PART I

QUANTUM OPTICS

1 Quantized fields

So far this course has generally treated light fields as classical objects which can be used to control quantum mechanical objects, such as atoms or nuclear spins. The only exception to this has been treatments of single photon systems, where we have described, for example, the photon's polarisation using kets such as $|H\rangle$ and $|V\rangle$. This approach is sometimes called *first quantization*. In this part of the course we will introduce a fully quantum mechanical treatment, called *second quantization*, in which light is treated as a quantum field.

1.1 Symmetries of wavefunctions

The extension of simple quantum mechanics to systems with multiple photons must be done with care, as it is necessary to consider the symmetry of the many body wavefunction. Photons are bosons, and so permitted wavefunctions must be symmetric with respect to exchange of particle labels. This means that for a system of two photons a product wave function such as

$$|\psi_1\rangle \otimes |\psi_2\rangle \tag{1.1}$$

is not a valid wavefunction, but instead the symmetrised form

$$|\psi_S\rangle = \frac{|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle}{\sqrt{2}} \tag{1.2}$$

must be used. In the same way spin-$\frac{1}{2}$ particles such as electrons are fermions, and their wave functions must take the antisymmetric form

$$|\psi_A\rangle = \frac{|\psi_1\rangle \otimes |\psi_2\rangle - |\psi_2\rangle \otimes |\psi_1\rangle}{\sqrt{2}}. \tag{1.3}$$

This approach can be extended to larger numbers of particles, using Slater determinants to calculate antisymmetric wavefunctions for fermions, and permanents[1] to calculate symmetric wavefunctions for bosons, followed by suitable normalisation. Care must be taken if two particles are in the same state: in this case the antisymmetric wavefunction vanishes, indicating that two fermions may not occupy the same quantum state. The symmetric wavefunction exists in all cases, but the normalisation factor changes depending on the number

[1] A Slater determinant uses the explicit expansion of a determinant in terms of cofactors to combine all the relabelled versions of a simple wavefunction with appropriate minus signs to ensure antisymmetry; the symmetric permanent is expanded in the same way, but omitting the minus signs.

of particles in the same state. For example for two identical particles in the same state

$$|\psi_S\rangle = |\psi_1\rangle \otimes |\psi_1\rangle, \tag{1.4}$$

and not

$$\frac{|\psi_1\rangle \otimes |\psi_1\rangle + |\psi_1\rangle \otimes |\psi_1\rangle}{\sqrt{2}} = \sqrt{2}\,|\psi_1\rangle \otimes |\psi_1\rangle \tag{1.5}$$

which would be obtained by naively replacing ψ_2 by ψ_1 in equation 1.2.

Until now we have largely ignored this point, raising the question of why it was safe to do so. This occurs because we have considered quantum systems which may be intrinsically indistinguishable but which can be made effectively distinguishable by considering extrinsic properties, such as their positions in space. This works for classical objects, but is problematic for quantum objects whose position is not perfectly defined. One simple way to think about this is that indistinguishability becomes important when the separation is comparable to the de Broglie wavelength of the particles in question. This explains why it is not possible to consider the two electrons in a lithium ion separately, but rather it is necessary to consider an antisymmetric two-electron wavefunction, but it is possible to treat two lithium ions in an ion trap as distinguishable systems.

Classical light waves can be thought of as containing very large numbers of photons, and in the case of a laser beam these will all be in much the same region of space with similar properties such as colour, polarisation and direction. In such cases explicitly symmetrising a multiphoton wavefunction will become terribly unwieldy. Instead we describe such states using occupation numbers, which simply list the number of particles in each quantum state. Thus we write equation 1.2 as

$$|1\rangle_{\psi_1}|1\rangle_{\psi_2}, \tag{1.6}$$

which indicates 1 particle in each quantum state, ψ_1 and ψ_2, without making the error of identifying which particle is in which state. In the same way equation 1.4 can be written as

$$|2\rangle_{\psi_1}|0\rangle_{\psi_2}, \tag{1.7}$$

indicating 2 particles in ψ_1 and 0 in ψ_2. As in general there can be a very large number of quantum states with no particles in them it is common to omit empty quantum states. It is also convenient to abbreviate labels, and so this state can be described more simply as $|2\rangle_1$.

This notation may initially appear to be little more than a way of avoiding symmetrisation problems. As we will soon see, however, it corresponds to a very natural way of describing light fields.

1.2 Light modes in a cavity

For simplicity we will initially confine ourselves to light fields in free space, although we will later consider the interaction of these fields with atoms or ions. We will, however, start by assuming that the light field is contained in some sort of cavity by perfectly reflecting mirrors. In this case a classical light field within the cavity is described by Maxwell's

equations in free space, with the cavity acting simply to impose boundary conditions that limit the wave to certain cavity modes, which correspond to standing waves, rather than the travelling waves usually considered in completely free space. However as the inside of the cavity is free space these standing waves are characterised by the usual phase velocity $v_p = \omega/k = c$.

We begin by considering a one-dimensional cavity of length L along the z-axis, and restrict ourselves to single longitudinal modes. The E field of the light must be perpendicular to the direction of the wave, and can be taken as defining the x-axis. As it cannot penetrate the perfectly reflecting walls we must have

$$\mathcal{E}_x = A \sin(\omega t) \sin(kz) \tag{1.8}$$

where $k = \kappa\pi/L$, with κ being a positive integer, to ensure that the wave fits within the cavity, and phases have been absorbed by choosing the origin of time. The magnetic field can then be found from Maxwell's fourth equation in free space

$$\nabla \wedge \mathbf{B} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.9}$$

which requires the magnetic field to lie along the y-axis, with

$$\mathcal{B}_y = \mu_0\epsilon_0 \frac{\omega}{k} A \cos(\omega t) \cos(kz). \tag{1.10}$$

The energy density of the field is given by

$$\mathrm{d}E = \tfrac{1}{2}\left(\epsilon_0\mathcal{E}_x^2 + \mathcal{B}_y^2/\mu_0\right) \mathrm{d}V \tag{1.11}$$

and integrating over the length of the cavity gives the field energy per unit area

$$E = \frac{A^2 L\epsilon_0}{4}\left[\sin^2(\omega t) + \mu_0\epsilon_0 \frac{\omega^2}{k^2}\cos^2(\omega t)\right] \tag{1.12}$$

where the two terms correspond to the energy stored in the electric and magnetic fields respectively. Since $\mu_0\epsilon_0 = 1/c^2 = k^2/\omega^2$ these two terms have the same amplitude, and the total energy of the standing wave simply oscillates between the electric and magnetic fields.

1.3 Quantizing the field

This oscillation of the energy between electric and magnetic fields is entirely analogous to the oscillation between potential and kinetic energy in a harmonic oscillator, suggesting that the correct way to quantize a light field is to treat it as a quantum harmonic oscillator, in which the electric and magnetic field strengths are replaced by non-commuting operators, just as is done for the position and momentum of a mass on a spring. Inspired by the ladder operator method we write

$$\hat{\mathcal{E}}_x = \mathcal{E}_0\left(\hat{a} + \hat{a}^\dagger\right)\sin(kz), \tag{1.13}$$

with the usual commutator

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{1.14}$$

We will soon see that choosing

$$\mathcal{E}_0 = \sqrt{\frac{\hbar\omega}{\epsilon_0 L}} \tag{1.15}$$

gives the expected energy scaling. As defined here the raising and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$, are dimensionless.

We also note that the electric field operator $\hat{\mathcal{E}}$ is Hermitian, and so will have real eigenvalues, as expected for a measurable quantity. The magnetic field operator must also be Hermitian, and takes the other Hermitian form

$$\hat{\mathcal{B}}_y = -\mathrm{i}\frac{\mathcal{E}_0}{c}\left(\hat{a} - \hat{a}^\dagger\right)\cos(kz). \tag{1.16}$$

From here on we will drop the "hats", and simply assume that $\mathcal{E}$ and $\mathcal{B}$ are Hermitian operators rather than numbers, while a and $a^\dagger$ are non-Hermitian operators.

Substituting these forms into equation 1.11 and integrating over the length of the cavity gives the energy per unit area as

$$\frac{L\epsilon_0\mathcal{E}_0^2}{4}\left[\left(a + a^\dagger\right)^2 - \left(a - a^\dagger\right)^2\right]. \tag{1.17}$$

After multiplying out the operators and substituting for $\mathcal{E}_0$ from equation 1.15 this simplifies to

$$\frac{\hbar\omega}{2}\left(aa^\dagger + a^\dagger a\right). \tag{1.18}$$

Finally use the commutator, equation 1.14, to write $aa^\dagger = 1 + a^\dagger a$, leading to the Hamiltonian

$$\mathcal{H} = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right). \tag{1.19}$$

1.4 Number states

The elementary properties of the quantum harmonic oscillator are described in most texts on quantum mechanics. In the context of quantum optics the eigenstates of the Hamiltonian, equation 1.19, are usually called the number states, $|n\rangle$, or the Fock states. These have the properties that

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}\,|n-1\rangle, \tag{1.20}$$

explaining the names raising and lowering operators. Here n must be a non-negative integer, and it is not possible to go down from the bottom state as $a|0\rangle = 0$. These states are eigenstates of the number operator

$$\hat{N}|n\rangle = n|n\rangle, \quad \text{where} \quad \hat{N} = a^\dagger a. \tag{1.21}$$

Now

$$\mathcal{H}|n\rangle = \hbar\omega\left(\hat{N} + \tfrac{1}{2}\right)|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)|n\rangle = E_n\,|n\rangle. \tag{1.22}$$

Comparing this with Einstein's model of light, which is treated as made up of photons each of energy $\hbar\omega$, it is immediately obviously that the number state $|n\rangle$ corresponds to a state of n photons. This equivalence also justifies the value used for $\mathcal{E}_0$, equation 1.15. The only slightly mysterious feature is the presence of the zero point energy, $E_0 = \frac{1}{2}\hbar\omega$. The properties of the state $|0\rangle$, known as the vacuum state, will prove to be very important, but for now we will simply ignore this point.

