Static Quark Model

Malcolm John
C4: particle physics

Units

	natural					*S.I.*
energy	MeV	$\Rightarrow$	MeV	$\times\, 1.6022\cdot 10^{-13}$	$\Rightarrow$	J
mass	MeV	$\Rightarrow$	MeV/c^2	$\times \dfrac{1.6022\cdot 10^{-13}}{(3\cdot 10^8)^2}$	$\Rightarrow$	kg
time	MeV $^{-1}$	$\Rightarrow$	$\hbar$/ MeV	$\times \dfrac{1.0546\cdot 10^{-34}}{1.6022\cdot 10^{-13}} \times 10^{12}$	$\Rightarrow$	ps
frequency	MeV	$\Rightarrow$	MeV/h	$\times \dfrac{1.6022\cdot 10^{-13}}{1.0546\cdot 10^{-34}} / 2\pi$	$\Rightarrow$	Hz
length	MeV^{-1}	$\Rightarrow$	$\hbar c$/ MeV	$\times \underbrace{\frac{1.0546\cdot 10^{-34}\times 3\cdot 10^8}{1.6022\cdot 10^{-13}}}_{197} \times 10^{15}$	$\Rightarrow$	fm
wavelength	MeV^{-1}	$\Rightarrow$	h/(MeV/c)	$\times\, 197 \times 2\pi$	$\Rightarrow$	fm
cross section	MeV^{-2}	$\Rightarrow$	$\hbar^2c^2$/ MeV^2	$\times\, (197)^2/100$	$\Rightarrow$	barn
						(1 b = 100 fm^2)

In transfering to and from MeV units, $\hbar c = 197$ MeV fm is useful.

Preface

There are several fundamental operations upon which we would expect physical behaviour not to depend. Translations in space; occurrence in time; rotation about an axis; reversal of spatial coordinates or the swapping of positive charge for negative charge. When the associated operators (Q) are applied to a wavefunction, $|\,\overline{\psi}\rangle = Q\,|\,\psi\rangle$ (the bar labels the transformed wavefunction), probabilities should not change:

$$\begin{aligned}\langle\overline{\phi}\,|\,\overline{\psi}\rangle = \langle\phi\,|\,Q^\dagger Q\,|\,\psi\rangle = \langle\phi\,|\,\psi\rangle\,,\\ Q^{-1} = Q^\dagger \quad \text{(Unitary condition)}\end{aligned} \tag{1}$$

Similarly, Q must not change the energy of the system,

$$\begin{aligned}\langle\overline{\phi}\,|\,H\,|\,\overline{\psi}\rangle = \langle\phi\,|\,Q^\dagger HQ\,|\,\psi\rangle = \langle\phi\,|\,H\,|\,\psi\rangle\,,\\ Q^{-1}HQ = H \quad\Rightarrow\quad HQ - QH = 0\,,\end{aligned} \tag{2}$$

the operator, Q, commutes with the Hamiltonian.

The set of all unitary transformations with the “Special" constraint that total probability equals one, constitute in mathematics, the SU(n) group. The n index is the number of degrees of freedom available under that transformation. The simplest non-trivial group is SU(2), which define spin-$\frac{1}{2}$ particles. SU(3) also features, especially in relation to the strong force.

The trivial group is the SU(1) group which contains just one element which must be its own inverse. Examples are the reversal of spatial coordinates, $P(\boldsymbol{x}) \longrightarrow -\boldsymbol{x}$, and charge-parity inversion, C(particle) $\longrightarrow$ antiparticle. Physics must to be invariant after two applications of such binary operators, so

$$P^2\,|\,\psi\rangle = I\,|\,\psi\rangle = \lambda^2\,|\,\psi\rangle$$

so, eigenvalues $\lambda = \pm 1$. Due to its simplicity, one would expect P to commute with the Hamiltonian so parity is a constant of the motion. If a state is an eigenstate of parity with a certain eigenvalue (± 1), any transition will preserve this eigenvalue. For this reason the parity and charge-parity quantum numbers, along with the angular momentum quantum number summarise the nature of a state with its J^{PC} (or J^P if not an eigenstate of C). Fermions and antifermions ($J = \frac{1}{2}$) have opposite intrinsic parity but defined $+1$ for fermions. The force mediators are vector constructs ($J = 1$), defined with -1 parity.

$$\begin{aligned}P\,|\,f\rangle &= +1\,|\,f\rangle \quad \text{for } f = e^-,\,\mu^-,\,q\\ P\,|\,\bar{f}\rangle &= -1\,|\,\bar{f}\rangle \quad \text{for } \bar{f} = e^+,\,\mu^+,\,\overline{q}\\ P\,|\,b\rangle &= -1\,|\,b\rangle \quad \text{for } b = \gamma,\,g,\,Z^0,\,W^\pm\,.\end{aligned} \tag{3}$$

The charge-parity operator, C, changes particle into antiparticle so few particles can be eigenstates of C. Charge parity is a useful conserved quantity when considering neutral bosons. The photon is a fundamental, neutral boson whose intrinsic charge quantum number is inferred by considering classical electromagnetism. The vector potential (A) associated with the photon is related to the electric field (E) and scalar potential (ϕ) by $\partial\boldsymbol{A}/\partial t = -\boldsymbol{E} - \nabla\phi$. Upon reversing charge, E and ϕ flip sign, so must the vector potential, $C\,|\,\gamma\rangle = -\,|\,\gamma\rangle$. In summary,

particle	f	$\bar{f}$	$\gamma,\,g,\,Z^0$	$W^\pm$
J^{PC}	$\frac{1}{2}^+$	$\frac{1}{2}^-$	1^{--}	$1^\pm$

QM reminders

Ehrenfest's theorem Consider the time variation of the expectation value of a quantum mechanical operator Q,

$$\begin{aligned}
\frac{d\langle Q\rangle}{dt} &= \frac{d}{dt}\langle\psi^* \mid Q \mid \psi\rangle \\
&= \langle\frac{\partial}{\partial t}\psi^* \mid Q \mid \psi\rangle + \langle\psi^* \mid \frac{\partial Q}{\partial t} \mid \psi\rangle + \langle\psi^* \mid Q \mid \frac{\partial}{\partial t}\psi\rangle \\
&= \langle+\frac{i}{\hbar}H\psi^* \mid Q \mid \psi\rangle + \langle\psi^* \mid Q \mid -\frac{i}{\hbar}H\psi\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle \\
&= \frac{i}{\hbar}\left[\langle\psi^* \mid [HQ - QH] \mid \psi\rangle\right] + \left\langle\frac{\partial Q}{\partial t}\right\rangle \\
\frac{d\langle Q\rangle}{dt} &= \frac{i}{\hbar}\underbrace{\left\langle[H,Q]\right\rangle}_{=0} + \cancel{\left\langle\frac{\partial Q}{\partial t}\right\rangle} \qquad Q \text{ has no explicit time dependence} \\
\langle Q\rangle &= const.
\end{aligned}$$

Ehrenfest's theorem shows that the expectation values of Q are constants of the motion and thus gives a quantum mechanical observable with good quantum numbers.

Angular momentum operators The angular momentum operators are Hermitian (observable quantum numbers) but are non-commutative,

$$[J_i, J_j] = i\,\varepsilon_{ijk}J_k\,. \tag{4}$$

so the three projections are not simultaneously observable. On the other hand, the Casimir operation,

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{5}$$

is a scalar so $[J^2, J_i] = 0$. The operator J^2 corresponds to the total angular momentum, which is simultaneously observable with one projection along an axis. By convention, the z axis is chosen. Next we calculate the eigenvalues of J^2. For this, it is useful to define raising(+) and lowering(−) operators,

$$\begin{aligned}
J_\pm &= J_x \pm iJ_y \qquad \text{such that,} \\
[J_z, J_\pm] &= \pm J_\pm\,, \\
\text{or}\quad J_zJ_\pm &= J_\pm(J_z \pm 1)\,.
\end{aligned}$$

The $J_\pm$ operations raise or lower the m quantum number through the quantised eigenvalues from the minimum to maximum values: $-j, -j+1, \ldots, j-1, j$, where j is identified at the total angular momentum quantum number. From Eq. 5,

$$\begin{aligned}
J^2 &= J_x^2 + J_y^2 + J_z^2 \\
&= \overbrace{J_-J_+ - i\underbrace{(J_xJ_y - J_yJ_x)}_{iJ_z}}^{} + J_z^2 \\
&= J_-J_+ + J_z(J_z+1)
\end{aligned} \tag{6}$$

Applying this result to the state, ψ where the projection along the z-axis is at it maximal value ($m = j$) such that $J_+ \mid \psi\rangle = 0$,

$$J^2 \mid \psi\rangle = j(j+1) \mid \psi\rangle\,. \tag{7}$$

The same conclusion is reached from applying the lowering first to the state $m = -j$, where $J_- \mid \psi\rangle = 0$.

1 Combinations and multiplets

Consider two particles with angular momentum j_a and j_b. The combination

$$\mid j, m\rangle = \mid j_a, m_a\rangle \mid j_b, m_b\rangle$$

has total angular momentum in a range of discrete values,

$$j = j_{min}, (j_{min}+1), \ \ldots\ , (j_{max}-1),\ j_{max},$$

where $j_{min} = |j_A - j_B|$ and $j_{max} = |j_A + j_B|$. For each value of j there are $2j+1$ states in an *irreducible representation* (multiplet). Consider for example, the combination of two spin-$\frac{1}{2}$ particles:

$$\begin{aligned}
(2\ \tfrac{1}{2}+1)(2\ \tfrac{1}{2}+1) &= 4 \text{ combinations}, \\
j_{min} &= \tfrac{1}{2} - \tfrac{1}{2} = 0, \\
j_{max} &= \tfrac{1}{2} + \tfrac{1}{2} = 1, \\
N_{states}(j_{min}) &= (2j_{min}+1) = 1, \\
N_{states}(j_{max}) &= (2j_{max}+1) = 3\,.
\end{aligned}$$

Hence to describe the multiplets arising from the combination of two spin-$\frac{1}{2}$ particles, we write,

$$2 \otimes 2 = 3 \oplus 1\,.$$

Combining two spin-$\frac{1}{2}$ particles by-hand

Combinations of two spin-$\frac{1}{2}$ particles can have total spin $j = 0$ or $j = 1, m \in \{-1, 0, 1\}$.

$$\begin{aligned}
\mid 1,1\rangle &= \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\
&= |\uparrow,\uparrow\rangle \\
\mid 1,0\rangle &= \sqrt{\tfrac{1}{2}} \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{2}} \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\
&= \sqrt{\tfrac{1}{2}}\,(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle) \\
\mid 1,-1\rangle &= \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \\
&= |\downarrow,\downarrow\rangle
\end{aligned}$$

$$\begin{aligned}
\mid 0,0\rangle &= \sqrt{\tfrac{1}{2}} \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{2}} \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\
&= \sqrt{\tfrac{1}{2}}\,(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)
\end{aligned}$$

The $j = 1$ triplet is symmetric under the interchange of particles. The singlet contains one $j = 0$ state which is deduced from requiring orthogonality to the $|1,0\rangle$ state. The singlet state is manifestly antisymmetric, meaning that a factor of -1 drops out upon the interchange of particle 1 and 2. This deconstruction explicitly demonstrates $2 \otimes 2 = 3 \oplus 1$.

Clebsch Gordan coefficients

The leading square-root coefficients are required for normalisation but although trivial in the above case, are not generally. They are the Clebsch Gordan coefficients who may be derived from successive application of raising and lowering operators. Raising the $m = j$ state or lowering the $m = -j$ state gives

zero but for the general case the action of raising/lowering is not null.

$$J_{\pm} \mid j,m\rangle = C_{\pm}(j,m) \mid j,m \pm 1\rangle \tag{8}$$

with the boundary condition that $C_{\pm}$ is required to be 0 for $J_{-} \mid j,-j\rangle$ and $J_{+} \mid j,+j\rangle$. To derive these constants two observations are made.

Eq. 6 shows effect of a raising, then lowering operation is a real number that is not in general, unity.

$$\begin{aligned} J_{-}J_{+} &= j(j+1) - m(m+1) \\ J_{+}J_{-} &= j(j+1) - m(m-1) \end{aligned} \tag{9}$$

Second, that J_x and J_y are Hermitian so J_+ is the Hermitian-conjugate of J_- (and vice-versa),

$$\begin{aligned} \Big| J_{-} \mid j,m\rangle \Big|^2 &= \langle j,m \mid J_{+}J_{-} \mid j,m\rangle \\ &= C_{-}^2 \langle j,m \mid j,m\rangle \;=\; C_{-}^2 \end{aligned}$$

where C_-^2 is real and defined in Eq. 9. Similarly,

$$\big| J_{+} \mid j,m\rangle \big|^2 = \langle j,m \mid J_{-}J_{+} \mid j,m\rangle \;=\; C_{+}^2$$

The $C_{\pm}$ are Clebsch-Gordan coefficients, defined as,

$$C_{\pm}(j,m) = \sqrt{j(j+1) - m(m \pm 1)}\,, \tag{10}$$

and are used extensively is to calculate the relative intensity (probability) of combined states.

Example: the $(j_1 = 1) \times (j_2 = 1)$ combination

Imagine the combination of two spin-1 particles, each with possible states: $m = -1, 0, 1$, into a system with total $j = 2$. Total $j = 1, 0$ is also possible but neglected in this example. The combination has five possible states, $m = -2, -1, 0, 1, 2$ but the composition of each state must be found by applying the lowering operator from the fully-stretch out state, calculating the Clebsch-Gordan coefficients from Eq. 10. First calculate various C-G coefficients,

$$\begin{aligned} J_{-} \mid 2,+2\rangle &= 2 \mid 2,+1\rangle \\ J_{-} \mid 2,+1\rangle &= \sqrt{6} \mid 2,0\rangle \\ J_{-} \mid 1,+1\rangle &= \sqrt{2} \mid 1,0\rangle \\ J_{-} \mid 1,-1\rangle &= 0 \end{aligned}$$

Then start from the “stretched-out" state successively applying the lowering operator to both sides by calculating the C-G coeficients dividing-through by any factor appearing on the LHC.

$$\begin{aligned} \mid 2,+2\rangle &= \mid 1,+1\rangle \mid 1,+1\rangle \;\text{(“stretched-out" state)} \\ \mid 2,+1\rangle &= \tfrac{1}{\sqrt{2}} \mid 1,0\rangle \mid 1,+1\rangle + \tfrac{1}{\sqrt{2}} \mid 1,1\rangle \mid 1,0\rangle \\ \mid 2,\;\;0\;\rangle &= \tfrac{1}{\sqrt{6}} \mid 1,-1\rangle \mid 1,+1\rangle + \sqrt{\tfrac{2}{3}} \mid 1,0\rangle \mid 1,0\rangle \\ &\qquad + \tfrac{1}{\sqrt{6}} \mid 1,1\rangle \mid 1,-1\rangle \\ \mid 2,-1\rangle &= \tfrac{1}{\sqrt{2}} \mid 1,-1\rangle \mid 1,0\rangle + \tfrac{1}{\sqrt{2}} \mid 1,0\rangle \mid 1,-1\rangle \\ \mid 2,-2\rangle &= \mid 1,-1\rangle \mid 1,-1\rangle \end{aligned}$$

Clebsch-Gordan coefficients for several common combinations are published in several places, notably in the PDG.

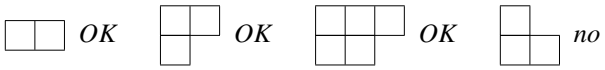
Young's Tableaux

Young's Tableaux is a tool of group mathematics used to calculate the number, dimension and symmetry of the irreducible representations. It is quicker than the ‘by-hand’ approach exemplified above for $2 \otimes 2$. Tableaux are constructed using the following rules:

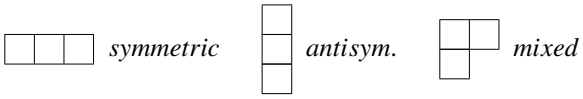
1. A tableau represents one SU(n) object and can take n possible labels,

   for SU(3), $\square \equiv \boxed{1}$, $\boxed{2}$ or $\boxed{3}$

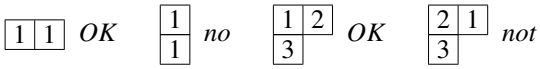
2. Objects are combined by adding tableaux to the right or bottom forming a top-left justified stack ensuring no row is longer than the one above it.

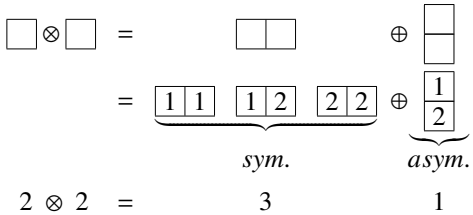
3. A single “horizontal" row configuration represents a multiplet of symmetric combinations. A single “vertical" column configuration represents an antisymmetric multiplet. All other configurations are of mixed symmetry.

4. When populating the tableaux with labels (1,2,3 etc...) to investigate the dimensionality of the multiplet, two more rules must be respected to avoid double-counting permutations:
   a. Moving right across a row of tableaux, labelsmay not decrease (it can stay the same);
   b. Moving down any column of tableaux, labels must increase (it can not stay the same).
   This last rule implied that there can be at-most n rows of tableaux in the SU(n) group.

Applying these rule to the example of two SU(2) particles recovers the by-hand result.



Combining three particles

When adding the third particle to the previous result, one notes that there is two ways that the third tableau may be added to the symmetric triplet, but only one way to add it to the asymmetric singlet without violating rule 4. The result is symmetric

quadruplet and two doublets of mixed symmetry. They retain separate labels $\mathcal{M}_S$ and $\mathcal{M}_A$ indicating the symmetry of the first two particle in the combination.

$$\square\otimes\square\otimes\square = \left(\square\square \oplus \begin{array}{|c|}\hline\\\hline\\\hline\end{array}\right)\otimes\square$$

$$= \underbrace{\square\square\square}_{sym.} \oplus \underbrace{\begin{array}{|c|c|}\hline\\\hline\\\hline\end{array}}_{\mathcal{M}_S} \oplus \underbrace{\begin{array}{|c|c|}\hline\\\hline\end{array}}_{\mathcal{M}_A}$$

$$= \boxed{1}\boxed{1}\boxed{1} \quad \boxed{1}\boxed{1}\boxed{2} \quad \boxed{1}\boxed{2}\boxed{2} \quad \boxed{2}\boxed{2}\boxed{2}$$
$$\oplus \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$$

Thus, $(2 \otimes 2) \otimes 2 = 4 \oplus 2 \oplus 2$ (11)

Combining three particles by hand

Start by adding a spin-up particle to the $|1,1\rangle$ state, then work down using the Clebsch-Gordan recipe.

