

Plasma Physics – Short Option

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Lecture 1

The Saha Equation and Debye Length

1.1 Introduction

As we increase the heat added to a solid, it will eventually undergo a phase transition to the liquid state. The addition of further heat causes it to become gaseous. Finally, as even more heat is added, the bonds binding electrons and nuclei are broken and the gas becomes an electrically conducting plasma, with an equal number of positive and negative charges. For this reason plasmas have often been described as the 'fourth state of matter'. Strictly speaking, this isn't quite true, because in general there is no discontinuity in thermodynamic variables or their gradients (which characterize a change of state) as a plasma is produced by heating up a gas. However, as we shall find as we pursue the course, plasmas have characteristics that make them very different indeed from normal gases and insulating liquids, and these characteristics are so unique, and to the physicist so fascinating, that we can perhaps forgive ourselves for calling plasmas the fourth state of matter, even if the strict definition of a distinct phase is not quite appropriate.

There are many motivations for studying plasmas, but I hope that it is not too difficult to persuade you of the importance of any of them. First and foremost, the vast majority of the visible universe is in the plasma state – including all of the stars, and much of the interstellar medium. It seems a bit strange to go through the whole of an undergraduate physics course without paying attention to the characteristics of 99% of the visible matter that exists! Instead, we mainly concentrate on solids, liquids, and gases – a very parochial view. Yes, that makes up most of our everyday experience, because that is what we meet on earth; but our little green and blue planet is the exception rather than the rule, and if we wish to have any understanding of the fundamentals of the Universe (which is what physicists tend to claim to do, at least at cocktail parties and in grant applications) then it would seem sensible to spend at least

the time allocated to a short option to peruse the physics of plasmas.

There are, as well, other, hard-nosed reasons for indulging in plasma physics as well. As we all know, the fossil fuel energy at our disposal is finite, and on some timescale an alternative to such sources of energy must be found. Although conventional nuclear power could fill the gap, the waste products produced in nuclear fission reactions are long-lived, and the political climate is such that the general public is loathe to accept nuclear (*fission*) power. There, is, though, another sort of nuclear power, not yet realised, but firmly based on plasma physics, that has given (at least some) hope of solving the problem: nuclear *fusion*. If we could get heavy hydrogen (deuterium and tritium) to fuse efficiently in the laboratory, then we could solve the energy crisis – there is plenty of deuterium in sea-water, and the products of the nuclear reaction tend to be short-lived (in terms of their half life) – i.e. in principle the waste is not such a problem. The difficulty, of course, is getting fusion to work. It all happens quite happily in the sun, and in the midst of an H-bomb, but otherwise it's a fairly tall order. If we want nuclei to fuse, then they need to be thrown together at high velocities in order to overcome their mutual electrostatic repulsion – this corresponds to heating the deuterium and tritium to high temperatures (about 100 million degrees), by which time it has reached the plasma state, and the heavy hydrogen nuclei swim in a bath of free electrons. Thus the nuclear fuel is a hot plasma. Controlling this plasma for long enough for the fusion reactions to occur is one of the most formidable problems in plasma physics today, and will remain so for some time. Research into nuclear fusion is one of the largest international scientific endeavours currently being undertaken, and one that we will revisit in a later lecture, once we have the relevant physics under our belts.

A plasma is an ionized gas, and thus one of the first things we need to determine is to what temperatures we need to heat the gas in order for it to be ionized. There are, as to most things in life, several different approaches to this calculation. The one I am going to take here is to make explicit use of the chemical potential, and it is the approach used in the second-year statistical mechanics course. Not only will this method enable us to work out the degree of ionization of a plasma, but it will also show the link between the statistical mechanics of plasmas and chemical reactions (after all, the chemical potential has that very name for a reason). Therefore, I am going to start with the general case of any chemical reaction, and determine the reaction constant, and then use this to show how it leads to the famous Saha equation for a plasma – the equation that enables us to determine the degree of ionization for a given temperature and density.

The chemical potential is the quantity that must be the same for two systems when particles can flow between them, and when chemical reactions take place, particles flow between systems (between the reactants and the products), and the formation of a plasma requires a flow of particles from the atomic state (where the electron and ion are combined) to a state where they are free from each other.

1.2 Conditions for 'Chemical Equilibrium'

So we will start with chemistry in the gas phase, and then use what we learn to form the Saha equation for plasmas. Let us have m different types of molecules, and let the total number of atoms within our container be fixed (atoms can transfer via chemical reactions between different types of molecules in the container, but they can't be created or destroyed, and can't get out of the container). Let us use the notation that the chemical symbol of the ith molecule is B_i. Just to make sure we understand our notation thus far, consider a container containing hydrogen, oxygen, and water molecules, all in the gas phase. Our chemical reaction is denoted

$$2H_2 + O_2 \rightleftarrows 2H_2O \tag{1.1}$$

and we have three types of molecules: B_1 is H_2, B_2 is O_2 and B_3 is H_2O. Now let b_i denote the coefficient of the B_i in the chemical reaction, where we use the convention that the *products* have positive b. So, in the above example. $b_1 = -2$, $b_2 = -1$, $b_3 = 2$, as we could have written

$$-2H_2 + -O_2 + 2H_2O = 0 \quad . \tag{1.2}$$

So for a general chemical reaction we can write

$$\sum_{i=1}^{m} b_i B_i = 0 \quad . \tag{1.3}$$

Let the number of molecules of B_i in the system be N_i. Clearly the number of such molecules will change during the chemical reaction until equilibrium is reached. However, they clearly have to change by converting into one of the other molecules in the system, and the changes in the numbers of molecules must be proportional to the b_i in order to conserve atoms. That is to say

$$\mathrm{d}N_i = \lambda b_i \quad , \tag{1.4}$$

where λ is some constant which is the same for all of the molecules.

If you take a look at section A.2 of appendix A you will find that particles flow between one system and another until the chemical potentials are equal. How does this generalise to the complicated case when there could be lots of different molecules in our chemical reaction? As we show in section A.2

$$dU = TdS - PdV + \sum_i \mu_i dN_i \quad , \tag{1.5}$$

so in a system comprising m components, our condition for equilibrium must be

$$\sum_{i}^{m} \mu_i dN_i = 0 \quad . \tag{1.6}$$

When we insert eqn(1.4) into (1.6) we recover the general condition for chemical equilibrium:

$$\sum_i^m b_i \mu_i = 0 \quad . \tag{1.7}$$

We will consider the situation when a reaction takes place at constant temperature in a container of constant volume. We do this because it is an easier case to treat than other conditions, and we can get some insight into how to deal with chemical reactions in general, as well as the ionization of a hydrogen plasma. The working is somewhat easier because we will find ourselves using the free energy F, and we recall that this thermodynamic potential is a relatively simple function of the partition function of the system: $F = -k_{\rm B}T \ln Z_N$.

Again, in appendix A we show that

$$\mu_i = \left(\frac{\partial F}{\partial N_i} \right)_{T,V,N_{j\neq i}} \tag{1.8}$$

– notice this embraces our conditions of constant temperature and volume. Thus our condition for chemical equilibrium, eqn (1.7), can now be written

$$b_i \sum_i \left(\frac{\partial F}{\partial N_i} \right)_{T,V,N_{j\neq i}} = \sum_i^m b_i \mu_i = 0 \quad . \tag{1.9}$$

1.3 Equilibrium for Ideal Gases

Consider a chemical reaction between ideal gases at constant temperature and volume. We must use the statistics of indistinguishable states, and thus the partition function of the system for the ith component, Z_{N_i}, which has a single-particle partition function $Z_{{\rm sp}_i}$, and contains N_i particles is

$$Z_{N_i} = \frac{(Z_{{\rm sp}_i})^{N_i}}{N_i!} \quad , \tag{1.10}$$

and given that $F = -k_{\rm B}T \ln Z_N$ the free energy is

$$F = -k_{\rm B}T \ln \left[\frac{(Z_{{\rm sp}_i})^{N_i}}{N_i!} \right] \quad . \tag{1.11}$$

Thus the chemical potential is

$$\begin{aligned} \mu_i &= \left(\frac{\partial F}{\partial N_i} \right)_{T,V,N_{j\neq i}} \\ &= -k_{\rm B}T \frac{\partial}{\partial N_i} \left(N_i \ln Z_{{\rm sp}_i} - \ln N_i! \right) \\ &= -k_{\rm B}T \frac{\partial}{\partial N_i} \left(N_i \ln Z_{{\rm sp}_i} - N_i \ln N_i + N_i \right) \\ &= -k_{\rm B}T (\ln Z_{sp_i} - \ln N_i) \quad , \end{aligned} \tag{1.12}$$

where we have employed Stirling's theorem. From eqns (1.9) and (1.12) we find that our condition for chemical equilibrium has become

$$\sum_i b_i k_{\rm B} T(\ln Z_{{\rm sp}_i} - \ln N_i) = 0 \tag{1.13}$$

that is to say,

$$\sum_i b_i \ln Z_{{\rm sp}_i} = \sum_i b_i \ln N_i \quad , \tag{1.14}$$

or to put it even more compactly,

$$\prod_i (Z_{{\rm sp}_i})^{b_i} = \prod_i (N_i)^{b_i} \quad . \tag{1.15}$$

It is remarkable that something as complicated as a chemical reaction can be written in so simple a form. This formula is known as *the law of mass action.*

What does this mean in practice? Well, it means that if we could calculate the partition functions of the reactants and the products, we would know what the equilibrium number distribution between reactants and products would be. In many ways this should not surprise us: if we know the partition function of a system, we know all of the equilibrium thermodynamics – and that should clearly also be able to encompass the equilibrium chemistry.

If we wished, we could now look at particular chemical reactions. I am not minded to do that, as I the object of this course is to study plasmas. However, before we leave chemistry, and even though we will not work out a reaction in detail, let us at least notice a general feature which may be familiar from the dim and distant past, if you ever studied A-level chemistry at school. If we consider the single particle partition functions of the reactants and products, it is clear is that they must be *some* function of volume and temperature (and independent of particle number, as they are single particle partition functions). Therefore our condition for equilibrium, eqn (1.15), can be written as

$$\prod_i (N_i)^{b_i} = N_1^{b_1} N_2^{b_2} \ldots N_m^{b_m} = K(T,V) \quad . \tag{1.16}$$

The quantity $K(T,V)$ is called the "equilibrium constant" of the reaction, and is the thing that dictates the equilibrium numbers of reactants and products. As I said, you may have come across such a concept before.

1.4 The Saha Equation

We have found the condition for chemical equilibrium, and noted that it leads to the concept of the equilibrium constant, which is simply a function of the single particle partition functions (for the reactions in the ideal gas phase). At this stage we could consider such a chemical reaction, and work out the reaction constant in detail. However, rather than do some chemistry, now is the time to show how all of the above is related to plasma physics.

At this stage we will consider the simplest plasma of all – the hydrogen plasma. Let us assume that we have *atomic* hydrogen at a high temperature (all of the molecules have dissociated). At sufficiently high temperatures collisions between the atoms will cause the electrons to be freed, leaving a free proton. Clearly this looks like a chemical reaction:

$$\begin{aligned} H &\rightleftarrows e^- + p^+ \\ e^- + p^+ - H &= 0 \quad . \end{aligned} \tag{1.17}$$

We are going to assume that the particles have a high temperature and low density. How hot and how dense, you ask, and why? Well, remember that almost all of our statistical mechanics arguments have concentrated on 'non-interacting' particles where we could ignore the forces between them – we are going to do the same here. We assume they are hot and not dense so that their thermal (kinetic) energy is much bigger than any electrostatic attraction or repulsion between them. That way we can still treat them as 'non-interacting' to a good approximation.

We will call the electron particle 1, the free proton particle 2, and the recombined hydrogen atom particle 3. Therefore for our plasma, $b_1 = 1$, $b_2 = 1$, and $b_3 = -1$. According to eqn (1.15) all we need to do is find out the single particle partition function of each of these particles, and we are home and dry, and should be able to find the equilibrium number of each. That is to say, from eqn (1.15) we now know that

$$\frac{Z_{\mathrm{sp}_{e^-}} Z_{\mathrm{sp}_{p^+}}}{Z_{\mathrm{sp}_H}} = \frac{N_{e^-} N_{p^+}}{N_H} \quad . \tag{1.18}$$

However, for hydrogen we know one other fact – the number of electrons produced must equal the number of protons, thus

$$\frac{N_{e^-}^2}{N_H} = \frac{Z_{\mathrm{sp}_{e^-}} Z_{\mathrm{sp}_{p^+}}}{Z_{\mathrm{sp}_H}} \quad . \tag{1.19}$$

We know that the partition functions look like $\sum_i g_i \exp(-\varepsilon_i / k_\mathrm{B} T)$ (or the equivalent integrals), so it is clearly important that when we work them out for each of the three particles we do so according to the same energy scale – that is to say we agree on what we mean by 'zero energy'. We will take zero energy to be the energy of a free electron and/or proton at rest. In that case, the single particle partition function of the electrons is obvious – it is just that of the ideal gas of electrons (note we will work in the limit of sufficiently low density that classical statistics hold):

$$Z_{\mathrm{sp}_{e^-}} = 2 \times V \left(\frac{2\pi m_e k_\mathrm{B} T}{h^2} \right)^{3/2} \quad , \tag{1.20}$$

where the factor of two takes into account (as usual) that the electron can have spin up or spin down with respect to some arbitrarily chosen z-axis. Similarly

for our free gas of protons, except now we replace the electron mass by the proton mass:

$$Z_{\mathrm{sp}_{p^+}} = 2 \times V \left(\frac{2\pi m_p k_\mathrm{B} T}{h^2} \right)^{3/2} \quad , \tag{1.21}$$

where we also have a factor of two for the proton – it is also a fermion with spin-1/2. So now we just need to work out the partition function of the hydrogen atom which is the result of an electron and a proton recombining. This is where we need to be careful. When an electron recombines with a proton to form the atom, the electron loses 13.6eV (the Rydberg) of energy. That is to say that the ground state of a stationary hydrogen atom is 13.6eV lower in energy than the energy of a stationary electron. So a stationary atom has an energy of *minus* the Rydberg. Given the equation for the partition function this leads us to conclude that the partition function of the atom is

$$Z_{\mathrm{sp}_H} = 4 \times \exp\left(\frac{-[-Ry]}{k_\mathrm{B} T} \right) \times V \left(\frac{2\pi m_H k_\mathrm{B} T}{h^2} \right)^{3/2} \quad , \tag{1.22}$$

where the factor of 4 takes into account that in the ground state of hydrogen, the electron can have spin up or down, but so can the proton.

We insert eqns (1.20), (1.21) and (1.22) into eqn (1.19) to find

$$\frac{N_{e^-}^2}{N_H} = \frac{V \left(\frac{2\pi m_e k_\mathrm{B} T}{h^2} \right)^{3/2} V \left(\frac{2\pi m_p k_\mathrm{B} T}{h^2} \right)^{3/2}}{\exp\left(\frac{+Ry}{k_\mathrm{B} T} \right) \times V \left(\frac{2\pi m_H k_\mathrm{B} T}{h^2} \right)^{3/2}} \quad . \tag{1.23}$$

Making the reasonable approximation that $m_p = m_H$ (they are only one part in 1836 different due to the electron in the atom) then eqn (1.23) reduces to

$$\frac{N_{e^-}^2}{N_H} = \exp\left(\frac{-Ry}{k_\mathrm{B} T} \right) V \left(\frac{2\pi m_e k_\mathrm{B} T}{h^2} \right)^{3/2} \quad , \tag{1.24}$$

alternatively, if we deal in terms of the number of particles per unit volume ($n = N/V$):

$$\frac{n_{e^-}^2}{n_H} = \exp\left(\frac{-Ry}{k_\mathrm{B} T} \right) \left(\frac{2\pi m_e k_\mathrm{B} T}{h^2} \right)^{3/2} \quad . \tag{1.25}$$

This famous equation of plasma physics is called the Saha equation.

We can simplify things even further. Let us say that the total number of atoms *plus* ions in the plasma is N_0 (i.e. ionized or un-ionized hydrogen). The degree of ionization is defined as ζ, that is to say $N_e = N_p = \zeta N_0$, and thus the number of un-ionized hydrogen atoms is $(1-\zeta)N_0$. Thus eqn (1.25) can be written as

$$\frac{\zeta^2}{(1-\zeta)} = f(V,T) \quad , \tag{1.26}$$

where

$$f(V,T) = \left(\frac{V}{N_0} \right) \exp\left(\frac{-Ry}{k_\mathrm{B} T} \right) \left(\frac{2\pi m_e k_\mathrm{B} T}{h^2} \right)^{3/2} \quad . \tag{1.27}$$

T (eV)	f(V,T)	ζ
0.6	2.01×10^{-3}	4.4×10^{-2}
0.8	0.9	0.6
1.0	37.4	0.975

Table 1. f and ζ as a function of temperature for $N_0/V = 10^{20}\mathrm{m}^{-3}$

Now eqn (1.26) is a quadratic equation in ζ with a solution,

$$\zeta = \frac{-f \pm \sqrt{f^2 + 4f}}{2} \quad . \tag{1.28}$$

We must take the positive root, as from physical considerations ζ must be a positive number, and thus

$$\zeta = \frac{\sqrt{f^2 + 4f} - f}{2} \quad . \tag{1.29}$$

It is interesting at this stage to put some numbers into eqns (1.29) and (1.27) to get some idea of what sort of temperatures are needed in order to ionize hydrogen. As an example, let us take $N_0/V = 10^{20}\mathrm{m}^{-3}$ (a typical density for magnetic confinement fusion, as we shall see later in the course). In Table 1 I have written down the values of $f(V,T)$ for different temperatures (note the temperatures are in units of eV, that is to say the temperature in kelvin would be k_B/e times these numbers). I have also shown the corresponding value of ζ from solving eqn (1.29).

A glance at Table 1 reveals something quite remarkable – the plasma is significantly ionized at a temperature which is considerably less than a Rydberg – in eV units it is more than half ionized at a temperature of 0.8 eV, even though the ionization potential is 13.6 eV. It is much, much easier to make a plasma than we might have originally envisaged. What is going on here?

1.5 Why Plasmas are 'Easy' to Ionize

The physical explanation for the ease of ionization lies in the concept of statistical weight and the density of states. Think of the extreme example of one atom in a box. There is only one state in which the electron can combine with the atom (well, actually two when we take into account spin). On the other hand, there are a multitude of translational quantum states where the electron can be free – the statistical weight of the continuum is very 'heavy' compared with the statistical weight of the atom, making ionization much more likely than we might have thought. The degree of ionization depends as an exponential function on temperature, and thus is very sensitive to it. The dependence on density is not so strong, but we see there is a tendency to less ionization as the density increases. Again there is a statistical weight explanation for this: if we

have a box of a certain volume, the density of the free electron states is fixed by the size of the box, but the more atoms we put in it, the more we weight the probability of the electron being combined with the proton to form the atom.

Rather than just waffling about statistical weight, we can, of course, do something a little more quantitative. Remember from your statistical mechanics that for free electrons there is one available quantum state for each volume element of size h^3 in phase space. Another way of saying the same thing was to say that the electrons started to compete to get into the same state when their de Broglie wavelength became comparable to the inter-electron separation, and this point marked the boundary between classical and quantum (Fermi-Dirac for electrons) statistics.

Following this line of argument we see that the ratio of the number of electrons to the actual number of thermally available states is equal to $n_e\lambda^3$, where we emphasize that λ is now the de Broglie wavelength, not the Debye length! This number, by definition, should be less than of order unity, and will be very small indeed for a low density plasma – the electron density is so low that there are many, many unoccupied quantum states for every occupied one. Given the vast range which this ratio will cover, we define a degree of degeneracy by the factor γ, where

$$\gamma = -\ln(n_e\lambda^3) \quad , \tag{1.30}$$

where the logarithm is used because it gives us a tractable number to cover the wide range, and the negative sign means that our factor γ is now always positive. So, γ is the logarithm of the ratio of the number of occupied free electron states to the number of unoccupied states.

Defining γ in this way also allows us to get a good handle on how easy it is to ionize the plasma. In order to do this we recall what the de Broglie length is for a thermal (classical) free electron. Remember that the argument runs that the classical kinetic energy is equated to the thermal energy:

$$\frac{p^2}{2m} \approx \left(\frac{h}{\lambda}\right)^2 \frac{1}{2m} \approx k_{\rm B}T \quad , \tag{1.31}$$

which, upon rearranging, yields

$$\lambda \approx \frac{h}{\sqrt{2\pi m k_{\rm B}T}} \tag{1.32}$$

where we have added the factor of order $\sqrt{2\pi}$ which is there by convention. With this definition of the de Broglie wavelength of an electron we see that eqn (1.25) can be written

$$\frac{n_e}{n_H} = \exp\left(\frac{-Ry}{k_{\rm B}T}\right)\left(\frac{2\pi m_e k_{\rm B}T}{h^2}\right)^{3/2}\frac{1}{n_e} = \exp\left(\frac{-Ry}{k_{\rm B}T}\right)\frac{1}{n_e\lambda^3} \quad . \tag{1.33}$$

Taking the log of both sides of eqn (1.33) we obtain

$$\ln\left(\frac{n_e}{n_H}\right) = -\frac{Ry}{k_{\rm B}T} + \gamma \quad . \tag{1.34}$$

Now, we define the point at which a good fraction of the atoms are ionized as being when $n_e = n_H$, i.e. we find that this occurs at a temperature such that

$$k_{\rm B}T = \frac{Ry}{\gamma} \quad , \tag{1.35}$$

that is to say the thermal energy in the free electrons needed for a plasma to be considerably ionized is not simply the ionization energy, but the ionization energy divided by our degree of degeneracy parameter; thus we see how the many free quantum states available to the electrons enter into this piece of physics.

So we have seen that we need a much lower temperature than what we might have initially expected in order to get a plasma ionized, but we have not yet mentioned how sharp the transition is – that is to say over what sort of temperature range do we go from neutral to ionized, rather than just finding the point where we are half ionized.

The rapidity as a function of temperature with which the ionization stage of the hydrogen changes from being neutral to singly ionized can be measured by the temperature range ΔT over which the ratio $n_{\rm e}/n_{\rm H}$ changes substantially. We note that $(\ln(n_{\rm e}/n_{\rm H}))$ is zero when we are half-ionized, and at this point we can call the temperature T_0, and $k_{\rm B}T_0 = Ry/\gamma$. We notice that if $(\ln(n_{\rm e}/n_{\rm H}))$ is -1 we are very weakly ionised, and if it is +1 we are strongly ionised. Now the point is that γ, being logarithmic, is a weak function of temperature, so there is a big change in ionisation at some new temperature, T, where $k_{\rm B}T = Ry/(\gamma{+}1)$. This in turn implies that the ionisation changes a lot over some temperature interval ΔT where

$$k_{\rm B}\Delta T = k_{\rm B}(T_0 - T) = \frac{1}{\gamma} - \frac{1}{\gamma+1} \approx \frac{1}{\gamma^2} \tag{1.36}$$

as $\gamma \gg 1$ for a non-degenerate plasma. That is to say, not only is the plasma about half ionized at a thermal energy of about Ry divided by γ, but we divide *that* temperature by about γ to get the temperature range over which we go from neutral to fully ionized.

1.6 The Heat Capacity of a Hydrogen Plasma

The analysis above about ionization also gives us a feel for how the heat capacity of the plasma will vary as a function of temperature. If we start off with $n_{\rm H}$ hydrogen atoms per unit volume at low temperature (i.e. not yet ionized), then the heat capacity per unit volume is clearly that of an ideal gas with that number density - that is to say $3n_{\rm H}k_{\rm B}/2$. At the other extreme, once we are fully ionized at high temperatures, we have twice as many ideal-gas type particles - as each hydrogen atom has dissociated into a free proton and free electron, so when the system is at these high temperatures we expect the heat capacity per unit volume to be $3n_{\rm H}k_{\rm B}$.

Clearly at the point where ionization is taking place (at the temperature of order Ry/γ mentioned above), then we are putting lots of energy into the system (because we need to give a Rydberg - 13.6 eV - of energy to each hydrogen atom to ionize it), but the temperature is not rising very much whilst that is happening - indeed we said that this ionization takes place over a temperature range ΔT which itself is of order $1/\gamma$ of the temperature at which ionization takes place. Thus we expect quite a large spike in the heat capacity around the temperature where ionization occurs, and the integral under the spike will equate to the 13.6 eV per atom required for ionization.

To make this clear, let's take an example to provide rough numbers. Say we have a hydrogen gas at a density such that ionization starts to occur at a temperature of 1.3 eV - i.e. about a tenth of a Rydberg. Then our argument is that we will go from un-ionized to ionized over a temperature range of order 0.13 eV around this 1.3 eV - so from un-ionized at 1.235 eV to fully ionized at 1.365 eV in this simple model. Up to a temperature of 1.235 eV the heat capacity will be $3n_{\rm H}k_{\rm B}/2$, and after 1.365 eV it will be $3n_{\rm H}k_{\rm B}$ as the system is fully ionized. In between, over the 0.13 eV temperature range, we need to give each atom 13.6 eV of energy to ionize it. That means the heat capacity, on average, over this region is of order $13.6/0.13\ n_{\rm H}k_{\rm B}/2$ - about 50 times higher than normal! Of course, this is in many ways obvious - we have to put in lots of heat energy to ionize the , without raising its temperature very much. In many ways it is analogous to latent heat in melting or vapourization.

In order to show that the above argument is not too far off, I have solved the Saha equation for a plasma with an initial density of hydrogen atoms of 10^{20} $\rm m^{-3}$. The results are shown in Fig. 1.1.

1.7 The Partition Function of an Atom

I have to confess here that there is a subtlety to the derivation of the Saha equation that I have conveniently ignored. Before, going further, I wish to say in my defence that the Saha equation we have derived *does* work extremely well under most circumstances. That said, we have conveniently glossed over an issue that the observant amongst you have no doubt noticed. We said that the electron recombined into the ground state to produce the hydrogen atom. But, I hear you ask, what about all the other states in the atom? Why did the electron not recombine into the $n = 2$ state of hydrogen, or $n = 3$, or $n = 56$? These other states will surely give extra statistical weight to the atom, won't they? Once again, these are very good questions, and I am now going to set out to show you why, in almost all cases of interest, we can ignore these other states when calculating the degree of ionization of the plasma, and we need only bother with the ground state.

First of all, let's think about the partition function of the hydrogen atom, but only take into account its internal electronic structure. What I mean by

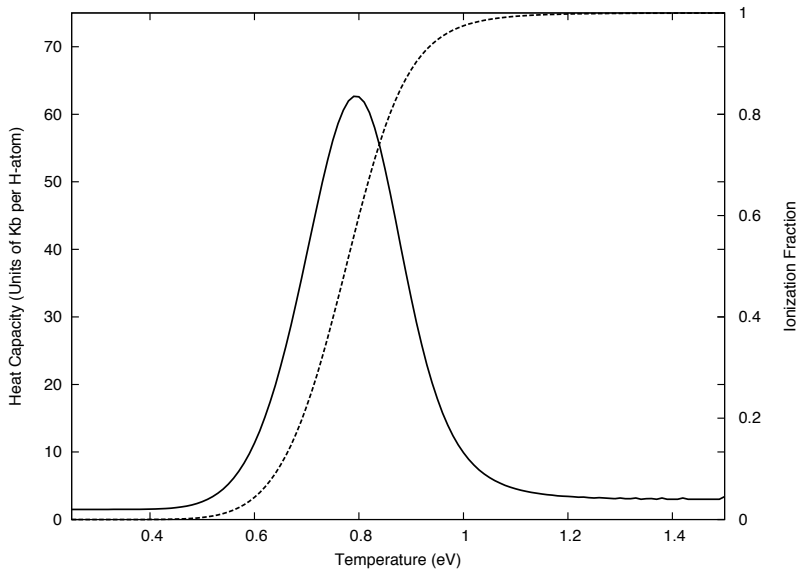


Figure 1.1: : The heat capacity and ionization of a hydrogren plasma as a function of temperature in units of $k_{\rm B}$ per particle, for an original hydrogen density of $10^{20}{\rm m}^{-3}$. Note the value is 3/2 at low temperature, 3 at high temperatures, and peaks as energy is given to ionize the atoms.

that is that when we said that the partition function of the hydrogen atom was

$$Z_{{\rm sp}_H} = 4 \times \exp\left(\frac{Ry}{k_{\rm B}T}\right) \times V \left(\frac{2\pi m_H k_{\rm B} T}{h^2}\right)^{3/2} \quad , \tag{1.37}$$

this breaks down into the translational motion of the atom, and its internal electronic structure – that is to say

$$Z_{{\rm sp}_H} = Z_{\rm electronic} \times Z_{\rm trans} \quad , \tag{1.38}$$

where

$$Z_{\rm trans} = V \left(\frac{2\pi m_H k_{\rm B} T}{h^2}\right)^{3/2} \quad , \tag{1.39}$$

and

$$Z_{\rm electronic} = 4 \times \exp\left(\frac{Ry}{k_{\rm B}T}\right) \quad . \tag{1.40}$$

Recall this electronic partition function comes about because we say that the ground state has energy $-Ry$, and a degeneracy of 4 because the proton can be

in 2 spin states, and so can the electron. If we are to include all the other states in the atom, we must somehow put them into this electronic partition function. How do we do this? Well, remember that the structure of hydrogen is such that the energy level with principal quantum number n is

$$E_n = -\frac{Ry}{n^2} \quad , \tag{1.41}$$

and the degeneracy of the level (as far as the electron is concerned) is $2n^2$ (there is still a need to multiply by another factor of 2 when we take into account the spin of the proton). Thus we conclude that, if all the energy levels are included,

$$Z_{\rm electronic} = 2 \times \sum_n 2n^2 \exp\left(\frac{Ry}{n^2 k_{\rm B} T}\right) \quad . \tag{1.42}$$

One does not need to reflect too long on eqn (1.42) to notice a little problem: as n gets large the exponential tends to 1, so the sum at that point looks like $2\sum_n 2n^2$, which diverges to infinity, as there is no limit on n. Oh dear, we have concluded that the electronic partition function of hydrogen is infinite! Where have we gone wrong? Well, nowhere really (at least from a mathematical point of view). However, from a physics point of view we have forgotten something. The point is as follows – we know that the radius of the electron in hydrogen scales as n^2, and thus the electron orbit gets larger and larger as n increases. If we have a gas of hydrogen atoms, at some point the electron from one atom, in a high n state, will overlap with that of the next atom – at that point, which atom does it belong to? Neither and none, of course – it is meaningless to talk of a bound state when the electron starts to overlap the next atom – in fact it becomes free. [1] Therefore in practice n does not run from 1 to ∞, but only up to some large number, and thus the partition function of the atom is finite. This is the first important feature to notice about the electronic component of the partition function. So, we have worked out why it is finite, but we have still not yet justified why it is usually a good approximation to just take the ground state, and ignore all the other ones, which is what we did when we derived the Saha equation.

The clue to this can be found by looking again at Table 1. Notice we predict that the plasma will be 97.5% ionized when the temperature is just 1.0 eV, and the density $10^{20}{\rm m}^{-3}$. In making this prediction, we only took into account the ground state, not the excited states of hydrogen. We are now in a position to work out how good an approximation this was. First of all, notice that for the density of $10^{20}{\rm m}^{-3}$, the average distance between atoms is about 0.2μm. Given that the radius of the electron in state n is $a_0 n^2$, this means that the maximum

[1] Note that in a slightly more sophisticated treatment the point at which the electron becomes 'free' is taken to be somewhere between the inter-atomic distance – or more particularly what is known as the ion sphere radius – and the Debye length. This leads to the well known Stewart-Pyatt model. However, the calculation performed here, where we say that the electron is free if it overlaps the next ion, illustrates the basic point that the number of terms in the partition function is finite.

possible value of n is about

$$n \approx \sqrt{\frac{2 \times 10^{-7}}{5 \times 10^{-11}}} \approx 60 \quad . \tag{1.43}$$

So the electronic partition function has about 60 terms in it. We might think this makes a big difference, but it does not. The point is as follows – the first term in eqn (1.42) – the one corresponding to the ground state which we have included up until now, is

$$2 \times \ 2 \times \exp\left(\frac{Ry}{k_{\rm B}T}\right) \quad . \tag{1.44}$$

How big are all the other terms compared with this? Only if they make a significant contribution to the overall electronic partition function will they make a difference. Well, when the plasma is 97.5 % ionized, at 1.0 eV, this first term is $4 \times \exp(13.6) = 3.2 \times 10^6$. The contribution to the partition function for the $n = 2$ state, from eqn (1.42) is $2 \times 8 \times \exp(13.6/4) = 479$, which is tiny in comparison with the first term. The 60th term will only be of order 1.4×10^4, and even when we sum together all the 59 terms corresponding to the excited states, they don't increase the electronic partition function by much compared with the initial term of size 3.2×10^6, and thus only including the ground state is a good approximation.

All this comes about because the plasma ionizes at very low temperatures compared with the ionization energy, owing, as we have said, to the large statistical weight of the continuum compared with the bound atom. Another way of looking at this is to say that it is much more likely for an electron in hydrogen to be ionized, than it is to be in the excited state. Notice, for example, that from table 1 the electron has a 60% chance of being ionized at 0.8 eV. However, from the Boltzmann distribution, the ratio of electrons in the $n = 2$ level of hydrogen, compared to the ground state, at this temperature, is

$$\begin{aligned}
\frac{N_{n=2}}{N_{n=1}} &= 2 \times \frac{\exp\left(\frac{Ry}{4k_{\rm B}T}\right)}{\exp\left(\frac{Ry}{k_{\rm B}T}\right)} \\
&= 2 \times \exp\left(-\frac{3Ry}{4k_{\rm B}T}\right) \\
&= 2 \times \exp\left(-\frac{3 \times 13.6}{4 \times 0.8}\right) \\
&= 5.8 \times 10^{-6} \quad ,
\end{aligned} \tag{1.45}$$

i.e. the population of the excited state is negligible compared with that of the ground state, and the population of free electrons.

1.8 Solution for Multielectron Atoms†

Up until now we have been dealing with hydrogen. As the hydrogen atom only has one electron, the solution has been particularly simple. We need to ask ourselves what happens when we have a multielectron atom? Clearly in principle several electrons can leave the atom, and we would suppose that there could be a distribution of charge states present within the system. Indeed, this is the case.

In the general case for multielectron atoms, there is no analytic solution, and we will need to find out the degree of ionization, and the populations of the various species of ions, computationally. Why is this? Well, let us study what is going on in the multielectron atom case. Consider an atom that we represent by the symbol A. If we then use the symbol A_Z to represent the ion that has lost Z electrons, the 'reactions' that take place are clearly of the form

$$\begin{aligned} A_0 &\rightleftarrows e^- + A_1 \\ &\cdots\cdots\cdots \\ A_Z &\rightleftarrows e^- + A_{Z+1} \qquad (1.46) \\ &\cdots\cdots\cdots \qquad (1.47) \\ A_{Z'-1} &\rightleftarrows e^- + A_{Z'} \quad , \end{aligned}$$

where A_0 represents the neutral atom (it has lost no electrons), and we denote by Z' the atomic number of the element, such that $A_{Z'}$ represents the bare nucleus. Notice that each of these reactions looks like eqn. (1.17), so for each reaction we can find an equation analogous equation to (1.18), that is to say,

$$\frac{Z_{\mathrm{sp}_{e^-}} Z_{\mathrm{sp}_{(Z+1)}}}{Z_{\mathrm{sp}_{(Z)}}} = \frac{N_{e^-} N_{Z+1}}{N_Z} \quad . \qquad (1.48)$$

where we must be careful to remember that the capital Z is being used to denote the partition function, whilst the sub-scripted Zs represent charge state. The electron has the same partition function as before (i.e. eqn. (1.20)), whilst the ratio of the partition functions of the ion with charge $Z+1$ to that with charge Z is given by

$$\frac{Z_{\mathrm{sp}_{(Z+1)}}}{Z_{\mathrm{sp}_{(Z)}}} = \frac{g_{(Z+1)}}{g_Z} \exp\left(\frac{I_Z}{k_{\mathrm{B}} T}\right) \quad , \qquad (1.49)$$

where g_Z is the degeneracy of the ground state of the ion with charge Z, and I_Z its ionization potential.

So what is the problem in working out the degree of ionization? Well, notice that the reason we could solve things analytically for hydrogen was that between eqn. (1.18) and (1.19) we made use of the fact that, because hydrogen only has one electron, we knew that the number of free electrons must equal the number of free protons, $N_{e^-} = N_{p^+}$. However, for a multielectron atom things become considerably more complicated, because the electrons present in the plasma come from the neutral atom being ionized, as well as the next stage being

ionized, and so on and so on – we can't equate the number of electrons to the number of any one of the ion stages. Indeed, it is obvious that the number of free electrons is given by

$$\begin{aligned} N_{e^-} &= N_1 + (2 \times N_2) + (3 \times N_3) + ... + (Z \times N_Z) + ... + (Z' \times N_{Z'}) \\ &= \sum_{i=0}^{Z'} i N_i \quad , \end{aligned} \tag{1.50}$$

where N_Z denotes the number of ions of charge Z.

To solve this problem computationally we can adopt the following approach. Let us assume we wish to find the degree of ionization for a given electron temperature and total ion density.

1. We make a guess at the electron number density (this can be done, for example using the Thomas-Fermi model.)

2. Using this electron density we work out all of the ratios of the charge stages using the set of equations (1.47).

3. Using these ratios of charge stages, and our given total ion density, we can employ eqn. (1.50) to find a new electron density.

4. If our new electron density agrees with our initial guess, we have a self-consistent solution to the Saha equation. If not, we raise or lower our electron density from our initial guess accordingly, and return to stage (1) until convergence is achieved within some pre-defined limits. Normally only a few iterations are necessary.

A simple computer program that solves the Saha equation for some multi-electron atoms is given on the web-site accompanying this course (it runs on a Macintosh computer under the Classic operating system). Typical output from the program is shown in Fig. 1.2, where we show the ion densities for an Aluminium plasma with a total ion density of 10^{20}cm^{-3}, as a function of temperature. It is a common feature of the ionization state of multielectron atoms that typically only about 3 ionization stages occur in any abundance for any given temperature.

Before we close this section, it is important to point out that the Saha equation has assumed thermodynamic equilibrium – that is to say the processes that cause ionization within the plasma are perfectly balanced by all of the processes that cause recombination. In real plasmas this is not necessarily the case, and the calculation can become far more complicated. We may touch on this briefly in a future lecture.

1.9 The Debye Length

Plasmas have a whole host of interesting properties, and in this section we are going to meet one of the most fundamental – the ability to shield electric fields

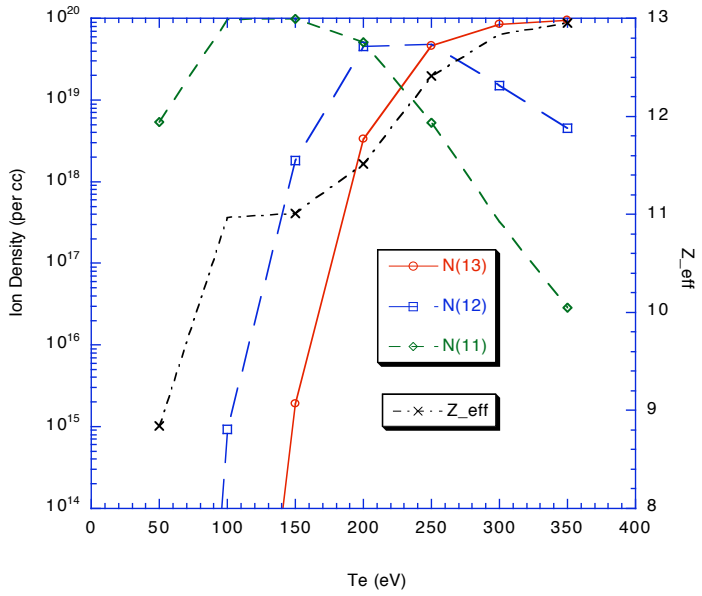


Figure 1.2: : Populations of the ionization states of Aluminium as a function of temperature for a total ion density of 10^{20}cm^{-3}. Z_{eff} is the average state of ionization.

– that is to say if a test charge is introduced into the plasma, the electrons and ions of the plasma move in such a way so as to shield its field.

Consider inserting a positively charged particle into a plasma, as shown in Fig. 1.3. If it were in a vacuum, the field surrounding the charge would fall off with distance r as $1/r^2$. In a plasma, however, the positively charged particle will tend to attract electrons towards it, and repel ions away from it. Therefore, from the point of view of a second test charge in the plasma, some distance from the first, the Coulomb field of the first particle will tend to be screened by the electrons attracted towards the first charge, and the ions repelled away from it. We will find that this screening process tends to make the field from the test charge look as though it is falling off exponentially, rather than $1/r^2$, and the $1/e$ distance of the fall off is known as the *Debye length*. It is one of the most important parameters that describe any given plasma.

Our approach to solving this problem is to assume that the electrons and

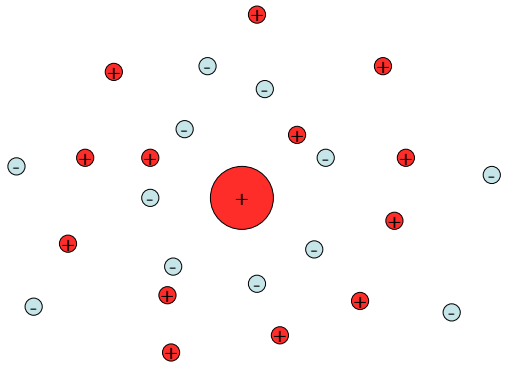


Figure 1.3: : When a positive test charge is introduced into a plasma, electrons are attracted towards it, whilst the ions are repelled.

the ions in the plasma have some temperature, T. We further assume that the overall mean number density of electrons and ions is equal, and we denote this number density by n_0. We then insert the (positive) test charge into the plasma. We then consider an electron that has been attracted to the vicinity of the test charge, and is a distance r from it. Because the test charge is there, the electron feels an extra potential, $\phi(r)$, (we use a scalar r, as clearly the problem will have spherical symmetry). Notice that we have not assumed that the extra potential that the electron feels is the the $1/r$ potential, as there will be other electrons between the one we are considering and the test charge that are also acting in the screening process. From Boltzmann's law we know that the number density of electrons that will be at this potential will be

$$n_e(r) = n_0 \exp\left(\frac{e\phi(r)}{k_{\rm B}T}\right) \approx n_0\left(1 + \frac{e\phi(r)}{k_{\rm B}T}\right) \quad , \tag{1.51}$$

where we have assumed that $e\phi \ll k_{\rm B}T$ (the reason for this assumption is discussed in the problem sheets). Thus the *excess* of electrons around the test charge, $n_e^+(r)$ over and above those that are normally there is

$$n_e^+(r) = n_0\left(\frac{e\phi(r)}{k_{\rm B}T}\right) \quad . \tag{1.52}$$

It is obvious that there will be a similar deficit of ions, $n_i^-(r)$. Therefore the

excess charge density $\rho(r)$ at distance r from the test charge, is

$$\rho(r) = e(n_i^-(r) - n_e^+(r)) = -2n_0 \left(\frac{e^2\phi(r)}{k_{\rm B}T} \right) \quad . \tag{1.53}$$

So we have an equation that links the charge density at a given point to the potential at a given point. How do we solve this to find the potential? Well, we notice that for self-consistency the potential itself is related to the charge density by Poisson's equation:

$$\nabla^2\phi(r) = -\frac{\rho}{\varepsilon_0} \quad . \tag{1.54}$$

Combining eqns (1.53) and (1.54) we find

$$\nabla^2\phi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r} \right) = \left(\frac{2n_0e^2}{\varepsilon_0 k_{\rm B}T} \right) \phi(r) \quad . \tag{1.55}$$

The above equation has the solution

$$\phi = \frac{A}{r} \exp\left(-\frac{\sqrt{2}r}{\lambda_{\rm D}} \right) \quad , \tag{1.56}$$

where

$$\lambda_{\rm D} = \sqrt{\left(\frac{\varepsilon_0 k_{\rm B}T}{n_0 e^2} \right)} \quad , \tag{1.57}$$

and A is a constant. Clearly at the origin the potential must simply be that due to the original test charge, Q, which fixes the constant, yielding the final solution

$$\phi = \frac{Q}{4\pi\varepsilon_0 r} \exp\left(-\frac{\sqrt{2}r}{\lambda_D} \right) \quad . \tag{1.58}$$

The length, $\lambda_{\rm D}$, is known as the Debye length, and is the typical length over which the potential due to the test charge is screened (we leave it to the reader to show that the $\phi(r)$ given in eqn (1.58) is the solution to (1.55)).

1.10 The Plasma Parameter

The above analysis of the Debye length – the length over which the field due to a test charge is shielded, implicitly assumes that there are a lot of electrons (and ions) doing the shielding. We implicitly assumed this by (a) using the Boltzmann equation, which will only be applicable over a large number of particles, as statistical mechanics is probabilistic in nature, and (b) in the manner in which we have assumed that $\phi(r)$ is a smoothly varying function. Therefore, in order for the above analysis to be valid, there must be many electrons that occupy a

sphere with a radius equal to the Debye length. This number, denoted $N_{\rm D}$, is known as the *plasma parameter*:

$$N_{\rm D} = n_0 \frac{4\pi}{3} \lambda_{\rm D}^3. \tag{1.59}$$

A 'good' plasma is one for which $N_{\rm D} \gg 1$. It so happens that the majority of plasmas in which we shall be interested are indeed 'good' in this sense.

1.11 Plasma Oscillations - The Plasma Frequency

Another of the most important, fascinating, (and by no means trivial) characteristics of plasmas is that they display a vast variety of wave phenomena; indeed, a lot of the first half of this course will be spent looking at such waves. There are a multitude of different sorts of oscillations that be supported in a plasma. Throughout the course as a whole we will only look at a subset of these. In this particular lecture, and in the next lecture, we will look at the most important of the lot – the plasma or Langmuir wave. The frequency of this particular wave is one of the most important parameters used to characterize a plasma.

We notice that electrons are nearly 2000 times lighter than protons. Therefore if an electric field exists within a plasma, then the electrons start to move (they are lighter), whereas the sluggish ions move far less in the same time. If the fields in the plasma oscillate rapidly, the amplitude of the oscillations of the electrons is about 2000 times greater than that of the ions (for a hydrogen plasma). Therefore, to a very good approximation, we can treat the ions as being stationary, and consider all of the response of the plasma to be due to the electrons. It must be said that there will be a few minor exceptions to this rule as we proceed through the course, but as a general rule it holds true. Ions are heavy, therefore slow, and can be considered to all intents and purposes to be static. It is the electrons that carry the currents.

So, let's look at plasma waves. Consider a plasma that is, overall, neutral (equal numbers of positive and negative charges). As shown in Fig. what we are going to do is somehow, artificially, pull all of the electrons to one side of the ions, and then let go. Clearly the electrons are going to be attracted back to the ions. The electrons accelarate towards the ions (the ions will move slightly towards the electrons, but as we have noted above, we can ignore this because they are so heavy compared with the electrons). The electron cloud moves faster and faster towards the ions until once more it completely overlaps it. However, the momentum built up in the electrons causes them to overshoot, and the electron cloud exits from the other side of the ions. The electric field built up slows down the electrons, until they are just as far to the right of the ions as they were to the left at the beginning, and back they go. Clearly this is an oscillatory process, and the frequency of oscillation of this most fundamental of waves (when electrons are displaced from their equilibrium positions) is known as the plasma frequency. Let us now work out what this frequency of oscillation actually is. We go back and consider Fig. 1.4.

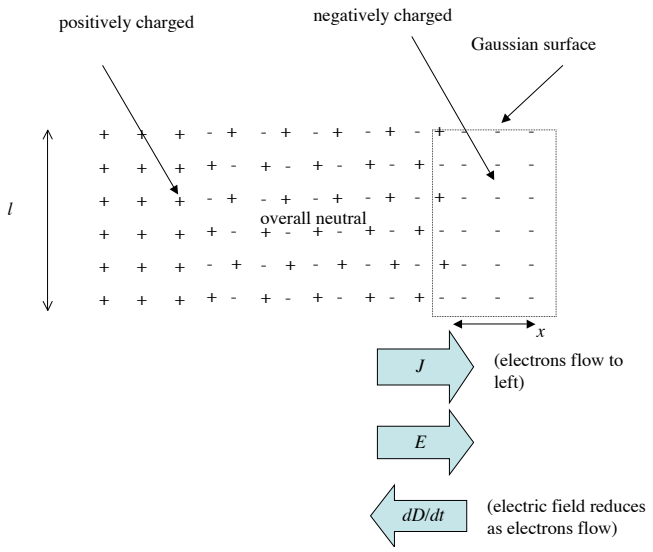


Figure 1.4: : Within the plasma a section of electrons has been displaced to the right giving rise to an electric field that causes them to move to the left. Note that the conduction current is in the opposite direction to the displacement current.

Consider a cube of plasma of side l. The number density of both the protons and the electrons is taken to be n. We displace the electrons to the right of the ions by a small distance x. They are attracted back towards the ions. We can work out the electric field by using Gauss' theorem. As the number densities of protons and electrons in the main part of the plasma are the same, we can take the field there to be zero. Thus taking the Gaussian surface shown in the diagram we deduce that the field at the point x is

$$El^2 = \frac{nel^2x}{\varepsilon_0} \quad , \quad E = \frac{nex}{\varepsilon_0} \tag{1.60}$$

Therefore

$$F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -eE = -\frac{ne^2}{\varepsilon_0}x \quad , \tag{1.61}$$

so the motion is simple harmonic with an angular frequency ω_p, where

$$\omega_\mathrm{p}^2 = \frac{ne^2}{\varepsilon_0 m} \quad . \tag{1.62}$$

A useful approximation to remember is that $f_\mathrm{p} = \omega_\mathrm{p}/2\pi \approx 9000\sqrt{n}$, where n is measured in units of cm^{-3}, and f_p in Hz. Thus in a tokamak, where $n \sim 10^{12}$ cm^{-3}, the plasma frequency is about 10 GHz. On the other hand, in the ionosphere, $n \sim 10^4$ cm^{-3}, the plasma frequency is about 1 MHz. In denser plasmas, such as those produced when high power lasers hit solid targets, the electron density can exceed $n \sim 10^{21}$ cm^{-3}, and the plasma frequency is about 3×10^{14}Hz – the frequency of optical light itself.

One thing you should notice immediately about the frequency of oscillations that we have derived is that there is no dependence on wavelength. The plasma frequency is the same whatever the wavelength of the oscillation (unlike light, for example, where $\omega = ck$ in vacuo). We will find out later in the course that this is an oversimplification of the physics. The problem we are solving at the moment is for a cold plasma – the motion we are giving the electrons is simply oscillatory, apart from that we have assumed they are stationary. Later in the course we will look at the (more realistic) case of the plasma having a finite temperature, and we will find then that the plasma (Langmuir) waves in the plasma do indeed depend on the k-vector, but we will still recover equation (1.62) in the low temperature limit.

The other thing that you will have noticed is that we have not taken into account the possibility that the electrons make collisions with the ions as they oscillate backwards and forwards – we have assumed that the electrons simply flow through the background ions, being attracted by the overall charge difference. At no point have we taken into account that individual electrons might be deflected by making a collision with an ion. Clearly on some timescale the electrons will collide with the ions. Indeed, later on in the lecture course (lecture 6) we will investigate this, and calculate the collision time, τ_ei, i.e. the average time between collisions of electrons and ions. The important point is that in general for a plasma the electrons undergo an oscillation on a timescale that is

far shorter than the time between collisions, that is to say $\omega_{\rm p}\tau_{\rm ei} \gg 1$. Therefore we are justified in ignoring the collisions when working out the frequency, and when working out the behaviour of the electrons on the plasma frequency timescale. Indeed, we shall find that very often we can ignore the collisions when working out a lot of the physics, and so-called *collisionless* behaviour is one of the hall-marks of plasma physics. Notice that we couldn't do the same in a normal gas: yes we can get waves in a gas (sound waves), but in a normal gas these rely on collisions. On the other hand, the *collective* motion of charges in a plasma (i.e. large numbers of electrons or ions bunching together) can give rise to large restoring forces which have nothing to do with any particular particle, but which result in the wave motion.

Another point to note is how the plasma frequency and Debye length are related: the Debye length, which we met in section 1.9, can be written

$$\lambda_{\rm D} = \sqrt{\left(\frac{\varepsilon_0 k_{\rm B} T}{n_0 e^2}\right)} = \left(\frac{k_{\rm B} T}{m}\right)^{1/2} \frac{1}{\omega_p} \approx \frac{v_{\rm th}}{\omega_p} \quad , \tag{1.63}$$

where $v_{\rm th}$ is the typical thermal velocity of the electrons. Thus, as well as being the screening length, the Debye length can thought of as the distance that a typical thermal electron travels during one period of a plasma wave. We will return to the importance of this statement later on in the course.

1.12 Summary of Lecture 1

1. The chemical potential is indeed related to chemistry, for it dictates the equilibrium concentration of reactants and products. The law of mass action states that
$$\prod_i (Z_{{\rm sp}_i})^{b_i} = \prod_i (N_i)^{b_i} \quad . \tag{1.64}$$
Thus all of the equilibrium chemistry is in the partition functions, as we would expect.

