Lectures on Ordinary Differential Equations

(Oxford Physics Papers CP3/4)

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These are the notes for my lectures on Ordinary Differential Equations for 1st-year undergraduate physicists, taught since 2018 as part of Paper CP3 at Oxford. They also include lectures on Normal Modes (part of Paper CP4), taught sunce 2021. I will be grateful for any feedback from students, tutors or (critical) sympathisers. The latest, up-to-date version of this file is available at `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/ODE/2018/ODELectureNotes.pdf`.

As typos, errors, and better ways of explaining things are found, I will update these notes in real time as lectures proceed. Please check back here for updates (the date tag of the version that you are reading appears above).

Below I give references to some relevant bits of various books and online lecture notes. Having access to these materials is not essential for being able to follow these lectures, but reading (these or other) books above and beyond your lecturer's notes is essential for you to be able to claim that you have received an education.

The problem sets suggested for your tutorials are at the end of this document. They too will be updated in real time if errors are found or new/better questions arise.

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> To play the good family doctor who warns about reading something prematurely, simply because it would be premature for him his whole life long—I'm not the man for that. And I find nothing more tactless and brutal than constantly trying to nail talented youth down to its "immaturity," with every other sentence a "that's nothing for you yet." Let him be the judge of that! Let him keep an eye out for how he manages.
>
> Thomas Mann, *Doctor Faustus*

1. The Language of the Game

It might not be an exaggeration to say that the ability of physical theory to make predictions and, consequently, both what we proudly call our "understanding of the world" and our technological civilisation hinge on our ability to solve differential equations—or, at least, to write them down and make intelligent progress in describing their solutions in qualitative, asymptotic or numerical terms. Thus, what you are about to start learning may well be the most "practically" important bit of mathematics for you as physicists—so be excited!

I shall start by introducing some terms and notions that will form the language that we will speak and by stating some general results that enable us to speak it meaningfully. Then I shall move on to methods for solving and analysing various kinds of differential equations.

1.1. *What is an ODE?*

[Literature: Pontryagin (1962, §1), Tikhonov *et al.* (1985, §§1.1-1.2), Arnold (2006, §§1.1-1.10), Yeomans (2014, §II.0), Tenenbaum & Pollard (1986, §3), Bender & Orszag (1999, §1.1)]

A *differential equation* is an equation in which the unknowns are functions of one or more variables and which contains both these functions and their derivatives. Physically, it is a relationship between quantities and their rates of change—in time, space or whatever other "independent variable" we care to introduce.

If more than one independent variable is involved, the differential equation is called a *partial differential equation (PDE)*. Those will not concern me—you will encounter them next term, in CP4, and also, in a proper way, in the second-year Mathematical Methods (Eßler 2009; Magorrian 2017; Lukas 2019).

It makes sense that this should happen after you have—hopefully—learned from me how to handle *ordinary differential equations (ODEs)*, which are differential equations that involve functions of a single independent variable. They have the general form

$$\boxed{F(x, y, y', \ldots, y^{(n)}) = 0}\,, \tag{1.1}$$

where x is the independent variable, y is the function (or "dependent variable"[1]) and y', y'', …, $y^{(n)}$ are its first, second, …, nth derivatives.

The *order* of an ODE is the order of the highest derivative that it contains.

There can be more than one function involved: $y_1(x)$, $y_2(x)$, …, in which case there would have to be more than one equation relating them and their derivatives (see §3.3).

I will denote the independent variable x, but also often t, when I particularly want to think of my ODE as describing the evolution of some physical quantity with time (or

[1] We shall see in what follows that what is an "independent" and what is a "dependent" variable can be negotiable as we manipulate ODEs. If y is a function of x, x is a function of y. See §1.4.



FIGURE 1. Gottfried Wilhelm von Leibniz (1646-1716), great German philosopher, inventor of calculus and of monads (intrigued? look them up!), Newton's (Fig. 11) *bête noire*.

just when I feel like it). A derivative is the instantaneous rate of change of a quantity with time (or the local rate of its change in space, or etc.). It is denoted, by convention,

$$\frac{\mathrm{d}y}{\mathrm{d}t} \equiv \dot{y} \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \equiv y'. \tag{1.2}$$

The tall-fraction notation, introduced by Leibniz (Fig. 1), underscores the derivative's meaning as the change in y per change in t (or x) and also signals that derivatives can (with caution) be handled as fractions, e.g.,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y'} \tag{1.3}$$

is the derivative of x with respect to y.