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$$
$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |\downarrow\downarrow\downarrow\rangle$$

Two more states come from subtracting angular momentum from $|1,1\rangle$ and requiring orthogonality to the $|\tfrac{3}{2},+\tfrac{1}{2}\rangle$ states:

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2\,|\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle)$$
$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{-1}{\sqrt{6}}(2\,|\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)$$

These states are of mixed symmetry. Finally, two states are derived from adding a particle to the two-particle singlet:

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle)$$
$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{-1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)$$

which is also of a mixed symmetry.

Application to SU(3)

The strong force has three charges of identical magnitude and is thought to exhibit perfect SU(3) symmetry. The three charges are imagined as colour,

$$\square = \boxed{1}\ \boxed{2}\ \boxed{3} = Red\ \ Blue\ \ Green$$

Unlike, electric charge, colour charge cannot be observed directly because all hadrons are neutral "colourless" combinations. It is the study of the patterns of colour combinations that much of the nature of the underlying force is inferred. The combination of two SU(3) particles is,

$$\square\otimes\square = \underbrace{\square\square}_{sym.} \oplus \underbrace{\begin{array}{|c|}\hline\\\hline\end{array}}_{asym.}$$

$$= \boxed{1}\boxed{1} \quad \boxed{1}\boxed{2} \quad \boxed{1}\boxed{3} \quad \boxed{2}\boxed{2} \quad \boxed{2}\boxed{3} \quad \boxed{3}\boxed{3}$$
$$\oplus \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2\\\hline 3\\\cline{1-1}\end{array}$$

Thus, $3 \otimes 3 = 6 \oplus 3$ (12)

Extend to three particles as per the Young's Tableaux rules:

$$\square\otimes\square\otimes\square = \left(\square\square \oplus \begin{array}{|c|}\hline\\\hline\end{array}\right) \otimes \square$$

$$= \underbrace{\square\square\square}_{sym.} \oplus \underbrace{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}}_{\mathcal{M}_S} \oplus \underbrace{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}}_{\mathcal{M}_A} \oplus \underbrace{\begin{array}{|c|}\hline\\\hline\\\hline\end{array}}_{asym.}$$

Thus, $(3 \otimes 3) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ (13)

For colour SU(3), the colour singlet plays an important role ensuring the baryon wavefunction is antisymmetric.

$$\begin{array}{|c|}\hline R\\\hline G\\\hline B\\\hline\end{array} = \frac{1}{\sqrt{6}}\Big(+|\,RGB\,\rangle - |\,GRB\,\rangle + |\,BRG\,\rangle - |\,RBG\,\rangle + |\,GBR\,\rangle - |\,BGR\,\rangle \Big). \tag{14}$$

2 Isospin

In the 1930s, the theoretical success of SU(2) and the similarity in mass of the proton and neutron inspired postulation that they were the same fundamental object, just with a different projection. Though this is long since known not to be the case, it is still a useful symmetry when dealing with hadronic physics. The concept is:

- $|u\rangle$ and $|d\rangle$ form a SU(2) doublet like spin-$\frac{1}{2}$

$$I_3\,|\,u\rangle = \tfrac{1}{2}\,|\,u\rangle, \quad I_3\,|\,d\rangle = -\tfrac{1}{2}\,|\,d\rangle$$

- Raising & lowering operators exist:

$$I_+\,|\,d\rangle = |\,u\rangle, \quad I_-\,|\,u\rangle = |\,d\rangle, \quad I_-\,|\,d\rangle = 0$$

- Total isospin is conserved in strong interactions.

Isospin conservation stems from the underlying symmetry of the SU(2) group. For the group properties of symmetry to apply, three assumptions are imposed:

1. u and d quarks have an identical [strong] coupling.
2. the strong force dominates inside hadrons.
3. u and d quarks have [essentially] identical mass.

The pattern of $\{u,d\}$ flavour multiplets for baryons is identical to the $2 \otimes (2 \otimes 2) = 4 \oplus 2 \oplus 2$ structure of $\{\frac{1}{2},-\frac{1}{2}\}$.

$$|\,I=\tfrac{3}{2},I_3=\tfrac{3}{2}\rangle = |\,uuu\rangle$$
$$|\,I=\tfrac{3}{2},I_3=\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(|\,duu\rangle + |\,udu\rangle + |\,uud\rangle)$$
$$|\,I=\tfrac{3}{2},I_3=-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(|\,ddu\rangle + |\,udd\rangle + |\,dud\rangle)$$
$$|\,I=\tfrac{3}{2},I_3=-\tfrac{3}{2}\rangle = |\,ddd\rangle$$

$$
\begin{aligned}
|\, I=\tfrac{1}{2}, I_3=\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\,|\,uud\rangle - |\,duu\rangle - |\,udu\rangle)\\
|\, I=\tfrac{1}{2}, I_3=-\tfrac{1}{2}\rangle &= \tfrac{-1}{\sqrt{6}}(2\,|\,ddu\rangle - |\,udd\rangle - |\,dud\rangle)\\
|\, I=\tfrac{1}{2}, I_3=\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(|\,udu\rangle - |\,duu\rangle)\\
|\, I=\tfrac{1}{2}, I_3=-\tfrac{1}{2}\rangle &= \tfrac{-1}{\sqrt{2}}(|\,udd\rangle - |\,dud\rangle)
\end{aligned}
$$

Spin-flavour wavefunction of the proton

The proton is the lowest-lying stable baryon; as $L=0$, the spatial part of the wavefunction, ψ_{space} is symmetric. The colour wavefunction, ψ_{colour} is always antisymmetric. Hence for the total wavefunction to a antisymmetric (as the proton is a fermion), $\psi_{spin}\psi_{flavour}$ must be symmetric. The proton has $S=\frac{1}{2}$ so must be built from the mixed symmetry wavefunctions because the symmetric quadruplet is for $S=\frac{3}{2}$ only. The correct combination is symmetric,

$$
\begin{aligned}
|p^{\uparrow}\rangle_{spin,flavour} \propto \Big(&\psi_{spin}(\mathcal{M}_S)\,\psi_{flavour}(\mathcal{M}_S)\\
&+\psi_{spin}(\mathcal{M}_A)\,\psi_{flavour}(\mathcal{M}_A)\Big)
\end{aligned}
\tag{15}
$$

where,

$$
\begin{aligned}
\psi_{spin}(\mathcal{M}_S) &= |\,\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2\,|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle)\\
\psi_{spin}(\mathcal{M}_A) &= |\,\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle)
\end{aligned}
$$

and with different notation, but identical logic:

$$
\begin{aligned}
\psi_{flavour}(\mathcal{M}_S) &= \tfrac{1}{\sqrt{6}}(2\,|\,uud\rangle - |\,udu\rangle - |\,duu\rangle)\\
\psi_{flavour}(\mathcal{M}_A) &= \tfrac{1}{\sqrt{2}}(|\,udu\rangle - |\,duu\rangle)
\end{aligned}
$$

Multiplying all these terms together and normalising gives the proton $(spin, flavour)$ wavefunction,

$$
\begin{aligned}
|\,p^{\uparrow}\rangle_{spin,flavour} = \tfrac{1}{\sqrt{18}}(\;&2\,|\,u^{\uparrow}u^{\uparrow}d^{\downarrow}\rangle - |\,u^{\uparrow}u^{\downarrow}d^{\uparrow}\rangle - |\,u^{\downarrow}u^{\uparrow}d^{\uparrow}\rangle\\
&+2\,|\,u^{\uparrow}d^{\downarrow}u^{\uparrow}\rangle - |\,u^{\uparrow}d^{\uparrow}u^{\downarrow}\rangle - |\,u^{\downarrow}d^{\uparrow}u^{\uparrow}\rangle\\
&+2\,|\,d^{\downarrow}u^{\uparrow}u^{\uparrow}\rangle - |\,d^{\uparrow}u^{\downarrow}u^{\uparrow}\rangle - |\,d^{\uparrow}u^{\uparrow}u^{\downarrow}\rangle\;).
\end{aligned}
$$

To confirm the symmetry, notice that each line is related to another by the swapping the position of any two particles.

Isospin of antiquarks

When dealing with antiquarks care must be taken of how charge conjugation is applied. We choose a convention which leave the Clebsch-Gordan coefficients the same as with quarks, though this introduces a sign convention (the Condon-Shortley convention),

$$
\begin{aligned}
I_-\,|\,\bar{d}\rangle &= -\,|\,\bar{u}\rangle,\\
I_+\,|\,\bar{u}\rangle &= -\,|\,\bar{d}\rangle,\\
I_+\,|\,\bar{d}\rangle &= I_-\,|\,\bar{u}\rangle = 0
\end{aligned}
\tag{16}
$$

For example, consider the SU(2) isospin combinations of $2\otimes\bar{2}=1\oplus 3$. The singlet is symmetric,

$$
\tfrac{1}{\sqrt{2}}(|d\bar{d}\rangle + |u\bar{u}\rangle)=|0,0\rangle
$$

and the antisymmetric triplet identifies three pions $(+,0,-)$,

$$
\begin{aligned}
|1,\;\;1\rangle &= |u\bar{d}\rangle &&= |\,\pi^{+}\rangle\\
|1,\;\;0\rangle &= \tfrac{1}{\sqrt{2}}(|d\bar{d}\rangle - |u\bar{u}\rangle) &&= |\,\pi^{0}\rangle\\
|1,-1\rangle &= -|\bar{u}d\rangle &&= |\,\pi^{-}\rangle\,.
\end{aligned}
$$

Experimentally, the use of SU(2) seems justified by the triplet of pions being [almost] degenerate in mass:

$$
\mathrm{m}(\pi^{+}) = \mathrm{m}(\pi^{-}) = 140\,\mathrm{MeV}/c^2, \quad \mathrm{m}(\pi^{0}) = 135\,\mathrm{MeV}/c^2.
$$

Example use of isospin: analysis of Δ decays

- The $\Delta(1232)$ baryons contain only u and d with total spin $S={}^{3}/_{2}$ and isospin $I={}^{3}/_{2}$.
- Baryon number conservation requires the Δ to decay to p or n, higher mass baryons are not accessible.
- As the strong interaction dominates, use isospin to understand relative rates remembering:

$$
\begin{aligned}
|p\rangle = |\,\tfrac{1}{2},\tfrac{1}{2}\rangle \qquad |n\rangle = |\,\tfrac{1}{2},-\tfrac{1}{2}\rangle\\
|\pi^{+}\rangle = |1,1\rangle \qquad |\pi^{0}\rangle = |1,0\rangle \qquad |\pi^{-}\rangle = |1,-1\rangle
\end{aligned}
$$

- Decompose $I=\frac{3}{2}$ state into possible combinations $I=\frac{1}{2}$ and $I=1$

$$
\begin{aligned}
\Delta^{++} &= |p\rangle|\pi^{+}\rangle\\
\Delta^{+} &= \tfrac{1}{\sqrt{3}}\,|n\rangle|\pi^{+}\rangle + \sqrt{\tfrac{2}{3}}\,|p\rangle|\pi^{0}\rangle\\
\Delta^{0} &= \tfrac{1}{\sqrt{3}}\,|p\rangle|\pi^{-}\rangle + \sqrt{\tfrac{2}{3}}\,|n\rangle|\pi^{0}\rangle\\
\Delta^{-} &= |n\rangle|\pi^{-}\rangle
\end{aligned}
$$

- One can deduce the ratio of branching fractions:

$$
\frac{\mathcal{B}(\Delta^{+}\to\pi^{0}p)}{\mathcal{B}(\Delta^{+}\to\pi^{+}n)} = \frac{|\langle\pi^{0}p\,|\,\Delta^{+}\rangle|^2}{|\langle\pi^{+}n\,|\,\Delta^{+}\rangle|^2} = 2
$$

.

- or the relative Δ cross sections of π^{-} and π^{+} scattering off a proton target at $\sqrt{s}\sim m(\Delta)$:

$$
\frac{\sigma(\pi^{+}p)}{\sigma(\pi^{-}p)} = \frac{|\langle\Delta^{++}\,|\,\pi^{+}p\rangle|^2}{|\langle\,\Delta^{0}\,|\,\pi^{-}p\rangle|^2} = 3\,.
$$

J^{PC} of hadrons

Parity and, where applicable, charge quantum numbers are often quoted, for a particular state, with its total angular momentum, $J=L+S$, as a J^{PC} number. The J^{P} of a particle is closely related to the spatial transformation properties of the state. 0^{-} are labelled *pseudoscalar* particles and 0^{+} *scalar*. 1^{-} are labelled *vector* particles and 1^{+} states are *axial vectors*.

- **Mesons** are $q\bar{q}$ bound states. As these have opposite parity, a ground state mesons will always have $P=-1$. Excited states bring additional parity factors according to $(-1)^{L}$.

$$
P_{meson} = (-1)^{L+1}
$$

C is defined (for light neutral mesons only) by interchanging $q \leftrightarrow \bar{q}$.

$$C_{meson} = (-1)^{L+S}$$

- **Baryons** contain three quarks and as such cannot be their own antiparticle; C is undefined. The calculation of baryon parity is more complex than for mesons because one must consider the angular momentum of a three-body system. The intrinsic parity of a baryon is $(+1)^3 = +1$; similarly it is -1 for an antibaryon. In full:

$$P_{baryon} = B\,(-1)^{L_{12}}(-1)^{L_3}$$

where B is baryon number, L_{12} is the angular momentum between quark 1 and 2; L_3 is the angular momentum of the third relative to the 1-2 pair.

Situations where consideration of J^{PC} prove insightful are,

- *J^{PC} of a π^0*
  Ground state: L=S=J=0. From the formula: $P = -1$, $C = +1$. So $J^{PC} = 0^{-+}$. It is a *pseudoscalar*. The dominant decay mode is $\pi^0 \to \gamma\gamma$. The conservation of charge conjugation is straight forward. Parity conservation requires that he overall $|\gamma\gamma\rangle$ is antisymmetric.

- *J^{PC} of a $\rho(770)$*
  Vector state with the quark spins aligned: L=0, S=1, J=1. From the formula: $P = -1$, $C = -1$. So $J^{PC} = 1^{--}$.

- *Is $\rho^0 \to \pi^+\pi^-$ an allowed decay?*
  Excited initial state, $S = J = 1$. L=0 so $P = -1$. Also from above, $C = -1$ so $J^{PC} = 1^{--}$. The pair of pions have intrinsic parity $+1$ from $(-1)_{\pi^+}(-1)_{\pi^-}$ and each $J = 0$. So producing them in a $L = 1$ state will simultaneously conserve J, P, C.

- *What about $\rho^0 \to \pi^0\pi^0$?*
  This would be similar to the previous example, requiring the pion pair to be in a $L = 1$ state. However, applying C to the final state has no effect: $C \mid \pi^0\pi^0\rangle = \mid \pi^0\pi^0\rangle$. $C = +1$. $J^{PC}(\pi^0\pi^0) = 1^{-+}$. So this process is forbidden by charge violation with the strong or EM force. Indeed, it has never been observed.

- $\rho^0 \to l^+l^-$
  Similar to $\rho^0 \to \pi^+\pi^-$ except l^+l^- system has intrinsic parity -1 from $(+1)_{l^-}(-1)_{l^+}$. Producing as final state with the spins aligned, $S = 1$ simultaneously conserves J, P, C. Note that, this process only proceeds via the electromagnetic force and is therefore only 10^{-4} as likely as the $\pi^+\pi^-$ mode.

- $\rho^0 \to \pi^0\gamma$
  The initial state is $J^{PC} = 1^{--}$. The photon: $J^{PC} = 1^{--}$ and the π^0 0^{-+}. To conserve parity as well as charge-parity the final state is $L = 1$ combining with the $S = 1$ of the photon match the $J = 1$ of the initial state. But, the photon carry parity $(-1)^L$ away from a system. So this EM decay is allowed.

- $\pi^0 \to \gamma\gamma$
  This EM decay is only way the π^0 can decay (except for extremely rare processes involving electrons). Its a pseudoscalar 0^{-+} to $\left[\left((1^{--})(1^{--})\right)_{S=1}\right]_{L=1}$ to simultaneously satisfy J^{PC}.

Strangeness

The hope that up-ness and down-ness was sufficient to described hadronic matter, died when a new particle, originally known as the V^0, was observed in 1948 (in Manchester). Most evidently, the displaced $\pi^+\pi^-$vertex meant this new particle was neutral and long-lived. By adding together the momentum of the daughter pions, the invisible particle was measured to have a mass $\sim \frac{1}{2}$ that of the proton, ruling out it being a baryon. Particles with this new long-lived property were assigned a new quantum number: strangeness.

$\tau - \theta$ puzzle Soon after, two long-lived, strange mesons were observed with indistinguishable masses and lifetimes.

- The "$\tau^{\pm}$" decayed isotropically to $\pi^{\pm}\pi^0$. By angular momentum conservation, this implies the $J(\tau) = 0$ and assuming parity conservation, $P(\tau) = +1$.

- The "$\theta^{\pm}$" decayed to $\pi^{\pm}\pi\pi$. $P(\pi) = -1, (-1)^3 = -1$. Again assuming parity conservation, the parity of the θ seemed in opposition to the *tau*. Suspicion that $J(\theta) = 1$ was tested by examination of the decay kinematics (in Oxford by Dick Dalitz) but they showed $L_{12} = L_3 = 0$, confirming $J(\theta) = 0$ and $P(\theta) = -1$.

Parity of an ensemble of pions

- The intrinsic parity of a quark is $+1$, an antiquark, -1. Hence a J=0 meson like the pion has $P = -1$.
- The parity of the two pion state is the product of their intrinsic parities and any angular momentum between them. However, by conservation of angular momentum from the initial "τ", $L = 0$:

$$P(\pi\pi) = (-1)(-1)(-1)^L = +1.$$

- The three pion system is more complicated because there are two angular momenta:
  1. the relative motion of the first two pions, L_{12}
  2. the motion of the third relative to the axis of the first two, L_3.
- However, because the overall angular momentum must be conserved from the "θ" ($J = 0$), $L_{12} \equiv L_3$.