2. The ionization of hydrogen is a little bit like a chemical reaction, in that we have a reactant (hydrogen) and products (electrons and protons). By applying the law of mass action we found that the ratio of the electron densities to the neutral hydrogen densities could be expressed as
$$\frac{n_{e^-}^2}{n_H} = \exp\left(\frac{-Ry}{k_{\rm B} T}\right) \left(\frac{2\pi m_e k_{\rm B} T}{h^2}\right)^{3/2} \quad . \tag{1.65}$$

3. Alternatively, we could define a degree of ionization, ζ, and then we found
$$\zeta = \frac{\sqrt{f^2 + 4f} - f}{2} \quad , \tag{1.66}$$
where
$$f(V,T) = \left(\frac{V}{N_0}\right) \exp\left(\frac{-Ry}{k_{\rm B} T}\right) \left(\frac{2\pi m_e k_{\rm B} T}{h^2}\right)^{3/2} \quad . \tag{1.67}$$

4. We worked out the fraction of ionized atoms for certain conditions, and found that hydrogen ionized at temperatures that were relatively small when compared with the Rydberg. We concluded that this was because the continuum was heavily weighted from a statistical point of view. This meant that, surprisingly, hydrogen started to ionize before the excited states got much population in them, and this also meant that only taking into account the ground state of hydrogen in the ionization process (that is to say in the electronic partition function) was a good approximation

5. The partition function of an atom is not infinite in practice, because at some point the electrons will have such a large orbit that they will be influenced by the next atom, and effectively become free.

6. For multielectron atoms the degree of ionization must be worked out computationally by an iterative procedure.

7. In a plasma the electrons and ions move so as to shield any test charge that is introduced. Thus the charge seen from any particular ion no longer falls off with the Coulomb law, but in a modified way such that the potential around a given charge is

$$\phi = \frac{Q}{4\pi\varepsilon_0 r}\exp\left(-\frac{\sqrt{2}r}{\lambda_{\rm D}}\right) \quad , \tag{1.68}$$

   where $\lambda_{\rm D}$, known as the Debye length, is given by

$$\lambda_{\rm D} = \sqrt{\left(\frac{\varepsilon_0 k_{\rm B} T}{n_0 e^2}\right)} \quad , \tag{1.69}$$

8. A 'good' plasma is one in which there are many particles in a sphere of radius the Debye length. This defines the *plasma parameter*, $N_{\rm D}$:

$$N_{\rm D} = n_0 \frac{4\pi}{3}\lambda_{\rm D}^3. \tag{1.70}$$

   A 'good' plasma is one for which $N_{\rm D} \gg 1$.

9. The natural oscillation frequency of the plasma – the frequency of oscillations of the electrons (for a cold plasma) is given by

$$\omega_{\rm p}^2 = \frac{ne^2}{\varepsilon_0 m} \quad . \tag{1.71}$$

Lecture 2

Electrostatic and Electromagnetic Waves

2.1 Electrostatic Waves

The plasma wave, the frequency of which we derived in the first lecture, is known as an *electrostatic wave*. From your studies of electromagnetism you will have come across *electromagnetic waves*, so it is clearly important to learn at this stage what an *electrostatic wave* is, and how the two differ. What is going on?

I will start by defining what an electrostatic wave is, and then we will go on to see why and how it exists: an electrostatic wave is one in which there can be an oscillating electric field, but there is no oscillating magnetic field. Now, given what you have learnt about electromagnetism to date, that statement ought to startle you. You will have studied the wave equation, and will be used to the idea of an oscillating electric field giving rise to an oscillating magnetic field giving rise to an oscillating electric field and so on and so on. So what is going on here?

In order to answer this question, let us write down two of Maxwell's equations, and remind ourselves of some of the simple vector algebra related to wave fields: Firstly we write down Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.1}$$

and then Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad . \tag{2.2}$$

Remember that in general if we have a vector such that

$$\mathbf{X} = \mathbf{X_0} \exp(\mathbf{i}[\omega \mathbf{t} - \mathbf{k} \cdot \mathbf{r}]) \quad , \tag{2.3}$$

where $\mathbf{X_0}$ is a constant vector then

$$\nabla \times \mathbf{X} = -i\mathbf{k} \times \mathbf{X} \quad , \tag{2.4}$$

$$\frac{\partial \mathbf{X}}{\partial t} = i\omega \mathbf{X} \quad , \tag{2.5}$$

$$\nabla \cdot \mathbf{X} = -i\mathbf{k} \cdot \mathbf{X} \quad , \tag{2.6}$$

and

$$\nabla(\nabla \cdot \mathbf{X}) = -\mathbf{k}(\mathbf{k} \cdot \mathbf{X}) \quad , \tag{2.7}$$

(if you have forgotten these relations, or not seen them before, convince yourselves of their validity.)

How can Maxwell's equations be consistent with our definition of an electrostatic wave – i.e. the electric field oscillates, but there is no magnetic field? Well, what we are going to do is to work out mathematically how this can be the case, and then see how this fits with our simple physical picture of electrons oscillating back and forth. We assume a wave-like solution to the electric field, i.e.

$$\mathbf{E} = \mathbf{E_0} \exp(\mathbf{i}[\omega \mathbf{t} - \mathbf{k} \cdot \mathbf{r}]) \quad , \tag{2.8}$$

and therefore, using eqns (2.2) and (2.4), we find

$$-i\mathbf{k} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad . \tag{2.9}$$

Thus, for there to be no oscillating magnetic field

$$\mathbf{k} \times \mathbf{E} = 0 \quad , \tag{2.10}$$

i.e. the k-vector of the wave is parallel to the electric field. But a cursory glance at our diagram in Fig 1.4 shows that this is indeed the case (make sure you can see this), so it makes sense! Thus, with our knowledge of Maxwell's equations, we can now make another definition of an electrostatic wave – it is one for which $\mathbf{k} \times \mathbf{E} = 0$: the k-vector is parallel to the electric field.

We have used Faraday's law to come to this conclusion, but there is also some pertinent physics to be learnt about electrostatic waves by considering Ampere's law (equation (2.1)). If an electrostatic wave doesn't have a magnetic field, then we deduce that in this case

$$\mathbf{J} = -\frac{\partial \mathbf{D}}{\partial t} \quad , \tag{2.11}$$

i.e. the conduction current is equal and opposite to the displacement current. Remember that $\mathbf{J}$ is the conduction current – the current that is really due to charges moving from one place to another, whereas the second term in equation (2.1), $\frac{\partial \mathbf{D}}{\partial t}$, is known as the displacement current – proportional to the rate of change in electric field. Thus what we have learnt is that in an electrostatic wave the conduction current and displacement current are intimately linked – they are equal and opposite.

Think once more about what happens when we pull our electrons away from the ions and then let go. The electrons start to rush back towards the ions. This is, of course, equivalent to a current (i.e. a $\mathbf{J}$). Recall that the direction

of current is defined to be in the opposite direction of that of the electrons. So, as the electrons rush to the left in Fig 1.4 (after we have let go), the current will flow to the right. We need to show that this oscillating conduction current is exactly equal and opposite to the oscillating displacement current. Now, this flow of conduction current is driven by the electric field. As the current flows to restore local charge neutrality the electric field decreases. In our diagram at the particular point in time shown the electric field is pointing to the right. It decreases as the electrons rush towards the ions, and thus $\frac{\partial \mathbf{D}}{\partial t}$ is a vector pointing to the left, in the opposite direction to the conduction current (which is what we need). Let's now show that they are not only opposite in direction, but equal in magnitude.

Consider a particular point in the plasma. At any instant in time the current is given by

$$\mathbf{J} = -ne\mathbf{v} = -ne\frac{\partial x}{\partial t} \quad . \tag{2.12}$$

What is the rate of decrease of the electric field? Well, from eqn (1.60) E is

$$E = \frac{nex}{\varepsilon_0} \tag{2.13}$$

and thus

$$\frac{\partial \mathbf{D}}{\partial t} = ne\frac{\partial x}{\partial t} = -\mathbf{J} \quad , \tag{2.14}$$

as required.

2.2 Electromagnetic waves

When we looked at electrostatic plasma waves, we were considering the possibility of oscillations in the electron density within the plasma. We thought of these as being set up by somehow artificially pulling the electrons to one side and 'letting go'. In reality they will simply exist within a plasma due to thermal fluctuations, just like sound waves are always excited in a solid simply due to the finite temperature. Remember that we found that the fluctuations in the electron density oscillated, which resulted in a fluctuation $\mathbf{J}$, and this fluctuating conduction current was both responsible for, and equal and opposite to, the fluctuating displacement current, $\frac{\partial \mathbf{D}}{\partial t}$. Read that again – it is a very important point – the fluctuating charge density *itself* caused all of the fluctuating electric field. Because of this the displacement current and conduction current were equal and opposite, and thus from Ampere's law no oscillating magnetic field was produced, and the wave was therefore *electrostatic*.

However, what happens if we shine light on our block of plasma? We know that light is an *electromagnetic* wave – there is an oscillating $\mathbf{E}$ field and an oscillating $\mathbf{B}$ field. Can the wave propagate in the plasma? Notice what is different here is that there is an oscillating magnetic field, and that changing magnetic field will (as we know for light waves) give rise to an oscillating electric field. That is to say that we now have a situation where the oscillating field is

related both to conduction currents and changing magnetic fields. This sounds as though it is going to get very complicated, but not really, we just have to keep our heads and apply Maxwell's equations properly.

We know that the electric field makes the electrons move – that is to say a current is produced. Basically, this is just Ohm's law:

$$\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E} \quad , \tag{2.15}$$

where ω is the frequency of oscillation, and we have taken into account the fact that the conductivity of the plasma can be frequency dependent. Thus Ampere's law reads

$$\nabla \times \mathbf{H} = \sigma(\omega)\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t} \quad . \tag{2.16}$$

We now take the same tack as we always do when trying to find out about electromagnetic radiation: we seek to find the wave equation. To do this we take the curl of Faraday's equation:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t} \quad , \tag{2.17}$$

i.e.

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t} \quad . \tag{2.18}$$

Now, when you derived the wave equation for light in a vacuum, this is the point where you get rid of the first term on the left hand side, and say $\nabla(\nabla \cdot \mathbf{E}) = 0$, because there are no free charges in a vacuum, and thus Gauss's law reduces to $\nabla \cdot \mathbf{E} = 0$, and then you insert Faraday's law (eqn (2.1)) into eqn (2.18)) to get the wave equation. Are we allowed to assume $\nabla \cdot \mathbf{E} = 0$ for electromagnetic waves propagating in a plasma? Well, the short answer is yes, but it is important that we understand why. We certainly can't make this assumption for electrostatic waves (where there is a charge separation), so why can we for electromagnetic ones?

Again, we can learn a lot from simple diagrams, and we will compare the situation for electrostatic and electromagnetic waves. Consider Fig 2.1, where I have drawn light of wavelength λ travelling through a plasma. The size of the plasma perpendicular to the direction of propagation of the plasma is L, and clearly we need $L \gg \lambda$ to treat the wave as plane and without the effects of diffraction. Note from the figure that the charge separation takes place at the edge of the plasma, and within the plasma there is no charge separation, and thus $\nabla \cdot \mathbf{E} = 0$.

So we see that the electrostatic waves really do produce a local separation in positive and negative charges on the scale of a wavelength, whereas electromagnetic waves do not, and for this to occur the k-vector must be perpendicular to the electric field - electromagnetic waves are transverse waves.

So, having shown that $\nabla \cdot \mathbf{E} = 0$ for electromagnetic waves in a plasma we

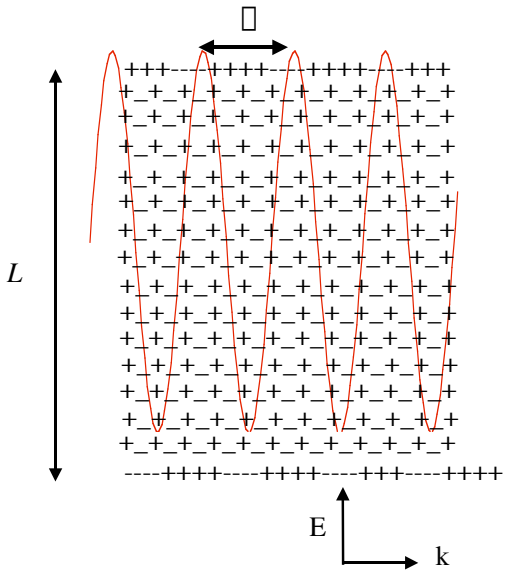


Figure 2.1: : When an electromagnetic wave propagates in a plasma, there is no local charge separation – only at the edges, which in a plasma of size $L \gg \lambda$, has no effect on the field within the plasma.

see that equation (2.18) reduces to

$$-\nabla^2\mathbf{E} = -\frac{\partial(\nabla\times\mathbf{B})}{\partial t} = -\mu_0\frac{\partial}{\partial t}\left(\mathbf{J}+\varepsilon_\mathbf{0}\frac{\partial\mathbf{E}}{\partial\mathbf{t}}\right) = -\mu_0\frac{\partial}{\partial t}\left(\sigma\mathbf{E}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad , \tag{2.19}$$

where we have made use of Ampere's law, eqn (2.16). Using our vector relationships, this can be written as

$$k^2 = -\mu_0\left(i\omega\sigma-\varepsilon_0\omega^2\right) = \frac{\omega^2}{c^2}\left(1-\frac{i\sigma}{\varepsilon_0\omega}\right) \quad . \tag{2.20}$$

That is to say, the plasma has a dielectric constant which is a function of frequency, $\varepsilon(\omega)$,

$$\varepsilon(\omega) = \left(1-\frac{i\sigma}{\varepsilon_0\omega}\right) \quad . \tag{2.21}$$

So, if we knew the conductivity of our plasma, we would be able to find the dielectric constant of the plasma, and thus the way in which electromagnetic waves can propagate within it. In order to find the conductivity, we will neglect collisions in the plasma between the electrons and the ions (as we did when we

studied the plasma waves) – we will simply assume that the electric field drives the electrons. If we can work out the relationship between the strength of the oscillating electric field, and the resultant velocity of the electrons, we will know the resultant current and hence conductivity. First of all we assume that we have an oscillating electric field:

$$\mathbf{E} = \mathbf{E}(\omega)\exp(i\omega t) \quad , \tag{2.22}$$

which results in an equation of motion for the electrons (remember the field drives the electrons in this case)

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -e\mathbf{E} = -e\mathbf{E}(\omega)\exp(i\omega t) \quad . \tag{2.23}$$

As the electric field is oscillatory, we assume the velocity of the electrons will also be, and thus we look for solutions of the form

$$\mathbf{v}(t) = \mathbf{v}(\omega)\exp(i\omega t) \quad . \tag{2.24}$$

Substituting equation (2.24) into (2.23) we obtain

$$im\mathbf{v}(\omega)\omega = -e\mathbf{E}(\omega) \quad . \tag{2.25}$$

This now allows us work to out the conductivity of the plasma. The current density in the plasma is proportional to the charge, the number of electrons, and how fast they are moving:

$$\mathbf{J}(\omega) = -ne\mathbf{v}(\omega) \quad . \tag{2.26}$$

From equation (2.25) and (2.26)

$$\mathbf{J}(\omega) = -ne\mathbf{v}(\omega) = -i\frac{ne^2}{m\omega}\mathbf{E}(\omega) \quad . \tag{2.27}$$

Therefore we have worked out the frequency-dependent conductivity of the plasma, $\sigma(\omega)$. The conductivity is defined by $\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$, i.e.

$$\sigma(\omega) = -i\frac{ne^2}{m\omega} \quad . \tag{2.28}$$

So we now have a formula for the conductivity that we can substitute back into eqn (2.21) to find the dielectric constant of the plasma. The result is

$$\epsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m\omega^2} = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} \quad . \tag{2.29}$$

Notice that this is a function of the plasma frequency, and also take note that the dielectric constant of the plasma is less than one. At very high frequencies the dielectric constant is close to unity, as in a vacuum. As the frequency, ω, of the electromagnetic radiation gets lower and lower, the value of the dielectric

constant gets less and less. When the frequency of the radiation reaches the plasma frequency, then the dielectric constant is zero, and at even lower frequencies it becomes negative. What does this imply? Well remember from your studies of electromagnetism and optics that the reflectivity, R, of a medium is given by

$$R = \left| \frac{1 - \sqrt{\varepsilon}}{1 + \sqrt{\varepsilon}} \right|^2 \quad . \tag{2.30}$$

Now, once $\epsilon(\omega)$ becomes negative, the square root of it is purely imaginary, and therefore the reflectivity is unity. That is to say, electromagnetic radiation cannot propagate at frequencies below the plasma frequency. If a wave with frequency below the plasma frequency is incident upon a the plasma boundary, it is reflected. For example, in the atmosphere there is a layer of plasma sited approximately 100 km above the surface of the earth called the ionosphere where there is a plasma with a number density of about 10^{10} m^{-3}. Therefore, radio waves with frequencies below 1MHz are reflected from the underside of it. This phenomenon has been used for many decades to send short wave radio signals around the globe, but is a bit of a hindrance for radio astronomers.

As another example, when a very high power laser is incident upon a target, it can heat the surface up to such an extent that a plasma is formed. These experiments are carried out in vacuum, and thus when the plasma is formed, it expands away from the target. Very quickly a situation is thus reached where the laser beam is penetrating through a plasma which is tenuous far above the surface of the target, but gets denser and denser the further towards the target surface it travels. The frequency of the light is a constant, but the plasma frequency is getting higher closer towards the target. Therefore there is a critical density of electrons in the plasma at a certain point above the target surface, where the laser light gets reflected. We will study such laser-plasma interactions in more detail in a future lecture.

2.3 An integrated Analysis

You might have got the impression from sections 2.1 and 2.2 that there are different ways of deriving the equations for the two different types of waves. This is not the case – all that has happened is that to start things off gently I have tried to give you an easy method of deriving the plasma waves. I now want to show you that both types of waves can be simply derived from Maxwell's equations (clearly this must be the case – both types of waves satisfy Maxwell's equations). The pertinent point is that all of the physics, and both types of waves, are described by equation (2.18) . This is because in the derivation of that equation we have not yet assumed that the waves are electromagnetic (we have not yet said $\nabla \cdot \mathbf{E} = 0$).

Equation (2.18) implies

$$-\mathbf{k}(\mathbf{k} \cdot \mathbf{E}) + k^2 \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \quad . \tag{2.31}$$

Using Faraday's law (eqn (2.1)) this becomes

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + k^2\mathbf{E} = -\mu_0\frac{\partial}{\partial t}\left(\mathbf{J} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \quad . \tag{2.32}$$

Substituting the relation between conduction current and field, eqn (2.27), into eqn (2.32) we obtain

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + k^2\mathbf{E} = -\mu_0\frac{\partial}{\partial t}\left(-i\frac{ne^2}{m\omega}\mathbf{E} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \quad , \tag{2.33}$$

which in turn leads to

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} = \frac{1}{c^2}\left(\omega_\mathrm{p}^2 - \omega^2\right)\mathbf{E} \quad , \tag{2.34}$$

where we have made use of eqn (2.5) and we have defined $\omega_\mathrm{p}^2 = ne^2/\varepsilon_0 m$. Rearranging, we obtain

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0 \quad , \tag{2.35}$$

Now we see the full physics of both types of waves revealed in the one equation. We already know that electrostatic waves are those in which $\mathbf{k}$ and $\mathbf{E}$ point in the same direction. In that case $\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = k^2\mathbf{E}$ and the frequency of the wave given by eqn (2.35) is

$$\omega = \omega_\mathrm{p} \quad . \tag{2.36}$$

On the other hand, for electromagnetic waves, where $\mathbf{k}\cdot\mathbf{E} = 0$, eqn (2.35) yields

$$\omega^2 = \omega_\mathrm{p}^2 + c^2k^2 \quad . \tag{2.37}$$

as expected.

Thus both types of waves can be expressed by the one equation (eqn (2.35)). Indeed, this is the approach we will take in future lectures when we study even more complicated wave scenarios (e.g. the waves that can exist in a plasma which is placed in a magnetic field): again we will derive one equation from Maxwell's equations, akin to eqn (2.35), and use it to derive all the possible waves, be they electrostatic or electromagnetic. Indeed, the general equation that we need, which can be deduced from eqn (2.32), is

$$(\omega^2\varepsilon(\omega) - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0 \quad . \tag{2.38}$$

That is to say, in more complicated situations we still need to find $\varepsilon(\omega)$, and once we have done so, solve the equivalent of eqn (2.38) with $\mathbf{k}$ and $\mathbf{E}$ parallel for the electrostatic waves, and with $\mathbf{k}\cdot\mathbf{E} = 0$ for electromagnetic waves.

2.4 Summary of Lecture 2

1. The natural oscillation frequency of the plasma – the frequency of oscillations of the electrons (for a cold plasma) is given by

$$\omega_{\rm p}^2 = \frac{ne^2}{\varepsilon_0 m} \quad . \tag{2.39}$$

2. The plasma waves we have looked at are electrostatic – the electric field oscillates but there is no oscillatory magnetic field. Both electromagnetic and electrostatic waves can be deduced from the same equation, in general

$$(\omega^2\varepsilon(\omega) - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0 \tag{2.40}$$

which in this particular case (a cold unmagnetized plasma) gives

$$(\omega^2 - \omega_{\rm p}^2 - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0 \quad , \tag{2.41}$$

which we solve with $\mathbf{k}$ and $\mathbf{E}$ parallel for the electrostatic waves, and with $\mathbf{k}\cdot\mathbf{E} = 0$ for electromagnetic waves.

Lecture 3

Single Particle Motion

3.1 Single Particle Motion

It does not take much thought to recognize that plasmas are complicated beasts. They are made up of positive and negative charges. As such, the motion of each charge present within the plasma is dominated by the electric and magnetic fields that it experiences – each particle experiences the Lorentz force:

$$\mathbf{F} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = q(\mathbf{E} + [\mathbf{v} \times \mathbf{B}]) \quad , \tag{3.1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields experienced by the particle which carries charge q. We ask ourselves where might these $\mathbf{E}$ and $\mathbf{B}$ fields come from? Well, we could consider a lump of plasma, and apply some external source of electric or magnetic field. This indeed often happens – for example the charged particles in space that are above the earth's atmosphere, but not more than a few earth radii away, will still experience the magnetic field of the earth and will respond to it. However, a moments thought will show us that the situation can become a lot more complicated than that. Consider a particular charged particle in the plasma. As all the other particles are charged, the one we have picked to follow will feel an overall electric field due to the position of all the other charged particles. It will also experience a magnetic field due to the motion of all the other charged particles. These fields dictate the motion of our charged particle, and cause it to move. But it gets worse, the motion of our chosen particle, and where it itself is positioned, contributes to the magnetic and electric fields that the other particles feel! Solving this sort of problem for all of the particles in the system self-consistently is clearly impossible.

Although it looks as though things get very messy very quickly, in my opinion this is one of the aspects of plasma physics that makes it a real physicists' subject(!) What I mean by that is that in order to make any headway it is essential that we think very carefully about the essential parts of the problem for a particular situation that we are studying, and make the appropriate approximations. We really have to do some physics (make the right approximations,

know why we are doing so, and have an idea of the applicability – or not – of the model to the various kinds of plasmas we could encounter). In this lecture we are going to consider what is known as single particle motion. This does exactly what it says on the tin. We take one particle, and apply electric and magnetic fields to it, and see what happens. Amongst other things, this approach will tell us about the physics of those situations where the externally applied electric and/or magnetic fields dominate the motion of the particles, rather than the fields that they themselves generate.

3.2 Particle in a uniform B field.

In your previous studies of electromagnetism you will have already encountered the problem of the motion of a charged particle in a uniform magnetic field (and no electric field), so we can get through this situation pretty quickly. Clearly in this case eqn (3.1) reduces to

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = q(\mathbf{v}\times\mathbf{B}) \quad . \tag{3.2}$$

i.e.

$$m\dot{\mathbf{v}} = q(\mathbf{v}\times\mathbf{B}) \quad , \tag{3.3}$$

i.e

$$\dot{\mathbf{v}} = \boldsymbol{\omega}_\mathrm{c}\times\mathbf{v} \quad , \tag{3.4}$$

where we have introduced the cyclotron frequency, $\boldsymbol{\omega}_\mathrm{c}$:

$$\boldsymbol{\omega}_\mathrm{c} = \frac{-q\mathbf{B}}{m} \quad . \tag{3.5}$$

So the electron cyclotron frequency, for which $q=-e$, is given by $\boldsymbol{\omega}_\mathrm{ce} = e\mathbf{B}/m_e$, whereas an ion of charge Ze and mass M has an ion cyclotron frequency $\boldsymbol{\omega}_\mathrm{ci} = -Ze\mathbf{B}/M$. Notice that the opposite signs of these two frequencies implies that the particles rotate in different senses in the magnetic field.

By convention we usually take the $\mathbf{B}$ field (to which $\boldsymbol{\omega}_\mathrm{c}$ is parallel or antiparallel) to lie along the z-axis. You will, no doubt, already be aware that the motion of a charged particle in this constant magnetic field is a helix, and you will have derived this many times before. However, just for the sake of completeness, let's go through this again. We remind ourselves that from eqn (3.4) that there is no component of force on the electron in the z direction due to the magnetic field (as $\mathbf{F}\propto(\mathbf{v}\times\mathbf{B})$, so its component of velocity in this direction, v_z, is constant. The same equation states that the force on the charged particle is always perpendicular to its velocity, so the constant magnetic field can do no work on the particle, and thus its kinetic energy and speed, v, are constant.

In the plane which is perpendicular to the magnetic field, the motion is circular, and added to the constant motion in the z direction, the overall motion

is a helix. One way to show this is to introduce the speed of the particle in the perpendicular plane: $v_\perp$. By conservation of energy

$$v_\perp = \sqrt{v^2 - v_z^2} \quad . \tag{3.6}$$

In vector form the perpendicular velocity clearly only contains the x and y components of the speed:

$$\mathbf{v}_\perp = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} \quad . \tag{3.7}$$

Now, from eqn (3.4)

$$\dot{v_x} = -\omega_c v_y \quad , \quad \dot{v_y} = \omega_c v_x \quad , \tag{3.8}$$

from which we note that v_x and v_y are $\pi/2$ out of phase. Differentiating once more w.r.t. time we find

$$\ddot{v_x} = -\omega_c^2 v_x \quad , \quad \ddot{v_y} = -\omega_c^2 v_y \quad , \tag{3.9}$$

which are the equations for simple harmonic motion of frequency ω_c. As the components are out of phase, we find

$$v_x = v_\perp \cos(\omega_c t + \phi) \quad , \quad v_y = v_\perp \sin(\omega_c t + \phi) \quad , \tag{3.10}$$

where ϕ is some arbitrary phase. Having found the solutions for the velocities of the particles, we can integrate eqns (3.10) w.r.t. time to find the equations for the coordinates:

$$x = x_0 + \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi) \quad , \tag{3.11}$$

$$y = y_0 - \frac{v_\perp}{\omega_c}\cos(\omega_c t + \phi) \quad , \tag{3.12}$$

$$z = z_0 + v_z t \quad , \tag{3.13}$$

where z_0 corresponds to the z coordinate of the particle at time zero, and x_0 and y_0 have entered as constants of integration. Notice that

$$(x - x_0)^2 + (y - y_0)^2 = \left(\frac{v_\perp}{\omega_c}\right)^2 \quad , \tag{3.14}$$

which is a constant. That is to say, the path in the perpendicular plane is circular, and the the centre of the circle in this plane is given by the coordinates (x_0, y_0), and the radius of the circle, known as the Larmor radius, is $r_L = v_\perp/|\omega_c|$.

3.3 Guiding centre drift due to a general constant force

In the previous section we found that the motion of the charged particle in a uniform magnetic field was helical. Evidently we would like to extend our

understanding to what happens in more complex situations – for example there may also be an electric field present, or perhaps we might want to be able to treat a situation where the magnetic field is not uniform. How do we approach such a problem? Well, the key to solving this problem is to note that we will often only be interested in the *average* motion of the particle. For example, in the case of the particle in a uniform magnetic field, although the particle follows a helical path, we know that the centre of the helix simply travels along the magnetic field with velocity v_z. If we only care about the time-averaged position (averaged over main inverse cyclotron frequencies), rather than exactly where the particle is in the x, y plane, then we know that the centre of the particle motion will be at position x_0, y_0, $v_z t$ after time t. This is the key to understanding more complex situations – rather than following the exact motion of the particle, we settle for following the centre of the particle motion – the so called guiding-centre.

So, let's see how this helps us work out the motion of the particle in more complex cases. We consider the situation where there is an another constant force, $\mathbf{F}$, acting on the particle in addition to the force due to the constant magnetic field. This extra force will change the motion of the particle so that the guiding centre is no longer necessarily constant in the $x-y$ plane – it is the new position of the guiding centre that we seek.

First of all we notice that if the extra force acts in the *same* direction as the magnetic field (the z direction), then all that happens is that the guiding centre of the electron starts to accelerate in the z direction – the circular motion in the perpendicular plane, and the average position in this plane, is unaffected. This can be seen by considering the new equation of motion for the particle. We find that the equivalent of eqn (3.4) now becomes

$$\dot{\mathbf{v}} = (\boldsymbol{\omega}_c \times \mathbf{v}) + \frac{\mathbf{F}}{m} \quad , \tag{3.15}$$

the first term on the right hand side, due to the magnetic field, which causes the circular motion, is perpendicular to the z direction, and thus this circular motion around the x and y components of the guiding centre are unaffected by a force in the z direction.

Therefore we see that it is trivial to treat the motion of a particle in the presence of an additional force in the z direction; clearly what is really interesting is the case where the new force has a component *perpendicular* to the magnetic field. Therefore, from here on in within this section, we will assume that $\mathbf{F}$ lies in the x, y plane (obviously in the general case we can split the components of the force into its z and planar components – the z component causes acceleration of the z component of the guiding centre, and we are about to find out what the effect of the planar component is). So, let's return to considering eqn (3.15) assuming $\mathbf{F}$ is in the $x-y$ plane, and we are thus only interested in the motion of the guiding centre in this plane. We differentiate eqn (3.15) once more with respect to time:

$$\ddot{\mathbf{v}} = (\boldsymbol{\omega}_c \times \dot{\mathbf{v}}) \quad , \tag{3.16}$$

where the force has dropped out as it is assumed to be constant in time.

Substituting eqn (3.15) into (3.16) we obtain

$$\begin{aligned}\ddot{\mathbf{v}} &= \boldsymbol{\omega}_{\rm c} \times \left((\boldsymbol{\omega}_{\rm c} \times \mathbf{v}) + \frac{\mathbf{F}}{m} \right) \\ &= \boldsymbol{\omega}_{\rm c}(\boldsymbol{\omega}_{\rm c} \cdot \mathbf{v}) - \mathbf{v}\omega_{\rm c}^2 + \frac{\boldsymbol{\omega}_{\rm c} \times \mathbf{F}}{m} \\ &= -\mathbf{v}\omega_{\rm c}^2 + \frac{\boldsymbol{\omega}_{\rm c} \times \mathbf{F}}{m} \quad . \end{aligned} \tag{3.17}$$

Let us study eqn (3.17) carefully. Notice that in the absence of this other constant force, we simply have the equation of motion of a simple harmonic oscillator – we would conclude that the components of the velocity in the $x-y$ plane simply oscillate with frequency $\omega_{\rm c}$. We knew that from our earlier analysis. However, what has the additional force done? Well, the force is a constant (we assumed that at the beginning of the section, we are not going to deal with time varying forces), so all that has happened is that we have added an additional constant vector to the right hand side, $(\boldsymbol{\omega}_{\rm c} \times \mathbf{F})/m$. The direction of this vector still lies in the $x-y$ plane, as it is perpendicular to $\omega_{\rm c}$. Therefore we see that in the $x-y$ plane the solution for the velocity of the particle is now the oscillatory component *plus* an extra velocity, which we will call the *drift velocity*, $\mathbf{v}_{\rm d}$, given by

$$\begin{aligned}\mathbf{v}_{\rm d} &= \frac{\boldsymbol{\omega}_{\rm c} \times \mathbf{F}}{m\omega_{\rm c}^2} \\ &= \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2} \quad . \end{aligned} \tag{3.18}$$

It is with this velocity that the guiding centre of the particle will drift in the $x-y$ plane in the presence of an additional force. We will now go on to look at several examples of such guiding centre drift under the presence of a variety of types of force.

3.4 Particle in uniform E and B fields.

If we add an electric field, **E**, in a perpendicular direction to the **B** field (remember the solution for the particle motion for the component of electric field along the **B** field will be trivial) then the additional force on the particle is $\mathbf{F} = q\mathbf{E}$, and thus, from eqn (3.18), the guiding centre will drift with a velocity

$$\mathbf{v}_{\rm d} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \quad . \tag{3.19}$$

It is important to note in this case that $\mathbf{v}_{\rm d}$ is completely independent of q, m, and $v_\perp$. So particles drift in the same direction, whatever their mass, whatever their charge. If the electric field is in the y direction, both electrons and ions drift in the x direction.

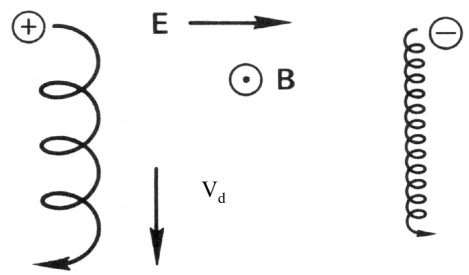


Figure 3.1: : In the presence of an electric field (as well as a magnetic field), both electrons and ions drift in the same direction.

It is instructive to have some physical picture of what is going on here. If the electric field lies along the y axis, why do both electrons and ions drift along the x axis? This phenomenon can be explained with reference to Fig. 3.1. Consider an ion in the electric field. First of all it gains energy from the electric field, and thus increases in $v_\perp$, and hence its Larmor radius, r_L increases. It curves around because of the magnetic field, but in the second half of the cycle it now loses energy due to the electric field, and r_L decreases. The overall effect over many cycles is a drift in the x direction. A negatively charged electron gyrates in the opposite sense, but it also gains energy from the electric field when travelling in the opposite direction to the ion, as it is oppositely charged. These two features cancel so as to make the electron drift in the same direction as the ion. Therefore the overall motion of charge particles in a magnetic field, with a perpendicular electric field, is a slanted helix. If there is a component of the electric field which lies along **B**, then the motion is a slanted helix with changing pitch, because of the acceleration in the z direction.

3.5 Grad-B drift

It would be rare to find a magnetic field that was perfectly uniform – so it behoves us to ask what happens to a charged particle in a non-uniform magnetic field. The simplest example we can consider is when the lines of force are completely straight, but their density changes, as shown in Fig. 3.2. Here, the

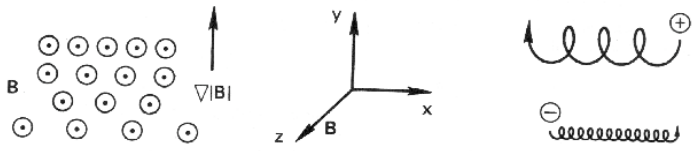


Figure 3.2: : A gradient in the magnetic field – here increasing in density in the y direction, with the field lines pointing along the z direction, causes drift along the x axis, but electrons and ions drift in opposite directions.

magnetic field lines always lie in the $z-$direction, but they increase in density in the $y-$direction. What do we expect to happen? Well, before deriving the formula for the drift in this particular case, let's see if we can work out what should happen simply by putting on our thinking caps. As the magnetic field density is changing, then the Larmor radius of the particles will be different, depending upon the y coordinate of the particle. As $r_{\rm L}$ is inversely proportional to B, we would expect the radius of curvature of the motion of the particles to be smaller for large y. Therefore, just as in the case of the electric field, we expect drift along the x axis. However, on this occasion, we expect electrons and ions to drift in opposite directions, as their sense of gyration is opposite. This motion is shown schematically in Fig. 3.2.

So, we have worked out the direction of drift, and the physical reason – now we need to see how to cope with this in terms of the mathematics. The trick here is to notice, as stated in eqn (3.18), that the direction of drift is perpendicular both to **B** and to the direction of the force. We have already convinced ourselves that the drift will be along the $x-$axis, and **B** lies along z by definition. Therefore we are looking for an extra force on the particles that appears to be in the y direction. Once we have found the magnitude of this force, we will have an expression for the drift velocity.

In order to find an approximate expression for this force in the y-direction we remind ourselves of the motion of the particle in a *uniform* field. From eqn (4.13)

$$\dot{v_y} = \omega_{\rm c} v_x \quad , \tag{3.20}$$

that is to say

$$\begin{aligned} F_y &= -q v_x B_z \\ &= -q v_\perp (\cos \omega_{\rm c} t) B_z \quad , \end{aligned} \tag{3.21}$$

where we have made use of eqn (3.10), and set $\phi = 0$ for convenience (as this just defines t_0.) Now we ask ourselves how to treat the situation where the

magnetic field varies slowly in magnitude along the y-axis. We assume that the magnitude of the field varies linearly, such that

$$B_z = B_0 + y\left(\frac{\partial B_z}{\partial y}\right) \quad . \tag{3.22}$$

We substitute eqn (3.22) into (3.21) to obtain

$$F_y = -qv_\perp(\cos\omega_c t)\left[B_0 + y\left(\frac{\partial B_z}{\partial y}\right)\right] \quad . \tag{3.23}$$

We assume that the y coordinate is not changed drastically by the variation in the field (this is tantamount to assuming that the Larmor radius is small compared with the scalelength of the gradient in the magnetic field), and so we can approximate y to be its value in a uniform field, as given in eqn (3.12) (and for convenience set $y_0 = 0$) to obtain

$$F_y = -qv_\perp(\cos\omega_c t)\left[B_0 - \frac{v_\perp}{\omega_c}(\cos\omega_c t)\left(\frac{\partial B_z}{\partial y}\right)\right] \quad . \tag{3.24}$$

This is the equation we were seeking. We ask ourselves what is the average value of F_y over a cycle? Clearly the first term on the right hand side averages to zero – this is just the normal force that oscillates. However, the second term does not average to zero, as the time average of $(\cos\omega_c t)^2$ over a cycle is $1/2$. We thus see that the time averaged force in the $y-$direction is

$$\bar{F_y} = qv_\perp^2\frac{1}{2\omega_c}\left(\frac{\partial B_z}{\partial y}\right) \tag{3.25}$$

It is this force that we must use to find the drift velocity. Substituting eqn (3.25) into (3.18) we find

$$\mathbf{v}_d = v_\perp^2\frac{1}{2\omega_c}\frac{\nabla B \times \mathbf{B}}{B^2} \quad . \tag{3.26}$$

Notice that this fits in with our original picture. It is a drift that is along the x axis for a gradient in the field along y, it is a drift in the opposite direction for electrons and ions (as ω_c is of opposite sign for each type of charge), and it is proportional to the magnitude of the gradient in B, as we might have expected.

3.6 Particle in a curved B field: curvature drift

Consider a set of magnetic field lines bent round into a circle, as shown in Fig. 3.3. In a vacuum Maxwell's equations imply that such a curved B field can't be constant in magnitude, but must fall off inversely with distance (for example, we could consider such a field as being produced by the current in an infinitely long wire at the centre of the circle, then we would see immediately that the field falls off with distance by invoking Ampére's law). Therefore, we see that in

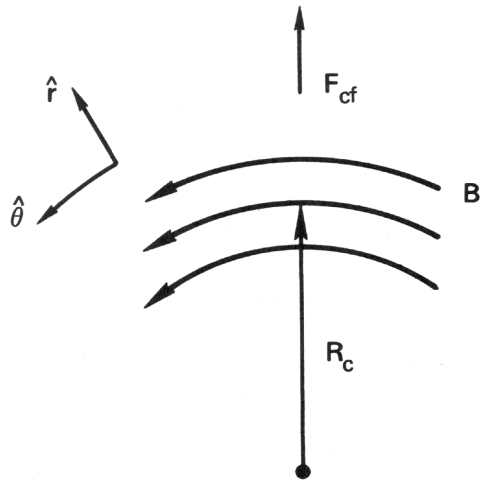


Figure 3.3: : A curved magnetic field also causes drift – in this case in the plane of the paper.

such a circular B field there must be a grad-B type drift due to the gradient in the field – and we can work that out using the results of section 3.5. However, there is also a drift which is simply due to the fact that the field lines are curved. We will now derive this so-called curvature drift, and then add on the associated grad-B drift, to find the total drift.

Consider the path of a charged particle in this circular field. Let the radius of curvature of the field lines be R. The particles will want to spiral round the magnetic field lines, and hence due to their velocity component tangential to the circle (i.e. along the magnetic field lines) they experience an effective centrifugal force, $\mathbf{F}_{\rm cf}$, due to their motion, $v_\parallel$, along the field lines, of

$$\mathbf{F}_{\rm cf} = m v_\parallel^2 \frac{\mathbf{R}}{R^2} \quad , \tag{3.27}$$

which, by the method with which you are now familiar, leads to a drift velocity given by

$$\mathbf{v}_{\rm d} = \frac{m v_\parallel^2}{qB^2} \frac{\mathbf{R} \times \mathbf{B}}{R^2} \quad . \tag{3.28}$$

This is the drift velocity due to the curvature itself. We noted at the beginning of this section that there will also be an associated grad-B drift, as the curved field in a vacuum cannot be uniform, and we noted that it was inversely proportional

to R, i.e.

$$|B| \propto \frac{1}{R} \quad , \quad \therefore \frac{\nabla|B|}{|B|} = -\frac{\mathbf{R}}{R^2} \quad . \tag{3.29}$$

Let's denote this extra drift due to the gradient by $\mathbf{v}_{\nabla\mathbf{B}}$. From eqns (3.29) and (3.26) we find

$$\begin{aligned} \mathbf{v}_{\nabla\mathbf{B}} &= -v_\perp^2 \frac{|B|}{2\omega_c} \frac{\mathbf{R}\times\mathbf{B}}{R^2 B^2} \\ &= \frac{m v_\perp^2}{2qB^2} \frac{\mathbf{R}\times\mathbf{B}}{R^2} \quad , \end{aligned} \tag{3.30}$$

which is in the same direction as the curvature drift. Therefore the overall drift velocity is

$$\mathbf{v}_{\text{total}} = \frac{m}{qB^2} \frac{\mathbf{R}\times\mathbf{B}}{R^2} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \quad , \tag{3.31}$$

The fact that the drifts due to curvature and due to the resultant gradient in field add together causes considerably difficulties when it comes to designing 'magnetic bottles' to hold plasmas for the purposes of magnetic confinement fusion. We will consider these problems in the relatively complicated geometry of a tokamak in a later lecture. For now, we will restrict our attention to the simpler case of a so-called magnetic mirror.

3.7 $\nabla B \parallel \mathbf{B}$: Magnetic Mirrors

We have already studied the effect of a non-uniform magnetic field on a charged particle in section 3.5. However, in that case the field lines were straight. We are now going to consider the case shown in Fig. 3.4, which is known (for reasons that will become apparent) as a magnetic mirror. We see that the field lines are cylindrically symmetric about the z axis. The field lines point mainly in the z direction but the magnitude of the field increases with z. Clearly the appropriate coordinates to use in this case are cylindrical: r, θ, z. As the field is symmetric about the $z-$axis we see immediately that $B_\theta = 0$ and $\partial/\partial\theta = 0$. However, as the field lines converge, there must be a component of the magnetic field in the radial direction, B_r. As we proceed we shall find that this radial component of the field can act so as to trap charged particles, and this is our first encounter with methods of confining plasmas by use of magnetic fields.

So, our first task is to find an expression for B_r. In order to do this we use the fact that $\nabla \cdot \mathbf{B} = 0$, which, in cylindrical coordinates, is

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \quad . \tag{3.32}$$

To make the problem tractable we will assume that the shape of the field is such that $\partial B_z/\partial z$ does not vary much from its value at $r = 0$ – this allows us to get

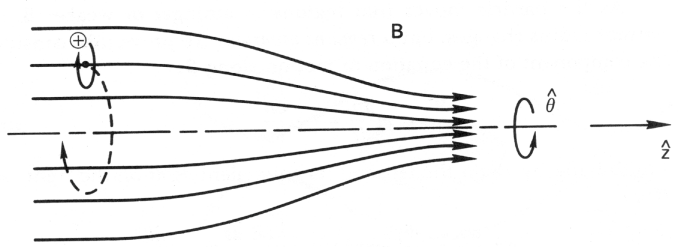


Figure 3.4: : The drift of a particle in the field of a magnetic mirror.

an approximate expression for B_r: from eqn (3.32)

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(rB_r) &= -\frac{\partial B_z}{\partial z} \\ \therefore rB_r &= -\int_0^r r\frac{\partial B_z}{\partial z}\mathrm{d}r \approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \\ B_r &= -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \quad . \end{aligned} \tag{3.33}$$

Notice that this is pretty much what we would have expected from our diagram: the radial component of the field points inwards towards the $z-$axis, and it gets larger with increasing r, as the angle the field lines make with the $z-$axis gets larger as the radius increases.

At the start of this section we said that this small, but finite, B_r causes the trapping of the particles. How does this occur? Well, consider a particle orbiting around the $z-$ axis with Larmor radius r_{L}. Its motion is perpendicular

to B_r, and therefore it feels a Lorentz ($\mathbf{v} \times \mathbf{B}$) force in the z direction given by

$$\begin{aligned} F_z &= qv_\perp \left(-\frac{1}{2} r_\mathrm{L} \left[\frac{\partial B_z}{\partial z} \right]_{r=0} \right) \\ &= -\frac{1}{2} \frac{m v_\perp^2}{B} \left[\frac{\partial B_z}{\partial z} \right]_{r=0} \\ &= -\mu \left[\frac{\partial B_z}{\partial z} \right]_{r=0} \quad , \end{aligned} \tag{3.34}$$

where μ is known as the *magnetic moment* of the charged particle. This is no different from the concept of magnetic moment that you have met previously: if a charge particle moves in a circular orbit, it has magnetic moment IA, where I is the current, and A the area enclosed by the orbit. Clearly a charge particle orbiting in a magnetic field encompasses an area πr_L^2, and carries a current $v_\perp/(2\pi r_\mathrm{L})$, and thus $\mu = q v_\perp r_\mathrm{L}/2$. Alternatively, we see from eqn (3.34) that another expression for the magnetic moment is

$$\mu = \frac{1}{2} \frac{m v_\perp^2}{B} \quad . \tag{3.35}$$

We have worked out the force on the particle when it is exactly on axis, but it can be shown that in general

$$F_z = -\mu \frac{\partial B_z}{\partial z} \quad . \tag{3.36}$$

It is this force, as we shall shortly see, that acts to confine (some of) the particles in the magnetic mirror. However, before we study that confinement in detail, let us consider the magnetic moment for a little longer. What happens to the magnetic moment as the particle moves from regions of high field to low field (or vice versa)? In order to answer this question notice that the force on the particle is along the z direction, i.e.

$$F_z = m \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu \frac{\partial B_z}{\partial z} \quad . \tag{3.37}$$

We multiply the left hand side of eqn (3.37) by $v_\parallel$, and the right hand side by its equivalent – $\mathrm{d}z/\mathrm{d}t$:

$$m v_\parallel \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\parallel^2 \right) = -\mu \frac{\partial B_z}{\partial z} \frac{\mathrm{d}z}{\mathrm{d}t} = -\mu \frac{\mathrm{d}B}{\mathrm{d}t} \quad . \tag{3.38}$$

It is important to note that in the above equation $\mathrm{d}B/\mathrm{d}t$ represents the time variation of B as seen by the particle because it is moving – we have not got a time varying $B-$field. Now, the energy of the particle must be conserved (we only have magnetic forces acting), and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\parallel^2 + \mu B \right) = 0 \quad . \tag{3.39}$$

Combining eqns (3.38) and (3.39) we find

$$-\mu\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(\mu B) = 0 \quad , \tag{3.40}$$

that is to say

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0 \quad . \tag{3.41}$$

Therefore the magnetic moment is a constant, and does not change as the particle moves from one region of field to another. It is known as the *first plasma invariant*.

From the invariance of μ we can get an idea of how the magnetic mirror works. Consider a charged particle that is moving from a region of low field to a region of high field. As μ is a constant, as the particle moves into areas of increasing B, then $v_\perp$ must also increase (from eqn (3.35)). However, from conservation of energy, this implies that $v_\parallel$, the velocity along the field lines, must decrease. If the magnetic field gets sufficiently high, eventually $v_\parallel$ becomes zero, and the particle is reflected. It is the F_z force that we derived above that causes this reflection. Particles with different initial velocities will get reflected at different places in the mirror. Some will get reflected right at the end, where the magnetic field is a maximum. However, even by that point, some particles may not have had their parallel components of velocity reduced to zero, and such particles will not be trapped.

Consider Fig. 3.4. Let us denote the magnetic field in the middle of the mirror by B_0, and that at the highest point of the field, the mirror point, by B_m. For given values of B_0 and B_m, we wish to find which particles escape, and which are trapped. Let the overall speed of the particle in the middle of the mirror be v_0 such that

$$v_0^2 = v_{\perp 0}^2 + v_{\parallel 0}^2 \quad . \tag{3.42}$$

By conservation of μ, at some other point within the mirror apparatus where the field is B

$$v_\perp^2 = \left(\frac{B}{B_0}\right) v_{\perp 0}^2 \quad , \tag{3.43}$$

and by conservation of energy the overall speed of the particle is always v_0. Thus from eqns (3.42) and (3.43) the parallel component of the velocity at some point other than the centre is given by

$$v_\parallel^2 = v_0^2 - v_\perp^2 = v_0^2\left(1 - \frac{B}{B_0}\frac{v_{\perp 0}^2}{v_0^2}\right) \quad . \tag{3.44}$$

So we see that particles are reflected (their parallel components of velocity reduced to zero) when the magnetic field,B_ref is such that

$$B_\mathrm{ref} = \frac{v_0^2}{v_{\perp 0}^2} B_0 \quad . \tag{3.45}$$

If the maximum magnetic field, $B_{\rm m} < B_{\rm ref}$, then clearly the particles will be lost. Thus we see that whether or not a particle is trapped depends on the maxima and minima in the fields, but also on the degree of perpendicular velocity component of the particles. If the initial perpendicular component is too small, then the particle will not be trapped (this makes sense: if the particle had no perpendicular component, it would simply travel along the field lines, and there would be no force to confine it). The particles are thus characterized by an angle, θ

$$\sin^2\theta = \frac{v_{\perp 0}^2}{v_0^2} = \frac{B_0}{B} \quad , \tag{3.46}$$

where θ is the pitch angle of the orbit in the weak field region. If this angle is too small, the particle is not confined. Thus in velocity space there exists a *loss cone*: particles with pitch angles less than $\theta_{\rm m}$, where

$$\sin^2\theta_{\rm m} = \frac{B_0}{B_{\rm m}} \tag{3.47}$$

are lost. Those with larger pitch angles of orbit are trapped by the mirror.

It is important to realise that trapped particles can still be lost if collisions alter their pitch angles such that the new angles put them into the loss cone. That said, mirror machines are used to confine plasmas. They also exist in nature – for example the particles trapped in the Van Allen belts. The earth's magnetic field is greatest at the poles, and thus particles that spiral around the field lines can bounce at the poles. We will look at this phenomenon in the problem sheets.

3.8 Summary of Lecture 3

1. In a uniform B field along the z direction the particle travels at a constant velocity along z, and executes circular motion in the $x-y$ plane at the cyclotron frequency,
$$\boldsymbol{\omega}_{\rm c} = \frac{-q\mathbf{B}}{m} \quad . \tag{3.48}$$
The overall motion is thus helical.

2. An additional constant force, perpendicular to $\mathbf{B}$, causes a drift of the guiding centre of the charged particle
$$\mathbf{v}_{\rm d} = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} \quad . \tag{3.49}$$

3. Thus in the presence of an electric field, $\mathbf{E}$, perpendicular to $\mathbf{B}$, the particles drift with velocity
$$\mathbf{v}_{\rm d} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \quad . \tag{3.50}$$
It is important to note in this case that $\mathbf{v}_{\rm d}$ is completely independent of q, m, and $v_\perp$. So particles drift in the same direction, whatever their mass, whatever their charge.

4. If there is a gradient in the magnetic field, the drift velocity is

$$\mathbf{v}_{\mathrm{d}} = v_{\perp}^2 \frac{1}{2\omega_{\mathrm{c}}} \frac{\nabla B \times \mathbf{B}}{B^2} \quad . \tag{3.51}$$

5. If the field lines are curved, there is a drift due to the curvature, which adds to the drift due to the gradient in the field, to give an overall drift velocity

$$\mathbf{v}_{\mathrm{total}} = \frac{m}{qB^2} \frac{\mathbf{R} \times \mathbf{B}}{R^2} \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right) \quad . \tag{3.52}$$

6. The magnetic moment of a charged particle in a B field is an invariant. It is given by

$$\mu = \frac{q v_{\perp} r_{\mathrm{L}}}{2} = \frac{1}{2} \frac{m v_{\perp}^2}{B} \quad . \tag{3.53}$$

7. From the invariance of μ we find that a magnetic mirror can confine particles with orbit pitches greater than a critical angle, θ_{m}, such that

$$\sin^2 \theta_{\mathrm{m}} = \frac{v_{\perp 0}^2}{v_0^2} = \frac{B_0}{B_{\mathrm{m}}} \quad . \tag{3.54}$$

Lecture 4

Waves in a cold magnetized plasma

4.1 Introduction

In lecture 1 we introduced one of the most important parameters of a plasma: the plasma frequency. In doing so we looked at the response of a set of (cold) electrons when they were displaced from the (assumed stationary) background ions. When we said the electrons were cold, we meant that the only motion of the electrons that we allowed was due to the electric fields set up by displacing them from the ions – we did not allow them to have any thermal motion that might have caused them to move from a peak to a trough in the plasma wave on a timescale comparable with the period of oscillation. We will treat such effects of finite temperature in a future lecture. However, we also assumed that the plasma was not subject to any external electric or magnetic fields – the electrons simply moved back towards the ions due to the electrostatic force between them, and were not deviated along the way by any other fields. The response of a plasma in the presence of a magnetic field is going to be the subject of this particular lecture.

