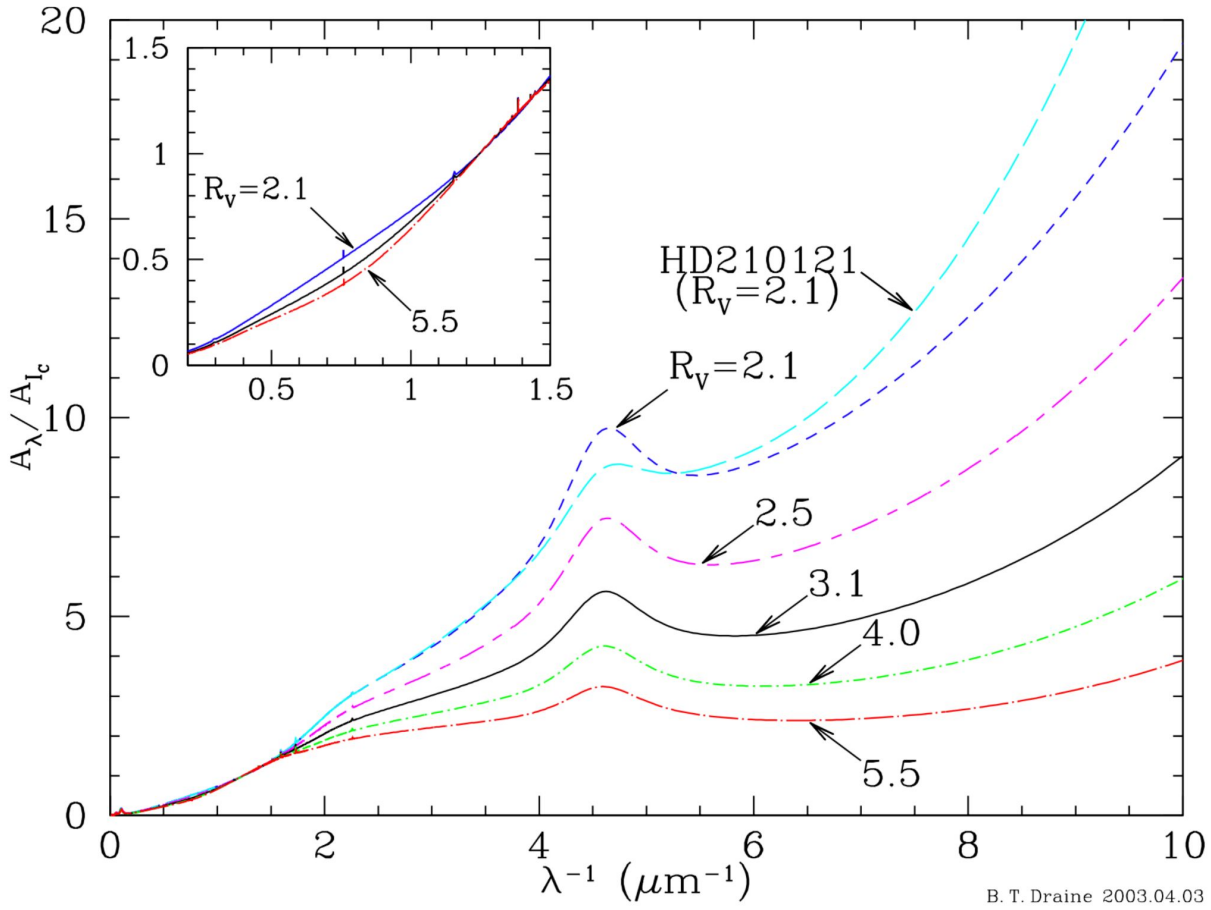
Dust optical depth



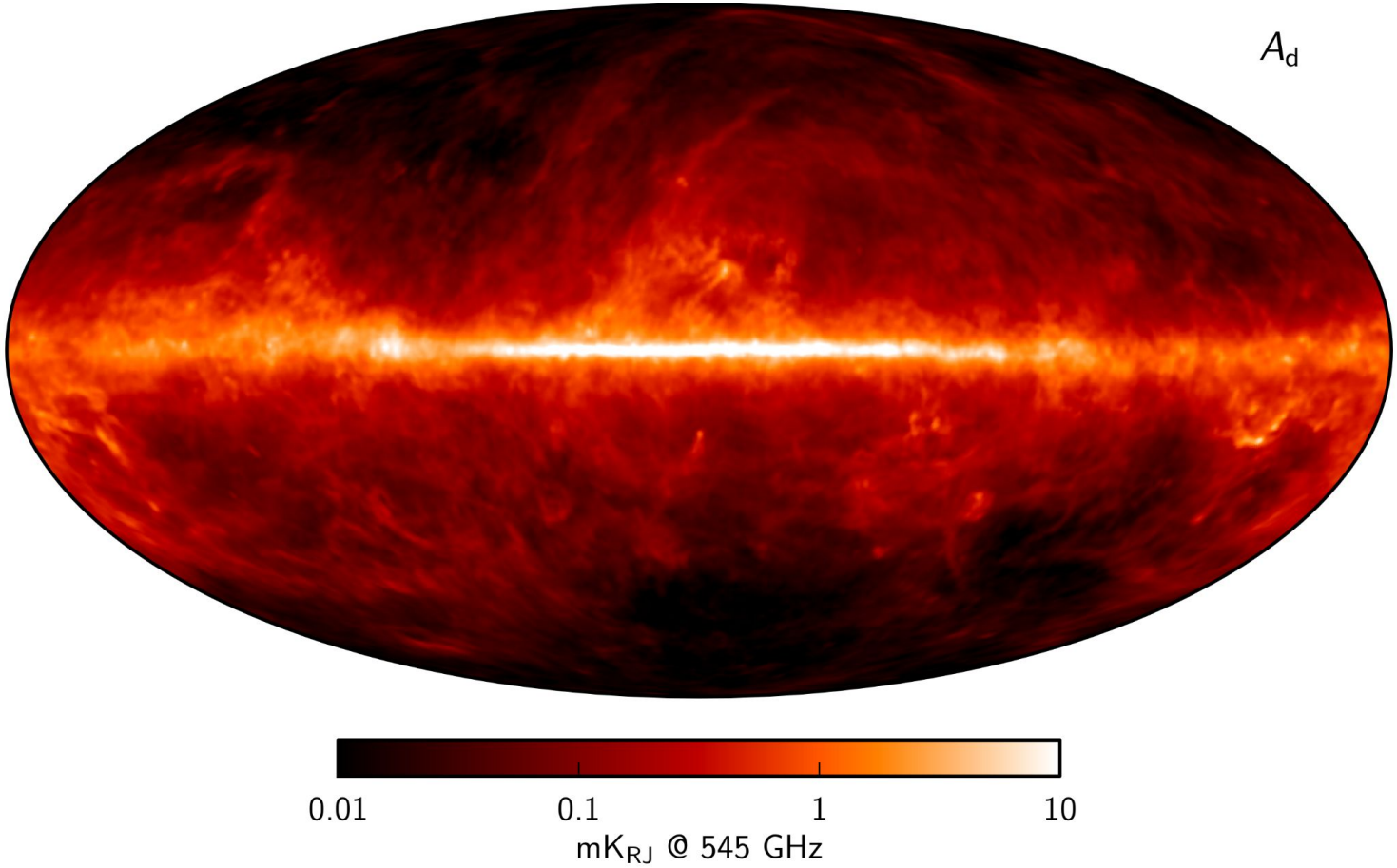