In the language of quantum optics the operator $a^\dagger$ is often called the creation operator, since it "creates" an additional photon which is added to the light field, while a is called the annihilation operator as it removes a photon from the light field.

1.5 Photon and field statistics

As we introduce new states of the light field we will also introduce ways of characterising them. One of the simplest is photon statistics, that is the mean number of photons found when measuring a light field, $\bar{N}$, and the standard deviation in this number, σ_N. For a general state $|\psi\rangle$ these are given by

$$\bar{N} = \langle\hat{N}\rangle = \langle\psi|\hat{N}|\psi\rangle \quad \text{and} \quad \sigma_N^2 = \langle\left(\hat{N} - \bar{N}\right)^2\rangle = \langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2. \tag{1.23}$$

For a number state the values are obviously $\bar{N} = n$ and $\sigma_N = 0$, so the photon number is perfectly defined. The electric and magnetic field strengths are not, however, well defined. The average values are obviously zero, as the amplitude depends on

$$\langle n|a + a^\dagger|n\rangle = \langle n|a|n\rangle + \langle n|a^\dagger|n\rangle = \sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle, \tag{1.24}$$

and the number states are orthonormal so the inner products are both 0. An equivalent argument hold for $\mathcal{B}$. The standard deviations are more complicated, with $\sigma_\mathcal{E}^2$ proportional to

$$\langle n|(a + a^\dagger)^2|n\rangle = \langle n|aa^\dagger + a^\dagger a|n\rangle = \langle n|2\hat{N} + 1|n\rangle = 2n + 1. \tag{1.25}$$

Thus

$$\sigma_\mathcal{E} = \sqrt{2n+1}\,\mathcal{E}_0\,\sin(kz), \tag{1.26}$$

and an equivalent expression applies for $\sigma_\mathcal{B}$. For brevity it is common to drop the amplitude and shape terms, and so write $\sigma_\mathcal{E} = \sigma_\mathcal{B} = \sqrt{2n+1}$.

These results are unsurprising when you consider the wavefunctions of the quantum harmonic oscillator eigenstates. These functions are all of well defined parity, reflecting the odd or even nature of n, and so positive and negative values are equally likely, leading to the expectation values of zero. The increasing variance with increasing n reflects the increasing width of the corresponding Hermite polynomials. The most surprising result is the non-zero variance in the fields for $|0\rangle$, showing that even the vacuum state *with no*

photons present exhibits fluctuating electric and magnetic fields, determined by the width of the Gaussian wavefunction ψ_0. This in turns reflects the fact that $\mathcal{E}$ and $\mathcal{B}$ are non-commuting operators, just like x and p in ordinary quantum mechanics.

Further reading

Photon statistics are discussed in many texts, but Fox (2006) and Vedral (2005) provide particulary useful introductions.

The ladder operator treatment of the quantum harmonic oscillator can be found in most elementary treatments of quantum mechanics, including both modern treatments such as Binney and Skinner (2014), and more traditional texts such as Gasiorowicz (2003). The extension to photons can be found in more advanced texts as well as specialist texts on quantum optics. As is often the case there are significant variations in the precise notations used.

Quantum optics can be seen as the simplest example of a quantum field theory, and can be extended to treat any quantum particle, not just photons, but that point is not explored here. For more details see Lancaster and Blundell (2014).

Exercises

1.1 Use a Slater determinant to antisymmetrise the wavefunction $|\psi_a\rangle \otimes |\psi_b\rangle \otimes |\psi_c\rangle$ for fermions, and a Slater permanent to symmetrise the same wavefunction for bosons. How many terms would there be in the corresponding expressions with six particles?

1.2 Show that equation 1.15 has dimensions of volts, rather than volts per metre as expected for an electric field. What corresponding error has been made in the following equations so that this all comes out correct in the end?

1.3 Find the commutator $[\mathcal{E}, \mathcal{B}]$.

2 Classical light

Having gone to all the trouble of developing a fully quantum theory of light fields it might seem odd to immediately seek to describe classical light, but this is important for two reasons. Firstly it is useful to explore classical situations where our instincts about what should happen are likely to fairly reliable, and to see how classical behaviour can emerge from an underlying quantum reality. Secondly, as we will discuss in a later chapter, the ability to control quantum systems with light relies on the use of classical fields, to avoid the generation of entanglement between the controlled system and the control field.

We will explore two situations: thermal radiation, which corresponds to light emitted by an ideal blackbody, and coherent light, which corresponds to light generated by an ideal laser. Finally we will briefly discuss how to model light outside a cavity, such as a laser beam.

2.1 Thermal states

Thermal states are simply incoherent mixtures of different number states, and take the general form

$$\rho_T = \sum_n P_n |n\rangle\langle n|. \tag{2.1}$$

The underlying probabilities P_n are non-negative numbers that sum to one, and are given by the Boltzmann distribution

$$P_n = \frac{\exp(-E_n/k_B T)}{Z} \tag{2.2}$$

with the partition function given by

$$Z = \sum_{n=0}^{\infty} \exp(-E_n/k_B T). \tag{2.3}$$

As we are using a second quantized representation the energies here represent the energies of the field number states, and there is no need to worry about the indistinguishability of the bosonic photons. Measuring energies from E_0 gives

$$P_n = \exp(-n\hbar\omega/k_B T)\left[1 - \exp(-\hbar\omega/k_B T)\right], \tag{2.4}$$

where Z has been calculated as the infinite sum of a convergent geometric progression.

These probabilities can then be used to calculate photon statistics (these calculations are explored in the exercises). For a thermal state

$$\bar{N} = \sum_n nP_n = \frac{\exp(-\hbar\omega/k_BT)}{1-\exp(-\hbar\omega/k_BT)} = \frac{1}{\exp(\hbar\omega/k_BT)-1} \tag{2.5}$$

while

$$\sigma_N^2 = \sum_n n^2P_n - \bar{N}^2 = \frac{\exp(-\hbar\omega/k_BT)}{[1-\exp(-\hbar\omega/k_BT)]^2} = \bar{N}+\bar{N}^2. \tag{2.6}$$

Some of these results may initially seem surprising. At all temperatures the most probable state of the system is $|0\rangle$, with the width of the probability distribution growing as the temperature increases. If photons were classical particles they would be expected to show a Poisson distribution, with the standard deviation of the number of photons equal to the square root of the mean, while for classical waves the standard deviation would be equal to the mean. Instead real photons exhibit particle-like statistics when the number of photons is small, but wave-like statistics when the number of photons is large.

The expectation values of the electric and magnetic fields are obviously zero for thermal states, as these are just weighted averages of values for number states, all of which are zero. As before these fields will vary around zero, but the details are not pursued here.

2.2 Coherent states

Coherent states are very different from thermal states. As their name suggests they are coherent superpositions of number states, but taking a very particular form, defined by a single complex number α. They are defined by

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.7}$$

where the exponential term before the sum acts to normalise the state. The notation $|\alpha\rangle$ is very similar to $|n\rangle$ used for number states, and confusion might seem inevitable, but in practice it is usually obvious which is meant. Note, however, that the coherent state $|0\rangle$ is uniquely identical to the number state $|0\rangle$.

Coherent states have the special property that they are right eigenvectors of the annihilation operator,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.8}$$

with eigenvalue α. Note that it is not a problem that the eigenvalues can be complex, as a is not Hermitian. This is also why a need not have left eigenvectors, and in particular $\langle\alpha|a$ is not a multiple of $\langle\alpha|$. Equivalently the creation operator $a^\dagger$ does not have right eigenvectors, but it does have left eigenvectors, defined by

$$\langle\alpha|a^\dagger = \langle\alpha|\alpha^*. \tag{2.9}$$

This result is most easily seen by taking the adjoint of both sides of equation 2.8.

The mean photon number is easy to calculate for coherent states as

$$\bar{N} = \langle\alpha|a^\dagger a|\alpha\rangle = \langle\alpha|\alpha^*\alpha|\alpha\rangle = |\alpha|^2, \tag{2.10}$$

where the eigenvalues are obtained by acting left with $a^\dagger$ and right with a. As discussed in the exercises the standard deviation is given by

$$\sigma_N = \sqrt{\bar{N}} \tag{2.11}$$

which is what one would expect as the photon number has a Poisson distribution,

$$P_n = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}, \tag{2.12}$$

which for large values of $|\alpha|$ is sharply peaked around $\bar{N}$.