In a cold, un-magnetized plasma, we found in lecture 1 that the dielectric constant of the plasma was a function of frequency, $\varepsilon(\omega)$,

$$\varepsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega^2} \quad . \tag{4.1}$$

In this lecture we wish to find a general expression for $\varepsilon(\omega)$ which is applicable even when there is a constant magnetic field present. We will then use this expression to find out the type of waves that can propagate in the plasma, be they electromagnetic or electrostatic.

It is important to state at the very start of this part of the course that this subject is complicated. It is not so much that the mathematics is difficult, it is more that the presence of an applied field allows a vast number of different

types of waves to propagate within the plasma, and it is easy to get lost amongst all the different types. Given the brevity of the course, I shall therefore only concentrate on a couple of the most important of the new waves that arise by magnetizing the plasma. You will be able to pick up very easily how to derive all the others if you so wished, and the majority of them are listed in any standard textbook on plasma physics (for a list of books, see the plasma physics website that can be accessed via the normal Oxford Physics site).

4.2 The Conductivity Tensor

The key to understanding the dielectric response of a magnetized plasma is to recognize that we no longer have a scalar conductivity and dielectric constant, but conductivity and dielectric tensors. This is by no means as scary as it sounds. Recall that in lecture 2 (where the plasma was unmagnetized) we related the current to the electric field in the following manner:

$$\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega) \quad , \tag{4.2}$$

that is to say we treated the conductivity as a scalar. That was fine because we knew that the electrons would move parallel to the electric field, and thus the current would always be parallel to the electric field. However, from our study of single particle motion in lecture 3 we are well aware that things are more complicated in the presence of a magnetic field. In this case an extra electric field can cause charged particles to drift in directions that are perpendicular both to the applied electric field and applied magnetic field. There is no way that the 'current', i.e. the motion of the electrons, is necessarily parallel to the direction of the electric field. That is why, in order to study magnetized plasmas, we allow the conductivity (and as we shall see, hence dielectric function), to be a tensor. We say

$$\mathbf{J} = \boldsymbol{\sigma} \cdot \mathbf{E} \quad . \tag{4.3}$$

The conductivity, $\boldsymbol{\sigma}$, is now a 3 x 3 matrix. It needs to be because each component of the current could dependent on any or all three of the components of the electric field. This implies that the dielectric response must be treated as a similar matrix (a tensor). The analogous equation to that of eqn (2.21) in lecture 2 is now [1]

$$\boldsymbol{\varepsilon} = \mathbf{I} - \frac{i}{\varepsilon_0 \omega}\boldsymbol{\sigma} \quad , \tag{4.4}$$

where $\mathbf{I}$ is the identity matrix. Similarly, the analogous equation to eqn (2.38), that will allow us to derive all of the electromagnetic and electrostatic waves, becomes

$$(\omega^2 \boldsymbol{\varepsilon} - c^2 k^2)\mathbf{E} + c^2 \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) = 0 \quad . \tag{4.5}$$

[1] Note that some textbooks will have a + sign on the right hand side of eqn (4.4) – it all depends on whether they have defined their fields to vary as $\exp(i\omega t)$ or $\exp(-i\omega t)$ – as long as one is consistent, it doesn't change the physics.

The way forward should now be clear. We need to work out the conductivity tensor, $\boldsymbol{\sigma}$, which, from eqn (4.4) gives us the dielectric tensor, $\boldsymbol{\varepsilon}$, and we then solve eqn (4.5) to find out the type of waves that are allowed in the magnetized plasma.

As usual, we are going to assume that the magnetic field is pointing in the z direction. In order to work out the conductivity tensor we need to find the relationship between the velocity of an electron and the applied oscillating electric field. Let's deal with the easy component first: the z component. The z component of the oscillating electric field will cause the electron to oscillate along the z axis. As this motion is parallel to the magnetic field, the magnetic field has no effect – it is just as though it was not there. This is exactly what we had in lecture 1 for an unmagnetized plasma, so for the z component eqn (2.28) must still hold – that is to say

$$\sigma_{zz} = -i\frac{ne^2}{m\omega} = -i\frac{\varepsilon_0\omega_{\rm p}^2}{\omega} \quad , \tag{4.6}$$

where the subscript zz means that this is the component of the 3 x 3 matrix that links the z component of the current to the z component of the electric field. Furthermore, it should be clear that this is the only linkage – the z component of the electric field never causes the electrons to move in the x or y directions, nor do the electric fields in the x and y directions ever cause motion in the z direction – they just complicate the motion in the $x-y$ plane because the motion is along the direction $(\mathbf{E}\times\mathbf{B})$. So we have already found out a lot about the conductivity tensor – it must look something like

$$\boldsymbol{\sigma} = \begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & -i\frac{\varepsilon_0\omega_{\rm p}^2}{\omega} \end{pmatrix} \quad . \tag{4.7}$$

We now need to resolve the question marks – what happens in the plane perpendicular to the magnetic field?

Well, we have all the tools at our fingertips to analyse this problem, as we have already dealt with it in the last lecture on single particle motion. From eqns (3.1) and (3.5) we know that the motion of the electrons in the direction perpendicular to the magnetic field is determined by the equation

$$\dot{\mathbf{v}}_\perp = -\frac{e}{m}\mathbf{E}_\perp + \boldsymbol{\omega}_{\rm ce}\times\mathbf{v}_\perp \quad , \tag{4.8}$$

where we have used the $\perp$ symbol to ensure that we remember that we are treating the motion perpendicular to the magnetic field. We differentiate eqn (4.8) w.r.t. time, noticing that now $\mathbf{E}$ is time dependent as it is an oscillating field (in the lecture on single particle motion, we only treated a constant electric field).

$$\ddot{\mathbf{v}}_\perp = -\frac{e}{m}\dot{\mathbf{E}}_\perp + \boldsymbol{\omega}_{\rm ce}\times\dot{\mathbf{v}}_\perp \quad , \tag{4.9}$$

Substituting eqn (4.8) into (4.9) we obtain

$$\begin{aligned}\ddot{\mathbf{v}}_\perp &= -\frac{e}{m}\dot{\mathbf{E}}_\perp + \boldsymbol{\omega}_{\rm ce}\times(-\frac{e}{m}\mathbf{E}_\perp + \boldsymbol{\omega}_{\rm ce}\times\mathbf{v}_\perp)\\ &= -\frac{e}{m}(\dot{\mathbf{E}}_\perp + \boldsymbol{\omega}_{\rm ce}\times\mathbf{E}_\perp) - \omega_{\rm ce}^2\mathbf{v}_\perp \end{aligned} \tag{4.10}$$

As the applied electric field is periodic, we apply a field $\mathbf{E}_\perp = \mathbf{E}_{\perp\mathbf{0}}\exp(i\omega t)$, and seek solutions for the velocity of the form $\mathbf{v}_\perp = \mathbf{v}_{\perp\mathbf{0}}\exp(i\omega t)$. With these assumptions, eqn (4.10) can be written

$$\mathbf{v}_{\perp\mathbf{0}}(\omega_{\rm ce}^2-\omega^2) = -\frac{e}{m}(i\omega\mathbf{E}_{\perp\mathbf{0}} + \boldsymbol{\omega}_{\rm ce}\times\mathbf{E}_{\perp\mathbf{0}}) \quad . \tag{4.11}$$

Looking at the individual components

$$v_{x0}(\omega_{\rm ce}^2-\omega^2) = -\frac{e}{m}(i\omega E_{x0} - \omega_{\rm ce}E_{y0}) \tag{4.12}$$

$$v_{y0}(\omega_{\rm ce}^2-\omega^2) = -\frac{e}{m}(i\omega E_{y0} + \omega_{\rm ce}E_{x0}) \quad . \tag{4.13}$$

This now allows us to work out the other components of the matrix that make up $\boldsymbol{\sigma}$, as $-ne\mathbf{v} = \boldsymbol{\sigma}\cdot\mathbf{E}$. Using this definition, and eqns (4.12) and (4.13), we find

$$\boldsymbol{\sigma} = \begin{pmatrix} i\frac{\varepsilon_0\omega\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & \frac{\varepsilon_0\omega_{\rm ce}\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & 0 \\ -\frac{\varepsilon_0\omega_{\rm ce}\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & i\frac{\varepsilon_0\omega\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & 0 \\ 0 & 0 & -i\frac{\varepsilon_0\omega_{\rm p}^2}{\omega} \end{pmatrix} \quad . \tag{4.14}$$

4.3 The Dielectric Tensor

Thus we find the full expression for the dielectric tensor that we originally sought:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 1+\frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & \frac{-i\omega_{\rm ce}}{\omega}\frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & 0 \\ \frac{i\omega_{\rm ce}}{\omega}\frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & 1+\frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} & 0 \\ 0 & 0 & 1-\frac{\omega_{\rm p}^2}{\omega^2} \end{pmatrix} \quad . \tag{4.15}$$

This can be written in a more managable form as

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \quad , \tag{4.16}$$

where

$$\varepsilon_1 = 1 + \frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} \tag{4.17}$$

$$\varepsilon_2 = \frac{\omega_{\rm ce}}{\omega}\frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2-\omega^2} \tag{4.18}$$

$$\varepsilon_3 = 1 - \frac{\omega_{\rm p}^2}{\omega^2} \tag{4.19}$$

4.4 Waves propagating parallel to the B field.

We are now in a position to find the waves that can exist in the plasma. Eqn (4.5) can be written

$$\mathbf{M} \cdot \mathbf{E} = 0 \quad , \tag{4.20}$$

where

$$\mathbf{M} = \frac{\omega^2}{c^2}\boldsymbol{\varepsilon} + \mathbf{kk} - k^2\mathbf{I} \quad . \tag{4.21}$$

We know that the solutions depend on the polarization of the wave (electrostatic waves have k-vectors along the direction of propagation, whereas electromagnetic waves have k-vectors perpendicular to the direction of propagation), so we need to keep track of the direction of k. We thus define

$$\mathbf{N} = \frac{c\mathbf{k}}{\omega} \quad . \tag{4.22}$$

With this definition, from eqns (4.21) and (4.16) we find

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - N_y^2 - N_z^2 & -i\varepsilon_2 + N_xN_y & N_xN_z \\ i\varepsilon_2 + N_xN_y & \varepsilon_1 - N_x^2 - N_z^2 & N_yN_z \\ N_xN_z & N_yN_z & \varepsilon_3 - N_x^2 - N_y^2 \end{pmatrix} \quad . \tag{4.23}$$

Now, if we define the waves to be propagating parallel to the field, $N_x = N_y = 0$, and thus in this case

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - N_z^2 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - N_z^2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \quad . \tag{4.24}$$

Non-trivial solutions to $\mathbf{M} \cdot \mathbf{E} = 0$ occur for the roots of $\det(\mathbf{M}) = 0$, i.e. when

$$\varepsilon_3[(\varepsilon_1 - N_z^2)^2 - \varepsilon_2^2] = 0 \quad . \tag{4.25}$$

This equation has three solutions. The first is

$$\varepsilon_3 = 0 \quad , \tag{4.26}$$

and the other two are

$$\varepsilon_1 - N_z^2 = \pm\varepsilon_2 \quad . \tag{4.27}$$

Let us now explore to what type of waves these solutions correspond.

The first solution is pretty obvious. If $\varepsilon_3 = 0$, then from eqn (4.19) then

$$\omega^2 = \omega_{\mathrm{p}}^2 \quad . \tag{4.28}$$

This must be our good old plasma wave, oscillating at the plasma frequency. Remember that we said this was an electrostatic wave. How do we show this now, with our more sophisticated analysis? Well, recall we have for the moment defined the direction of propagation of the wave to be along the $z-$axis (we

set N_x and N_y to zero). An electrostatic wave would be one which had the oscillating electric field pointing along the *same* direction as the k-vector. If we substitute eqn (4.26) back into (4.24) we see that $\mathbf{M} \cdot \mathbf{E} = 0$ is satisfied for electric fields of the form $(0, 0, E_z)$, so the wave is indeed electrostatic. Notice that we should not be surprised to find our normal plasma waves along the direction of the magnetic field. We know that the electrons are not deflected when they travel parallel to a magnetic field, so we should indeed recover the same result for these electrostatic waves as we did for the unmagnetized plasma.

It has probably occurred to you that the other two solutions, the ones indicated by eqn (4.27), might correspond to the *electromagnetic* waves that propagate in this magnetized plasma. Indeed, you would be correct, and we will examine these waves in more detail in just a moment. Before we do so, let us consider whether or not we would expect these electromagnetic waves to be exactly the same as those that we found in the unmagnetized case? Recall that we have defined these waves to be travelling along the z direction, the direction of the magnetic field. If they are to be electromagnetic, the electric fields must be transverse, i.e. in the x and/or y directions. Now, if an electron is accelerated in the $x - y$ plane by an electric field, then the magnetic field along the z direction will certainly now play a role in its dynamics, as the magnetic force on a charged particle is proportional to $(\mathbf{v} \times \mathbf{B})$. Therefore we would *not* necessarily expect the solutions for the electromagnetic waves to be the same as for the unmagnetized plasma. We would, however, expect any solutions we get to agree with the unmagnetized case if we set $B = 0$. Given that preamble, let us now examine eqn (4.27) more closely.

There are two solutions (because of the $\pm$ sign). First consider

$$\varepsilon_1 - \varepsilon_2 = N_z^2 \quad . \tag{4.29}$$

As we have defined k to lie along z, we know that $N_z^2 = c^2k^2/\omega^2$. Substituting eqns (4.17) and (4.18) into (4.29) and rearranging we find

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega + \omega_{\rm ce})} \quad . \tag{4.30}$$

Notice that this does indeed give us back our familiar electromagnetic wave for an unmagnetized plasma (if there is no B field, then $\omega_{\rm ce} = 0$). I leave it as an exercise for you to show that if we take the other sign in eqn (4.27) we find the second solution:

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega - \omega_{\rm ce})} \quad , \tag{4.31}$$

which again agrees with what we would expect in the absence of a magnetic field.

Let us now prove that these two solutions do indeed correspond to electromagnetic waves, and learn something about their polarization. If we substitute

$\varepsilon_1 \pm \varepsilon_2 = N_z^2$ into eqn (4.24) we find

$$\mathbf{M} = \frac{\omega^2}{c^2} \begin{pmatrix} \pm\varepsilon_2 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \pm\varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \quad . \tag{4.32}$$

For this $\mathbf{M}$ we find that for $\mathbf{M} \cdot \mathbf{E} = 0$, then the electric field is of the form $(E_x, \mp iE_y, 0)$. So the waves are indeed electromagnetic (the electric field is perpendicular to the k-vector). However, unlike the unmagnetized case, where the wave was linearly polarized, these waves are circularly polarized. The wave corresponding to the dispersion relation given in eqn (4.30) is a left-handed wave, whereas the other is right-handed.

Let us consider the left-handed wave in more detail. First of all, notice that if ω is very large compared with $\omega_{\rm p}$ and $\omega_{\rm ce}$, then we get the same dispersion relation for light in a vacuum – clearly at very high frequency there is no time in one period to interact with the plasma oscillations or the gyratory motion of the electrons. On the other hand, as ω is reduced, there comes a point where $k = 0$, and the waves are of infinitely long wavelength. This frequency represents a *cut off*, because below this certain frequency (which we shall denote by $\omega_{\rm LC}$), k is imaginary and thus waves cannot propagate. From eqn (4.30) we see that this cut off occurs when

$$\begin{aligned} 1 - \frac{\omega_{\rm p}^2}{\omega(\omega + \omega_{\rm ce})} &= 0 \\ \omega^2 + \omega_{\rm ce}\omega - \omega_{\rm p}^2 &= 0 \\ \omega_{\rm LC} &= \frac{-\omega_{\rm ce} + \sqrt{\omega_{\rm ce}^2 + 4\omega_{\rm p}^2}}{2} \end{aligned} \tag{4.33}$$

So much for the left-handed wave. What about the right-handed wave? Its dispersion relation is a little bit more complicated than that of the left handed wave. First of all we notice once again that at the very highest frequencies, we recover $\omega = ck$ as expected. Now let us consider very, very low frequencies. If we rearrange eqn (4.31) we see that it can be written

$$c^2k^2 = \omega^2 - \frac{\omega^2\omega_{\rm p}^2}{\omega(\omega - \omega_{\rm ce})} \quad . \tag{4.34}$$

Now if ω is very small – less than both $\omega_{\rm p}$ and $\omega_{\rm ce}$ – then we find

$$c^2k^2 \approx \frac{\omega\omega_{\rm p}^2}{(\omega_{\rm ce} - \omega)} \quad . \tag{4.35}$$

Two things are immediately apparent. Firstly it appears that $k \propto \sqrt{\omega}$ at low frequencies. Secondly, we notice that as ω approaches $\omega_{\rm ce}$ the k-vector becomes infinite – the wavelength tends to zero: the system has hit a resonance. Physically what is happening here is that the wave frequency equals the electron

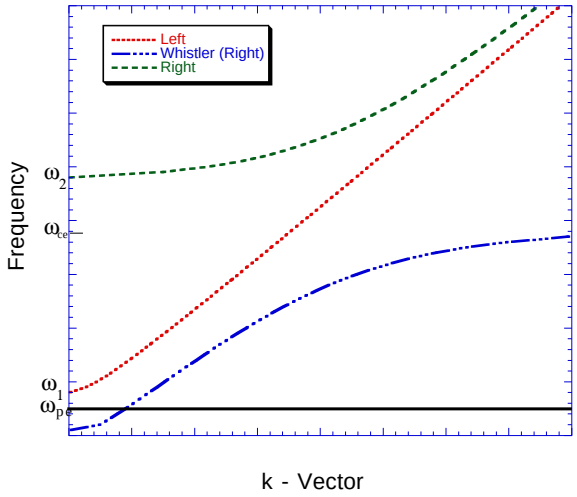


Figure 4.1: : The $\omega - k$ relation for waves that propagate parallel to the magnetic field.

cyclotron frequency. The cyclotron motion of the electron is right-handed just like the wave, and that is why resonance occurs. As for the very low frequency behaviour – i.e. our conclusion that $k \propto \sqrt{\omega}$, a word of warning. Throughout this lecture we have worked out the dielectric tensor by taking into account the motion of electrons: we have completely neglected the motion of the ions. We recall that this is usually a valid approach because the electrons are so light compared with the ions, that we normally think of the ions as not having time to respond. However, at very low frequencies they *do* have time to move, and thus they must also be taken into account at very low frequencies. We will return to this subject in section 4.7.

Returning to our right-handed electromagnetic wave, we see that if ω is just above $\omega_{\rm ce}$, then k is imaginary, so no waves propagate. However, as ω is increased even further eventually the first term on the right hand side of eqn (4.34), ω^2, exceeds the second term on the right hand side, and thus waves can once more propagate. This cutoff frequency, $\omega_{\rm RC}$, to the right handed wave can be found by setting $k = 0$ in eqn (4.34), which leads to

$$\omega_{\rm RC} = \frac{\omega_{\rm ce} + \sqrt{\omega_{\rm ce}^2 + 4\omega_{\rm p}^2}}{2} \tag{4.36}$$

As an aid to remembering all these different modes and dispersion relations, they are plotted on an $\omega - k$ diagram in Fig. 4.1.

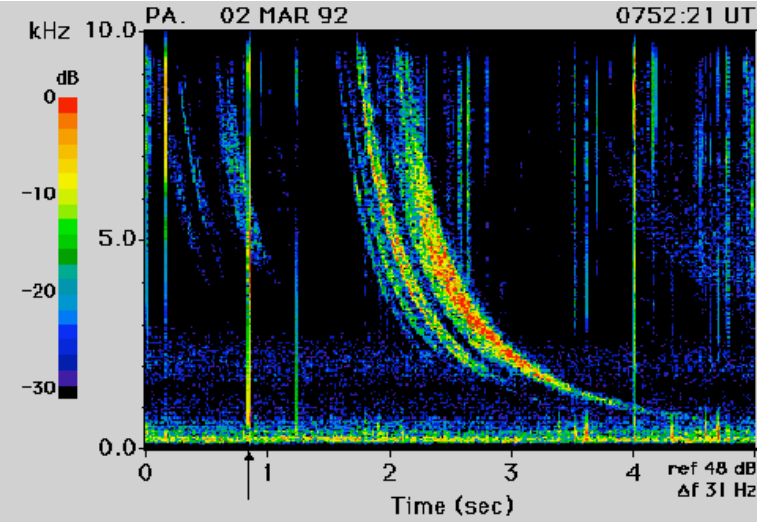


Figure 4.2: : An experimental record of frequency as a function of time showing the presence of whistler waves.

4.5 Whistler Modes

The right-handed electromagnetic wave that we have been discussing arises as a phenomenon in the earth's ionosphere that can actually be 'heard'. If one tunes to very low frequency radio waves – a few kHz up to a maximum of around 100 kHz, one often records a series of descending glide tones, starting at a high frequency, and getting lower and lower over a timescale of a few seconds. The radio frequency is usually sufficiently low that, once amplified, the tones can be heard directly on a loudspeaker (the higher frequency tones can easily be heterodyned if necessary, as happens with normal AM emissions). A typical spectrum of such a whistler signal is shown in Fig. 4.2.

The origin of these waves is shown in schematic form in Fig. 4.3. A lightning strike in one hemisphere creates radio waves across a wide range of frequencies. The right handed component of these waves, launched into the ionosphere and magnetosphere, travel along the earth's magnetic field lines to the other hemisphere, where they are detected. Because of the dispersion relation given in eqn (4.31), different frequencies travel at different group velocities – the higher the frequency the faster the wave. Thus an observer records a descending glide tone. A systematic study of these wave modes can yield valuable information about the conditions in the magnetosphere.

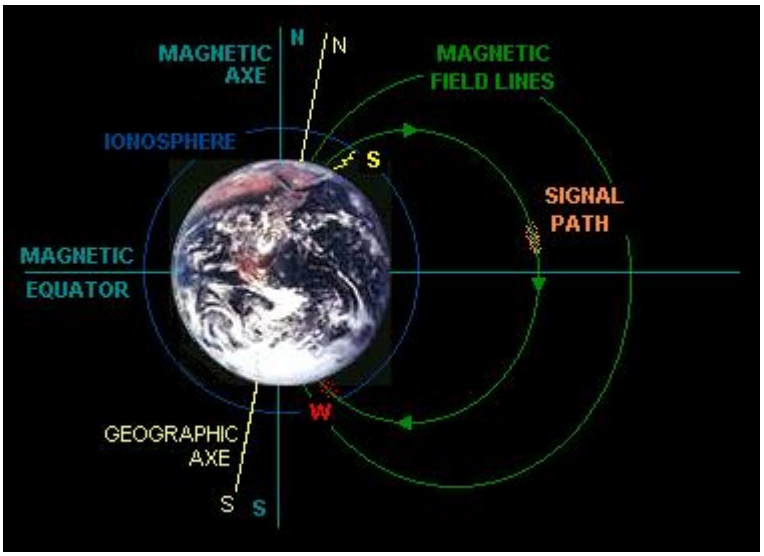


Figure 4.3: : Whistler waves are caused by lightning strikes in one hemisphere creating a spectrum of waves that travel along the field lines to the opposite hemisphere.

4.6 Solutions for waves propagating perpendicular to the B field.

By now you will have no doubt realized that the addition of a magnetic field gives rise to an extremely rich range of wave phenomena in plasmas. We have just looked at the waves that can propagate parallel to the magnetic field: the electrostatic plasma wave, the left-handed electromagnetic wave, and the lower branch (whistler mode) and upper branch of the right-handed electromagnetic wave. You will probably be relieved to learn that for the sake of brevity (this is, after all, called a short option), we will not explore in detail (and nor will you be expected to know about) the waves that propagate perpendicular to **B**. That said, it is worthwhile pointing out how one would find these waves. We would retrace all we have done in the previous section, but instead of setting $N_x = N_y = 0$, and keeping N_z non-zero, we would choose either N_x *or* N_y to be non-zero, and set the other components of **N** to be zero. We would then proceed with the whole analysis once more. Notice that we can pick either N_x or N_y to be non-zero, as there should be no physical difference between them.

Such a procedure does indeed lead to new electromagnetic modes – a so-called O-mode (ordinary), and two extraordinary (X-mode) branches which you can learn about in any standard textbook on plasma physics.

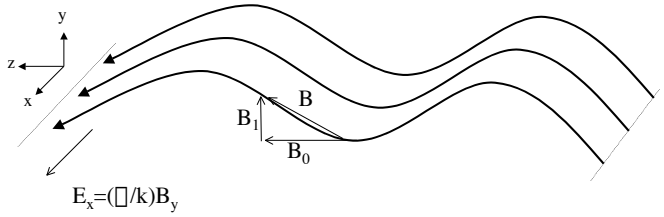


Figure 4.4: : Alfvén waves correspond to oscillations of the magnetic field as well as the particles – the particles appear to be 'embedded' in the field.

4.7 Alfvén Waves

Towards the end of section 4.4, when discussing the right-handed electromagnetic wave at very low frequency, I warned that the frequency could get so low that we would need to take into account the response of the ions. How do we do this? Well, it's really not too difficult. As ions are charged particles, they too will have a 'plasma' frequency, and a cyclotron frequency. The cyclotron frequency of an ion, $\omega_{\rm ci}$ is much lower than that of an electron, because of the higher mass. It is also of opposite sign to the cyclotron frequency of an electron, as it is of opposite charge (see the definition of cyclotron frequency in lecture 2). In order to work out the ion plasma frequency we note that if we have n electrons, and the ions are of charge Z, then the number of ions is $n_i = n/Z$. Therefore the ion plasma frequency is

$$\begin{aligned}\omega_{\rm pi}^2 &= \frac{(n/Z)(Ze)^2}{\varepsilon_0 M} \\ &= \frac{nZe^2}{\varepsilon_0 M} \quad .\end{aligned} \tag{4.37}$$

Note that from now on we will denote the normal electron plasma frequency by $\omega_{\rm pe}$, to distinguish it from the ion plasma frequency.

Now, the ions must move in an electric field, just like the electrons, and thus will give rise to a conductivity, and hence contribute to the dielectric tensor – but the only difference is that their cyclotron frequency is the opposite sign to the electron cyclotron frequency. All this will do to the expressions we have derived above is to add an extra term. That is to say we can take ion motion

into account just be redefining ε_1, ε_2 and ε_3:

$$\varepsilon_1 = 1 + \frac{\omega_{\rm pe}^2}{\omega_{\rm ce}^2 - \omega^2} + \frac{\omega_{\rm pi}^2}{\omega_{\rm ci}^2 - \omega^2} \tag{4.38}$$

$$\varepsilon_2 = \frac{\omega_{\rm ce}}{\omega}\frac{\omega_{\rm pe}^2}{\omega_{\rm ce}^2 - \omega^2} - \frac{\omega_{\rm ci}}{\omega}\frac{\omega_{\rm pi}^2}{\omega_{\rm ci}^2 - \omega^2} \tag{4.39}$$

$$\varepsilon_3 = 1 - \frac{\omega_{\rm pe}^2}{\omega^2} - \frac{\omega_{\rm pi}^2}{\omega^2} \tag{4.40}$$

We will investigate the implications of this for the very low frequency part of the right handed wave. We don't really need to go through all of the algebra – it should be clear that including the ions just changes the dispersion relation we found in eqn (4.31) to

$$\begin{aligned} n^2 = \frac{c^2k^2}{\omega^2} &= 1 - \frac{\omega_{\rm pe}^2}{\omega(\omega - \omega_{\rm ce})} - \frac{\omega_{\rm pi}^2}{\omega(\omega + \omega_{\rm ci})} \\ &= 1 + \frac{\omega_{\rm pe}^2 + \omega_{\rm pi}^2}{(\omega_{\rm ce} - \omega)(\omega_{\rm ci} + \omega)} \quad , \end{aligned} \tag{4.41}$$

where we have made use of the easily-verifiable relation

$$\frac{\omega_{\rm pe}^2}{\omega_{\rm ce}} = \frac{\omega_{\rm pi}^2}{\omega_{\rm ci}} \quad . \tag{4.42}$$

If we consider eqn (4.41) at very low frequencies, recognizing that the electron plasma frequency is always far greater than the ion plasma frequency (electrons are lighter than ions), we find that the low frequency limit is

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{\omega_{\rm pe}^2}{\omega_{\rm ce}\omega_{\rm ci}} \quad . \tag{4.43}$$

Substituting the values of $\omega_{\rm pe}$, $\omega_{\rm ce}$ and $\omega_{\rm ci}$ into eqn (4.43) we obtain

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{n_i\mu_0 M c^2}{B^2} \quad . \tag{4.44}$$

The quantity $B^2/n_i\mu_0 M$, which is the ratio of twice the energy density in the magnetic field to the mass density of the plasma, has the dimensions of velocity squared. We therefore define a velocity, $V_{\rm A}$, called the Alfvén velocity,

$$V_{\rm A} = \sqrt{\frac{B^2}{n_i\mu_0 M}} \quad , \tag{4.45}$$

such that eqn (4.44) can be written

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{c^2}{V_{\rm A}^2} \quad , \tag{4.46}$$

which, upon rearrangement can be written

$$\frac{\omega}{k} = \frac{V_{\rm A}}{\sqrt{1+V_{\rm A}^2/c^2}} \approx V_{\rm A} \quad , \tag{4.47}$$

as typically $V_{\rm A} \ll c$. Thus these low frequency waves propagate with the Alfvén velocity, and are, unsurprisingly, known as Alfvén waves. If we did the analysis for the left-handed waves, we would also find that at very low frequencies a new branch appears with exactly the same velocity (and this new branch exists up to the ion cyclotron frequency, $\omega_{\rm ci}$). Thus, in practice a combination of the two means we can have linearly polarized Alfvén waves.

It is instructive to try and get some sort of physical feel for the nature of these waves. Remember that they are low frequency electromagnetic waves, with a k-vector pointing along the background magnetic field, but oscillating electric and magnetic fields perpendicular to this $z-$axis. Let us suppose that we deal with a linearly polarized wave, with the oscillating B field along y, and thus the electric field along x. When a small oscillating B_y field in the y direction is superimposed on the background field (which we shall call $\mathbf{B_0}$), then the overall magnetic field appears rippled, as shown in Fig. 4.4. As the phase velocity of the wave is ω/k, the ripple moves up and down, with a velocity $(\omega/k)(|B_y/B_0|)$. As this is an electromagnetic wave, there is an oscillating electric field along the x direction. The magnitude of this oscillating electric field can be related easily to the magnitude of the oscillating magnetic field:

$$\nabla \times \mathbf{E} = \dot{\mathbf{B}} \quad , \quad \therefore E_x = \frac{\omega}{k} B_y \quad . \tag{4.48}$$

So we see that the magnetic field lines move up and down with a velocity $|E_x/B_0|$.

Now consider what is happening to the plasma particles. Because there is an oscillating electric field, at any instant the particles themselves are moving with a drift velocity (see lecture 3)

$$\mathbf{v}_{\rm d} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{E_x}{B_0} \quad . \tag{4.49}$$

Thus we conclude that in these low frequency Alfvén waves the plasma particles move with exactly the same velocity as the ripples in the field lines. It is as though the plasma itself was 'stuck' to the field lines – and when the field lines are 'plucked' to make them oscillate, the plasma itself moves completely in sympathy.

4.8 Summary of Lecture 4

1. In the presence of a magnetic field, the dielectric response of the plasma must be represented by a tensor. To find the allowed waves, we must solve

$$(\omega^2 \boldsymbol{\varepsilon} - c^2 k^2)\mathbf{E} + c^2 \mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0 \quad , \tag{4.50}$$

   where $\boldsymbol{\varepsilon}$ is now a tensor.

2. By considering the velocities of the electrons, we deduced that the tensor was given by

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \quad , \tag{4.51}$$

where

$$\varepsilon_1 = 1 + \frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2 - \omega^2} \tag{4.52}$$

$$\varepsilon_2 = \frac{\omega_{\rm ce}}{\omega} \frac{\omega_{\rm p}^2}{\omega_{\rm ce}^2 - \omega^2} \tag{4.53}$$

$$\varepsilon_3 = 1 - \frac{\omega_{\rm p}^2}{\omega^2} \tag{4.54}$$

3. We solved for the waves that propagated along the z axis – parallel to the magnetic field. The electrostatic wave, $\omega = \omega_{\rm p}$ was unchanged, but the electromagnetic wave was now left and right circularly polarized (electromagnetic waves also propagate perpendicular to the magnetic field, but we skipped over those for the sake of brevity).

4. The left hand wave had a dispersion relationship

$$n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega + \omega_{\rm ce})} \quad , \tag{4.55}$$

and did not exist below the cutoff frequency

$$\omega_{\rm LC} = \frac{-\omega_{\rm ce} + \sqrt{\omega_{\rm ce}^2 + 4\omega_{\rm p}^2}}{2} \tag{4.56}$$

5. The right hand wave had the dispersion relationship

$$n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega - \omega_{\rm ce})} \quad , \tag{4.57}$$

and exists up to $\omega_{\rm ce}$, where there is a resonance, and then does not exist again until it attains a frequency of

$$\omega_{\rm RC} = \frac{\omega_{\rm ce} + \sqrt{\omega_{\rm ce}^2 + 4\omega_{\rm p}^2}}{2} \tag{4.58}$$

The lower frequency portion, below $\omega_{\rm ce}$, gives rise to the phenomenon of whistler waves.

6. At extremely low frequencies we need to include the response of the ions. We then find at very low frequencies the phenomenon of Alfvén waves: the

field lines and the plasma itself oscillate in frequency, with a dispersion relation

$$\frac{\omega}{k} = \frac{V_{\rm A}}{\sqrt{1 + V_{\rm A}^2/c^2}} \approx V_{\rm A} \quad , \tag{4.59}$$

where

$$V_{\rm A} = \sqrt{\frac{B^2}{n_i \mu_0 M}} \quad . \tag{4.60}$$

Lecture 5

Astrophysical Plasmas

5.1 Introduction

It is bad enough having to try to compress the whole of plasma physics into twelve, short lectures, without then being given the additional problem of trying to say something meaningful about astrophysical plasmas in just one, or at most two, lectures. However, we clearly cannot go through the course without mentioning something about plasmas and the wider cosmos, especially given that, as far as we know (and for the moment ignoring dark matter, dark energy, and other dark horses) the vast majority of the visible universe is in the plasma state.

Given these constraints what I am going to attempt to do is to pick three basic, but related, issues in plasma physics just to give you a flavour of how we can apply some of the things we have learnt. Along the way I would like to say something more about the statistical mechanics of plasmas, lest you finish the course with the completely misleading impression that, armed with the Saha and Boltzmann equations, you can work out the degree of ionization and state of excitation of atoms within 'real' plasmas.

What we are going to do is the following: I am going to show you how we can work out (a) the electron number density in the interstellar medium then (b), using this number density, we shall use our knowledge of waves travelling through plasmas to work out the distance to a pulsar and (c) armed with the two pieces of knowledge above, we will work out the approximate strength of the galactic magnetic field. I should stress that the beginning of this chapter that I am not an astrophysicist, so I will not be going into great detail in any of these topics.

5.2 Different Types of Equilibrium

Up until now the only physics I have mentioned which was related to the ionization state of a plasma was the Saha equation. One might at first imagine

that that is all you need to know, because the Saha equation tells you the degree of ionization of a plasma at for a given density and temperature, and then you might think that you could work out the populations of the excited atomic states in a given ionization stage simply by applying Boltzmann's equation (i.e $n_2/n_1 = (g_2/g_1)\exp(-(E_2 - E_1)/kT)$). However, unfortunately, life is not that simple

5.2.1 Thermodynamic Equilibrium

The problem with the Saha equation for the ratio of ionization stages and Boltzmann's equation for the ratio of excited state populations to ground state populations is that these equations are only really vaid for plasmas in complete thermodynamic equilibrium, A system in thermal equilibrium is one within which the statistical mechanics of the atoms, ions *and* the radiation are describable by a single temperature, T over a the space comprising the system. In this case the radiation is blackbody, the ionization states of the atoms follow the Saha equation, and the populations of the energy levels obey the Boltzmann equation. It is important to emphasize that this is a highly idealized situation which is almost impossible to *observe* in practice, even though approximations to it might *exist* in certain places. It is not difficult to convince oneself that TE is hard to achieve for a whole system: most systems have a boundary (the thing we look at), and radiation is escaping from this boundary to our detector, beit our eye or a spectrometer. There will always, therefore, be a region of plasma at the edge of the system which is radiating photons to the surrounding, but *not* receiving any from the other side of the boundary. This edge region will therefore, in practice, be cooler than the rest of the system, such that it is not possible to describe the *whole* system by a single temperature, and thus, strictly speaking, TE does not apply. However, within the system, far from the boundary, it may be possible to approximate the physics by the TE approximation as long as the system is large enough that it is optically thick over all photon energies – that is to say, whatever the energy of the photon emitted within the plasma, it will more than likely be re-absorbed before it has much of a chance of getting to the boundary and escaping. Of course, we normally observe the radiation from the boundary. For example, when looking at the optical radiation from the sun, it looks blackbody in form, but the corresponding temperature is that of the part of the sun about 1 optical depth deep – there will be a thin region closer to us which is slightly cooler.

For this reason one of the best methods of making quasi-blackbody radiation is by use of a *hohlraum* – the heated box with a hole in it. In this case the vast majority of the photons leaving a point on the inside surface of the box hit another part of the inside surface, thus keeping the photons in thermal equilibrium with the atoms making up the side of the ox. Only a very, very few escape from the box through the hole, and thus the cooling of the inside surface due to loss of photons is negligible – its a bit like making a system with an artificially small surface.

5.2.2 Local Thermodynamic Equilibrium

Another term that we often use (as an approximation) to describe a system is to say that it is in local thermodynamic equilibrium (LTE). Within this approximation we assume that the ionization states and populations of the atoms are described by the Boltzmann and Saha formulae, but using a *local* temperature, and that temperature can vary slowly as a function of space throughout the system. Furthermore, we assume that the emissivity and opacity of the material can also be described by the same local temperature at each point. However, that is *not* to say that the radiation field is Planckian locally, as we allow the radiation field to deviate from a thermal field. Essentially what we mean by this is that the system could be optically thin – i.e. emitting line radiation, rather than blackbody radiation. The obvious question is if radiation is escaping easily from the plasma, so that the radiation is no longer thermal (blackbody), how can the energy levels be populated according to Boltzmann's law. That is the subject of the next section.

5.3 Collisional Rates

How can we achieve the situation where the emissivity and opacity are determined by a local temperature, as in LTE, when the radiation field is not necessarily Planckian? We require the populations of the levels to be determined by the laws of equilibrium statistical mechanics – i.e. the levels must be populated according to the Boltzmann distribution. The way this comes about is by collisional processes exciting and de-exciting the levels. As long as the collisional transitions between the levels, populating and de-populating them, are much faster than any of the radiative rates (i.e the spontaneous emission determined by the Einstein A value, and the absorption and stimulated emission determined by the B values), then the levels will still be well described by Boltzmann's law.

You will recall that in your statmech and/or atomic course you worked out the relationship between the Einstein A and B coefficients by employing the *principle of detailed balance* – the upward radiative rates had to equal the downward radiative rates. The same sort of procedure applies for the collisional rates. Consider both collisional and radiative processes taking place in a two-level system as shown in Fig. 5.1. We assume that the collisions that are populating and depopulating the levels are due to electrons. Clearly the rate at which a level is populated will now depend on the electron density, N_e. The collisional rate coefficient, C, will clearly also be some function of temperature, and will somehow be related to the atomic physics of the levels. For now we note that the rate of populating a level is denoted by CN_e. Thus in complete thermal equilibrium we would write

$$N_1\rho(\omega_{1,2})B_{1,2} + N_1N_eC_{1,2} = N_2A_{2,1} + N_2\rho(\omega_{1,2})B_{2,1} + N_2N_eC_{2,1} \quad , \quad (5.1)$$

where $\rho(\omega_{1,2})$ is blackbody radiation at a temperature T. Now eqn (5.1) must hold over all possible conditions of density and temperature – and this is the

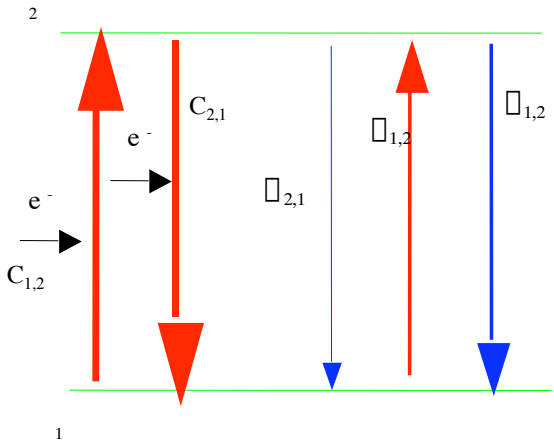


Figure 5.1: In a two-level system the levels can be populated by both collisional and radiative processes.

important point. Notice, therefore, that if we assume extremely low electron densities, at some point we much get into a regime when the collisional terms, those like $N_1 N_e C_{1,2}$, are negligible compared with all of the radiative rates. In that case, we are back to the situation you have seen before where you derived the relationship between the Einstein A and B coefficients, and we see that the radiative rates on their own must be in balance with each other. We could, of course, go to the opposite extreme – that is to say consider what happens at very high electron densities. There must come a point where the collisional rates, both up and down, must dominate over all of the radiative rates, and thus we see that the collisional rates must be in balance on their own as well, that is to say we conclude

$$N_1 N_e C_{1,2} = N_2 N_e C_{2,1} \quad , \tag{5.2}$$

and thus

$$\frac{C_{1,2}}{C_{2,1}} = \frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left(-\frac{\Delta E_{1,2}}{k_{\rm B} T}\right) \quad . \tag{5.3}$$

This is the relationship between the upward and downward collisional rates, in the same fashion as we derived the relationship between the A and B coefficients. Of course, it doesn't tell us what the rates actually are, no more than we yet know how to calculate A (or B) – but it does tell us that once we calculate the upward rate, we know the downward rate (or vice versa).

We are now in a position to understand the conditions that underly the LTE approximation. As long as the system is dense enough the only rates that will really matter will be the collisional ones, and these rates will keep the levels populated according to Boltzmann, and thus the emissivities and

opacities (which are a function of the populations) will be determined by the same temperature as that of the atoms/ions. The fact that the radiation field is weak does not alter the populations. Note that LTE could also apply within a very large plasma if the density is low and the collisional rates are very small. In this case, the plasma is so large that the radiation field builds up and up, until it becomes blackbody, and now it is the radiation that will drive the system into have populations determined by Boltzmann (though of course this cannot occur at the boundaries where the radiation escapes, as there is no inverse process to the emission at that point).

5.3.1 Coronal Equilibrium

The beauty of LTE is that it enables us to calculate relative populations of the atomic states without any knowledge of the cross-sections or transition probabilities (although radiative rates are required to derive the radiation field). The next most important form of equilibrium after LTE is known as coronal equilibrium (so-called because it is applicable to the sun's corona) and is relevant when the temperature is high and the density very low (for the sun's corona these numbers are about 100 eV and 10^8 cm^{-3}) *and* the system sufficiently optically thin that the radiation escapes easily. This approximation also works reasonably well far out in a laser-produced plasma, well below the critical density (which is why this region is known as the coronal region), and (as we shall see) also works well in the interstellar medium, which maybe not that hot, but is of such a low electron density that the approximation still holds. The radiation density in this case is also low (as the plasma is assumed to be optically thin), and it is thus assumed that upward transitions in a given ion are due to electron collisions, whereas the downward transitions are due to spontaneous radiation – the upward transitions are due to collisions because radiation escapes the plasma before it reaches sufficient intensity to cause pumping of the level, but the downward rate is assumed to be due to radiation because the density is too low for the collisional rate to be important. The steady-state balance between the two allows the excited-state population fractions to be calculated. Obviously in this case (in contrast to LTE) we need to know what the rates actually are. For example, consider two atomic levels within a particular ionization stage, and we label the upper level as 2,and the lower level as 1. Let the downward radiative rate be A_{21}, the upward collisional rate C_{12}, and the downward collisional rate C_{21}. If the electron density is N_e then the approximation of coronal equilibrium boils down to the statement

$$A_{21} \gg N_e C_{21} \quad , \tag{5.4}$$

and we equate the upward and downward rates according to

$$N_2 A_{21} = N_1 N_e C_{12} \quad , \tag{5.5}$$

so that

$$\frac{N_2}{N_1} = \frac{N_e C_{12}}{A_{21}} \quad . \tag{5.6}$$

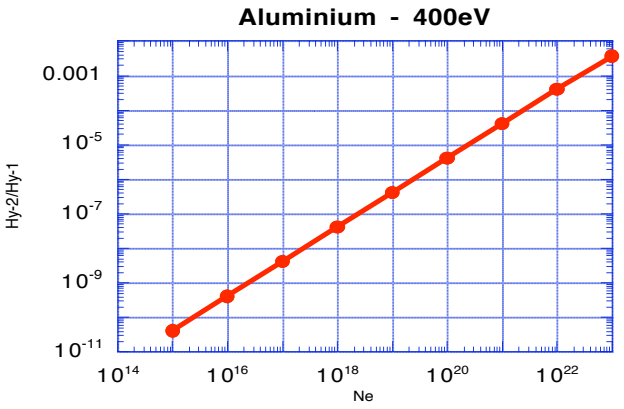


Figure 5.2: Ratio of the population of hydrogenic aluminium ions in the state $n = 2$ to those in the state $n = 1$ at a temperature of 400 eV as a function of electron density (in units of cm^{-3}).

We notice two things about the population ratio of the upper to lower state. Firstly, it is proportional to the electron density. Detailed calculations support this analytic result – in Fig. 5.2 we plot the ratio of the population of the $n = 2$ level hydrogenic aluminium to the ground state, $n = 1$, at a temperature of 400 eV. Secondly it should be clear that the ratio must be less than the Boltzmann ratio for the local temperature. We can see this by noting that to have the Boltzmann ratio, by the principle of detailed balance, the denominator on the right hand side of eqn (5.6) would need to be $N_e C_{21}$, but the whole assumption of coronal equilibrium, i.e. eqn (5.4) is that A_{21} is much bigger than this rate. The Boltzmann ratio for the states shown in Fig. 5.2 at 400 eV would be 0.054.

5.4 Non-LTE

LTE and Coronal equilibrium are obviously two limits of a general case where collisional and radiative rates compete – the general case being known as non-LTE (or collisional-radiative). In this general case we must take into account all of the relevant rates, and solve for the steady-state ionization balance and excited state populations. This is a complicated procedure that obviously must be performed computationally. What one must do in effect is solve a set of coupled equations, such as eqn (5.1), between all possible levels (taking into account ionization and recombination rates into the continuum, for example). However, as the system may well be optically thin (or thick, depending on the transition of interest), one must attempt to find a method of keeping track of the

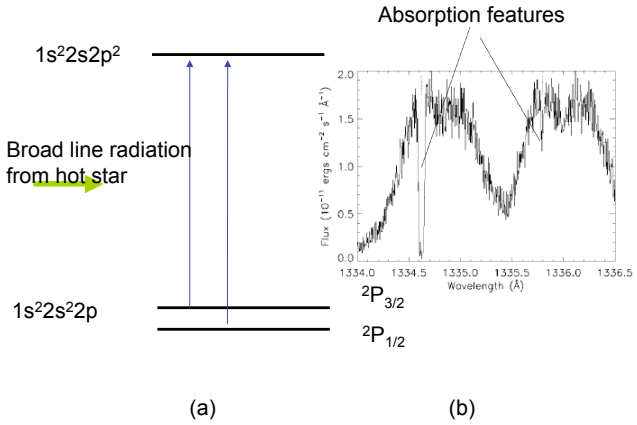


Figure 5.3: (a) Line radiation from a hot star (so that the lines are Doppler broadened) excites relatively cold (CII) ions from their ground state producing (b) the absorption spectrum. As the electrons in the fine structure states are excited to the same upper state, a measurement of the relative absorbed radiation provides a measurement of the ratio of the populations in the fine structure levels.

effect of the radiation on the populations. Solving the problem self-consistently – i.e. having complete consistency between the populations and the computed radiation fields is a very difficult task, and often approximations are made. The best known approximation being the use of an 'escape factor'. In this approximation, we note that as the thickness of the plasma is increased for a given density, the probability that a photon will be reabsorbed before escaping from the plasma increases. One of the simplest (though approximate) methods of dealing with this is to treat this reabsorption by reducing the effective A-value.

5.5 The Interstellar Electron Number Density

We are now in a position to discuss one method which is used by astrophysicists to determine the interstellar electron number density. We saw in eqn (5.6) that the ratio of the populations in two states for coronal equilibrium is proportional to the electron number density – this is the key to the method. The interstellar medium is so rarefied, and the radiation field so weak, that coronal equilibrium is a very good approximation.

I am going to illustrate the technique by reference to measurements that have been made using singly-ionized Carbon (CII). The configuration for CII is, as one would expect, $(1s^2\ 2s^2\ 2p)$. As the final electron is a p electron this

ground state is going to be split in two by fine structure – i.e the true ground state has $J = 1/2$, and just above it, split by fine structure, is going to be the $2p$ state with $J = 3/2$. As the first state above the true ground state is only slightly above it in energy (fine-structure splitting), there will be an appreciable fraction of CII ions in this state. In fact the number will be determined by eqn (5.6) – the collision rate in this case will be high, because there is little energy to be transferred between the two states. What this means, then, is that if we can measure the populations in the $2p_{1/2}$ and $2p_{3/2}$ states, we would have a measurement of N_e (because the A rate can be calculated from atomic physics, and so can the collisional rate – the collisional rate is actually a function of temperature, but a sufficiently weak function that it is know with reasonable confidence).

The way that the ratio of the $2p$ populations is found is via absorption spectroscopy, as shown in schematic form in Fig. 5.3. A spectrometer is directed towards a star or other object which is emitting on the resonance line of CII (the first true excited state, i.e. neglecting the fact that the 'ground' state is split by fine structure, has configuration ($1s^2\ 2s\ 2p^2$)). The energy difference between the two fine-structure levels in the ground configuration and this upper level is such that the transitions occur at 1335 and 1336 Å. The emission lines from the star are fairly broad, as the star is hot, and the lines are therefore Doppler broadened. As this emission passes through the interstellar medium it encounters a much colder region, which is also very rarefied. However, the temperatures are such that it is still possible to have quite a few singly-ionized Carbon atoms in this region, though of course they pretty much all be in the ground state. Thus the radiation from the star is reabsorbed by the ground state CII ions in the interstellar region, and gives rise to absorption features, such as those shown in Fig. 5.3(b). [1] Note that these absorption features correspond to transitions from the two lower fine-structure levels, i.e. the $1s^2\ 2s^2\ 2p_{1/2}$ and $1s^2\ 2s^2\ 2p_{3/2}$ states up to *the same* upper level with the configuration $1s^2\ 2s\ 2p^2$. We know the A values for these transitions, and so the relative strength of the transitions tells us the ratio of the populations in the fine-structure levels (knowing the A and C values between them) – exactly what we need to determine N_e.

Of course, in practice it is found that the number varies depending upon where you look, and there is more than one type of atom which you can use to make the measurement, but when all is said and done, it is found that the typical electron density in the interstellar medium seems to lie between 10^4 and 10^5 m^{-3}.

5.6 Distance to Pulsars

Although radio emission from pulsars was first discovered thirty years ago, the mechanism which creates the radiation is not well understood. It is generally believed that the high brightness temperature, which is characteristic of pulsar

[1] See Brian Wood and Jeffrey Linsky, The Astrophysical Journal, 474, L39-L42, 1997.

pulses, must be the result of a coherent emission process. Relativistic plasma emission mechanisms in which waves are generated through an instability and converted into escaping radiation are one of the favoured emission mechanisms.

We can use what we have already learnt in chapter 2 to make estimates of the distance between the earth and a pulsar – or, more strictly speaking we can calculate the product of that distance and the average electron number density, $n_{\rm e}$, found within the interstellar medium. However, we have just seen there are other methods for making reasonable estimates of $n_{\rm e}$, and so we can be left with an estimate of the earth-pulsar separation.

We recall from equation 2.29 section 2.2 that

$$\epsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega^2} \quad , \tag{5.7}$$

or, another way of putting this is to state the dispersion relation

$$\omega^2 = \omega_{\rm p}^2 + c^2k^2 \quad , \tag{5.8}$$

as $\epsilon = n^2 = c^2k^2/\omega^2$, where n is the refractive index.

