PLASMA PHYSICS PROBLEM SHEET

JUSTIN WARK

(1) Show that 1eV corresponds to a thermal energy, $k_{\rm B}T$ equivalent to a temperature of order 1.1×10^4K. The Saha equation states that for atomic hydrogen the number of free electrons per unit volume, n_e, is related to the number of hydrogen atoms, n_H by

$$\frac{n_{e^-}^2}{n_H} = \exp\left(\frac{-Ry}{k_{\rm B}T}\right)\left(\frac{2\pi m_e k_{\rm B}T}{h^2}\right)^{3/2} \quad .$$

where Ry is the Rydberg. The degree of ionization, ζ, is defined as the number of electrons divided by the number of protons (whether the protons are free or within the atom). Explicitly calculate ζ for the conditions stated in the notes – i.e. a total proton density of $10^{20}{\rm m}^{-3}$ and temperatures of 0.6, 0.8 and 1.0eV. [You can take the value of the Rydberg to be 13.6eV.]

(2) Consider two infinite parallel plates both held at a constant potential ϕ_0. The plates are a distance L apart and between them is situated an electron gas (i.e. a gas composed solely of electrons) of uniform number density n_e and temperature T. The plates are covered in an insulating material so that no current can flow into them from the electron gas. Show that the potential between the plates (before the electrons respond to it) varies as a quadratic function $\phi = a_0 + a_1 x + a_2 x^2$ of the distance x from the plane half-way between the two plates (so in this coordinate system, the plates are at $x = \pm L/2$). Find the coefficients a_0, a_1, a_2. Calculate the value of L such that the energy required to move a particle from one of the plates to the midplane is equal to the average kinetic energy of the particles. Comment on your result.

(3) Show that in SI units the Debye length is equal to $69\sqrt{(T/n)}$ m. What do you understand by the term a *good* plasma? Under what conditions are we justified in treating the electrons and ions within a plasma as ideal gases, and how is this related to the concept of a *good* plasma? For each of the following plasmas determine the Debye length, and whether or not the plasma is 'good'. (a) the ionosphere: $\log_{10} n_e = 9, \log_{10} T_e = 2.5$, (b) the solar corona: $\log_{10} n_e = 13, \log_{10} T_e = 6.5$, (c) a nuclear explosion: $\log_{10} n_e = 26, \log_{10} T_e = 6$, (d) a metal: $\log_{10} n_e = 29, \log_{10} T_e = 2.5$. The density and temperature values given are in ${\rm m}^{-3}$ and K.

(4) What is the Plasma parameter, N_D. List as many reasons as you can for its importance in plasma physics.

(5) Show that the motion of particles of charge q and mass m in a uniform magnetic field $\mathbf{B}$ is helical.

Assume that the particles experience an additional force, $\mathbf{F}$, that is perpendicular to the magnetic field. Show that the guiding centre of each of the particles drifts with a velocity given by

$$\mathbf{v}_{\mathrm{d}} = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2}.$$

Give a qualitative explanation of why the magnetic field in a tokamak is designed to be not purely toroidal.

In order to achieve thermonuclear fusion a heavy hydrogen plasma of temperature 10^8 K and ion number density n_{i} must be confined for a time τ such that $n_i\tau \geq 10^{20}\mathrm{sm}^{-3}$. Make a reasoned estimate of typical required confinement times within a tokamak, stating any assumptions that you make.

(6) The ionosphere is about 70 km above the surface of the earth. Show that due to the force of gravity, the electrons and ions drift in opposite directions, setting up a net current, but that the majority of the current is in this case carried by the ions. Assuming that the ionosphere is a proton-electron plasma, and that the Earth's magnetic field at this height is about 3×10^{-5} Tesla, what is the drift velocity of each species?

