A2 optics problem set I

1. A light ray is incident at angle θ_i on the surface of a glass plate with plane, parallel faces separated by distance d. Part of the light is reflected from the front surface, and part is reflected from the rear surface, as shown in the figure. The refractive index in the incident medium is n_1, the refractive index inside the glass is n_2.

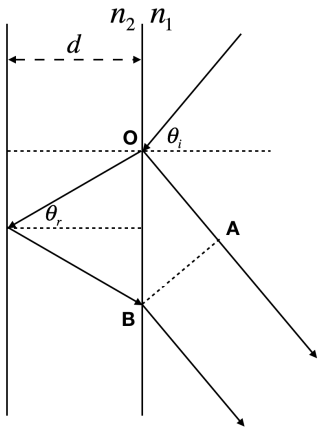
(a) Write down an expression for the angle of reflection of the ray reflected from the rear surface and argue that the rays emerging at A and B are parallel.

(b) Find the difference in time taken by the light rays reflected from the front and rear surfaces to reach points A and B, respectively, expressed as a function of the angle of refraction, θ_r.

(c) Hence find the phase difference between the plane waves reflected from the front and rear surfaces when a plane wave of wavenumber k_0 in vacuum is incident on the plate at angle θ_i, where $k_0 \equiv 2\pi/\lambda_0$ for vacuum wavelength λ_0. You should assume that the wave that is reflected from the rear surface experiences a phase shift of π radians on reflection, arising from the electromagnetic boundary conditions at the surface.

(d) Does the phase difference in (c) increase or decrease as the angle of incidence θ_i is increased?

(e) Suppose that we bring the reflected wavefronts to a focus on a screen. Assuming that the amplitudes of the two wavefronts are equal, give an expression for the intensity observed on the screen, as a function of parameter $\delta = 2k_0 n_2 d \cos\theta_r$, and sketch the functional form of the intensity as a function of δ.



2. Consider a pair of parallel, long, narrow slits (meaning that you can treat this as a one-dimensional problem) separated by distance d, that are illuminated at normal incidence by a plane wavefront of monochromatic light with wavelength λ.

(a) Obtain an expression for the Fraunhofer diffraction pattern, which is the variation in intensity as a function of viewing angle θ, when the observer is a large distance, $D \gg d$, from the slits, making and stating suitable approximations and assumptions.

(b) Calculate the one-dimensional Fourier transform of a pair of delta functions located at x-coordinate $x = -d/2$ and $x = +d/2$. Compare your expression with the result of (a) to suggest the link between the Fourier transform and Fraunhofer diffraction (no need to obtain a formal proof at this stage).

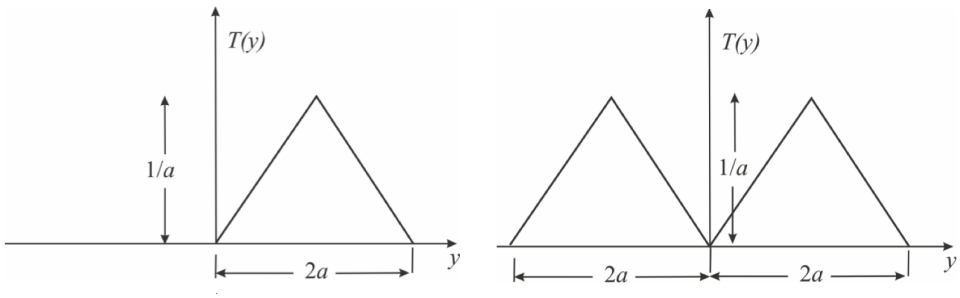
(c) Calculate the Fourier transform of a "top hat" function $f(x)$, defined as

$$f(x) = \begin{cases} 1/w, & -w/2 < x < w/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(d) State without proof the convolution theorem, including a mathematical description (see the Mathematical Methods course).

(e) Argue that a pair of slits of finite width, w and separated by distance d may be represented mathematically as a pair of delta functions separated by d, convolved with a top hat function of width w. Use this statement, together with the convolution theorem, to find an expression for the Fraunhofer diffraction pattern of such a pair of finite-width slits.

3. The two panels of the figure show: (left panel) a single triangular function $T(y)$; and (right panel) a double-triangular function.