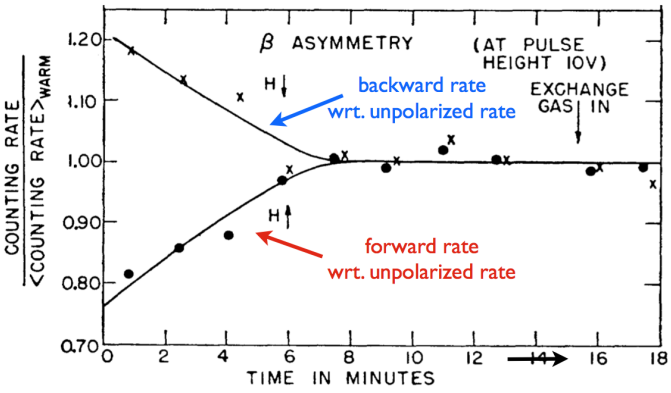
$$P(\pi\pi\pi) = (-1)(-1)(-1)(-1)^{L_{12}}(-1)^{L_3} = -1.$$

In a 1956 literature review Lee and Yang concluded there was no evidence of parity conservation in weak processes. They correctly postulated in a paper entitled, *Question of Parity Conservation in Weak Interactions* that the "τ" and "θ" mesons were the same particle (now called the $K^{\pm}$) and that parity is

not a conserved quantity of the weak force, in stark contrast to the strong force, electromagnetism and common sense.

The confirmation of parity violation came from a dedicated, and renown experiment by Madame Wu and collaborators. The essential details are,

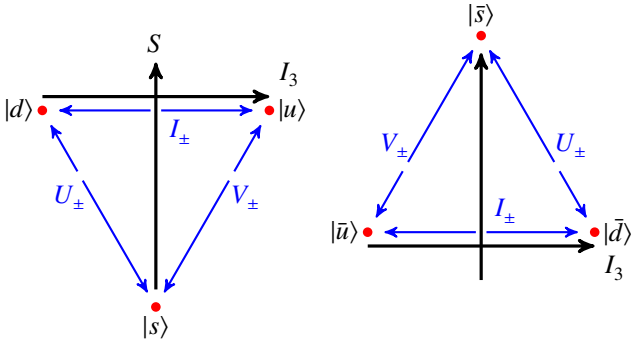
- $^{60}_{27}$Co is an artificial isotope that beta-decays to $^{60}_{28}$Ni with a half-life of 5.27 years.
- It has spin=5 whereas ^{60}Ni has spin=4 hence the electron and antineutrino of the beta decay have J=1.
- The cobalt sample was cooled to 0.01K (!) in a magnetic field such that all their spins were aligned.
- This spin alignment is vital because in order to inspect properties of parity, a reference direction is needed, and the only “intrinsic" direction for the initial state is the spin axis.
- An electron counter was mounted above the sample and depending on the polarity of the magnetic field, it records the electron flux in, or opposite to, the direction of the Cobalt-60's spin.
- The data is shown below. It is seen that there is a deficit in electrons in the direction of the spin.



3 SU(3) flavour

With a new flavour quantum number, the applicability of SU(3) is tested for three eigenvectors,

$$|u\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |d\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad |s\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$



The isospin (*I-spin*) transformation is still valid, but it is joined by *U-spin* and *V-spin* which navigate the (s,d) and (s,u) doublets respectively. The complete set of raising and lowering operators are:

$$\begin{aligned}
I_+|d\rangle &= |u\rangle & I_-|u\rangle &= |d\rangle \\
V_+|u\rangle &= |s\rangle & V_-|s\rangle &= |u\rangle \\
U_+|s\rangle &= |d\rangle & U_-|d\rangle &= |s\rangle \\
I_+|\bar{u}\rangle &= -|\bar{d}\rangle & I_-|\bar{d}\rangle &= -|\bar{u}\rangle \\
U_+|\bar{d}\rangle &= |\bar{s}\rangle & U_-|\bar{s}\rangle &= |\bar{d}\rangle \\
V_+|\bar{s}\rangle &= -|\bar{u}\rangle & V_-|\bar{u}\rangle &= -|\bar{s}\rangle
\end{aligned}$$

The Condon-Shortley convention is used for the antiquarks which gives rise to the ‘extra’ minus signs.

SU(3) mesons

Young's tableau for the SU(3) antiquark triplet is the antisymmetric reciprocal of the quark triplet defined according to

$$q_i = \epsilon_{ijk} q_k q_j \tag{17}$$

where a negative ϵ_{ijk} indicates the antiparticle. Explicitly,

SU(3) quarks = $\square$ SU(3) antiquarks = $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

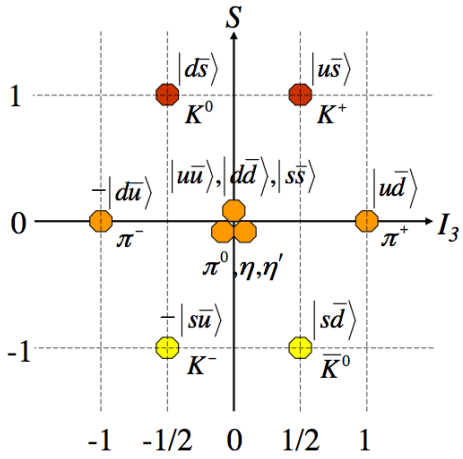
$\square = u \;,\; d \;,\; s$ $\qquad \begin{smallmatrix}\square\\\square\end{smallmatrix} = \begin{smallmatrix}s\\d\end{smallmatrix} \;,\; \begin{smallmatrix}u\\s\end{smallmatrix} \;,\; \begin{smallmatrix}d\\u\end{smallmatrix} = \bar{u} \;,\; \bar{d} \;,\; \bar{s}$

Thus Young's tableaux of quark-antiquark combinations are,

$$\square \otimes \square = \underbrace{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}_{\bar{1}} \oplus \underbrace{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{8}$$

$$\bar{3} \otimes 3 = \bar{1} \oplus 8$$

We now have the information to extend beyond the previously-observed $\{\pi^+, \pi^0, \pi^-\}$ triplet of SU(2) and identify all the SU(3) $q\bar{q}$ states. We start with the simplest case of $J = 0$ pseudoscalar mesons.



The six states with non-zero I and non-zero S are have unambiguous quark content. The other three need thought. The flavourless, symmetric singlet, known as the η' is

$$\eta' = \tfrac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$$

The remaining two are part of the octet and are deduced with rotations within the group,

$$
\begin{aligned}
I_- \,|\, u\bar{d}\rangle &= \;|\, d\bar{d}\rangle - |\, u\bar{u}\rangle \quad + X_{\text{flavourless}} \\
U_- \,|\, d\bar{s}\rangle &= \;|\, s\bar{s}\rangle + |\, d\bar{d}\rangle \quad + Y_{\text{flavourless}} \\
V_- \,|\, u\bar{s}\rangle &= \;|\, s\bar{s}\rangle - |\, u\bar{u}\rangle \quad + Z_{\text{flavourless}}
\end{aligned}
$$

These equations are not linearly independent so the convention is to choose one to be the established, isospin-triplet partner of the π^+ and π^-,

$$\pi^0 = \tfrac{1}{\sqrt{2}}(|d\bar{d}\rangle - |u\bar{u}\rangle)\,. \tag{18}$$

The last member of the octet is deduced by requiring its wavefunction be orthogonal to both the η' the π^0, and normalised to 1.

$$
\begin{aligned}
\eta \quad &\cdot \quad \pi^0 \quad = 0 \\
\begin{pmatrix}\alpha|u\bar{u}\rangle \\ \beta|d\bar{d}\rangle \\ \gamma|s\bar{s}\rangle\end{pmatrix} \quad &\cdot \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix}|u\bar{u}\rangle \\ -|d\bar{d}\rangle \\ 0\end{pmatrix} = 0 \quad \Rightarrow \beta = \alpha \\
\eta \quad &\cdot \quad \eta' \quad = 0 \\
\begin{pmatrix}\alpha|u\bar{u}\rangle \\ \beta|d\bar{d}\rangle \\ \gamma|s\bar{s}\rangle\end{pmatrix} \quad &\cdot \quad \tfrac{1}{\sqrt{3}}\begin{pmatrix}|u\bar{u}\rangle \\ |d\bar{d}\rangle \\ |s\bar{s}\rangle\end{pmatrix} = 0 \quad \Rightarrow \gamma = -2\alpha \\
\eta \quad &\cdot \quad \eta \quad = 1 \quad \Rightarrow \alpha^2 + \alpha^2 + (-2\alpha)^2 = 1 \\
&\qquad\qquad \alpha = \tfrac{1}{\sqrt{6}}
\end{aligned}
$$

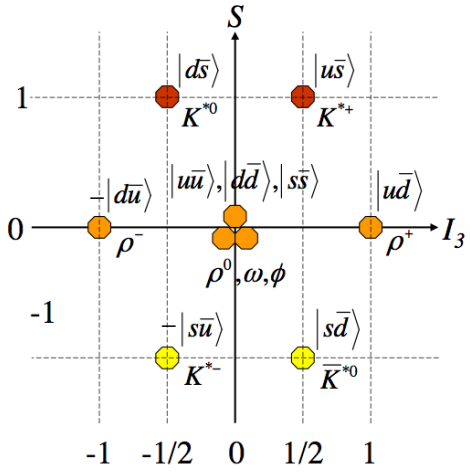
Thus, $$|\eta\rangle = \tfrac{1}{\sqrt{6}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle\right) \tag{19}$$

I	I_3	S	Meson	Composition	Decay	Mass (MeV)
1	1	0	π^+	$\lvert u\bar{d}\rangle$	$\pi^+ \to \mu^+\nu_\mu$	140
1	−1	0	π^-	$\lvert \bar{u}d\rangle$	$\pi^- \to \mu^-\bar{\nu}_\mu$	140
1	0	0	π^0	$\frac{1}{\sqrt{2}}\left(\lvert d\bar{d}\rangle - \lvert u\bar{u}\rangle\right)$	$\pi^0 \to \gamma\gamma$	135
½	½	1	K^+	$\lvert u\bar{s}\rangle$	$K^+ \to \mu^+\nu_\mu$	494
½	−½	1	K^0	$\lvert d\bar{s}\rangle$	$K^0 \to \pi\pi(\pi)$	498
½	−½	−1	K^-	$\lvert \bar{u}s\rangle$	$K^- \to \mu^-\bar{\nu}_\mu$	494
½	½	−1	$\bar{K}^0$	$\lvert \bar{d}s\rangle$	$K^0 \to \pi\pi(\pi)$	498
0	0	0	η	$\frac{1}{\sqrt{6}}\left(\lvert d\bar{d}\rangle + \lvert u\bar{u}\rangle - 2\lvert s\bar{s}\rangle\right)$	$\eta \to \gamma\gamma$	549
0	0	0	η'	$\frac{1}{\sqrt{3}}\left(\lvert d\bar{d}\rangle + \lvert u\bar{u}\rangle + \lvert s\bar{s}\rangle\right)$	$\eta' \to \eta\pi\pi$ or $\rho\gamma$	958

The $J = 1$ vector mesons exhibit a similar pattern of states with the notable difference that “octet-singlet" mixing occurs amongst the $I = 0,\ S = 0$ states. Experimentally it is seen that the singlet state ($\phi(1020)$) decays largely to kaons the other ($\omega(782)$) to pions (i.e. it has lost its $s\bar{s}$ content).

$$
\begin{aligned}
|\rho^0\rangle &= \tfrac{1}{\sqrt{2}}(|d\bar{d}\rangle - |u\bar{u}\rangle) & \mathcal{B}(\rho^0 \to \pi^+\pi^-) &= 100\% \\
|\omega\rangle &\approx \tfrac{1}{\sqrt{2}}(|d\bar{d}\rangle + |u\bar{u}\rangle) & \mathcal{B}(\omega \to \pi^+\pi^-\pi^0) &= 90\% \\
|\phi\rangle &\approx |s\bar{s}\rangle & \mathcal{B}(\phi \to K\overline{K}) &= 84\% \\
& & \mathcal{B}(\phi \to \pi^+\pi^-\pi^0) &= 15\%
\end{aligned}
$$

This is further breaking of the pattern anticipated by the SU(3) symmetry and more evidence that the s quark mass is significantly different to that of the u and d quarks.



I	I_3	S	Meson	Composition	Decay	Mass (MeV)
1	1	0	ρ^+	$\lvert u\bar{d}\rangle$		
1	−1	0	ρ^-	$\lvert \bar{u}d\rangle$	$\rho \to \pi\pi$	776
1	0	0	ρ^0	$\frac{1}{\sqrt{2}}\left(\lvert d\bar{d}\rangle - \lvert u\bar{u}\rangle\right)$		
½	½	1	K^{*+}	$\lvert u\bar{s}\rangle$		
½	−½	1	K^{*0}	$\lvert d\bar{s}\rangle$	$K^* \to K\pi$	892
½	−½	−1	K^{*-}	$\lvert \bar{u}s\rangle$		
½	½	−1	$\bar{K}^{*0}$	$\lvert \bar{d}s\rangle$		
0	0	0	ω	$\frac{1}{\sqrt{2}}\left(\lvert d\bar{d}\rangle + \lvert u\bar{u}\rangle\right)$	$\omega \to \pi^+\pi^-\pi^0$	783
0	0	0	ϕ	$\lvert s\bar{s}\rangle$	$\phi \to KK$	1019

SU(3) baryons

The baryon wavefunction is made of four parts:

$$\Psi = \psi_{space}\psi_{spin}\psi_{flavour}\psi_{colour}\,.$$

We consider ground state baryons that have no orbital angular momentum ($L = 0$) such that ψ_{space} is symmetric. The colour wavefunction ψ_{colour} is identical for all baryons and always antisymmetric. Hence, for the overall fermionic wavefunction to be antisymmetric, $\psi_{spin}\psi_{flavour}$ must be symmetric. The possible combinations for ψ_{spin} are:

$$2 \otimes (2 \otimes 2) = 8 = 3 \oplus 3 \oplus 1 \oplus 1 = \underbrace{4}_{sym.} \oplus \underbrace{2}_{\mathcal{M}_S} \oplus \underbrace{2}_{\mathcal{M}_A}$$

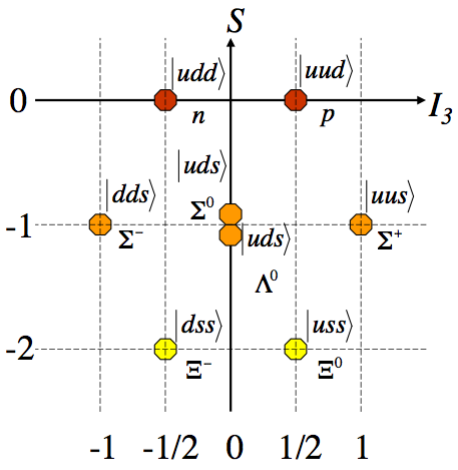
where we recall the four symmetric states are spin-$\frac{3}{2}$ states, whereas the other are spin-$\frac{1}{2}$. The SU(3) irreducible representation for three particle combinations is:

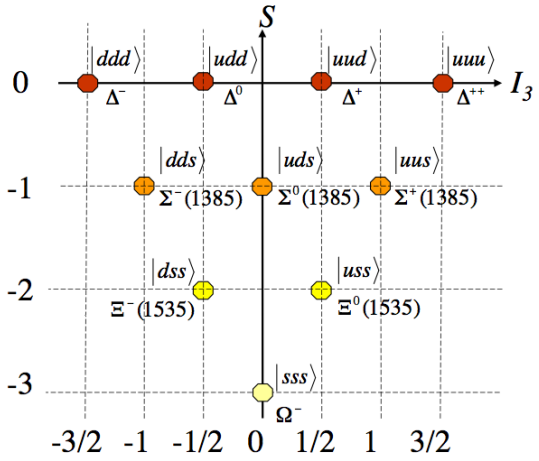
$$3 \otimes 3 \otimes 3 = \underbrace{10}_{sym.} \oplus \underbrace{8}_{\mathcal{M}_S} \oplus \underbrace{8}_{\mathcal{M}_A} \oplus \underbrace{1}_{asym.}$$

So the symmetric (SU(3),SU(2)) combinations must be,

$$\frac{1}{\sqrt{2}}[(\ 8\ ,\ 2\ ) + (\ 8\ ,\ 2\ )] \quad \text{and} \quad (\ 10\ ,\ 4\ )$$
$$\mathcal{M}_A\ \mathcal{M}_A \qquad \mathcal{M}_S\ \mathcal{M}_S \qquad\qquad sym.\ sym.$$

The $J = \frac{1}{2}$ triplet and $J = \frac{3}{2}$ decuplet are





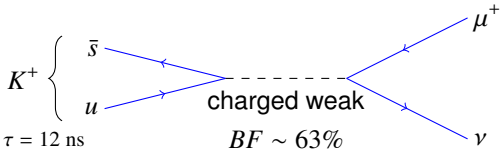
Note that the Δ^{++} is manifestly symmetric in $\psi_{flavour}\psi_{spin}\psi_{space}$. This observation was the driving force for an additional degree or freedom : *colour charge*. Also, the quark model was used to predict the $S = -3\Omega^-$. It was verified in 1964 when a new particle was observed to decay via three weak decays. The weak decay is identified by particles “flying" an appreciable distance; this implying longer lifetimes than the EM or strong interactions could describe. The production and decay chain is:

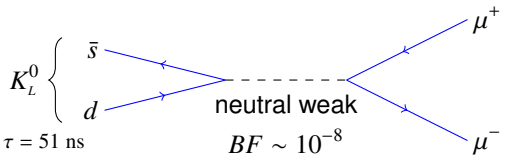
$$K^- p \rightarrow \Omega^- K^+ K^0_s$$
$$\Omega^- \rightarrow \Xi^0 \pi^- \quad .... \quad \Xi^0 \rightarrow \Lambda^0 \pi^0 \quad .... \quad \Lambda^0 \rightarrow p\pi^-$$

where each decay is a strangeness-changing weak decay.

4 The November Revolution

By 1970, the quark model was established as an excellent explanation for the patterns of discovered hadronic states. Though it was still not clear why Nature had need for a third, apparently redundant type of quark. There were other mysteries with the new strange particles, for example why the neutral decays were neither as likely at their charged counterparts, but not zero either.