(7) Consider a 10keV energetic ion in the Van Allen belts about 10^3km above the Earth's surface. At this point the magnetic field is of order 10^{-6} Tesla and its variation with distance can be reasonably approximated as a dipole. Estimate the ∇B drift speed of this particle, and compare it with the gravitational drift speed.

(8) In a magnetized plasma, the eigenmodes for electromagnetic waves for waves travelling parallel to the magnetic field correspond to left and right-handed circularly polarized light, and the refractive index, n, is given by

$$n_{\mp}^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega\mp\omega_{\mathrm{ce}})} \quad ,$$

where the $-$ sign corresponds to the right handed wave, and the $+$ sign the left handed wave. Show that there is a frequency below which LH waves cannot propagate, whereas for RH waves there is a gap in frequency-space within which propagation is not possible.

Demonstrate that for very low frequency RH waves the group velocity divided by the speed of light in vacuo is $2\sqrt{\omega\omega_{\mathrm{ce}}}/\omega_{\mathrm{pe}}$. Lightening strikes at one of the earth's poles can lead to radiation in the frequency range of 1-10 KHz being produced. Given typical magnetic fields surrounding the earth in the ten microtesla range, and ionospheric electron densities of order $10^9\mathrm{m}^{-3}$, make an estimate of the typical group velocities of such waves. Given that they follow the earth's magnetic field lines from one pole to another, and thus must travel several earth radii, make a rough order of magnitude estimate of the differences in arrival times between the 1kHz and 10KHz waves. [NB - these are rough estimates for

illustrative purposes – in practice there are large variation in magnetic fields and electron densities in the regions of interest.]

(9) Given the formula in the previous question for the refractive indices for left and right handed polarized light travelling through a plasma, demonstrate that linearly polarized light has its plane of polarization rotated as it travels through a magnetized plasma. For an electron density of $10^{20}\mathrm{m}^{-3}$, and magnetic field of 1T, what length of plasma is required to rotate the plane of polarization of 1THz radiation by 90 degrees?

(10) Why are plasmas often described as 'collisionless'?

An electron in a plasma scatters from ions of charge Ze. Show that, for an electron with initial velocity v, and impact parameter with respect to an ion, b the magnitude of the change in velocity of the ion during a single small-angle deflection is given approximately by

$$\Delta v \approx \frac{Ze^2}{2\pi\varepsilon_0 mbv} \quad .$$

State why the above formula is not valid for small values of b, and make an estimate of the smallest physically meaningful impact paramater, $b_{\min}$, used in an analysis of electron-ion collisions. What determines the maximum physically meaningful value for the impact parameter, $b_{\max}$? Determine an expression for $b_{\max}$ as a function of the plasma temperature and number density. Show that for an electron undergoing multiple small angle collisions the average time taken for it to be deflected through an angle of $\pi/2$ is approximately

$$\tau \approx \frac{2\pi\varepsilon_0^2 m^2 v^3}{n_i Z^2 e^4 \ln(b_{\max}/b_{\min})} \quad .$$

Demonstrate that a plasma containing a large number of particles in a Debye sphere is 'collisionless'.

(11) In the two-fluid model of a plasma it is assumed that

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0 \quad ,$$

$$\frac{\partial}{\partial t}(n_e u_e) + \frac{\partial}{\partial x}(n_e u_e^2) = -\frac{n_e e E}{m} - \frac{1}{m}\frac{\partial P_e}{\partial x} \quad .$$

where n_e is the electron number density, u_e the mean velocity of the electron fluid, E the electric field, and P_e the electron pressure. Give a physical justification for each of the above two formulae.

Small amplitude plane waves in the electron density exist within the plasma. Show that the resultant amplitude of the fluctuations in the electron pressure, $\tilde{P}$, are related to the amplitude of the fluctuations in electron number density, $\tilde{n}_e$, by

$$\tilde{P} = 3\tilde{n} k_{\mathrm{B}} T_e \quad ,$$

where T_e is the electron temperature. State any assumptions that you make.