The time evolution of coherent states is also straightforward to evaluate by considering the time-dependent Schrödinger equation in the number state basis. For a system beginning in a number state, $|\psi(0)\rangle = |n\rangle$,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle = \hbar\omega(n+\tfrac{1}{2})\,|\psi\rangle, \tag{2.13}$$

so

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\omega t}|n\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}\left(\mathrm{e}^{-\mathrm{i}\omega t}\right)^n|n\rangle. \tag{2.14}$$

For a superposition state, the first term is a global phase while the second term indicates a state-dependent phase. However for the particular form of a coherent state, equation 2.7, this phase combines naturally with the α^n term in the superposition, so that

$$|\alpha(t)\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}\,|\alpha(0)\mathrm{e}^{-\mathrm{i}\omega t}\rangle. \tag{2.15}$$

A coherent state remains a coherent state under free evolution, with a fixed value of $|\alpha|$, so that the average photon number remains constant, but with a phase that accumulates at angular frequency ω.

The interpretation of α is further clarified by calculating electric and magnetic fields. These will now be non-zero, precisely because coherent states are superpositions of adjacent states. As before we concentrate on dimensionless values, and find that

$$\langle\mathcal{E}\rangle \propto \langle\alpha|a + a^\dagger|\alpha\rangle = \alpha + \alpha^* = 2\,\mathrm{Re}(\alpha), \tag{2.16}$$

while

$$\langle\mathcal{B}\rangle \propto -\mathrm{i}\langle\alpha|a - a^\dagger|\alpha\rangle = -\mathrm{i}(\alpha - \alpha^*) = 2\,\mathrm{Im}(\alpha). \tag{2.17}$$

In the same units the standard deviations of the field strengths are

$$\sigma_\mathcal{E} = \sigma_\mathcal{B} = 1, \tag{2.18}$$

independent of α. This leads to a very natural way of depicting a coherent state, shown in Fig. 2.1, as a circular blob on an Argand diagram, centred on α, with the real axis corresponding to $\mathcal{E}$, the imaginary axis to $\mathcal{B}$, and the size of the blob indicating the uncertainties in these field strengths. The time evolution of $|\alpha\rangle$ corresponds to the rotation of the blob around the origin at a frequency ω. The total energy of the light field, which corresponds to

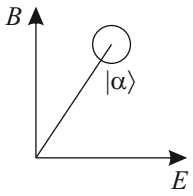


Fig. 2.1 Representation of the coherent state $|\alpha\rangle$ on an Argand diagram depicting the electric and magnetic fields. The state is represented by a blob centred on the value α, with the blob indicating the intrinsic uncertainty in the electric and magnetic field strengths.

the photon number, remains fixed, but the energy oscillates between electric and magnetic fields.

Note that the size of the fields depends on the magnitude of α, and as the energy is quadratic in the field strengths the average photon number should depend on $|\alpha|^2$ as previously calculated. The Poisson distribution of photon number means that in the limit of large values of $|\alpha|$ coherent states tend towards classical light waves with a well defined intensity, such as the output of a laser. For small values of $|\alpha|$, however, the quantised nature is far more visible.

2.3 Travelling wavepackets

So far we have only considered standing waves in cavities: although we have described a range of different states of the light field, these are all superpositions or mixtures of number states of a single mode, and so all have the same spatial pattern. This can provide a good model of simple light fields within cavities, but does not immediately resemble light in, for example, a laser beam.

To describe such cases it is necessary to consider superpositions of different modes. As each mode will evolve at its own characteristic frequency such superpositions will evolve in time, and it is simple to construct, for example, an excitation that will oscillate from end to end within a cavity. The possible mode frequencies are given by $\omega = \kappa\pi c/L$, where κ is a positive integer, and so for a long cavity the modes will be very finely spaced. Light in free space can be conveniently treated as occupying an infinitely long cavity, giving rise to a continuous spectrum, and so it is possible to construct, for example, Gaussian wavepackets.

Any travelling wavepacket will inevitably have a spread of frequencies, and thus an uncertainty in the energy of individual photons, but this is nothing more than the conventional time-frequency (and thus time-energy) Fourier relationship. This uncertainty can be made arbitrarily small by choosing a light source with a very long coherence time (a very sharp frequency), at the cost of losing location information within the beam. Fundamentally this changes very little in the analyses above, and from now on we will simply ignore it.

Further reading

Coherent states are described in all books on quantum optics, such as Fox (2006), Vedral (2005), and Walls and Milburn (2010). The approach can also be used with a ordinary quantum harmonic oscillator as discussed in Lvovsky (2018).

Exercises

2.1 Prove the value of $\bar{N}$ for a thermal state shown in equation 2.5. You should use the formula for the infinite sum of a convergent arithmetico-geometric progression

$$\sum_{k=1}^{\infty} k\, r^k = \frac{r}{(1-r)^2}.$$

2.2 Prove the value of σ_N^2 for a thermal state shown in equation 2.6. You should use the formula

$$\sum_{k=1}^{\infty} k^2\, r^k = \frac{r(1+r)}{(1-r)^3}.$$

2.3 Show that for a coherent state the expectation value of $\hat{N}^2$ is given by

$$\langle\alpha|\hat{N}^2|\alpha\rangle = |\alpha|^4 + |\alpha|^2,$$

and hence prove equation 2.11.

3 Operator methods

Although we have adopted a second quantized representation, in which we describe a system by the number of excitations in each mode rather than by the states of individual particles (a concept which makes little sense for identical bosons in the same mode), we are still describing systems in terms of their quantum states. It can be more useful to take another step of abstraction and concentrate on the operators corresponding to these states rather than the states themselves.

This is most easily demonstrated for number states, which are integer excitations from the vacuum state. Recalling the elementary property of the creation operator

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \tag{3.1}$$

we can write

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \tag{3.2}$$

where $|0\rangle$ is the vacuum state for the mode under consideration. We can then describe another mode (which in a cavity could be another allowed frequency, but in free space could simply be a beam travelling in a different direction) by using a different raising operator, say $b^\dagger$. Thus we can describe a state with one photon in each mode as

$$|1\rangle_a|1\rangle_b = a^\dagger|0\rangle_a b^\dagger|0\rangle_b, \tag{3.3}$$

where we have drawn a careful distinction between the two vacuum states. But this notation becomes incomplete if other modes could be present (we have failed to specify that there are 0 photons in some other mode, c), or hopelessly baroque if all potentially excited modes are listed. For this reason we simply write $|0\rangle$ to stand for the vacuum state of *every* mode, and so can write

$$|1\rangle_a|1\rangle_b = a^\dagger b^\dagger|0\rangle. \tag{3.4}$$

As the vacuum state will be present on the right hand side of every expression it carries no significance, and any number state is simply equivalent to the corresponding set of creation operators. For excitations above $|1\rangle$ it is strictly necessary to ensure that a state is correctly normalised, but in practice this can often be neglected.

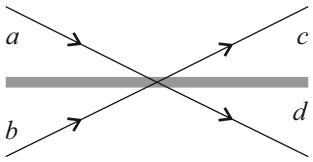


Fig. 3.1 Input modes, a and b, and output modes, c and d, at a 50:50 beam splitter.

3.1 Beam splitters

Now we consider what happens when two photons impinge on a 50:50 beam splitter. We treat the beam splitter as having upper and lower input modes, a and b, and corresponding output modes, c and d. The action on photons in mode a can be summarised in the wavefunction description as

$$|1\rangle_a \rightarrow \left(|1\rangle_c + |1\rangle_d\right)/\sqrt{2}, \tag{3.5}$$

while on action on mode b is

$$|1\rangle_b \rightarrow \left(|1\rangle_c - |1\rangle_d\right)/\sqrt{2}. \tag{3.6}$$

The phase difference between the actions on the two modes is sometimes ascribed to different phase shifts on reflection for different paths through the physical beam splitter, but its origin is far more fundamental, simply reflecting the unitary nature of the process. Quantum states representing the different input modes, a and b, are orthogonal, and so the output states must also be orthogonal, and this can only be achieved for equally weighted superpositions if there is a phase shift of π between the relative phases of modes c and d in the two output superpositions. The actual phases will depend on details of construction, but any device must ultimately be equivalent to the actions listed above.

In the operator description the action of the beamsplitter can be rewritten as

$$a^\dagger|0\rangle \rightarrow \left(c^\dagger|0\rangle + d^\dagger|0\rangle\right)/\sqrt{2}, \quad b^\dagger|0\rangle \rightarrow \left(c^\dagger|0\rangle - d^\dagger|0\rangle\right)/\sqrt{2}, \tag{3.7}$$

but as previously noted we can neglect the common $|0\rangle$ term, and so write this as

$$a^\dagger \rightarrow \frac{c^\dagger + d^\dagger}{\sqrt{2}}, \quad b^\dagger \rightarrow \frac{c^\dagger - d^\dagger}{\sqrt{2}}. \tag{3.8}$$

Some authors also drop the normalisation at this stage, but here we leave it in.