As the medium is dispersive, waves of different frequencies travel at different group velocities, and we can use this to determine distances. For example, the pulses of radiation emitted from radio pulsars can be very short indeed, and this in turn means that the pulse must contain a range of radio frequencies (the spread of frequencies, $\Delta\omega$, must exceed the transform limit $1/\tau$, where τ is the duration of the pulse). These different frequency components of the pulse travel through the interstellar medium, which has a frequency-dependent refractive index, and the frequency components arrive at different times, with the high frequency component arriving first – that is to say we find on earth that the pulse appears to be chirped.

The mathematics of this is very simple. Let us consider the group velocity, $v_{\rm g}$ of an electromagnetic wave travelling through the plasma:

$$v_{\rm g} = \frac{{\rm d}\omega}{{\rm d}k} = \frac{1}{2\sqrt{\omega_{\rm p}^2 + c^2k^2}} 2c^2k \tag{5.9}$$

$$= \frac{c\sqrt{\omega^2 - \omega_{\rm p}^2}}{\omega} \quad . \tag{5.10}$$

Now let there be waves of two slightly different frequencies, ω_1 and ω_2, which have two different group velocities, v_2 and v_1.

$$v_2 - v_1 = c\left(\frac{\sqrt{\omega_2^2 - \omega_{\rm p}^2}}{\omega_2} - \frac{\sqrt{\omega_1^2 - \omega_{\rm p}^2}}{\omega_1}\right) \tag{5.11}$$

$$\approx c\left[\left(1 - \frac{\omega_{\rm p}^2}{2\omega_2^2}\right) - \left(1 - \frac{\omega_{\rm p}^2}{2\omega_1^2}\right)\right] \tag{5.12}$$

$$= \frac{\omega_{\rm p}^2 c}{2}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) \tag{5.13}$$

where we have assumed $\omega_1, \omega_2 \gg \omega_{\rm p}$ (i.e. we are dealing with a rarefied plasma). We define

$$\Delta\omega \equiv \omega_2 - \omega_1 \tag{5.14}$$

and

$$\omega = \frac{\omega_1 + \omega_2}{2} \quad , \tag{5.15}$$

and then eqn (5.13) can be written

$$v_2 - v_1 = \frac{\omega_{\rm p}^2 c}{2}\left[\frac{1}{(\omega - \frac{\Delta\omega}{2})^2} - \frac{1}{(\omega + \frac{\Delta\omega}{2})^2}\right] \tag{5.16}$$

$$= \frac{\omega_{\rm p}^2 c}{2}\left[\frac{(\omega + \frac{\Delta\omega}{2})^2 - (\omega - \frac{\Delta\omega}{2})^2}{(\omega - \frac{\Delta\omega}{2})^2(\omega + \frac{\Delta\omega}{2})^2}\right] \quad . \tag{5.17}$$

As $\Delta\omega$ is small, we neglect its higher order terms to find

$$v_2 - v_2 \approx \frac{\omega_{\rm p}^2 c}{2}\left(\frac{2\omega\Delta\omega}{\omega^4}\right) \tag{5.18}$$

$$= \frac{\omega_{\rm p}^2 c\Delta\omega}{\omega^3} \quad . \tag{5.19}$$

This is the expression we have been seeking - an expression for the difference in group velocity for two closely spaced frequencies as they travel through a medium with (relatively low) plasma frequency.

Now consider our pulsar emitting a pulse of radiation at time $t = 0$. That pulse contains a range of frequencies; let us pick two of them, ω_2 and ω_1, and find out the difference in their time of arrival, $t_1 - t_2$ at the earth (which will be the measured quantity). Let us assume that the pulse travels through the interstellar medium, which has an overall average plasma frequency (due to the free electrons in the medium) $\omega_{\rm p}$. If the distance between the earth and the pulsar is L, then

$$t_1 - t_2 = \frac{L}{v_1} - \frac{L}{v_2} = L\left(\frac{1}{v_1} - \frac{1}{v_2}\right) = L\frac{v_2 - v_1}{v_1 v_2} \quad . \tag{5.20}$$

Now, as $\omega \gg \omega_{\rm p}$, $v \approx c$, and thus we find

$$L \approx \frac{\Delta t c^2}{(v_2 - v_1)} \quad . \tag{5.21}$$

Substituting eqn (5.19) into eqn (5.21) we obtain

$$L = \frac{\Delta t c \omega^3}{\omega_{\rm p}^2 \Delta\omega} \quad . \tag{5.22}$$

For example, there is a particular pulsar that emits radiation at a frequency of order 1400 MHz. Over a spread in frequencies of about 40 MHz there is a

time delay of about 20 ms in the arrival time of the pulses. In order to get a distance from this, we need to estimate $\omega_{\rm p}$, which we can do from estimates of the free electron number density in the interstellar medium, and typical figures for this are about 3×10^4 m^{-3} (as we saw from absorption spectroscopy). You will find that if you insert these numbers into eqn (5.22), then you will conclude that L is about 1.7×10^{19} m, which is about 1800 light years.

5.7 The Galactic Magnetic Field

Within the Milky Way there is a weak and largely distorted magnetic field which has a strength of about 5×10^{-10} T. The obvious question is how do we know. The answer is once more related to pulsars, and to our knowledge of the electron density in the interstellar medium. Although there will not be space to go into detail, one thing you need to know about the emission of radiation from pulsars is that it often has a degree of linear polarization. The reason for this is presumably that much of the emission is coming from the slowing down of relativistic jets of particles as they themselves traverse magnetic fields. It is this polarized light that we see on earth. However, as it travels through the interstellar medium, not only do the different frequencies travel at different velocities, as we have seen a above, but as the interstellar medium also contains a magnetic field (the origin of which is still an area of very active research), the plane of polarization of the radiation is rotated due to the Faraday effect (which we look at in more detail below). We shall find that by observing the angle of the plane of polarization as a function of the wavelength of the emission, λ, we will be able to make an estimate of the strength of the magnetic field within the galaxy.

5.7.1 The Faraday Effect

As it happens, we have already derived nearly all of the formulae we need to discuss the Faraday effect. You will recall that in lecture 4 we discussed waves in a cold magnetized plasma (cold in the sense that kv_{th} is small compared with $\omega_{\rm pe}$). For waves travelling parallel to the magnetic field (which is the case we will consider here) we found in eqns (4.30) and (4.31) two possible solutions, with refractive indices given by

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega \pm \omega_{\rm ce})} \quad , \tag{5.23}$$

and that the solution with the positive sign in the denominator was a left-handed circularly polarized wave, whilst that with the negative sign was a right-handed circularly polarized wave.

Now the Faraday effect, you will recall, arises because we can break down a linearly polarized light wave into equal amplitude left and right circularly polarized components, but if these two components travel through a medium where each component has a different refractive index, they travel at different

velocities, so the plane of polarization of the linearly polarized light will rotate. That is exactly what goes on in the rotation of the polarization of the linear light emitted by the pulsar.

Let us assume once more that the refractive indices are close to unity – that is to say again we assume that we are dealing with a tenuous plasma where both $\omega_{\rm ce}$ and $\omega_{\rm p} \ll \omega$. Then the refractive index for the left-handed, n_+, and right-handed wave, n_- can be approximated via a binomial expansion as

$$n_\pm \approx 1 - \frac{\omega_{\rm p}^2}{2\omega^2}\left(1 - \pm\frac{\omega_{\rm ce}}{\omega}\right) \quad . \tag{5.24}$$

Now the left-handed wave has x and y components of the electric field that are related by

$$E_{x({\rm LH})} = E_0 \cos\left(\frac{\omega}{c}(n_+ z - ct)\right) \tag{5.25}$$

$$E_{y({\rm LH})} = E_0 \sin\left(\frac{\omega}{c}(n_+ z - ct)\right) \quad , \tag{5.26}$$

whilst the right-handed wave has x and y components

$$E_{x({\rm RH})} = E_0 \cos\left(\frac{\omega}{c}(n_- z - ct)\right) \tag{5.27}$$

$$E_{y({\rm RH})} = -E_0 \sin\left(\frac{\omega}{c}(n_- z - ct)\right) \quad , \tag{5.28}$$

We recall that linearly polarized light is constructed from adding these two circularly polarized components together – in doing so we should find that the x and y components now have the same time dependence. So, let us add together the two x components, i.e. eqn (5.26) and (5.28) to obtain

$$E_{x(Lin)} = E_0\left[\cos\left(\frac{\omega}{c}(n_+ z - ct)\right) + \cos\left(\frac{\omega}{c}(n_- z - ct)\right)\right] \tag{5.29}$$

$$= 2E_0 \cos\left(\frac{\omega}{c}(\bar{n} z - ct)\right)\cos\left(\frac{\omega}{2c}(n_+ - n_-)z\right) \tag{5.30}$$

where $\bar{n} = (n_+ - n_-)/2$. Similarly when we sum the y components of the two circularly polarized waves we obtain

$$E_{y(Lin)} = E_0\left[\sin\left(\frac{\omega}{c}(n_+ z - ct)\right) - \sin\left(\frac{\omega}{c}(n_- z - ct)\right)\right] \tag{5.31}$$

$$= 2E_0 \cos\left(\frac{\omega}{c}(\bar{n} z - ct)\right)\sin\left(\frac{\omega}{2c}(n_+ - n_-)z\right) \quad , \tag{5.32}$$

where for both components we have made use of the normal trigonometric equations for the sum of cosines and sines. So, as we expect, we get linearly polarized light (both components oscillate as cosine waves), but the angle of the plane of polarization rotates as the wave propagates. The angle, θ, is given by

$$\theta = \tan^{-1}\left(\frac{E_{y(Lin)}}{E_{x(Lin)}}\right) \tag{5.33}$$

$$= \frac{\omega}{2c}(n_+ - n_-)z \quad , \tag{5.34}$$

From eqn (5.24) we see that

$$n_+ - n_- = \frac{\omega_{\rm p}^2 \omega_{\rm ce}}{\omega^3} \quad , \tag{5.35}$$

and substituting this into eqn (5.34) we obtain for the angle of rotation

$$\begin{aligned} \theta &= \frac{\omega_{\rm p}^2 \omega_{\rm ce}}{\omega^3} \frac{\omega}{2c} z & (5.36) \\ &= \left(\frac{n_e e^2}{\varepsilon_0 m_e} \right) \frac{eB}{m_e} \frac{1}{2c\omega^2} z & (5.37) \\ &= \frac{e^3 n_e B}{8\pi^2 \varepsilon_0 m_e^2 c^3} \lambda^2 z \quad , & (5.38) \end{aligned}$$

where λ is the wavelength of the radiation.

The important thing that we notice is that the angle of rotation depends on the electron density (which we will have measured by absorption spectroscopy), the length through which the radiation will have travelled (which we will have measured by the dispersion of the radio waves from the pulsar), and, you've guessed it, the magnetic field – which is what we are after. Now, of course, we don't know at what angle the linearly polarized light was emitted, so how do we know the change in angle? Well, of course, what we do is to measure the angle as a function of λ^2, as it will have rotated a different amount for each wavelength. In astrophysics, all of the factors in front of the λ^2 in eqn (5.38) are known as the rotation measure – that is to say

$$\theta = RM \ \lambda^2 \tag{5.39}$$

where

$$RM = \frac{e^3 n_e B}{8\pi^2 \varepsilon_0 m_e^2 c^3} z \quad . \tag{5.40}$$

RM clearly has units of radians m^{-2}.

We emphasize once more that along with the dispersion measure (DM), discussed in section 5.5, we now have a way to back out the intergalactic magnetic field. I have put a paper on the course website which discusses this procedure for over 200 pulsars,[2] and is the evidence by which we can estimate a mean magnetic field of order 5×10^{-10} T.

It should be noted that Faraday rotation in plasmas can also be seen in other circumstances – for example radio waves passing through the Earth's ionosphere are also subject to Faraday rotation. At 435 MHz (UHF), one should expect in the order of 1.5 complete rotations of the wavefront as it transits the ionosphere. This effect (which tends to vary on the time of day) must be taken into account when measuring the RM due to the galactic field. We also can observe the same

[2] J. L Han, R. N Manchester, G. J Qiao (1999) Pulsar rotation measures and the magnetic structure of our Galaxy Monthly Notices of the Royal Astronomical Society 306 (2), 371380. doi:10.1046/j.1365-8711.1999.02544.x

effect in magnetic confinement fusion plasmas. A quick application of eqn (5.38) will show that one could rotate $\lambda = 0.08$m radiation through $\pi/2$ radians in a plasma of typical fusion type electron densities (say $10^{19}\mathrm{m}^{-3}$) in a distance of about 0.1 m.

5.8 Summary of Lecture 5

1. Real plasmas are often not in complete thermal equilibrium, and as processes are not balanced, the Saha and Boltzmann equations do not hold.

2. For very low density plasmas a further type of equilibrium is called coronal equilibrium. In this case the downward rate is dominated by radiation, and this radiation escapes the plasma. As the radiation field is thus weak, the only thing that can cause excitation is electron-ion collisions. These two processes are in balance, and as a consequence the ratio of the populations of two levels in coronal equilibrium is given by

$$\frac{N_2}{N_1} = \frac{N_e C_{12}}{A_{21}} \quad . \tag{5.41}$$

3. The plasma in the interstellar medium is in coronal equilibrium. For certain systems one can determine the ratio of populations in two closely spaced levels by absorption spectroscopy. The rates can be calculated, leading to a measurement of the electron density, which is typically found to lie in the range 10^4 to 10^5 m^{-3}.

4. A pulse of radio emission from a pulsar is so short that it contains a range of frequencies. The dispersion of the interstellar medium causes these different frequencies to arrive at the earth at slightly different times. For frequencies centred around ω, with difference $\Delta\omega$, the distance to the pulsar as a function of this time difference is

$$L = \frac{\Delta t c \omega^3}{\omega_\mathrm{p}^2 \Delta\omega} \quad . \tag{5.42}$$

   thus a measure of Δt gives the distance to the pulsar, if we know the electron density, e.g. from absorption spectroscopy.

5. The emission from pulsars is also often linearly polarized, and thus can be decomposed into its two constituent circularly polarized components. There is a weak magnetic field throughout the galaxy which magnetizes the interstellar medium, and causes these two components to travel with different velocities: the result is Faraday rotation of the plane of polarization. The angle of rotation scales as λ^2:

$$\theta = RM\ \lambda^2 \tag{5.43}$$

where

$$RM = \frac{e^3 n_e B}{8\pi^2 \varepsilon_0 m_e^2 c^3} z \quad . \tag{5.44}$$

Thus knowing n_e (spectroscopy), and L (dispersion) we can find the galactic magnetic field by measuring the rotation angle as a function of λ. Typical values are of order 5×10^{-10} T.

Lecture 6

Collisions and Waves in Warm Plasmas

6.1 Collisions

In lecture 1 we introduced two of the most important parameters of a plasma: the plasma frequency and the Debye length. When discussing the derivation of the plasma frequency we assumed that we could ignore the collisions between the electrons and the ions, because the time scale for a collision was far greater than the oscillation period. What we will do now is show that this is generally the case (at least for most plasmas of interest). So, we need to derive the collision frequency.

Once again we will consider the ions to be static, and concentrate on the motion of the electrons. To make things as simple as possible we will consider an electron approaching an ion with some velocity, v, due to its thermal motion. Clearly it is attracted to the ion and its path is deviated as shown in Fig. 6.1. You might already have met the scattering of one charged particle by another previously in the physics course – this is simply Rutherford scattering – but let's go through it in sufficient detail to remind ourselves about what is going on.

6.2 Rutherford Scattering

The approach I am going to take to deriving the scattering of the electron by the ion is rather hand-waving – but it will suit our purpose in as much as I hope it will bring out the physics of what is going on, and at the same time get us quickly to an answer which is correct within a constant of order unity. Considering Fig. 6.1 once more, we assume the impact parameter of the electron to be b – by impact parameter we mean the distance of closest approach of the electron to the ion if a force did not act, as shown in the diagram. Clearly

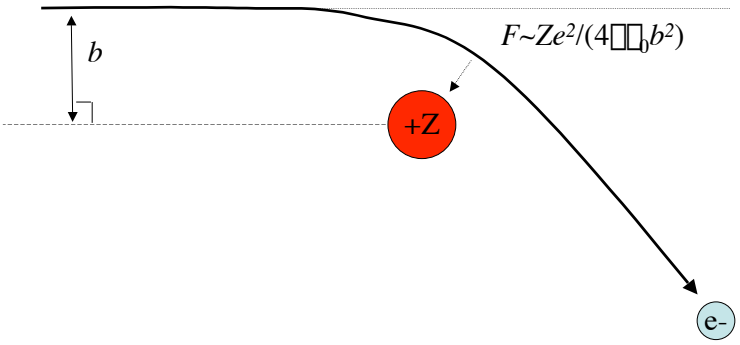


Figure 6.1: : The path of an electron, with velocity v, and impact parameter b, being scattered by an ion of charge Ze. The average force the electron feels during a time $2b/v$ is approximately $Ze^2/(4\pi\varepsilon_0 b^2)$.

the electron will be attracted to the ion, will accelerate towards it, and have its direction changed. It will gain potential energy as it flies past the ion, but as it then heads off in a slightly deviated direction, this potential energy will be converted back to kinetic energy, and overall its final velocity will equal its initial velocity (assuming an infinitely massive ion), though the *direction*, $\Delta\mathbf{v}$ will change. Although we could use our knowledge of mechanics and central force fields to calculate this change exactly for a given v and b, there is a simple method that gets roughly the right answer. The simple method is to note that the deviation comes about because of the force exerted on the electron by the ion. Although this force is constantly changing in both magnitude and direction as the electron flies past, on average it acts in the direction shown in the diagram, and it is appreciable (i.e. a reasonable fraction of its maximum value) when the electron is within about a distance $\approx b$ of the ion. Thus if we wave our hands we can say that the electron feels an average force, F, of magnitude,

$$F \approx \frac{Ze^2}{4\pi\varepsilon_0 b^2} \quad . \tag{6.1}$$

The electron feels this force as it flies past the ion, which takes a time which is approximately $\Delta t \approx 2b/v$ (clearly this is a rough estimate, as it accelerates towards the point of closest approach, and then decelerates afterwards). Anyway, if we take this to be the time for which the force acts, we can immediately work out the change in velocity, Δv, as force equals rate of change of momentum:

$$\begin{aligned} F &= m\frac{\Delta v}{\Delta t} \\ \frac{Ze^2}{4\pi\varepsilon_0 b^2} &= m\Delta v\frac{v}{2b} \quad . \end{aligned} \tag{6.2}$$

Rearranging eqn (6.2) we find

$$\Delta v = \frac{Ze^2}{2\pi\varepsilon_0 mbv} \quad . \tag{6.3}$$

This is roughly the magnitude of the change in velocity of an electron of initial velocity v undergoing Rutherford scattering at an impact parameter b from an ion of charge Z.

6.3 The Collision time and Coulomb Logarithm

Now, how does our analysis of Rutherford scattering help us derive an electron-ion collision time? To answer this question we better make sure we know what we mean by a collision. If the electron has a large impact parameter it is not going to be deviated very much in angle as it flies past the ion. So, is this 'a collision'? Well, again, what do we mean by a collision? When we have met the concept of a collision in kinetic theory it is clear that we mean a significant deviation in the path of the particle. Indeed, when we talk about collisions of electrons in metals determining the resistivity, we think of the electron, having been accelerated in the electric field, coming to rest at the point of collision (with a defect or phonon or whatever). Thus it is clear that what we mean by a collision is the complete deviation of the motion of the particle in its original direction. In a plasma this corresponds not to stopping the particle, but the scattering of it through an angle of $\pi/2$.

However, usually when an electron flies past an ion it is not scattered through $\pi/2$ radians, but through a much smaller angle. Indeed, the chances of being scattered through $\pi/2$ in a single interaction with an ion are usually very small indeed, as this requires a small impact parameter. What actually happens in practice is that the electron is scattered through a small angle by one ion, and then another small angle by another ion, and so on and so on. One might think that this would mean that it never 'collides' at all, because the first scattering might shift it to the left, the next to the right, and so on, so that it never gets scattered through $\pi/2$ (i.e. the average scattering angle is zero). However, although it is indeed true that the average scattering angle is zero, this does not mean that no collisions take place.

Imagine you were walking home from the pub after too much to drink (I recognize that this is a totally hypothetical situation which would never apply to any student, but suspend your disbelief for a while, and stay with me). You stagger from lamp-post to lamp-post, wandering off in slightly the wrong direction each time. It is true that on average your staggering should have no effect, inasmuch as you are just as likely to stagger to the left as to the right, but this does not mean that there is no chance at all of you ending up at some point going at right angles to the way home. Indeed, there will be some average, definable, timescale on which it is likely that you will be heading towards ChristChurch, when you meant to be heading towards Trinity (perish the thought). It is very much like the distribution functions we have

met before in statistical mechanics. In a set of random events the *average* value of a parameter might be zero, but it is the deviation from this average value that is important – i.e. the *root-mean-square* of the value (the average value of a centralized Gaussian is zero, but it is its width that tells you how likely you are to deviate from that average value). The same principle applies here, what we need to do is find the root-mean-square deviation in the velocity. We then try to work out how long it takes for this root-mean-square of the change in the velocity to *equal* the original velocity, as this will correspond to a deflection through an angle of $\pi/2$ – when the particle is deflected through that angle, one component of its velocity has gone from 0 to v, whilst the orthogonal component has undergone the opposite change. This is the manner in which we shall proceed.

Thus, from eqn (6.3),

$$(\Delta v)^2 = \frac{Z^2 e^4}{4\pi^2 \varepsilon_0^2 m^2 b^2 v^2} \quad . \tag{6.4}$$

At what rate does $(\Delta v)^2$ change? We need to know this in order to work out the collision time. Well it changes every time there is a collision, and the rate of encounters is $n_i \sigma v$ where n_i is the ion number density, σ the cross section for the collision, and v the velocity. The cross section, σ, is a function of the impact parameter, b. If the electron has an impact parameter that lies between b and $b + \mathrm{d}b$, then the cross section for this collision must be $2\pi b\ \mathrm{d}b$. That is to say, we conclude

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} < (\Delta v)^2 > &= \int 2\pi b\ \mathrm{d}b\ n_i v (\Delta v)^2 \\ &= \int 2\pi b\ \mathrm{d}b\ n_i v \frac{Z^2 e^4}{4\pi^2 \varepsilon_0^2 m^2 b^2 v^2} \\ &= \int \frac{n_i Z^2 e^4\ \mathrm{d}b}{2\pi \varepsilon_0^2 m^2 v b} \quad . \end{aligned} \tag{6.5}$$

Now, for the moment, we have not put any limits on the integral, and that is what we must now ponder. Let us recall that we are integrating over all of the possible impact parameters available to the electron. One might initially think that these should thus range from 0 to ∞, but this is not the case. If the electron has a very large impact parameter with respect to some particular ion, there may well be some electrons (and or ions) between it and the ion we are considering. In this sense the electric field of the ion we wish to consider will be shielded by all the other charges, and after a certain distance the ion of interest will have little influence on our electron – it would be incorrect to treat it as feeling the full Coulomb force of that ion. What is this distance beyond which the Coulomb field is not felt as strongly? Well, we have already met it in our very first lecture – yes, it's the Debye length, λ_D, where

$$\lambda_\mathrm{D} = \sqrt{\left(\frac{\varepsilon_0 k_\mathrm{B} T}{n_0 e^2} \right)} \quad . \tag{6.6}$$

So the Debye length represents the largest possible meaningful, impact parameter. What about the smallest possible impact parameter, $b_{\min}$? Notice that in eqn (6.3) our approximations give $\Delta v \propto b^{-1}$, thus in our approximation the change in magnitude of velocity would be infinite for an electron incident head-on to an ion: that can't be correct – we know that it will simply return back along its original path, and the magnitude in the change in the velocity will be $2v$. Therefore we say that the smallest meaningful impact parameter in this rough analysis [1] is one for which $\Delta v \approx v$, i.e. from eqn (6.3)

$$b_{\min} = \frac{Ze^2}{2\pi\varepsilon_0 m v^2} \quad . \tag{6.7}$$

Having found the limits to the integral,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} < (\Delta v)^2 > &= \int_{b_{\min}}^{\lambda_{\mathrm{D}}} \frac{n_i Z^2 e^4 \, \mathrm{d}b}{2\pi\varepsilon_0^2 m^2 v b} \\ &= \frac{n_i Z^2 e^4}{2\pi\varepsilon_0^2 m^2 v} \ln(\Lambda) \quad , \end{aligned} \tag{6.8}$$

where $\ln(\Lambda)$ is known as the *Coulomb Logarithm*, and

$$\begin{aligned} \Lambda &= \frac{\lambda_{\mathrm{D}}}{b_{\min}} \\ &= \sqrt{\left(\frac{\varepsilon_0 k_{\mathrm{B}} T}{n_0 e^2}\right)} \frac{2\pi\varepsilon_0 m v^2}{Ze^2} \\ &\approx \frac{3\pi n_0 \lambda_{\mathrm{D}}^3}{Z} \approx \frac{3\pi N_{\mathrm{D}}}{Z} \quad . \end{aligned} \tag{6.9}$$

That is to say, Λ is approximately the plasma parameter – the number of particles in a Debye sphere. As it only enters as the logarithm, it does not have a very large influence on the collision time – typical numbers for $\ln(\Lambda)$ range from a few up to 30 or so.

We are getting close to the answer we originally sought – the collision time, τ. For a ninety-degree collision we expect $(\Delta v)^2$ to be of order v^2. Thus from eqn (6.8)

$$\begin{aligned} \frac{1}{\tau} v^2 &= \frac{n_i Z^2 e^4}{2\pi\varepsilon_0^2 m^2 v} \ln(\Lambda) \\ \therefore \quad \tau &= \frac{2\pi\varepsilon_0^2 m^2 v^3}{n_i Z^2 e^4 \ln(\Lambda)} \quad . \end{aligned} \tag{6.10}$$

[1] We have treated the scattering event as though it were completely classical, whereas we know in reality that everything on the smallest scale must be treated by quantum mechanics, and thinking along those lines we would realise that we should really represent the electron as a wave, with a wavelength equal to its de Broglie wavelength, given by $\approx h/(mv)$. Indeed, as we can think of the electron as always being 'spread out' over this sort of distance, we can see that it is impossible to meaningfully define an impact parameter smaller than this, and thus $b_{\min}$ is usually taken to be the *larger* of h/mv or the classical value given by eqn (6.7). The classical value is the one that is most often relevant to the plasmas we shall encounter, so we persist in using it as the lower limit in the integral in eqn (6.5).

In all our analysis up to this point we have been dealing with an electron with particular velocity v. Often in plasmas (although it must be said not always) the electrons will have a Maxwellian distribution. If we average v^3 over a Maxwellian, and define the electron thermal velocity $v_e = \sqrt{k_{\rm B}T_e/m}$, then we find

$$\tau \approx 6.4 \frac{2\pi\varepsilon_0^2 m^2 v_e^3}{n_i Z^2 e^4 \ln(\Lambda)} \quad . \tag{6.11}$$

6.4 Collisions and the Plasma frequency

We are now in a position to justify our statement that there is only a very small chance that an electron will be scattered by an ion during one oscillation of a plasma wave (which was why we ignored collisions in our original analysis). In order to be able to ignore collisions on the timescale of the plasma frequency, we require $\omega_{\rm pe}\tau \gg 1$. Now

$$\begin{aligned}
\omega_{\rm pe}\tau &= \sqrt{\frac{n_e e^2}{\varepsilon_0 m}} 6.4 \frac{2\pi\varepsilon_0^2 m^2 v_e^3}{n_i Z^2 e^4 \ln(\Lambda)} \\
&= 6.4 \sqrt{\frac{n_e e^2}{\varepsilon_0 k_{\rm B} T_e}} \frac{2\pi\varepsilon_0^2 (k_{\rm B}T_e)^2}{n_e Z e^4 \ln(\Lambda)} \\
&= 6.4 \frac{1}{\lambda_{\rm D}} 2\pi n_e \frac{\lambda_{\rm D}^4}{Z \ln\Lambda} \\
&= 6.4 \quad 2\pi \quad \frac{3}{4\pi} \frac{N_{\rm D}}{z \ln\Lambda} \\
&\approx \frac{10 N_{\rm D}}{Z \ln\Lambda} \quad .
\end{aligned} \tag{6.12}$$

Thus we discover some very interesting physics. Given that $\ln\Lambda$ is not a large number, (as I have already indicated, typically of order 10), then we see that we can ignore collisions on the timescale of a plasma period if the plasma parameter is large – if there are a large number of particles in a Debye sphere. Way back in lecture 1 this was how we defined a 'good' plasma – one for which a Debye length was a meaningful concept. We have now discovered that such a plasma can invariably be treated as being 'collisionless'. When we call a plasma collisionless, of course we don't mean that no collisions take place, we mean that they take place on a timescale that is long compared with a plasma period. However, this is a very, very, important point, as it means that there are potential means of interacting with plasmas, or them interacting with things, in a 'collisionless' way. To make the point clearer it is best to use an example: if I shine laser light on certain types of plasmas, I can excite plasma waves when the frequency of the laser equals the plasma frequency (this is a complex process, and my aim is not to go into details, I am just using the example to illustrate the point). The important point is that clearly in this process I am getting energy into the plasma, but doing so on a timescale short compared with an oscillation

time. The laser energy is going into lots of electrons in their collective motion (the wave), and if I looked at any particular electron, during this particular absorption process, it would not have time to be deflected much by the ions. Therefore the energy absorption mechanism itself would be *collisionless*. This is in stark contrast to the way in which we normally think of energy being absorbed by matter. These properties of acting in ways that are *collisionless* are fundamental to the understanding of many types of plasma phenomena.

6.5 The Bohm-Gross frequency

Recall that when we derived the frequency of oscillations of electron plasma waves, $\omega_{\rm pe}$, in lecture 1, we found that there was no dependence on wavelength. We stated even then, in anticipation of this section, that this was an oversimplification of the physics. If a plasma has a finite temperature then individual electrons have velocities – their random thermal velocities, that are in addition to any velocity set up by the electric field caused by displacing the electrons from the ions. In this section we will look at these finite-temperature effects on the plasma waves. Before we dive into the mathematics, let us put our thinking caps on as physicists, and try to get a feel for how this might alter the situation.

Firstly, it should be reasonably obvious that not a lot should happen at very long wavelengths (i.e. small k-vectors). In the absence of temperature, the wave will have the frequency $\omega_{\rm pe}$. It is evident that in order to make an influence on the wave dynamics we need typical electrons, with their thermal velocity, v_e, to travel from one part of the wave (say a peak), to the opposite part, (the trough) *before* the wave has time to undergo one oscillation. If the wavelength is very long, then clearly the thermal electrons cannot travel this distance in the time available. Following this line of reasoning, we expect some modification to the dispersion relation (the relationship between ω and k) when an electron can travel a wavelength in what was (without temperature effects), one period of oscillation. That is to say, we expect the frequency of the wave could be very different when $kv_e \approx \omega_{\rm pe}$. Let's keep this concept in mind as we make our way through this particular section.

In order be able to take into account the effects of temperature, we need a description of the plasma which will allow us to deal with the thermal component, as well as the electric field produced. The way we do this is to treat the plasma as a fluid. Actually, we treat it as *two* fluids – a fluid of ions interpenetrating a fluid of electrons. We use two fluid theory, as we know that the heavy ions are pretty much stationary on the timescale of a plasma period, and it is the lighter electrons that oscillate in this background sea of positive ions.

From your work in atmospheric physics you will already be familiar with the equations of fluid mechanics. First we write down the very familiar equation of continuity:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0 \quad . \tag{6.13}$$

where u_e is the mean velocity of the electrons.

Secondly we consider the fluid equation which essentially states that the force on the fluid equals its rate of change of momentum:

$$\frac{\partial}{\partial t}(n_e u_e) + \frac{\partial}{\partial x}(n_e u_e^2) = \frac{F}{m} \quad . \tag{6.14}$$

What possible forces are acting on the electrons? Well, we are going to look at plasma waves once more, but in an unmagnetized plasma, so there will not be any magnetic forces. However, somehow we are going to use these equations to find out about plasma oscillations, and when we pull the electrons away from the ions there certainly is an electric field (which is why the electron density oscillates – the electric field is the restoring force). Therefore we better take into account electric fields in F. As we have emphasised above we also need to take into account the thermal force, which, from your previous studies of fluid mechanics, you will know is the gradient of the electron pressure, P_e. Thus we conclude that the relevant form of eqn (6.14) should be

$$\frac{\partial}{\partial t}(n_e u_e) + \frac{\partial}{\partial x}(n_e u_e^2) = -\frac{n_e e E}{m} - \frac{1}{m}\frac{\partial P_e}{\partial x} \quad . \tag{6.15}$$

In order to study the oscillations implied by the above equations, we take the time derivative of eqn (6.13), the spatial derivative of eqn (6.15), and eliminate the term $\partial^2(n_e u_e)/\partial t \partial x$ to obtain

$$\frac{\partial^2 n_e}{\partial t^2} - \frac{\partial^2}{\partial x^2}(n_e u_e^2) - \frac{e}{m}\frac{\partial}{\partial x}(n_e E) - \frac{1}{m}\frac{\partial^2 P_e}{\partial x^2} = 0 \quad . \tag{6.16}$$

How do we use eqn (6.16) to find out about waves in the system? Well, we use something called the *method of linearization*. Waves in the plasma are fluctuations in density about the mean density. So we say that the electron density at any point is the mean density, n_0, plus some fluctuation, $\tilde{n}$:

$$n_e = n_0 + \tilde{n} \quad . \tag{6.17}$$

Now, if we have a slab of plasma with no fluctuations, then the mean electron velocity is zero. Thus we see that u_e only exists as a fluctuation,

$$u_e = \tilde{u} \quad , \tag{6.18}$$

and this is also true of the electric field – there is no overall field that oscillates without fluctuations:

$$E = \tilde{E} \quad . \tag{6.19}$$

On the other hand, the pressure, like the density, does have some mean value, P_0, so we write

$$P_e = P_0 + \tilde{P} \quad . \tag{6.20}$$

In the method of linearization we insert eqns (6.17 - 6.20) into eqn (6.16). Linearization means that we ignore products of small terms (e.g. we ignore

terms in $\tilde{n}\tilde{u}$ because as a product of two fluctuations they must be very small compared with, for example $n_0\tilde{u}$). The result is

$$\frac{\partial^2\tilde{n}}{\partial t^2} - \frac{n_0 e}{m}\frac{\partial\tilde{E}}{\partial x} - \frac{1}{m}\frac{\partial^2\tilde{P}}{\partial x^2} = 0 \quad . \tag{6.21}$$

We now need to find out how to relate $\tilde{E}$ and $\tilde{P}$ to $\tilde{n}$ – if we could do that we would have an equation for the deviation in the density from the mean that we could solve to find out the frequency of the oscillations. The electric field, $\tilde{E}$, comes about because of the fluctuations in the electron density. From Gauss's law

$$\frac{\partial\tilde{E}}{\partial x} = \frac{-\tilde{n}e}{\varepsilon_0} \quad . \tag{6.22}$$

What about the pressure variation? Well, this again must be proportional to the density variation, as pressure is proportional to density. But the exact relationship will depend on the equation of state that we consider to be valid for the electrons. Now, in the plasmas with which we are concerned we are certainly in the classical, rather than quantum-statistical regime. This might lead us to suppose that we should simply use $P = nk_{\mathrm{B}}T$. However, this is not necessarily the case. The point is that if the fluctuations are very fast, then the compression and expansion of a particular part of the plasma during the wave motion is so fast that the appropriate relationship to use is the adiabat, not the isotherm. The oscillations are so fast that any temperature rise on compression, for example, has no time to conduct away. Of course, this depends on the frequency of the oscillation, and on the wavelength of the oscillation (as that dictates the lengthscale on which heat should flow). The upshot of all this is that as a rule of thumb one uses the normal isothermal equation of state, $P = nk_{\mathrm{B}}T$, when the phase velocity of the wave ω/k is slow compared with the thermal velocity, v_e of the electrons. If $\omega/k \gg v_e$ then we use an adiabatic equation of state:

$$\frac{P}{n^\gamma} = \text{constant} \quad . \tag{6.23}$$

It is these fast group velocity waves which are the most interesting in many cases, and so we will assume this adiabatic equation of state. Thus we need to determine the coefficient γ. Aha – I hear you say – I know that from thermodynamics and statistical mechanics, it's 5/3. Wrong! But wrong for quite a subtle reason. Normally when we have an adiabatic (or isothermal) expansion in a gas, the gas changes density (obviously), but in doing so the distance between the atoms changes in all three directions. However, we are studying what happens in a plane wave in a plasma – we are only interested in the motion of the electrons back and forth along a particular direction (the direction of propagation of the wave, as this is an electrostatic wave) – the problem is one dimensional. Now in a plane wave in a normal gas, the oscillating compression would *still* give rise to a change in distance between the atoms in all three directions, but it doesn't in a plasma. Why? Well, the big difference is, as we have found out in section 6.4, is that plasmas are collisionless – electrons do not collide during an

oscillation period. Therefore, as the electrostatic field pulls them in a particular direction, and reduces the distance between them in that direction, there are no scattering events to turn this 1-D compression into 3-D compression. This is very different from the situation in a standard neutral gas, where any sounds waves come about *due* to collisions, and thus the regions of expansion and rarefaction in the wave correspond to changes in the distance between the particles in all 3 directions.

So, we have convinced ourselves that we need to use an adiabatic equation of state, and it needs to be 1 dimensional. What is γ in this case? We need to recall how we find γ in general. Remember that the single particle partition function of an ideal gas (taking into account the translational motion in all three dimensions) is

$$Z_{\rm sp} = V\left(\frac{2\pi m k_{\rm B} T}{h^2}\right)^{3/2} \quad . \tag{6.24}$$

We use the fact that the partition function cannot change in an adiabatic change (remember the partition function represents the number of states thermally available to the system – this needs to stay constant to keep the entropy constant, as entropy is related to the number of ways of arranging the particles, and thus the number of available states). So, we conclude that for the normal 3-D case, during an adiabatic expansion $VT^{3/2}$ is a constant. When we combine this with the ideal gas equation of state, $PV = RT$, we find $PV^{5/3}$ must be constant, i.e. $\gamma = 5/3$. Following the same line of reasoning we can work out what happens for 1-D compression. In 1-D, $Z_{\rm sp} \propto T^{1/2}$, and thus $VT^{1/2}$ is constant along an adiabat. When combined with the ideal gas equation of state we find PV^3 is constant, i.e. in 1-D $\gamma = 3$, and equation (6.23) becomes

$$\frac{P}{n^3} = \text{constant} \quad . \tag{6.25}$$

We are now in a position to work out how the fluctuations in pressure are related to the fluctuations in density, as equation (6.25) implies

$$P_e = P_0 + \tilde{P} = C(n_0 + \tilde{n})^3 \quad , \tag{6.26}$$

thus, by performing a binomial expansion (recalling that $\tilde{n}$ is small) we find

$$\begin{aligned} P_0 + \tilde{P} &= C n_0^3 \left(1 + \frac{\tilde{n}}{n_0}\right)^3 \\ &= C n_0^3 \left(1 + 3\frac{\tilde{n}}{n_0}\right) \\ \therefore \tilde{P} &= 3C n_0^2 \tilde{n} \quad . \end{aligned} \tag{6.27}$$

However, we see from eqn (6.26) that by setting the fluctuations to be zero and employing the ideal gas equation of state

$$P_0 = C n_0^3 = n_0 k_{\rm B} T_e \quad , \tag{6.28}$$

and thus $Cn_0^2 = k_{\rm B}T_e$. Therefore eqn (6.27) becomes

$$\tilde{P} = 3\tilde{n}k_{\rm B}T_e \quad . \tag{6.29}$$

This is the relation between the pressure fluctuations and the density fluctuations that we have been seeking.

We substitute eqns (6.29) and (6.22) into eqn (6.21) to obtain an equation solely in terms of the variations in the plasma electron density:

$$\frac{\partial^2\tilde{n}}{\partial t^2} + \frac{n_0e^2}{\varepsilon_0 m}\tilde{n} - \frac{3k_{\rm B}T_e}{m}\frac{\partial^2\tilde{n}}{\partial x^2} = 0 \quad , \tag{6.30}$$

which can be written

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\rm pe}^2 - 3v_e^2\frac{\partial^2}{\partial x^2}\right)\tilde{n} = 0 \quad , \tag{6.31}$$

where $v_e = \sqrt{k_{\rm B}T_e/m}$ and is referred to as the electron thermal velocity. The physical arguments we put forward at the start of this section led us to believe that the plasma waves would now have a frequency that depended upon a wavelength. Thus we assume solutions to eqn (6.31) to be of the form $\tilde{n} \propto e^{[i\omega t - kx]}$. Hence

$$\omega^2 = \omega_{\rm pe}^2 + 3k^2v_e^2 \quad . \tag{6.32}$$

This new frequency for the plasma waves which takes into account the finite thermal correction is known as the *Bohm-Gross frequency*, after those who first derived it. Notice that it is pretty much what we expected from our discussion at the start of this section. As we would have hoped, for a cold plasma we recover our old plasma frequency, but the frequency of the waves increases with temperature, and the change in frequency becomes a significant fraction of the plasma frequency when the thermal velocity of the electrons is such that they travel a distance corresponding to a wavelength during one plasma oscillation.

6.6 Ion acoustic waves

In the previous section we had our first encounter with using fluid (actually two fluid) theory to find out how finite temperature influences wave frequencies and dispersion relations: we found a thermal correction to the frequency of plasma waves. At this juncture it is useful to remind ourselves that in a neutral gas (rather than a plasma) the sound waves that exist come about due to collisions between the atoms, and this sound velocity is constant, and proportional to $\sqrt{T/M}$. Thus it is plausible that at finite temperature similar sorts of waves could be present in a plasma – that is to say waves corresponding to movement of the *ions* due to finite temperature effects (the previous section having simply described oscillations of the electrons). Indeed this is the case, and the derivation of the frequency of these so-called ion-acoustic waves follows a very similar path to those of the electrons.

Firstly, we note that these ions must also obey the fluid equations, and thus there will be a set of equations for the motion of the ions which are analagous to eqns (6.13) and (6.15). By inspection these equations must be

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0 \quad . \tag{6.33}$$

$$\frac{\partial}{\partial t}(n_i u_i) + \frac{\partial}{\partial x}(n_i u_i^2) = +\frac{n_i Z e E}{M} - \frac{1}{M}\frac{\partial P_i}{\partial x} \quad . \tag{6.34}$$

where we have replaced $-e$ in eqn (6.15) by $+Ze$ (the charge on the ion), and the electron mass, m, by the ion mass, M. We follow exactly the same procedure as we did for the ion waves: we take the temporal derivative of eqn (6.33), the spatial derivative of (6.34), and eliminate the term $\partial^2(n_i u_i)/\partial t \partial x$ to obtain the equivalent of eqn (6.16):

$$\frac{\partial^2 n_i}{\partial t^2} - \frac{\partial^2}{\partial x^2}(n_i u_i^2) + \frac{Ze}{M}\frac{\partial}{\partial x}(n_i E) - \frac{1}{M}\frac{\partial^2 P_i}{\partial x^2} = 0 \quad . \tag{6.35}$$

The way ahead should now be clear – we are once more going to use the method of linearization. We say that there are small fluctuations in the density ($n_i = n_{i0} + \tilde{n}$), velocity ($u_i = \tilde{u}_i$), electric field ($E = \tilde{E}$) and pressure ($P_i = P_{i0} + \tilde{P}_i$). Substituting these expressions into eqn (6.35), and once again neglecting products of perturbed quantities, we obtain an equation for the ions, which is analogous to eqn (6.21) for the electrons:

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} + \frac{n_{i0} Z e}{M}\frac{\partial \tilde{E}}{\partial x} - \frac{1}{M}\frac{\partial^2 \tilde{P}_i}{\partial x^2} = 0 \quad . \tag{6.36}$$

We again assume a 1-D adiabatic equation of state for the ions – that is to say $\tilde{P}_i = 3k_{\rm B}T_i\tilde{n}_i$ and obtain

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} + \frac{n_{i0} Z e}{M}\frac{\partial \tilde{E}}{\partial x} - \frac{3k_{\rm B}T_i}{M}\frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0 \quad . \tag{6.37}$$

All that remains in order to find the waves is to obtain an expression for the gradient of the electric field. This is where we must be careful. Recall we are trying to find the oscillations supported by the ions in the plasma. It should be evident that because the ions are so much heavier than the electrons, whenever the ions move, the electrons react extremely quickly to try to eliminate any electric field set up by the motion of the ions with respect to them – the electrons are far more mobile than the ions because they are far, far lighter. This is the clue that helps us find an expression for the gradient of the electric field. As the electrons are so light, but the frequency of oscillation in which we are interested so low, the motion of the electrons is such as if they were almost massless. In this case, as we can see from eqn (6.15), the force equation for the electrons becomes

$$n_e e E = -\frac{\partial P_e}{\partial x} \quad , \tag{6.38}$$

and hence

$$n_e e\tilde{E} = -k_{\rm B}T_e\frac{\partial\tilde{n}_e}{\partial x} \quad , \tag{6.39}$$

where we have assumed $\tilde{P} = \tilde{n}k_{\rm B}T_e$ – that is to say we have assumed an *isothermal* equation of state for the electrons: the ion waves will be so slow (because of their large mass) that the fast moving electrons easily have time to equilibriate their temperature, and the isothermal equation of state is more suitable for them.

We make use of the facts that $n_e = Zn_{i0}$ and $\tilde{n}_e = Z\tilde{n}_i$, and differentiate both sides of eqn (6.39) with respect to x to obtain

$$n_{i0}Ze\frac{\partial\tilde{E}}{\partial x} = -Zk_{\rm B}T_e\frac{\partial^2\tilde{n}_i}{\partial x^2} \quad . \tag{6.40}$$

Substituting eqn (6.40) into (6.37) we find

$$\frac{\partial^2\tilde{n}_i}{\partial t^2} - \frac{Zk_{\rm B}T_e}{M}\frac{\partial^2\tilde{n}_i}{\partial x^2} - \frac{3k_{\rm B}T_i}{M}\frac{\partial^2\tilde{n}_i}{\partial x^2} = 0 \quad , \tag{6.41}$$

i.e.

$$\frac{\partial^2\tilde{n}_i}{\partial t^2} - \left(\frac{Zk_{\rm B}T_e + 3k_{\rm B}T_i}{M}\right)\frac{\partial^2\tilde{n}_i}{\partial x^2} = 0 \quad . \tag{6.42}$$

once again we assume wavelike solutions of the form $\tilde{n}_i \propto e^{[i\omega t - kx]}$, and find

$$\omega = \pm k v_{\rm ia} \quad , \tag{6.43}$$

where $v_{\rm ia}$, the so-called ion-acoustic frequency is given by

$$v_{\rm ia} = \sqrt{\left(\frac{Zk_{\rm B}T_e + 3k_{\rm B}T_i}{M}\right)} \quad . \tag{6.44}$$

These low frequency waves (low in comparison with electron waves because of the greater ion mass) are the analogue of the sound waves in normal neutral gases. The ions provide the inertia for the waves, and the plasma pressure provides the restoring force.

Notice that we have been careful to define the electron temperature, T_e, and the ion temperature, T_i, separately. The time for collisions between electrons and ions in a plasma is often so long that the two species can have significantly different temperatures (particles with close to equal masses transfer momentum between themselves more efficiently than particles of very different mass, thus the electrons tend to equilibriate amongst themselves to provide a fluid with a temperature associated with a Maxwellian distribution, and so do the ions, but the time for equilibrium to be reached between the two species can be a lot longer). Because of this effect, one curious outcome of eqn (6.44) is that it is possible to observe ion acoustic waves even at zero ion temperature. Indeed, low temperature ion acoustic waves are often seen in plasmas, as it is relatively common to find $T_i \ll T_e$, as it is usually easier to heat the electrons than the ions (again because of the mass difference).

6.7 Summary of Lecture 6

1. The time taken for an electron to undergo a collision with an ion via multiple small angle scattering is given by

$$\tau \approx 6.4 \frac{2\pi\varepsilon_0^2 m^2 v_e^3}{n_i Z^2 e^4 \ln(\Lambda)} \tag{6.45}$$

2. $\ln\Lambda$ is known as the *coulomb logarithm*, where

$$\begin{aligned}\Lambda &= \frac{\lambda_{\rm D}}{b_{\rm min}} \\ &= \sqrt{\left(\frac{\varepsilon_0 k_{\rm B} T}{n_0 e^2}\right)} \frac{2\pi\varepsilon_0 m v^2}{Z e^2} \\ &\approx \frac{3\pi n_0 \lambda_{\rm D}^3}{Z} \approx \frac{3\pi N_{\rm D}}{Z} \quad .\end{aligned} \tag{6.46}$$

   That is to say, Λ is approximately the plasma parameter – the number of particles in a debye sphere. As it only enters as the logarithm, it does not have a very large influence on the collision time – typical numbers for $\ln(\Lambda)$ range from a few up to 30 or so.

3. If we compare the electron-ion collision time with a plasma period we find

$$\omega_{\rm pe}\tau \approx \frac{10 N_{\rm D}}{Z \ln\Lambda} \quad . \tag{6.47}$$

   Thus if we have a good plasma ($N_{\rm D} \gg 1$), then the plasma is also *collisionless*.

4. At finite temperature the thermal pressure term alters the frequency of plasma waves. We used the fluid equations and the method of linearization to show that the electron waves in a plasma at a finite temperature have the dispersion relation

$$\omega^2 = \omega_{\rm pe}^2 + 3k^2 v_e^2 \quad , \tag{6.48}$$

   where $v_e = k_{\rm B} T_e / m$ and is referred to as the electron thermal velocity. In deriving this we noted that the electrons were compressed and rarefied adiabatically, but in 1-D $\gamma = 3$.

5. Also finite temperature effects allows waves in the ions to occur. Once more by using the fluid equations and the method of linearization we found

$$\omega = \pm k v_{\rm ia} \quad , \tag{6.49}$$

   where $v_{\rm ia}$, the so-called ion-acoustic frequency is given by

$$v_{\rm ia} = \sqrt{\left(\frac{Z k_{\rm B} T_e + 3 k_{\rm B} T_i}{M}\right)} \quad . \tag{6.50}$$

In this case the wave is of such low frequency that we used the isothermal equation of state for the electrons, although still used an adiabatic equation of state for the ions.

Lecture 7

The Vlasov Equation

7.1 Introduction

At this juncture it is useful to recap where we have got to, and to remind ourselves about the techniques we have developed to investigate the physics of plasmas. In particular, if we think about how we have studied waves in plasmas, we recall that we have studied waves in cold plasmas by considering single particle motion, and we have just studied waves in warm, finite-temperature, plasmas by the application of fluid theories. There are clearly even more detailed descriptions of the dynamics for plasmas with finite temperatures. For example, rather than simply assign some temperature T to the plasma, we could try to keep track of the actual distribution of the velocities, $f(\mathbf{v})$. However, we can consider an even more detailed picture than this – a fuller picture considers not only the distribution of particles in velocity space, but in the six-dimensional phase space, which is characterised by the three velocity co-ordinates and three cartesian spatial co-ordinates: $f(\mathbf{v}, \mathbf{r})$. A completely homogeneous plasma extending over all space would be characterized by a Maxwellian distribution. Presumably if waves exist within a plasma, they will somehow alter the distribution function (clearly they must do, a wave alters the velocities of particles in a way that is periodic in real space). Thus it is not unreasonable to suppose that there must be yet another method of deriving the dispersion relations for waves that can exist within a plasma by studying the equations governing the distribution function. Indeed, this is the case, and is the subject of the following sections.

7.2 The Vlasov Equation

If we are to study waves in plasmas by considering the distribution function, we need to make that function dependent upon time, as well as six-dimensional phase space: $f(\mathbf{v}, \mathbf{r}, t)$. The distribution function defines the number of particles, $\mathrm{d}N$ whose position lies within the small volume element $\mathrm{d}^3\mathbf{r}$ at position $\mathbf{r}$, and

whose velocity lies within the small volume element $\mathrm{d}^3\mathbf{v}$ in volume space at velocity $\mathbf{v}$, at time t:

$$\mathrm{d}N(\mathbf{v},\mathbf{r},t) = f(\mathbf{v},\mathbf{r},t)\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{v} \quad . \tag{7.1}$$

Clearly the integral of the distribution function over the whole of real space must produce the familiar distribution function in velocity space –

$$f(\mathbf{v},t) = \int f(\mathbf{v},\mathbf{r},t)\mathrm{d}^3\mathbf{r} \quad , \tag{7.2}$$

and the integral of the distribution function in phase space over the whole of phase space must give us the total number of particles in the plasma –

$$N = \int\int f(\mathbf{v},\mathbf{r},t)\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{v} \quad . \tag{7.3}$$

What more can we say about this distribution function in phase space? Well, one thing is clear – as we cannot create or destroy plasma particles, whatever happens to the distribution function over time, it must conserve particles. That is to say, there must be an equation of continuity which applies to it. Recall that we have met such a concept before, when we said that for a fluid we must conserve particle number. For a fluid this led to the continuity equation (6.13):

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0 \quad . \tag{7.4}$$

where u_e was the mean velocity of the electrons. There must be an analogous equation for the distribution function of particles in phase space, as we know they are conserved. Indeed there is. If we study eqn (7.4), we see it is of the form

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e \dot{x}) = 0 \quad , \tag{7.5}$$

Thus, as phase space contains both velocity as well as real space, the analogous equation of continuity is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (f\dot{\mathbf{r}}) + \frac{\partial}{\partial \mathbf{v}} \cdot (f\dot{\mathbf{v}}) = 0 \quad , \tag{7.6}$$

as the flux of particles in phase space occurs in both cartesian and velocity space.