By use of the above formulae, show that plasma waves of frequency ω and wavevector k, obey the dispersion relation

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_e^2 \quad ,$$

where $\omega_{pe}^2 = ne^2/\varepsilon_0 m$, and $v_e = \sqrt{k_\mathrm{B} T/m}$. Give a physical explanation for the deviation of the frequency of the wave from that of a cold plasma when $kv_e/\omega \approx 1$.

(12) In a given plasma, waves of wavelength 15 mm have frequencies of 45.2 MHz, while those of 10 mm have frequencies of 51 MHz. Deduce the electron temperature and number density of the plasma. Which of the waves would you expect to damp most quickly?

(13) In order for a fusion reaction between deuterium and tritium to produce energy gain the product of the ion number density, n, and confinement time, τ, must satisfy the Lawson criterion, $n\tau \geq 10^{20}$ s m^{-3}, for a reaction taking place at a temperature of 10 keV. Show that for a sphere of plasma confined by its own inertia the Lawson criterion can be expressed as

$$\rho r \geq A \text{ gcm}^{-2}$$

where ρ is the density of the plasma, r its radius, and A a constant of order unity.

Current high power laser systems for fusion applications can deliver pulses of laser light with energies of a few MJ. Show that such energies are insufficient to produce energy gain unless the plasma has a density approximately two orders of magnitude greater than that of the density of normal liquids at atmospheric pressure. Assuming the electrons within the plasma are Fermi degenerate, estimate the pressure of such a compressed plasma. Given that the ablation pressure that can be generated by lasers for fusion applications is 100 Mbar, outline the means by which the pressures required for fusion might be achieved.

Discuss briefly the principal difficulties in achieving energy gain in inertial confinement fusion.

(14) The continuity equation for a distribution function $f(\mathbf{v}, \mathbf{r}, t)$ for particles in a plasma is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (f\dot{\mathbf{r}}) + \frac{\partial}{\partial \mathbf{v}} \cdot (f\dot{\mathbf{v}}) = 0 \,,$$

where $\mathbf{v}$ is velocity and $\mathbf{r}$ is position.

Using this equation, derive the Vlasov equation, which holds for the electrons and ions separately:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} \left(\mathbf{E} + (\mathbf{v} \times \mathbf{B})\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \,,$$

where $\mathbf{E}$ is the electric field present within the plasma and $\mathbf{B}$ the magnetic field.

Using the above approximation, consider a periodic perturbation f_1, of angular frequency ω and wavevector $\mathbf{k}$, to the Maxwellian distribution function f_0 of the

electrons:

$$f_1 = f_{1(\mathbf{k},\omega)} \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

Assuming that this perturbation causes a similarly periodic perturbation to the electric field within the plasma, derive the dispersion relation for plasma waves propagating along the z-axis of an unmagnetized plasma:

$$1 + \frac{e^2}{\varepsilon_0 m_\mathrm{e} k_z} \int \frac{\frac{\partial f_0}{\partial v_z}}{\omega - k_z v_z} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z = 0$$

What physical effect is associated with the pole in the integral, and at what electron velocities would you expect it to be significant?

PLASMA PHYSICS PROBLEM SHEET - ANSWERS

JUSTIN WARK

(1) To get 1 eV in Kelvin, $T = k_B/e = 1.1 \times 10^4$. The values of ζ can be found in the notes – for these conditions you should get in % 4.4, 60, and 97.5 respectively.