Now we consider what happens when two photons enter the two input modes, a and b, simultaneously. If the two photons are distinguishable then nothing special should occur, but if the photons are genuinely indistinguishable then interference effects have to be included. Even if the two photons have the same frequency and arrive at exactly the same time we can still use polarisation as a useful label: if the two photons are orthogonally

polarised, say H and V, then they should behave as distinguishable particles, but if they have the same polarisation, say H, then interference can occur.

We begin by analysing the case of two distinguishable polarisations, with the initial state

$$a_H^\dagger b_V^\dagger |0\rangle, \tag{3.9}$$

and apply the beam splitter transformations in equation 3.8, to get

$$\left(\frac{c_H^\dagger + d_H^\dagger}{\sqrt{2}}\right)\left(\frac{c_V^\dagger - d_V^\dagger}{\sqrt{2}}\right)|0\rangle = \left(\frac{c_H^\dagger c_V^\dagger + d_H^\dagger c_V^\dagger - c_H^\dagger d_V^\dagger - d_H^\dagger d_V^\dagger}{2}\right)|0\rangle. \tag{3.10}$$

Squaring the amplitudes of the four output states we see that there is a 25% chance of each of four distinguishable outcomes: both photons leaving in the upper arm, one photon leaving in each arm with the vertical photon at the top, one photon leaving in each arm with the horizontal photon at the top, and both photons leaving in the lower arm. This is what one would naively expect, with each of the four distinguishable outcomes happening with equal probability.

If the two photons have the same polarisation, however, they become indistinguishable and now the outcome is quite different. Following through

$$a_H^\dagger b_H^\dagger |0\rangle \rightarrow \left(\frac{c_H^\dagger + d_H^\dagger}{\sqrt{2}}\right)\left(\frac{c_H^\dagger - d_H^\dagger}{\sqrt{2}}\right)|0\rangle = \left(\frac{c_H^\dagger c_H^\dagger + d_H^\dagger c_H^\dagger - c_H^\dagger d_H^\dagger - d_H^\dagger d_H^\dagger}{2}\right)|0\rangle, \tag{3.11}$$

but as operators on different modes commute, so that $c_H^\dagger d_H^\dagger = d_H^\dagger c_H^\dagger$, the middle two terms cancel, leaving only

$$\left(\frac{c_H^\dagger c_H^\dagger - d_H^\dagger d_H^\dagger}{2}\right)|0\rangle, \tag{3.12}$$

and only two outcomes are now possible: either the photons both leave in the upper arm, or both leave in the lower arm. Note that the polarisation now plays no role in the calculation, and can simply be dropped if preferred.

It might seem that the normalisation in equation 3.12 is wrong, but it is important to include the normalisation in equation 3.2 correctly. Acting with the creation operators on the vacuum state gives

$$\frac{\sqrt{2}|2\rangle_c - \sqrt{2}|2\rangle_d}{2} = \frac{|2\rangle_c - |2\rangle_d}{\sqrt{2}}, \tag{3.13}$$

and so the final state is indeed correctly normalised.

This effect, in which the two identical photons always leave the beam splitter in the same mode, is called the Hong–Ou–Mandel effect, and is the simplest example of *bunching*, in which identical photons (or bosons in general) tend to leave a beam splitter together. It is easily observed by placing a photon detector in each arm and looking for *coincidences*, where both detectors detect a photon at the same time. This experiment has the advantage that it does not require photon number resolving detectors, which can distinguish between 1 and 2 photons in the same mode, but only simple detectors capable of distinguishing between no photons and some photons. If the set up is perfect then photons with different polarisations will lead to coincident clicks 50% of the time and clicks in only one detector

50% of the time, split equally between the upper and lower detectors, while photons with the same polarisation should give no coincidences. More generally the lack of coincidences can be used as a measure of the indistinguishability of the two photons.

For the Hong–Ou–Mandel effect to be seen it is vital that the two photons arrive at the detector at the same time, so that their wavepackets overlap. If the two photons arrive at different times then the two photons are distinguishable, and each photon will be found in the upper or lower arm with independent probabilities of 50%. Experimentally this can be ensured by putting variable delays before the two input arms, and adjusting the relative delay to maximise the Hong–Ou–Mandel dip that occurs when the two photons overlap. The depth of the dip indicates the indistinguishability of the two photons, while the shape and width of the dip depends on the shapes of the photon wavepackets. In effect a single spatial mode can be split into multiple temporal modes as long as these are split by times large compared with the temporal width of the photon's wavepacket.

3.2 Coherent states

Coherent states can also be described in a similar language by using the displacement operator,

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \tag{3.14}$$

which can be shown to displace the vacuum state in the complex plane to produce a coherent state,

$$D(\alpha)|0\rangle = |\alpha\rangle. \tag{3.15}$$

Note that the term inside the exponential in the definition of D is anti-Hermitian, and so D must be unitary.

To make further progress we must derive some identities. First we note that although a and $a^\dagger$ do not commute, their commutator is just a number and so if we write

$$D = \mathrm{e}^{A+B}, \quad A = \alpha a^\dagger \quad B = -\alpha^* a, \tag{3.16}$$

then since

$$[A,[A,B]] = [B,[A,B]] = 0 \tag{3.17}$$

it follows that D can be rewritten as a finite product[1] of exponentials,

$$D = \mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B \mathrm{e}^{-[A,B]/2}. \tag{3.18}$$

Reinserting the values from equation 3.18 gives

$$D(\alpha) = e^{-|\alpha|^2/2} \mathrm{e}^{\alpha a^\dagger} \mathrm{e}^{-\alpha^* a}, \tag{3.19}$$

[1] In general the exponential of the sum of two non-commuting operators must be expanded as an infinite product of successive commutators, but as explored in the problems the property in equation 3.17 means that the product terminates after only three terms.

and acting on the vacuum state gives

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{3.20}$$

as before.

As D is unitary its adjoint is equal to its inverse, and by direct calculation from equation 3.14 it follows that

$$D^{-1}(\alpha) = D^\dagger(\alpha) = D(-\alpha), \tag{3.21}$$

and so

$$D(-\alpha)|\alpha\rangle = D^\dagger(\alpha)D(\alpha)|0\rangle = |0\rangle. \tag{3.22}$$

While one might be tempted to believe that displacements can simply be combined by adding the corresponding values, this is not quite true. Clearly

$$D(\alpha+\beta) = \exp\left((\alpha+\beta)\,a^\dagger - (\alpha^*+\beta^*)\,a\right) = \exp\left((\alpha a^\dagger - \alpha^* a) + (\beta a^\dagger - \beta^* a)\right) \tag{3.23}$$

and as the commutator of the two terms is just a number,

$$[\alpha a^\dagger - \alpha^* a, \beta a^\dagger - \beta^* a] = -\alpha\beta^*[a^\dagger, a] - \alpha^*\beta[a, a^\dagger] = \alpha\beta^* - \alpha^*\beta = 2\mathrm{i}\,\mathrm{Im}(\alpha\beta^*) \tag{3.24}$$

we can write the combined displacement using equation 3.18 to get

$$D(\alpha+\beta) = D(\alpha)D(\beta)\mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(\alpha\beta^*)}. \tag{3.25}$$

Thus in addition to adding the displacement parameters it is necessary to correct for any relative phase.

The result can be used to find the scalar product between two coherent states. We have

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(\beta)D(\alpha)|0\rangle = \langle 0|D(-\beta)D(\alpha)|0\rangle, \tag{3.26}$$

and using equation 3.25 this can be rewritten as

$$\langle\beta|\alpha\rangle = \langle 0|D(\alpha-\beta)|0\rangle\mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(\alpha^*\beta)}. \tag{3.27}$$

The square modulus of this scalar product is just the probability of finding 0 photons in the coherent state $|\alpha-\beta\rangle$, and so

$$|\langle\beta|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}. \tag{3.28}$$

Note that none of the coherent states are fully orthogonal to any other coherent state, although any two states become approximately orthogonal when α and β are very different. The coherent states form an overcomplete continuous complex valued basis, in contradistinction to the number states which from a complete integer valued basis. As before this ultimately reflects the fact that they are not eigenstates of a Hermitian operator.

The displacement operator also gives a simple method for calculating what happens when a coherent state strikes a beam splitter. If a coherent state $|\alpha\rangle$ arrives on input port a of the beam splitter then the initial state can be written as

$$|\psi_i\rangle = |\alpha\rangle_a = D_a(\alpha)|0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle. \tag{3.29}$$

The effect of the beam splitter is to replace $a^\dagger$ by $(c^\dagger + d^\dagger)/\sqrt{2}$ as before, and so the state after the beam splitter is

$$|\psi_f\rangle = \exp\left(\alpha(c^\dagger + d^\dagger)/\sqrt{2} - \alpha^*(c + d)/\sqrt{2}\right)|0\rangle \tag{3.30}$$

but since operators on different modes commute the exponential can be written as the product of two terms

$$\exp\left(\alpha c^\dagger/\sqrt{2} - \alpha^* c/\sqrt{2}\right) \times \exp\left(\alpha d^\dagger/\sqrt{2} - \alpha^* d/\sqrt{2}\right) \tag{3.31}$$

and so

$$|\psi_f\rangle = D_c(\alpha')D_c(\alpha')|0\rangle = |\alpha'\rangle_c|\alpha'\rangle_d \tag{3.32}$$

with $\alpha' = \alpha/\sqrt{2}$. The beam splitter splits a coherent state into a product of two coherent states, one in each output arm, with the amplitude, and thus the light intensity, split equally between the two outputs.