Equation (7.6) can be simplified slightly, as we note that the second term on the left hand side can be written

$$\frac{\partial}{\partial \mathbf{r}} \cdot (f\dot{\mathbf{r}}) = \frac{\partial}{\partial \mathbf{r}} \cdot (f\mathbf{v}) = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + f\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}) = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} \quad , \tag{7.7}$$

as the $\mathbf{r}$ and $\mathbf{v}$ coordinates are completely independent (although the *number* of particles with velocity $\mathbf{v}$ can depend on $\mathbf{r}$, that is *not* to say that $\mathbf{v}$ depends on $\mathbf{r}$, just as the coordinates x and y in cartesian space are orthogonal.)

Can we simplify the third term on the left hand side of eqn (7.6) in the same way? That is to say, what can we say about

$$\frac{\partial}{\partial \mathbf{v}} \cdot (f\dot{\mathbf{v}}) = \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} + f\frac{\partial}{\partial \mathbf{v}} \cdot (\dot{\mathbf{v}}) \quad . \tag{7.8}$$

Is the second term on the right hand side once again zero? The answer, for a plasma, is yes, for the following reason. If we consider the forces on individual particles within the plasma, we note that the equation for the Lorentz force is given by

$$\dot{\mathbf{v}} = \frac{q}{m}(\mathbf{E} + (\mathbf{v} \times \mathbf{B})) \quad , \tag{7.9}$$

which means that

$$\frac{\partial}{\partial \mathbf{v}} \cdot (\dot{\mathbf{v}}) = 0 \quad , \tag{7.10}$$

and thus

$$\frac{\partial}{\partial \mathbf{v}} \cdot (f\dot{\mathbf{v}}) = \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} \quad . \tag{7.11}$$

Combining eqns (7.6) through (7.11) yields the *Vlasov Equation*:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + (\mathbf{v} \times \mathbf{B})) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad . \tag{7.12}$$

This equation holds for the electrons and the ions separately.

The Vlasov equation forms the basis of a kinetic theory description of the plasma, but in order to construct a self-consistent set of equations we need to know where the electric and magnetic fields come from, what they represent, and how to derive them. First of all it is important to note that these **E** and **B** fields act on the whole group of particles at a particular position in space, **r**. Therefore they cannot be representing the fine-scale fields that are present in binary collisions between particles – on the contrary, they must represent macroscopic fields averaged over a large number of particles. As such, they have two possible sources. First of all, they could represent a set of external fields that we produce by, for example, placing the whole plasma in an external magnetic field. Secondly, they could be the fields generated by collective effects within the plasma (for example the electric field associated with plasma waves, where the periodic motion of many, many particles gives rise to periodic electric fields).

It is this second case with which we will be interested here. What can we say about the electric fields in this case, where they are generated 'internally'. Well, the fields must obey Maxwell's equations, and thus Gauss' law:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \sum_{\alpha} \frac{q_\alpha}{\varepsilon_0} \int f_\alpha(\mathbf{r}, \mathbf{v}, t) \mathrm{d}^3\mathbf{v} \quad , \tag{7.13}$$

where α is used to denote the species – electrons or ions.

We will now proceed to show how we can use the Vlasov equation, along with the self-consistent electric fields, to solve real physical problems, giving the example (once more) of how to derive the dispersion relationship for waves in a plasma.

7.3 Waves in an Unmagnetized Plasma (Again)

When waves exist in a plasma, two things happen which are of import when thinking in the terms defined by the Vlasov equation. First of all, there must be some change to the distribution function from the normal function (which we take to be the Maxwell-Boltzmann distribution function for both electrons and ions). If we make the usual approximation that the ions are heavy and slow, whilst the electrons are light and fast, we can see that we are justified in saying that plasma waves don't change the distribution function of the ions, but must somehow constititute a small modification to the distribution function of the electrons. We denote the normal, unperturbed distribution function of the electrons by f_0, and denote the perturbation to it by $f_1(\mathbf{r}, \mathbf{v}, t)$. Secondly, we know that these plasma waves give rise to electric fields within the plasma (indeed, it is these electric fields that provide the restoring force for the electrons that results in the wave motion). In the absence of the waves, the fields of the ions cancel out the fields of the electrons, and there are no net electric fields. Thus the electric fields that are present within the plasma come totally from this f_1 term, and applying Gauss' law, as in eqn (7.13) –

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{e}{\varepsilon_0} \int f_1(\mathbf{r}, \mathbf{v}, t) \mathrm{d}^3\mathbf{v} \quad . \tag{7.14}$$

So, our distribution function is $(f_0 + f_1)$, and applying the Vlasov equation for the undisturbed and disturbed distribution functions we obtain

$$\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{r}} = 0 \quad . \tag{7.15}$$

and

$$\frac{\partial (f_0 + f_1)}{\partial t} + \mathbf{v} \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}) \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{v}} = 0 \quad . \tag{7.16}$$

We subtract eqn (7.15) from (7.16) to obtain

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial (f_1)}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}) \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{v}} = 0 \quad . \tag{7.17}$$

Given that $\mathbf{E}$ is a small quantity that depends on f_1, we can ignore terms in $\mathbf{E}f_1$ as they are effectively quadratic in f_1 (just as we used the theory of linearisation when solving the fluid equations), which finally provides

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0 \quad . \tag{7.18}$$

Our goal is to use this equation to derive the dispersion relation for plasma waves. The way we do this is to recognise that when waves exist within the plasma, this will be reflected in a periodicity in both the electric field and the perturbed part of the distribution function, that is to say

$$\mathbf{E} = \mathbf{E}_{\mathbf{k},\omega} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t]) \tag{7.19}$$

$$f_1 = f_{1(\mathbf{k},\omega)} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t]) \quad . \tag{7.20}$$

We substitute eqns (7.19) and (7.20) into (7.18) to obtain

$$-i\omega f_{1(\mathbf{k},\omega)} + i\mathbf{k}\cdot\mathbf{v} f_{1(\mathbf{k},\omega)} - \frac{e}{m}\mathbf{E}_{\mathbf{k},\omega}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0 \quad , \tag{7.21}$$

which, upon rearrangement, becomes

$$f_{1(\mathbf{k},\omega)} = \frac{ie}{m}\frac{1}{(\omega - \mathbf{k}\cdot\mathbf{v})}\mathbf{E}_{\mathbf{k},\omega}\cdot\frac{\partial f_0}{\partial \mathbf{v}} \quad . \tag{7.22}$$

Finally, we substitute eqn (7.22) back into the other relation we have between the electric field and the distribution function, that is to say Gauss' law, as given by eqn (7.14) –

$$\mathbf{k}\cdot\mathbf{E}_{\mathbf{k},\omega} = -\frac{e^2}{\varepsilon_0 m}\mathbf{E}_{\mathbf{k},\omega}\cdot\int\frac{\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}}\mathrm{d}^3\mathbf{v} \quad . \tag{7.23}$$

Let us make things easier by assuming that our plasma waves are propagating along the z axis. We know that for electrostatic waves this means that $\mathbf{E}$ and $\mathbf{k}$ then lie along the z direction. Thus, if v_z is the component of the velocity along this direction, then for a non-zero electric field eqn (7.23) requires

$$1 + \frac{e^2}{\varepsilon_0 mk}\int\frac{\frac{\partial f_0}{\partial v_z}}{\omega - kv_z}\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z = 0 \quad . \tag{7.24}$$

7.4 Waves in a Warm Plasma

Equation (7.24) is clearly going to give rise to a dispersion relationship, as it contains both ω and k. In order to solve this equation, we simply need to insert the undisturbed distribution function, f_0, which we know to be the Maxwell-Boltzmann distribution function –

$$f_0 = \frac{n_0}{\pi^{3/2}}\left(\frac{m}{2k_\mathrm{B}T}\right)^{3/2}\exp\{-m(v_x^2+v_y^2+v_z^2)/(2k_\mathrm{B}T)\} \quad . \tag{7.25}$$

We insert this form of f_0 into eqn (7.24) and obtain

$$1 - \frac{2n_0e^2}{\pi^{3/2}\varepsilon_0 mk}\left(\frac{m}{2k_\mathrm{B}T}\right)^{5/2}\int\frac{v_z e^{-\frac{mv_z^2}{2k_\mathrm{B}T}}}{\omega - kv_z}\mathrm{d}v_z\int e^{-\frac{mv_x^2}{2k_\mathrm{B}T}}\mathrm{d}v_x\int e^{-\frac{mv_y^2}{2k_\mathrm{B}T}}\mathrm{d}v_y = 0 \quad . \tag{7.26}$$

The integrals with respect to v_x and v_y are both equal to $(2\pi k_\mathrm{B}T/m)^{1/2}$, and thus

$$1 - \frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_\mathrm{B}T}\right)^{3/2}\int\frac{v_z e^{-\frac{mv_z^2}{2k_\mathrm{B}T}}}{\omega - kv_z}\mathrm{d}v_z = 0 \quad . \tag{7.27}$$

We notice that there is a problem in performing this integral. The denominator goes to zero at the velocity $v_z = \omega/k$. The proper way to perform such an

integral, where there is a singularity, is to carry out the integral in the complex plane. However, there is still information that we can extract in a certain limit, as long as we are able to accept a bit of a hand-waving argument, and a sleight of hand. What we do is say that we will live with the fact that there is a singularity, but say to ourselves that perhaps this singularity can be conveniently ignored as long as it happens at a large value of v_z. We hope that this will work, because if v_z is large compared with the thermal velocity, v_e, i.e. $(k_{\rm B}T/m)^{1/2}$, then the exponential term in the integral in eqn (7.27) will reduce its effect. As you can see, this is very handwaving, but it is ultimately justified in that it does give the same result (for this limit of the singularity occuring at large v_z) for the full analysis where the integral is properly performed in the complex plane.

Thus, for those regions in velocity space where the exponential term has any significant value, we are making the assumption that in these regions $kv_z \ll \omega$. In these regions we can therefore make use of the binomial expansion

$$\frac{1}{\omega - kv_z} = \frac{1}{\omega}\left(1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + ...\right) \quad . \tag{7.28}$$

Of course, this expansion is not valid in those regions where v_z is high, but we conveniently ignore this, given that the exponential factor will insure that in those regions there will not be much contribution to the integral.

Therefore, eqn (7.27) now can be written

$$1 - \frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_{\rm B}T}\right)^{3/2}\frac{1}{\omega}\int v_z\left(1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + ...\right)e^{-\frac{mv_z^2}{2k_{\rm B}T}}{\rm d}v_z = 0 \quad . \tag{7.29}$$

Now as the integral runs from $-\infty$ to $+\infty$, then any odd function will integrate to zero, and thus we see that the terms that matter are

$$1 - \frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_{\rm B}T}\right)^{3/2}\frac{1}{\omega}\int v_z\left(\frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^3 + ...\right)e^{-\frac{mv_z^2}{2k_{\rm B}T}}{\rm d}v_z = 0 \quad . \tag{7.30}$$

Upon integration we find

$$1 - \frac{n_0e^2}{\varepsilon_0 m}\frac{1}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{k_{\rm B}T}{m} + ...\right) = 0 \quad , \tag{7.31}$$

which can be written

$$1 - \frac{\omega_{\rm pe}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_e^2 + ...\right) = 0 \quad , \tag{7.32}$$

where $v_e = \sqrt{k_{\rm B}T/m}$.

Before we go any further, remember that in order to derive this result we had to make the assumption that we were in the region where $kv_e \ll \omega$. We

therefore see that the approximate solution to eqn (7.32) is $\omega \approx \omega_{\rm pe}$, which in turn allows us to find the approximate solution

$$\omega^2 = \omega_{\rm pe}^2 + 3k^2 v_e^2 \quad . \tag{7.33}$$

Notice that this is precisely the dispersion relationship that we derived from a consideration of the fluid equations (it is identical to equation (6.32)), yet it has been derived by a completely different method. Clearly our new method will also give us the higher order corrections as well, if we calculate the other terms in the expansion.

7.5 Landau Damping: Physical Picture

So far we have conveniently ignored the fact that there is a pole in integral at the velocity $v_z = \omega/k$. As stated earlier, to properly take into account this pole, we need to perform the integral in the complex plane. However, before we do any quantitative work, let us discuss the physics behind this pole. There is clearly some sort of resonance going on here when the velocity of the electron, v_z, is equal to the phase velocity of the plasma wave. We know that a plasma wave sets up an oscillating electric field, but electrons that have such velocities, moving with the phase velocity of the plasma wave, don't experience a rapidly changing electric field, but see it pretty much as static – they are, after all, moving along with the wave. Furthermore, those electrons with velocities just slightly less than the resonant condition will start to accelerate in this almost static field, whereas those with velocities slightly more than the resonant condition will be decelerated by the field of the wave. Without thinking about the equations for a moment, we can see that the particles that are accelerated by the waves must take energy from the waves (helping to damp them), whereas particles with energy greater than the phase velocity of the wave must give up energy to the wave (causing it to grow), as they are decelerated in the electric field of the wave. This principle is the underlying idea of *Landau damping*: waves are damped in a plasma due to this process whereby particles resonant with the wave are accelerated. Given that particles can be both accelerated or decelerated by the field of the wave, why does the wave damp? Well, the answer is quite trivial – the zeroth order distribution function for the particles is Maxwellian – $\exp(-mv_z^2/(k_{\rm B}T))$ – therefore for any particular velocity (i.e. the phase velocity of the wave, ω/k), there are always slightly more electrons with a velocity just less than the phase velocity of the wave than there are with velocities slightly greater than the phase velocity of the wave – hence overall there is a transfer of energy from the wave to the resonant particles, and the wave damps.

So, that is, in a handwaving form, the physics of the pole in the integral. We now would like to calculate this damping coefficient. How do we do this? One way, clearly, is to perform this integral in the complex plane, and show how it leads to the damping. For the stout of heart we do this can be found in any standard textbook on plasma physics, and I would recommend reading such a descrption, as the derivation of this damping rate by Landau was one of

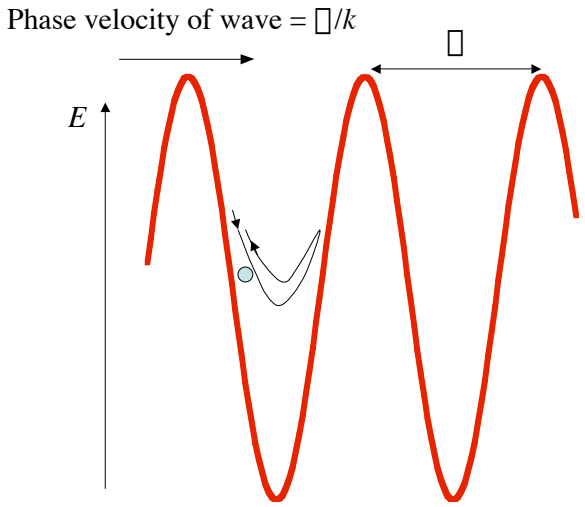


Figure 7.1: : Particles that travel with a velocity close to the phase velocity, ω/k, of the wave are trapped, and oscillate in the field of the wave. Depending upon their original velocity with respect to the wave they can be accelerated or decelerated by the electric field of the wave. The height of the potential of the wave is of order $E\lambda/2$.

the great triumphs of theoretical plasma physics. However, just as we used a handwaving way to ignore the pole when deriving the dispersion relation for the plasma waves, we will also follow a handwaving approach to deduce the damping rate of the waves due to the resonant particles (we will also do a handwaving analysis of what happens in the complex plane in the next section).

As we start to wave our hands, it will be useful to keep in mind Figure 7.1. First of all we notice that the electric field of the wave will trap a certain set of particles that have a particular range of velocities with respect to the phase velocity of the wave. If a particle is travelling more slowly than the wave it will be accelerated by the wave, up the other side of the wave, and then decelerated – and the trapped particles will oscillate between two crests of the wave. The difference in velocity between the particle and the crest of the wave is clearly $v - \omega/k$, we denote this difference by, Δv, such that

$$\Delta v = |v - \omega/k| \quad . \tag{7.34}$$

Our first task is to determine the range of Δv such that particles will remain trapped. Clearly it will depend on the amplitude of the wave. If we say that the wave has an amplitude of electric field E, then the height of the potential associated with the wave will be something of the order of $E\lambda/2$. If we are moving along in the frame of reference of the wave, the particles have a kinetic energy of order $m\Delta v^2$, and to get over the crest of the wave this kinetic energy must exceed the height of the potential barrier. Therefore, within factors of 2 and π etc., we see that the trapped particles are such that

$$m\Delta v^2 < \frac{eE}{k} \tag{7.35}$$

$$\Delta v \approx \left(\frac{eE}{mk}\right)^{1/2} \quad . \tag{7.36}$$

These particle oscillate backwards and forwards. As they are accelerated they take energy from the wave, and it is damped, and as they are decelerated they give energy back to the wave, and it grows. However, as we noted above, at the beginning of the process there are more particles moving slower than the wave than faster (because of the Maxwell distribution), so on average more particles are being accelerated than decelerated, and the wave damps. As the wave damps, the electric field associated with it must reduce, and the faster particles eventually have enough energy to get over the crest of the wave, and become untrapped. Of course, this could happen, with equal probability, as they are climbing up the slope of the field of the wave from either direction, and thus they emerge from being trapped within the wave with equal probabilities of moving faster or slower than the wave.

Roughly how many more particles are there, amongst the ones which are trapped, that have velocities slower than the wave than have velocities greater than the wave? Consider Figure 7.2 – we see that if the trapped particles have a range of velocities Δv, then the excess of the slower ones over the faster ones, n_{v+} is of order

$$n_{v+} \approx \Delta v \frac{\partial f}{\partial v}\Delta v = \frac{\partial f}{\partial v}(\Delta v)^2 \quad . \tag{7.37}$$

In the laboratory frame, the particle has velocity $mv^2/2$, but when it becomes untrapped, the velocity is changed by Δv, so the energy of the particle changes by $mv\Delta v$. Every time the particles oscillate over one period, energy is thus lost from the wave, and the electric field of the wave is reduced. In order to work out the rate of damping of the wave, we consider how much the energy density in the wave is reduced each period of oscillation of the particles. The energy density of an electric field is $\varepsilon_0 E^2/2$, and thus the power, P, lost by the wave is

$$P = \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon_0 \frac{E^2}{2}\right) = \varepsilon_0 E \frac{\mathrm{d}E}{\mathrm{d}t} \quad , \tag{7.38}$$

so during one oscillation period, τ,

$$P = \varepsilon_0 E \frac{\delta E}{\tau} \quad . \tag{7.39}$$

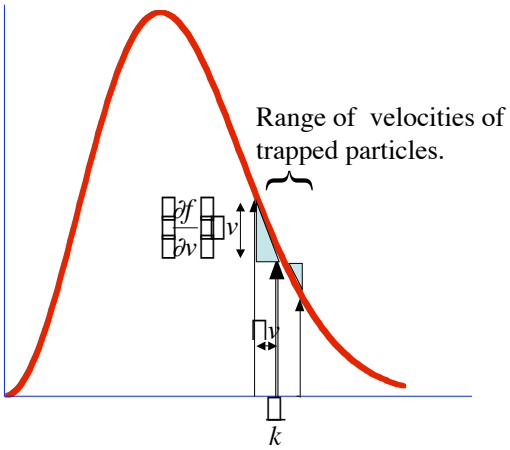


Figure 7.2: : In a Maxwellian distribution there are more trapped particles travelling slower than the wave than travelling faster than it – it is this that leads to damping of the wave.

What is one oscillation period? Well in a time t a particle will acquire a velocity eEt/m. The time taken to go from one crest to the other, τ, will be something of order half the wavelength divided by the velocity difference between the particle and the wave, i.e.

$$\tau \approx \frac{\lambda}{2\delta v} = \frac{\pi}{k\delta v} \quad . \tag{7.40}$$

We are now in a position to put everything together: the power lost by the wave is given by

$$\text{Power} \quad = \quad (\text{No. of particles}) \times (\text{EnergyLost}) \frac{1}{(\text{Time})} \tag{7.41}$$

$$P = \varepsilon_0 E \frac{\delta E}{\tau} = \left(\frac{\partial f}{\partial v} (\Delta v)^2 \right) \times (mv\Delta v) \times \left(\frac{k\Delta v}{\pi} \right) \tag{7.42}$$

$$\approx \left(\frac{\partial f}{\partial v} \right) mv\Delta v^4 k \quad . \tag{7.43}$$

Substituting eqn (7.36) into eqn (7.43) we obtain

$$P \approx \left(\frac{\partial f}{\partial v} \right) \frac{e^2 E^2}{mvk} \quad . \tag{7.44}$$

We are now in a position to work out how quickly the wave is damped by the resonant particles. We assume that the wave is damped at a rate γ such that

$$E = E_0 \exp(-\gamma t) \tag{7.45}$$

thus

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\gamma E \therefore \gamma = -\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{\varepsilon_0}\frac{P}{E^2} \quad . \tag{7.46}$$

When we compare eqn (7.44) and (7.46) we find

$$\gamma = \left(\frac{\partial f}{\partial v}\right)\frac{e^2 v}{\varepsilon_0 m k} = \left(\frac{\partial f}{\partial v}\right)\frac{\omega_{\mathrm{pe}}^2 v}{n_e k} \quad .$$

And given, once more, that $v = \omega k \approx \omega_{\mathrm{pe}} k$ we find

$$\gamma = \left(\frac{\partial f}{\partial v}\right)\frac{\omega_{\mathrm{pe}}^3}{k^2 n_e} \quad . \tag{7.47}$$

This simple analysis is for a 1-D situation, and thus using a 1-D Maxwellian, i.e. $f = (n_e/\sqrt{\pi})(m/2k_{\mathrm{B}}T)^{1/2}\exp(-mv_z^2/k_{\mathrm{B}}T)$ we find that

$$\gamma \approx \frac{1}{\sqrt{\pi}}\frac{\omega_{\mathrm{pe}}^3}{k^2}\left(\frac{m}{2k_{\mathrm{B}}T}\right)^{3/2} v_z \exp\left(-\frac{mv_z^2}{2k_{\mathrm{B}}T}\right) \quad . \tag{7.48}$$

We note that $v_z = \omega/k \approx \omega_{\mathrm{pe}}/k$ and thus

$$\gamma \approx \frac{1}{\sqrt{8\pi}}\frac{\omega_{\mathrm{pe}}^4}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{m\omega_{\mathrm{pe}}^2}{2k^2 k_{\mathrm{B}}T}\right)$$
$$\gamma \approx \frac{1}{\sqrt{8\pi}}\frac{\omega_{\mathrm{pe}}^4}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{1}{2k^2\lambda_{\mathrm{D}}^2}\right) \quad . \tag{7.49}$$

where v_e is the thermal velocity of the electrons, $\sqrt{k_{\mathrm{B}}T/m}$. Thus we see that the waves that are heavily damped are those with wavelengths of order, or shorter than, a Debye length.

7.6 Landau Damping: Quantitative Calculation†

A simple physical argument led us to an approximate value of the Landau damping rate – eqn (7.47). We obtained this damping rate just by thinking about the physics, without in any way bothering ourselves with the mathematics of how to deal with the pole in the integral. In this section I will outline a method of dealing with the pole, which leads to the correct rate, but again is not sufficiently rigorous to prevent a mathematician from undergoing cardiac arrest.

We go back to eqn (7.27):

$$1 - \frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_{\rm B}T}\right)^{3/2}\int \frac{v_z e^{-\frac{mv_z^2}{2k_{\rm B}T}}}{\omega - kv_z}{\rm d}v_z = 0 \quad . \tag{7.50}$$

Recall that we used this equation to work out the dispersion relation between ω and k, but in the process conveniently ignored the pole in the integral. By making the assumption that the velocity of the wave was large compared with the thermal velocity, and simply ignoring the pole, we obtained the dispersion relation for waves in a warm plasma, eqn (7.32):

$$1 - \frac{\omega_{\rm pe}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_e^2 + ...\right) = 0 \quad , \tag{7.51}$$

We now wish to find out what happens when we take the pole into account, rather than simply leave it out. The problem, of course, is that the integral 'blows up' at $v_z = \omega/k$. How do we deal with this? If you have previously studied how to do integrals in the complex plane, you will be aware of how to do this, but I will take a slightly less rigorous approach that gets the correct answer.

What we do is notice that the pole occurs at one point, so to deal with it, we only need to integrate over an infinitessimally small region around it, i.e. we need to evaluate

$$\int_{\frac{\omega}{k}-\delta v_z}^{\frac{\omega}{k}+\delta v_z} \frac{v_z e^{-\frac{mv_z^2}{2k_{\rm B}T}}}{\omega - kv_z}{\rm d}v_z \quad . \tag{7.52}$$

However, as the range of the integral is so small, v_z can be treated as a constant in the numerator, i.e.

$$\int_{\frac{\omega}{k}-\delta v_z}^{\frac{\omega}{k}+\delta v_z} \frac{v_z e^{-\frac{mv_z^2}{2k_{\rm B}T}}}{\omega - kv_z}{\rm d}v_z = v_z e^{-\frac{mv_z^2}{2k_{\rm B}T}}\int_{\frac{\omega}{k}-\delta v_z}^{\frac{\omega}{k}+\delta v_z} \frac{1}{\omega - kv_z}{\rm d}v_z \quad . \tag{7.53}$$

By making the substitution $u = \omega - kv_z$, we see that the integral, I, can be written

$$\begin{aligned} I &= \int_{\frac{\omega}{k}-\delta v_z}^{\frac{\omega}{k}+\delta v_z} \frac{1}{\omega - kv_z}{\rm d}v_z \\ &= \frac{1}{k}\int_{-\delta u}^{+\delta u}\frac{{\rm d}u}{u} \\ &= \frac{1}{k}\left(\ln(\delta u) - \ln(-\delta u)\right) \\ &= \frac{1}{k}\left(\ln(\delta u) - \ln(\delta u\ e^{i\pi})\right) \\ &= \frac{1}{k}\left(\ln(\delta u) - \ln(\delta u) - \ln(e^{i\pi})\right) \\ &= -\frac{i\pi}{k} \quad . \end{aligned} \tag{7.54}$$

Thus we see that if we consider eqns (7.50) and (7.51), then if we include the pole in our analysis, eqn (7.32) should be written as

$$\begin{aligned} 1-\frac{\omega_{\rm pe}^2}{\omega^2}\left(1+3\frac{k^2}{\omega^2}v_e^2+...\right)+i\frac{\pi}{k}\frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_{\rm B}T}\right)^{3/2}v_z e^{-\frac{mv_z^2}{2k_{\rm B}T}}=0 \\ 1-\frac{\omega_{\rm pe}^2}{\omega^2}\left(1+3\frac{k^2}{\omega^2}v_e^2+...\right)+i2\sqrt{\frac{\pi}{8}}\frac{\omega_{\rm pe}^2}{k^2}\left(\frac{1}{v_e^3}\right)v_z e^{-\frac{mv_z^2}{2k_{\rm B}T}}=0 \quad , \end{aligned} \tag{7.55}$$

and, noticing that $v_z = \omega/k$ this becomes

$$1-\frac{\omega_{\rm pe}^2}{\omega^2}\left(1+3\frac{k^2}{\omega^2}v_e^2+...\right)+i2\sqrt{\frac{\pi}{8}}\frac{\omega_{\rm pe}^2}{k^3}\left(\frac{1}{v_e^3}\right)\omega e^{-\frac{m\omega^2}{2k^2k_{\rm B}T}}=0 \quad . \tag{7.56}$$

In order to determine the damping rate we make the further assumption that both it and the thermal correction to the real part of the frequency are small, thus $\omega \approx \omega_{\rm pe}$, so that

$$\begin{aligned} 1-\frac{\omega_{\rm pe}^2}{\omega^2}+i2\sqrt{\frac{\pi}{8}}\frac{\omega_{\rm pe}^3}{k^3}\left(\frac{1}{v_e^3}\right)\omega\exp\left(-\frac{m\omega_{\rm pe}^2}{2k^2k_{\rm B}T}\right)=0 \\ 1-\frac{\omega_{\rm pe}^2}{\omega^2}+i2\sqrt{\frac{\pi}{8}}\frac{\omega_{\rm pe}^3}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{1}{2k^2\lambda_{\rm D}^2}\right)=0 \quad . \end{aligned} \tag{7.57}$$

Using the fact that the damping (imaginary part of ω) is small, we can use the binomial theorem to determine that

$$\gamma = -{\rm Im}(\omega) = \sqrt{\frac{\pi}{8}}\frac{\omega_{\rm pe}^4}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{1}{2k^2\lambda_{\rm D}^2}\right) \quad . \tag{7.58}$$

We note that this more complete analysis (which agrees with the accepted result), is the same (within a factor of π) as that found from our simpler physical arguments - i.e. that shown in eqn (7.49).

7.7 Plasma Accelerators

We cannot leave this section without saying a little bit about plasma accelerators, as not only do these work on some of the principles outlined in this chapter, but they constitute an important part of the research work carried out in the department (specifically by Simon Hooker's group). Indeed, Simon Hooker and co-workers recently managed to accelerate electrons to 1 GeV using a plasma just 3cm in length – a feat so spectacular that it was even reported by the economist, given that it would take a conventional accelerator about 100 m to do the same job.

The idea is shown schematically in Fig. 7.3. What one does is take an intense laser pulse, of order 100-fs in duration, and focus it into a plasma. The

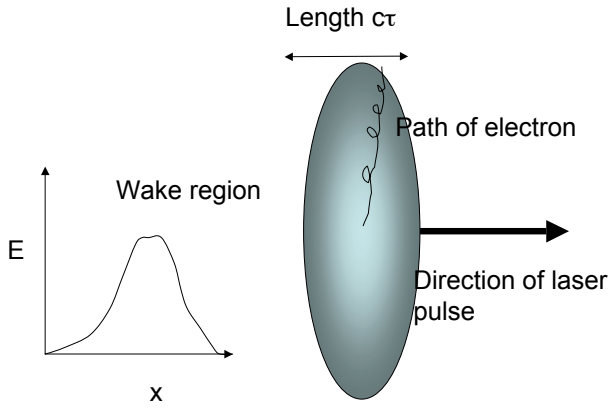


Figure 7.3: : An intense laser pulse can expel electrons, leaving behind it a 'wake' almost devoid of electrons, of length $c\tau$, where τ is the duration of the laser pulse. Any residual electrons trapped in this wake can be accelerated to highly relativistic energies.

electrons feel the electric field of the laser and start to oscillate, thus gaining oscillatory energy (we call this their quiver energy). Clearly the electrons in the centre of the laser pulse, where the laser intensity is greatest, feel a greater electric field, and thus oscillate with a larger amplitude than those at the edge – thus across the pulse there is a gradient in the quiver energy. Now, we know that a gradient in potential is the same as a force (with a minus sign), and thus electrons get expelled from the region in the middle of the laser pulse. A simple way of thinking about this is to note that due to the gradient in the electric field across the pulse, they never return to their original position during one laser cycle, as shown in Fig. **. Thus, as the pulse travels through the plasma, it expels electrons, and leaves a 'wake' behind it, which is almost completely devoid of electrons, and of length of order $c\tau$, where τ is the duration of the laser pulse. However, this wake that is nearly devoid of electrons will have a huge electric field within it, and any electrons that *are* trapped within it will get accelerated along with the laser pulse, and thus to very high velocities and energies.

We can then use eqn (1.60) to work out what sort of electric field is found behind the propagating pulse. If the system is completely devoid of electrons,

but the ambient density is n, then the field must be of order

$$E \approx \frac{nec\tau}{\varepsilon_0} \quad . \tag{7.59}$$

Now, the electron density cannot be too high, or else the pulse will not travel close to the speed of light (recall the refractive index for a plasma, although less than one, still results in a group velocity less than light speed). Typically an electron density of about $10^{23}\mathrm{m}^{-3}$ is used, for which eqn (9.42) predicts a field of about 5×10^{10} Vm^{-1}. This is a truly enormous field. Recall that the mass of an electron is 0.5 MeV, and thus we would only require about 10μm to accelerate an electron to relativistic speeds! This fits in reasonably well with my statement at the start of this section, that the Hooker group has measured GeV electrons from a 3cm length of plasma, as this corresponds to a field of $3 \times 10^{10}\mathrm{Vm}^{-1}$.

One might ask what the limits are on the acceleration that can be achieved? Well, there is a lot that can be said about this, but the main limit is that eventually as the electrons become very relativistic there will be a slight difference between their velocity and that of the light pulse. Another way of thinking about this is that they will have gone from the peak of the wake, to the trough, and thus a new stage of acceleration will be needed to get the electrons bake into phase with the wake.

7.8 Summary of Lecture 7

1. Just as there is a continuity equation for particles, there is also a continuity equation for the distribution function. This equation is known as the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + (\mathbf{v} \times \mathbf{B})) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad . \tag{7.60}$$

   This equation holds for the electrons and the ions separately.

2. Gauss's law can be written in terms of the distribution functions of the electrons and ions:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \sum_\alpha \frac{q_\alpha}{\varepsilon_0} \int f_\alpha(\mathbf{r}, \mathbf{v}, t) \mathrm{d}^3\mathbf{v} \quad , \tag{7.61}$$

   where α is used to denote the species – electrons or ions.

3. In order to analyse waves in plasmas starting from the distribution functions, we make a small periodic perturbation, f_1 to the distribution function of the electrons (we assume the distribution function of the ions remains a Maxwellian, as they are so sluggish). This approach leads to the following relation between the frequency, ω, of the waves, and their k-vector:

$$1 + \frac{e^2}{\varepsilon_0 mk} \int \frac{\frac{\partial f_0}{\partial v_z}}{\omega - kv_z} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z = 0 \quad . \tag{7.62}$$

4. Eqn (7.62) has a pole at $v_z = \omega/k$. We conveniently ignore this by assuming that the wave is such that the pole occurs at a velocity high compared with a thermal velocity, so that at the point where it occurs $\frac{\partial f_0}{\partial v_z}$ is negligible. Within this approximation we find the same dispersion relation as we found using the fluid model, but could work out the higher order terms if we wanted to:
$$\omega^2 = \omega_{\rm pe}^2 + 3k^2 v_e^2 \quad . \tag{7.63}$$

5. We associated the pole with particles travelling close with velocities close to the phase velocity of the wave. These particles experience an electric field that is almost constant in time, and are thus accelerated or decelerated by the wave. The trapped particles oscillate within two peaks of the wave, but because of the Maxwellian distribution function, there must originally have been more particles travelling slower than the wave (which are accelerated) than travelling faster than the wave. Therefore energy is transferred from the wave to the particles, and the waves damp. As they damp, some particles escape and are no longer trapped. Using this simple physical picture we managed to obtain an approximate expression for the damping coefficient, γ, of the electric field of the wave. A fuller analysis which actually took into account the mathematics of how to deal with the pole led to the following expression for the damping coefficient:
$$\gamma = -{\rm Im}(\omega) = \sqrt{\frac{\pi}{8}} \frac{\omega_{\rm pe}^4}{k^3} \left(\frac{1}{v_e^3}\right) \exp\left(-\frac{1}{2k^2\lambda_{\rm D}^2}\right) \quad . \tag{7.64}$$

   Note that the waves that are heavily damped are those with wavelengths of order, or shorter than, a Debye length.

Lecture 8

Magnetic Confinement Fusion

8.1 Fusion

One of the most promising technological uses for plasma physics is the creation of fusion energy. We consider below what conditions are necessary within a high-temperature plasma for fusion to be achieved. Clearly some of the ions involved in the fusion process must have enough thermal energy so that if they are, just due to random motion, approaching each other head-on, they have enough kinetic energy to overcome their mutual coulomb repulsion, and get close enough together for the attractive nuclear forces to start to dominate: this in turn means we need to heat the plasma to hundreds of millions degrees Kelvin. The Maxwellian-averaged reaction rates for several isotopes are plotted as a function of ion temperature in Fig. 8.1. The fusion reaction with the highest cross section is the Deuterium-Tritium reaction

$$\mathrm{D} + \mathrm{T} \rightarrow \alpha + n \quad , \tag{8.1}$$

where the alpha particle is ejected with an energy of 3.5 MeV, and the neutron with an energy of 14.1 MeV. Deuterium is available in sea-water in quantities that make its extraction commericially viable. Whilst Tritium is rarer, it can be manufactured during the fusion process itself from Lithium (which is very abundant – you all carry a sizeable chunk of it in you mobile phone battery), via the reaction

$$\mathrm{Li}^6 + n \rightarrow \mathrm{T} + \alpha + 4.8\mathrm{MeV} \quad . \tag{8.2}$$

Like fission power, fusion can produce enormous amounts of energy for a small amount of fuel. 1 kg of fuel per day would produce a sustained power output of 1 GW.

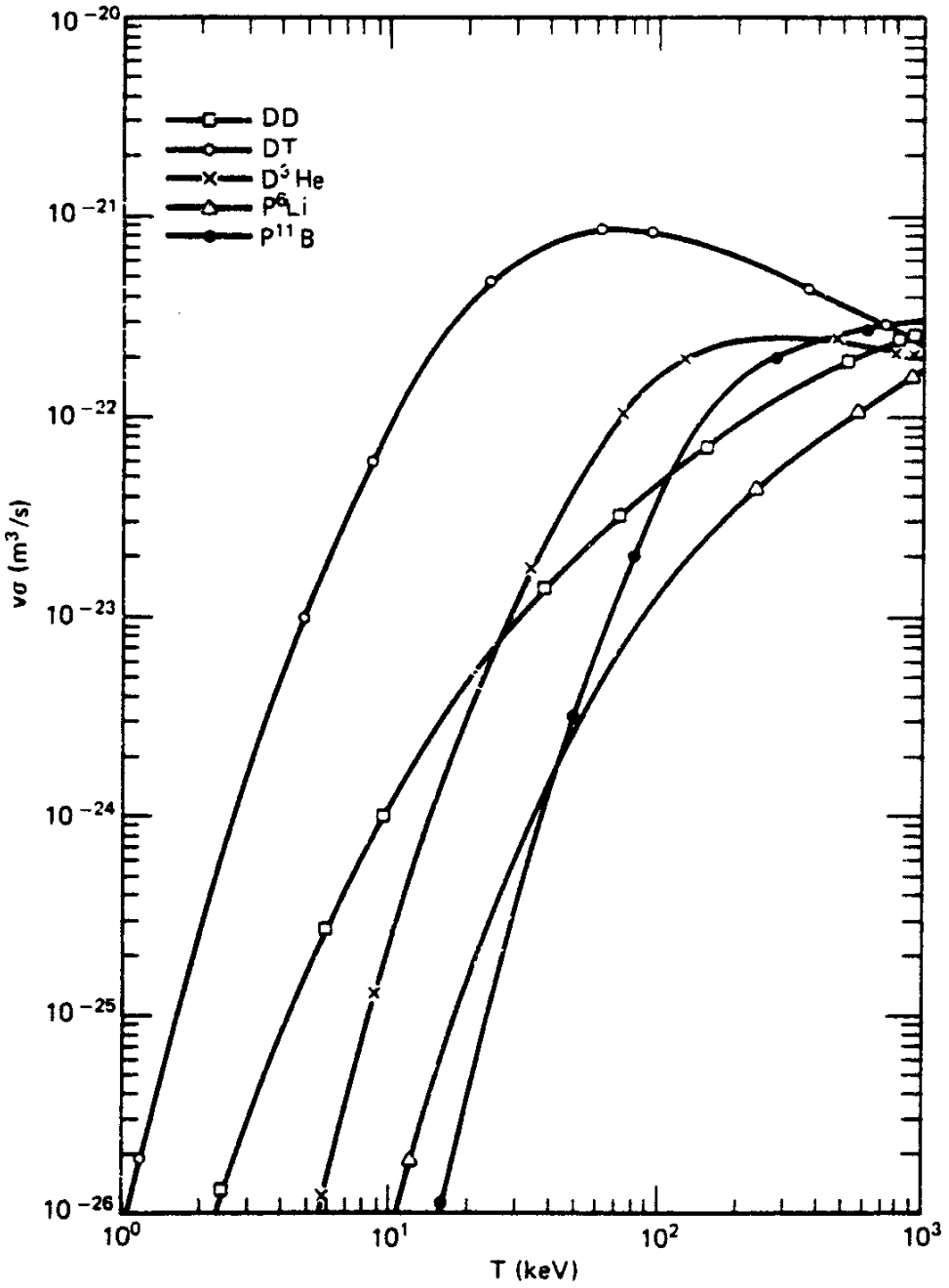


Figure 8.1: : Fusion cross section as a function of temperature for a number of isotopes of hydrogen. Note the largest cross section is for the D-T reaction.

8.2 The Lawson Criterion

In order to achieve energy gain the reacting nuclei must be confined long enough for a sufficient number of them to fuse together. The fusion energy released per unit volume, $E_{\rm f}$, in a D-T reaction can be written as

$$E_{\rm f} = n_{\rm D} n_{\rm T} \sigma W \tau \quad , \tag{8.3}$$

where n is the number density of the ions, σ the Maxwellian-averaged reaction rate, W the energy released per reaction, and τ the confinement time. Assuming an equal number of Deuterium and Tritium ions, $n_{\rm D} = n_{\rm T} = n/2$,

$$E_{\rm f} = \frac{n^2 \sigma W \tau}{4} \quad . \tag{8.4}$$

The energy required, $E_{\rm t}$ to heat the fuel to thermonuclear temperature, is given by

$$E_{\rm t} = \frac{3}{2} n k_{\rm B} T_i + \frac{3}{2} n_e k_{\rm B} T_e = 3 n k_{\rm B} T \quad , \tag{8.5}$$

where we have assumed $T_e = T_i$ for convenience.

In order to get energy gain, we need to get more energy out from the fusion reaction than we put in to heat up the plasma, i.e. we require $E_{\rm f} > E_{\rm t}$:

$$n\tau > \frac{12 k_{\rm B} T}{\sigma W} \quad . \tag{8.6}$$

This condition is known as the Lawson criterion. As we would expect, the density-confinement time product is a constant. A low density plasma needs to be confined for a long time in order for enough fusion reactions to occur in order for the fusion energy out to exceed the energy we put in to heat up the plasma in the first place. The denser the plasma, the shorter the necessary confinement time. Given the reaction rate shown in Fig. 8.1, which has a value of approximately 10^{-22} m^{3}s at a temperature of 10 keV, and that the energy produced per reaction is 17.6 MeV, we find that we require $n\tau > 10^{20}$s m^{-3}.

8.3 Introduction to Magnetic Fusion

In order to achieve fusion we have seen that we must create a hot plasma – at a temperature of order 10 keV, which corresponds to over 10^8 K. We have also seen that this plasma must be contained for at least a certain length of time in order for fusion to take place, this confinement time being determined by the Lawson criterion. There are two main approaches to trying to achieve fusion. The first, which we shall briefly study in this Lecture, is the method by which the hot plasma is confined by means of magnetic fields, and which is known as magnetic confinement fusion. As we shall see, this approach shows the potential to confine sparse plasmas (with particle number densities of order 10^{20} m^{-3}), and requires confinement times of order seconds (as can be seen from

the Lawson criterion). A second approach is to create plasmas which are almost a thousand times denser than a normal fluid, and use the inertia of the system to provide confinement – that is to say the nuclear reaction takes place before the plasma has time to blow apart. This second approach, inertial confinement fusion, is the subject of later lectures.

Before proceeding any further, it is useful to get some rough estimates of the numbers involved in magnetic fusion. We already know from our analysis of the fusion reaction that we need to confine a plasma at 10^8 K. Furthermore, it should be clear that we wish to confine as dense a plasma as we possibly can – not only with this reduce the time required to confine it to achieve fusion, but the power output of any final fusion plant (if such a thing is possible) will, for a given volume of plasma, scale as the square of the density. Given typical magnetic fields that are available in the laboratory, what sort of plasmas might we be able to confine? Well, the hot plasma will exert a pressure, and, if left unchecked, will wish to expand due to this pressure. It will be the magnetic forces that will check this expansion, and so we might suppose that we can confine a plasma such that the magnetic pressure is equal to the thermal pressure, and indeed this is a reasonable zeroth order approximation. Recall that pressure is equivalent to energy density, and that the energy density of a magnetic field is $B^2/2\mu_0$, thus we might expect to be able to confine a plasma with number density n such that

$$\frac{B^2}{2\mu_0} \approx nk_{\mathrm{B}}T \quad . \tag{8.7}$$

We recall that we can only produce fields of a Tesla, or few Tesla, in the laboratory, and thus, knowing that $T = 10^8$ K, it becomes clear that the maximum number density in the plasma will be of order 10^{20} m^{-3}, as stated above. Notice that such a plasma, being so sparse, is not at a high pressure even though it is very hot – the pressures involved are of order 10^5 N m^{-2}, that is to say of order an atmosphere.

It should be said that the above analysis is highly over-simplified. The magnetic field in any confinement device actually penetrates the plasma (the plasma is diamagnetic, but cannot exclude all of the field), and in fact all we can say is that in a stable configuration the total pressure, thermal plus magnetic, is constant, i.e.

$$\frac{B^2}{2\mu_0} + \Sigma nk_{\mathrm{B}}T = \text{constant} \quad . \tag{8.8}$$

8.4 Magnetic Mirror Revisited

One method of confining a plasma is with a magnetic mirror, which we studied in section 3.7. Recall that this device worked essentially on the principle of conservation of angular momentum. Charged particles rotate about field lines, and thus as the field lines become more concentrated, the rotational energy must increase at the expense of the kinetic energy along the field lines.

Recall that we denoted the magnetic field in the middle of the mirror by B_0, and that at the highest point of the field, the mirror point, by B_m. The particles are characterized by an angle, θ

$$\sin^2\theta = \frac{v_{\perp 0}^2}{v_0^2} = \frac{B_0}{B} \quad , \tag{8.9}$$

where θ is the pitch angle of the orbit in the weak field region, $v_{\perp 0}$ the velocity component perpendicular to the field lines in the centre of the mirror, and v_0 their total velocity at this point. If this angle is too small, the particle is not confined. Thus in velocity space there exists a *loss cone*: particles with pitch angles less than θ_m, where

$$\sin^2\theta_\mathrm{m} = \frac{B_0}{B_\mathrm{m}} \tag{8.10}$$

are lost. Those with larger pitch angles of orbit are trapped by the mirror. It is important to realise that trapped particles can still be lost if collisions alter their pitch angles such that the new angles put them into the loss cone. Although large magnetic mirrors have been built and investigated, it is probably fair to say that they are not considered a viable route forward for fusion, as they inevitably lose particles, and thus provide a limited confinement time.

8.5 The Z Pinch

In a magnetic mirror, the magnetic field lines are mainly along the axis. The Z-pinch operates in a different manner. Consider a cylindrical column of plasma through which a current is flowing, as shown in Fig. 8.2. The current itself not only heats the plasma, but generates a magnetic field that goes round the cylinder as shown. This field acts so as to confine the plasma further – the electrons have a velocity due to the current flow, and the $\mathbf{v}\times\mathbf{B}$ force pushes the electrons towards the centre of the cylinder. The ions feel a force in the same direction of equal magnitude (although they are oppositely charged, due to the current they travel in the opposite direction to the electrons). However, the electrons have a greater acceleration owing to their smaller mass – and thus the compression is due to the electrons travelling to the centre, and then dragging the ions due to electrostatic forces.

The Z-pinch serves as a good example of how we can work out the sorts of currents necessary for fusion relatively easily. We have already seen that to confine plasmas at fusion temperatures and densities we require magnetic fields of order Teslas. From Ampere's law the magnetic field just outside a cylinder of radius a carrying a current I is

$$B = \frac{\mu_0 I}{2\pi a} \quad . \tag{8.11}$$

Thus we see that even for a relatively small plasma of just a cm radius, we could require greater than 0.1 MA to confine the plasma. However, the main

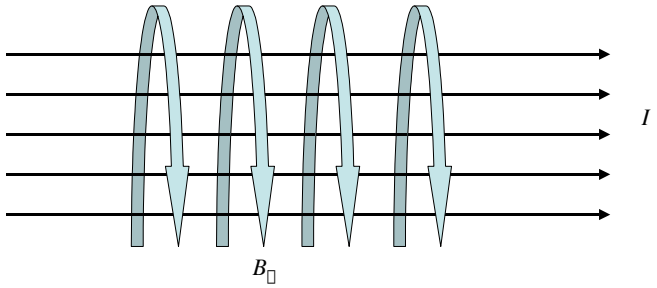


Figure 8.2: : In a Z-pinch the current along a cylindrical column of plasma generates a magnetic field. The $\mathbf{v} \times \mathbf{B}$ force confines the plasma.

disadvantage of the Z pinch is that it is unstable – the two most common instabilities being the bizarrely named sausage and kink instability. We will discuss these, albeit in a mainly qualitative way, as they are of relevance to other potential methods of confining plasmas magnetically.

8.5.1 The Sausage Instability

Up to now we have assumed that when a current passes through a Z pinch the magnetic field will keep it confined (and indeed compress it further), evenly along its length. However, as we shall show, this is not a stable configuration, and the sausage instability gets its name from the tendency of a Z pinch plasma to pinch down to a small radius at various points along its length, making the plasma look like a string of sausages in a butcher's shop. The confinement is poor, and fusion cannot be achieved, because the system only becomes hot and dense at a few specific points along the pinch, and thus the energy out cannot exceed the electrical energy put in to make the plasma in the first place.

In order to gain an idea of the physics of what goes on we refer to Fig. 8.3. Consider a region of the Z-pinch that has a radius slightly smaller than that of the surrounding region – this could occur just due to some random fluctuations. The magnetic field at the edge of the plasma must be given (assuming cylindrical symmetry) by $B = \mu_0 I/2\pi r$. Therefore, this region will have a larger magnetic field outside it, as it has a smaller radius, and therefore this region will feel a larger magnetic pressure than the surrounding region, and will tend to be

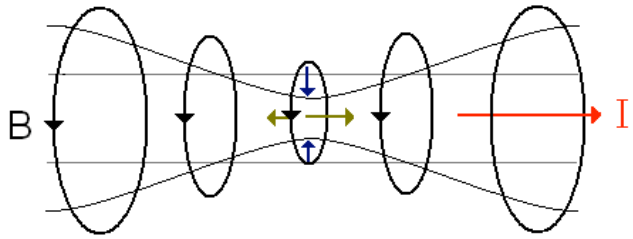


Figure 8.3: : In the sausage instability we consider a region of the plasma which, just from random fluctuations, has a slightly smaller radius than the rest of the plasma. This region has an increased magnetic field outside it as the current must be constant. The increased magnetic field compresses this region further, increasing the field, and thus the situation grows as an instability.

compressed more. As it is compressed further, the magnetic field rises (because r is decreasing), and so further compression takes place. We thus see that the whole plasma is unstable to this sort of situation, and we are unsurprised to learn that such a simple operation of a Z-pinch is not a good way to directly achieve fusion (although it was one of the first ways attempted). As the instability develops, plasma gets squirted sideways (due to the increased pressure in the compressed region), but although it can get quite hot, and some fusion reactions can occur, such reactions only ever take place over a very small region of the overall plasma (the squashed bits, where the sausages pinch off), that energy gain is completely infeasible.

8.5.2 The Kink Instability

Z-pinches are also unstable to the so-called kink-instability, shown schematically in Fig. 8.4. If a kink should develop within the plasma, again simply from some random noise, then the magnetic field lines shown will be closer together on the inside of the kink that on the outside. The magnetic pressure, $B^2/2\mu_0$, therefore acts to increase the size of the kink, and the instability grows. Eventually, this would lead to the plasma being pushed against the walls of any container, leading to a cooling of the plasma via contact with the walls, and high Z impurities from the walls entering the plasma further cooling it by radiation.

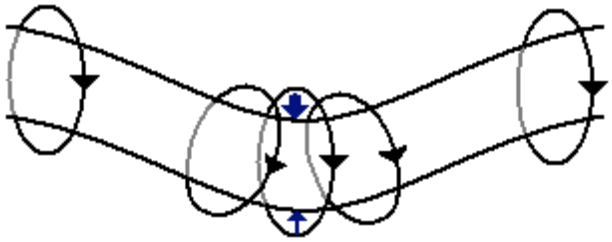


Figure 8.4: : In the kink instability we consider a region of the plasma which, just from random fluctuations, is slightly 'kinked', as shown above. This kink will grow as an instability as described in the text.