(2) We consider a thin slab of the plasma between x and $x = \delta x$, and consider the increase in the electric field from $E(x)$ to $E(x + \delta x)$. We see that

$$\frac{dE}{dx} = \frac{-ne}{\epsilon_0}$$

therefore

$$E = -\frac{d\phi}{dx} = \frac{-nex}{\epsilon_0} + C$$

$$\phi = \frac{nex^2}{\epsilon_0} - Cx + D$$

We now put in the boundary conditions that $\phi = 0$ at $x = \pm L/2$ to find $C = 0$ and $D = -neL^2/4\epsilon_0$, so that our solution for ϕ is

$$\phi = \frac{nex^2}{\epsilon_0} - \frac{neL^2}{4\epsilon_0}$$

which is of the form requested in the question. Thus the energy to move an electron to the centre, $|e\phi|$, which is equal to the thermal energy, is

$$k_B T = \frac{ne^2L^2}{4\epsilon_0}$$

We notice that the distance here is of order the Debye length!

(3) The Debye Length is

$$\sqrt{\frac{\epsilon_0 k_B T}{ne^2}}$$

Put in the numbers, and we find that in SI this is 69 $\sqrt{(T/n)}$ m. A good plasma is one with a large number of electrons in a sphere of radius the Debye Length. To treat the electrons and ions as ideal gases, we require the thermal energy of the particles to be large compared with their Coulomb interactions. Their thermal energy is $k_B T$, and if there are of order n particles per unit volume, (so the typical distance between them is $n^{-1/3}$) this is the same as saying

$$k_B T \gg \frac{e^2}{4\pi\epsilon_0 n^{-1/3}}$$

which we note is the same as saying

$$n\left(\frac{\epsilon_0 k_{\rm B} T}{ne^2}\right)^{3/2} \gg 1$$

i.e. there are many particles in a Debye Sphere.

We now list the Debye length and N_D for the plasmas given. (a) ionosphere, $\lambda_D = 0.039$m, $N_D = 2.4 \times 10^5$ - a good plasma (b) solar corona $\lambda_D = 0.039$m, $N_D = 2.4 \times 10^9$ - a good plasma (c) nuclear explosion $\lambda_D = 6.9 \times 10^{-9}$m, $N_D = 137$ - still a good plasma. (d) a metal $\lambda_D = 3.9 \times 10^{-12}$m, $N_D = 2.4 \times 10^{-5}$ - this is not a good plasma, as we would expect, as the binding energies in a solid are by definition much higher than the thermal energies (in fact the situation is more complicated than this, as the electrons are actually degenerate, and we should probably use a Fermi temperature rather than room temperature, but it turns out that even if we do this a metal is not a 'good' plasma in the normal sense.).

(4) The plasma parameter is the number of particles in a Debye Sphere. It is important because if it is large it means (a) we can use the spherically-symmetric-average potential approach to determining the debye length in the first place. (b) as shown above if it is large it means that the thermal energy is much greater than the coulomb energy, so ideal gas partition functions can be used (c) it turns out that within factors of order unity it is the value of Λ in the Coulomb logarithm that partly determines the electron-ion collision time, and hence the transport properties and (d) within factors of order unity it is the number of plasma oscillations the electrons undergo before a 'collision' - i.e. being scattered through $\pi/2$ via multiple small angle scatters.

(5) Helical motion and the formula for drift are derived in the notes. In a purely toroidal field we will get both grad-B and curvature drift, which will be in opposite directions for the electrons and ions, thus setting up an electric field. This in turn leads to a drift due to **E**, where the ions and electrons march off in the same direction and drift into the wall, thus stopping confinement.

Large magnetic fields are of order 1 - 10 Tesla. The pressure that a magnetic field can exert must be of order the energy density, i.e. $B^2/2\mu_0$. Given the temperature of 10^8, K, this magnetic pressure corresponds to the pressure of an ideal gas with a number density of in the region of $10^{20}{\rm m}^{-3}$. Hence we conclude that the required confinement times for satisfying the Lawson criterion must be of the order of seconds.

(6) We start with

$$\mathbf{v}_{\rm d} = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}.$$

We note that due to gravity the electrons experience a force $m_e g$ and the protons $m_p g$. Hence the drift velocity depends on the sign of q, and is proportional to the mass of the particles. Thus the electrons and the protons drift in opposite directions due to gravity, but the protons travel 1836 times faster, and

thus dominate the current. At 70km height we still have g close to 9.8 ms^{-2} (the radius of the earth is 6400 km), and putting in the numbers we find a drift velocity of 3.3 mm s^{-1} for the protons, and this divided by 1836 for the electrons.