Dust optical depth



Dust optical depth

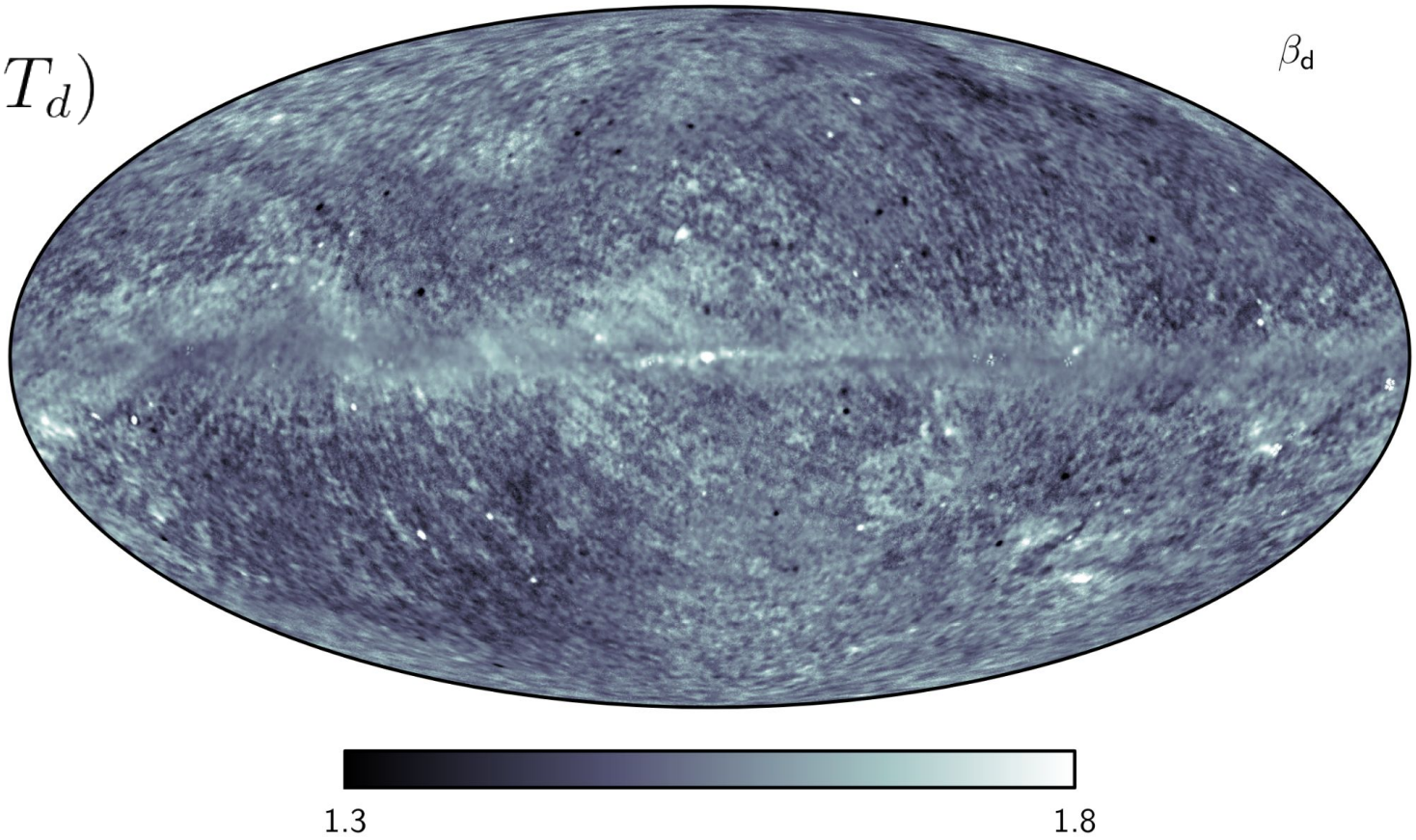