8.6 Tokamaks

One of the difficulties with the mirror machine is that a certain set of particles will always be lost from the system – it does not confine well. As noted above, there is a critical angle below which particles are free to leave the system, and this corresponds to particles with high momentum along the field lines. One obvious way to get around this sort of problem – i.e. non confinement along a field line – is to bend the mirror, or Z-pinch, into a torus, so that the plasma has no 'end'. However, simply bending a plasma into a torus still leaves behind several problems. Firstly, such a plasma is still unstable with respect to the sausage and kink instabilities. Secondly, such a plasma has an additional difficulty, which is shown in schematic form in Fig. 8.5. Recall what we learnt in section 3.6 – we found that a particle in a curved magnetic field, such as a torus, where the magnetic field varies as $1/r$, drifts. We found that the drift velocity was given by

$$\mathbf{v}_{\text{total}} = \frac{m}{qB^2}\frac{\mathbf{R}\times\mathbf{B}}{R^2}\left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2}\right) \quad , \tag{8.12}$$

that is to say that the drift is perpendicular to the radius of curvature and to the magnetic field – i.e. in the vertical direction in our diagram. However, we note that this drift velocity is proportional to the charge, and the ions and electrons thus drift in different directions as shown. This opposing drift of ions and electrons sets up an electric field, which gives rise to further drift, because we know that charged particles drift in the presence of an electric and magnetic field – that was our subject of study in section 3.4, where we found that the

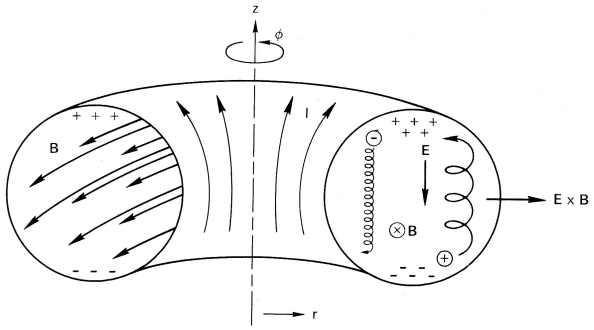


Figure 8.5: : A schematic diagram of a the magnetic field and forces acting in a simple torus.

drift velocity in this case was

$$\mathbf{v}_{\rm d} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \quad . \tag{8.13}$$

It is important to note in this case that $\mathbf{v}_{\rm d}$ is completely independent of q, m, and $v_\perp$. So particles drift in the same direction, whatever their mass, whatever their charge. Indeed, the implication for the torus is that the particles now drift towards the outside of the torus, and this drift will take place until the plasma hits the walls of the container.

The tokamak, put forward by the Russian scientist Artzimovich, is an attempt solve the problems outlined above. Although tokamaks look promising for fusion applications, it would be misleading to give the impression that all of the physics of such systems are still understood. The aim of this section is simply to give some very simple concepts concerning the advantages of the tokamak configuration.

The magnetic field configuration within a tokamak is shown schematically in Fig. 8.6. The most important feature of a tokamak is that it contains both poloidal and toroidal magnetic fields. The toroidal field – going the long way round the torus, is generated by external coils, as would be done in a simple torus. However, we know that on its own this is unstable, as discussed above. However, we also generate a poloidal field – i.e. rings of magnetic field that form shapes similar to the coils that are producing the toroidal field. This poloidal field is created by a current that we generate within the plasma, travelling in the toroidal direction (this is not as complicated as it sounds – see the diagram). The current is generated within the plasma by means of a large transformer.

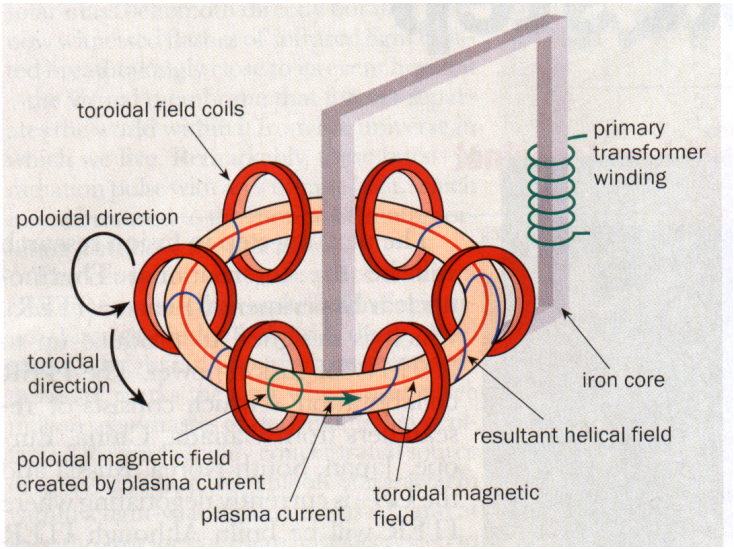


Figure 8.6: : A schematic diagram of a tokamak.

Using the transformer we can generate an extremely large $\partial B/\partial t$ though the middle of the torus, which in turn generates a large electromotive force around the torus, driving the current.

What are the advantages of adding a poloidal field to the toroidal field? Well, let us first consider the problem of particle drift. Recall that one of the difficulties with a simple torus was that the curvature drift caused particles of opposite charge to separate, giving rise to an electric field. The particles then drifted in the electric field, causing the torus to expand and hit the walls of the container. The poloidal field can overcome this problem – as is shown in Fig. 8.7. The combination of the two fields, toroidal and poloidal, means that the magnetic field lines are twisted as they go around the torus. That is to say that a magnetic field line on one side of the torus, on the outside (shown as point P on the diagram), is at position P' on the opposite side of the torus. This overcomes the particle drift problem, and we can see this in two different ways. Recall that in our model of the simple torus (rather than a tokamak), the ions drifted vertically upwards, and the electrons downwards due to the curvature drift. Well, if magnetic field lines that were at the top of the torus on one side of it are now towards the bottom, we have effectively connected the top of the torus to the bottom – it is as though we have shorted out the electric field that was being generated. An alternative way of looking at this is in terms of the single particle motion. On the right hand side in Fig. 8.7 the ion drifting upwards at point P will be moving away from the centre of the torus. On reaching point P', this drift component of its velocity now points towards the centre of the torus. Thus we conclude that as long as the thermal velocities of the particles are large compared with the drift velocities, then averaged over many cycles around the

torus, the drifts should cancel out, and the plasma remain stable.

The tokamak also provides some degree of stability against the sausage instability. The easiest way to see this is to go back and reanalyse the sausage instability for a Z-pinch, but now allow there to be a magnetic field along the pinch, as well as around it, as shown in Fig. 8.8 (a): clearly this is equivalent to the tokamak but neglects the curvature (we have already seen how the tokamak overcomes the drift problems due to curvature). We assume that the cylinder of plasma has a sharp boundary. In order to simplify the analysis we assum that the main magnetic field inside the plasma is longitudinal, B_z, and outside the plasma the main field is azimuthal, $B_\theta = \mu_0 I_z/2\pi r$. When the plasma radius a is changed by a small amount δa, the conservation of magnetic flux in the $z-$ direction implies

$$\begin{aligned} \delta(B_z \pi a^2) &= 0 \\ \therefore \delta B_z \pi a^2 + B_z \pi 2a\delta a &= 0 \\ \therefore \delta B_z &= -B_z \frac{2\delta a}{a} \quad . \end{aligned} \tag{8.14}$$

whilst conservation of current implies

$$\delta B_\theta = -B_\theta \frac{\delta a}{a} \quad . \tag{8.15}$$

The longitudinal magnetic field inside the plasma acts so as to prevent the instability, because a contraction of the plasma increases its component of the magnetic pressure. The difference in the two magnetic pressures, $\delta p_{\rm m}$, is thus given by

$$\begin{aligned} \delta p_{\rm m} &= \delta \left(\frac{B_z^2}{2\mu_0} - \frac{B_\theta^2}{2\mu_0} \right) \\ &= \frac{B_z \delta B_z}{\mu_0} - \frac{B_\theta \delta B_\theta}{2\mu_0} \quad . \end{aligned} \tag{8.16}$$

Substituting eqns (8.14) and (8.15) into eqn (8.16) we obtain

$$\delta p_{\rm m} = -\frac{B_z^2}{\mu_0}\frac{2\delta a}{a} + \frac{B_\theta^2}{\mu_0}\frac{\delta a}{a} \quad . \tag{8.17}$$

Thus the plasma is stable against the sausage instability as long as the following condition is met:

$$B_z^2 > \frac{B_\theta^2}{2} \quad . \tag{8.18}$$

A z-pinch with a magnetic field along the z-axis can also be configured so as to be stable against the kink instability, as shown in Fig. 8.8 (b). We will not go into the details of the physics of this, as it is quite complicated. It suffices to say that the B field in the $z-$ direction acts to give the plasma a degree of 'tension', and therefore there is a restoring force, like the restoring force of a stretched

string. A detailed analysis of this situation leads to the *Kruskal-Shafranov* limit, which states that the plasma is stable as long as

$$\left|\frac{B_\theta}{B_z}\right| < \frac{2\pi a}{\lambda} \quad , \tag{8.19}$$

where λ is the wavelength of the mode. When we apply this condition to a tokamak, where the longest wavelength mode is $2\pi R$, where R is the radius of the ring, is that the so-called $q-$factor must satisfy

$$q \equiv \frac{B_{\rm t}}{B_{\rm p}}\frac{a}{R} > 1 \quad , \tag{8.20}$$

where $B_{\rm t}$ is the toroidal field, and $B_{\rm p}$ the poloidal field.

One of the purposes of the current in the plasma is to create the poloidal field that helps overcome all of the instability problems that we have mentioned above. It is important to state that there overall physics is more complicated than the rather simplistic picture provided, but it catches a flavour of the main points. Of course, as well as making the plasma stable and confined, we also need to heat it to thermonuclear temperatures. Part of this heating occurs by the generation of the current within the plasma – the plasma has a finite resistance, and thus there is a certain amount of so-called ohmic heating, due to I^2R. Clearly we wish to use as high a value of current per unit area as we possibly can, in order to have as much heating as possible to get the plasma as hot as possible. However, there are two main constraints to this approach. Firstly, a large current per unit area implies a large value of $B_{\rm p}/a$, and thus a large current requires a low q. Although $q = 1$ is the theoretical limit, in practice stable operation with q less than 2 or even 3 is hard to achieve. Secondly, as the current is increased, and as the plasma gets hotter, the heating becomes less effective. This is because the resistance of the plasma decreases as the temperature increases. The resistance of a plasma will be proportional to the electron-ion collision time, and we found in section 6.3, that this scales as $T_e^{-3/2}$ (see eqn (6.11)). The net result is that fusion temperatures cannot be achieved by ohmic heating alone. Therefore, once the plasma has been confined by the toroidal current/poloidal field, additional heating mechanisms are introduced. Two of the most common of these are neutral beam injection and RF heating. In neutral beam heating ions are accelerated in a separate machine outside the tokamak, and the beam neutralized with electrons, and then shot into the plasma at ultra-high velocities. Radio frequency heating relies on resonantly exciting particular wave modes within the plasma with very high power radio waves. As these waves damp they convert their energy into thermal energy, thus heating the plasma.

8.7 Current Status

Significant international collaboration has taken place over many years into fusion research. Perhaps the best known tokamak in the UK is the one housed at

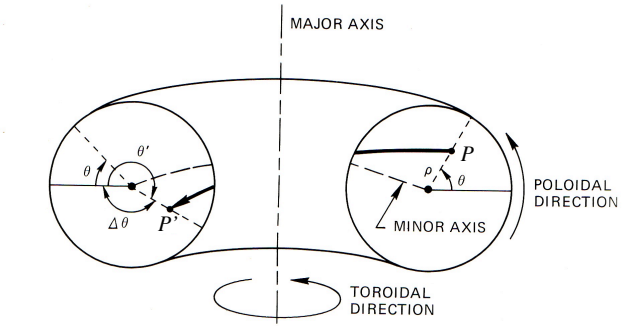


Figure 8.7: : A schematic diagram of a the magnetic field and forces acting in a tokamak. A magnetic field line on one side of the torus, towards the outside of the torus, is twisted towards the inside of the torus on the opposite side.

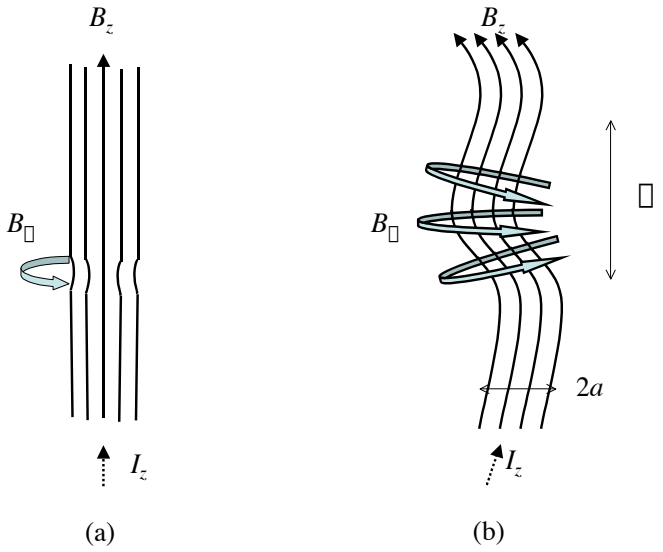


Figure 8.8: : A schematic diagram of the (a) sausage and (b) kink instability in a Z-pinch with a field along the pinch, B_z, as well as the B_θ induced by the current.

Culham, known as JET – the Joint European Torus. Confinement times of order seconds have been achieved on JET, and although breakeven has not quite been reached, the fusion power output has come close to the input power. Research on this and other machines has led to the point where there is increasing confidence that a fusion reactor can be made to work, though there are still significant obstacles to overcome. The next big step forward in fusion research will be the International Thermonuclear Experimental Reactor (ITER), and at the time of writing a vigorous debate is taking place over the siting of this machine – either in France or Japan. ITER is designed to produce up to 500 MW of fusion power. However, as well as getting the reactor to work in principle, significant research still needs to be undertaken to develop the materials that can withstand the high fluxes of particles that do impinge on the chamber walls.

8.8 Summary of Lecture 8

1. The fusion reaction with the highest cross section is the D-T reaction. Deuterium is plentiful in sea-water, and Tritium can be bred from Lithium.

2. To achieve fusion, the particles must be heated to high temperatures to overcome electrostatic repulsion between the nuclei. The plasma must be sufficiently dense, and the particles held for sufficient time, for enough fusion energy to be released to overcome the energy required to heat the plasma. This limitation on the density time product is known as the Lawson criterion.

3. Magnetic mirrors can confine particles, but particles travelling along the field lines are lost. Z-pinches can also confine due to the magnetic field set up by the current passing through them: however, they are unstable to the sausage and kink instabilities.

4. A field along the Z-pinch can stabilize to a certain extent the sausage and kink instability. To prevent losses from the ends, the ends of the pinch are joined to form a torus. The use of both toroidal and poloidal fields can stabilize the plasma against the sausage and kink instabilities, and is the field configuration in a tokamak.

5. A tokamak is characterized by its q factor:

$$q \equiv \frac{B_{\rm t}}{B_{\rm p}} \frac{a}{R} > 1 \quad , \tag{8.21}$$

   A large current (to heat the plasma) requires a small q, but 2 or 3 is difficult in practice. This limits the heating current. However, as the plasma gets hotter the ohmic heating becomes less efficient. Supplementary heating methods, such as neutral beam injection and RF heating are also used.

Lecture 9

Laser-Produced-Plasmas and ICF

9.1 Introduction

Plasmas can be created by shining extremely intense laser beams onto solid targets. The most common laser used in the field of laser-matter interactions is the Neodymium glass laser, which emits at a wavelength of 1.053 μm, though the light can be frequency doubled, or tripled, by the use of anharmonic conversion crystals. The physics of the high power lasers themselves is a field of research in its own right, and we will not investigate it further here. For the current discussion is suffices to know that the irradiances (power per unit area) that can be produced by the use of such lasers are truly enormous. Typically such large laser systems can produce pulses of laser light containing tens or hundreds of kJ of energy in pulses of order nanoseconds. When focussed by huge lenses down to focal spots with diameters of order a few hundred microns, the irradiances produced are of order 10^{14} to 10^{15} Wcm^{-2}.[1]

When such a laser beam is incident upon a solid immediately a small fraction of it is absorbed and a plasma is created at the surface. This plasma can further absorb laser energy (by a process we shall discuss below) and thus gets heated to a very high temperature. The experiments are carried out under vacuum, and thus the hot plasma expands away from the solid surface, into the vacuum, creating a low density region, known as the corona. By conservation of momentum, the expansion of this plasma into the vacuum must result in a force being exerted on the rest of the target. Clearly as the plasma expands its density drops. Once the plasma is created, the laser energy can no longer be deposited close to the solid target surface. Recall that the refractive index, μ,

[1] Although this is not an S.I. unit, it is the one in common use in the research field for historical reasons.

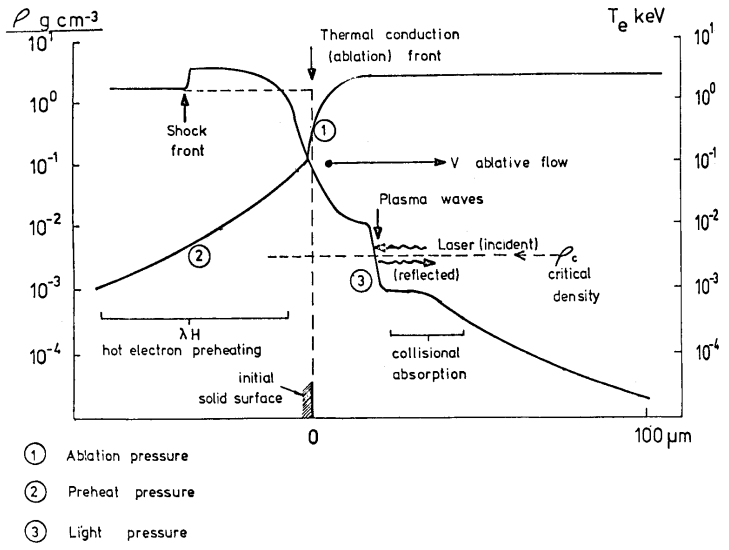


Figure 9.1: : Typical density and temperature profile as a function of distance from a laser-irradiated target.

of a plasma (see eqn (2.29)) is given by

$$\mu = \sqrt{\varepsilon(\omega)} = \sqrt{1 - \frac{ne^2}{\varepsilon_0 m\omega^2}} = \sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}} \quad , \tag{9.1}$$

where n is the number density of the electrons. Hence laser light can only propagate up to a certain density (being absorbed as it travels), known as the critical density, $n_{\rm c}$, at which point it is reflected. The surface of the plasma corresponding to this density is known as the critical surface. As the absorption increases as a function of density, but the laser cannot propagate beyond the critical surface, we expect the temperature of the plasma to be a maximum at the critical density. Between the critical surface and the solid the temperature must therefore drop. Heat from the hot plasma at the critical surface is conducted down this temperature gradient, and this conducted heat generates more plasma at the solid surface, keeping the process going. If a roughly steady-state situation is achieved, the velocity of the plasma must increase as a function of distance from the target, to conserve mass flow. A schematic diagram of the process is shown in Fig. 9.1. It can be seen that the temperature in the corona (which is what we call the region with densities below the critical density) is roughly constant – although the plasma is expanding (and hence cooling) in this region, this is somewhat compensated for by the fact that some laser energy is being deposited here.

The pressure also rises as we get closer to the target surface (although the

temperature is dropping, the density is rising), and reaches a maximum at the point close to which the plasma is created. This position is known as the ablation surface.

In practice the calculation of the hydrodynamic motion of the plasma is extremely complicated, and is carried out by computational techniques. However, there are several order of magnitude estimates we can apply that will give us some idea of what sort of temperatures, pressures, and velocities are associated with this process, and we will consider such a simple model in section 9.3. Before we discuss the ablation process, we first consider the mechanism by which the laser energy is absorbed at densities below, and up to, the critical density.

9.2 Inverse Bremsstrahlung Absorption

When a laser beam propagates in a plasma it causes the electrons to oscillate. If the peak electric field of the laser is E_0, and the frequency of the laser is ω, then the acceleration of the electron is given by

$$\ddot{x} = -\frac{eE_0}{m}\sin(\omega t) \quad . \tag{9.2}$$

Integrating we find

$$\dot{x} = \frac{eE_0}{m\omega}\cos(\omega t) \quad , \tag{9.3}$$

and thus the time-averaged kinetic energy of the electron oscillating in the field of the laser is

$$\overline{\frac{1}{2}m\dot{x}^2} = \frac{e^2E_0^2}{4m\omega^2} \quad . \tag{9.4}$$

Note that the totally oscillatory energy is actually twice this - just like any simple harmonic oscillator, the energy is divided between kinetic energy and potential energy.

Consider a set of isolated electrons subjected to such a laser field. As the laser field is turned on the electrons start to oscillate, and therefore themselves radiate with the same frequency as the laser. In steady state, therefore, no energy is lost from the laser field.

The situation in a real plasma, however, is different – the electrons are not isolated. As the electrons oscillate up and down in the laser field (we could consider the light beam to be linearly polarized along the z axis) they also have their thermal energy and associated velocity. Due to their random thermal velocities they will collide with the ions in the plasma. During these collisions, the extra laser-induced ordered oscillatory velocity along the z axis gets randomized, and thus the laser energy is transferred to the thermal energy of the electrons. This process is known as inverse bremsstrahlung absorption, and our task now is to find the absorption coefficient, κ, defined such that for light of intensity I

$$I = I_0\exp(-\kappa x) \quad , \tag{9.5}$$

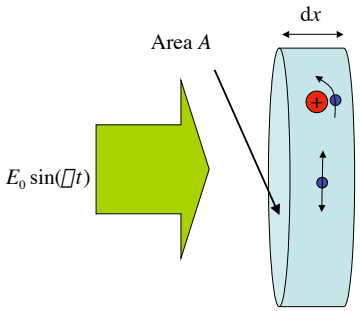


Figure 9.2: : Isolated electrons oscillate in the laser field. However, the random thermal velocities ensure that the electrons collide with ions, and this oscillatory motion is thus thermalized.

i.e.

$$\kappa = -\frac{1}{I}\frac{\mathrm{d}I}{\mathrm{d}x} \quad . \tag{9.6}$$

A very simple and hand-waving approach (which gets the correct answer to within a factor of two) to work out the absorption coefficient is as follows. Consider the diagram shown in Fig. 9.2. Light of intensity I (corresponding to peak electric field E_0) is incident upon a plasma containing n_e electrons per unit volume. The plasma has an area A and length $\mathrm{d}x$. Due to the laser field, and using eqn (9.4), the total extra oscillatory energy (kinetic+potential) of the electrons within this volume is

$$U_e = n_e \left(\frac{e^2 E_0^2}{2m\omega^2}\right) A\mathrm{d}x \quad . \tag{9.7}$$

This oscillatory energy is converted to normal thermal energy due to the collisions of the electrons with the ions. We know the time-scale for these collisions – it is simply the electron-ion collision time, τ_{ei}, that we derived in eqn (6.11) of section 6.3. Therefore during some time t, we expect the fraction of the oscillatory energy that is converted to thermal energy to be $\approx t/\tau_{\mathrm{ei}}$. Thus the change in oscillatory energy, dU_e, in time t, which we take to be equal to the change in energy of the laser field, $\mathrm{d}U_{\mathrm{L}}$, is given by

$$\mathrm{d}U_e = \mathrm{d}U_{\mathrm{L}} = -\frac{t}{\tau_{\mathrm{ei}}} n_e \left(\frac{e^2 E_0^2}{2m\omega^2}\right) A\mathrm{d}x \quad . \tag{9.8}$$

We need to know how this energy loss compares with the laser energy fed into the cylinder during time t, which is simply

$$U_{\rm L} = IAt \quad . \tag{9.9}$$

The absorption coefficient, from eqn (9.8) and (9.9) is given by

$$\begin{aligned}\kappa &= -\frac{1}{I}\frac{{\rm d}I}{{\rm d}x} \\ &= -\frac{1}{U_{\rm L}}\frac{{\rm d}U_{\rm L}}{{\rm d}x} \\ &= \frac{1}{IAt}\frac{t}{\tau_{\rm ei}} n_e \left(\frac{e^2 E_0^2}{2m\omega^2}\right) A \\ &= \frac{1}{I}\frac{1}{\tau_{\rm ei}} n_e \left(\frac{e^2 E_0^2}{2m\omega^2}\right) \quad .\end{aligned} \tag{9.10}$$

We note that the electric field of the laser, E_0, is related to the intensity via the Poynting vector:

$$I = N = \frac{1}{2}\sqrt{\varepsilon_{\rm r}}\varepsilon_0 E_0^2 c \quad . \tag{9.11}$$

Substituting eqn (9.11) into (9.10) we obtain

$$\kappa = \frac{1}{\tau_{\rm ei}}\left(\frac{n_e e^2}{\varepsilon_0 m\omega^2}\right)\frac{1}{c\sqrt{\varepsilon_{\rm r}}} \quad . \tag{9.12}$$

If we denote the plasma frequency associated with the local electron density by $\omega_{\rm p}$, this can be written

$$\kappa = \frac{1}{c\tau_{\rm ei}}\left(\frac{\omega_{\rm p}^2}{\omega^2}\right)\left(1 - \frac{\omega_{\rm p}^2}{\omega^2}\right)^{-1/2} \quad . \tag{9.13}$$

The true value of the absorption coefficient, evaluated by a more rigorous method, gives the same answer.

Equation (9.13) tells us some very important physics. We know that light cannot propagate in a plasma if its frequency exceeds the local plasma frequency, and the above equation shows us that as the density of the plasma increases, such that the plasma frequency becomes closer to the laser frequency, the absorption coefficient goes up very steeply. This is indeed, the case. Furthermore, as the critical density is higher for a higher plasma frequency, we would expect high frequency, short wavelength laser light to be more efficiently absorbed when a high power laser is incident upon a target. Although the calculation of the expected amount of absorbed energy is far from trivial, as the density of the plasma varies as a function of distance, it is a general rule that shorter wavelength light is more efficiently absorbed. As an example of this Fig. 9.3 shows the fraction of the absorbed light as a function of laser intensity for a variety of different laser wavelengths. This is one, but by no means the

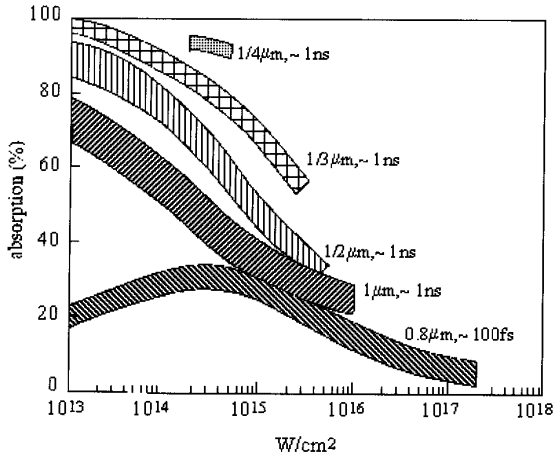


Figure 9.3: : Experimentally determined absorption fraction as a function of laser irradiance (W cm^{-2}) for various laser wavelengths. Note the trend that shorter wavelengths are more efficiently absorbed, and that the absorbed fraction tends to reduce at higher intensities – the plasma gets hotter during the laser pulse, which increases the electron-ion collision time.

only, reason why short wavelength light is used in fusion applications (typically the output from the high power infrared lasers, which emit around 1μm, is frequency-tripled by the use of anharmonic conversion crystals). It should also be noted from Fig. 9.3 that the fraction of the laser light absorbed decreases as the laser irradiance increases. The higher the irradiance, the hotter the plasma produced at the surface of the target: we recall that the electron-ion collision time, which features in the absorption coefficient in eqn (9.13), is proportional to $T_e^{-3/2}$, and thus a hotter plasma is less absorbing.

9.3 Ablation Model†

Let us now return to Fig. 9.1 from the introductory section, where we provided a schematic diagram of how we might expect the plasma parameters to vary as a function of distance from the target surface. This diagram is meant to depict what we might find if steady-state flow can be achieved. A full treatment of the analytic solutions to the plasma parameters (a few of which exist), would be far beyond the scope of this course. Instead, we will concentrate on looking at a very simple physical model which will provide us with sufficient tools to at least work out orders of magnitudes for the relevant plasma parameters.[2]

If we have a steady state situation, then the amount of material being removed – we use the term ablated – from the target, is constant during the laser

[2]This model, one of the simplest we can imagine for such a complicated process, was put forward by C. Fauquignon and F. Floux, Phys. Fluids **13**, 386 (1970)

pulse, i.e. the rate of mass flow at any point in the plasma, slightly out from the ablation surface, is a constant:

$$\text{flux of mass} = \rho u \quad , \tag{9.14}$$

where ρ is the plasma density, and u its flow velocity. Thus is we denote conditions just outside the critical surface by the subscript $_{\rm c}$ (i.e. slightly less dense than critical – for reasons that will become apparent), and those at the ablation surface by the subscript $_{\rm a}$, we find

$$\rho_{\rm a} u_{\rm a} = \rho_{\rm c} u_{\rm c} \quad . \tag{9.15}$$

Similarly we assume that the flux of momentum is also a constant:

$$P_{\rm a} + \rho_{\rm a} u_{\rm a}^2 = P_{\rm c} + \rho_{\rm c} u_{\rm c}^2 \quad , \tag{9.16}$$

(note the pressure comes into the above equation – momentum flux means rate of flow of momentum per unit area, i.e. force per unit area, i.e. pressure). Furthermore, we assume that the energy flux is constant. This equation is slightly trickier to construct. First we note that the plasma is expanding into a vacuum – its internal energy, pressure, and density are all changing as a function of distance, but remain constant at a particular point in space during steady state. Thus it looks like an isenthalpic throttling process – recall $H = U + PV$. We are looking at the flow of energy, and hence we assume that the flux of enthalpy is constant. However, it more complicated than this, because that does not take into account the energy tied up in the kinetic energy of the plasma – the velocity of mass flow. So we would conclude that enthalpy flow plus kinetic energy flow is constant. Furthermore, we must also take into account that the laser energy is being absorbed, at a rate we will denote by W per unit area. Most of this absorption will take place around the critical density (due to the higher density, and the lower group velocity of the light at critical – see eqn (9.13)). This absorbed energy flows down the temperature gradient, and in steady state must be a constant. As we have denoted by the subscript $_c$ the position just a little lower in density than critical, no energy is absorbed in this region, but it flows down the temperature gradient and is absorbed at the ablation surface. Thus our equation for conservation of energy flow reads

$$\frac{1}{2}\rho_{\rm a} u_{\rm a}^3 + H_{\rm a}\rho_{\rm a} u_{\rm a} + W = \frac{1}{2}\rho_{\rm c} u_{\rm c}^3 + H_{\rm c}\rho_{\rm c} u_{\rm c} \quad . \tag{9.17}$$

In order to make the analysis easier, we are going to simplify things by assuming that the plasma obeys an ideal gas equation of state, and therefore the enthalpy is given by

$$H = \left(\frac{\gamma}{\gamma - 1}\right)\frac{P}{\rho} \quad . \tag{9.18}$$

where γ is the normal ratio of heat capacities (5/3) – which, as you can imagine, is a gross oversimplification of the real physics.

In order to close the above equations, resulting in a physical model, we need to make some further assumptions. The first assumption we will make is that the critical density is much less than the density at which the material is actually being ablated away, i.e $\rho_{\rm a} \gg \rho_{\rm c}$. This is a pretty reasonable assumption: the critical density, determined by the frequency of the laser light, corresponds to about an electron number density of $10^{27}{\rm m}^{-3}$ for light of wavelength 1μm, and thus a mass density of order 1.67×10^{-3} g cm^{-3} – about a thousandth of solid density. Whilst the density at which the ablation front is situated cannot be equated with solid density, it is certainly large compared with critical density.

Our second assumption is the following:

$$W \gg \frac{1}{2}\rho_{\rm a} u_{\rm a}^3 + H_{\rm a}\rho_{\rm a} u_{\rm a} \quad . \tag{9.19}$$

How do we justify this? Well, remember that W represents the energy flow due to the heat (carried by the electrons) flowing down the critical surface towards the solid surface of the target. At some point that energy starts to eat away at the target, causing it to ionize and expand. At some point, that will be happening when the material is effectively stationary, $u_a \approx 0$, and the only source of energy flux is the thermal flux, W.

We make one final assumption. We have assumed that the laser energy is deposited at the critical surface, and we are assuming steady-state flow conditions. The electrons are carrying the heat down the temperature gradient to the target surface, but this conduction is taking place within a flowing environment, and the pressure is being applied within a flowing plasma. However, in a flowing fluid, information cannot travel faster than the speed of sound. Thus we make the assumption that the velocity of the plasma at the critical surface, where the laser energy is principally absorbed, is given by the sound velocity of the ions. Depending on the model this is taken to be either the isothermal speed of sound, or the adiabatic speed of sound. As we only seek order of magnitude estimates, it matters little which one we take – we will assume the adiabatic assumption (as was done in the work described in the footnote). Therefore we assume

$$u_{\rm c} = \left(\frac{\gamma P_{\rm c}}{\rho_{\rm c}}\right)^{1/2} \quad . \tag{9.20}$$

We are now in a position to solve the above equations and construct our model. From eqns (9.21) and (9.19)

$$W = \frac{1}{2}\rho_{\rm c} u_{\rm c}^3 + H_{\rm c}\rho_{\rm c} u_{\rm c} \quad . \tag{9.21}$$

Substituting eqn (9.20) and (9.18) into (9.21) we obtain

$$W = \frac{\rho_{\rm c}}{2}\left(\frac{\gamma P_{\rm c}}{\rho_{\rm c}}\right)^{3/2} + \left(\frac{\gamma}{\gamma-1}\right) P_{\rm c}\left(\frac{\gamma P_{\rm c}}{\rho_{\rm c}}\right)^{1/2} \tag{9.22}$$

$$W = \frac{1}{\sqrt{\rho_{\rm c}}} P_{\rm c}^{3/2}\sqrt{\gamma}\left(\frac{\gamma}{2} + \frac{1}{\gamma-1}\right) \quad . \tag{9.23}$$

Taking $\gamma = 5/3$, upon rearrangement we find

$$P_{\rm c} \approx 2\rho_{\rm c}^{1/3} W^{2/3} \quad . \tag{9.24}$$

At the ablation surface the velocity tends to zero, and thus, from eqn (9.16)

$$P_{\rm a} = P_{\rm c} + \rho_{\rm c} u_{\rm c}^2 \quad . \tag{9.25}$$

Substituting eqn (9.20) into (9.26) we see that

$$P_{\rm a} = (1+\gamma)P_{\rm c} = \frac{8}{3}P_{\rm c} \quad , \tag{9.26}$$

and thus, from eqn (9.24), we finally find that the ablation pressure – the pressure applied to the target, is

$$P_{\rm a} \approx \frac{16}{3}\rho_{\rm c}^{1/3} W^{2/3} \quad . \tag{9.27}$$

For a given absorbed energy from the laser, this is now a pressure we can evaluate. The critical density, $\rho_{\rm c}$, will depend on the wavelength of the laser light, as we recall the critical number density is just defined as the point where the plasma frequency equals the frequency of the light:

$$\omega^2 = \frac{n_{\rm c} e^2}{\varepsilon_0 m_e} \quad . \tag{9.28}$$

The mass density, $\rho_{\rm c}$, is given by $n_i M$, where n_i is the number density of ions of mass M. Let us assume that we are dealing with targets of reasonably low atomic number, such that the ions at the critical density are fully-ionised, so that $n_i = n_e/Z$, and roughly speaking $M \approx 2Zm_p$ – i.e. we assume nuclei with roughly equal numbers of protons and neutrons. Therefore we find

$$\rho_{\rm c} = 2n_{\rm c} m_p \quad , \tag{9.29}$$

note, importantly, eqn (9.29) tells us that the mass density at the critical surface is independent of the atomic number of the target.

Combining eqns (9.27), (9.28), and (9.29), we obtain

$$P_{\rm a} \approx \frac{16}{3}\left(\frac{2\varepsilon_0 m_e m_p \omega^2}{e^2}\right)^{1/3} W^{2/3} \quad . \tag{9.30}$$

In this research field we often quote W in units of W cm^{-2}, the wavelength of the laser, λ, in microns, and pressure in units of Mbar. In these units eqn (9.30) can be written

$$P_{\rm a} \approx C\lambda^{-2/3}\,(W)^{2/3} \ \text{Mbar} \quad , \tag{9.31}$$

where $C \approx 2.8 \times 10^{-9}$. That is to say, if we shine a laser with wavelength 1 μm onto a target with an irradiance of $2{\times}10^{14}$ W cm^{-2}, half of which is absorbed, we would expect to obtain a pressure of $2.8 \times 10^{-9} \times (10^{14})^{2/3}$, i.e. 6 Mbar.

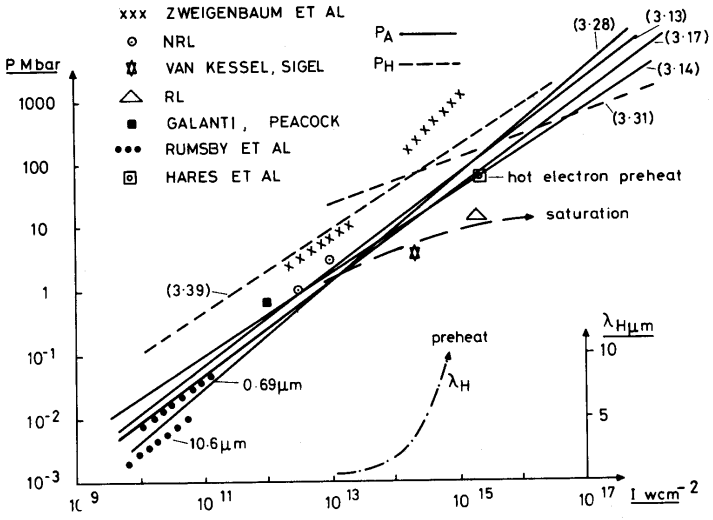


Figure 9.4: : Experimentally determined ablation pressure (Mbar) as a function of laser irradiance (W cm^{-2}).

Some experimental results are shown in Fig. 9.4. [3] Notice that, although simple, our model has produced answers which are indeed in the right ballpark. Not only is the pressure we predict roughly correct, but the scaling of the pressure as a function of the laser irradiance is also fairly consistent.

We are now in a position to use the model to find out other typical parameters within the laser produced plasma. For example, we can now calculate the expected temperature at the critical surface. First, let us estimate the temperature of such a plasma. The pressure at the critical surface is 3/8 of that at the ablation surface, and

$$P_{\rm c} = n_{\rm c} k_{\rm B} T_{\rm c} \quad , \tag{9.32}$$

which, combined with eqn (9.31) leads to

$$k_{\rm B} T_{\rm c} = B \lambda^{4/3} \left(\frac{W}{10^{16}} \right)^{2/3} \quad , \tag{9.33}$$

where $k_{\rm B}T_{\rm c}$ is in units of keV, and $B \approx 70$. In order to get a feeling for the orders of magnitude involved, we once more use the example of a 1μm laser with an irradiance of 2×10^{14} W cm$-$2, half of which is absorbed – we find a plasma temperature at the critical surface of $\approx$3 keV. By similar considerations we find

[3] Some of these results were obtained by a particularly simple method: the laser is shone onto a target suspended like the bob at the end of a pendulum. The recoil momentum was measured by noting the amplitude of motion of the pendulum after being hit, and assuming $PAt = Mv$.

that the velocity of the plasma at the critical surface for these conditions is of order a few times 10^5 ms^{-1}, i.e. several hundred km per second!

Finally, we can use this model to estimate the rate at which mass is ablated from a solid target, as this is simply $\rho_c u_c$ – for the above conditions this turns out to be a number of order 10^6 kg m^{-2} s^{-1}. That is to say, for a solid target with the density close to that of water or plastic (1 g cm^{-3}), during a 1 nanosecond laser pulse we would expect to ablate a layer approximately 1 μm thick from the surface. The above numbers and scaling laws are worth keeping in mind: high power lasers, incident upon solid targets at irradiances of order 10^{14} W cm^{-2} produce plasmas with keV temperatures, apply pressures of several Mbar, and ablate away microns of material within a nanosecond, such that at the critical surface it is moving at a velocity of several hundred km per second.

Finally, it is important to stress that the model given above is extremely simplistic, and is best used as an aid to understanding the main physical processes, and a rule of thumb in planar geometry, rather than anything more. Many laser-plasma experiments are performed with the irradiation of spheres, resulting in spherically-symmetric flow, which result in a different set of analytic scaling laws. In practice the situation is further complicated by the fact that the absorption changes throughout the laser pulse, and clearly is not all deposited at the critical surface. We have also ignored the energy that goes into ionizing the material etc. The best representations of this process we can obtain rely on complex time-dependent hydrodynamic calculations, and even these have limitations.

9.4 Inertial Confinement Fusion

Although plasmas can be confined by magnetic fields, there is another method by which they can be confined, at least for short time-scales, which on the face of it is far simpler. Imagine a sphere of hot plasma sitting in a vacuum. What happens? Well, clearly as it is at a high pressure it will expand into the vacuum. However, it cannot do this infinitely fast, it can't expand faster than the speed of sound. Thus, at least for a short time – which we might define as the time taken for it to double its radius – the plasma is confined by its own inertia, and unsurprisingly this is known as inertial confinement. Clearly this confinement time is going to be short, but if the plasma is dense enough, we might still be able to achieve enough fusion reactions during the disassembly process.

Let us consider a sphere of plasma of radius r, which is heated to such a temperature that the sound speed is C_s. From the argument given above we can approximate the confinement time, τ, as

$$\tau \approx \frac{r}{4C_s} \quad . \tag{9.34}$$

The number density of the ions is related to the mass density of the plasma – $n = \rho/M$, where M is the mass of an ion. Therefore another way of writing the

Lawson criterion is

$$n\tau = \frac{\rho r}{4C_{\rm s}M} > 10^{20} \quad \text{s m}^{-3} \quad . \tag{9.35}$$

If we insert the sound speed at 10 keV into the above equation, then we find this form of the Lawson criterion can be written as $\rho r > 0.6$ g cm^{-2}.

9.4.1 The Need for Compression

We are now in a position to work out how much laser energy we require to produce a sphere of plasma at 10 keV which satisfies the Lawson criterion, such that we just achieve energy gain from a fusion reaction. We substitute eqn (9.34) into eqn (8.4) to obtain

$$E_{\rm f} = \frac{n^2\sigma W r}{16C_{\rm s}} \approx E_{\rm t} = 3nk_{\rm B}T \quad . \tag{9.36}$$

Rearranging, we find an expression for the radius of the plasma

$$r = \frac{48k_{\rm B}TC_{\rm s}}{n\sigma W} \quad . \tag{9.37}$$

Now E_f is the fusion energy released per unit volume. If we wish to use a laser pulse, containing a total energy U, to heat up the sphere to achieve fusion, then for a sphere of radius r the amount of energy required will be

$$U = \frac{4\pi r^3}{3}E_{\rm f} \quad . \tag{9.38}$$

Substituting eqns (9.36) and (9.37) into (9.38) we find

$$U = 4\pi\left(\frac{48k_{\rm B}TC_{\rm s}}{n\sigma W}\right)^3 nk_{\rm B}T \quad . \tag{9.39}$$

If we evaluate this once more at 10 keV we find

$$U \approx 4\times10^{68}\frac{1}{n^2} \text{ J} \quad . \tag{9.40}$$

Let us imagine that somehow we could get some deuterium/tritium in the form of a liquid, and (somehow) could heat it quickly with a laser beam to form a 10 keV plasma. Liquid DT has a number density, n of order 5×10^{29} m^{-3} (a density, ρ, of order 1 g cm^{-3}). Inserting this value into eqn (9.40) shows us that we would require a total laser energy far in excess of 10^9 J (!). This is more than a thousand times greater than any laser every built. Indeed, it is only very recently that lasers, with nanosecond pulselengths, but with MJ energies have started to be constructed. Even then there are still many problems, because all of the above analysis was very handwaving, and didn't take into account any losses or inefficiencies. Furthermore, this is the energy we need to put in to the system, just to get the same amount of energy out, thus we would be getting

at least 1GJ of energy out of the plasma as a fusion reaction in a single laser pulse. This is a truly enormous amount of energy, which would be difficult for a vacuum chamber to withstand on a repeated basis. One GigaJoule is equivalent to about 0.2 tons of TNT. Clearly if we are heating D-T at liquid density, and the Lawson criterion is that $\rho r > 0.6$ g cm^{-2}, the radius of our D-T plasma would need to be about 0.6 cm – the size of a marble: we would effectively be making a miniature H-bomb of marble dimensions. All these considerations might lead us to conclude that laser fusion is a complete pipe-dream.

However, that isn't quite true. The point is that the laser energy required scales as the inverse square of the density. If we could somehow create and heat a plasma at a very high density, the total energy requirements come down to something more realistic. This is indeed what is being attempted in inertial confinement fusion (ICF). Compression of the fuel by, say, a factor of 100, would, at least in principle, bring the required laser energy down by a factor of 10^4, to say 100 kJ – a number within that achievable with present technology. [4] The basic idea is shown schematically in Fig. 9.5. We take a spherical shell (which can be made of plastic) of a few hundred microns to a couple of millimetres in diameter, and perhaps tens of microns thick, and illuminate it with many laser beams from all directions (in fusion research, a single laser beam from the oscillator is split into more and more beams as it passes through amplifiers, eventually allowing the focussing down of many beams onto the target from many directions). The beams create a plasma at the surface, which expands into the vacuum, applying pressure to the shell (as we have seen in section 9.3). The shell accelerates like a spherical rocket, and implodes, compressing the DT to a very high density – many, many times liquid density. The laser ablation process is timed such that at the peak compression a shock wave converges to the centre of the DT fuel, causing a high-temperature spark to initiate a fusion reaction in the centre of the target. The energy released from this small hot-spot region in the form of alpha particles heats the surrounding fuel, in a phase known as alpha-burn. As the reaction takes place at a very high density, the Lawson criterion is satisfied for a pellet-core of a reasonable size (tens to hundreds of microns), which does not need such a large amount of energy to fuse, and corresponding less energy is produced per laser shot, and thus the reaction can take place in a more controlled manner.

It is the spherical geometry of the implosion which is crucial to the success of this approach. Compressing a plasma to hundreds or thousands of times solid density requires an enormous amount of pressure (of order 10^{10} bar, as we shall see below). From section 9.3 we know that high power lasers, in the operational region of interest, only apply pressures of order Mbar to tens of Mbar. The spherical implosion, however, acts to 'amplify' the applied ablation pressure. The shell of the spherical target is accelerated by the ablation pressure throughout the course of the laser pulse, and when convergence occurs at the centre, the kinetic energy due to the motion of the shell is converted back into

[4] Because of inefficiencies in the process, current understanding and detailed simulations show that a laser with energy of order 1 MJ will be required, as is being built at Lawrence Livermore National Laboratory (the NIF laser), and the MegaJoule laser in Bordeaux.

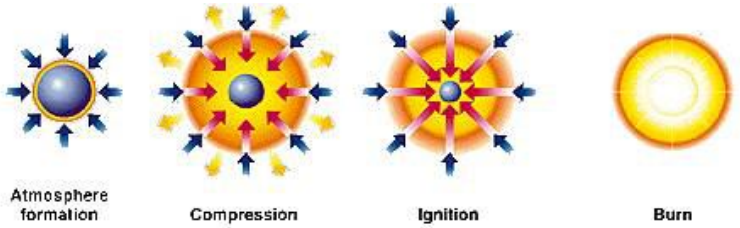


Figure 9.5: : In inertial confinement fusion high-power lasers irradiate a hollow shell, containing deuterium and tritium, from all sides. Acceleration of the shell causes amplification of the pressure upon convergence at the centre, leading to a high density.

energy of compression, resulting in a high pressure.

9.5 Pressure required for Compression

We have noted that in order to achieve inertial confinement fusion using lasers with parameters within those provided by current technology we must compress the D-T fuel. In order to estimate the pressures which must be generated, we must take into account the fact that the plasma in the compressed state will be so dense that the electrons within it will obey quantum (Fermi-Dirac) rather than classical statistics (if we heat it too much as it is compressed, it might obey classical statistics, but the best we can do is to compress it adiabatically, and hope this keeps it sufficiently cool – thus this is a lower estimate of the required pressure). We recall that the pressure of a Fermi gas, $P_{\rm F}$ is given by

$$P_{\rm F} = \frac{2}{5} n E_{\rm F} = \frac{2}{5} \frac{\hbar^2 n}{2m} \left(3\pi^2 n\right)^{2/3} \quad , \tag{9.41}$$

where E_F is the fermi energy, and n the number of electrons per unit volume. In normal hydrogen we have an electron density of about 6×10^{29} m^{-3}, and thus we need to compress the fuel to an electron density of order 6×10^{31} m^{-3} , which corresponds to a fermi energy of 550 eV, and a pressure of 20,000 MBar. This is the sort of pressure we need to achieve if fusion is to occur with the reduced laser energy allowed by the compression technique.

9.6 Pressure Amplification

The spherical compression effectively amplifies the ablation pressure. Consider a spherical hollow capsule, of initial radius r_0, and initial thickness Δr, to which we apply an ablation pressure $P_{\rm a}$. We assume that when the walls converge at the centre of the sphere, the kinetic energy acquired by the shell is converted to an energy of compression (i.e. a pressure). We make an order of magnitude estimate of the amplification factor as follows.

The initial acceleration, a, can be calculated, as the total integrated magnitude of the 'force' on the shell (regardless of its direction) is its mass multiplied by the acceleration:

$$F = P_{\rm a} 4\pi r_0^2 = \rho_0 4\pi r_0^2 \Delta r a \quad , \tag{9.42}$$

where ρ_0 is the initial density. As we are seeking order of magnitude estimates, we assume this acceleration is constant during the implosion. The final velocity of the shell is

$$v^2 = 2as \quad , \tag{9.43}$$

where s is the distance travelled by the shell. The final radius of the sphere is small, so we assume that $s = r_0$, the initial radius of the sphere. Thus, from eqns (9.42) and (9.43) we see that the final kinetic energy is

$$\begin{aligned} \frac{1}{2}mv^2 &= mas \\ &= m\frac{P_{\rm a}}{\rho_0 \Delta r} r_0 \\ &= V P_{\rm a} \left(\frac{r_0}{\Delta r}\right) \quad . \end{aligned} \tag{9.44}$$

As pressure is energy per unit volume, if this kinetic energy is converted into energy of compression adiabatically, then the stagnation pressure, $P_{\rm s}$ is

$$P_{\rm s} = \frac{1}{V}\frac{1}{2}mv^2 = P_a \left(\frac{r_0}{\Delta r}\right) \quad . \tag{9.45}$$

The ablation pressure has been amplified by a factor which is simply the initial radius of the sphere, divided by the width of the shell wall – the so-called 'aspect-ratio'. In ICF it is anticipated that the ablation pressure will be of order 100Mbar (0.35 μm light will be used at an irradiance of order 10^{15} Wcm^{-2}). Thus, with this simple analysis we would conclude that in order to achieve the pressures necessary for fusion we need to use targets with aspect ratios of order 200. In practice this number is predicted to be slightly smaller than this – somewhere around 50 or so. This difference of a factor of 4 is partly due to the fact that the mass of the shell reduces during the implosion, due to ablation, increasing the velocity of the rest of the shell for a given ablation pressure. Also, we should recall that all of the above calculations are very rough, and intended only to give a feel for orders of magnitude, and the underlying physics.

9.7 Limitiations on Aspect Ratio: The R-T Instability

In the previous section we have found that the pressure of the compressed core can be greater than the ablation pressure applied by the laser by a factor of order the aspect ratio. Therefore we wish to use large aspect ratio targets. However, there is a severe limitation on the aspect ratio that can be used in practice

due to a hydrodynamic instability known as the Rayleigh-Taylor instability. The physics underlying this instability will be described in more detail below, but essentially the effect of the instability is to cause the shell wall to break up before compression is achieved. The target can never be made absolutely perfectly spherical (there will always be some degree of surface roughness), and the target can never be subject to perfect spherical illumination (we shine laser light at it from all sides, using many beams, but without an infinite number of beams we could never achieve perfect spherical illumination). Therefore, beit due to the target itself, or due to the illumination, there will always be some deviation from perfect spherical symmetry. The Rayleigh-Taylor instability is an instability that causes these deviations from perfect spherical symmetry to grow exponentially as the target is being compressed. If these deviations become as large as the shell width itself, the target has essentially broken up, and full compression is not achieved. We will find below that this places a limitation on the maximum aspect ratio that can be safely used.

Before we show how the instability limits the useful aspect ratio, let us explain further the nature of the R-T instability, and explore how it comes about in laser fusion targets. The R-T instability is a classical instability that arises in fluid dynamics. Imagine taking a glass of water, and trying to balance a layer of liquid mercury on the top of the water. Although this would be tough to do in practice, there is nothing stopping us doing this in principle. The heavy fluid (mercury), would exert a force on the water underneath, increasing the pressure in the water, and thus the mercury would be supported, as shown in Fig. 9.6. However, the situation is extremely unstable. If there is any perturbation to the interface the mercury would rapidly fall to the bottom, and we would get the situation that we might intuitively expect – i.e. the denser fluid at the bottom. This is the R-T instability – a heavy fluid supported by a light fluid in a gravitational field. Another way of putting this is to say that the instability arises when the pressure gradients in the system are in the opposite direction to the density gradients (the light fluid has a higher pressure, caused by the weight of the heavy fluid on top).

The R-T instability occurs in laser-produced plasmas around the ablation surface. The high pressure plasma that is acting on the rest of the target is clearly lower in density than the rest of the target, and thus at the ablation surface the R-T instability can develop because the pressure gradient and the density gradient are opposed: the dense target is being accelerated by the less-dense plasma.