Example. Perhaps the most famous and most important ODE in the world is

$$\dot{y} = ay, \tag{1.4}$$

where a is some constant. It describes a situation in which the rate of growth (when a is real and positive) or decay (when $a < 0$) of some quantity is proportional to the quantity itself, e.g., the growth of a population whose rate of increase is proportional to the number of individuals in it (i.e., a monogamous population; what is the ODE for a polygamous population?). This equation is *first-order*. It is also *linear*.

Generally, an ODE is *linear* if it has the form

$$\boxed{a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)}\,, \tag{1.5}$$

or, to use Einstein's convention of implied summation over repeated indices, here meant to run from 0 to n,

$$a_i(x)y^{(i)} = f(x). \tag{1.6}$$

When $f(x) = 0$, the linear ODE is called *homogeneous* [of which (1.4) is an example], otherwise it is *inhomogeneous*. Homogeneous linear ODEs have the important property that they do not change under an arbitrary rescaling $y \to \lambda y\ \forall \lambda \neq 0$.[2] We shall see (§4) that linear ODEs have many nice properties that make them much easier to solve than most nonlinear ones—and they also turn out to be useful in the analysis of the latter (§§3.5.3, 8).

[2]Nonlinear ODEs can also have this property, in which case they are easier to solve: see Q2.6.

An **example** of *nonlinear* ODE is

$$\dot{y} = ay^2, \tag{1.7}$$

which describes the evolution of a radically polygamous (or rather "omnigamous") population, whose rate of increase is proportional to the number of all possible pairings.

Let me return to (1.4). You can probably guess quite easily what its solution is:

$$y = Ce^{at}, \tag{1.8}$$

where C is an arbitrary constant.[3] The emergence of C highlights an important property of ODEs: typically, *they have an infinite number solutions, parametrised by arbitrary constants*.[4] We shall see that the number of constants should be equal to the order of the equation, but some more work is needed to make this obvious.

The solution (1.8) has been written in terms of the familiar exponential function. But what is that? In fact, it is possible to introduce e^x by definition as the function whose derivative is equal to the function itself.

Exercise 1.1. Ponder or investigate how this can be done.

Thus, in a sense, (1.8) is the solution of (1.4) by definition. This illustrates a general principle: so called "elementary functions" can be thought of as nothing but solutions of some oft-encountered and useful ODEs. When an ODE cannot be solved in elementary functions—or, more generally, *in quadratures*, which means in terms of integrals of combinations of elementary functions—and most ODEs cannot be so solved!—one has four options:

(i) Introduce a new function, defined as the solution of the previously unsolved ODE, and study it in some way: e.g., work out a power series for it (see Q2.9), plot it, tabulate it, make an app for computing it. These, usually in application to second-order linear ODEs, are called *special functions* (a good textbook on them is Lebedev 1972).

(ii) Find approximate, or *asymptotic*, solutions (a graduate-level textbook: White 2010; a cult-status, wild romp: Bender & Orszag 1999; a short classic introduction: Erdélyi 2003; see also Tikhonov *et al.* 1985, §7).

(iii) Find numerical solutions (a short introduction: Tikhonov *et al.* 1985, §6.1; an ultimate "how-to" guide: Press *et al.* 2002; a mathematical theory: Samarskii 2001).

(iv) Study solutions qualitatively (§§3.5 and 8).

There is another way in which (1.4) is special: it is *resolved with respect to derivative*, i.e., it is of the general form

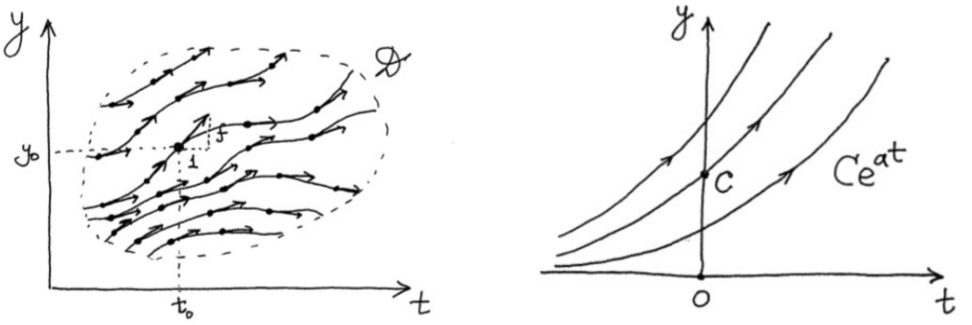
$$\boxed{\dot{y} = f(t, y)}\,. \tag{1.9}$$

This type of ODE allows for a vivid way of thinking of the multiplicity of an ODE's solutions, which we shall now discuss.