Galactic dust



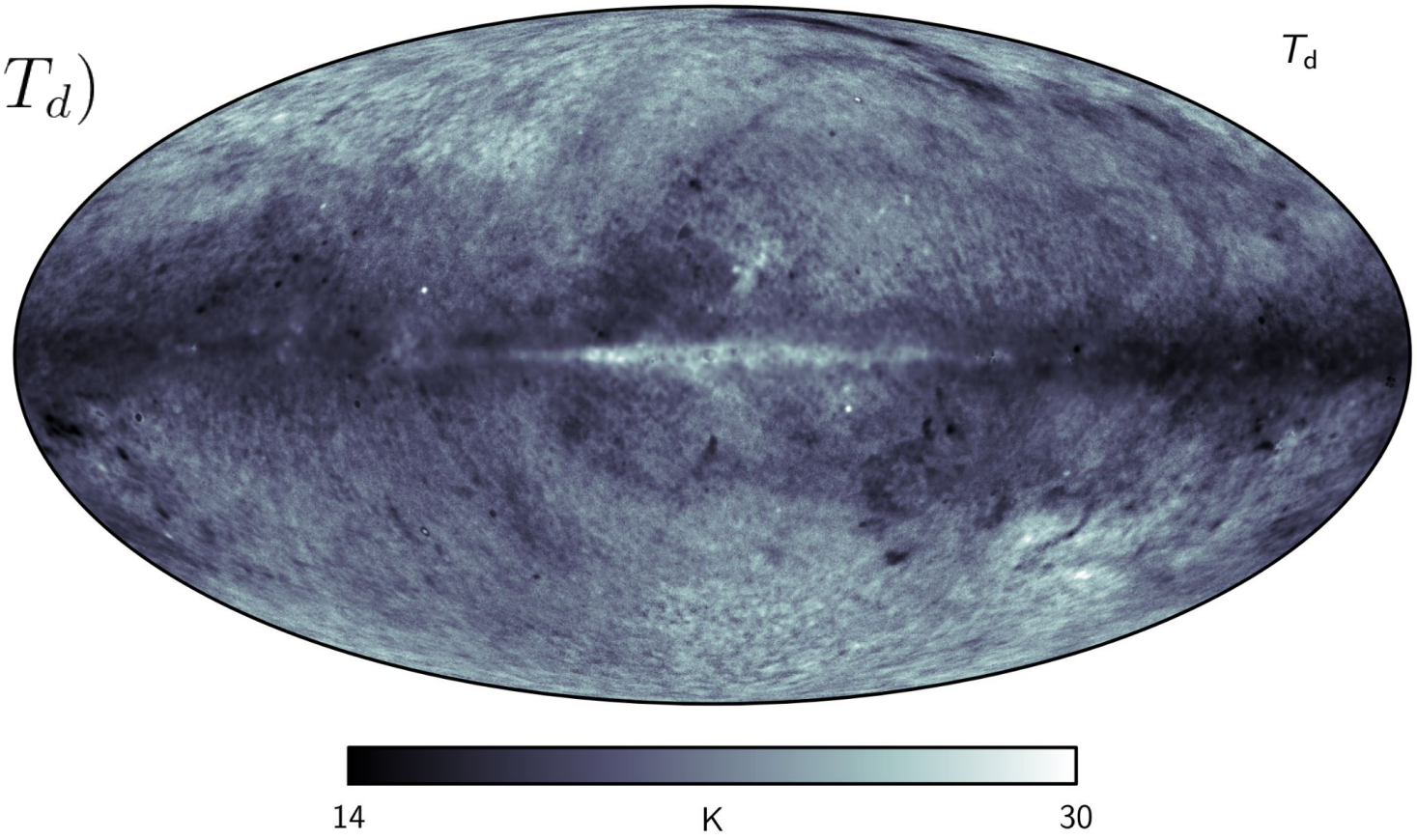
Galactic dust