(a) Show that the single triangular function in the left panel may be constructed as the convolution of a top hat function with itself, and give expressions for the width and location of the top hat function.

(b) Calculate the Fourier transform of the top hat from (a), including accounting for its location. Then, use the convolution theorem to deduce the Fraunhofer diffraction pattern from the aperture with the triangular transmission function.

(c) The resulting Fraunhofer diffraction pattern appears not to depend on the offset of the triangular function from the origin. Explain why this is expected.

(d) Now consider an aperture whose transmission as a function of y is given by the right panel of the figure. Deduce the Fraunhofer diffraction pattern of an aperture with this transmission function. Estimate the height of the secondary maximum and sketch the diffracted intensity as a function of $\delta = ka \sin\theta$.

(e) Find where the diffracted intensity first falls to zero. This location corresponds to a zero value in the Fourier transform of the transmission function: on the diagram of the double triangular function, sketch the Fourier mode whose amplitude is zero.

4. A simple pinhole "camera obscura" is made by placing a viewing screen in a darkened room, a distance D away from a small, square pinhole, of width a, in an otherwise opaque primary screen. The viewing screen is distant from the pinhole, $D \gg a$.

(a) Calculate the two-dimensional Fourier transform of the pinhole's transmission function, assuming this to be a two-dimensional top hat function in the xy plane:

$$T(x,y) = \begin{cases} 1/a^2 & -a/2 < x < a/2 \text{ and } -a/2 < y < a/2 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Hence estimate the size of the Fraunhofer diffraction pattern, $s_{\rm F}$, that would result if the pinhole were illuminated by a plane wave with wavelength λ, when that pattern is projected on the viewing screen. Sketch how the size of that pattern varies with the size of the pinhole.

(c) A small, bright object outside the room, a distance u away, is imaged through the pinhole onto the viewing screen. Use geometrical ray optics to find the blurring of the image, $s_{\rm R}$, caused by the finite size of the pinhole, and sketch how the blurring varies with the size of the pinhole.

(d) As an approximation, the observed image size may be regarded as arising from the convolution of the geometrical ray optics blurring with the blurring caused by Fraunhofer diffraction. Assuming that the observed size is given approximately by the quadrature sum of the individual effects,

$$s_{\rm total}^2 \simeq s_{\rm F}^2 + s_{\rm R}^2 \,, \tag{3}$$

find the size of the pinhole that yields the sharpest image.

A2 optics problem set II

5. A simple two-lens astronomical telescope is used to observe a distant planet. The telescope is focussed to produce an image at infinity, suitable for viewing by eye, when the eye is placed at the exit pupil of the telescope. The primary, or objective, lens has a diameter $D = 5\,\text{cm}$ and a focal length $f_P = 30\,\text{cm}$. The telescope is equipped with two choices for the secondary, or eyepiece, lens, with focal lengths $f_s = 18\,\text{mm}$ and $60\,\text{mm}$.

(a) Which eyepiece is the best choice for viewing by eye?

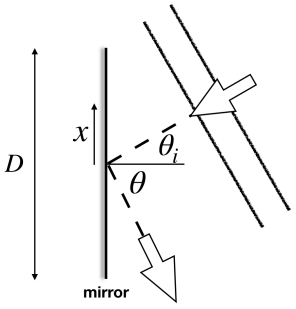
(b) What magnification would be obtained?

(c) How far from the eyepiece lens is the exit pupil? Give the focal length and location of a third field lens that could make the exit pupil positioned just behind the eyepiece.

(d) What is the angular resolution of the telescope, expressed in arcsec (seconds of arc), at a typical wavelength for visible light? If a star is observed using this telescope (with the best eyepiece), estimate the size of the diffraction pattern on the retina of the eye. The spacing between rod or cone photoreceptor cells in the human retina is about 2μm: comment on this dimension relative to the size of the diffraction pattern.

(e) Repeat this calculation for the diffraction pattern of the unaided human eye, for the case where the pupil diameter is 1 mm (bright light) or 7 mm (fully dark adapted) and again comment on your result.