---

[3] It is obvious that there should be a multiplicative constant in the solution of any homogeneous ODE because of the possibility of an arbitrary rescaling of y.

[4] In Q1.8, you get to play with an "inverse" problem: given an infinite, parametrised set of functions, find an ODE whose solutions they are.



(a) Integral curves of (1.9) (b) Integral curves of (1.4)

FIGURE 2. Integral curves of (1.9), an ODE that is resolved with respect to derivative: (a) general case, (b) integral curves (1.8) of (1.4). The domain where $f(t, y)$ is defined is (a) $\mathscr{D}$, (b) the entire plane $\mathbb{R}^2$. The ODE's direction field is $(1, f)$, shown by arrows in (a). The initial condition is (t_0, y_0).

1.2. *Integral Curves and the Cauchy Problem*

[Literature: Pontryagin (1962, §1), Tikhonov *et al.* (1985, §1.1), Arnold (2006, §1.3), Tenenbaum & Pollard (1986, §§4-5), Bender & Orszag (1999, §1.2)]

Let us suppose that the function $f(t, y)$ is specified in some domain $\mathscr{D}$ within the $\mathbb{R}^2$ plane (t, y). Through each point of this domain, let us draw a line whose slope is $f(t, y)$, i.e., whose direction vector is $(1, f)$ (if you like, you can normalise it by $\sqrt{1 + f^2}$). This is called the *direction field* of the ODE (1.9). Then solutions of (1.9) are curves $y(t)$ that are at each point in $\mathscr{D}$ tangent to its direction field (Fig. 2a). They are called *integral curves*. *Solving an ODE* means finding all its integral curves.

For example, the integral curves of (1.4), parametrised by C in (1.8), are graphs of the exponential function in the (t, y) plane that intersect the y axis at $y = C$ (Fig. 2b). We can pick a single one of these curves, i.e., make the solution *unique* if we append to (1.4) an additional requirement that $y(t = 0) = y_0$ for some given y_0. Then the desired curve (1.8) is the one with $C = y_0$.

More generally, an *initial condition (IC)*—which, if we were speaking in terms of a spatial variable x, rather than time t, we would call a *boundary condition (BC)*—is the statement that

$$\boxed{y(t_0) = y_0} \quad \text{for some } (t_0, y_0) \in \mathscr{D}. \tag{1.10}$$

Equivalently, it is the statement that the integral curve that we seek must pass through the point (t_0, y_0).

The problem of finding the solution of (1.9) that satisfies the initial condition (1.10) is called the *initial-value problem*, or *Cauchy problem* (Fig. 3).

Does the Cauchy problem always have a solution? If we can find one, is it the only one or are there others? In other words, is there an integral curve that passes through every point $(t_0, y_0) \in \mathscr{D}$ and can these curves ever intersect?



FIGURE 3. Baron Augustin-Louis Cauchy (1789-1857), great French mathematician, aristocrat and rabid right-winger.

1.3. *Existence and Uniqueness*

[Literature: Pontryagin (1962, §§1, 2.4, 20), Tikhonov *et al.* (1985, §2.2), Arnold (2006, §§2.1-2.3), Tenenbaum & Pollard (1986, §58), Coddington (1990, Ch. 5)]

The answer to these questions is given by the following *existence and uniqueness theorem.*

Theorem 1. *Let $f(t,y)$ and $\partial f/\partial y$ exist and be continuous functions on some open domain*[5] *$\mathscr{D} \subset \mathbb{R}^2$. Then*

(a) $\forall (t_0, y_0) \in \mathscr{D}$, $\exists \Delta t$ such that the Cauchy problem

$$\dot{y} = f(t,y), \quad y(t_0) = y_0 \tag{1.11}$$

has a solution in the interval $\mathscr{I} = [t_0 - \Delta t, t_0 + \Delta t]$.[6]

(b) This solution is unique, i.e., if $y_1(t)$ and $y_2(t)$ are solutions of (1.11) on the intervals $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively, then $y_1(t) = y_2(t)$ $\forall t \in \mathscr{I}_1 \cap \mathscr{I}_2$ (they are the same in the intersection of the intervals where they are solutions).