$$I_\nu \propto \nu^{\beta_d} B_\nu(T_d)$$

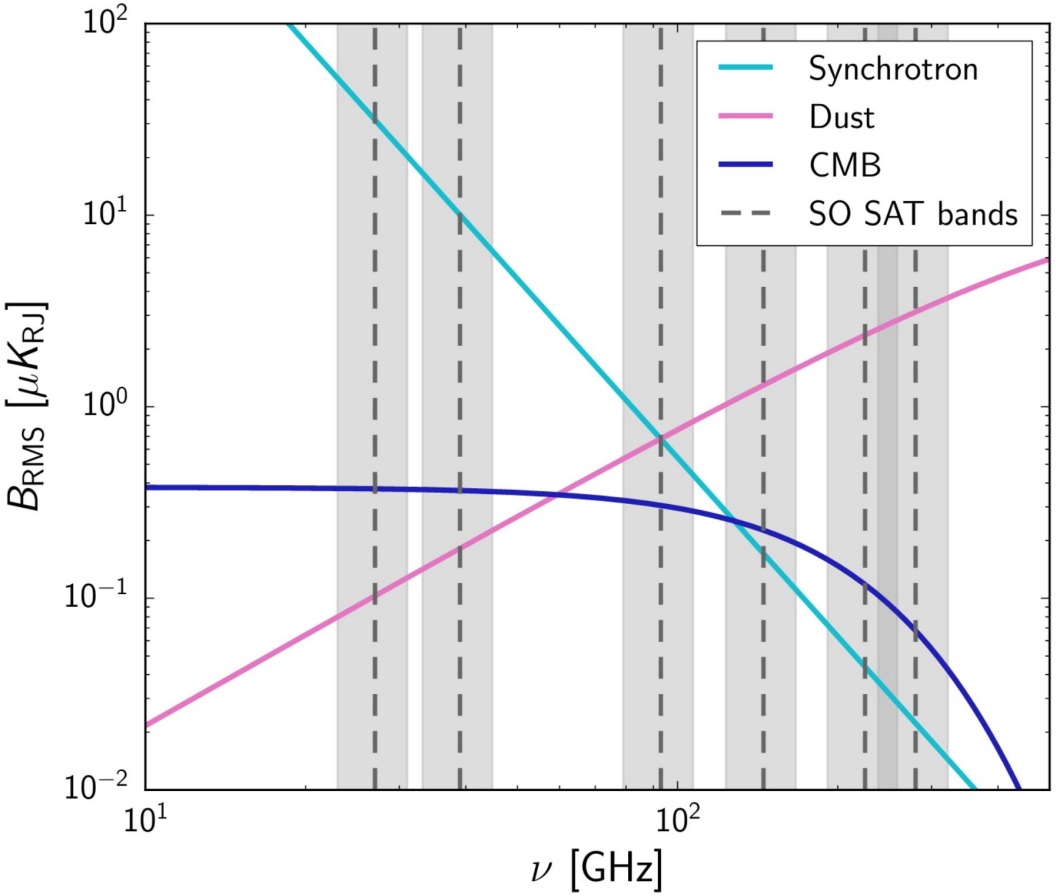


Galactic dust