(7) Firstly, the gravitional drift speed will be similar to that found in the previous question (a small correction for the slightly lower acceleration due to gravity). However, grad-B drift scales as

$$\mathbf{v}_\mathrm{d} = v_\perp^2 \frac{1}{2\omega_\mathrm{c}} \frac{\nabla \mathbf{B} \times \mathbf{B}}{B^2}$$

which, recalling that $\omega = qB/m$, means that

$$v_\mathrm{d} = \frac{U}{q} |\frac{\nabla B}{B^2}|$$

where $U = mv^2/2$ is the energy of the particle. We are told that the magnetic field of the earth can be approximated as a dipole, that is to say

$$B(r) = \frac{B_0}{r^3} \quad \nabla B(r) = -\frac{3B_0}{r^4}$$

so that

$$|\frac{\nabla B(r)}{B(r)^2}| = \frac{3r^2}{B_0} = \frac{3}{rB(r)}$$

and

$$v_\mathrm{d} = \frac{3U}{qrB(r)}$$

Inserting the values of U, r and $B(r)$ given in the question (but recalling that the relevant r is (6400+1000) km, we find a drift velocity of $4 \times 10^3 \mathrm{ms}^{-1}$. The period of the drift east-west around the earth will be just over 3 hours at this radius. We note the grad -B drift dominates for ions of this energy!

(8) The first part of this question is bookwork (that is to say, the refractive indices are derived in full in the notes). Now for the RH wave

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega - \omega_\mathrm{ce})}$$

We know that we are dealing with extremely small ω, and thus

$$\frac{c^2k^2}{\omega^2} \approx \frac{\omega_p^2}{\omega\omega_c}$$

Now the group velocity is $d\omega/dk$, and

$$2c^2k \; dk = \frac{\omega_p^2 d\omega}{\omega_c}$$

hence

$$\frac{1}{c}\frac{d\omega}{dk} = \frac{2\sqrt{\omega\omega_c}}{\omega_p}$$

For the electron density in the ionosphere we find ω_p is about 1.6×10^6 rads s^{-1}, and for a 10^{-5} tesla field ω_c is the same. This gives a velocity for 1Khz (6300

rads s^{-1}) of 0.063 of the speed of light, i.e. $1.9 \times 10^7 ms^{-1}$, and $\sqrt{10}$ times this for the 10kHz radiation. If we assume the light has to travel three or four earth radii (seems reasonable, let's say 20,000 km), the delay time between the waves is roughly of order a second or so. This is in the right ballpark, as can we seen from the diagram showing whistler modes in the notes.

(9) Faraday rotation is derived in the notes, and we find that the angle of rotation is

$$\begin{aligned} \theta &= \frac{\omega_{\rm p}^2 \omega_{\rm ce}}{\omega^3} \frac{\omega}{2c} z \\ &= \left(\frac{n_e e^2}{\varepsilon_0 m_e} \right) \frac{eB}{m_e} \frac{1}{2c\omega^2} z \\ &= \frac{e^3 n_e B}{8\pi^2 \varepsilon_0 m_e^2 c^3} \lambda^2 z \quad , \end{aligned}$$

where λ is the wavelength of the radiation. In this case we have THz radiation (there was a typo in the original question, as GHz left handed waves cannot propogate!) For THz radiation $\lambda = 3 \times 10^{-4}$m, and we find that the length to rotate through $\pi/2$ radians is 0.43m.