Thus, we are guaranteed that we can solve the initial-value problem for at least some time after (and before) t_0 and that if we have found a solution, we can rest assured that it is the only one. We are in business!

Example. Let me illustrate how some of this can be broken and, therefore, why one should not be dismissive about formulating mathematical results precisely and watching for all the stated conditions to be satisfied. Consider the following Cauchy problem:

$$\dot{y} = y^{2/3}, \quad y(0) = 0. \tag{1.12}$$

Clearly, $y(t) = 0$ is a solution. But $y(t) = (t/3)^3$ is also a solution, as can be verified by direct substitution. The two integral curves intersect at $t = 0$ (Fig. 4)! What has gone

[5] $\mathscr{D}$ being an open domain means that around $\forall$ point $\in \mathscr{D}$, there is a circle of some radius such that all points within the circle are also $\in \mathscr{D}$. Thus, if you think of $\mathscr{D}$ as some bounded area, it cannot include its own boundary. Obviously, the whole space $\mathbb{R}^2$ is very much an open domain.

[6] Formally, this means that $\exists$ a function $y(t)$ such that $y(t_0) = y_0$ and $\forall t \in \mathscr{I}$, $(t, y(t)) \in \mathscr{D}$, $\dot{y}(t)$ is continuous and $\dot{y}(t) = f(t, y(t))$.

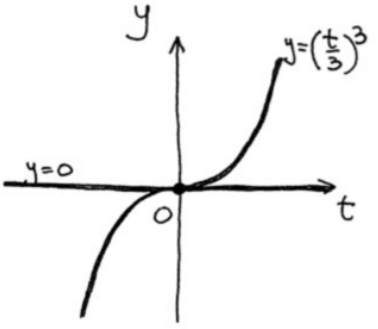


FIGURE 4. Non-uniqueness: two solutions of (1.12), shown as bold lines, intersect at the origin.

wrong? This is easy to see:

$$f(t,y)=y^{2/3} \quad \Rightarrow \quad \frac{\partial f}{\partial y}=\frac{2}{3}\,y^{-1/3}=\infty \quad \text{at} \quad y=0. \tag{1.13}$$

Thus, $\partial f/\partial y$ does not exist at $y=0$, the conditions of Theorem 1 are violated, and so the Cauchy problem is under no obligation to have a unique solution.

1.4. *Parametric Solutions*

Before I move on, let me generalise slightly what I have just done. Casting aside the bigoted language of "dependent" (y) and "independent" (x or t) variables, let me treat everybody equally and write an ODE in the so-called *symmetric form:*

$$\boxed{P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=0}\,. \tag{1.14}$$

We shall see (§2) that ODEs always end up in this form before getting integrated. Clearly, any ODE of the form (1.9) can be recast in the form (1.14) ($Q=-1$, $P=f$), but not necessarily vice versa because $Q(x,y)$ may well vanish in some places, preventing one from dividing by it and simply recasting (1.14) as

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{P(x,y)}{Q(x,y)}\equiv f(x,y). \tag{1.15}$$

To be precise, what we shall assume is that P and Q are continuous functions in some domain $\mathscr{D}\subset\mathbb{R}^2$ and that $|P(x,y)|+|Q(x,y)|>0\;\forall(x,y)\in\mathscr{D}$, i.e., that they do not vanish simulatneously. This means that in some parts of $\mathscr{D}$, where $Q\neq 0$, our equation (1.14) can be written as (1.15), an ODE resolved with respect to $\mathrm{d}y/\mathrm{d}x$, and in others, where $P\neq 0$, it can be resolved with respect to $\mathrm{d}x/\mathrm{d}y$:

$$\frac{\mathrm{d}x}{\mathrm{d}y}=-\frac{Q(x,y)}{P(x,y)}. \tag{1.16}$$

The symmetric form (1.14) is a compact way of writing the two equations (1.15) and (1.16) together.