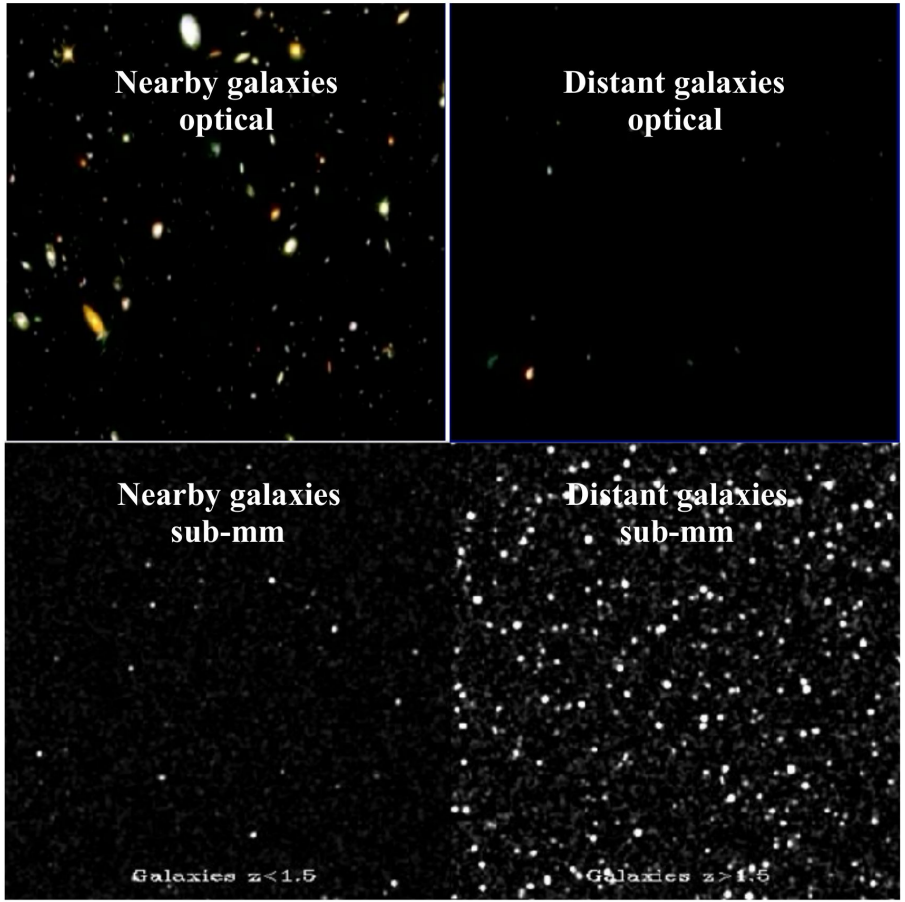
$$I_\nu \propto \nu^{\beta_d} B_\nu(T_d)$$