6. Consider a plane wavefront incident at angle θ_i onto a plane mirror as shown.



(a) Treating this as a 1D Fraunhofer diffraction problem, write down an expression for the phase of the diffracted wavefront as a function of distance x from the mirror centre and angle of diffraction θ, and hence deduce the Fraunhofer intensity distribution as a function of θ and θ_i.

(b) Show that the diffracted intensity has a maximum at an angle that agrees with the law of reflection.

(c) Calculate the full angular width to zero intensity of the maximum of the diffraction pattern. Give the value of this angular width for a typical wavelength of optical light and for a mirror size $D = 5\,\text{cm}$, and thus explain why, for most purposes, we can treat reflection from a plane mirror as a ray problem, ignoring diffraction effects.

7. A grating with N narrow slits of spacing d is illuminated normally with monochromatic light of wavelength λ, and the p^{th} order principal maximum is observed at an angle θ_p.

(a) Show that the diffracted intensity at angle θ is given by

$$I(\delta) = \frac{I_0}{N^2}\frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}\,,$$

where $\delta = kd\sin\theta = \frac{2\pi}{\lambda}d\sin\theta$ and I_0 is a normalisation constant.

(b) Show that the adjacent minima occur at angles $\theta_p \pm \Delta\theta_p$ where

$$\Delta\theta_p = \frac{\lambda}{Nd\cos\theta_p}$$

(hint: assume $\Delta\theta_p$ is small).

(c) Compare your result from part (b) with the width of the diffracted intensity in Q6(c) and comment on your result - what determines the angular width of the maximum of the diffraction pattern?

(d) Show that the dispersion of the grating in p^{th} order is

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{p}{d\cos\theta}$$

that the instrumental width in wavelengths is $\Delta\lambda = \lambda/Np$ and that the resolving power is $R = Np$.

(e) Does the resolving power change if the grating is used in a medium of high refractive index?

(f) Now consider the case where the slits have a finite width a. Use the convolution theorem to predict the effect on the grating's diffraction pattern, and sketch the diffraction pattern for the cases $d = 2a$ and $d = 3a$.

8. A reflection grating, with 500 lines per mm, is illuminated by a plane wave at angle of incidence $\theta_i = 17.46°$.

(a) At what angles can the second-order diffraction pattern be measured near a wavelength of $\lambda = 600\,\mathrm{nm}$? Draw a diagram showing the angle of incidence and the angles of diffraction maxima for all possible orders.

(b) The efficiency of a reflection grating can be greatly improved by inclining the the diffracting elements at a blaze angle θ_B. What are the optimum blaze angles for this grating, in second order? Indicate these angles on your diagram.

(c) What is the longest wavelength of light that can be measured in the diffracted spectrum with this grating in second order, at this angle of incidence?

A2 optics problem set III

9. Give a brief account of the principal features of a Michelson interferometer used with visual observation and illuminated by an extended monochromatic source. Explain what configurations of the mirrors will give (i) circular fringes, (ii) straight, equally spaced fringes. State where the fringes are localised and why. Why in case (ii) does the mirror spacing have to be small?

10. A Michelson interferometer is set up to give circular fringes when illuminated by light of wavelength λ.

(a) What will be the order of interference, p_0, for the fringe on the axis when the difference in the lengths of the arms of the interferometer is t?

(b) What condition must be fulfilled to give a bright circular fringe at angle α to the axis when the interferometer is illuminated by an extended source of wavelength λ?

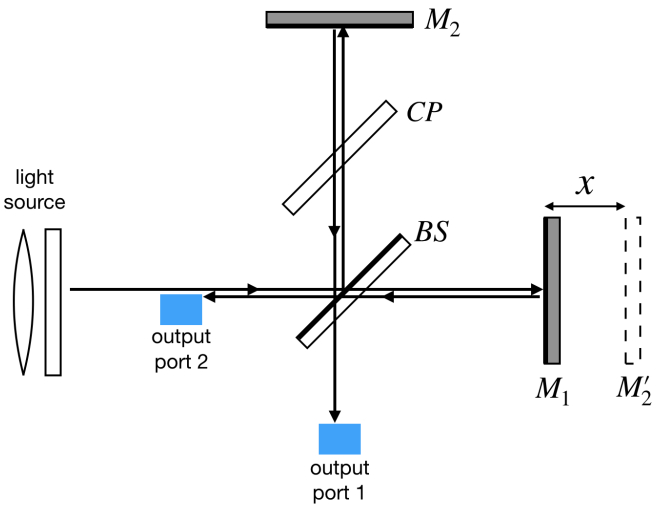
(c) Explain why the order of interference p of a fringe at angle α to the axis is less than p_0.