Solutions of (1.14) can in general be thought of as determined *parametrically* in terms of an auxiliary variable t: namely, $(x(t),y(t))$ is a solution of (1.14) if it is specified in some interval $\mathscr{I}\subset\mathbb{R}$, and, $\forall t\in\mathscr{I}$, $x(t)$ and $y(t)$ are continuously differentiable, $(x(t),y(t))\in\mathscr{D}$, and

$$P(x(t),y(t))\frac{\mathrm{d}x}{\mathrm{d}t}+Q(x(t),y(t))\frac{\mathrm{d}y}{\mathrm{d}t}=0. \tag{1.17}$$

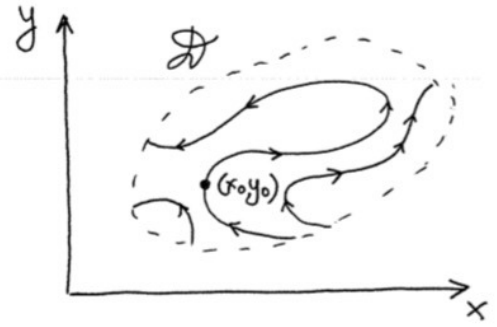


FIGURE 5. Integral curves of (1.14), an ODE in symmetric form. The direction field is $(-Q, P)$. The initial condition is (x_0, y_0).

One may be able to write such a parametric solution as y vs. x or as x vs. y in some sub-intervals of $\mathscr{I}$, provided $\dot{x} \neq 0$ or $\dot{y} \neq 0$ in these sub-intervals. For example, if $\dot{x} \neq 0$ in some sub-interval, then $x = x(t)$ can be inverted there to give $t = t(x)$. Then $y = y(t(x))$ is the y vs. x solution.

Geometrically, we can think of $(\dot{x}, \dot{y})$ as components of a velocity field on $\mathscr{D} \subset \mathbb{R}^2$, or, formally, $\mathbb{R}^3$ if the third component is set to 0. Then (1.14) says

$$\begin{pmatrix} -Q \\ P \\ 0 \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} = 0, \tag{1.18}$$

i.e., the velocity field $(\dot{x}, \dot{y})$ is everywhere parallel to the vector field $(-Q, P)$. This means that $(-Q, P)$ is the direction field of our equation and the integral curves in $\mathscr{D}$ are everywhere tangent to this direction field (cf. §1.2, where $Q = -1$ and $P = f$). The realm of possibilities here is slightly richer than for (1.9) because now the integral curves can be vertical (Fig. 5). This was not allowed for (1.9) lest we had $y' = \infty$ in some places.

The Cauchy problem in this set up is the problem of finding an integral curve that passes through some given point $(x_0, y_0) \in \mathscr{D}$.

The existence and uniqueness result still holds because it is local—for any point (x_0, y_0), we can always find a neighbourhood where either $P \neq 0$ or $Q \neq 0$, so (1.14) can be written as either (1.16) or (1.15) and we are back to Theorem 1.

Example. Let me illustrate two points: how an ODE in the form (1.9) is usefully recast as (1.14) and how the existence theorem, in general, only works locally, rather than on the entire $\mathscr{D}$. Let us go back to (1.7), the equation of omnigamous procreation, and set up a Cauchy problem for it:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay^2, \quad y(t_0) = y_0. \tag{1.19}$$

This can be rearranged to read

$$\frac{\mathrm{d}y}{y^2} = a\,\mathrm{d}t, \tag{1.20}$$

which has the symmetric form (1.14) with $P = a$ and $Q = -1/y^2$. Anticipating the integration techniques to be properly discussed in §2, we integrate (1.20):

$$\int_{y_0}^{y} \frac{\mathrm{d}y}{y^2} = a \int_{t_0}^{t} \mathrm{d}t \quad \Rightarrow \quad y = \frac{a^{-1}}{t_\mathrm{c} - t}, \quad t_\mathrm{c} = t_0 + \frac{a^{-1}}{y_0}. \tag{1.21}$$

This solution blows up at a finite time t_c and does not exist for $t \geqslant t_\mathrm{c}$ (Fig. 6), even though the

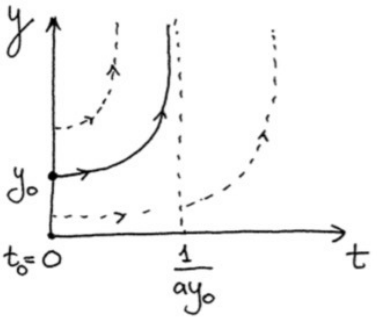


FIGURE 6. Explosion: integral curves of (1.19).

equation itself is perfectly well defined on the entire plane (t, y). At whatever point (t_0, y_0) on the plane we set the initial condition, the solution will always exist in some neighbourhood of this point, but not globally.