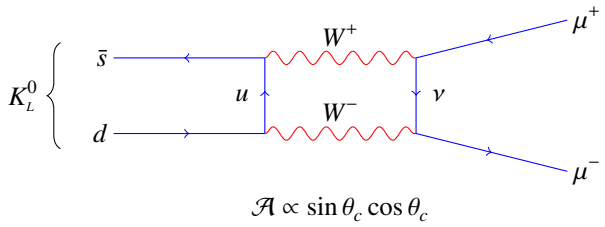


GIM suppression

It was known that the eigenstates of the charged weak interaction were not the same as the mass eigenstates. Cabibbo parameterised this by introducing an ‘angle’ that described how much the d and s mass eigenstates contributed to the weak interaction with the u:

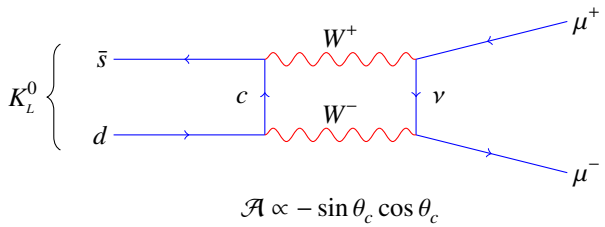
$$(u) \quad \leftarrow \text{weak force} \rightarrow \quad (d\cos\theta_c + s\sin\theta_c)$$

However, this could not explain why dimuon decays were so rare; experiment measured $\mathcal{B}(K^0_L \rightarrow \mu^+\mu^-) \approx 10^{-8}$ when the Feynman diagram was straightforward,



$$\mathcal{A} \propto \sin\theta_c \cos\theta_c$$

In 1970 the “GIM mechanism" (Glashow, Illiopoulos, Maiani) introduced a $+\frac{2}{3}$ partner to the s-quark. A second up-type quark give a second diagram that contributes with an [almost] equal and opposite contribution.



$$\mathcal{A} \propto -\sin\theta_c \cos\theta_c$$

The new model, with two generation of doublets becomes:

$$\begin{pmatrix} u \\ c \end{pmatrix} \quad \leftarrow \text{weak force} \rightarrow \quad \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

The reason this process happens at all is because the cancelling-out of these two diagrams is not exact because the u and c quarks have different mass. And from a consideration of kaon mixing, m_c was estimated at ≈ 1.5 GeV/c^2.

Charmonium

Direct evidence of charm arrived in November 1974 from two experiments. One collaboration called it the J; the other the ψ.

- e^+ e^- collisions at centre of mass energy $\sqrt{s} \approx 3$ GeV/c^2 at Stanford, CA.
- 28 GeV protons on a beryllium target at Brookhaven, NY.

Quantum numbers

First question: how much u-ness and d-ness is there in this new state? The J/ψ isospin assignment comes from observing:

$$\mathcal{B}\,(J/\psi \to \rho^+\pi^-) = \mathcal{B}\,(J/\psi \to \rho^0\pi^0) = \mathcal{B}\,(J/\psi \to \rho^-\pi^+)$$

Clebsch Gordan coefficients shows the J/ψ has zero isospin.

Clebsch-Gordan coefficents	m_1	m_2	$\lvert J,M\rangle$ = $\lvert 2,0\rangle$	$\lvert 1,0\rangle$	$\lvert 0,0\rangle$
$\lvert\rho^+\pi^-\rangle = \lvert 1,1\rangle\,\lvert 1,-1\rangle$	1	$^-1$	$\sqrt{1/6}$	$\sqrt{1/2}$	$\sqrt{1/3}$
$\lvert\rho^0\pi^0\rangle = \lvert 1,0\rangle\,\lvert 1,0\rangle$	0	0	$\sqrt{2/3}$	0	$-\sqrt{1/3}$
$\lvert\rho^-\pi^+\rangle = \lvert 1,-1\rangle\,\lvert 1,1\rangle$	$^-1$	1	$\sqrt{1/6}$	$-\sqrt{1/2}$	$\sqrt{1/3}$

At Stanford, they examined $e^+e^- \to \mu^+\mu^-$ with $\sqrt{s} \approx \mathrm{m}(J/\psi)$ so the final state is produced via either a J/ψ resonance decaying electromagnetically or via a virtual photon. The data shows a 'dip' on the low-mass side of the resonance which implies the photon and the J/ψ process are interfering, which means the J/ψ has the same $J^{PC} = 1^{--}$ as the photon.

Width

Consider Breit-Wigner formula:

$$\sigma(E) = \sigma_{max}\frac{\Gamma_i\Gamma_f}{(E-M)^2+\Gamma^2/4} \qquad (20)$$
$$\text{where}\quad \sigma_{max} = \pi\lambda\!\!\!^{-2}\frac{2J+1}{(2s_1+1)(2s_2+1)} \text{ and } \lambda\!\!\!^{-} = \frac{\hbar c}{pc}\,.$$

For $e^+e^- \to J/\psi \to e^+e^-$, $J=1,\ s_1=s_2=\frac{1}{2},\ \Gamma_f=\Gamma_i=\Gamma_{e^+e^-}$.

$$\sigma_{ee} = \tfrac{3}{4}\pi\lambda\!\!\!^{-2}\frac{\Gamma^2_{e^+e^-}}{(E-M)^2+\Gamma^2/4}$$

This can be integrated to find the total area under the cross section curve using the substitution, $E - M = \frac{\Gamma}{2}\tan\theta$.

$$\begin{aligned}\int \sigma_{ee}d(E-M) &= \tfrac{3}{4}\pi\lambda\!\!\!^{-2}\frac{\Gamma^2_{e^+e^-}}{(\tan^2\theta)\Gamma^2/4+\Gamma^2/4}d(E-M)\\ &= 3\pi\lambda\!\!\!^{-2}\frac{\left(\frac{\Gamma_{e^+e^-}}{\Gamma}\right)^2}{\tan^2\theta+1}d(E-M)\end{aligned}$$

Then,

$$d(E-M) = \tfrac{\Gamma}{2}\,d\tan\theta \quad , \quad d(\tan\theta) = (\tan^2\theta+1)\,d\theta$$
$$\text{and}\qquad \int_{-\infty}^{\infty} d(E-M) \mapsto \int_{-\pi/2}^{\pi/2} d\theta$$

$$\begin{aligned}\int \sigma_{ee}d(E-M) &= 3\pi\lambda\!\!\!^{-2}\left(\frac{\Gamma_{e^+e^-}}{\Gamma}\right)^2\frac{\Gamma}{2}\int_{-\pi/2}^{\pi/2}d\theta\\ &= \frac{3\pi^2\lambda\!\!\!^{-2}}{2}\left(\frac{\Gamma_{e^+e^-}}{\Gamma}\right)^2\Gamma\end{aligned}$$

From the measurements made by Richter *et al.* we can take:

- the area under the $e^+e^- \to J/\psi \to e^+e^-$ peak is ≈ 400 nb MeV (approx. a triangle, height: 100 nb, base: 8 MeV)
- the selection efficiency of $|\cos\theta| < 0.6$ on a $(1+\cos^2\theta)$ distribution $\sim 50\%$;
- $\Gamma_{e^+e^-}/\Gamma = \mathcal{B}(J/\psi \to e^+e^-) \approx 6\%$

Thus:

$$\Gamma = \frac{800\,\text{nb MeV}}{\frac{3}{2}\pi^2\lambda\!\!\!^{-2}}(6\%)^{-2}$$
$$\text{using}: \lambda\!\!\!^{-} = \frac{\hbar c}{pc} = \frac{197\ \text{MeV fm}}{1548\ \text{MeV}} = 0.127\ \text{fm}$$

and taking care of the area units, $\Gamma \approx 92.7$ keV .

This is much narrower than the established $J=1$ mesons. For example, $\Gamma(K^*) \approx 50$ MeV and $\Gamma(\rho) \approx 150$ MeV. This energy is far below the experimental resolution; hence the width cannot be measured directly and this calculation is necessary to determine the width. Such a narrow width ("long lifetime") means the J/ψ can not to be interpreted as some high-mass combination of u,d,s. Instead, the J/ψ heralded the discovery of the 4th quark, "charm", in a bound $c\bar{c}$ state.

Charmonium spectrum

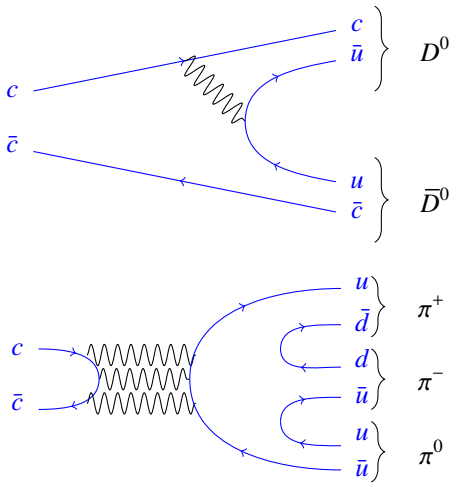
Soon after, more $J^{PC} = 1^{--}$ $c\bar{c}$ states was found:

- $\psi(2S)$ $\Gamma = 320$ keV. $\quad \mathcal{B}(\psi(3686) \to J/\psi\ \pi^+\ \pi^-) \approx 50\%$
- $\psi(3S)$ $\Gamma = 27.3$ MeV. $\quad \mathcal{B}(\psi(3770) \to D^0\ \bar{D}^0) \approx 85\%$

Decays of the J/ψ must be via the annihilation of the $c\bar{c}$. In the case of a strong-force, the transition must involve a minimum of three gluons because:

- a single gluon is not colourless so not colour-conserving. A colourless exchange is possible with two gluons (e.g. $R\bar{B} + \bar{R}B$) but this is forbidden for the following reason.
- a gluon is not a C eigenstate. It does not conserve the charge quantum number $C|J/\psi\rangle = -1$, $C|gg\rangle = +1$. A three gluon exchange is the minimum number of gluons that can propagate a colourless C-odd transition.

This implies the decays depends on α_s^6. The suppression of the strong decay of $q\bar{q}$ states is known as the "OZI-rule". Also seen with the ϕ $(s\bar{s})$. Running of the strong coupling constant means OZI-suppression is more significant than with the lighter mesons.



Charmonium decays are governed by kinematics: The J/ψ is below the kinematic threshold for production of the lowest-mass open-charm mesons (the D mesons),

$$m(J/\psi) \;<\; m(\psi') \;<\; 2m(D^0) \;<\; m(\psi'')$$

The $\psi(3770)$ falls apart rapidly (large width) to a $D^0\overline{D}^0$ pair is via a single gluon exchange. ($\propto \alpha_s^2$)

In considering the whole charmonium spectrum, a comparison to positronium (bound states of e^+e^-) can be made,

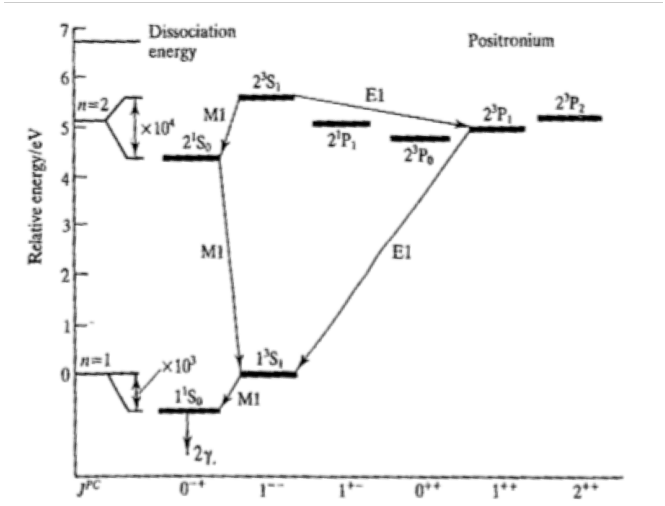
- Their energy levels are predicted by the non-relativistic Schrodinger equation in analogous fashion to and hydrogen atom but with reduced mass, $m_e/2$.
- Singlets $^1\mathrm{S}_0$ and $^3\mathrm{S}_1$ split by spin-spin interactions (aligned/not-aligned).
- Three P states have $L = 1$ and split by L⊕S alignment.

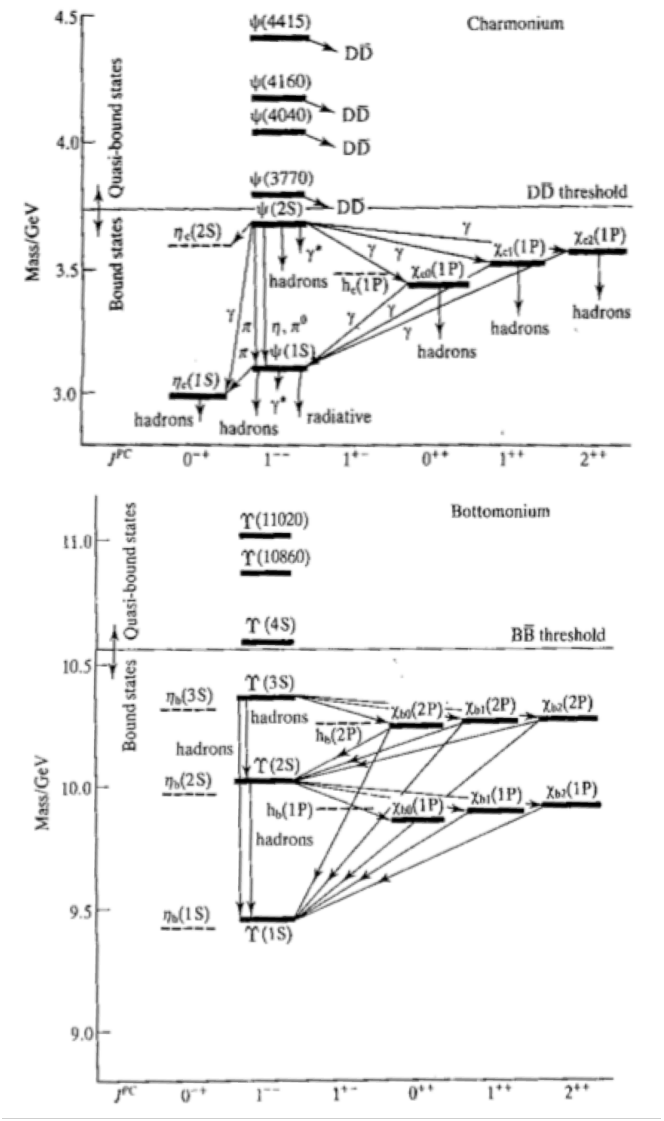
Inspecting the energy levels of positronium and charmonium shows striking similarity providing further evidence that the resonances above $m > 3\,\mathrm{GeV}/c^2$ are bound $q\bar{q}$ states. Differences to positronium are:

- strength of α_s leads to larger splitting from spin-spin and spin-orbit interactions
- difference in the relative energies comes from the fundamental difference in the potential. Unlike the the α_{EM}/r dependence in positronium, charmonium is proportional to $-\alpha/r$ but $\alpha_s/r + \kappa r$. (At short distances a coulomb-like potential, at large distances the confinement dominates)

In 1977, an excess of events were observed around 10 GeV. Using the e^+e^- collider at SLAC, three narrow resonances were observed. $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$. In a strong parallel to charmonium, a broader resonance, $\Upsilon(4S)$, also found to decay to $B^0\overline{B}^0$ pairs, each with a mass around 5.3 GeV/c^2.

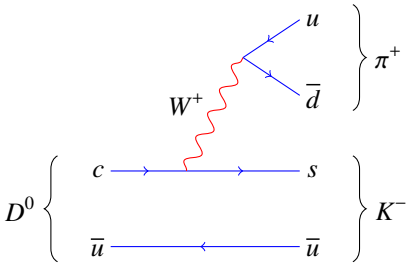
Until the early 80s, searches for $t\bar{t}$ resonances were conducted, but none were found. The top quark was later discovered at Fermilab with a mass close to 175 GeV. At such high mass, so much phase space is available to the quark that it decays before hadronising into $t\bar{t}$.





Open charm

Just as the ϕ is the $J^{PC} = 1^{--}$, vector $|s\bar{s}\rangle$ state and the kaons are "open-strange" states, $|s\bar{q}\rangle$ ($q \in \{u, d\}$), the D mesons are the "open-charm" equivalents to the J/ψ mesons. As the lowest-mass charm mesons, they decay by the weak force only, with a characteristically long lifetime before decaying my changing flavour, usually $c \to s$.



The higher mass of the charm quark and the same quark-generation coupling means charm lifetimes are shorter than kaon lifetimes, $\tau(K) \sim 1 - 500\,\mathrm{ns}$ $\tau(D) \sim 1\,\mathrm{ps}$. Similarly, a larger number of charmed hyperons are expected within quark model. Using Young's tableaux, the SU(3) baryon octet and decuplet become a 12-plet and 20-plet, respectively in SU(4) and several states, including the triplely-charmed Ω_{ccc}^{+++} are yet to be observed.

Relativistic Kinematics

Malcolm John
C4: particle physics

Fermi's Golden Rule calculates the transition rate of an initial state to specific final state under the effect of some perturbation. Its simple formalism presents an integral over the available phase-space $\rho(E_f)$, of the dynamics of the perturbation.

$$\Gamma_f = \hbar W_{fi} = 2\pi \int \underbrace{|\mathcal{T}_{fi}|^2}_{\text{dynamics}} \underbrace{d\rho(E_f)}_{\text{kinematics}} \tag{1}$$

Here, W_{fi} is the rate: the number of transitions per unit time;

- $\mathcal{T}_{fi}$ is the *transition matrix element* describing the dynamics (coupling strength, spin dependency, theory model...):

$$\mathcal{T}_{fi} = \langle \phi_f \mid P \mid \phi_i \rangle ,$$

- $d\rho(E_f)$ is the number of states available *per unit energy in the final state*: the phase-space factor.

In particle physics, transition rate is called *partial width* where the word "partial" refers to one of many possible transitions. The *total width* of an initial state is the sum of all its partial widths. As the total (or *natural*) width is the total rate (multiplied by $\hbar$) it is an expression of the lifetime of the initial state,

$$\Gamma_f = \hbar W_{fi} \quad \text{with} \quad \Gamma = \sum_f \Gamma_f \qquad \Gamma = \frac{\hbar}{\tau}, \tag{2}$$

In nuclear physics, τ might be anywhere from 10^{-5} to 10^{+12} seconds but particle physics deals with shorter time-scales. In the region of $\sim 10^{-18}$s for electromagnetic interactions, $\sim 10^{-23}$s for the strong nuclear force and up to 10^{-13}–10^{-8}s for weak transitions. For the shortest lifetimes, the uncertainty principle means there is a natural variation in the state's energy (mass),

$$\Delta E \Delta t \sim \hbar \quad \Rightarrow \quad \Delta M c^2 = \frac{\hbar}{\tau} = \quad \Rightarrow \quad \Delta M c^2 = \Gamma. \tag{3}$$

The *natural width*, Γ, is the width of a pure Breit-Wigner resonance curve. For the short-lived resonances ($\tau < 10^{-18}$s), it is appropriate to quote a width rather than a lifetime:
$\Gamma = 1.2\,\text{keV} \equiv \tau = 5.5 \times 10^{-19}\text{s} \equiv 1.8 \times 10^{18}$ transitions/second.