9.7.1 R-T Growth Rate†

As the R-T instability is a fluid instability, it will hardly come as a surprise to you to find that we use the fluid equations to describe it, and to determine its growth rate. We will consider a heavy fluid, of density ρ_2, on top of a light fluid of density ρ_1, in a gravitational field. We use the equations of conservation of particle number and of momentum – essentially the same equations as (6.13)

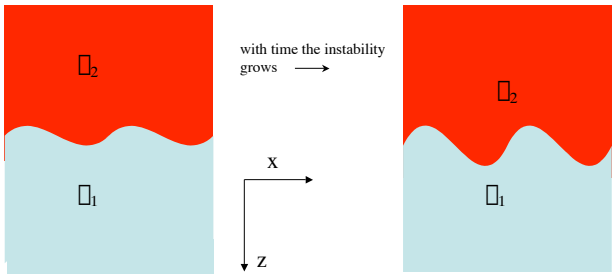


Figure 9.6: : A heavy fluid, of density ρ_2, can be placed on top of a less dense fluid of density ρ_1 – the pressure in the less dense fluid is greater, and hence the dense fluid is supported. However, if the interface is unstable, and perturbations to it will grow exponentially. This is the Rayleigh-Taylor instability, which occurs whenever the density and pressure gradients in a fluid are opposed.

and (6.14).

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 \tag{9.46}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - \rho g \hat{\mathbf{z}} \quad , \tag{9.47}$$

where g is the acceleration due to gravity, P the pressure, and $\mathbf{u}$ the velocity of the fluid. In equilibrium the fluid is at rest, so $\mathbf{u} = 0$. However the pressure and the density depend on z, i.e. $P = P(z)$, and $\rho = \rho(z)$. As we have done on previous occasions, we are going to use the method of linearization to look at the perturbations to this system: we perturb the pressure and the density, and give the fluid some non-zero velocity –

$$P \to P + \delta P \tag{9.48}$$

$$\rho \to \rho + \delta \rho \tag{9.49}$$

$$\mathbf{u} \neq 0 \quad . \tag{9.50}$$

Although the equilibrium values of P and ρ are simply functions of z, the perturbations will obviously depend on both x and z (if we think of ripples on the interface in one direction only then the problem becomes 2-D). Inserting eqns (9.48), (9.49) and (9.50) into eqn (9.46) we find

$$\frac{\partial (\rho + \delta \rho)}{\partial t} + \mathbf{u} \cdot \nabla (\rho + \delta \rho) = 0 \quad . \tag{9.51}$$

Keeping only the terms linear in the perturbation, and subtracting eqn (9.46) from (9.51) we obtain the equation of conservation of mass for the perturbation

terms:

$$\frac{\partial(\delta\rho)}{\partial t} + u_z \frac{\partial \rho}{\partial z} = 0 \quad , \tag{9.52}$$

where u_z is the z-component of the velocity of the fluid.

We are now going to carry out the same procedure with the equation for continuity of momentum: we insert eqns (9.48), (9.49) and (9.50) into eqn (9.47), and by keeping the terms linear in the perturbation, and subtracting the unperturbed values, we find

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial(\delta P)}{\partial x} \tag{9.53}$$

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial(\delta P)}{\partial z} - g(\delta\rho) \quad . \tag{9.54}$$

in the x and z directions respectively.

Equations (9.52), (9.53) and (9.54) are the equations that determine the motion and growth of the perturbed terms. We assume a wavelike solution, growing at a rate γ – that is to say we look for solutions to the above three equations of the form

$$\mathbf{u} = \mathbf{u_0} \exp(ikx) \exp(\gamma t) \tag{9.55}$$

$$\delta\rho = \delta\rho_0 \exp(ikx) \exp(\gamma t) \tag{9.56}$$

$$\delta P = \delta P_0 \exp(ikx) \exp(\gamma t) \quad . \tag{9.57}$$

Substituting the wave-like solution for the density perturbation, eqn (9.56), into the equation of conservation of mass for the perturbation, eqn (9.52), we obtain

$$\gamma\delta\rho = -u_z \frac{\partial \rho}{\partial z} \quad . \tag{9.58}$$

And substituting the wave-like solution for the velocity and pressure perturbations, eqns (9.55) and (9.57), into the x and z components of the equation of conservation of momentum for the perturbation, eqns (9.53) and (9.54), we obtain

$$\gamma\rho u_x = -ik\delta P \tag{9.59}$$

$$\gamma\rho u_z = -\frac{\partial(\delta P)}{\partial z} - g\delta\rho \quad . \tag{9.60}$$

These three equations, (9.58), (9.59), and (9.60), now need to be solved to give the growth rate of the instability, γ, for a given pressure profile, $P(z)$, and density profile, $\rho(z)$. The amplitude of the perturbations we finally seek clearly grows in the z direction, yet eqn (9.59) contains a term in the velocity in the x-direction. In order to get all of our equations in terms of the co-ordinate z, we assume that the fluid is incompressible (even though its density at a given point can change, this happens because a part of the fluid from an originally

different position has flowed into that point in space, not that because any piece of the fluid has changed its density). As the fluid is incompressible,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \tag{9.61}$$

$$iku_x + \frac{\partial u_z}{\partial z} = 0 \quad . \tag{9.62}$$

We use the expression for u_x in eqn (9.62) in eqn (9.59) to obtain

$$\delta P = -\frac{\gamma\rho}{k^2}\frac{\partial u_z}{\partial z} \quad , \tag{9.63}$$

and substituting this expression for δP, along with the expression for $\delta\rho$ from eqn (9.58), into eqn (9.60), we obtain the result we have been seeking

$$\frac{\partial}{\partial z}\left(\rho\frac{\partial u_z}{\partial z}\right) = k^2\rho u_z - \frac{gk^2}{\gamma^2}u_z\frac{\partial\rho}{\partial z} \quad . \tag{9.64}$$

This is the equation that describes the evolution of the fluid after the perturbation.

However, as well as the information we have gleaned from the conservation equations, there are certain boundary conditions we can apply. Let us take $z = 0$ to be the initial boundary between the two fluids, which have different densities. Clearly u_z is continous across this boundary (no gap opens up between the two fluids - the fluid just either side of the interface must move at the same velocity). Furthermore, the spatial derivative, $\partial u_z/\partial z$, must also be continuous across the interface by applying the same condition to u_x, and considering eqn (9.62). Although the velocity and its spatial derivative are continuous across the interface, other quantities may not be (for example, the density itself is clearly not continuous). Therefore if we integrate eqn (9.64) across an infinitessimally small segment which crosses the boundary, we obtain the constraint

$$\Delta\left(\rho\frac{\partial u_z}{\partial z}\right) = -\frac{gk^2}{\gamma^2}u_z\Delta(\rho) \quad , \tag{9.65}$$

where

$$\Delta f \equiv f(0)_+ - f(0)_- \tag{9.66}$$

is the jump in the quantities across the boundary, where the + and - signs denote the limits from the positive and negative sides. That is to say, for fluids 1 and 2,

$$\rho_2\left(\frac{\partial u_z}{\partial z}\right)_2 - \rho_1\left(\frac{\partial u_z}{\partial z}\right)_1 = -\frac{gk^2}{\gamma^2}u_z(\rho_2 - \rho_1) \quad . \tag{9.67}$$

We assume that the z component of the velocity grows with the same wavevector, k, but increases exponentially rather than oscillates, i.e.

$$u_z = A\exp(kz) + B\exp(kz) \quad . \tag{9.68}$$

Given that u_z is continuous at the boundary, but must be zero at $\pm\infty$, we see that

$$u_z = A\exp(kz) \qquad (z<0) \tag{9.69}$$
$$u_z = A\exp(-kz) \qquad (z>0) \quad . \tag{9.70}$$

Substituting these equations into eqn (9.67) we obtain,

$$-(\rho_2+\rho_1)k = -\frac{gk^2}{\gamma^2}(\rho_2-\rho_1) \quad , \tag{9.71}$$

which upon rearrangement, gives the growth rate for the instability between two fluids in a gravitational field:

$$\gamma = \sqrt{\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)gk} \quad . \tag{9.72}$$

9.7.2 Limitations on Aspect Ratio

Given the expression for the growth rate, eqn (9.72), we are now in a position to see how the R-T instability limits the useful aspect ratio for a target. For the moment we will assume the worse possible case for the growth rate – that the 'fluid' being accelerated is so much dense than the one that is pushing on it, that the growth rate is simply equal to $\sqrt{ka}$. If there is some perturbation to the compression process – perhaps the shell is not a perfect sphere to start off with, or, as there are a finite number of laser beams illuminating the target, the pressure being applied has not got perfect spherical symmetry – then as soon as this perturbation is introduced, it will start to grow via the R-T instability. It grows exponentially, and can potentially become so large that the shell breaks up before compression is achieved.

Given the growth rate, we see that the perturbation that will grow most quickly is the one with the largest value of k – i.e. the shortest wavelength mode. However, a particular mode cannot keep growing for ever – in the end it must saturate (our formula for the growth rate was for small amplitude perturbations in the linear regime) – typically this saturation will happen when the amplitude of the mode has reached around the same value as its wavelength. Thus, if we are worried about target breakup, the most dangerous, fastest growing mode is the one with a value of k of order $1/\Delta r$, and for this mode we cannot let the total number of $e-$foldings by which the perturbation grows exceed some set amount – i..e we cannot let γt exceed some constant. As a simple example, imagine we can manufacture targets with shell thickness of order 50 μm, which have a surface roughness of 100 Å(and the surface roughness, rather than the illumination, was the cause of the problem). Then we could only live with $\log(5\times10^{-5}/10^{-8}) = 8.5$ $e-$foldings of the instability before shell break-up. That is to say

$$\gamma t = \sqrt{ka}t = \sqrt{\frac{a}{\Delta r}}\; t \le 8.5 \quad . \tag{9.73}$$

However, the acceleration of the target is related to the total distance travelled (the shell radius) before compression after time t:

$$a = \frac{2r}{t^2} \quad . \tag{9.74}$$

Substituting eqn (9.74) into (9.73) we obtain

$$\sqrt{\frac{2r}{\Delta r}} \leq 8.5 \quad , \tag{9.75}$$

that is to say we need the aspect ratio of the target to be less than about 36. We recall that we have stated above that we need aspect ratios of about 50 in order to get the pressure amplification via compression – and thus on this rough estimate it seems that inertial confinement fusion is not possible. However, things may not be quite this bad. Our estimate of the growth rate of the R-T instability being equal to $\sqrt{ka}$ is an overestimate: this would be the rate if the ablation surface was equivalent to a classical instability with one fluid very much heavier than the other. In fact, there is a density gradient at the ablation surface, and this acts to somewhat reduce the growth rate. Furthermore, the very fact that material is being ablated reduces the growth rate partially as well. [5] Nevertheless, the overall message is clear and valid: in order to get ICF to work we need compression in order to amplify the ablation pressure. The amplification factor is about the aspect ratio of the target, and thus we would like to use high aspect ratios. However, the number of $e-$foldings of the R-T instability between the start of the acceleration of the target and its convergence to the centre is proportional to the square-root of the aspect ratio, and thus if the aspect ratio is too high, the target will break up before compression is achieved. The situation is exacerbated if our targets are not sufficiently smooth, or the laser irradiation deviates significantly from spherical symmetry (which it always will, as we can only illuminate the target with a finite number of beams), as these departures from true spherical behaviour are the initial seed of the instability. Achieving sufficient uniformity of illumination is far from easy, and overcoming this issue has been one of the key obstacles in realising laser-driven inertial confinement fusion, and a significant amount of research has gone into measuring the growth rate of the instability, to check that the computer calculations are correct, and to then set limits on what degree of non-uniformity of illumination, and/or roughness of targets we can live with, without the R-T instability causing shell-breakup before peak compression.

9.8 Summary of Lecture 9

1. High power lasers, incident on solid targets, generate plasmas that expand

[5] Detailed computer simulations of the expected growth rate have led to an expression known as the Takabe formula, which states that the growth rate is $\alpha\sqrt{ka} - \beta k v_a$, where α and β are constants, and v_a is the velocity of the plasma at the ablation surface – i.e. it is reduced from the classical value both by a density gradient effect, and by the ablation process itself 'eating away' at the surface where the instability is growing.

into the vacuum. For irradiation of solid targets simple analytic models predict that the ablation pressure, P_a, scales as the laser irradiance, as $I^{2/3}$. Typically, we can obtain an ablation pressure of order 100 Mbar with 0.35μm light at an irradiance of 10^{15} Wcm^{-2}.

2. At these irradiances the laser light is absorbed principally by inverse-bremsstrahlung – the electrons oscillate in the field of the laser, but this energy is thermalized due to collisions with the ions. This process is more efficient at higher densities. The laser cannot propagate beyond the so-called critical surface, where the laser frequency equals the plasma frequency, therefore the absorption process is more efficient for short wavelength laser light

3. In order to achieve fusion, the ions must be confined for a sufficient time; this leads to the Lawson criterion. For inertial confinement fusion this can be written $\rho r > 0.6$ g cm^{-3}.

4. The laser energy required to heat a heavy hydrogen pellet to 10 keV (the temperature required for fusion), is enormous for solid density hydrogen, but scales as the inverse square of the compression – thus we need to compress the fuel to achieve fusion.

5. Compression is achieved by spherically illuminating, and thus accelerating, a hollow target containing the D-T fuel. Convergence on axis converts the kinetic energy of the shell to a high pressure that is required for the compression. The ablation pressure is thus amplified by a factor of order the aspect ratio (initial radius of the target divided by the shell thickness).

6. There is a limitation on the value of the aspect ratio that can be safely used – small perturbations to perfect spherical symmetry grow due to the Rayleigh-Taylor instability. This places strict constraints on the perfection of the targets, and the degree of departure from perfect spherical illumination that can be tolerated.

Lecture 10

Resonance Absorption and Parametric Instabilities

10.1 Introduction

In the previous chapter we considered the absorption of laser light within a plasma by the mechanism of inverse bremsstrahlung. The electrons oscillated in the electric field of the laser, and this ordered motion was converted to thermal energy via collisions with ions. However, we have not thus far considered the possibility that the energy within the laser light could be directly transferred to waves within the plasma – be they plasma waves or ion-acoustic waves. It is with such conversion processes that we will occupy ourselves during this chapter, as we study the topics of resonance absorption and parametric instabilities. Both of these topics are of relevance to ICF.

In the previous chapter we learnt that laser light can be absorbed in a plasma up until a certain electron number density, where the laser frequency is equal to the plasma frequency – we called this density the critical density, n_c. This is true for light incident normally on a plasma. However, as we shall find below, if light is incident upon a plasma at an angle θ with respect to the direction of the density gradient, refraction effects imply that it can only propagate up to an electron density of $n_c \cos^2\theta$, and thus is reflected at a density a little lower than the critical density. A schematic diagram of the path of the light is shown in Fig. 10.1. For light incident upon the plasma with p-polarization the electric field points along the density gradient at the point of reflection. This oscillating electric field is clearly causing the electrons to oscillate backwards and forwards along the density gradient, and thus can cause a real separation in charge leading to plasma waves. We know that the frequency of the plasma waves matches that of the laser (ignoring the thermal corrections) at the critical density. Although the light is reflected before this point, we shall find that a component of the field tunnels between the reflection point and the critical surface (much like an evanescent wave), and thus can resonantly excite very large amplitude plasma

waves at the critical surface. The energy of the laser is thus transferred to plasma waves, and therefore there is another method by which laser energy can be absorbed by the plasma.

One might initially suppose that this extra absorption mechanism is a bonus – for ICF applications we clearly wish to couple as much of the laser light as possible into the plasma to maximize the efficiency of the whole process. However, it is not quite that simple, and in fact resonance absorption is, in general, very detrimental to ICF. The laser energy can indeed be coupled into plasma waves by this resonant mechanism, and these waves can be driven to very large amplitudes. As we have seen in lecture 7, plasma waves lose their energy via Landau damping, which involves accelerating trapped particles to the phase velocity of the plasma wave. Usually this phase velocity is larger than the average thermal velocity of the bulk of the electrons. Thus the end result of resonance absorption is that we couple laser energy into electrons which have velocities several times larger than the average velocity – so-called 'hot electrons'. These hot electrons travel further in the plasma before undergoing a collision (consider the velocity dependence of the electron-ion collision frequency that we found in eqn (6.11)). Indeed, they can travel so far without a collision that in ICF applications they can pass through the shell of the target, and deposit some of their energy into the centre of the target during the laser implosion. This is extremely detrimental because, as we noted, we need to compress the target, and the minimum energy needed to compress it will be when it is isentropically compressed – if it is heated before peak compression it will be far harder to compress it to the required density. For this reason resonance absorption must be kept to a minimum. When we study this absorption mechanism in detail in section 10.4, we will find that the degree of resonance absorption is a strong function of laser wavelength – long wavelengths giving rise to more resonance absorption, and this provides another reason why short wavelength lasers are generally used for ICF applications.

There is a second method by which laser light can interact with waves within the plasma which is also of import. Waves will always exist within the plasma due to thermal fluctuations, and the laser light will interact with these waves with some amplitude which can be determined. The system must obey the Manley-Rowe relations (which simply means that we must conserve energy and momentum). That is to say, if the incident light has angular frequency ω_0 and wavevector $\mathbf{k_0}$, and it generates waves within the plasma with angular frequency ω_w and wavevector $\mathbf{k}_\mathrm{w}$, then the scattered light wave will have an angular frequency ω_s and wavevector $\mathbf{k}_\mathrm{s}$ such that

$$\begin{aligned} \omega_0 &= \omega_\mathrm{w} + \omega_\mathrm{s} \\ \mathbf{k_0} &= \mathbf{k}_\mathrm{w} + \mathbf{k}_\mathrm{s} \quad . \end{aligned} \tag{10.1}$$

The plasma waves that are generated by this mechanism have an associated oscillating electric field. We shall find that this electric field of the oscillating electrons associated with the wave can beat with the oscillating electric field of the laser to further enhance the amplitude of the waves – a positive feedback

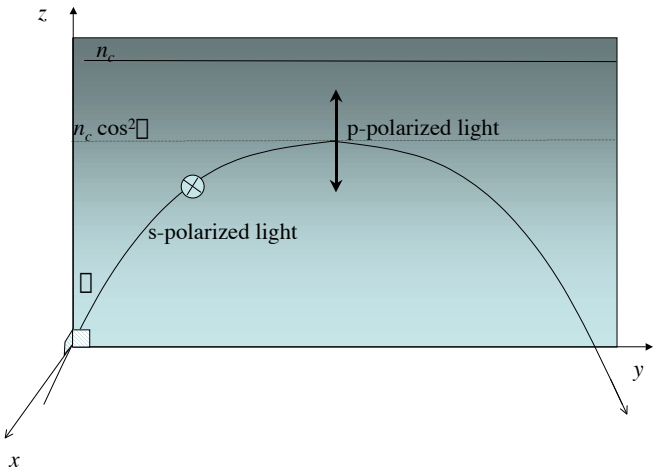


Figure 10.1: : A schematic diagram showing the path of a light ray obliquely incident upon an inhomogeneous slab of plasma. Note that for p-polarized light the oscillating electric field points along the density gradient at the point of reflection.

loop results, which is known as a parametric instability. We will find that above a certain laser irradiance a large fraction of the laser light can be converted into plasma waves (which is known as stimulated Raman scattering – SRS), and into ion-acoustic waves (known as stimulated Brillouin scattering – SBS). As SRS results in the generation of large-amplitude plasma waves, it is detrimental to ICF for the same reason as resonance absorption is detrimental – it too results in the generation of hot electrons that can heat the target before peak compression. We will also find that SBS is a nuisance, but for a different reason. This instability results in some laser energy being transferred to ion waves, but due to the energy and momentum conservation laws, we find that much of the scattered light propagates away from the target, and the overall coupling of laser energy into the plasma is reduced. It is important to note that other laser light-wave interactions exist, and we will mention them in passing, but not treat them in detail. We now consider each of the mechanisms mentioned above in turn.

10.2 Resonance Absorption

We wish to consider the interaction of p-polarized light of angular frequency ω on a plasma with a density gradient. As the density varies as a function of z,

so does the dielectric function of the plasma:

$$\varepsilon(z) = 1 - \frac{\omega_{\rm pe}^2(z)}{\omega^2} \quad . \tag{10.2}$$

We note that $\varepsilon = 0$ when the laser frequency matches the plasma frequency – i.e. at the critical surface. That light will generate some sort of resonance phenomenon at the critical surface can be seen by considering Poisson's equation. As the plasma is overall neutral

$$\nabla \cdot (\varepsilon \mathbf{E}) = \varepsilon \nabla \cdot \mathbf{E} + \nabla\varepsilon \cdot \mathbf{E} = 0 \quad , \tag{10.3}$$

hence

$$\nabla \cdot \mathbf{E} = -\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial z} E_z \quad . \tag{10.4}$$

Thus there is a resonance at the critical surface where $\varepsilon = 0$, and according to this very simply model, the gradient of the electric field becomes infinite at this point. In practice, of course, this does not happen: we have treated ε as a real function, whereas we know that in fact it has an imaginary component (e.g. Landau damping, or collisional absorption), and this prevents this singularity. However, it is still generally the case that the calculation of the electric field at the critical surface in the case of p-polarized light is non-trivial. We will thus first consider the interaction of s-polarized light with the plasma, where resonance absorption does not take place as the electric field oscillates along a direction where there is no gradient in density. We will then use the knowledge we gain from this study to construct a simplified model of the interaction of the p-polarized light with the plasma.

10.3 Obliquely Incident S-Polarized Light†

We consider s-polarized light incident upon a plasma with a density gradient in the z direction, as shown in Fig 10.1. As the light is s-polarized the electric field oscillates in the $\hat{\mathbf{x}}$ direction, $\mathbf{E} = E_x\hat{\mathbf{x}}$. We see from the wave equation for light and eqn (2.21) that

$$\begin{aligned} \nabla^2 E_x + \frac{\omega^2}{c^2}\varepsilon(z) E_x &= 0 \\ \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon(z) E_x &= 0 \quad . \end{aligned} \tag{10.5}$$

Note that there is no variation of any of the quantities with x within the problem, and thus we have dropped the $\partial^2/\partial x^2$ term in the above equation. Furthermore, the k-vector of the light has components in the y and z directions but as the refractive index changes only as a function of z, the k-vector in the y direction must be conserved, i.e. $k_y = (\omega/c)\sin\theta$. The k-vector in the z direction will

change, but as $k_z z$ is simply a function of z, we include this phase factor into the amplitude E, so that

$$E_x = E(z) \exp\left(\frac{i\omega y \sin\theta}{c}\right) \quad . \tag{10.6}$$

Substituting eqn (10.6) into (10.5) we obtain

$$\frac{\mathrm{d}^2 E(z)}{\mathrm{d}z^2} + \frac{\omega^2}{c^2}\left(\varepsilon(z) - \sin^2\theta\right) E(z) = 0 \quad . \tag{10.7}$$

It is clear that the light is now reflected at the point where $\varepsilon(z) = \sin^2\theta$. This is a manifestation of Snell's law, $\mu \sin\theta =$ constant, where the refractive index $\mu = \sqrt{\varepsilon}$. Before being incident on the edge of the plasma, $\mu = 1$. Given the form of $\varepsilon(z)$ – eqn (10.2) – we see that the reflection takes place at a point in the plasma where the electron plasma frequency $\omega_{\mathrm{pe}} = \omega \cos\theta$.

A general analytic solution for the propagation of the ray for an arbitrary density gradient cannot be found. However, there is an analytic solution for the case of a linear density gradient, and we will use this case to illustrate the pertinent physics. We assume that up until $z \leq 0$ the light propagates in a vacuum, and $z = 0$ defines the vacuum-plasma interface. The electron density varies linearly such that

$$n_e = \frac{n_{\mathrm{c}} z}{L} \quad , \tag{10.8}$$

where n_{c} is the critical density where the plasma frequency equals that of the laser, and L is some constant scale-length. Substituting eqn (10.8) into (10.7) we obtain

$$\frac{\mathrm{d}^2 E(z)}{\mathrm{d}z^2} + \frac{\omega^2}{Lc^2}\left(L - z - L\sin^2\theta\right) E(z) = 0 \quad . \tag{10.9}$$

With a change of variable

$$\zeta = \left(\frac{\omega^2}{Lc^2}\right)^{1/3} (z - L\cos^2\theta) \quad , \tag{10.10}$$

eqn (10.9) can be written

$$\frac{\mathrm{d}^2 E(z)}{\mathrm{d}\zeta^2} - \zeta E(z) = 0 \quad . \tag{10.11}$$

This equation is known as the Stokes differential equation. Its solutions are known as the Airy functions, A_i and B_i. We will not concern ourselves here with the methods by which these solutions are found, as it is quite involved. It is far more important simply to be aware of the form of the functions – they are plotted in Fig. 10.2. As the wave cannot propagate beyond the critical density, we exclude the B_i solution, as it diverges. We also need to be aware that for large negative values of ζ an approximate form of $A_i(\zeta)$ is given by

$$A_i(-\zeta) = \frac{1}{\sqrt{\pi}\zeta^{1/4}} \cos\left(\frac{2}{3}\zeta^{2/3} - \frac{\pi}{4}\right) \quad , \tag{10.12}$$

and for large positive values of ζ it can be approximated by

$$A_i(\zeta) = \frac{1}{\sqrt{\pi}|\zeta|^{1/4}} \exp\left(-\int_0^\zeta \sqrt{|\zeta|}\mathrm{d}\zeta\right) \quad . \tag{10.13}$$

We notice that these asymptotic solutions are of the form we might expect – up until the point of reflection, we have a cosine solution, which can therefore be decomposed into the incoming and reflected waves On the other hand, beyond the turning point, the amplitude of the electric field decays exponentially into the plasma.

We can work out the decay constant of the field into the plasma beyond the reflection point from eqn (10.13). The field falls off as $\exp(-\beta\zeta)$ where

$$\beta = \int_0^\zeta \sqrt{|\zeta|}\mathrm{d}\zeta = \frac{2}{3}\zeta^{3/2} \quad . \tag{10.14}$$

Now, to find out how the field falls off with z we notice that $\zeta = 0$ at $z = L\cos^2\theta$, i.e. the reflection point, and at the critical surface

$$\zeta = \left(\frac{\omega^2}{Lc^2}\right)^{1/3}(L - L\cos^2\theta) = \left(\frac{\omega^2}{Lc^2}\right)^{1/3} L\sin^2\theta \quad . \tag{10.15}$$

Thus in terms of z the field falls off as $\exp(-\beta' z)$, where

$$\beta' = \frac{2}{3}\left[\left(\frac{\omega^2}{Lc^2}\right)^{1/3} L\sin^2\theta\right]^{3/2} = \left(\frac{2\omega L}{3c}\right)\sin^3\theta \quad . \tag{10.16}$$

This is our main result for this section – the electric field is reflected, but penetrates beyond the reflection point, decaying exponentially with the decay constant β'.

10.4 P-Polarized Light - Resonance Absorption†

The solution for p-polarized light is considerably more complicated than that for the s-polarized light. First of all, if we analysed the situation in terms of the electric field, we would need to keep track of both the y and z components of the field. Therefore it is easier to solve the situation for the magnetic field (which is now oscillating in the x-direction), and then use our knowledge of this to calculate what happens to the electric field. However, even then things are not so simple, as the equation for the magnetic field for a p-polarized electric field is not the same as that as for the equation for an s-polarized electric field.

We start from Maxwell's equations, assuming light oscillating as $\exp(i\omega t)$:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -i\omega\mu_0\mathbf{H} \tag{10.17}$$

$$\nabla \times \mathbf{H} = i\varepsilon\varepsilon_0\omega\mathbf{E} \quad . \tag{10.18}$$

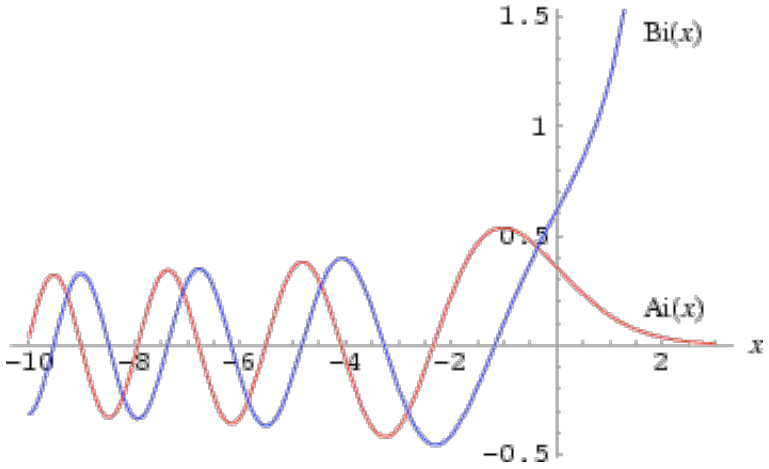


Figure 10.2: : A plot of the Airy functions $A_i(x)$ and $B_i(x)$ as a function of x.

Thus

$$\nabla \times (\nabla \times \mathbf{H}) = -\nabla^2 \mathbf{H} = i\varepsilon_0 \omega (\nabla \times [\varepsilon \mathbf{E}]) \quad . \tag{10.19}$$

We recall that ε is a function of z, and thus

$$-\nabla^2 \mathbf{H} = i\varepsilon_0 \omega (\varepsilon [\nabla \times \mathbf{E}] + [\nabla \varepsilon] \times \mathbf{E}) \quad . \tag{10.20}$$

Substituting eqns (10.17) and (10.18) into (10.20) we obtain

$$\nabla^2 \mathbf{H} + \frac{\omega^2}{c^2} \varepsilon \mathbf{H} + \frac{\nabla \varepsilon}{\varepsilon} \times (\nabla \times \mathbf{H}) \quad . \tag{10.21}$$

As $\mathbf{H}$ only has an x-component, and we assume that its dependence on y is $\exp(i\omega y \sin\theta/c)$, we find that eqn (10.21) can be written:-

$$\frac{\partial^2 H}{\partial z^2} - \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial z} \frac{\partial H}{\partial z} + \frac{\omega^2}{c^2} \left(\varepsilon - \sin^2 \theta \right) H = 0 \quad . \tag{10.22}$$

We notice that this is the same as our equation for the electric field in s-polarisation, except for the second term on the left hand side. Ignoring this second term is equivalent to assuming that the scale-length of the plasma, L is large compared with the wavelength of light. In this approximation we can still assume that the magnetic field falls off exponentially, once again as $\exp(-\beta' z)$. However, the electric field is related to the magnetic field by

$$E_z = \frac{1}{\varepsilon} B_x \sin\theta c \quad , \tag{10.23}$$

and hence

$$E_z \propto \frac{1}{\varepsilon} \sin\theta \exp\left[\left(-\frac{2\omega L}{3c} \right) \sin^3 \theta \right] \quad , \tag{10.24}$$

Notice that if ε was completely real the electric field would be infinite at the critical surface. We treat this point in a moment. but in the meantime let us imagine keeping the scale-length of the plasma with respect to the wavelength of light fixed (i.e. $\omega L/c$ fixed), and varying the angle of incidence. Then if we define $\tau = (\omega L/c)^{1/3} \sin\theta$, we see

$$E_z \propto \tau \exp(-2\tau^3/3) \quad . \tag{10.25}$$

This is a function that has a maximum at about $\tau = 0.8$, and thus we would expect that as we vary the angle for a given scale-length plasma, we would obtain a maximum in the electric field at the critical surface when

$$(\omega L/c)^{1/3} \sin\theta = 0.8 \quad . \tag{10.26}$$

As we noted above, the electric field has a singularity at the critical surface in the approximation that ε is real. In practice however, it will have some imaginary component due to damping of the wave. In is interesting to note, as we shall show below, that the amount of energy absorbed is independent of the damping mechanism – it could be collisional damping (inverse bremsstrahlung) or landau damping – the fraction absorbed would be the same. In practice, as the electric fields become so large, landau damping dominates, and thus this resonance absorption process results in the trapping and consequent acceleration of electrons – the so-called 'hot-electrons' to which we referred in the introduction.

To show that the absorbed energy is independent of the damping process and damping frequency let us consider the equation of motion of an electron in the oscillating field of the laser in the presence of damping:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{e}{m}\mathbf{E} - \nu\mathbf{u} \quad , \tag{10.27}$$

where $\mathbf{u}$ is the velocity of the electron, and ν an effective collision frequency related to some mechanism that we have not specified. Assuming the fields vary harmonically in times as $\exp(i\omega t)$ we find

$$\mathbf{u} = \frac{ie\mathbf{E}}{m(\omega - i\nu)} \quad . \tag{10.28}$$

Now the current density, $\mathbf{J}$, is given by $\mathbf{J} = -ne\mathbf{u} = \sigma\mathbf{E}$, i.e

$$\sigma = \frac{i\omega_{\rm pe}^2 \varepsilon_0}{(\omega - i\nu)} \quad . \tag{10.29}$$

We know the relationship between the conductivity, σ, and the dielectric function, ε – see eqn (2.21). Hence we find that in the presence of damping

$$\varepsilon = 1 - \frac{\omega_{\rm pe}^2}{\omega(\omega - i\nu)} \quad , \tag{10.30}$$

and hence

$$\varepsilon = 1 - \frac{\omega_{\rm pe}^2}{(\omega^2 + \nu^2)} - \frac{i\nu\omega_{\rm pe}^2}{\omega(\omega^2 + \nu^2)} \quad , \tag{10.31}$$

and

$$|\varepsilon|^2 = \left(1 - \frac{\omega_{\rm pe}^2}{(\omega^2+\nu^2)}\right)^2 + \frac{\nu^2\omega_{\rm pe}^4}{\omega^2(\omega^2+\nu^2)^2} \quad , \tag{10.32}$$

which, under the assumption that $\nu \ll \omega$, and that we once more have a linear density gradient, can be written

$$|\varepsilon|^2 = \left(1 - \frac{z}{L}\right)^2 + \left(\frac{\nu}{\omega}\right)^2 \frac{z^2}{L^2} \quad . \tag{10.33}$$

One might think that we need to take into account the fact that ν is itself density dependent. However, as $\nu \ll \omega$, the second term on the right hand side of eqn (10.33) only contributes appreciably to the value of $|\varepsilon|^2$ when $z \approx L$. Not only does this mean that we need not take into account the density dependence of this damping term, it also means we can further approximate $|\varepsilon|^2$ as

$$|\varepsilon|^2 = \left(1 - \frac{z}{L}\right)^2 + \left(\frac{\nu}{\omega}\right)^2 \quad . \tag{10.34}$$

The absorbed energy flux, $I_{\rm abs}$ is given by

$$I_{\rm abs} = \int_0^\infty \nu \frac{1}{2}\varepsilon_0 E_z^2 {\rm d}z \quad . \tag{10.35}$$

We know that the electric field is only appreciable over the narrow region of the resonance. Over this width

$$E_z \approx \frac{E_{\rm r}}{\varepsilon} \quad , \tag{10.36}$$

where $E_{\rm r}$ is roughly a constant (as the resonant region is so narrow). Hence the absorbed energy flux can be written

$$I_{\rm abs} = \frac{1}{2}\nu\varepsilon_0 E_{\rm r}^2 \int_0^\infty \frac{{\rm d}z}{|\varepsilon|^2} = \frac{1}{2}\nu\varepsilon_0 E_{\rm r}^2 \int_0^\infty \frac{{\rm d}z}{(1-z/L)^2 + (\nu/\omega)^2} \quad . \tag{10.37}$$

We can evaluate the integral by making the substitution $(1-z/L) = (\nu/\omega)\tan\theta$. We find

$$\begin{aligned} I_{\rm abs} &= \frac{1}{2}\nu\varepsilon_0 E_{\rm r}^2 \left(-\frac{L\nu}{\omega}\right)\left(\frac{\omega}{\nu}\right)^2 \int_{\pi/2}^{-\pi/2} {\rm d}\theta \\ &= \frac{1}{2}\varepsilon_0\omega L\pi E_{\rm r}^2 \quad . \end{aligned} \tag{10.38}$$

Note that this is independent of ν, and therefore of the absorption mechanism. The reason for this is that the smaller the damping term, the higher the electric field at the critical surface, but the region over which it is high becomes narrower. These two effects cancel out, to make the total energy absorbed independent of the damping rate. More detailed calculations of the fraction of the energy that is absorbed show that when the resonance condition is exactly satisfied, approximately half of the energy initially reaching the turning point is

absorbed. In practice, energy is absorbed by inverse bremsstrahlung before the laser reaches the turning point. Furthermore, as an incident laser beam contains a range of angles of incidence (depending on the F-number of the lens), this fraction is reduced from that at exact resonance.

A diagram showing some of the first experimental results to confirm resonance absorption is shown in Fig. 10.3. [1] Notice that the absorption of s-polarized light simply falls off as the laser deviates from normal incidence, whereas the absorption of p-polarized light is both greater than that for s-polarized for most angles (they are the same at normal incidence, where no distinction between polarizations exists) and peaks at an angle of about 25 degrees. Given that the laser light used in this experiment had a wavelength of just over 1-μm, a peak in absorption at this angle implies – using eqn (10.26) – that the scale-length of the plasma is of order the wavelength of the light itself. We will discuss why the scale-length was so short in this particular experiment in the next section.

10.5 Ponderomotive Steepening

The hot electrons associated with the landau damping and wavebreaking of waves at the critical surface induced by resonance absorption are, as we have noted, detrimental to ICF. Thus, to keep resonance absorption to a minimum we wish to work with plasmas where most of the light energy has been absorbed by inverse bremsstrahlung (producing thermal electrons) before it reaches the critical surface. This is one reason for choosing short wavelength light as the driver. From our above analysis we also see that it is desirable for the plasma to have a scale-length which is long compared with the wavelength of light, for in this case resonance absorption can only take place over a very narrow range of angles close to normal incidence. However, as has been inferred from the experimental data shown in Fig. 10.3, there can be circumstances when the plasma scale-length is of order the wavelength of the incident light itself. Why is this?

Well, recall from section 9.2 that the time-averaged oscillatory energy of an electron in the electric field of the laser is given by

$$\overline{\frac{1}{2}m\dot{x}^2} = \frac{e^2E_0^2}{4m\omega^2} \quad . \tag{10.39}$$

If there is a gradient in the electric field of the laser then there will be a gradient in the oscillatory energy per unit volume, and hence an associated force on the electrons. This force is known as the ponderomotive force, $F_{\rm p}$. It is via this force that the photon pressure is imparted to the medium. Thus

$$F_{\rm p} = -\frac{e^2}{4m\omega^2}\nabla E_0^2 \quad . \tag{10.40}$$

[1] These diagrams are taken from the paper entitled "Polarization and Angular Dependence of 1.06-μm Laser-Light Absorption by Planar Plasmas" by K.R. Manes *et al.*, Physical Review Letters, **39**, 281 (1977)

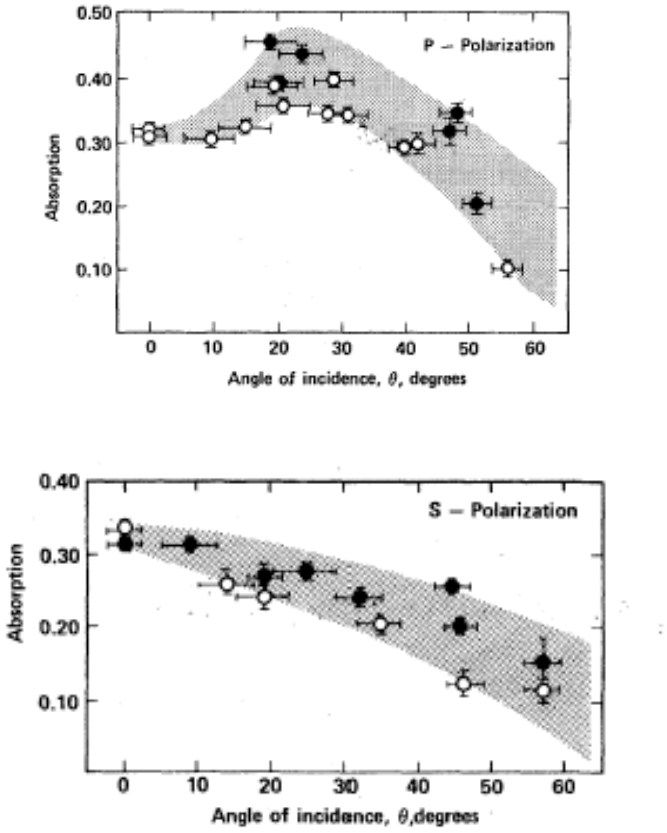


Figure 10.3: : A plot of the absorbed laser energy as a function of angle between the target normal and the incident-beam k-vector for (top plot) p-polarized light and, (lower plot) s-polarized light. Different symbols represent different laser irradiances between about 2 and $9 \times 10^{15}\mathrm{Wcm}^{-2}$.

We notice that for a given gradient in the intensity of the laser, the force is proportional to λ^2. The gradient in the intensity of the laser field is going to be greatest around the critical surface, as it is here that the electric field peaks for p-polarized light. The ponderomotive force pushing on the plasma at the critical surface can alter the plasma – the force is pushing towards the ablation surface, and causes the gradient in the electron density to steepen at the critical surface. This steepening of the density gradient increases the resonance absorption, and eventually the scale-length can become comparable to the laser wavelength, as shown in the above experimental data. The wavelength scaling of the ponderomotive force is yet another reason for using short-wavelength drivers.

The hot electrons that are produced by resonance absorption can be diagnosed by observing the associated x-ray emission that they produce as they collide with ions. The thermal electrons within the plasma produce a bremsstrahlung emission, but with a spectrum falling off in energy as $\exp(-E/k_{\rm B}T)$, where T is the normal thermal temperature of the plasma. On the other hand, the hot electrons produce X-rays with a much higher characteristic temperature, $T_{\rm hot}$, and the broad-band X-ray spectrum observed from the target often contains traces of both temperatures, and the relative intensity of the spectra can provide information about the number of hot electrons, and the fraction of laser energy absorbed via resonance absorption.

10.6 Parametric Instabilities

We have noted in the section on resonance absorption that a density gradient in a plasma gave rise to the possibility of the laser field producing real separations of charge, leading to plasma waves – the density gradient allowed for the coupling of laser light into modes of oscillation of the plasma. Small density fluctuations will always exist within a plasma due to, for example, ion-acoustic waves, and hence we conclude that there is, in principle at least, a mechanism by which laser light can be coupled into waves within the system.

In a parametric instability this coupling can become unstable due to feedback, and a variety of different coupling scenarios are possible. Let us consider an intense laser, of frequency ω_0, propagating within the plasma. We assume that there is some oscillation (at this stage we do not specify in particular what it is – but it could be, for example, an ion acoustic wave or a plasma wave) within the plasma which has a frequency ω_1. This wave simply exists within the plasma due to noise. It will beat with the incoming laser to produce oscillations with frequency $\omega_2 = \omega_0 - \omega_1$ and $\omega_3 = \omega_0 + \omega_1$. It could be that the new wave produced with frequency ω_2 is not heavily damped, and it can re-beat with the light to produce the wave with frequency ω_1 once more, and thus the ω_1 mode is enhanced. Provided that these potentially enhanced modes, driven by the laser, receive energy from the laser faster than they lose it by damping, they will grow.

There are four common parametric instabilities that occur in laser produced

plasmas. Firstly, the light can produce a plasma wave and an ion wave (known as the parametric instability). Secondly it can produce ion acoustic waves and scattered light (the Brillouin instability). Thirdly it can produce plasma waves and scattered light (the Raman instability), and finally, it can decay into two different plasma waves (the two plasmon instability). There is a formulism common to all of these which we adopt below. We will then discuss some of the instabilities in a little more detail.

Let us assume light of amplitude A_0 and frequency ω_0 is incident upon a plasma. Within the plasma can exist two different waves of amplitudes A_1 and A_2, with frequencies ω_1 and ω_2, and with damping rates Γ_1 and Γ_2 respectively. Note that we do not specify the nature of these waves at this point – they could be electron plasma waves, ion acoustic waves, or scattered light waves. In the absence of any driver, the equations governing the waves are

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} + \Gamma_1 \frac{\mathrm{d}A_1}{\mathrm{d}t} + \omega_1^2 A_1 = 0 \tag{10.41}$$

$$\frac{\mathrm{d}^2 A_2}{\mathrm{d}t^2} + \Gamma_1 \frac{\mathrm{d}A_2}{\mathrm{d}t} + \omega_2^2 A_2 = 0 \quad . \tag{10.42}$$

We then switch on the laser – the pump wave. We assume that this produces some coupling between the waves, and that this coupling is proportional to both the amplitude of the pump wave, and the amplitude of the coupled wave. That is to say in the presence of the pump

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} + \Gamma_1 \frac{\mathrm{d}A_1}{\mathrm{d}t} + \omega_1^2 A_1 = c_1 A_2 A_0 \tag{10.43}$$

$$\frac{\mathrm{d}^2 A_2}{\mathrm{d}t^2} + \Gamma_1 \frac{\mathrm{d}A_2}{\mathrm{d}t} + \omega_2^2 A_2 = c_2 A_1 A_0 \quad , \tag{10.44}$$

where c_1 and c_2 are constants. We make the further assumption that A_0, the laser amplitude, is unaffected by the growth of the two waves (hence we study just the initial stage of any instability that may arise). Notice that these equations are non-linear, and therefore in order to solve them we are not allowed simply to replace the temporal derivative by $i\omega$. Clearly this must be the case – we are saying that A_1 is linked to A_2, and therefore it cannot simply be oscillating as $\exp(i\omega_1 t)$, but must have some of the frequency of A_2 mixed in with it. As each of the waves has an amplitude that can exist, potentially, at several different frequencies the appropriate method of solution is to note that each wave amplitude in eqns (10.43) and (10.44), which in those equations are functions of time, $A(t)$, can be represented in the frequency domain by a Fourier transform:

$$A(\omega) = \int_{-\infty}^{\infty} \exp(i\omega t) A(t) \mathrm{d}t \quad . \tag{10.45}$$

We express the incident laser field as

$$A_0 = E_0 \cos(\omega_0 t) = \frac{E_0}{2} [\exp(i\omega_0 t) + \exp(-i\omega_0 t)] \quad . \tag{10.46}$$

Thus taking the Fourier transform of both sides of eqn (10.43) we obtain

$$[\omega^2 - \omega_1^2 + i\Gamma_1]A_1(\omega) + \frac{c_1 E_0}{2}[A_2(\omega+\omega_0) + A_2(\omega-\omega_0)] = 0 \quad , \tag{10.47}$$

which is often written

$$D_1(\omega)A_1(\omega) + \frac{c_1 E_0}{2}[A_2(\omega+\omega_0) + A_2(\omega-\omega_0)] = 0 \quad , \tag{10.48}$$

where $D_i(\omega) = \omega^2 - \omega_i^2 + i\omega\Gamma_i$. So we see that wave 1 at some frequency ω couples with wave 2 at frequencies $(\omega+\omega_0)$ and $(\omega-\omega_0)$. Therefore, to obtain the coefficients $A_2(\omega+\omega_0)$ and $A_2(\omega-\omega_0)$ we need to take Fourier transforms of eqn (10.44) at these frequencies. This leads to the following pair of equations:

$$D_2(\omega-\omega_0)A_2(\omega-\omega_0) + \frac{c_2 E_0}{2}[A_1(\omega) + A_1(\omega-2\omega_0)] = 0 \tag{10.49}$$

$$D_2(\omega+\omega_0)A_2(\omega+\omega_0) + \frac{c_2 E_0}{2}[A_1(\omega) + A_1(\omega+2\omega_0)] = 0 \quad . \tag{10.50}$$

At this point in our analysis it is useful to start to consider what these waves might possibly represent. In almost all cases of interest, one of the excited modes (let us pick mode 1) will have a frequency close to that of the incoming laser light, and thus $A_1(\omega)$ will be much larger than $A_1(\omega \pm 2\omega_0)$. For example, in Brillouin scattering the frequency of an ion acoustic wave is so small compared with that of the light, that the scattered light (our mode 1) satisfies $\omega_0 \approx \omega \approx \omega_1$. Furthermore, it is clear that in order for an instability to develop, the pump laser must transfer energy to the waves within the plasma – that is to say the frequency of mode 2 must be less than that of the incoming light simply from energy conservation. Hence we conclude that $A_2(\omega-\omega_0)$ is finite, but we can safely neglect terms in $A_2(\omega+\omega_0)$. Taking into account all of these assumptions, we see that eqns (10.48) and (10.49) are the pertinent ones, and writing these in matrix form, and ignoring the terms mentioned above, we find

$$\begin{pmatrix} D_1(\omega) & c_1E_0/2 \\ c_2E_0/2 & D_2(\omega-\omega_0) \end{pmatrix} \begin{pmatrix} A_1(\omega) \\ A_2(\omega-\omega_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad . \tag{10.51}$$

For this equation to have a non-trivial solution, the determinant must be zero:

$$D_1(\omega)D_2(\omega-\omega_0) - \frac{c_1c_2E_0^2}{4} = 0 \quad . \tag{10.52}$$

We make the approximations $\omega_0 - \omega \approx \omega_2$, $\omega + \omega_1 \approx 2\omega_1$ and find that eqn (10.52) can be approximated as

$$-4\omega_1\omega_2\left(\omega - \omega_1 + i\frac{\Gamma_1}{2}\right)\left(\omega - \omega_0 + \omega_2 + i\frac{\Gamma_2}{2}\right) - \frac{c_1c_2E_0^2}{4} = 0 \quad . \tag{10.53}$$

Now, to find out whether or not an instability will grow, we break down the frequency ω into its real and imaginary parts: $\omega = \omega_{\rm R} + i\Gamma$. Equating the real part of eqn (10.53) to zero we obtain

$$(\omega_{\rm R} - \omega_1)(\omega_{\rm R} - \omega_1 - \Delta) + \psi E_0^2 = \left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right) \quad , \tag{10.54}$$

where

$$\Delta \equiv \omega_0 - \omega_1 - \omega_2 \tag{10.55}$$

$$\psi \equiv \frac{c_1 c_2}{16\omega_1\omega_2} \quad . \tag{10.56}$$

Equating the imaginary parts of eqn (10.53) we obtain

$$\omega_{\rm R} = \omega_1 + \frac{(\Gamma + \frac{1}{2}\Gamma_1)\Delta}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2} \quad . \tag{10.57}$$

Now that we have two equations involving $\omega_{\rm R}$, it can be eliminated: we substitute eqn (10.57) into (10.54) to obtain

$$E_0^2 = \frac{1}{\psi}\left\{(\Gamma + \frac{1}{2}\Gamma_1)(\Gamma + \frac{1}{2}\Gamma_2)\left[1 + \frac{\Delta^2}{(2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2)^2}\right]\right\} \quad . \tag{10.58}$$

In order for the instability to grow, Γ must be positive. [2] From eqn (10.58) we see that there is some threshold electric field for this to be the case, and the value of the threshold field will occur when $\Gamma = 0$, i.e. when

$$E_{0,\rm th}^2 = \frac{4\omega_1\omega_2\Gamma_1\Gamma_2}{c_1 c_2}\left[1 + \frac{4\Delta^2}{(\Gamma_1 + \Gamma_2)^2}\right] \quad . \tag{10.59}$$

This threshold intensity will be a minimum when exactly on resonance – i.e. when $\Delta = 0$ – so the minimum intensity is

$$E_{0,\rm min}^2 = \frac{4\omega_1\omega_2\Gamma_1\Gamma_2}{c_1 c_2} \quad . \tag{10.60}$$

We notice that the threshold for the instability to grow is proportional to the damping terms of both the daughter waves, as we might expect. Furthermore, the threshold is lowered if there is strong coupling between the modes – again this is physically reasonable. The growth rate of the instability will also depend on these parameters, but will maximise when the damping coefficients of the daughter waves, Γ_1 and Γ_2, are a minimum (i.e. zero), and we are on resonance:

$$\Gamma_{\rm max} = E_0\sqrt{\left(\frac{c_1 c_2}{16\omega_1\omega_2}\right)} \quad . \tag{10.61}$$

In summary of this section, if we assume that light can generate modes of excitation within a plasma, and then mode 1 can be enhanced by beating between the light and mode 2, then we have a parametric instability, with the feedback giving rise to unstable growth of the waves if some threshold intensity is reached. We will now consider in a little more detail two of the most common types of instability which are of import to ICF applications.

[2] Our definition of the Fourier transform implies fields oscillating as $\exp(-i\omega t)$, whereas often in this course we have used the convention $\exp(+i\omega t)$.

10.6.1 Stimulated Brillouin Scattering†

Stimulated Brillouin Scattering (SBS) is extremely detrimental to laser fusion. In this instability the incoming laser light generates ion acoustic waves and scattered light of slightly lower frequency than the incoming laser . The reason it is so detrimental can be understood with reference to Fig. 10.4, where we plot in $\omega - k$ space the dispersion relation for light propagating in a plasma, and for the ion acoustic waves, for the situation where light is scattered back towards the original direction of incidence of the laser. This figure represents pictorially the Manley-Rowe relations – i.e. the conservation of energy (frequency) and momentum (k-vector) for the process – the parallelograms demarking various points where the relations are obeyed.