In physical terms, we have just encountered the interesting phenomen of *explosion*—defined as a quantity becoming infinite in finite time. In the context of omnigenous populations, this suggests that some regulating effect, which is not in the simple model (1.19) will switch on before the time t_c comes—Victorian morality or contraception or exhaustion of the food supply... (cf. Q5.4).

It is clear that we are ready to start solving things. Thus, before I shower you with more generalities, let me teach you some methods for actually solving ODEs.

2. Some Methods for Solving First-Order ODEs

"Solving" an ODE in a narrow, literal sense, is "integrating" it—and "integration" means integration, literally.[7] So here is the first, most basic method.

2.1. *Equations in Full Differentials*

[Literature: Pontryagin (1962, §2A), Arnold (2006, §6.6), Yeomans (2014, §II.5), Binney (2002, §5.2), Tenenbaum & Pollard (1986, §9)]

Consider a first-order ODE in symmetric form (1.14). It is easy to integrate if the left-hand side is a full differential of some function, viz., if $\exists\, \Phi(x, y)$ such that

$$\forall (x, y) \in \mathscr{D}, \quad P(x, y) = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad Q(x, y) = \frac{\partial \Phi}{\partial y}. \tag{2.1}$$

Then (1.14) is called an *equation in full differentials* and becomes

$$\mathrm{d}\Phi(x, y) = 0 \quad \Rightarrow \quad \boxed{\Phi(x, y) = C}\,, \tag{2.2}$$

where C is an artbitrary *constant of integration*. The level sets of $\Phi(x, y)$ are the integral curves of (1.14). The equation has been "integrated".

How do we check in general whether the relations (2.1) hold for any function Φ? The necessary condition for that is easy to obtain: if these relations do hold, then,

[7] Keep in mind, however, that in learning how to "integrate" ODEs, we are dealing with a small minority of them. Here is an example of a ludicrously simple equation that nevertheless cannot be solved in quadratures: $y' = y^2 + x$, a Riccati equation (cf. §2.6).

differentiating the first of them with respect to y and the second with respect to x, we find that these derivatives must be equal:

$$\forall (x,y) \in \mathscr{D}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{2.3}$$

The fact that this is a necessary condition means that you can always use a failure of (2.3) to rule out the existence of Φ. In the event, if $\mathscr{D}$ is simply connected (meaning that any two points within it can be connected by a continuous curve all of which lies in $\mathscr{D}$), one can prove that (2.3) is also the sufficient condition for (2.1).

Example. Q1.2(a).

Example. Consider

$$2xy\mathrm{d}x + (x^2 - y^2)\mathrm{d}y = 0. \tag{2.4}$$

It is instantly clear that (2.3) is satisfied, so we are in business. It must be the case that

$$\frac{\partial \Phi}{\partial x} = 2xy \quad \Rightarrow \quad \Phi = x^2 y + \phi(y), \tag{2.5}$$

where $\phi(y)$ is the integration "constant", as far as x dependence is concerned, but really a function of y. We find it from

$$\frac{\partial \Phi}{\partial y} = x^2 - y^2 \quad \Rightarrow \quad \cancel{x^2} + \phi'(y) = \cancel{x^2} - y^2 \quad \Rightarrow \quad \phi(y) = -\frac{y^3}{3} + C. \tag{2.6}$$

Thus, the solution is

$$\Phi(x,y) = x^2 y - \frac{y^3}{3} = C \tag{2.7}$$

(different C here than in the previous line).

2.1.1. *Integrating Factor*

[Literature: Tenenbaum & Pollard (1986, §10)]

Suppose now that (2.3) is *not* satisfied. Do we give up? Not necessarily. There is sometimes a miraculous cure called an *integrating factor:* it is a function $\Lambda(x,y) \neq 0$ such that

$$\Lambda(x,y)\,[P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y] = 0 \tag{2.8}$$

is an equation in full differentials, i.e.,

$$\frac{\partial \Lambda P}{\partial y} = \frac{\partial \Lambda Q}{\partial x}, \quad \text{or} \quad Q\frac{\partial \Lambda}{\partial x} - P\frac{\partial \Lambda}{\partial y} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\Lambda. \tag{2.9}$$

Generally speaking, (2.9) is a PDE (first-order) and is not necessarily any easier to solve than the original ODE (1.14), but remember that you do not require generality here—all you need is a particular solution of (2.9). If you can deduce/intuit/guess it, you are done.[8] Good luck!