The fraction of events for which an initial state decays to final state f is the branching fraction. It is defined as the ratio of partial width to the total width,

$$\mathcal{B}(i \to f) = \frac{\Gamma_f}{\Gamma}. \tag{4}$$

It is important to measure both the total width (or equivalently, lifetime) and branching fraction when comparing to a theory of the fundamental interaction. For example, the K^0_S and K^0_L (which have the same quark content) have branching fractions to the semileptonic final state, $\pi^+ e^- \nu$,

$$\frac{\mathcal{B}(K^0_L \to \pi e \nu)}{\mathcal{B}(K^0_S \to \pi e \nu)} = \frac{4.06 \times 10^{-2}}{7.04 \times 10^{-4}} \approx 575$$

Naively, it would seems the fundamental force behind the transition to $\pi^+ e^- \nu$ is much stronger in K^0_L decays. However, one must also consider that the K^0_L lives for a longer time,

$$\frac{\Gamma(K^0_L)}{\Gamma(K^0_S)} = \frac{\tau(K^0_S)}{\tau(K^0_L)} = \frac{8.95 \times 10^{-11}}{5.12 \times 10^{-8}} \approx \frac{1}{575}.$$

The correct conclusion is that the semileptonic partial width is the same. The two kaons have almost-identical mass so the phase-space factor is the same, thus the fundamental force driving the transition is the same for the K^0_S and K^0_L. The difference in lifetime due to the fact that there are other decays, with higher transition probabilities, that are only available to the K^0_S.

1 Lorentz invariant phase space

In order to use branching fraction measurements and Eq. 1 to accurately assess the dynamics of a theoretical model, the phase space factor must readily calculable. Phase space is the number of states available per unit of energy in the final state. Imagine a cube of sides L containing one of the final-state particles with quantised momentum, $p = \hbar k$ (though take $\hbar = 1$) and $k = n\,2\pi/L$, $n \in \mathbb{Z}^+$.

$$\begin{aligned} & p_x = \frac{2\pi n_x}{L} \qquad p_y = \frac{2\pi n_y}{L} \qquad p_z = \frac{2\pi n_z}{L} \\ \text{or,} \quad & n_x = \frac{L p_x}{2\pi} \qquad n_y = \frac{L p_y}{2\pi} \qquad n_z = \frac{L p_z}{2\pi} \end{aligned} \tag{5}$$

Each (p_x, p_y, p_z) momentum state resides in the elemental volume $(2\pi/L)^3 = (2\pi)^3/V$ in "momentum space", so that:

$$\text{total momentum space in volume V} = \frac{(2\pi)^3}{V} \tag{6}$$

Now let's normalise to one particle per *spatial* volume V and rewrite "total momentum space" more mathematically as the integral over all momenta:

$$N_1 \quad = \quad \frac{1}{\frac{(2\pi)^3}{V}} \int dp_x\, dp_y\, dp_z \,\Big|\, V \quad = \quad \frac{1}{(2\pi)^3} \int d^3p$$

Scaling up to n particles, and because total momentum conservation in the transition will constrain the n^{th} particle, the number of available states is that of $n-1$ free particles:

$$N_{n-1} = \frac{1}{(2\pi)^{3(n-1)}} \int \prod_{j=1}^{n-1} d^3 p_j$$

From which the number of states per unit energy, is:

$$d\rho(E) = \frac{dN_{n-1}}{dE} = \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{j=1}^{n-1} d^3 p_j \tag{7}$$

Dirac δ-functions

The Dirac δ-function is function of a continuous variable x, whose value is zero everywhere for $x = 0$, where its value is one. The integral of a Dirac δ-function is also one.

$$\delta(x) = \begin{cases} 1 & x=0 \\ 0 & \text{elsewhere.} \end{cases} \qquad \int \delta(x)\, dx = 1.$$

In the case that the the δ-function applies to a function,

$$\delta\left[f(x)\right] = \left|\frac{df}{dx}\right|_a^{-1} \delta[x-a] \tag{8}$$

Then using δ-functions for total momentum ($\boldsymbol{P}$) conversation,

$$\begin{aligned} \sum_{j=1}^{n-1} \boldsymbol{p}_j \;+\; \boldsymbol{p}_n \;&=\; \boldsymbol{P} \\ \int d^3p_n\, \delta\left[\boldsymbol{P} - \sum_{j=1}^{n} \boldsymbol{p}_j\right] \;&=\; 1 \\ \text{similarly}: \int dE\, \delta\left[E - \sum_{j=1}^{n} E_j\right] \;&=\; 1 \end{aligned}$$

The density of states of n non-relativistic particles is,

$$\begin{aligned} d\rho(E) \;&=\; \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{j=1}^{n-1} d^3p_j \times \int d^3p_n \\ &\qquad \delta\left[\boldsymbol{P} - \sum_{j=1}^{n} \boldsymbol{p}_j\right] \times \int dE\, \delta\left[E - \sum_{j=1}^{n} E_j\right] \\ &=\; (2\pi)^3 \int \prod_{j=1}^{n} \frac{d^3p_j}{(2\pi)^3}\, \delta\left[\boldsymbol{P} - \sum_{j=1}^{n} \boldsymbol{p}_j\right] \delta\left[E - \sum_{j=1}^{n} E_j\right] \end{aligned} \tag{9}$$

Impose Lorentz invariance

If the n particles are moving relativistically a problem arises. The normalisation used in Eq. 6 is one particle in volume V.

$$\int |\, \phi \,|^2\, dV = 1$$

At relativistic speeds, Lorentz contraction in direction of motion will increase the particle density by γ. So a redefinition is needed such that a wavefunction ψ that has normalisation $\propto \sqrt{\gamma}$. The usual convention is to **2E** particles/unit-volume:

$$\begin{aligned} \int |\, \psi \,|^2\, dV &= 2E \\ \phi &= \frac{1}{\sqrt{2E}}\, \psi \end{aligned} \tag{10}$$

So the matrix element accommodating the Lorentz invariant wavefunction becomes, $\mathcal{M}_{fi} = \langle\, \psi_f \,|\, P \,|\, \psi_i \,\rangle$, so

$$|\, \mathcal{T}_{fi} \,|^2 = \prod_{i=1}^{m} \frac{1}{2E_i} \prod_{j=1}^{n} \frac{1}{2E_j} \,|\, \mathcal{M}_{fi} \,|^2, \tag{11}$$

for m particles in the initial state and n in the final state.

The Golden Rule in a Lorentz invariant form becomes,

$$\Gamma_{fi} = (2\pi)^4 \prod_{i=1}^{m} \frac{1}{2E_i} \int |\, \mathcal{M}_{fi} \,|^2 \underbrace{\prod_{j=1}^{n} \frac{d^3p_j}{(2\pi)^3(2E_j)}\, \delta\left[\vec{P} - \sum_{j=1}^{n} \vec{p}_j\right] \delta\left[E - \sum_{j=1}^{n} E_j\right]}_{\text{Lorentz invariant phase space}} \tag{12}$$

where the $1/(2E_i)$ terms are taken out the integration over the *final* state momenta. The transition rate is inversely proportional to energy of the initial state as expected for a high-energy decaying particle whose decay rate is slowed by time dilation.

Checking the invariance of Lorentz-invariant phase space

Choose a boost in z,

$$p_z' = \gamma(p_z - \beta E) \qquad E' = \gamma(E - \beta p_z)$$

Differentiate E with respect to p_z:

$$\frac{dE}{dp_z} = \frac{d}{dp_z}\left(\sum_{i=xyz} p_i^2 + m^2\right)^{\frac{1}{2}} = p_z\left(\sum_{i=xyz} p_i^2 + m^2\right)^{-\frac{1}{2}} = \frac{p_z}{E}$$

Differentiate p_z' with respect to p_z and substitute in the above.

$$\begin{aligned} \frac{dp_z'}{dp_z} \;&=\; \frac{d}{dp_z}\, \gamma(p_z - \beta E) \\ &=\; \frac{\gamma(E - \beta p_z)}{E} = \frac{E'}{E} \end{aligned}$$

Thus we confirm that the quotient dp/E is Lorentz invariant.

2 Two-body decay

As the final state integral is Lorentz invariant, we are free to choose any frame. The zero-momentum frame is convenient as this frame has the decaying particle at rest: $\vec{P} = 0\,,\ E_i = M$.

$$\begin{aligned} \Gamma_{fi} \;=\; \frac{(2\pi)^4}{2M} \int |\, \mathcal{M}_{fi} \,|^2\, \frac{d^3p_1}{(2\pi)^3(2E_1)} \frac{d^3p_2}{(2\pi)^3(2E_2)} \\ \times\, \delta\left[\boldsymbol{p}_1 + \boldsymbol{p}_2\right] \delta\left[M - E_1 - E_2\right] \end{aligned} \tag{13}$$

Use one of the δ functions to remove one integral, gather constants and switch to polar coordinates, $d^3p \mapsto\ p_1^2\, dp_1 d\Omega$,

$$\Gamma_{fi} = \frac{1}{32\pi^2 M} \int |\, \mathcal{M}_{fi} \,|^2\, \frac{p_1^2 dp_1 d\Omega}{E_1 E_2} \delta\left[M - E_1 - E_2\right]$$

Observe that $E_2^2 = m_2^2 + p_1^2$ because $p_2 = -p_1$,

$$\Gamma_{fi} = \frac{1}{32\pi^2 M} \int |\, \mathcal{M}_{fi} \,|^2\, g(p_1)\, \delta\left[f(p_1)\right]\, dp_1 d\Omega\,,$$

where $g(p_1) = p_1^2/(E_1 E_2)$

and $f(p_1) = M - \sqrt{(m_1^2 + p_1^2)} - \sqrt{(m_2^2 + p_1^2)}\,.$

Using eq. 8,

$$\delta\left[f(p_1)\right] = \left|\frac{df}{dp_1}\right|^{-1}_{p^*} \delta\left[p_1 - p^*\right] ,$$

where p^* denotes the value of p_1 that conserves momentum. Then the function $g(p_1)$ can be easily integrated,

$$\int g(p_1)\,\delta[f(p_1)]\,dp_1 = \left|\frac{df}{dp_1}\right|^{-1}_{p^*} \underbrace{\int g(p_1)\,\delta(p_1 - p^*)\,dp_1}_{g(p^*)}$$
$$= \left|\frac{df}{dp_1}\right|^{-1}_{p^*} \frac{(p^*)^2}{E_1 E_2}$$

The inverted modulus term is,

$$\frac{df}{dp_1} = -\frac{p_1}{\sqrt{m_1^2 + p_1^2}} - \frac{p_1}{\sqrt{m_2^2 + p_1^2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1 + E_2}{E_1 E_2}$$

$$\text{Thus} \quad \left|\frac{df}{dp_1}\right|^{-1}_{p^*} = \frac{1}{p^*}\frac{E_1 E_2}{E_1 + E_2}$$

Giving:

$$\Gamma_{fi} = \frac{1}{32\pi^2 M}\ \frac{E_1 E_2}{p^*(E_1 + E_2)}\ \frac{(p^*)^2}{E_1 E_2}\ \int |\,\mathcal{M}_{fi}\,|^2\,d\Omega$$

And as $E_1 + E_2 = M$ by energy conservation,

$$\boxed{\Gamma_{fi} = \frac{|p^*|}{32\pi^2 M^2}\int |\,\mathcal{M}_{fi}\,|^2\,d\Omega} \tag{14}$$

Finally p^* is obtained from energy conservation in the CoM,

$$\begin{aligned}
M &= E_1 + E_2 \\
M &= E_1 + \sqrt{m_2^2 + p^{*2}} \\
M &= E_1 + \sqrt{m_2^2 + E_1^2 - m_1^2} \\
(M - E_1)^2 &= E_1^2 + m_2^2 - m_1^2 \\
M^2 - 2ME_1 + E_1^2 &= E_1^2 + m_2^2 - m_1^2 \\
2ME_1 &= M^2 + m_1^2 - m_2^2 \\
4M^2 p^{*2} + 4M^2 m_1^2 &= (M^2 + m_1^2 - m_2^2)^2
\end{aligned}$$

Giving,

$$\boxed{\begin{aligned} p^* &= \frac{1}{2M}\sqrt{(M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2} \\ \text{or}\ \ p^* &= \frac{1}{2M}\sqrt{[M^2 - (m_1 + m_2)^2][|M^2 - (m_1 - m_2)^2]} \end{aligned}} \tag{15}$$

Example: $W^- \to e^-\bar{\nu}$. The matrix element is,

$$|\mathcal{M}_{W^-\to e^-\bar{\nu}}|^2 = \left|m_W\, g\, d^1_{m',m}\right|^2$$

where $m_W = 80.4\,\text{GeV}$ is the mass of the W boson and g is the weak coupling constant and $d^1_{m',m}$ is the rotation matrix for spin 1 boson. The final-state spin projection is $m' = -1$. There are three possible values for $m = -1, 0, 1$ but they give the same result and we average over the three, so just picking one: $d^1_{-1,-1} = \frac{1}{2}(1 - \cos\theta)$. Eq. 15 simplifies because the lepton masses are negligible compared to the $W^\pm$ mass,

$$\begin{aligned}
p^* &= \frac{1}{2m_W}\sqrt{[m_W^2 - (m_e + m_\nu)^2][m_W^2 - (m_e - m_\nu)^2]} \\
p^* &\simeq \frac{m_W}{2}
\end{aligned}$$

Using the main result from Eq. 14,

$$\begin{aligned}
\Gamma_{W^-\to e^-\bar{\nu}} &= \frac{m_W/2}{32\pi^2 m_W^2}\int \left|m_W g \tfrac{1}{2}(1 - \cos\theta)\right|^2 d\Omega \\
&= \frac{m_W g^2}{128\pi}\int \left|(1 - \cos\theta)^2\right| d(\cos\theta) \\
&= \frac{m_W g^2}{128\pi}\underbrace{\left[\tfrac{1}{3}(1 - \cos\theta)^3\right]^{-1}_{\cos\theta = 1}}_{8/3} \\
&= \frac{m_W g^2}{48\pi}
\end{aligned} \tag{16}$$

The total width of the W^- boson is measured to be $2.09 \pm 0.04\,\text{GeV}$ and its branching fraction to the $e^-\bar{\nu}_e = 10.7\%$ so the partial width $\Gamma_{W^-\to e^-\bar{\nu}} 0.224\,\text{GeV}$ hence the weak coupling constant is $g = 0.65$. Note that the weak force g is higher than electromagnetic coupling, $g_e = \sqrt{4\pi\alpha} = 0.31$ for $\alpha = \frac{1}{137}$.

3 Dalitz plot

The three-body decay rate is more complicated than the two-body case but an interesting observation can be made without performing the full calculation. First we consider an infinitesimal volume of the three-body phase space, $d\rho_3$ for a initial state particle of mass M. Stripping out the constant factors,

$$d\rho_3 \propto \frac{d^3p_1 d^3p_2 d^3p_3}{E_1 E_2 E_3}\ \delta\left(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3\right)\ \delta\left(E_1 + E_2 + E_3 - M\right)$$

As in the 2-body case, one of the integrals over d^3p_i is removed with the momentum δ-function.

$$d\rho_3 \propto \frac{d^3p_1 d^3p_2}{E_1 E_2 E_3}\,\delta\left(E_1 + E_2 + E_3 - M\right). \tag{17}$$

Then change to spherical co-ordinates,

$$\begin{aligned}
d^3p_1 d^3p_2 &= p_1^2 \sin\theta_1 dp_1 d\theta_1 d\phi_1\ p_2^2 \sin\theta_2 dp_2 d\theta_2 d\phi_2 \\
&= p_1^2 p_2^2 dp_1 dp_2 \sin\theta_1 d\theta_1 \sin\theta_{12} d\theta_{12} d\phi_1 d\phi_2 .
\end{aligned}$$

where we choose to define the polar angle of the second particle with respect to the first particle's angle: $d\Omega_2 = \sin\theta_{12} d\theta_{12}\phi_2$. And several terms, $\int d\theta_1 d\phi_1 d\phi_2 = 8\pi^2$ are absorbed into the constant of proportionality. Now consider the back-to-back systems, p_3 vs. $(p_1 + p_2)$:

$$p_3^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\theta_{12}$$

Given a $|\vec{p_1}|$ and $|\vec{p_2}|$, p_3 depends only on θ_{12}. Differentiating,

$$2p_3 dp_3 = 2p_1 p_2 \sin\theta_{12} d\theta_{12}$$
$$\text{thus} \quad \sin\theta_{12} d\theta_{12} \propto \frac{p_3}{p_1 p_2} dp_3$$

Putting this into Eq. 17,

$$d\rho_3 \propto \frac{1}{E_1E_2E_3}p_1^2p_2^2dp_1dp_2\frac{p_3}{p_1p_2}dp_3\;\delta(E_1+E_2+E_3-M)$$
$$\propto \frac{p_1p_2p_3}{E_1E_2E_3}dp_1dp_2dp_3\;\delta(E_1+E_2+E_3-M)$$

Perform a variable change by differentiating $E = \sqrt{p^2+m^2}$: $p_i dp_i = E_i dE_i$ and remove one term using the δ-function.

$$d\rho_3 \propto dE_1\,\cancel{dE_2}\,dE_3\;\cancel{\delta(E_1+E_2+E_3-M)} \tag{18}$$

We start by writing the Lorentz sum of m_1 and m_2 explicitly in terms of E_3 and p_3:

$$\begin{aligned}M_{12}^2 &= (E_1+E_2)^2-(p_1+p_2)^2\\ &= (M-E_3)^2-p_3^2\\ &= M^2-2ME_3+E_3^2-p_3^2\\ &= M^2+m_3^2-2ME_3\end{aligned} \tag{19}$$

And differentiating,

$$d\left(M_{12}^2\right) \propto dE_3$$
$$\text{And similarly,}\quad d\left(M_{23}^2\right) \propto dE_1$$

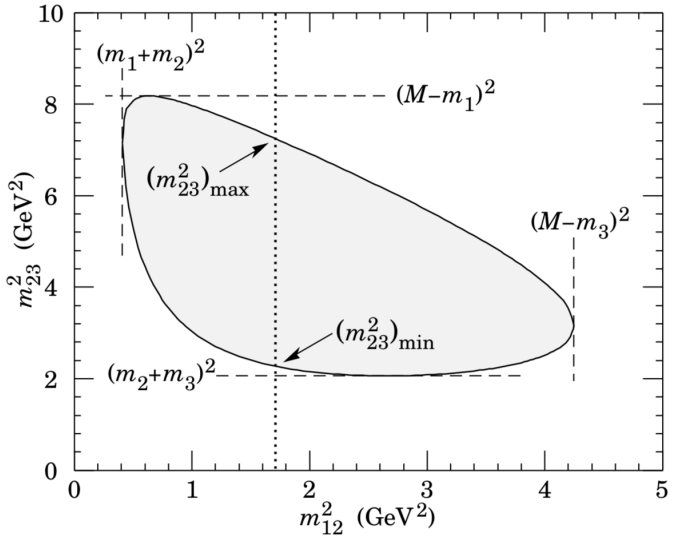
Finally plugging these relationships into Eq. 18 obtains,

$$\boxed{d\rho_3 \propto d\left(M_{12}^2\right)\,d\left(M_{13}^2\right)} \tag{20}$$

The important conclusion is that the final-state phase space in a three-body decay is proportional to the product of the invariant-mass-squared of decay products $(1+2)$ and $(3+3)$ (or $(2+3)$). This means:

- phase space is uniform across the 2D distribution of M_{12}^2 vs. M_{13}^2 (or vs. M_{23}^2).
- the “dynamics", like spin dependence in the matrix element show-up as structure on the 2D plane.