3.3 Squeezed light

As discussed in the previous chapter, the coherent states all have the same standard deviations for the electric and magnetic field strengths, with no dependence on α, and in natural units these two uncertainties are the same. It can be shown that the product of these two standard deviations is limited by an uncertainty relation, and the coherent states achieve this minimum uncertainty bound.

There are, therefore, no states which simultaneously have lower uncertainties in both the electric and magnetic fields, but it is possible to prepare states with lower uncertainties in one at the cost of higher uncertainties in the other. States which achieve an uncertainty in one field strength below that in coherent states are called squeezed states, whether or not these states achieve the minimum bound on the product of the uncertainties.

Squeezed coherent states, which do achieve the minimum uncertainty product, are characterised by two parameters, and can be though of as being prepared by first squeezing the vacuum state and then displacing the result:

$$|\alpha, \epsilon\rangle = D(\alpha)S(\epsilon)|0\rangle, \tag{3.33}$$

where the squeeze operator

$$S(\epsilon) = \exp\left(\tfrac{1}{2}\epsilon^* a^2 - \tfrac{1}{2}\epsilon a^{\dagger 2}\right) \tag{3.34}$$

is unitary. The resulting state is centred at α, but the uncertainties in the electric and magnetic fields are now described by an ellipse rather than a circle. Like α the squeezing parameter ϵ is a complex number in general, whose magnitude indicates the degree of squeezing and whose phase determines the direction. It is common, however, to consider only real values of ϵ, which correspond to squeezing either the real component (electric field) or the imaginary component (magnetic field) depending on the sign of ϵ.

Instead of squeezing the real or imaginary components one can consider squeezing the

magnitude or the phase. States with a squeezed magnitude have a better determined photon number than coherent states, at the cost of increased phase uncertainty, and exhibit sub-Poissonian photon number statistics. The limiting example of such squeezing is provided by the number states, which have perfectly defined photon numbers and completely undefined phases. As discussed above the number states have on average no electric or magnetic field strength accompanied by an enhanced uncertainty in both values.

Squeezed states have important applications in precision measurement, for example in improving the sensitivity of interferometers. Analogous methods can also be applied to atomic states, with important applications in atomic clocks. However we will not consider the generation or detection of squeezed states further.

Further reading

The original Hong–Ou–Mandel paper, Hong et al. (1987), is a fairly easy read. A more sophisticated treatment of operator methods for analysing coherent and squeezed states can be found in Walls and Milburn (2010).

Exercises

3.1 Write down an operator which creates a photon in mode b which is polarised at an angle θ, using a sum of the previous creation operators $b_H^\dagger$ and $b_V^\dagger$. Hence calculate the probability of observing a coincidence between this photon and an otherwise identical photon in mode a with polarisation H when they are simultaneously incident on a beam splitter.

3.2 Use the Baker–Campbell–Hausdorff formula

$$\exp(X)\exp(Y) = \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] + \tfrac{1}{12}[[X,Y],Y] + \ldots\right)$$

to justify 3.19 and hence derive equation 3.20.

3.3 Prove equation 3.21.

3.4 Use the linear series approximation to an operator exponential to evaluate $S(\epsilon)|0\rangle$ for small real values of ϵ. Hence find expressions for $\sigma_{\mathcal{E}}$ and $\sigma_{\mathcal{B}}$, and show that their product is unchanged in this limit. (Take care to avoid evaluating terms that will not contribute at first order.)

4 Light–matter interactions

Having described a quantum light field in some detail we now switch to the interaction between this light field and an atom or ion.

The semi-classical treatment of such systems is well known, and was covered in detail in the first part of this course. The atom can be treated as a two level system, with a ground state $|g\rangle$ and an excited state $|e\rangle$, separated by an energy gap $\hbar\omega_0$, with the zero of the energy scale conveniently placed between the two levels. The light is treated as an electric field oscillating at a frequency ω. This interacts with the instantaneous dipole moment of the atom, leading to a Hamiltonian

$$\mathcal{H} = \hbar\omega_0 \frac{\sigma_z}{2} + \hbar\Omega \left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right) \frac{\sigma_x}{2} \tag{4.1}$$

where Ω, the Rabi frequency, parameterises the strength of the interaction and a cosinusoidal oscillation of the electric field is written as the sum of two counter-rotating terms. If the light is near resonance, so $\omega \approx \omega_0$, then it is convenient to work in an interaction representation, so that the interaction term rotating in the right direction is static, while the other term now varies at 2ω. The rotating wave approximation consists in neglecting this rapidly varying term, leaving a constant Hamiltonian

$$\mathcal{H}_i = \frac{\hbar}{2} \begin{pmatrix} \delta & \Omega \\ \Omega & \delta \end{pmatrix} \tag{4.2}$$

which is straightforward to analyse for fixed values of Ω and δ. Note that the two-level atom approximation will be valid as long as the light field is close to resonance with the transition of interest and far from resonance with all other transitions in the atom.

A fully quantum mechanical treatment will be very different, as the electric field will be described by operators rather than by numbers. We begin by assuming a two-level atom in a cavity, with the light field described as a quantum harmonic oscillator.

4.1 The Jaynes–Cummings Hamiltonian

The Jaynes–Cummings Hamiltonian for the interaction between a two level atom and a quantum light field starts by writing out the Hamiltonian for the atom and light field explicitly (the so-called free Hamiltonian) and then quantising the interaction term in the obvious way. By tradition the energy zeros for the atom and the light field are set to states $|g\rangle$ and $|0\rangle$ respectively: while this differs from the semi-classical treatment above this is

of no fundamental significance as only energy differences are physically meaningful. The Hamiltonian then takes the form

$$\mathcal{H} = \hbar\omega_0 \, |e\rangle\langle e| + \hbar\omega \, a^\dagger a + \hbar\frac{\Omega}{2}\,\sigma_x\left(a + a^\dagger\right) \tag{4.3}$$

noting that

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e| = \sigma^+ + \sigma^- \tag{4.4}$$

is a combination of raising (σ^+) and lowering (σ^-) operators on the two level atom. These raising and lowering operators are not Hermitian, and do not have a complete set of eigenvectors. The interaction strength has been parameterised so that Ω will have a convenient interpretation.

To make further progress we write out this Hamiltonian explicitly in a basis corresponding to the product of the two states of the atom and the number states of the field. In the absence of the interaction the Hamiltonian is diagonal

$$\mathcal{H}_0 = \hbar\omega_0 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \hbar\omega \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.5}$$

combining the atomic excitation energy term $\hbar\omega_0$ with an infinite ladder of light field number states, separated by $\hbar\omega$. The interaction is purely off-diagonal:

$$\mathcal{H}_1 = \frac{\hbar\Omega}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \sqrt{2} & \cdots \\ 1 & 0 & 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.6}$$

Note that these off-diagonal terms can only connect number states differing by precisely one photon, and the $\sqrt{n}$ dependence reflects the fact that $a|n\rangle = \sqrt{n}|n-1\rangle$ as usual.

To put these two components together we will use a truncated harmonic oscillator basis for the light field, considering only the two lowest number states, $|0\rangle$ and $|1\rangle$ to get

$$\mathcal{H} = \hbar \begin{pmatrix} 0 & 0 & 0 & \Omega/2 \\ 0 & \omega_0 & \Omega/2 & 0 \\ 0 & \Omega/2 & \omega & 0 \\ \Omega/2 & 0 & 0 & \omega_0 + \omega \end{pmatrix}. \tag{4.7}$$

The interaction term will be small compared with the main diagonal terms, and so it is appropriate to consider a perturbation theory approach. If the light is approximately resonant

with the atomic transition, so $\omega \approx \omega_0$, the energy gap between the two middle terms in $\mathcal{H}_0$ will be far smaller than the gap between the first and last terms, and so the perturbation will be far more effective in connecting the middle terms than the outer ones. Indeed directly on resonance, where $\omega = \omega_0$, the two middle terms become degenerate, and the interaction term can no longer be treated with simple perturbation theory.

Because of this dominance of the mixing of the middle terms it is sufficient in practice to use a simplified Hamiltonian in which the outer two terms are simply dropped. We thus write the Jaynes–Cummings Hamiltonian as

$$\mathcal{H} = \hbar\omega_0 \left|e\right\rangle\!\left\langle e\right| + \hbar\omega\, a^\dagger a + \frac{\hbar\Omega}{2}\left(\sigma^+ a + \sigma^- a^\dagger\right). \tag{4.8}$$

The off-diagonal terms retained correspond to the excitation of the atom and the simultaneous absorption of a photon from the light field, and the de-excitation of the atom and the simultaneous emission of a photon into the light field. Retaining these two processes, which in the absence of the interaction are approximately energy conserving, while dropping the two processes which correspond to large changes in energy, is precisely analogous to making the rotating wave approximation in semi-classical treatments.