The dispersion relations for the incoming laser (ω_0, k_0), scattered light (ω_s, k_s), and ion-acoustic waves (ω_i, k_i) are

$$k_0^2 c^2 = \omega_0^2 - \omega_{\mathrm{pe}}^2 \tag{10.62}$$

$$k_\mathrm{s}^2 c^2 = \omega_\mathrm{s}^2 - \omega_{\mathrm{pe}}^2 \tag{10.63}$$

$$k_\mathrm{i}^2 V_{\mathrm{ia}}^2 = \omega_\mathrm{i}^2 \quad . \tag{10.64}$$

We note that because the frequency of an ion wave is so much less than that of a photon, very little energy is transferred to the ion-wave, and thus very little energy is absorbed by the plasma. However, as can be seen from the diagram, the scattered light can have a k-vector in the opposite direction to that of the incident light – i.e. if it generates these ion waves, it simply gets reflected back from the target. Therefore SBS can act to significantly reduce the fraction of laser energy absorbed – that is the main reason why it is so detrimental to ICF.

As the light cannot have a frequency lower than the plasma frequency, there is a minimum value of k_i which can backscatter the light, and this will produce backscattered light such that k_s is just less than zero, i.e. $k_\mathrm{s} \approx 0$, $\omega_\mathrm{s} \approx \omega_{\mathrm{pe}}$, $k_0 \approx k_\mathrm{i}$, and $\omega_0 \approx \omega_\mathrm{i} + \omega_{\mathrm{pe}}$. The dispersion relation for the light is

$$k_0^2 c^2 = \omega_0^2 - \omega_{\mathrm{pe}}^2 \quad , \tag{10.65}$$

but because $k_\mathrm{i} \approx k_0$,

$$k_{\mathrm{i,min}}^2 c^2 \approx \omega_0^2 - \omega_{\mathrm{pe}}^2 \tag{10.66}$$

$$k_{\mathrm{i,min}}^2 c^2 \approx (\omega_\mathrm{i} + \omega_{\mathrm{pe}})^2 - \omega_{\mathrm{pe}}^2 \tag{10.67}$$

$$k_{\mathrm{i,min}}^2 c^2 \approx \omega_\mathrm{i}^2 + \omega_{\mathrm{pe}}^2 + 2\omega_\mathrm{i}\omega_{\mathrm{pe}} - \omega_{\mathrm{pe}}^2 \tag{10.68}$$

$$. \tag{10.69}$$

As $\omega_\mathrm{i} \ll \omega_{\mathrm{pe}}$, this leads to

$$k_{\mathrm{i,min}} \approx 2\frac{V_{\mathrm{ia}}}{c^2}\omega_{\mathrm{pe}} \quad , \tag{10.70}$$

and therefore

$$\omega_{\mathrm{i,min}} \approx 2\frac{V_{\mathrm{ia}}^2}{c^2}\omega_{\mathrm{pe}} \quad . \tag{10.71}$$

Eqn (10.66) can now be written

$$4\frac{V_{\rm ia}^2}{c^2} = \frac{n_{\rm c}}{n_e} - 1 \quad , \tag{10.72}$$

and thus we see that SBS can take place at all densities from the lowest density all the way up to an electron density of

$$n_e = n_{\rm c}\left(1 + 4\frac{V_{\rm ia}^2}{c^2}\right)^{-1} \quad , \tag{10.73}$$

i.e. almost all the way to the critical density. Light scattered at these high densities will have a photon energies around the plasma frequency – i.e. very close to ω_0. At the lowest densities we see from the dispersion diagram that $k_{\rm i} \approx 2k_0$, and then the dispersion relation for the scattered light becomes

$$\frac{k_{\rm i}^2 c^2}{4} = \omega_{\rm s}^2 - \omega_{\rm pe}^2 \approx \omega_0^2 - \omega_{\rm pe}^2 \tag{10.74}$$

$$\frac{\omega_{\rm i}^2 c^2}{4V_{\rm ia}^2} = \omega_0^2\left(1 - \frac{\omega_{\rm pe}^2}{\omega_0^2}\right) = \omega_0^2\left(1 - \frac{n_e}{n_{\rm c}}\right) \tag{10.75}$$

$$\frac{\omega_{\rm i}}{\omega_0} = \frac{2V_{\rm ia}}{c}\sqrt{1 - \frac{n_e}{n_{\rm c}}} \quad . \tag{10.76}$$

That is to say the greatest fractional shift in the light frequency is of order $2V_{\rm ia}/c$, which is a function of temperature and ion mass, but for a high Z ion, and assuming $A/Z = 2$, implies

$$\left(\frac{\omega_0 - \omega_{\rm s}}{\omega_0}\right)_{\rm max} \approx 5 \times 10^{-5} T_e^{1/2} \quad , \tag{10.77}$$

where the temperature is measured in eV. Thus for a temperature of order 1 keV, and 1-μm laser light, the maximum shift in the wavelength of the scattered light will be of order 15 Å. This is consistent with experimental data, as can be seen from the measurements shown in Fig. 10.5. In practice the shifts can be larger than this owing to the fact that the plasma has some flow velocity, and thus there can be an associated Doppler shift, as the dispersion relations should be solved in the frame of the moving fluid.

Thus far we have not considered in any detail the actual values of the threshold intensities or the growth rates. In order to do this we clearly need to gain some understanding of the coupling constants, c_1 and c_2, as well as the damping rates Γ_1 and Γ_2. Unfortunately, although the coupling constants can be relatively easily determined (and we will look at them, as they inform us of the underlying physics) the damping rates are more illusive for real plasmas. For example, in SBS the ion waves will be mainly damped via Landau damping for a perfectly homogeneous plasma. However, in reality, the plasma has a large density and velocity gradient, and thus the damping of the instability usually has far more to do with the gradients in the real system than the fundamental

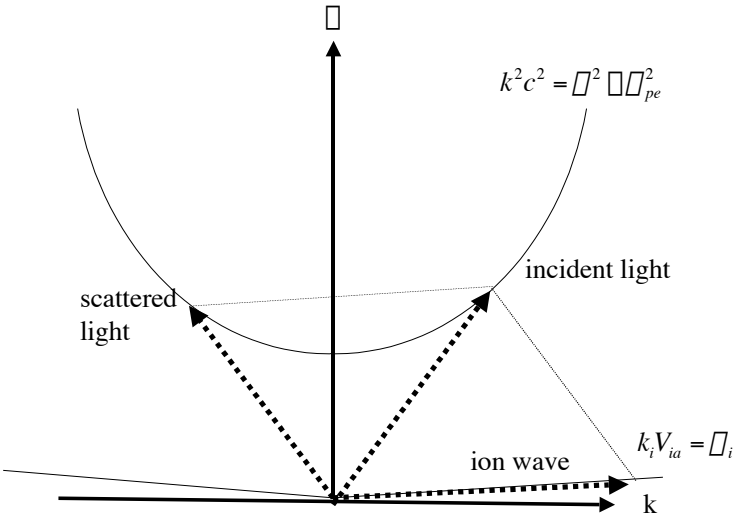


Figure 10.4: : Dispersion relation for light propagating in a plasma and for ion acoustic waves showing how energy and momentum are conserved during SBS.

damping mechanism of the mode. Therefore we will restrict ourselves to studying the coupling constants in order to understand better the physical mechanism through which the instability arises.

Let us denote the amplitude of the incident laser wave by E_0, that of the scattered wave by E. Furthermore, we assume that the ion acoustic wave gives rise to fluctuations in the ion density, $n_{\rm ia}$, around the background density, n_0, such that

$$n_{\rm ia} = n_{\rm a}\cos(k_{\rm a}x) \quad . \tag{10.78}$$

As the oscillations in the ions are slow, the electrons are able to follow, keeping overall charge neutrality. Therefore there is also a periodic variation in the electron density, which is associated with the ion acoustic wave. We call this amplitude of the electrons $n_{\rm ea}$, and $n_{\rm ea} = n_{\rm ia}$.

We can derive certain relations between the amplitudes of the incident and scattered electromagnetic waves and the ion acoustic waves simply from considerations of conservation of energy and of momentum. First of all we consider conservation of energy. The spatially averaged energies of the incident and reflected waves are $\varepsilon_0 E_0^2/2$ and $\varepsilon_0 E^2/2$ respectively. Let us now work out the spatially averaged energy associated with the ion wave. We use the normal theory for sound waves, and assume that the ion-acoustic wave has a sufficiently

low frequency that the equation of state that is obeyed is isothermal. First we define the condensation, s

$$s = \frac{n_{\rm ia}}{n_0} = -\frac{\int {\rm d}V}{V_0} \quad . \tag{10.79}$$

The excess pressure is periodic, with the total pressure given by

$$P = P_0 + p = P_0 \left(1 + \frac{n_{\rm ia}}{n_0}\right) \quad , \tag{10.80}$$

where p is the excess pressure on top of the ambient pressure P_0. Note that $p = P_0 s$. The compressive energy increase at any point associated with the wave is given by

$$\Delta U = -\int p {\rm d}V = \int P_0 V_0 s {\rm d}s = \frac{1}{2} P_0 V_0 s^2 \quad . \tag{10.81}$$

By equipartition there will be an equal amount of kinetic energy associated with the movement of the ions in the wave, but spatially averaging the energy will bring in another factor of 1/2, and thus the spatially averaged energy per unit volume of the sound wave is given by

$$U_{\rm a} = \frac{1}{2} P_0 s^2 = \frac{1}{2} k_B T_e \frac{n_{\rm a}^2}{n_0} \quad . \tag{10.82}$$

We have assumed that the laser heats up the electrons, and is often the case in laser-plasma interactions, although the electrons have time to equilibriate, they do not have enough time during the laser pulse to transfer much energy to the ions, so that the electron temperature is much higher than the ion temperature.

Now that we have expressions for the spatially averaged energy per unit volume for all three of the waves we can write down the equation of conservation of energy:

$$\frac{1}{2}\varepsilon_0 E_0^2 = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2} k_B T_e \frac{n_{\rm a}^2}{n_0} \quad . \tag{10.83}$$

Let us now turn our attention to finding a formula expressing the conservation of momentum density. This is most easily done by noting that we are dealing with waves, and waves have energy that is quantized in units of $\hbar\omega$, but each quantum has momentum $\hbar\mathbf{k}$. Therefore the momentum density of a wave must be proportional to the energy density multiplied by $(\mathbf{k}/\omega)$. For the situation we are studying this implies

$$\frac{1}{2}\varepsilon_0 E_0^2 \left(\frac{\mathbf{k_0}}{\omega_0}\right) = \frac{1}{2}\varepsilon_0 E^2 \left(\frac{\mathbf{k}}{\omega}\right) + \frac{1}{2} k_B T_e \frac{n_{\rm a}^2}{n_0} \left(\frac{\mathbf{k}_{\rm a}}{\omega_{\rm a}}\right) \quad . \tag{10.84}$$

We substitute the expression for E_0 from the energy conservation equation, eqn (10.83), into our equation for momentum conservation, (10.84) , to obtain

$$\varepsilon_0 E^2 \left(\frac{\mathbf{k_0}}{\omega_0} - \frac{\mathbf{k}}{\omega}\right) = \frac{k_{\rm B} T_e n_{\rm a}^2}{n_0} \left(\frac{\mathbf{k}_{\rm a}}{\omega_{\rm a}} - \frac{\mathbf{k_0}}{\omega_0}\right) \quad . \tag{10.85}$$

Let us work out what happens at densities reasonably low compared with the critical density. In this case we recall that $\mathbf{k} \approx -\mathbf{k_0}$, and the speed of the light is c, i.e. $\omega/k \approx \omega_0/k_0 \approx c$. Furthermore, as the frequency of the ion acoustic wave is so low compared with that of the electromagnetic waves we can assume $\omega \approx \omega_0$, and $(1/V_{\rm ia}) \gg (1/c)$. Using these approximations in eqn (10.85) we find

$$\frac{2\varepsilon_0 E^2}{c} = \frac{k_{\rm B} T_e n_{\rm a}^2}{n_0}\left(\frac{1}{V_{\rm ia}}\right)$$

$$E = \left(\frac{c}{2\varepsilon_0 n_0}\sqrt{M k_{\rm B} T_e}\right)^{1/2} n_{\rm a} \quad , \tag{10.86}$$

where M is the mass of the ion. Thus, simply from considerations of conservation of energy and of momentum we have obtained a relation between the amplitude of the scattered electromagnetic wave and the amplitude of the ion wave. What we really seek is how these waves grow as a function of time. To ascertain this we need to use Maxwell's wave equation to find the link between the driving field, E_0, and the fields of the daughter waves.

We recall the expression for the wave equation when a current is present:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t} \quad . \tag{10.87}$$

As the waves are transverse ($\mathbf{k} \cdot \mathbf{E} = 0$) the first term on the left hand side of eqn (10.87) is zero, i.e.

$$-\nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t} \quad . \tag{10.88}$$

How do we use eqn (10.88) to inform us of the physics of brillouin scattering? Well, let us first consider what this equation represents in the absence of any ion wave fluctuations within the plasma. In this case it simply represents the propagation of the incident wave, E_0, within the plasma. If we derived an expression for the electron current we would just, once more, recover the dielectric function of the plasma. However, the situation we actually wish to discuss is that in which there is an ion wave present within the system. We thus conclude that the ion wave – the periodic variation in the ion (and hence electron) density is what gives rise to the scattered electromagnetic wave, E. The extra periodic current due to the ion wave is given by

$$\mathbf{J} = -e n_{\rm a} \mathbf{v_e} \cos(k_{\rm a} x) \quad , \tag{10.89}$$

but we need to remember that the velocity of the electron is still that provided by the driving laser – that is to say the motion of the electrons is determined by the amplitude of the driving field according to

$$m_e \frac{\partial v_e}{\partial t} = -eE_0 \cos(k_0 x - \omega_0 t)$$

$$v_e = -\frac{eE_0}{m_e \omega_0} \sin(k_0 x - \omega_0 t) \quad . \tag{10.90}$$

Substituting eqn (10.90) into eqn (10.89) we obtain

$$J = \frac{n_{\rm a} e^2 E_0}{2 m_e \omega_0} \left\{ \sin[(k_0 + k_{\rm a})x - \omega_0 t] + \sin[(k_0 - k_{\rm a})x - \omega_0 t] \right\} \quad . \tag{10.91}$$

The wave that is backscattered by the ion acoustic wave is represented by the term oscillating in space according to $(k_0 - k_{\rm a})$. Using this term from the current in eqn (10.91), we see that the rate of change of current to use in the wave equation for the scattered radiation is

$$\frac{\partial J}{\partial t} = -\frac{n_{\rm a} e^2 E_0}{2 m_e} \cos[(k_0 - k_{\rm a})x - \omega_0 t] \approx -\frac{n_{\rm a} e^2 E_0}{2 m_e} \cos(k_0 x + \omega_0 t) \quad , \tag{10.92}$$

where we have made use of the fact that $\mathbf{k_0} - \mathbf{k}_{\rm a} = \mathbf{k} \approx -\mathbf{k_0}$. Hence the wave equation that describes the evolution of the amplitude of the scattered wave is

$$\frac{\partial^2 E}{\partial t^2} + k^2 c^2 E = \frac{\omega_{\rm pe}^2 n_{\rm a} E_0}{2 n_0} \cos(k_0 x + \omega_0 t) \quad . \tag{10.93}$$

This is the expression that we have been seeking. As we expect for a parametric instability, it tells us that the backscattered wave, E, evolves due to a linking between the drive wave, E_0, and the other mode – the ion acoustic wave $n_{\rm a}$. Notice that we can now immediately see from eqn (10.93) that the first of our coupling constants, let us call it c_1, is given by

$$c_1 = \frac{\omega_{\rm pe}^2}{2 n_0} \quad . \tag{10.94}$$

Eqn (10.93) is the equation of a driven, undamped, simple harmonic oscillator. It is undamped because we have not considered any mechanism that might damp either the scattered wave or the ion wave. We note that for the resonant case when the frequency of E is the same as E_0, $\omega = \omega_0$, then the solution for E is

$$E = \frac{\omega_{\rm pe}^2 E_0 t}{4 \omega_0} \frac{n_{\rm a}}{n_0} \sin(k_0 x + \omega_0 t) = E(t) \sin(k_0 x + \omega_0 t) \quad . \tag{10.95}$$

where

$$E(t) = \frac{\omega_{\rm pe}^2 E_0 t}{4 \omega_0} \frac{n_{\rm a}}{n_0} \quad . \tag{10.96}$$

Therefore

$$\frac{{\rm d}E(t)}{{\rm d}t} = \frac{\omega_{\rm pe}^2 E_0}{4 \omega_0} \frac{n_{\rm a}}{n_0} \quad . \tag{10.97}$$

But recall that the amplitude of the ion acoustic wave is proportional to the amplitude of the scattered electromagnetic wave (we found this by conservation of energy and momentum of the waves). Thus by substituting eqn (10.86) into eqn (10.97) we obtain

$$\frac{{\rm d}E(t)}{{\rm d}t} = \frac{\omega_{\rm pe}^2 E_0}{4 \omega_0 n_0} \left(\frac{c}{2 \varepsilon_0 n_0} \sqrt{M k_{\rm B} T_e} \right)^{-1/2} E \quad . \tag{10.98}$$

Hence the scattered wave (and also therefore the acoustic wave) grows exponentially with a growth rate $\Gamma_{\rm brill}$, where

$$\Gamma_{\rm brill} = \frac{\omega_{\rm pe}^2 E_0}{4\omega_0 n_0}\left(\frac{c}{2\varepsilon_0 n_0}\sqrt{M k_{\rm B} T_e}\right)^{-1/2} \quad . \tag{10.99}$$

This expression for the growth rate can be simplified to

$$\Gamma_{\rm brill} = \frac{1}{2\sqrt{2}}\frac{\omega_{\rm pe} v_{\rm os}\sqrt{m_e/M}}{\sqrt{cV_{\rm ia}}} \quad , \tag{10.100}$$

where $v_{\rm os}$ is the peak oscillatory velocity of the electron in the laser field. We can also express the growth rate as a function of the incident laser intensity:

$$\Gamma_{\rm brill} = \frac{1}{2}\frac{\omega_{\rm pe}^2}{\omega_0}\frac{\sqrt{I_0}}{\sqrt{n_0 c^2 V_{\rm ia} M}} \quad . \tag{10.101}$$

It is important to note that this peak growth rate can be extremely large. For example, if we take light of wavelength 1-μm, at an intensity of 10^{14} W cm^{-2} incident upon a carbon plasma at a temperature of 1 keV, and with an electron density of 10^{20} cm^{-3}, we obtain a growth rate of a little over 10^{12} s^{-1} – the amplitude of the scattered wave grows by 1 e-folding within a picosecond. We do not need many such $e-$foldings to produce a significant amount of scattered light. Although we have not discussed what the noise level of the ion acoustic wave actually is, it is not unreasonable to pick a lower limit that is the inverse of square root of the total number of particles, which may be of order, say, 10^{16}. If we suppose such noise at the 10^{-8} level, we only need about 18 $e-$foldings to reach saturation. Thus we see that SBS can grow to saturation levels in a matter of a few picoseconds – well within the pulse length of a nanosecond experiment. It is important to stress, however, that we have not derived a quantitative expression for the threshold, as this often depends on gradients within the system. Also, within this particular analysis, we have not thus far obtained an expression for the coupling constant between the two electromagnetic waves, that causes the ion acoustic wave to grow. However, this can be done by comparing eqns (10.61), (10.94) and (10.99). After some algebra we find

$$c_2 = \frac{8\varepsilon_0}{M}\frac{\omega_{\rm pe}^2}{c^2} \quad . \tag{10.102}$$

In summary of this section, we have found that Brillouin scattering at the intensities used in laser plasma interactions can have extremely high growth rates if certain thresholds are exceeded, and due to the energy and wavevector matching considerations, this instability can take place over almost all densities up to the critical density. Actual calcluations of the threshold intensities are complex, and at present a great deal of effort is being expended in determining the thresholds, especially in large, fairly homogeneous plasmas, where the scalelengths are long. Larger plasmas will be used in the latest fusion programs. Thus

far it seems that the lasers should be able to operate at below SBS threshold, and the interested reader is directed to the work of Seka *et al.* [3].

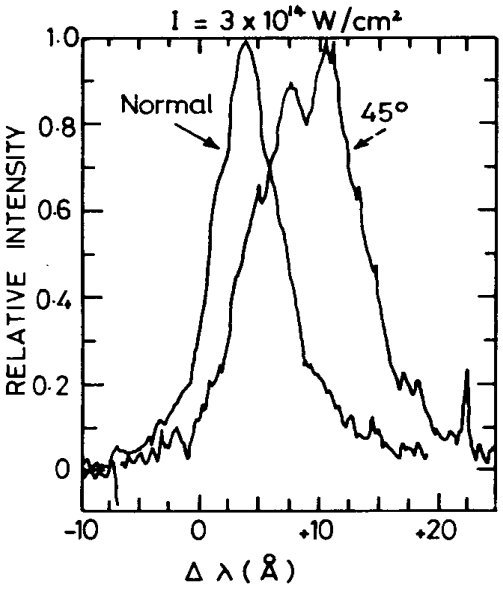


Figure 10.5: : Backscattered light due to SBS from the interaction of $1-\mu$m light with a gold target. The spectra are taken at normal incidence and at 45 degrees to the target. As the plasma is flowing towards the spectrometer the light is blue-shifted due to Doppler effects. Measurements at two different angles allow the separation of the shift due to the Doppler motion (found to be 17.6 Å) and that due to Brillouin scattering (22 Å). Spectra taken from M.D. Rosen *et al.*, Livermore Laboratory Preprint, UCRL-82146 (1978).

10.6.2 Stimulated Raman Scattering†

Having derived an expression for the growth rate of ion acoustic waves (when the intensity is above threshold), we are in a position to write down immediately the expression for the growth of SRS. By analogy with eqn (10.101), the maximum growth rate of the Raman instability is laser intensity:

$$\Gamma_{\rm Raman} = \frac{1}{2}\frac{\omega_{\rm pe}^2}{\omega_0}\frac{\sqrt{I_0}}{\sqrt{n_0 c^2 v_{\rm ph} m_e}} \quad , \tag{10.103}$$

where $v_{\rm ph}$ is the phase velocity of the plasma wave, as determined from the Bohm-Gross relation. Once more we can show that the instability has more than

[3] W. Seka *et al.*, Phys. Rev. Lett. **89**, 175002 (2002).

adequate time to grow during a typical nanosecond laser pulse, if the threshold intensity is exceeded. However, SRS cannot take place over the same range of densities as SBS. A light wave cannot propagate in a plasma if its frequency is less than the local plasma frequency. When the incident wave creates a plasma wave, the scattered wave has a lower frequency, and clearly if the frequency of this scattered wave were lower than the local plasma frequency, it could not propagate. From such a consideration we see that the instability can only take place if $\omega_0 \geq 2\omega_{\rm pe}$, and hence can only take place up to electron densities such that $n_e \leq n_{\rm c}/4$. We say that SRS takes place below the 'quarter-critical surface'. The scattered light can therefore, in principle, have frequencies ranging from that of the incident light all the way down to half of this original frequency. As we noted at the beginning of this chapter, SRS is detrimental for ICF because the large amplitude plasma waves can trap and accelerate electrons, just as the large amplitude waves caused by resonance absorption create hot electrons. Pre-heating of the fusion fuel by these hot electrons makes the compression of the fuel far more difficult. Recent work shows, however, that the levels of both instabilities should not be significant under conditions relevant for fusion, if care is taken with the smoothness of the laser beams etc. [4] Finally, we note from eqn (10.60) that the threshold intensity for these instabilities is proportional to the frequency of the incident light, and thus once more we find it advantageous to use short wavelength lasers as the drivers for ICF.

10.7 Summary of Lecture 10

1. When light is incident obliquely onto a plasma which has a density gradient, the light is refracted, and is reflected at the point where the electron plasma frequency $\omega_{\rm pe} = \omega \cos\theta$, where θ is the angle between the k-vector of the light and the gradient of the plasma density.

2. For s-polarized light, the electric field decays from its value at the point of reflection, and is still finite at the critical surface. However, in this polarization the field is parallel to the surface of the target, and no charge separation occurs.

3. For p-polarized light, the electric field at the critical surface is parallel to the surface normal, along the density gradient, charge separation can take place, and there is a resonance (as $\varepsilon = 0$). In the absence of damping the field is infinite. In the presence of a small damping term, energy is absorbed in the form of large amplitude plasma waves. This is known as resonance absorption. Resonance absorption is detrimental to ICF as it can generate hot electrons that pre-heat the fusion fuel.

4. Coupling between the incident laser light and modes of oscillation within the plasma can lead to parametric instabilities – two of the most important

[4] See work by S. P. Regan *et al.*, Physics of Plasmas **6**, 2072 (1999).

of which are stimulated brillouin scattering (SBS) and stimulated raman scattering (SRS).

5. In SBS the incident laser light causes ion acoustic waves to grow exponentially, if a certain threshold intensity is exceeded. The maximum posssible growth rate is

$$\Gamma_{\rm brill} = \frac{1}{2}\frac{\omega_{\rm pe}^2}{\omega_0}\frac{\sqrt{I_0}}{\sqrt{n_0 c^2 V_{\rm ia} M}} \quad . \tag{10.104}$$

   This can saturate within picoseconds. SBS is detrimental to fusion as most of the incident light is scattered back out of the plasma, and the laser energy is lost and not absorbed.

6. In SRS the incident laser generates the exponential growth of plasma waves. This can happen at densities below that of the quarter critical surface. SRS is harmful for fusion for the same reason that resonance absorption is harmful – the large amplitude plasma waves can generate hot electrons.

Appendix A

The Grand Canonical Ensemble

A.1 Introduction

I have to confess at this time to having performed a 'sleight of hand' earlier on when deriving the formulae for the Bose-Einstein and Fermi-Dirac distributions in the introductory chapter. You will recall that the final formulae included a term, μ, the so-called chemical potential. Now, this appeared in a rather arbitrary way, without full justification. You might also have noticed that we never derived a partition function for Fermi-Dirac, or Bose-Einstein systems. Furthermore, for the systems where we did derive a partition function, we assumed that we had a fixed number of particles, N, and a fixed energy U. A moments thought should convince you that there might be systems where such an approach will not work. For example, even if we have a fixed number of monatomic ideal gas particles in a box, surely there is some small probability that the particles in the box will have a little more, or a little less energy than $3Nk_BT/2$, as there is some chance that the atoms making up the walls of the box might give up a little more or less of the energy that they possess to the gas atoms. Or consider an even more complex case: consider a cubic centimetre of air in the room, and draw an imaginary box around it. Although you might know the average density and total energy of all of the air in the room, there will be some small fluctuations in the exact number of particles, and the exact total energy of the particles within the given cubic centimetre. Evidently we are not going to be able to learn anything about such fluctuations merely by taking an approach that assumes that the total energy and total particle number is always fixed: a more sophisticated approach is necessary.

So we see that our approach so far has not allowed us to deal with fluctuations. However, there is another aspect of the problem that we have also ignored: interactions. If you look back to the review in chapter **??** you will notice that we said things like "let there be n_1 particles in the energy level ϵ_1..."

– i.e. we designated particular particles to have particular energies, and these energies were found by solving Schrödinger's equation for a *single* particle in the potential of interest. That is all very well and good if the particles don't interact (no attraction or repulsion), but as soon as we do have some energy of interaction (such as we do, for example, in a plasma, though we conveniently ignore them in the treatment given in these notes) then we are stuck, because the wavefunctions for the particles in the potential cannot simply be written as products of single-particle wavefunctions. What are we going to do?

The more sophisticated approach is the method of ensembles. This is not an easy subject, yet because it does things 'properly', it rightly forms the backbone of more rigorous approaches to the subject of statistical mechanics. Let us start by getting an idea of ensembles. Up until now we have been considering boxes, or sub-systems of N atoms with total energy U. Consider many, many such sub-systems - the whole lot taken together is called an ensemble. There are three types of ensembles that we can imagine:-

1. An ensemble of sub-systems, with each sub-system containing a fixed number of atoms, and having fixed energy. This is called the *microcanonical ensemble*, and is in fact the ensemble that we have considered up to now in these notes, although we have not hitherto referred to it explicitly.

2. Imagine a sub-system in thermal contact with a very large reservoir, so energy can flow between each sub-system and its reservoir. The total energy of the sub-system+reservoir is kept fixed, but the energy of the particular sub-system itself is not fixed as it can exchange energy with its reservoir. However, the sub-system still has a fixed number of particles within it. Now consider replicating many, many such sub-systems + their associated reservoirs - such an ensemble is known as the *canonical ensemble*.

3. Imagine a sub-system again in contact with a very large reservoir, but now such that both energy and particles can flow between the sub-system and the reservoir. The total energy and the total number of particles of the sub-system+reservoir is fixed, but the energy and number of particles in the sub-system itself can vary as it can exchange energy and particles with the reservoir. Now consider replicating many, many such sub-systems and their associated reservoirs - such an ensemble is known as the *grand canonical ensemble* - which sounds like a great name for a physicist's jazz band.

By considering the grand canonical approach we shall see that we arrive at a much more complete, and more satisfying, derivation of the Bose-Einstein and Fermi-Dirac formulae in which the chemical potential is explicitly introduced at the beginning, and we will also get a simple introduction to how we deal with fluctuations. Thus this appendix can be thought of as a bridge between the simplified form of statistical mechanics used in the review chapter and the more complete theory needed at a higher level.

A.2 The Chemical Potential

Before we set about redefining the statistical arguments needed to solve this problem, we wish to consider for the moment a particular bit of macroscopic classical thermodynamics. Normally we are accustomed to thinking of the internal energy U as being a function of heat and work, expressed, for instance, by a relationship involving functions of state, S and V, thus

$$U = U(S, V) \tag{A.1}$$

The most common expression of this relationship is

$$dU = TdS - PdV \tag{A.2}$$

Now this equation is only valid if we define a thermodynamic system in which the amount of material contained is held constant. But suppose now we allow for the possibility that N_i particles of a substance i are added to the system. Clearly this will affect the internal energy, and so we can rewrite equation (A.1) by

$$U = U(S, V, N_i) \tag{A.3}$$

This functional relationship can then be written as

$$dU = \left(\frac{\partial U}{\partial S}\right)_{V,N_i} dS + \left(\frac{\partial U}{\partial V}\right)_{S,N_i} dV + \sum_i \left(\frac{\partial U}{\partial S}\right)_{S,V,N_{j\neq i}} dN_i \tag{A.4}$$

By comparison with equation (A.2), we can then write

$$dU = TdS - PdV + \sum_i \mu_i dN_i \tag{A.5}$$

where the chemical potential is given by

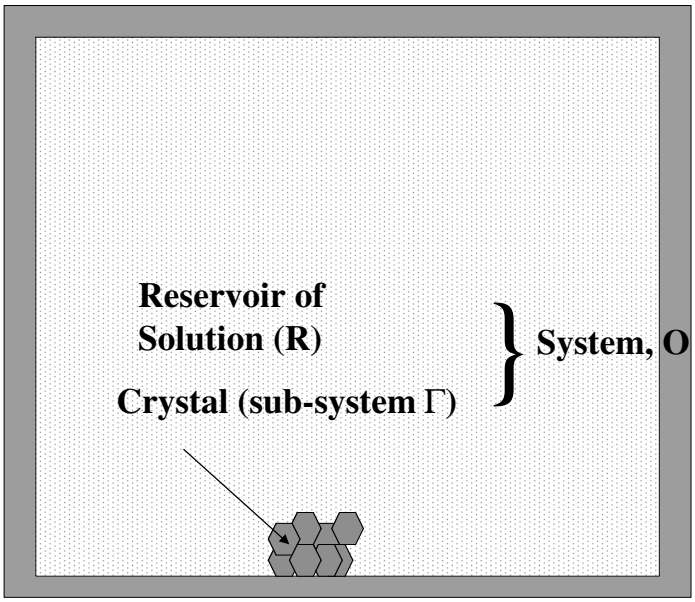
$$\mu_i = \left(\frac{\partial U}{\partial N_i}\right)_{S,V,N_{j\neq i}} \tag{A.6}$$

The same procedure can be used for the other standard thermodynamic potential equations for enthalpy H, Gibbs free energy G and Helmholtz free energy F, to give the following equivalent definitions of the chemical potential

$$\mu_i = \left(\frac{\partial U}{\partial N_i}\right)_{S,V,N_{j\neq i}} = \left(\frac{\partial H}{\partial N_i}\right)_{S,P,N_{j\neq i}} = \left(\frac{\partial G}{\partial N_i}\right)_{T,P,N_{j\neq i}} = \left(\frac{\partial F}{\partial N_i}\right)_{T,V,N_{j\neq i}} \tag{A.7}$$

This shows that the chemical potential is a quantity that describes how the various thermodynamic potentials change when particles are added to the system under consideration. This is an important result for thermodynamics because it leads further to a clean definition of what is meant by equilibrium. Thus suppose there is an equilibrium set up between two phases α and β of a material; we expect to find a balance between the effects of particles of the α phase moving to the β phase and *vice versa*, so that the equilibrium is defined simply by

$$\mu_\alpha = \mu_\beta \tag{A.8}$$



Consider a huge system O comprising a small subsystem Γ (shown here as a collection of growing crystals) and a reservoir R (a large amount of solution). As both energy and particles can flow between the crystal and solution, many of these systems repeated would form a *grand canonical ensemble.*

A.3 Ensembles and Probabilities

We now leave the world of macroscopic thermodynamics and return to the micro-world of statistical assemblies of particles, in order to try to link the two by using the grand canonical ensemble - that is an ensemble of assemblies of sub-systems and reservoirs, where both energy and particles can flow between the two. We denote the sub-system in which we are interested by the symbol Γ, and the larger reservoir by R, rather like in Fig. A.3. Taken together they form a very large system O. For example the sub-system could be a crystal growing out of a solution, whilst the reservoir could be an enormous volume of the solution. It should be re-emphasised that the sub-system Γ and the reservoir R allow for exchange of thermal energy as well as for numbers of particles.

Our sub-system is a quantum entity. As I said earlier, up to now we have thought of it as having n_1 particles in the single-particle energy level ε_1, n_2 in the single-particle energy level ε_2 and so on – we have assumed that the particles don't interact. However, even if the particles don't interact we can also describe the sub-system in terms of its energy levels *as a whole.* The total energy of the

sub-system, E_s is related to the single particle energies by

$$E_s = n_1\varepsilon_1 + n_2\varepsilon_2 + \ldots = \sum_i n_i\varepsilon_i \tag{A.9}$$

and the total number of particles in the sub-system is

$$N_s = \sum_i n_i \tag{A.10}$$

It is important to recognise that this is an equally valid way of representing the quantum state of the sub-system – there is a quantum state of the whole sub-system with energy E_s (if is another state with the same overall energy, it is clearly degenerate), containing N_s particles.

Now, how do things change if the particles *do* interact? Well, the energy levels of the whole sub-system will *still* be quantized, and they will *still* be some function of all of the particles in the system – but the thing that changes is that that function will *not* be a simple sum of non-interacting single-particle energies. This should be pretty obvious – if we have a gas of electrons (like we did for a metal) if we ignore the interactions between the electrons (as we did) then we can treat their energy as being simply the sum of the kinetic energies of each of the electrons, and therefore we solved Schrödinger's equation for each one separately. On the other hand, if we had tried to take into account interactions, we would have needed lots of terms like $1/|r_i - r_j|$ in the Schrödinger equation to take into account the electrostatic repulsion between the electrons – and we would have needed to work out the wavefunction for all of the particles in the system at the same time – we could not have assumed that the wavefunction of the N particles could be expressed in terms of the products of the single particle wavefunctions.

So, we conclude that we can still proceed for interacting particles, but we need to know the quantum levels of the sub-system as a whole. Clearly if the particles do interact, calculating what these quantum levels are is going to be significantly tougher than calculating the energies of simple non-interacting particles and then summing them. However, for the moment, we will put aside the problem of *how* we calculate these energies, we will just assume that somewhere some poor soul has done so. How do we proceed?

Well, we first ask ourselves what is the probability of finding our sub-system in a particular many-particle quantum state? Well, we know overall that the total large system, comprising our sub-system and the reservoir does have fixed energy, U, and particle number, N. If our sub-system is in a particular quantum state with energy E_s and particle number N_s, then the energy of the reservoir must be $(U - E_s)$, and it must contain $(N - N_s)$ particles.

If we look at the whole ensemble, what is the probability of finding a sub-system in this particular quantum state? Well, it must equal the number of ways that the reservoir can have energy $(U - E_s)$ and particle number $(N - N_s)$. In order to determine this number of ways we return to Boltzmann's original postulate that one can express macroscopic entropy in terms of numbers of

microstates, thus

$$S = k_B \ln W \qquad \text{or} \qquad W = \exp(S/k_B) \tag{A.11}$$

The number of arrangements W is in turn proportional to the probability P of being in a particular state. Returning to equation (A.5) and considering now finite changes we can write that the change in entropy of the reservoir is given by

$$\Delta S = \frac{1}{T}(\Delta U + P\Delta V - \mu\Delta N) \tag{A.12}$$

We will consider changes at constant volume, so will drop the term in ΔV. Thus by allowing energy ΔU and particles ΔN to transfer from the reservoir R to the particular sub-system Γ, the entropy of the reservoir is reduced by the amount

$$\Delta S_R = \frac{1}{T}(\Delta U - \mu\Delta N) \tag{A.13}$$

This means that the entropy S_R of the reservoir after the this process is given by

$$S_R = S_{max} - \frac{1}{T}(\Delta U + -\mu\Delta N) \tag{A.14}$$

and so , using equation (A.11), we then get the probability distribution

$$P \propto \exp\left[S_{max} - \frac{1}{T}(\Delta U - \mu\Delta N)\right]/k_B \tag{A.15}$$

or

$$P = A\exp\left[-\frac{1}{k_B T}(\Delta U - \mu\Delta N)\right] \tag{A.16}$$

where A is a constant.

Recalling that our assembly has energy E_s and particle number N_s, we can say that the probability of finding an assembly in this particular quantum state is thus

$$P = A\exp\left[-\frac{1}{k_B T}(E_s - N_s\mu)\right] \tag{A.17}$$

where A is some constant. To find the absolute probability we must normalize: if we sum over all of the possible energies that the system can have, and over all of the possible particle numbers, the total probability must sum to 1. Thus

$$P = \frac{\exp\left[-\frac{1}{k_B T}(E_s - N_s\mu)\right]}{\sum_{N=0}^{\infty}\sum_{s(N)}\exp\left[-\frac{1}{k_B T}(E_{s(N)} - N\mu)\right]} \tag{A.18}$$

which can be written

$$P = \frac{\exp\left[-\frac{1}{k_B T}(E_s - N_s\mu)\right]}{\Xi} \tag{A.19}$$

where Ξ, which is known as the grand partition function, is given by

$$\Xi = \sum_{N=0}^{\infty} \sum_{s(N)} \exp\left[-\frac{1}{k_B T}(E_{s(N)} - N\mu)\right] \tag{A.20}$$

It is important to recognize the importance of eqn(A.17). When we started off these lectures we said that the macrostate that we would observe *was* the one with the most microstates. In doing so, we ignored fluctuations, and our justification for doing this was that the fluctuations were going to be small. What we are saying in eqn(A.17) is that we know the probability of a particular quantum state occuring. We could clearly maximise this to find the most likely occurance, as we did before, but, as we will show in section A.8 knowing the detailed probabilities now allows us to find out the chances of having a given fluctuation around the most probable outcome.

It is also important to re-emphasize that eqn(A.19) works even when we have interactions between the particles – it is just that actually calculating the energies E_s of a particular state in this case is difficult, precisely because of the interactions. However, for the moment let us go back and revisit non-interacting particles once more, and see where the above reasoning leads us....

A.4 The Fermi-Dirac and Bose-Einstein Distributions

Assuming non-interacting particles allows us to re-derive FD and BE statistics in a new and more formally complete way – a wonderful piece of magic resulting from adopting the Grand Canonical formalism – and a both beautiful and subtle piece of physics. Up to now in our discussion of the grand canonical ensemble, we have talked in terms of a sub-system in contact with a large reservoir: we gave as an example a crystal in a very large volume of solution. That is to say we thought in terms of a real piece of 'stuff' (the crystal) in thermal and particle-flow contact with another piece of 'stuff' (the solution). However, let us consider something like a gas of particles (Bosons or Fermions). We know that a box of such particles has particular quantum states and energies (the solutions to Schrödinger's equation for a 3-D infinite square well when the particles don't interact). Why don't we denote one of these quantum states to be the sub-system, and call all the other quantum states, with their associated particles and thus energy, the reservoir? This is the subtle thought which is the stroke of genius. There is nothing stopping us doing this as particles and energy can certainly flow into our quantum state (when it becomes occupied) or flow out of it when a particle transfers to one of the other quantum states.

By thinking along these lines we can make use of the probability distribution (A.17) to calculate the mean occupation numbers of energy levels containing either fermions or bosons. If the quantum state we are interested in (the one we call the sub-system) is associated with the single-particle energy level ε, then if it has n particles in it, clearly the energy of this 'sub-system' is $n\varepsilon$ *as long as*

the particles don't interact. Thus in this case we can write the mean occupation number of the quantum state as

$$\bar{n} = \frac{\sum nP}{\sum P} = \frac{\sum n \exp[-n(\varepsilon - \mu)/k_B T]}{\sum \exp[-n(\varepsilon - \mu)/k_B T]} \tag{A.21}$$

where the sum goes over all the possible integer values of n. For convenience, we shall write this as

$$\bar{n} = \frac{\sum n x^n}{\sum x^n} \tag{A.22}$$

where

$$x = \exp[-(\varepsilon - \mu)/k_B T] \tag{A.23}$$

A.5 Fermi-Dirac

Consider the application of equation (A.22) to the case of Fermi-Dirac statistics. As we saw earlier, particles obeying such statistics conform to the Pauli exclusion principle, i.e. it is not possible to have more than one particle in the same state at the same time. This means that the values of n possible in equation (A.22) can only be either 0 or 1. Thus

$$\bar{n} = \frac{0 + x}{1 + x} = \frac{1}{x^{-1} + 1} = \frac{1}{\exp[(\varepsilon - \mu)/k_B T] + 1} \tag{A.24}$$

A.6 Bose-Einstein

In this case, there is no limit to the number of particles that we can place in a particular quantum state with energy ε, and so we now have to sum the complete series in equation (A.22). Thus we have

$$\bar{n} = \frac{0 + x + 2x^2 + 3x^3 + 4x^4 + \ldots}{1 + x + x^2 + x^3 + x^4 + \ldots} \tag{A.25}$$

We note that the geometric series S is

$$S = 1 + x + x^2 + x^3 + x^4 + \ldots = (1 - x)^{-1} \tag{A.26}$$

and taking the factor x out of the top line we get

$$\bar{n} = \frac{1}{S} x(1 + 2x + 3x^2 + 4x^3 \ldots) = (1 - x)x(1 + 2x + 3x^2 + 4x^3 \ldots) \tag{A.27}$$

The remaining series to sum on the r.h.s. is a harmonic series. We note that

$$1 + 2x + 3x^2 + 4x^3 \ldots = \frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + \ldots) = \frac{\mathrm{dS}}{\mathrm{d}x} = (1 - x)^{-2} \tag{A.28}$$

Substituting this into equation (A.30) results in

$$\bar{n} = \frac{(1 - x)x}{(1 - x)^2} = \frac{1}{x^{-1} - 1} = \frac{1}{\exp[(\varepsilon - \mu)/k_B T] - 1} \tag{A.29}$$

You should now appreciate that both these distribution formulae have appeared with the chemical potential in place naturally and without the need to insert it artificially at the end. This is a very impressive demonstration of the power of using the full treatment afforded by the Grand Canonical formalism.

A.7 The Canonical Ensemble

The difference between the canonical ensemble and the grand canonical one is that in the canonical case we do not allow particle flow between our sub-system and the reservoir (though we do allow energy flow). If we look back through our argument in section A.3, it is clear that all this will do is remove the terms in μ. Therefore, if we let the energy of particular quantum states in the system be E_s (and once more they need not at this stage necessary be broken down into single-particle energies – interactions are treatable in principle, if not easily in practice) then now the analogous equation to eqn(A.17) is

$$P = A \exp\left[-\frac{1}{k_B T} E_s\right] \tag{A.30}$$

where A is once more some constant. Again by normalizing this we see that the absolute probability of being in a particular quantum state is

$$P = \frac{\exp\left[-\frac{1}{k_B T} E_s\right]}{Z_c} \tag{A.31}$$

where Z_c is the canonical partition function

$$Z_c = \sum_s \exp\left[-\frac{1}{k_B T} E_s\right] \tag{A.32}$$

A.8 Introduction to Fluctuations

Way back in the first chapter, where we reviewed simple statistical mechanics, we said that the macrostate that we would observe *was* the one with the most microstates. Clearly making such an assumption rules out the possibility of studying fluctuations. Presumably there must be some method of working out the statistics of fluctuations in a thermal system, and indeed there is. As an example we consider a sub-system described by the canonical ensemble. That is to say we take boxes of fixed numbers of particles in contact with some much larger heat bath. There must be some probability that the sub-system will have more or less energy than the mean value for all the sub-systems, as energy can be exchanged between the sub-system and the reservoir. How do we analyse such fluctuations? Well, we note that we now have a formula for the probability of the sub-system having a certain energy: for the canonical system this is eqn (A.31). Therefore we can work out both the mean energy, and the mean square

energy – these are important because mathematically the variance of the energy, Var$U$, is defined as

$$\mathrm{Var}U =< (U- < U >)^2 >=< U^2 > - < U >^2 \tag{A.33}$$

Note that in the following we use the notation $\beta = 1/k_{\mathrm{B}}T$. The mean energy is given by

$$\begin{aligned} < U > &= \sum_s E_s P(E_s) \\ &= \frac{\sum_s E_s \exp\left(-E_s/k_{\mathrm{B}}T\right)}{Z_c} \\ &= -\frac{1}{Z_c}\frac{\partial Z_c}{\partial \beta} \\ &= -\frac{\partial \ln Z_c}{\partial \beta} \quad . \end{aligned} \tag{A.34}$$

The mean square energy is given by

$$\begin{aligned} < U^2 > &= \sum_s E_s^2 P(E_s) \\ &= \frac{\sum_s E_s^2 \exp\left(-E_s/k_{\mathrm{B}}T\right)}{Z_c} \\ &= \frac{1}{Z_c}\frac{\partial^2 Z_c}{\partial \beta^2} \quad . \end{aligned} \tag{A.35}$$

Having worked out these two quantities, it is instructive to work out the the heat capacity C (you will see why as we go along). It is given by

$$\begin{aligned} C &= \frac{\mathrm{d}\overline{U}}{\mathrm{d}T} \\ &= \frac{\mathrm{d}\beta}{\mathrm{d}T}\frac{\mathrm{d}\overline{U}}{\mathrm{d}\beta} \\ &= -\frac{1}{k_{\mathrm{B}}T^2}\frac{\mathrm{d}\overline{U}}{\mathrm{d}\beta} \\ &= \frac{1}{k_{\mathrm{B}}T^2}\frac{\partial^2 \ln Z_c}{\partial \beta^2} \quad . \end{aligned} \tag{A.36}$$

But notice that this can be written

$$\begin{aligned} C &= \frac{1}{k_{\rm B}T^2}\frac{\partial^2 \ln Z_c}{\partial \beta^2} \\ &= \frac{1}{k_{\rm B}T^2}\frac{\partial}{\partial \beta}\left(\frac{1}{Z_c}\frac{\partial Z_c}{\partial \beta}\right) \\ &= \frac{1}{k_{\rm B}T^2}\left\{\frac{1}{Z_c}\frac{\partial^2 Z_c}{\partial \beta^2} - \frac{1}{Z_c^2}\left(\frac{\partial Z_c}{\partial \beta}\right)^2\right\} \\ &= \frac{1}{k_{\rm B}T^2}\left\{\frac{1}{Z_c}\frac{\partial^2 Z_c}{\partial \beta^2} - \left(\frac{\partial \ln Z_c}{\partial \beta}\right)^2\right\} \end{aligned} \tag{A.37}$$

Substituting eqns (A.34) and (A.35) into eqn (A.37) we find

$$\begin{aligned} C &= \frac{1}{k_{\rm B}T^2}(< U^2 > - < U >^2) \\ &= \frac{{\rm Var}U}{k_{\rm B}T^2} \quad . \end{aligned} \tag{A.38}$$

Thus the variance of U is $Ck_{\rm B}T^2$. The standard deviation, ΔU, is defined as $\sqrt{|{\rm Var}U|}$. Thus the fractional fluctuation in energy is given by

$$\frac{\Delta U}{< U >} = \frac{\sqrt{|Ck_{\rm B}T^2|}}{< U >} \tag{A.39}$$

What does this mean in practice? Well, take for example a monatomic ideal gas. We know that $C = 3Nk_{\rm B}/2$, and $< U >= 3Nk_{\rm B}T/2$. Inserting these values into eqn(A.39) we find

$$\frac{\Delta U}{< U >} = \sqrt{\frac{2}{3}}\frac{1}{\sqrt{N}} \tag{A.40}$$

Now for typical systems $N \approx 10^{23}$ and thus the fluctuations are negligible. Notice this is going to be a general result – the heat capacity is proportional to the number of particles, and so is the energy, so the fractional fluctuations in energy go as $N^{-1/2}$.

Clearly there must be a method using the grand partition function (where both energy and particle number vary), to find how the fluctuations in the *number* of particles in a given volume. This is correct, and the stout of heart may wish to look into this in more advanced text books on statistical mechanics. The purpose of this section has simply been to give you a taster of how we treat fluctuations.

You will also notice that although we said that in principle our treatment in this appendix could deal with interactions between particles (because we worked in terms of the quantized energies of the sub-system as a whole, rather than single particles), we still have not tackled such a problem, which is outside the scope of these notes.

A.9 Summary of Appendix A

1. Up until this lecture, we ignored the possibility of dealing with fluctuations – to do this we need to employ ensembles.

2. We also start to think in terms of the energy of a system, or sub-system as a whole. This must also be quantized. Thinking like this allows us to include (in principle) interactions between the particles, though working out what the quantized levels actually are may be tough.

3. We derived the grand canonical partition function which allows us to treat sub-systems where both particles and energy can flow between them and the reservoir. By thinking of an individual quantum state as such a sub-system, we re-derived the Fermi-Dirac and Bose-Einstein distributions in a more elegant way.

4. As an introduction to fluctuations, for a canonical ensemble, where the energy can fluctuate, but not particle number, we showed that the fractional fluctuations in energy scaled as $N^{-1/2}$.

Appendix B

Useful Integrals

$$\int_0^\infty x^{2n+1} \mathrm{e}^{-a^2x^2} \mathrm{d}x = \frac{n!}{2a^{2n+2}} \tag{B.1}$$

$$\int_{-\infty}^\infty x^{2n} \mathrm{e}^{-a^2x^2} \mathrm{d}x = \frac{(2n)!\pi^{1/2}}{n!(2a)^{2n}a} \tag{B.2}$$

$$\int_0^\infty \frac{x^{1/2}}{\mathrm{e}^x - 1} \mathrm{d}x = 2{\cdot}32 \tag{B.3}$$

$$\int_0^\infty \frac{x^{3/2}}{\mathrm{e}^x - 1} \mathrm{d}x = 1{\cdot}78 \tag{B.4}$$

$$\int_0^\infty \frac{x^3}{\mathrm{e}^x - 1} \mathrm{d}x = \frac{\pi^4}{15} \tag{B.5}$$

Appendix C

Physical Constants

Constants are given to 4 or 5 significant figures.

Electron rest mass		m_e	$9.109 \times 10^{-31}\,\mathrm{kg}$
Proton rest mass		M_p	$1.6726 \times 10^{-27}\,\mathrm{kg}$
Electronic charge		e	$1.6022 \times 10^{-19}\,\mathrm{C}$
Speed of light in free space		c	$2.9979 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Planck's constant		h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$h/2\pi =$	$\hbar$	$1.0546 \times 10^{-34}\,\mathrm{J\,s}$
Boltzmann's constant		k_B	$1.3807 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Molar gas constant		R	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro's number		N	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Standard molar volume			$22.414 \times 10^{-3}\,\mathrm{m^3\,mol^{-1}}$
Bohr magneton		μ_B	$9.274 \times 10^{-24}\,\mathrm{A\,m^2}$ *or* $\mathrm{J\,T^{-1}}$
Nuclear magneton		μ_N	$5.051 \times 10^{-27}\,\mathrm{A\,m^2}$ *or* $\mathrm{J\,T^{-1}}$
Stefan's constant		σ	$5.671 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Gravitational constant		G	$6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Proton magnetic moment		μ_p	$2.7928\,\mu_N$
Neutron magnetic moment		μ_n	$-1.9130\,\mu_N$
Standard acceleration due to gravity		g	$9.807\,\mathrm{m\,s^{-2}}$
Permeability of free space		μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Permittivity of free space		ε_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$

Other data and conversion factors

1 angstrom	Å	$10^{-10}\,\mathrm{m}$
1 pascal	Pa	$1\,\mathrm{N\,m^{-2}}$
1 standard atmosphere		$1.0132 \times 10^{5}\,\mathrm{Pa\ (N\,m^{-2})}$
1 electron volt	eV	$1.6022 \times 10^{-19}\,\mathrm{J}$
	eV/k_B	$1.1604 \times 10^{4}\ \mathrm{K}$
Wavelength of 1 eV photon		$1.2399 \times 10^{-6}\,\mathrm{m}$