Such 2D distributions are called Dalitz plots after their inventor, Dick Dalitz. Below is an illustrative example of a Dalitz plot[1] for the three-body final state $M_{123} = \pi K^0 p$ for $M = 3$ GeV.



[1] Taken from the PDG Kinematics review

4 Rotations

Let's rotate the coordinate system about the y-axis by an infinitesimal rotation, ϵ such that point $\boldsymbol{a}$ becomes point $\overline{\boldsymbol{a}}$,

$$\overline{\boldsymbol{a}} = \rho_y(\epsilon)\,\boldsymbol{a}$$
$$\begin{pmatrix}\overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix} = \begin{pmatrix}\cos\epsilon & & -\sin\epsilon\\ & 1 & \\ \sin\epsilon & & \cos\epsilon\end{pmatrix}\begin{pmatrix}x_a\\ y_a\\ z_a\end{pmatrix}$$

Imposing rotational invariance, the wavefunction in the rotated coordinates is,

$$\overline{\psi}\begin{pmatrix}\overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix} = \psi\begin{pmatrix}x_a\cos\epsilon - z_a\sin\epsilon\\ y_a\\ z_a\cos\epsilon + x_a\sin\epsilon\end{pmatrix}$$

Then using the first terms of the [co]sine Taylor expansion,

$$\begin{aligned}\overline{\psi}(\overline{\boldsymbol{a}}) &= \psi(\boldsymbol{a}) - \epsilon z_a\frac{\partial\psi(\boldsymbol{a})}{\partial x} + \epsilon x_a\frac{\partial\psi(\boldsymbol{a})}{\partial z} + \ldots\\ &\simeq \left[1-\epsilon\left(z_a\frac{\partial}{\partial x} - x_a\frac{\partial}{\partial z}\right)\right]\psi(\boldsymbol{a})\\ &= \underbrace{\left[1-\frac{i}{\hbar}\epsilon J_y\right]}_{R(\epsilon)}\psi(\boldsymbol{a}) \quad \text{as: } J_y = \underbrace{-i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]}_{(\boldsymbol{r}\wedge\boldsymbol{p})_y}\end{aligned}$$

Successive infinitesimal rotation are applied to perform a macroscopic rotation, This is the property of a Lie group,

$$\begin{aligned}R(\beta) &= \lim_{n\to\infty}\left(1-\frac{i}{\hbar}\frac{\beta}{n}J_y\right)^n\\ &= \exp\left(-\frac{i}{\hbar}\beta J_y\right).\end{aligned} \tag{21}$$

Dropping the $\hbar$ that cancel, the general 3D definition is,

$$R(\boldsymbol{\theta}) = \exp\left(i\boldsymbol{J}\cdot\boldsymbol{\theta}\right). \tag{22}$$

$$\text{with,}\qquad \boldsymbol{J} = -i\frac{\partial R(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\bigg|_{\theta=0} \tag{23}$$

In matrix form, the generator $J_i = -i\,\varepsilon_{ijk}\,x_j\partial_k$ is,

$$J_x = -i\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\quad J_y = -i\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\quad J_z = -i\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} \tag{24}$$

which, as expected, obey the quantum mechanical commutation relation for angular momentum,

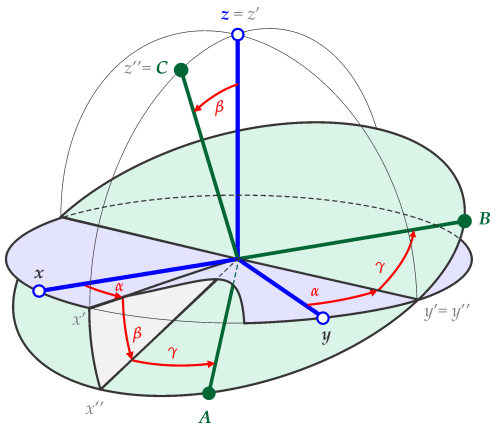
$$[J_i, J_j] = i\,\varepsilon_{ijk}J_k\,. \tag{25}$$

Any combination of successive rotations can be reduced to a combination of the three J_x, J_y, J_z generators that form a closed, complete symmetric group of special ($\det R = 1$) orthogonal ($R^T = R^{-1}$), SO(3) transformations of 3 dimensions. Orthogonality is the real vector space analogue of unitarity.

The $d^j_{m',m}$ matrix

Generic rotation can be parameterised by three Euler angles defining three successive rotations. Most usefully :

1. rotation by angle α about the *original* z-axis.
2. rotation by angle β about the *intermediate* y-axis.
3. rotation by angle γ about the *final* z-axis.



For a wavefunction with angular momentum j being rotated such that its z-projection changes from m to m', the probability could be written,

$$\begin{aligned} P^{j}_{m',m}(\alpha\beta\gamma) &= \left| \langle j,m' \mid e^{-\frac{i}{\hbar}\gamma J_z} e^{-\frac{i}{\hbar}\beta J_y} e^{-\frac{i}{\hbar}\alpha J_z} \mid j,m\rangle \right|^2 \\ &= \left| \; e^{-\frac{i}{\hbar}(\gamma m' + \alpha m)} \; \langle j,m' \mid e^{-\frac{i}{\hbar}\beta J_y} \mid j,m\rangle \right|^2 \\ &= \left| \, d^{j}_{m',m}(\beta) \right|^2 , \end{aligned}$$

where,

$$d^{j}_{m'm}(\beta) = \langle j,m' \mid e^{-\frac{i}{\hbar}\beta J_y} \mid j,m\rangle \tag{26}$$

The change of the z-projection ($m \to m'$) only occurs during the rotation around $\hat{y}$. The two z-rotations lead just to a phase shift that is unobservable in the modulus-square. The state $\mid j,m\rangle$ rotates into a linear combination on the $(2j+1)$ $\mid j,m'\rangle$ states.

We calculate the $d^{j}_{m',m}$ matrix the specific case of $j=1$ and apply it to the realistic case of $e^+e^- \to \mu^+\mu^-$. There are 3 z-projections $(-1,0,1)$ so there are 3×3 matrix elements in this case.

1. From inspecting the expansion of $e^{-i\beta J_y}$ (with $\hbar = 1$):

$$e^{-i\beta J_y} = 1 - i\beta J_y - \frac{1}{2!}\beta^2 J_y^2 + \frac{i}{3!}\beta^3 J_y^3 + \frac{1}{4!}\beta^4 J_y^4 + \ldots$$

2. Look separately for solutions of $J_y^{2n+1} \mid j,m\rangle$ and $J_y^{2n} \mid j,m\rangle$
3. From the raising/lowering operators $J_\pm = J_x \pm iJ_y$ and Clebsch-Gordan coefficient, $C_-(1,1) = \sqrt{2}$.

$$\begin{aligned} J_+ - J_- &= J_x + iJ_y - J_x + iJ_y \\ J_y &= -\frac{i}{2}[J_+ - J_-] \\ \text{so, } J_y \mid 1,1\rangle &= -\frac{i}{2}\left[\cancel{C_+(1,1)\cdot 0} - C_-(1,1) \mid 1,0\rangle\right] \\ &= \frac{i}{\sqrt{2}} \mid 1,0\rangle \end{aligned}$$

4. Operate again with J_y; $C_+(1,0) = C_-(1,0) = \sqrt{2}$,

$$\begin{aligned} J_y^2 \mid 1,1\rangle &= \frac{i}{\sqrt{2}} J_y \mid 1,0\rangle \\ &= \frac{1}{2\sqrt{2}}(C_+(1,0) \mid 1,1\rangle - C_-(1,0) \mid 1,-1\rangle) \\ &= \frac{1}{2}(\mid 1,1\rangle - \mid 1,-1\rangle) \end{aligned}$$

5. And again:

$$\begin{aligned} J_y^3 \mid 1,1\rangle &= \frac{1}{2} J_y(\mid 1,1\rangle - \mid 1,-1\rangle) \\ &= \frac{1}{2}\left(\frac{-i}{2}[J_+ - J_-] \mid 1,1\rangle - \right. \\ &\qquad \left. \frac{-i}{2}[J_+ - J_-] \mid 1,-1\rangle\right) \\ &= \frac{i}{4}(\sqrt{2} \mid 1,0\rangle + \sqrt{2} \mid 1,0\rangle) \\ &= \frac{i}{\sqrt{2}} \mid 1,0\rangle \\ &= J_y \mid 1,1\rangle \end{aligned}$$

6. Note the cyclical pattern and conclude:

$$\begin{aligned} J_y^{2n+1} \mid 1,1\rangle &= \frac{i}{\sqrt{2}} \mid 1,0\rangle \\ J_y^{2n} \mid 1,1\rangle &= \frac{1}{2}(\mid 1,1\rangle - \mid 1,-1\rangle) \end{aligned}$$

thus

$$\begin{aligned} \langle 1,1 \mid J_y^{2n+1} \mid 1,1\rangle &= 0 \\ \langle 1,-1 \mid J_y^{2n+1} \mid 1,1\rangle &= 0 \\ \langle 1,0 \mid J_y^{2n+1} \mid 1,1\rangle &= \tfrac{i}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \langle 1,1 \mid J_y^{2n} \mid 1,1\rangle &= \tfrac{1}{2} \\ \langle 1,-1 \mid J_y^{2n} \mid 1,1\rangle &= -\tfrac{1}{2} \\ \langle 1,0 \mid J_y^{2n} \mid 1,1\rangle &= 0 \end{aligned}$$

7. Using these relations the $d^{j}_{m'm}$ elements are,

$$\begin{aligned} d^1_{11} &= \langle 1,1 \mid e^{-i\beta J_y} \mid 1,1\rangle \\ &= \langle 1,1 \mid 1 + \cancel{(-i\beta J_y)} + \frac{1}{2!}(-i\beta J_y)^2 + \\ &\qquad \cancel{\frac{i}{3!}(-i\beta J_y)^3} + \frac{1}{4!}(-i\beta J_y)^4 + \ldots \mid 1,1\rangle \\ &= 1 - \frac{1}{2!}\frac{\beta^2}{2} + \frac{i}{4!}\frac{\beta^4}{2} - \ldots\ldots \\ &= \frac{1}{2}(1 + (1 - \frac{1}{2!}\beta^2 + \frac{1}{4!}\beta^4 - \ldots)) \\ &= \frac{1}{2}(1 + \cos\beta) \end{aligned}$$

$$\begin{aligned} d^1_{-11} &= \langle 1,-1 \mid e^{-i\beta J_y} \mid 1,1\rangle \\ &= \frac{1}{2}(1 - (1 - \frac{1}{2!}\beta^2 + \frac{1}{4!}\beta^4 - \ldots)) \\ &= \frac{1}{2}(1 - \cos\beta) \end{aligned}$$

8. Elements are related by,

$$d^1_{m',m} = (-1)^{m-m'} d^1_{m,m'} = d^1_{-m,-m'} .$$

The complete $d^1_{m',m}$ matrix is,

$$\begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix} . \tag{27}$$

5 Cross section

Cross section relates to the likelihood of an interaction. The etymology stems from the early days of nuclear scattering where the nuclei presents a finite area to the incoming electron probe. Nowadays the idea of a physical area is just an analogy (think point particles) though cross section retains dimensions $[\mathrm{L}^2]$.

Imagine a uniform-density (ρ), stream (cross-sectional area A) of particles flowing towards a barrier at speed v. The number of particle per unit time passing the barrier is $\rho A v$ and the flux – the the number of particles per unit time per unit area – is ρv. Defining the stream to be one particle per unit volume, the flux equals the speed v. The interaction rate with the barrier, W_{fi} $[\mathrm{T}^{-1}]$, will be the flux, $v[\mathrm{L}^{-2}\mathrm{T}^{-1}]$ multiplied by a quantity with the units $[\mathrm{L}^2]$: the cross section.

$$W_{fi} = \nu\,\sigma$$

The unit of cross section is barn, $1\mathrm{b} = 10^{-28}\,\mathrm{m}^2$. In modern collider physics, processes are measured in micro- (μb), nano- (nb) or even picobarn (pb), depending on their rarity.

Consider the usual case: $1+2 \to 3+4$ scattering. The two particles (1 and 2) approach each other from opposite direction so the flux is $v_1 + v_2$ (v_2 could be zero). Using the two-body decay formula, Eq. 13, the cross section is written:

$$\sigma = (2\pi)^4 \frac{1}{v_1+v_2}\frac{1}{2E_1 2E_2}\int |\mathcal{M}_{fi}|^2 \frac{d^3p_3}{(2\pi)^3(2E_3)}\times \frac{d^3p_4}{(2\pi)^3(2E_4)}\delta[\boldsymbol{p}_3+\boldsymbol{p}_4]\,\delta[E-E_3-E_4] \tag{28}$$

The result for the two body decay can be recycled except the m_i^2 term must be reconsidered.

- one factor of m_i had equated to the initial state mass, M. Here, this returns to E_1 and is joined by a factor $2E_2$.
- second factor of m_i was the sum over final state energies. Here, this is the total centre of mass energy, $\sqrt{s}$.

Hence the cross section is written,

$$\boxed{\sigma = \frac{1}{2\mathcal{F}}\cdot\frac{|p_f^*|}{32\pi^2\sqrt{s}}\int |\mathcal{M}_{fi}|^2\, d\Omega^*} \tag{29}$$

where Ω^* is the solid angle in the centre of mass frame. and $\mathcal{F}$ is the Lorentz invariant flux term,

$$\mathcal{F} = (v_1+v_2)E_1E_2 \tag{30}$$

Initially the use of $v_1 + v_2$ as the relative speed of two particles was questionable but it is vindicated in the following.

Demonstrating the Lorentz invariance of the flux term

Consider the square of the fourmomentum scalar product of the two initial-state particles travelling towards each other,

$$(p_1\cdot p_2)^2 = (E_1E_2 - -|p_1||p_2|)^2$$
$$= E_1^2E_2^2 + |p_1|^2|p_2|^2 + 2E_1E_2|p_1||p_2|$$

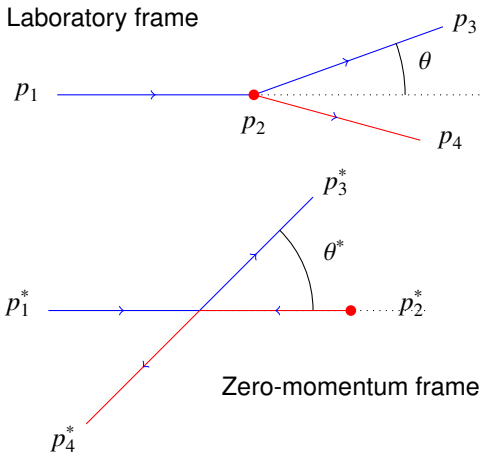
The scalar product is Lorentz invariant by definition. Next we introduce another term, the product of the two invariant masses, also manifestly Lorentz invariant.

$$\begin{aligned}(p_1\cdot p_2)^2 - m_1^2m_2^2 &= E_1^2E_2^2 + |p_1|^2|p_2|^2 + 2E_1E_2|p_1||p_2| \\ &\quad - (E_1^2-|p_1|^2)(E_2^2-|p_2|^2) \\ &= \cancel{E_1^2E_2^2} + \cancel{|p_1|^2|p_2|^2} + 2E_1E_2|p_1||p_2| \\ &\quad - \cancel{E_1^2E_2^2} + E_1^2|p_2|^2 + E_2^2|p_1|^2 - \cancel{|p_1|^2|p_2|^2} \\ &= (E_2|p_1| + E_1|p_2|)^2 \\ &= E_1^2E_2^2\left(\frac{|p_1|}{E_1}+\frac{|p_2|}{E_2}\right)^2 \\ &= [E_1E_2(v_1+v_2)]^2 = \mathcal{F}^2 \qquad \left(\tfrac{pc}{E}=\beta\right)\end{aligned}$$

which is the square of $\mathcal{F}$ in Eq. 30.

Cross section with a fixed target

Our understanding of the atomic nucleus came largely via scattering of a light probe particle (electron) off fixed targets made of protons and neutrons as imagine in these sketches.