We have justified this approximation by appealing to a truncated harmonic oscillator basis, but it is clear that this approach extends nicely to the full basis. The retained terms connect nearly degenerate pairs of terms of the form $|e\rangle|n-1\rangle$ and $|g\rangle|n\rangle$, with interaction terms $\hbar\Omega\sqrt{n}/2$, and the evolutions is confined to these subspaces. Neglecting the other two terms will give a small error, but this easily shown by perturbation theory to be of the order of Ω/ω times smaller than the main interaction.

4.2 Cavity quantum electrodynamics

Cavity quantum electrodynamics (cavity QED) is the experimental study of the Jaynes–Cummings Hamiltonian. In its simplest form, a single atom is prepared in the excited state $|e\rangle$ and then made to pass through a small high quality optical cavity with a cavity mode resonant with the atomic transition to state $|g\rangle$. We will initially assume that the cavity has no light within it, and so can be described by the vacuum state $|0\rangle$.

When the atom is outside the cavity the interaction between the atom and the light field is negligible, and the initial state $|e\rangle|0\rangle$ is an eigenstate of the Hamiltonian. When the atom enters the cavity, however, the resonant light mode is tightly confined within the cavity, and its electric field, and thus its interaction strength, is correspondingly increased. Assuming exact resonance the Hamiltonian in the truncated harmonic oscillator basis has the form

$$\mathcal{H} = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega & \Omega/2 & 0 \\ 0 & \Omega/2 & \omega & 0 \\ 0 & 0 & 0 & 2\omega \end{pmatrix} \tag{4.9}$$

and the eigenstates and eigenvalues can simply be read off: the two outer states $|g\rangle|0\rangle$ and

$|e\rangle|1\rangle$ remain eigenstates, with energies of 0 and $2\hbar\omega$ respectively, but the inner states $|e\rangle|0\rangle$ and $|g\rangle|1\rangle$ are split to give *dressed states*

$$|\pm\rangle = \frac{|e\rangle|0\rangle \pm |g\rangle|1\rangle}{\sqrt{2}} \tag{4.10}$$

with energies $\hbar(\omega \pm \frac{1}{2}\Omega)$. The gap of $\hbar\Omega$ between these two (previously degenerate) levels is called the vacuum Rabi splitting.

The movement of the atom in and out of the cavity can be easily modelled using the sudden approximation. The atom enters the cavity in the state $|e\rangle|0\rangle$ but this is suddenly no longer an eigenstate, but instead an equal superposition of the eigenstates $|+\rangle$ and $|-\rangle$. Inside the cavity the new eigenstates pick up phases, and after a time t the state of the system is

$$\mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\mathrm{e}^{-\mathrm{i}\Omega t/2}|+\rangle + \mathrm{e}^{+\mathrm{i}\Omega t/2}|-\rangle}{\sqrt{2}}\right). \tag{4.11}$$

At this point the atom leaves the cavity and we can return to the original eigenbasis. If the state of the atom is measured after it leaves the cavity then

$$P_e(t) = \cos^2(\Omega t/2) = \tfrac{1}{2}(1 + \cos(\Omega t)), \tag{4.12}$$

that is the atom undergoes Rabi flopping. The evolution time t can be controlled by choosing the transverse speed of the atom as it enters the cavity. This way of controlling the state of the atom is fully coherent, and the actual state of the system before measurement is an entangled superposition of the internal states of the atom and the number states of the cavity field.

The situation is very similar if the cavity contains photons before the atom enters. If the initial state of the system is $|e\rangle|n\rangle$ then Rabi flopping will occur between this state and $|g\rangle|n+1\rangle$ with a Rabi frequency of $\sqrt{n+1}\Omega$. If the initial state is $|g\rangle|n\rangle$ then Rabi flopping will again occur, now between this state and $|e\rangle|n-1\rangle$ with a Rabi frequency of $\sqrt{n}\Omega$. This formula is consistent with the fact that $|g\rangle|0\rangle$ will not flop (there is no lower state to go to), as the predicted flopping frequency is 0.

It is interesting to note that the initial state $|e\rangle|n\rangle$ will flop faster than the state $|g\rangle|n\rangle$, reflecting the higher photon number states accessible. Equivalently a given number state will be more efficient at causing de-excitation (emission) than excitation (absorption), and in particular the vacuum state can only induce de-excitation. This observation indicates the origin of spontaneous emission, which we will explore in more detail later.

4.3 Coherent light

The generation of entanglement when atoms interact with number states in cavities is a very interesting result in its own right, but indicates a potential major problem for quantum control. If number states were used to implement logic gates in trapped ion quantum computers, then the resultant entanglement between the ion internal states and the control

would inevitably lead to rapid decoherence. To control a qubit reliably it is vital that the control fields do not become entangled with the system!

The solution is, of course, to use coherent states. We will not analyse the process in detail, but it suffices to note that while removing a photon from a number state or adding one to it produces a different, and therefore orthogonal, number state, this is not true for coherent states. Removing a photon from a coherent state leaves it unchanged, as $|\alpha\rangle$ is an eigenstate of the annihilation operator. This is not true for adding a photon, most obviously because $a^\dagger|\alpha\rangle$ cannot have any component in the vacuum state $|0\rangle$, but even so the resultant state is not orthogonal to the original, and when the magnitude of α is very large the difference becomes effectively negligible. The initial state will evolve into a superposition

$$|\psi\rangle \approx \Big(c_g(t)|g\rangle + c_e(t)|e\rangle\Big)\,|\alpha\rangle \tag{4.13}$$

where we can approximately factor the state into atom and field components, indicating the lack of significant entanglement.

The final question to consider is how the Rabi flopping rate depends on α. For number states we have seen that the flopping rate depends on $\sqrt{n}$, but coherent states are superpositions of number states and so different components would be expected to induce flopping at different rates. However, as discussed previously, the mean photon number for a coherent state is $\bar{N} = |\alpha|^2$, and the distribution is sharply peaked around this value, with $\sigma_N = \sqrt{\bar{N}}$, which for large values of $|\alpha|$ becomes negligible in comparison. For this reason we expect the flopping rate to be well defined for intense coherent states, and to be proportional to $|\alpha|$.

Further reading

The Jaynes–Cummings model is discussed in Vedral (2005), Fox (2006), and Walls and Milburn (2010). For a detailed discussion of cavity QED see Haroche and Raimond (2006) by two of the founders of the field.

Exercises

4.1 Complete the diagonalisation of the full Hamiltonian in the truncated basis, equation 4.7, assuming on-resonance excitation. Does your result mean that it is safe to completely neglect the off-diagonal terms dropped in the Jaynes–Cummings Hamiltonian?

4.2 Complete the calculation leading to equation 4.12, finding the quantum state before the measurement. Calculate the minimum time τ required to convert an atom in initially in state $|e\rangle$ into state $|g\rangle$.

4.3 Use matrix exponentials to find the propagator corresponding to equation 4.9 (make sure to use the structure of the matrix to simplify your calculation) and thus derive equation 4.12 directly.

4.4 Calculate the final state of the atom and the field after passing an atom through a cavity for time τ in the following cases: (a) the atom is initially in the superposition state $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle$ and the cavity is initally in state $|0\rangle$; (b) the atom is initially in state $|g\rangle$ and the cavity is in the state produced by (a).

5 The zero-point energy and fields

While many of the results discussed above might seem surprising, perhaps the most counterintuitive is the nature of the vacuum state $|0\rangle$, corresponding to a mode without any photons in it. In the absence of photons one might expect there to be no electric or magnetic fields, but while this is true on average there are still fluctuating fields, with standard deviations (in the usual dimensionless units) given by $\sigma_{\mathcal{E}} = \sigma_{\mathcal{B}} = 1$. This result can be deduced either from considering the properties of number states, or from the properties of coherent states, as the number state $|0\rangle$ is also the coherent state $|0\rangle$.

These fluctuating fields can have real physical consequences, some of which stem directly from the fields themselves, but some of which can be traced to the corresponding field energy. As these zero-point fields exist for every mode in the system their effects depend on the mode structure imposed by the cavity or other environment in which the atom is placed.

5.1 Spontaneous emission

The most direct result of these fields is spontaneous emission. In semi-classical treatments stimulated absorption and emission are induced by components of electric fields oscillating resonantly with an atomic transition, but as the name suggests spontaneous emission has no immediately obvious cause. Here quantum optics comes to the rescue, as the fluctuating zero-point fields provide a simple explanation of spontaneous transitions, while the absence of the corresponding spontaneous absorption process is easily explained by energy conservation.

Spontaneous emission occurring in the presence of stimulated processes is also easy to understand, as emission induced by a number state $|n\rangle$ occurs at a rate proportional to $\sqrt{n+1}$, while absorption occurs at a rate of $\sqrt{n}$. While this difference is most dramatic for the vacuum state $|0\rangle$, a corresponding effect occurs for any other number state, and by extension for coherent states.