(d) Derive an expression for the radius, r_p, of the p^{th}-order fringe formed in the focal plane of a lens of focal length f. Hence show that the difference in radii of the p^{th} and $(p+1)^{th}$–order fringes is given by

$$r_p^2 - r_{p+1}^2 = \frac{\lambda f^2}{t}$$

(e) The path length of the arm containing the moving mirror is initially larger than the path in the arm containing the fixed mirror. The path difference t is slowly decreased towards the zero path difference condition. Describe what happens to the fringes as the path difference decreases and then increases as the moving mirror moves past the position of zero path difference.

11. Consider the Michelson interferometer shown schematically below.



(a) Explain why the interference fringes are dark when viewed at output port 2, if the fringes are bright at output port 1, and vice versa (hint: consider the phase shift introduced on reflection at the beamsplitter, which depends on whether the reflection is internal or external).

(b) Show that energy is conserved.

(c) Draw a simple phasor diagram to represent the combination of the two beams at output port 1, when the phase difference between the beams is δ. Draw the corresponding phasor diagram for the combination at output port 2, and show that the combined beams at ports 1 and 2 have a phase difference of $\pi/2$ with respect to each other (i.e. port 1 compared with port 2), if the beams along each arm have equal amplitude.

12. A Michelson interferometer is set up to give circular fringes and illuminated with light in the spectral range $800-900\,\mathrm{nm}$ by a caesium light source. The intensity $I(x)$ at the centre of the interference pattern is recorded as a function of the distance x of the moving mirror from that corresponding to zero path difference. It is found to have the form

$$I(x) = I_0\left[3 + 3\cos(K_1 x)\cos(K_2 x) - \sin(K_1 x)\sin(K_2 x)\right] ,$$

where $K_1 = 1.44 \times 10^7\,\mathrm{m}^{-1}$, $K_2 = 3.48 \times 10^5\,\mathrm{m}^{-1}$ and I_0 is a constant. Show that this can be written as the sum of the patterns due to two monochromatic spectral components. Hence determine, for the two caesium atomic lines in this wavelength range:

(a) their mean wavenumber, $\bar{\nu}$,

(b) their wavenumber separation, $\Delta\bar{\nu}$,

(c) their relative intensities.

Make a sketch of the interference pattern over the range $-10 < x < 10\,\mu\mathrm{m}$.

When the interference pattern is recorded over a much larger range of path difference, it is found that the periodic terms in the pattern are in fact multiplied by a function

$$f(x) \propto \exp(-K_3 x^2) .$$

Suggest what might cause this effect (hint: use your knowledge of the convolution theorem).

13. The Fabry-Perot interferometer.

(a) Show that the transmission function of a lossless Fabry-Perot interferometer for monochromatic light of wavelength λ is given by

$$I(\delta) = I_0\left[1 + \frac{4F^2}{\pi^2}\sin^2(\delta/2)\right]^{-1}$$

where I_0 is the incident intensity and δ is a function of the parameters of the instrument: give expressions for δ and for F in terms of the reflectance R of the reflecting plates. The reflecting plates may be assumed to have equal R.

(b) Give the equivalent expression for the reflection function.

(c) Show that the sum of the transmitted and reflected intensities is equal to the incident intensity (conservation of energy) for a lossless Fabry-Perot interferometer.

14. A spectral line is known to consist of two equally intense components with a wavenumber separation $\Delta\bar{\nu} < 20\,\mathrm{m}^{-1}$. The Fabry-Perot fringes produced by this line are photographed, using a plate separation $d = 25\,\mathrm{mm}$. The diameters of the smallest rings are found to be 1.82, 3.30, 4.84, 5.57, 6.60 and 7.15 mm. Explain why this experiment does not allow $\Delta\bar{\nu}$ to be determined uniquely, but gives two possible values. Estimate those values. Suggest a further experiment that could be carried out to resolve the ambiguity.