Laboratory frame	Zero-momentum frame
$p_1 = (E_1, 0, 0, E_1)$	$p_1^* = (E^*, 0, 0, E^*)$
$p_2 = (M, 0, 0, 0)$	$p_2^* = (E^*, 0, 0, -E^*)$
$p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)$	$p_3^* = (E^*, E^*\sin\theta^*, 0, E^*\cos\theta^*)$

where p_4 is constrained by total fourmomentum conservation and $p_1^2 = p_3^2 = m^2 \approx 0$ in the relativistic case. To relate the two frames, Lorentz-invariant *Mandelstam variables* are used,

$$\begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_1-p_3)^2 \\ u &= (p_1-p_4)^2. \end{aligned} \tag{31}$$

Most relevant here is $t = p_1^2 + p_3^2 - 2p_1\cdot p_3$ which is,

$$\begin{aligned} \text{Lab:}\quad t &= \cancel{2m^2} - 2E_1E_3 + 2|p_1||p_3|\cos\theta \\ &= -2E_1E_3 + 2E_1E_3\cos\theta \\ \text{ZMF:}\quad t &= \cancel{2m^2} - 2E_1^*E_3^* + 2|p_i^*||p_f^*|\cos\theta^* \\ \text{from which,}\quad dt &= 2\,|p_i^*|\,|p_f^*|\,d(\cos\theta^*) \end{aligned} \tag{32}$$

because the magnitude of momentum and energy in the ZMF depend only on the masses, not the scattering angle. Realising that the azimuthal integral will the same in both frames: $\int d\phi = \int d\phi^* = 2\pi$, we have,

$$d\Omega^* = d(\cos\theta^*)d\phi^* = \frac{dt d\phi}{2\,|p_1^*|\,|p_f^*|}$$

Developing from Eq. 29,

$$\sigma = \frac{1}{2\mathcal{F}}\frac{1}{32\pi^2\sqrt{s}}\frac{\cancel{|p_f^*|}}{2\,|p_1^*|\,\cancel{|p_f^*|}}\int |\mathcal{M}_{fi}|^2\,dt\,d\phi \tag{33}$$

where p_i^* is calculable from Eq. 15 given the masses and centre of mass energy, $\sqrt{s}$,

$$|p_i^*| = \frac{1}{2\sqrt{s}}\sqrt{[s-(\cancel{m}+M)^2][s-(\cancel{m}-M)^2]}$$
$$2\sqrt{s}|p_i^*| = s - M^2\,. \tag{34}$$

and the invariant, s, can be calculated in the lab frame,

$$\begin{aligned} s &= (E_1+M)^2 - |p_1|^2 \\ &= \cancel{E_1^2} + M^2 + 2ME_1 - \cancel{E_1^2} \text{ as } |p_1| = E_1,\ m\to 0 \\ s - M^2 &= 2ME_1\,. \end{aligned} \tag{35}$$

The flux term, $\mathcal{F}$, is most simply calculated in the Lab frame: $\nu_2 = 0$, $E_2 = M$, $\nu_1 = \beta = |p_1|/E_1$:

$$\begin{aligned} (\nu_1+\nu_2)E_1E_2 &= \frac{|p_1|}{E_1}E_1M \\ &= M|p_1| \;\rightarrow\; ME_1\,,\ m\to 0\,. \end{aligned} \tag{36}$$

Using the identities in Eqs. 34, 35 and 36, Eq. 33 becomes an expression evaluated in the lab frame,

$$\sigma = \frac{1}{128\pi^2(ME_1)^2}\int |\mathcal{M}_{fi}|^2\,dt\,d\phi \tag{37}$$

The last step is to convert $dt\,d\phi \to d(\cos\theta)d\phi = \Omega$. By energy and momentum conservation, $p_1 + p_2 = p_3 + p_4$, so invariant t may be written in two ways:

$$\begin{aligned} t &= (p_1-p_3)^2 &&= (p_4-p_2)^2 \\ &&&= 2M^2 - 2ME_4 \\ &&&= 2M^2 - 2M(E_1+M-E_3) \\ t &&&= 2M(E_3-E_1) \qquad (38) \\ dt &&&= 2M\,dE_3 \qquad (39) \end{aligned}$$

Equating the definitions of t from Eq. 38 and Eq. 32,

$$\begin{aligned} 2M(E_3-E_1) &= -2E_1E_3 + 2E_1E_3\cos\theta \\ E_3 &= \frac{ME_1}{M+E_1(1-\cos\theta)} \\ \frac{dt}{d(\cos\theta)} = 2M\frac{dE_3}{d(\cos\theta)} &= \frac{2M^2E_1^2}{[M+E_1(1-\cos\theta)]^2} \end{aligned}$$

Thus substituting this result into Eq. 37,

$$\boxed{\sigma_{\text{fixed target}} = \frac{1}{64\pi^2}\int |\mathcal{M}_{fi}|^2\frac{1}{[M+E_1(1-\cos\theta)]^2}\,d\Omega} \tag{40}$$

This function is strongly peaked towards low scattering angle in the laboratory ($\cos\theta \to 1$), as seems appropriate for a high energy probe and a stationary target.

Cross section in a collider

In the typical case, colliding beams have equal and opposite momentum, $\boldsymbol{p}_1 = -\boldsymbol{p}_2$ so the centre of mass frame is coincident with the lab frame, $|p_1| = |p_2| = |p_i^*|$ and $|p_3| = |p_4| = |p_f^*|$. The flux term becomes,

$$\begin{aligned} ,\mathcal{F} \;=\; (\nu_1+\nu_2)E_1E_2 &= \left(\frac{|p_1|}{E_1}+\frac{|p_2|}{E_2}\right)E_1E_2 \\ &= |p_i^*|E_2 + |p_i^*|E_1 \\ &= |p_i^*|\sqrt{s} \end{aligned} \tag{41}$$

And due to the zero-momentum frame construction of Eq. 29, it immediately follows that cross section at a collider is:

$$\boxed{\sigma_{\text{ZMF collider}} = \frac{1}{64\pi^2 s}\frac{|p_f^*|}{|p_i^*|}\int |\,\mathcal{M}_{fi}\,|^2\,d\Omega^*\,.} \tag{42}$$

For a given beam energy, colliders reach much higher centre of mass (CoM) energy compared to a fixed target machine,

$$s_{fixed} = (E_p+m_p)^2 - |\vec{p}|^2 = 2m_p^2 + 2m_pE_p$$

$$s_{CM} = 4E_p^2$$

$$\begin{aligned} \frac{\sqrt{s}_{CM}}{\sqrt{s}_{fixed}} &\approx \sqrt{\frac{2E_p}{m_p}} \quad \text{where } m_p << E_p \\ &\approx 120 \quad \text{for 7 TeV protons (LHC)} \end{aligned}$$

which is a why colliders are preferred over fixed-target experiments for searches at the energy frontier.

Hadron colliders

Hadronic colliders benefit from a high energy reach and high production cross sections. The disadvantages of a hadron environment are large backgrounds and an unknown initial state. Though the proton energy is precisely known, it is impossible to know what momenta any given quark has at the moment of collision. Nevertheless, the final states of interest are copiously produced and are studied in detail. The event rate for a process is calculated from the cross section:

$$W_{X\to Y} = \sigma_{X\to Y}\times\mathcal{L} \tag{43}$$

where $\mathcal{L}$ is the luminosity (flux factor) in units of $\text{cm}^{-2}\text{s}^{-1}$. Total event yields are estimated from the integrated luminosity,

$$N_{X\to Y} = \mathcal{L}_{\text{int}}\,\varepsilon\,\sigma_{X\to Y} \quad \text{where} \quad \mathcal{L}_{\text{int}} = \int \mathcal{L}dt\,, \tag{44}$$

where ε is the efficiency to reconstruct and select the decay of interest. Very roughly, this is $O(1\%)$; the vast majority of the decays are not reconstructed at the LHC, a large number are removed in the background rejection. Typical yields for analysis are estimated in the following table,

Process	$\sigma(\sqrt{s} = 8\,\text{TeV})$	$\mathcal{L}_{int}$ by 2019	ε	N_{events}
$c\bar{c}$	6 mb	9 fb^{-1} (LHCb)	1%	54 billion
$b\bar{b}$	300 μb	9 fb^{-1} (LHCb)	1%	2.7 billion
W^+W^-	100 nb	140 fb^{-1} (ATLAS)	1%	140 million
$H_{125\ \text{GeV}/c^2}$	10 pb	140 fb^{-1} (ATLAS)	1%	14,000

$e^+ \rightarrow\leftarrow e^-$

Electron-positron colliders provide collisions with a precise, tuneable energy but at the cost of a much reduced cross section (0.1 nb instead of < 100 mb). They are much-used in the precise study of the electromagnetic, and later weak interaction.

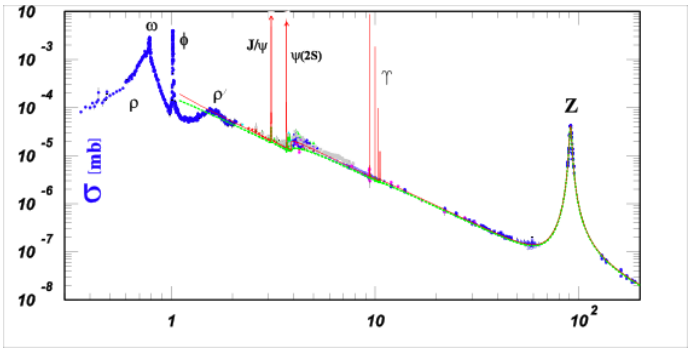
To determine the probability of rotating the photon of an electromagnetic interaction the Wigner rotation matrix is used. The $J = 1$ system, assuming it is initially prepared in a $|\ 1, +1\rangle$ state, may be found in a combination of $|\ 1, +1\rangle$ and $|\ 1, -1\rangle$ states, according to an angular dependance given by the Eq. 27. This is relevant to relativistic $e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-$ collisions.

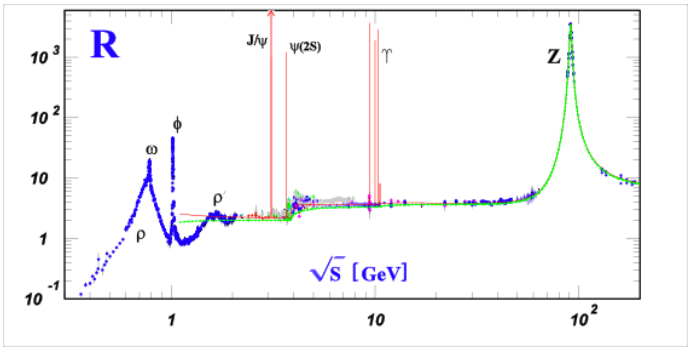
$$\begin{aligned} P_{e^+e^-\rightarrow\mu^+\mu^-} &\propto \left|d^1_{1,1}\right|^2 + \left|d^1_{-1,1}\right|^2 \\ &\propto \left|\tfrac{1}{2}(1+\cos\theta)\right|^2 + \left|\tfrac{1}{2}(1-\cos\theta)\right|^2 \\ &\propto 1 + \cos^2\theta\,, \end{aligned} \tag{45}$$

where θ is the scattering angle in the zero-momentum frame. The constant of proportionality is the electromagnetic charge to the fourth power, e^4 where $e = \sqrt{4\pi\alpha(\varepsilon_0\hbar c)_{=1}}$. Concentrating on the usual case where the momenta of the leptons is much higher than their masses, (Eq. 42) reduces to:

$$\begin{aligned} \sigma(e^+e^- \rightarrow \mu^+\mu^-) &= \frac{1}{64\pi^2 s}\int (4\pi\alpha)^2(1+\cos^2\theta)\,d\Omega \\ &= \frac{\alpha^2}{4s}\underbrace{\int (1+\cos^2\theta)\,d\Omega}_{2\pi\left[\cos\theta + \frac{\cos^3\theta}{3}\right]_{\theta=\pi}^{0}} \\ &= \frac{4\pi\alpha^2}{3s}\,(\varepsilon_0\hbar c)^2 \qquad 8/3 \end{aligned}$$

This simple expression has only dependence on the CoM energy-squared. This $1/s$ dependance is evident in the plot below which collates data from thirty years of e^+e^- experiments.





It should be remembered that this applies to the simplest electromagnetic process: the exchange of a virtual photon between two pairs of leptons. In addition to the basic virtual photon exchange, hadronic resonance appear as spikes and a wide Z^0 Bright Wigner around 90 GeV/c^2 is clear.

As well as muons $q\bar{q}$ pairs can be produced in e^+e^- annihilation. Their cross section derivation is similar except quarks have fractional charge and possess colour charge,

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = \sum_{\text{quarks}} \frac{4\pi}{3s}(\alpha \times q_i)^2 \times n_{col}$$

where n_{col} is the number of possible colour-charges and the sum runs over all the energetically-allowed quarks. The physics in the matrix element is performed by dividing-out the phase space and constant factors,

$$R = \frac{\sigma(\text{e}^+\text{e}^- \rightarrow \text{hadrons})}{\sigma(\text{e}^+\text{e}^- \rightarrow \mu^+\mu^-)} \quad = \quad n_{col}\sum_i q_i^2$$

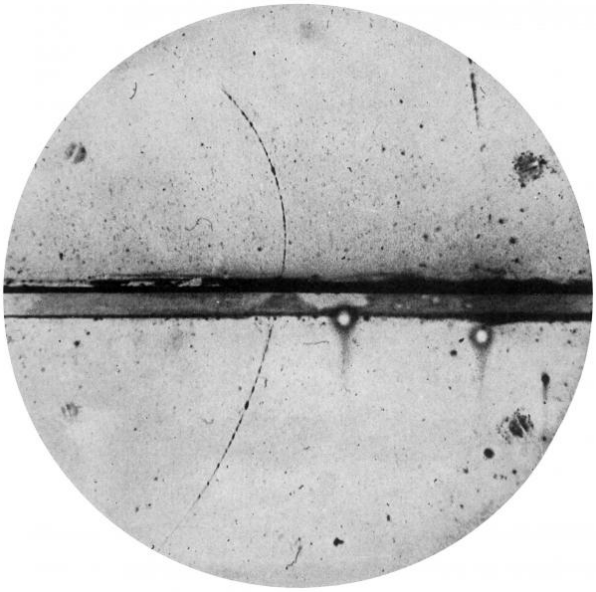
The study of R as a function of energy was an important validation of the quark model. Up to the ψ resonances, only three quark species can be produced so,

$$R_{uds} = (\tfrac{2}{3}^2 + \tfrac{1}{3}^2 + \tfrac{1}{3}^2)\,n_{col} = \frac{6}{9}\,n_{col}$$

Above this threshold, and again above the Υ resonances,

$$R_{udsc} = \frac{10}{9}\,n_{col} \qquad R_{udscb} = \frac{11}{9}\,n_{col}$$

The data is consistent with $n_{col} = 3$.



The Oxford MPhys Course

C4: Particle Physics

Problem Set Zero

Answers provided for self-checking on Sunday, beginning of Wk 1

1.

In an inertial frame, a particle has a momentum $\mathbf{p}$ and total energy E. In a second inertial frame the particle has momentum $\mathbf{p'}$ and total energy E'.

(a) Assuming only the formulae for the Lorentz transformations of momentum and energy[1], show that:

$$(E')^2 - (\mathbf{p}')^2 = E^2 - \mathbf{p}^2.$$

(b) Pions can decay via the reaction $\pi^+ \to \mu^+ \nu_\mu$. Show that the energy of the neutrino in the rest frame of the pion is given by:

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}.$$

[1] We will use natural units $\hbar = c = 1$ in this course.

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(c) Pions with energy E_π in the laboratory frame ($E_\pi \gg m_\pi$) decay via the above reaction. Show that the maximum energy of the neutrinos in the laboratory frame, E_ν^{max}, is given by:

$$E_\nu^{max} = E_\pi \frac{m_\pi^2 - m_\mu^2}{m_\pi^2}.$$

(d) These neutrinos collide with stationary protons in a hydrogen target. Find the minimum value of E_π required to produce D^0 mesons via the following reaction:

$$\nu_\mu + p \to \nu_\mu + p + D^0 + \bar{D}^0,$$

using $m_\mu = 106$ MeV, $m_\pi = 140$ MeV, $m_p = 938$ MeV, $m_{D^0} = 1865$ MeV.

2.

This question discusses the basic ideas of the quark model.

(a) How does the quark model account for the spins and parities of the pseudoscalar ($J^P = 0^-$) mesons? In particular, how does this model account for the observed multiplet of such mesons composed of u, d and s quarks and their anti-quarks?

(b) There are three distinct neutral vector mesons, ρ^0, ω, ϕ. The ρ^0 and the ω decay to pions while the ϕ seldom decays to pions. The ρ^0 is almost degenerate in mass with positive and negative charged partners. The ω is a meson with no such charged partners. Deduce the quark composition of these mesons. [Masses: ρ^0, $770\,\mathrm{MeV}/c^2$; ω, $782\,\mathrm{MeV}/c^2$; ϕ, $1019\,\mathrm{MeV}/c^2$.]

(c) How is it possible to have mesons with spin-parity $J^P = 1^+$ in the quark model?

(d) Order the lifetimes of the ρ^0, K^0 and π^0 mesons, explaining your reasoning.

3.

The next two questions involve the Breit-Wigner resonance formula. The formula represents the energy dependence of the total cross section $\sigma(i \to f)$ for unpolarised scattering between a two-body initial state i to a final state f, in the vicinity of a resonance of rest-mass energy M, spin J and total width Γ.

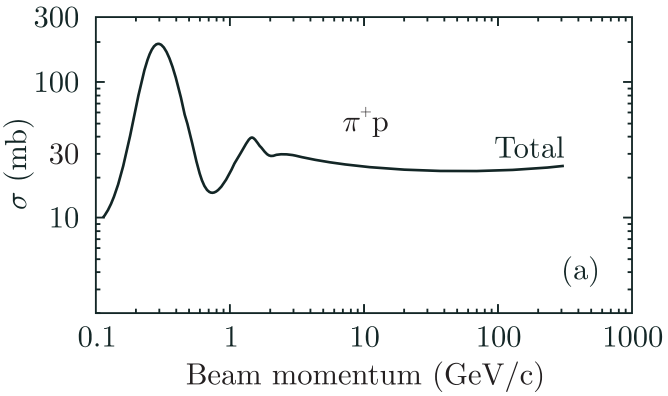
$$\sigma(i \to f) = \pi \lambda\!\!\!^{-2} g \frac{\Gamma_i \Gamma_f}{[(E-M)^2 + \Gamma^2/4]},$$

where $\lambda\!\!\!^{-} = \dfrac{\hbar c}{pc}$, $g = \dfrac{2J+1}{(2s_a+1)(2s_b+1)}$, p is the magnitude of the centre-of-mass momentum of the initial state particles, s_a, s_b are their spins and Γ_i, Γ_f the initial and final state partial widths.

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(a) Using the quark model for hadrons, give the spins, parities and quark compositions of the lowest lying baryon states, i.e. the states in which the quarks have no orbital angular momentum.

(b) The figure shows the π^+p total scattering cross section as a function of the incident pion beam momentum, p_π, in the laboratory frame in which the proton target is at rest.



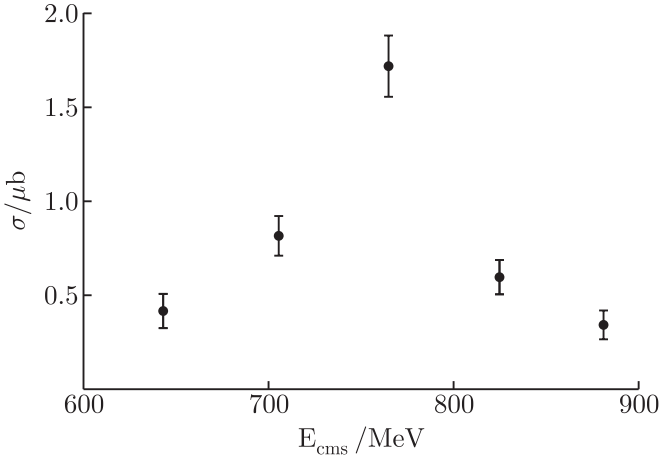
There is a prominent resonance at a beam momentum of about 300 MeV/c. Why is it reasonable to assume that the scattering at this energy is elastic? Hint: additional pion production is negligible and therefore the reaction must be elastic and therefore the initial and final states are the same.