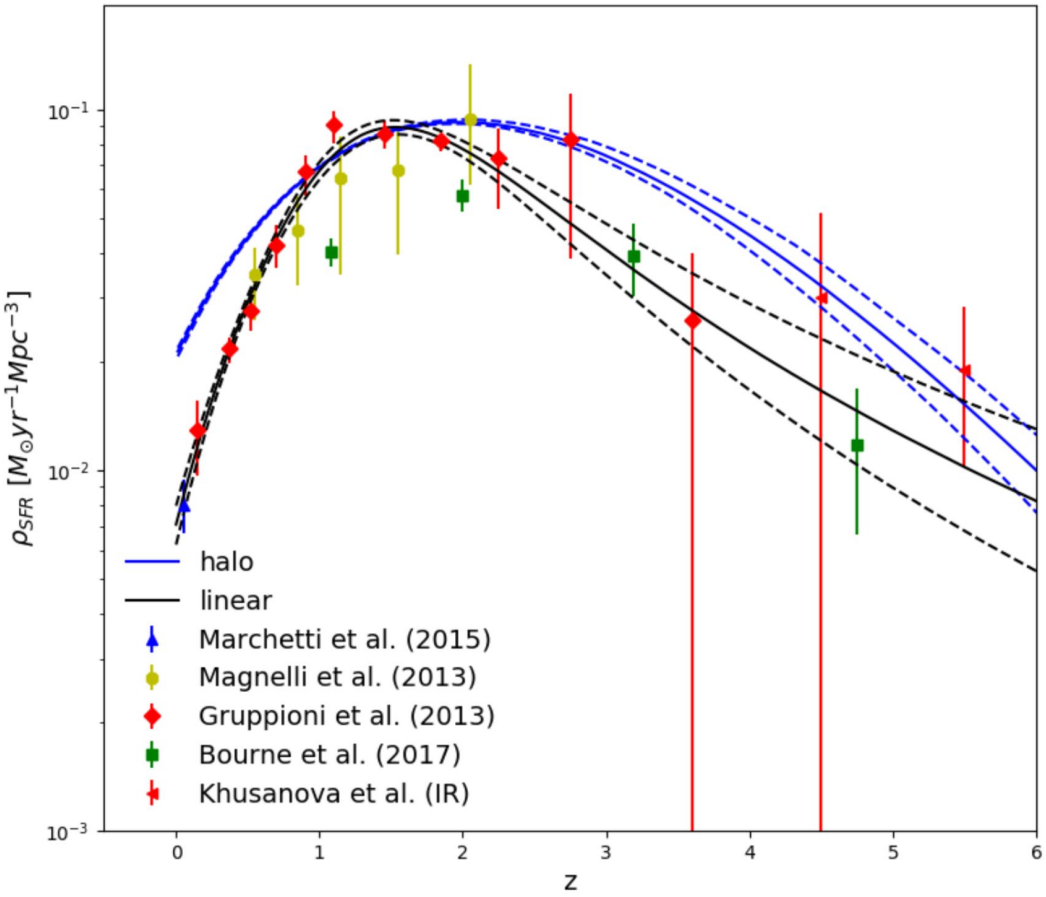
Galactic dust



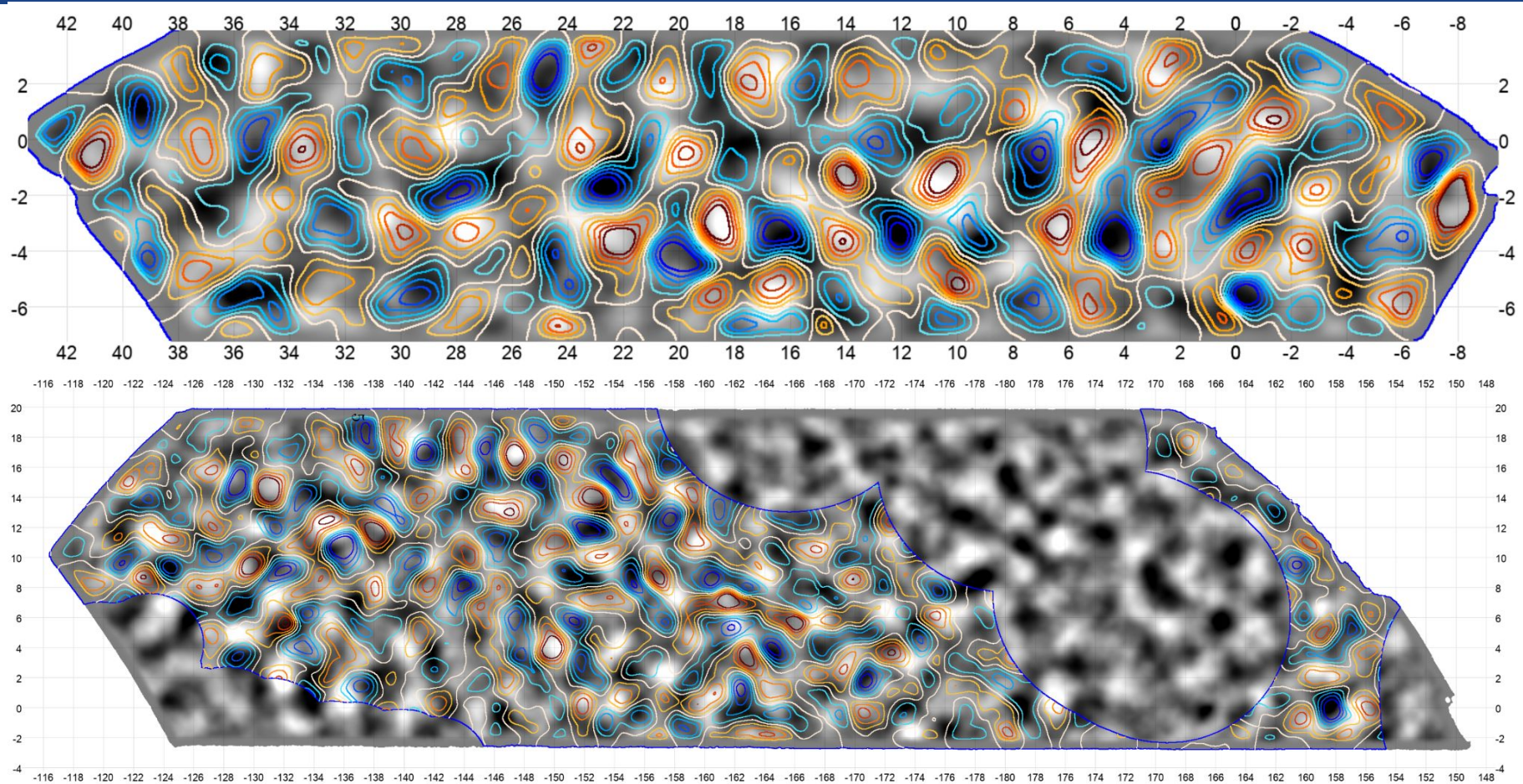
Cosmic dust


Nearby galaxies
optical
Distant galaxies
optical
Nearby galaxies
sub-mm
Distant galaxies
sub-mm
Galaxies z<1.5
Galaxies z>1.5

Cosmic infrared background



Cosmic infrared background



Credit: Atacama Cosmology Telescope