(10) The answer to this question is in the notes. Plasmas are collisionless in the sense that they can exhibit physical phenomena (plasma waves) that do not require collisions, but collective behaviour, and the oscillation period of a plasma wave is very much shorter than the time taken for an electron to be scattered through 90 degrees by multiple small angle scatters. In the notes we derive the collision time, noting that it is due to multiple rutherford scatters with maximum impact parameters of the Debye length. From this we derive the collision time, and when we multiply it by the plasma frequency we find a number, that within factors of order unity, is the plasma parameter (see the notes).

(11) The formulae represent conservation of particle number, and conservation of momentum. Linearization of these equations is covered in the notes. The important point is that the waves are adiabatic, but we must remember that they are not just plane waves, but truly 1-D, in the sense that the distance between the electrons changes in one dimension only, as the plasma is collisionless. This means that along an isentrope, no longer can we say that $VT^{3/2}$ is a constant (as in a normal gas), but that $VT^{1/2}$ is a constant. Along with the ideal gas equation of state this leads to PV^3 being constant, and thus for fluctuations in pressure

$$\tilde{P} = 3\tilde{n} k_{\rm B} T_e \quad ,$$

The derivation of the Bohm Gross frequency is covered in the notes. However, we expect the deviation of the frequency of the wave from that of a cold plasma when $kv_e/\omega \approx 1$ because physically this means that a typical thermal electron has enough time during one plasma period to travel from a peak to a trough.

(12) The Bohm Gross formula for plasma waves in a finite temperature plasma is

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_e^2 \quad ,$$

We have two equations with the two unknowns ω_{pe} and v_e:

$$(2\pi \times 45.2 \times 10^6)^2 = \omega_{pe}^2 + 3(2\pi/0.015)^2 v_e^2 \quad ,$$

$$(2\pi \times 51 \times 10^6)^2 = \omega_{pe}^2 + 3(2\pi/0.01)^2 v_e^2 \quad ,$$

Solving for the plasma frequency and thermal velocity we find $\omega_{pe} = 2.45 \times 10^8$rads s^{-1}, $v_e = 2 \times 10^5$ms^{-1}, which corresponds to an electron number density of 1.88×10^{13}m^{-3} and temperature of 2640 K. We expect the shorter wavelength modes to damp most quickly – these are the ones that give a greater chance for the electrons to move from a peak to a trough during one oscillation – or another way to look at these is that they have wavelengths closer to the Debye length, so have larger Landau damping.

(13) The ρr criterion derivation can be found in the notes. MJ lasers are insufficient to produce gain in liquid DT. To show this we set $\rho = 1$gcm^{-3}, and see we need a radius of about 0.6cm. If we calculate the number of atoms within the radius, and give each of them the 10keV needed for fusion, we end up with energies in the GJ, rather than MJ, regime. SEe the notes for how compression reduces the required energy, by reducing the radius, and the total number of atoms. We find we need compressions of a facto of a 100, however, as shown in the notes, this leads to pressures of the fermi degenarate system of order 20,000 MBar, about a factor of 100 or so bigger than ablation pressures. We get round this by pressure amplification - ablatively imploding a hollow shell makes the stagnation pressure in the centre roughly equal to the ablation pressure multiplied by the aspect ratio $r/\Delta r$ of the shell. However, this in turn introduces the main difficulty - the Rayleigh-Taylor instability. It so happens that this grows as $\exp(\gamma t)$ where $\gamma = \sqrt{ka}$, where k is the wavevector of the mode, and a the acceleration. As discussed in the notes, the most damaging mode has $k = 1/\Delta r$, and the end result is that the RT instability places a constraint on the maximum allowed aspect ratio before the target breaks up.

(14) Derivation of the Vlasov equation is in the notes, as is the derivation of the dispersion relation. Also discussed in the notes is the pole in the integral. This represents the resonance between the particles and the wave, when the particles velocity matches the phase velocity of the wave. The pole leads to Landau damping, and the wave is damped as in a Maxwellian distribution there are more slow electrons than fast ones. We expect the damping to be more and more significant for the waves with shorter and shorter wavelengths.