It might be objected that these processes simply result in coherent Rabi flopping, rather than the exponential decay that characterises conventional spontaneous emission, but this is easily resolved by adding cavity loss processes. So far we have assumed that the optical cavity is perfect, but in reality no cavity is truly perfect and photons will always leak out, either to the sides or through imperfect reflection at the end mirrors. This leads to an exponential decay in any light fields within the cavity, and if the light decay is rapid in comparison with the Rabi flopping (weak coupling) then the whole process appears as a

simple decay of the excited state. In the case of strong coupling this decay is superimposed on an underlying flopping process.

Spontaneous emission in free space can be treated by first letting the cavity length go to infinity, and then assuming that any photon emitted is immediately lost and never interacts with the atom again. The infinite cavity length means that the electric field strength, and thus the Rabi flopping rate, from any individual mode is now infinitesimal, but there are now an infinite number of modes which are almost resonant with the transition, and integrating over this continuum once again leads to a finite result. The photon loss then acts as before, once again resulting in exponential decay.

Spontaneous decay is frequently enhanced in cavities, because the confined cavity modes are far more strongly coupled to the atomic levels than modes corresponding to travelling waves in free space. However if the cavity is small enough that the cavity modes are well separated then it can be arranged that none of these modes are anywhere near resonance with the atomic transitions. In such cases it can become almost impossible for an atom to spontaneously decay as the light modes it must emit into are completely suppressed by the cavity.

5.2 The Casimir effect

The Casimir effect arises from the energy of the zero-point fields: the energy of an electromagnetic fields depends on the squared of the electric and magnetic field strengths, and so a randomly fluctuating field will have a positive energy. This energy is familiar as the zero-point energy of the quantum harmonic oscillator, $E_0 = \frac{1}{2}\hbar\omega$. When applied to a single cavity mode there is nothing particularly new about this in quantum optics, and since only energy differences are physically meaningful it is possible to simply shift the origin of the energy scale so that this zero-point energy effectively disappears.

Care must be taken, however, because a cavity can support multiple modes: if a mode is supported at ω then another mode will exist at 2ω, and in general at every integer multiple $n\omega$. This sharpens concerns about the zero-point energy, as every mode will have its own zero-point energy $\frac{1}{2}n\hbar\omega$, and there are naively an infinite number of such modes, with ever increasing zero-point energies. As this energy cannot apparently be accessed in any way (it is not possible to take a quantum system below its ground state) does it exist in any meaningful sense? The divergence as n tends to infinity can be sidestepped by noting that there will be some fundamental limit, imposed by the Planck scale if nothing else, beyond which higher frequency modes are no longer possible but this still leaves room for an awfully large quantity of inaccessible energy. The situation is even worse in free space, where the spacing between adjacent modes becomes infinitesimal, restoring the troubling infinite divergence.

The zero-point energy should, however, have a simple consequence in regions of space confined between two flat conducting plates, corresponding to a cavity. In this case the region *between* the plates has a lowest permitted frequency, with all frequencies below this cut off, and so the zero-point energy density is lower between the plates than outside them.

If the plates are brought closer together then the low-frequency cutoff is raised to higher energies, and so the energy density is reduced still further. This leads to a force which acts to push the two plates together.

Calculating the Casimir force is subtle, as it is necessary to consider the difference between two diverging series, and the precise calculation depends on the geometry chosen. The force can also be derived without explicitly invoking quantum fluctuations, and has been claimed to be simply a relativistic correction to the van der Waals forces between any two polarizable objects. An analogy is sometimes drawn with interaction forces that can occur between two ships in the presence of waves, where the suppression of waves in the region between the ships acts to suck the ships together, but once again it is easy to confuse such forces with other more direct interactions between moving ships.

5.3 The Lamb shift

The Lamb shift refers to a very small difference between the energies of the 2s $^2S_{1/2}$ and 2p $^2P_{1/2}$ levels in the hydrogen atom. The 2p $^2P_{1/2}$ level is split from the 2p $^2P_{3/2}$ level by spin–orbit coupling, but a careful calculation making two further relativistic corrections (the kinetic energy relativistic correction and the Darwin term) within first quantized quantum mechanics indicates that the two levels with $j = \frac{1}{2}$ should be degenerate, and the same conclusion can be reached more directly using Dirac's formulation of relativistic quantum mechanics, which indicates that the energy levels of the hydrogen atom should depend only on n and j. Within quantum electrodynamics, however, there is an additional interaction, whose principal effect is to raise the energies of s electrons, so that the 2s $^2S_{1/2}$ level lies above the 2p $^2P_{1/2}$ level.

This shift was first detected by Willis Lamb and his graduate student Robert Retherford who used microwave techniques to measure the direction transition between these nearly degenerate levels. The interpretation is limited by the existence of two other interactions of a broadly similar size: the nuclear volume correction, arising from the fact that the central proton is not truly a point charge, and the nuclear hyperfine interaction arising from the proton's spin. Although Lamb type shifts are present in all atomic systems it is only in hydrogenic atoms and ions that the spectrum can be both measured and calculated to sufficient precision that the effect can be unambiguously identified.

Further reading

Spontaneous decay and weak and strong coupling to cavity modes is explored in Fox (2006). The Casimir effect and the Lamb shift are discussed by Vedral (2005), and more detailed treatments of the Lamb shift in spectroscopy can be found in Foot (2005) and van der Straten and Metcalf (2016).

6 Geometric phases

In this final section we leave quantum optics and address a general feature in the quantum mechanics of discrete systems. Sometimes known as the geometric phase, the phenomenon is easily observed in a range of two level quantum systems, such as spin-$\frac{1}{2}$ particles and the polarization states of light. The phenomenon was greatly elucidated by Michael Berry, but precursors were discovered in many areas of science, most notably by Shivaramakrishnan Pancharatnam in optics and by Christopher Longuet-Higgins within theoretical chemistry.

6.1 Global phases

The conventional mathematical description of quantum states appears to contain an unnecessary *global phase*. Mathematically the state $|\psi_1\rangle$ can be written in the general form

$$e^{i\phi_1}|\psi_1\rangle \tag{6.1}$$

where the value of ϕ_1 has no physically detectable consequences and cannot be determined by any means at all. This global phase must be sharply distinguished from the relative phase in a superposition: in the general state of a single qubit

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle \tag{6.2}$$

the *relative phase* of α and β is a physically meaningful quantity, but the *absolute phases* have no meaning, as they can be changed by applying a global phase shift. This phenomenon is exploited in the standard Bloch sphere picture of the qubit, where we choose to make α real and positive.

Global phases cannot be measured because we have no direct access to the wave function itself, but only to the square modulus values for overlap integrals of the form

$$|\langle\psi_2|\psi_1\rangle|^2 = \langle\psi_2|\psi_1\rangle\langle\psi_1|\psi_2\rangle \tag{6.3}$$

where every ket $|\psi_1\rangle$ is matched by a corresponding bra $\langle\psi_1|$, and since equal and opposite global phase shifts are applied to the two contribution any global phase will always cancel.

Since global phases cannot be detected, why should we worry about them? Simply because some phases which might be thought of as global phases can in fact be turned into relative phases, and thus physically real. The traditional way to do this is via an interferometer experiment, in which a quantum state is impinged on a beam splitter, creating a superposition of two possible paths, which are subsequently recombined on a second beam

splitter. If a "global" phase is applied to the system in only one path then it becomes a relative phase which can be detected by interference effects. The same effect can be achieved by using gates to apply "global" phases to a target qubit conditional on the state of a control qubit: this is equivalent to the interferometer with the target qubit corresponding to the quantum system and the control qubit to the interferometer path. Simplest of all, an apparently global phase can take different values for different eigenstates of a system, and so be directly detectable as a relative phase. Once again this can be recast as an interferometry experiment, with the choice of eigenstate now corresponding to the path information.

6.2 Berry Phase

The Berry phases arises in situations where the quantum system's Hamiltonian is varied adiabatically, so that if the quantum system starts in a particular eigenstate of the original Hamiltonian then it will always remain in the corresponding eigenstate of the time varying Hamiltonian. A simple example is provided by a spin-$\frac{1}{2}$ particle in a magnetic field along the z-axis, for which the Hamiltonian $\mathcal{H}_z$ has the form

$$\mathcal{H}_z = \hbar\omega\sigma_z/2. \tag{6.4}$$

The spins states can be described by a qubit, and if the system starts in the eigenstate $|0_z\rangle$ then it evolves simply by acquiring phases

$$|0_z(t)\rangle = e^{-i\omega t/2}\,|0_z\rangle. \tag{6.5}$$

Now suppose the magnetic field is turned slowly (for example, by physically turning the magnet) from lying along the z-axis towards the x-axis. As long as the turning rate is very slow in comparison with ω the process will be *adiabatic*, and the final state of the system will be given by

$$e^{i\phi}\,|0_x\rangle \tag{6.6}$$

where $|0_x\rangle$ is the corresponding eigenstate of the new Hamiltonian $\mathcal{H}_x$, and ϕ is some phase, to be determined.