(c) Show that at the maximum of a resonance in an elastic reaction, $\sigma_{\max} = 4\pi g \lambda^2$.

(d) Using the data shown in the figure determine the mass and spin of the π^+p resonance. What is the most likely quark composition of this resonance?

4.

The figure shows the cross section σ for the production of pion pairs as a function of centre of mass energy E_{cms} in e^+e^- annihilation.



(a) Use this figure to estimate the mass and width of the ρ^0 meson resonance. Estimate the lifetime of the ρ^0.

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(b) Consider the following decay modes of the ρ^0: (a) $\rho^0 \to \pi^0\pi^0$; (b) $\rho^0 \to \pi^+\pi^-$; (c) $\rho^0 \to e^+e^-$. Explain why mode (a) is forbidden. Using the data shown in the figure, estimate the branching ratio for mode (c).

(c) An e^+e^- collider has a centre of mass energy of 800 MeV and a luminosity of $10^{35}\,m^{-2}\,s^{-1}$. The efficiency of the detector for charged or neutral particles is 0·95. Estimate the rate of detection of ρ^0 events, explaining your assumptions.

(d) How could the decay products of the ρ^0 be detected and their energy or momenta be measured?

5.

A quantum system has Hamiltonian $H = H_0 + V$ where H_0 is a time-independent energy operator and V is a perturbing interaction.

(a) If the state of the system at time t is written in the form,

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(t)|u_n\rangle e^{-i\varepsilon_n t/\hbar},$$

where the $|u_n\rangle$ are orthonormalised eigenstates of H_0 with energies ε_n, derive the set of differential equations satisfied by the coefficients $a_n(t)$. Show that,

$$\sum_{n=0}^{\infty} |a_n(t)|^2 = constant.$$

(b) At time $t = 0$ a particle is in a state with wavefunction $|u_i\rangle$. This particle is scattered under the influence of the perturbation V (where the perturbation is a step function from time 0 to t) to final states $|u_f\rangle$. It may be assumed that the amplitude of the final state remains small at all times and that the matrix elements $V_{fi} = \langle u_f | V | u_i \rangle$ vary only slowly with the energies ε_f of the final states $|u_f\rangle$. Find an expression for the probability of observing final state $|u_f\rangle$ at time t.

(c) Show that the transition rate, γ, is given approximately by

$$\gamma = \frac{2\pi}{\hbar} |V_{fi}|^2 \rho(\varepsilon_f),$$

where $\rho(\varepsilon_f)$ is the density of final states and $\varepsilon_f \approx \varepsilon_i$.

6.

The cross section for scattering between two states $i \to f$ is given by,

$$\text{cross section} = \frac{\text{transition rate}}{\text{incident flux}},$$

where the transition rate is given by Fermi's Golden Rule:

$$\text{transition rate} = \frac{2\pi}{\hbar} |\langle u_f | V | u_i \rangle|^2 \rho(\varepsilon_f).$$

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(a) Show that for particles with non-relativistic energies, the density of final states can be written as,

$$\rho(E) = \frac{W}{(2\pi)^3}\frac{m}{\hbar^2}k\,\mathrm{d}\Omega,$$

where V is the potential, W is the volume enclosing the states, $\hbar k$ is the momentum, $\mathrm{d}\Omega$ is an element of solid angle, and m is the particle mass.

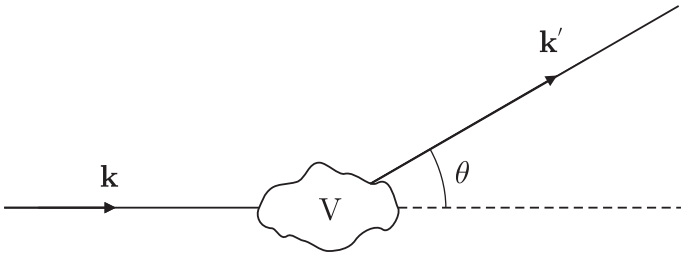
(b) Show that the flux of incident particles is given by $\frac{\hbar k}{mW}$.

(c) Lastly show that the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ is given by,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int V(\mathbf{r}) e^{i(\mathbf{k}_i - \mathbf{k}_f)\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{r}\right|^2.$$

7.

A particle of mass m and 3-momentum $\mathbf{k}$ is elastically scattered from a static potential of limited range V – as shown in the figure.



The Born approximation for the elastic scattering cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int V(\mathbf{r})\mathrm{e}^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} d^3\mathbf{r}\right|^2.$$

[This was derived in the previous question.] The integral required is the Fourier transform (or momentum-space representation) of the potential function, define $U(q^2) = \int V(r)\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}$, where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$.

(a) For a potential of the form $V(r) = \frac{g^2}{r}\mathrm{e}^{-\mu r}$ where g and μ are constants show that

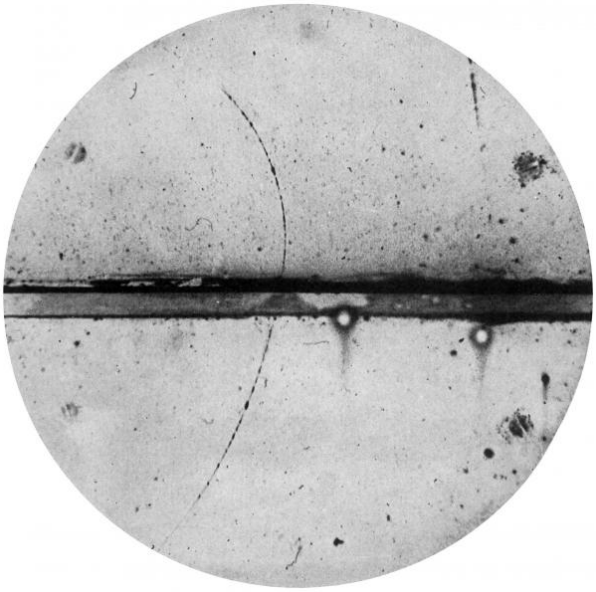
$$U(q^2) = \frac{4\pi g^2}{q^2 + \mu^2} \quad \text{and} \quad q^2 = 4k^2\sin^2\theta/2,$$

where θ is the scattering angle and $k = |\mathbf{k}| = |\mathbf{k}'|$.

(b) For a Yukawa potential $\mu = Mc/\hbar$ where M is the mass of the exchanged force quantum. Show that in the limit $M \to 0$ the resulting formula for the differential cross section is that for Rutherford scattering. [Note that if one attempts to calculate with the Coulomb potential directly, the integral diverges.]

Sketch a few examples for $M \neq 0$ to show how the variation of $\dfrac{d\sigma}{d\cos\theta}$ with $\cos\theta$ depends on the relative magnitude of k^2 and μ^2.

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The Oxford MPhys Course

C4: Particle Physics

Problem Set One

1.

This question concerns the use of natural units. Throughout the course we will use "natural units" in which we set $\hbar = c = 1$. This convention is widely used in particle physics and removes clutter from the equations. In these units particle masses and decay widths are given in units of GeV (the convention is to simply use GeV rather than GeV/c^2 as we have set $c = 1$). However it is often necessary to convert the result of a calculation from natural units, into practical units so that we can compare our calculations with measured quantities. This question gives examples of these conversions.

(a) The mass of the proton is $m_p = 938$ MeV. What is the mass in units of kg?

(b) In natural units the total width of a resonance is related to the lifetime by $\Gamma = 1/\tau$. The total width of the Z is $\Gamma_Z = 3$ GeV. Determine the lifetime of the Z in seconds.

(c) The cross section for νp scattering is given approximately by $\sigma = G_F^2 m_p E_\nu$, where E_ν is the energy of the neutrino. Calculate the value of σ in units of b for $E_\nu = 1$ MeV. ($1b = 10^{-28} m^2$.)

(d) The momentum p for a particle moving in a plane perpendicular to the axis of a uniform magnetic field, $\mathbf{B}$, with a charge, q, moves in a trajectory with a radius of curvature, R, such that $p = BqR$. Consider the trajectory of a particle with

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charge equal to $|e|$. B and R are measured in SI units but we wish to express p in units of GeV. Determine the appropriate conversion constant k, such that $p = kBqR$ with p in GeV. Hint: 1 Tesla = 1Vsm^{-2}.

(e) In the theory of neutrino oscillations, we have a dimensionless phase given by

$$\phi = \frac{\Delta m^2 L}{4E}$$

We measure Δm^2 in units of eV, L in km and E in MeV. Determine the conversion constant to be used.

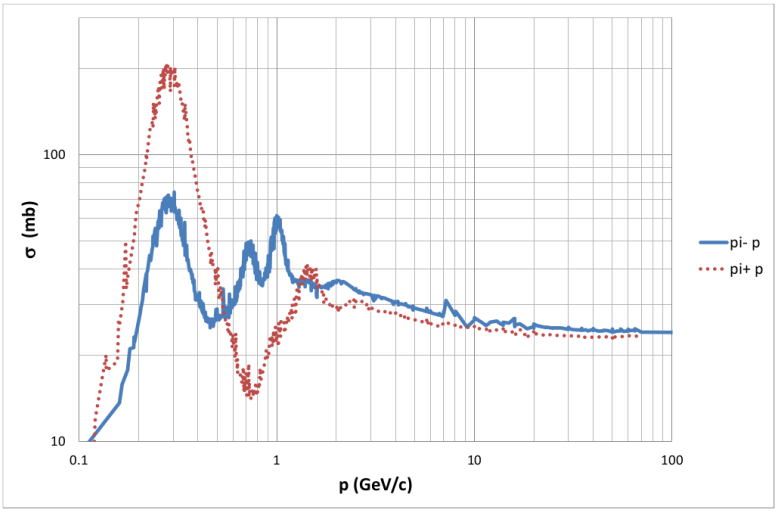
2.

Consider the production of the Δ resonance in $\pi^+ p \to \Delta^{++}$ and $\pi^- p \to \Delta^0$. The Δ resonance exists in 4 charge states ($\Delta^{++}, \Delta^+, \Delta^0$ and Δ^-).

(a) Explaining all the assumptions you need to make, estimate the ratio of cross sections

$$R = \frac{\sigma(\pi^+ p \to \Delta^{++})}{\sigma(\pi^- p \to \Delta^0)}$$

. Compare your answer with the measured data shown in the figure and discuss the origin of any discrepancies.



(b) Estimate the ratio of branching ratios

$$R = \frac{BR(\Delta^+ \to \pi^0 p)}{BR(\Delta^+ \to \pi^+ n)})$$

.

3.

Consider two particles $|l_1, m_1\rangle$ and $|l_2, m_2\rangle$ forming a combined state $|l, m\rangle$.

(a) Assume $l_1 = l_2 = 1$. Starting from $|2, 2\rangle = |1, 1\rangle\,|1, 1\rangle$, use the ladder (shift) operators to find all the possible normalised $l = 2$ $|l, m\rangle$ states in terms of the $|l_1, m_1\rangle$, $|l_2, m_2\rangle$ states.

(b) Find also the normalised $l = 0$ and $l = 1$ states.

4. Consider the three Euler's angles α, β, and γ. It can be shown that the rotation

$$R(\vartheta) = R(\alpha)R(\beta)R(\gamma)$$

can be written as

$$R = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}.$$

(a) Defining the matrix elements $d^j_{m'm}$ as

$$d^j_{m'm}(\beta) = \left\langle jm' \left| e^{-i\beta J_2} \right| jm \right\rangle,$$

find the elements of the rotation matrix

$$\mathcal{D}^j_{m'm}(\alpha, \beta, \gamma) = \langle jm' \,|R|\, jm \rangle$$

in terms of $d^j_{m'm}$.

(b) Calculate $d^j_{m'm}$ for the case $j = 1$.

(c) Determine the form of the CMS angular distribution for an interaction in which the initial state is $J = 1,\ m = 1$ and the outgoing state is $J = 1,\ m' = 1$.

5. The cross sections for the processes

$$\begin{aligned} e^+ + e^- &\rightarrow \mu^+ + \mu^- \ and \\ e^+ + e^- &\rightarrow hadrons \end{aligned}$$

show prominent resonances at centre of mass energies of 3·097 GeV (ψ), 3·686 GeV (ψ') and 3·770 GeV, with widths of 68 keV, 243 keV and 24 MeV, respectively. The measured ratio R of the cross sections,

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)},$$

has a value of ≈ 2 below 3 GeV and $\approx 3{\cdot}5$ at 5 GeV.

(a) Show that these data imply the existence of a fourth quark (charm) in addition to the u, d, and s quarks. Suggest values for the charge and mass of the charm quark c. Explain the structure and widths of the three resonances.

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(b) In a small proportion of cases it is observed that

$$\psi' \to \gamma\gamma\psi,\ \psi \to \mu^+\mu^-.$$

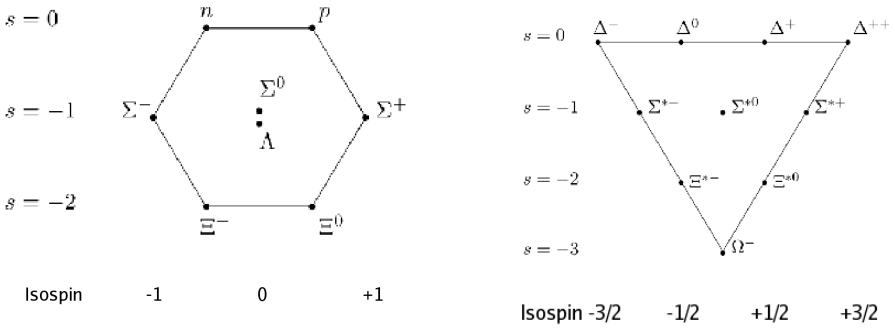
Assuming that the first decay takes place via the chain,

$$\psi' \to \chi\gamma_1,\ \chi \to \psi\gamma_2,$$

find in terms of the masses $m_{\psi'}$ and m_χ expressions for the energy of γ_1 in the ψ' centre of mass, the energy of γ_2 in the χ centre of mass and the speed of the χ in the ψ' centre of mass.

(c) When the production and decay of the ψ' is studied at an e^+e^- collider, the ψ' is produced at rest in the laboratory. The momentum of each γ and the momentum of the ψ are measured in each event. However, it is not known which of the two γ rays is γ_1 and which is γ_2 for each individual event. Suggest how you might resolve this ambiguity in the data, without knowing a precise value for m_χ. Deduce the mass and speed of a χ state for which the momentum of γ_1 in the ψ' centre of mass is $0{\cdot}172\,\mathrm{GeV}/c$.

6. In nature we see the baryons split into two structures when the particles are plotted on a graph with "Y" (a quantity related to "strangeness") on one axis and isospin (I) on the other axis. One of the structures is an octet that forms a hexagonal shape and the other is a decuplet that forms a triangular shape. Measurements indicate that the total spin of all the baryons in the octet are $J = \frac{1}{2}$, while all of the baryons in the decuplet have total spin $J = \frac{3}{2}$.



(a) Give a detailed explanation of the source of this structure.

(b) The Δ^+ baryons decay into either a $\pi^+ n$ or a $\pi^0 p$. Calculate the branching ratios of these two decay modes.

(c) Consider the decay modes of the Δ^+ baryon if it had just enough mass to decay to two pions and either a proton or neutron. Predict the relative decay rates to $\pi^+\pi^- p$, $\pi^+\pi^0 n$, and $\pi^0\pi^0 p$, if the two pions form an $I = 2$ pair.

(d) Why is it necessary to specify the isospin of the pion pair to get a definite prediction?

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7. The general n-body phase space is:

$$R_n(E) = \frac{1}{(2\pi\hbar)^{3n}} \int_1 \cdots \int_n \prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta^3 \left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P} \right) \delta \left(\sum_{i=1}^{n} E_i - E \right),$$

where $\mathbf{P}$ is the total 3-momentum and E the total energy.

(a) Show that the Lorentz invariant two-body phase space for particles with masses m_1 and m_2 is

$$R_2(E) = \frac{\pi}{(2\pi\hbar)^6} \frac{p}{E},$$

where p is the magnitude of the 3-momentum and E the total energy in the two-particle centre-of-mass frame.

(b) Show that,

$$p = \frac{\left\{ \left[E^2 - (m_2 - m_1)^2 \right] \left[E^2 - (m_2 + m_1)^2 \right] \right\}^{\frac{1}{2}}}{2E}.$$

(c) How does R_2 behave as $E \to \infty$? What is the threshold energy and how does R_2 behave as E approaches the threshold energy from above? Give a physical interpretation for the latter for the equal mass case $m_1 = m_2 = m$.

8. Three resonances are observed in the process $e^+e^- \to$ hadrons at centre-of-mass energies $\sqrt{s}$ between 3 and 4GeV: the J/ψ at $\sqrt{s} = 3{\cdot}097$GeV, the ψ' at $\sqrt{s} = 3{\cdot}686$GeV and the ψ'' at $\sqrt{s} = 3{\cdot}771$GeV with widths of 0.09, 0.34 and 23MeV, respectively.

(a) Explain the nature of these resonances.

(b) Complete the following,

- Infer the isospin of the J/ψ from the observation that the branching fractions to $\rho^+\pi^-$, $\rho^-\pi^+$ and $\rho^0\pi^0$ are equal.
- Infer the J^{PC} of the J/ψ from the observation that the cross-section for $e^+e^- \to \mu^+\mu^-$ decreases more rapidly than $1/s$ when $\sqrt{s}$ is just below the resonance.
- Explain the much larger width of the ψ'' compared to the ψ' and J/ψ.

(c) Three decays of the type $\psi' \to \chi\gamma$ are observed in which χ is a hadron. In the rest-frame of the ψ' the photon energies are 132, 184 and 261MeV. What are the quark compositions, masses and J^{PC} of the hadrons χ?

(d) Drawing an analogy with positronium, give the J^{PC} of the other hadrons that have the same quark composition as the ψ and χ hadrons and also have orbital angular momentum quantum number $L \leq 1$.

(e) The decay $J/\psi \to \eta_c\gamma$ is observed with a branching fraction much less than that for $\psi' \to \chi\gamma$. The η_c is not resonantly produced in $e^+e^- \to$ hadrons and decays electromagnetically to two photons. Use the J^{PC} of the η_c to explain these observations.

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