15. Use the expression in Q13(a) to find the instrument width and resolving power of the Fabry-Perot etalon, as follows.

(a) Maxima of the Airy function occur when $\delta = 2p\pi$, where p is a (large) integer (the use of p here is not the same as in the notes). Show that the values of δ at which the intensity has dropped to 50% of maximum near the p^{th} order of interference are

$$\delta = 2\pi(p \pm 1/(2F))$$

(b) Show that two spectral lines for which the p^{th} maximum of one falls on the $(p+1)^{th}$ maximum of the other, the Free Spectral Range, are separated by

$$\Delta\bar{\nu}_{\mathrm{FSR}} = \frac{1}{2t}$$

(c) From (a) and (b), show that the instrumental width (FWHM, full width at half maximum intensity), of the etalon is given by

$$\Delta\bar{\nu}_{\mathrm{INST}} = \frac{1}{2Ft}$$

(d) Hence show that the resolving power is Fp.

(e) How are the above relations modified if the etalon is filled with a medium of refractive index n?

A2 optics problem set IV

16. Consider a simple two-level atom.

(a) Write down expressions for the transition rates of absorption, stimulated emission and spontaneous emission, in terms of the Einstein coefficients.

(b) Show that it is not possible to achieve the population inversion required for laser action solely by radiative pumping between the two levels, in the steady state (hint: write down an expression for the total rate of change of the number density of atoms in the upper energy level, assuming only radiative transitions contribute).

(c) Show that steady-state laser action requires $A\tau < 1$, where τ is the lifetime for electrons in the lower energy level of the laser transition and A is the Einstein coefficient for spontaneous emission from the upper to the lower level.

17. Polarised light:

(a) Explain what is meant by polarised light and describe the types of polarisation that are possible.

(b) Explain how a uniaxial, birefringent material of length l, with ordinary and extra-ordinary refractive indices, n_o and n_e, respectively, can be used to change the relative phase of two orthogonal components of the electric field in a light wave of wavelength λ. Describe briefly the principle of operation of a polarising device made from a birefringent material.

(c) Demonstrate analytically that unpolarised light can be converted to circularly polarised light by passing the radiation through a linear polariser followed by a quarter-wave plate at a suitable orientation.

(d) A beam of initially unpolarised light passes through this system, and is then reflected back normally through it from a plane mirror. Describe the state of polarisation of the beam in each part of its path through the system.

(e) Show that any elliptically polarised light can be converted to plane polarised light using a suitably oriented quarter-wave plate. Is the converse also true? Show also that plane polarised light remains plane polarised after passing through a half-wave plate, but with the direction of the electric vector altered. How would you use a half-wave plate to change this direction by 60°?

18. A beam of light is elliptically polarised with the major axis vertical.

(a) A beam of light consisting of a mixture of elliptically polarised and unpolarised light is passed through a linearly polarising filter. The maximum of the transmitted light intensity is observed when the transmission axis of the filter is vertical, and is twice the minimum intensity. In a second experiment, the beam is passed through a quarter-wave plate with the fast axis vertical followed by the polarising filter. The maximum is now observed when the transmission axis is at 26.56° to the vertical. Calculate the ratio of the intensities of the polarised to unpolarised components of the light.

(b) How could the handedness of the elliptically polarised component be determined?

19. A beam of initially unpolarised light is incident normally on a piece of polaroid polarising sheet, A. The transmitted light is then falling on a second similar sheet, B. A and B are set with their axes at right angles.

(a) A third polaroid sheet, C, is placed between the first two, with its axis at an angle ϕ to that of sheet A. What fraction of the incident energy passes through the system, as a function of the angle ϕ?

(b) Instead of C, a cell containing a uniaxial birefringent material, with a difference in its $o-$ and $e-$ refractive indices $\Delta n = n_o - n_e$, is placed between A and B (i.e. the system now comprises A–cell–B). Describe how the optic axis of the material should be orientated to maximise the intensity of light passing through the system. If the cell has a length of 1 cm, in the direction of propagation of light through the system, what value of Δn is required?