The obvious answer for the phase is to use the *dynamical phase* given by $\delta = -\omega\tau/2$, where τ is the time taken to turn the field. This answer superficially makes sense because the phase evolution of a fixed state depends only on its energy, here parameterised by ω, and the energy does not change as the field is rotated. However this ignores an important subtlety: how should one define the phase relationship between basis states in different directions? One cannot simply determine this phase experimentally, for example by interference, because states in different directions cannot be directly compared.

The solution to this problem is to sweep the field in two stages: first it is turned from z to x, and then it is turned back from x to z again. It now no-longer matters how the phase is defined at each point on the sweep: the final state is measured in the same basis as the initial state, and the resulting phase shift is well defined. Similarly the relative phase between basis states at two points on the sweep does not matter as any phase change on the

way out is simply reversed on the way back. In this situation the final phase depends only on the dynamical phase, with $\phi = 2\delta = -\omega\tau$. More generally we can write

$$|0_z(2\tau)\rangle = \exp\left(-i\int_0^{2\tau} E(t)/\hbar\,\mathrm{d}t\right)|0_z\rangle. \tag{6.7}$$

where we have allowed for the fact that the energy value, and hence ω, might also change adiabatically during the sweep.

That might seem to be the whole story, and for a long time this was believed to be the case, but considering more complex trajectories leads to a more complex result. Consider next rotating the field along a more complex three-step path: first from z towards x, then along the equator from x towards y, and finally back up from y towards z again. For this path

$$|0_z(3\tau)\rangle = \exp(i\gamma)\,\exp\left(-i\int_0^{3\tau} E(t)/\hbar\,\mathrm{d}t\right)|0_z\rangle. \tag{6.8}$$

where γ, the Berry phase, depends only on the geometry of the path, and not on how long it takes, and so is an example of a geometric phase.

Determining the value of γ requires a more careful consideration of the relative phases of basis states along the path. The simplest way to do this uses the concept of *parallel transport*, which adopts the obvious strategy that there should be no change in phase as one moves from one basis state to any immediately adjacent one, so that the *local* phase change is always zero. The subtlety here, however, is that the path followed by the system lies on the surface of a sphere, which has a fundamental global curvature that cannot be neglected, which means that local parallel transport can have non-trivial effects.

The traditional way to think about this is to model the reference phase at each point by a vector in a plane tangential to the sphere at that point. For simplicity we will start at the z axis, at the north pole of the sphere, and align the initial vector parallel with the direction of the first step of the sweep, pointing towards the x axis. Throughout this first stage we keep the reference vector pointing towards the x axis, that is always pointing south. In the second step, moving from x to y, we do not turn the vector but continue to keep the reference pointing due south. In the third and final stage of the sweep the reference still always points south, but now points towards the y axis. Thus when we get back to the north pole we find that although the reference vector has always been held locally fixed the overall effect is that it has turned through 90° through the whole process, reflecting the curvature of the surface.

The description above is purely classical, but a very similar conclusion applies to quantum systems. For a spin-$\frac{1}{2}$ particle with eigenstates labelled by $m_s = \pm\frac{1}{2}$ it turn out that the geometric phase is given by $\gamma = -m_s\Omega$, where Ω is the solid angle that the path subtends at the origin (the origin is important here because it is a singular point at which the two eigenstates are degenerate.) Note that for the first path we considered ($z \to x \to z$) the solid angle is zero, and so no geometric phase is seen, while for the second path, which surrounds one octant of the whole 4π surface angle of the sphere, the phase is $\gamma_{\pm 1/2} = \mp\pi/4$.

6.3 Pancharatnam phase and other precursors

Although Pancharatnam did not develop a systematic treatment of geometric phases his 1956 paper on interference between light beams with different polarization states prefigures some of the key results. Polarization states can be conveniently described using the Poincaré sphere, which is equivalent to the Bloch sphere used for spin-$\frac{1}{2}$ particles. This is not surprising as any two orthogonal polarization states, such as horizontal and vertical, or left and right circular polarization, form the basis for a qubit.

Pancharatnam was interested in overlap integrals of the form $\langle\psi_2|\psi_1\rangle$, but as previously noted the phase relationships between different states are not simple to define. The simplest way to avoid this is to perform measurements that return the square modulus of such overlap integrals, but Pancharatnam noticed that the problem could also be side stepped by combining overlap integrals in groups of three. In particular the product

$$\langle\psi_2|\psi_1\rangle\langle\psi_3|\psi_2\rangle\langle\psi_1|\psi_3\rangle \tag{6.9}$$

is unaffected by global phase changes as every state occurs as both a bra and a ket, with equal and opposite phases.

Another precursor is found in the 1975 work of Longuet–Higgins within theoretical chemistry, whose ideas were subsequently expanded by others, most notably by Mead and Truhlar in 1978 who were studying the chemical reaction between atomic and molecular hydrogen. The quantum mechanical description of this system of three identical particles is complicated by degeneracies, which can lead to a singular point in parameter space, analogous to the degeneracy that occurs for the spin states of a spin-$\frac{1}{2}$ particle at zero magnetic field.

6.4 The Aharonov–Bohm effect

Perhaps the most important precursor of Berry's phase is the Aharonov–Bohm effect, which they described in 1959, although the effect was actually first discovered by Ehrenberg and Siday in 1949. Aharonov and Bohm considered an electron interferometer experiment in which the two electron paths pass on opposite sides of a long tightly-wound solenoid coil. If a current flows through the coil then a large magnetic field will be generated inside the coil, but the field outside the coil, where the electron paths lie, will be negligible. Surprisingly it can be shown that the interference pattern is sensitive to the magnetic potential, and so depends on the current in the coil even in the absence of detectable fields.

The effect arises because the location of the solenoid acts as a potential singularity, but now the size of the singularity depends on the magnetic potential, and the two paths that pass on each side of the singularity develop a consequent phase difference. In this case even the geometry of the two paths does not matter, but simply the fact that they pass on opposite sides of the singularity. This is most easily seen by considering a loop path, corresponding to travelling backwards along one path and then forwards along the other,

returning to the starting point. This path of necessity winds around the singularity, and the *winding number* (the number of times it loops round the singularity before returning to the original position) is a topological invariant.

6.5 Subsequent developments

Since Berry's seminal papers which pulled these disparate ideas together, geometric phases have been identified, generalised, and used in many different areas of physics. In particular Aharonov and Anandan realised that the restriction to adiabatic sweeps in Berry's origonal paper was unnecessary and an equivalent geometric phase occurs in any situation in which a quantum state was moved smoothly (but not necessarily adiabatically) along some closed path. In particular if a qubit initially in state $|0\rangle$ is moved along some path subtending a solid angle Ω on the Bloch sphere then it acquires a geometric phase $\frac{1}{2}\Omega$, while a qubit initially in state $|1\rangle$ which experiences the same sequence of operations will follow a related path and will acquire the opposite geometric phase $-\frac{1}{2}\Omega$. By linearity a qubit in a general superposition state will acquire a relative phase of Ω between the two components of the superposition, which is equivalent to a rotation around the z-axis of the Bloch sphere. Note that here the relevant angle is defined by the path of the quantum state on the Bloch sphere, and not by the motion of the Hamiltonian driving the system, which can change discontinuously.

The Aharonov–Anandan phase can provide a convenient way of implementing single-qubit z rotation gates in quantum information technologies, such as NMR, where the natural gate set is limited to rotations around axes in the xy plane. Applying a π rotation around the x axis followed by a $-\pi$ rotation around an axis in the xy plane at an angle of ϕ from the x axis drives a qubit in state $|0\rangle$ along an "orange segment" path which subtends a solid angle of $\Omega = 2\phi$. (Note that a $-\pi$ rotation around an axis at an angle of ϕ is easily implemented using a π rotation around an axis at an angle of $\phi + \pi$.) On a general state this pair of pulses thus implements a $2\phi_z$ rotation, as is easily proved by direct calculation.

A special case of the above arises when the quantum state $|0\rangle$ is driven along a great circle going all the way around the Bloch sphere, for example by applying two NOT gates (π_x rotations) in succession. In this case the solid angle subtended is $\Omega = 2\pi$, and the geometric phase is π for $|0\rangle$ and $-\pi$ for $|1\rangle$. Since $e^{i\pi} = e^{-i\pi} = -1$ the geometric phase is now a truly global phase, and was seen previously as an example of the spinor behaviour of a qubit under a 2π rotation.

As described above the Aharonov–Anandan phase is used to perform single-qubit logic gates in some implementations of quantum information processing, and the controlled application of spinor behaviour is used in the Cirac–Zoller implementation of two-qubit logic gates in ion traps. The use of geometric phases to generate controlled-logic has also been explored within NMR. However for quantum information processing a much more interesting idea is to use topological phases, analogous to the Aharonov–Bohm effect. Such topological phases can in principle be completely insensitive to both the dynamics and geometry of the path implementing, providing exceptional robustness, but although these

ideas seem theoretically very promising their experimental implementation has proved extremely challenging.

Further reading

Geometric phases are described briefly in many texts on quantum mechanics or quantum optics. Berry (1988) provides a gentle introduction, while an extensive collection of key theoretical and experimental papers can be found in Shapere and Wilczek (1989). Controlled geometric phases in NMR are discussed in Jones et al. (2000).

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