In what follows, you will see an endless procession of examples of integrating factors being used (see, e.g., §2.4.1—perhaps the most famous—and certainly the most examinable!—such example).

Example. Q1.2(b).

[8] Sometimes it is useful to look for a solution in the form $\Lambda = \Lambda(x)$ or $\Lambda = \Lambda(y)$. **Example.** Q1.5(a).

Example. This one illustrates the merits of opportunism. Consider

$$(x^2+y^2+y)\mathrm{d}x - x\mathrm{d}y = 0. \tag{2.10}$$

Spot that $y\mathrm{d}x - x\mathrm{d}y = -x^2\mathrm{d}(y/x)$, so the above is

$$(x^2+y^2)\mathrm{d}x - x^2\mathrm{d}(y/x) = 0. \tag{2.11}$$

The integrating factor is $\Lambda = 1/(x^2+y^2)$. Indeed, multiplying by it gives us an equation in full differentials:

$$\mathrm{d}x - \frac{\mathrm{d}(y/x)}{1+(y/x)^2} = 0 \quad \Rightarrow \quad x - \arctan(y/x) = C. \tag{2.12}$$

Q1.5(b) is a more sophisticated **example** of this kind of skulduggery.

2.2. *Separable Equations*

[Literature: Pontryagin (1962, §2.1), Tikhonov *et al.* (1985, §2.1.1), Arnold (2006, §2.6), Yeomans (2014, §II.1), Tenenbaum & Pollard (1986, §6)]

A particularly simple and very useful special case of an ODE in full differentials is

$$\boxed{P(x)\mathrm{d}x + Q(y)\mathrm{d}y = 0}\,. \tag{2.13}$$

This is known as a *separable ODE*, referring to the possiblity (in fact, the manifest reality) of x and y entering separately. It very clearly does satisfy (2.3) and can be integrated instantly:

$$\int \mathrm{d}x\, P(x) + \int \mathrm{d}y\, Q(y) = C. \tag{2.14}$$

If we are solving a Cauchy problem and thus are looking for a solution that passes through some point (x_0, y_0), the answer obviously is

$$\int_{x_0}^{x} \mathrm{d}x\, P(x) + \int_{y_0}^{y} \mathrm{d}y\, Q(y) = 0. \tag{2.15}$$

Most cases of successful integration of ODEs culminate in the equation under scrutiny being reduced to (2.13) via various manipulations involving rearranging things and/or changing variables.

Example. Q1.2(c)

2.2.1. *Separable Equations and Composite Solutions*

Here is a simple example of rearranging things. Suppose our equation is

$$\boxed{P_1(x)Q_1(y)\mathrm{d}x + P_2(x)Q_2(y)\mathrm{d}y = 0}\,. \tag{2.16}$$

Let us divide it through by $Q_1(y)P_2(x)$ (an example of integrating factor):

$$\frac{P_1(x)}{P_2(x)}\,\mathrm{d}x + \frac{Q_2(y)}{Q_1(y)}\,\mathrm{d}y = 0. \tag{2.17}$$

The variables are separated, the equation is begging to be integrated.

You must be careful, however, about the possibility that $Q_1(y) = 0$ at some $y = y_0$ or/and $P_2(x) = 0$ at some $x = x_0$, in which case the reduction of (2.16) to (2.17) is only allowed when $y \neq y_0$, $x \neq x_0$. Besides the solutions of (2.17), there are then other

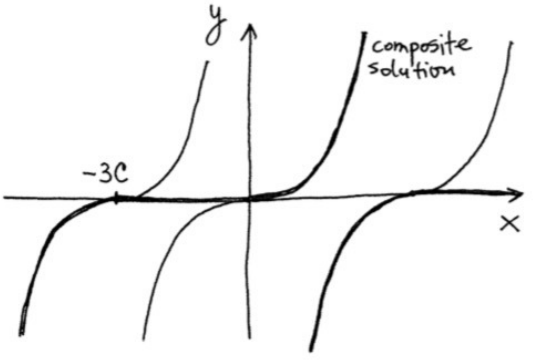


FIGURE 7. Integral curves of (2.18): a family of cubics and the axis $y = 0$. Examples of composite solutions are shown as bold lines.

solutions of (2.16): clearly, if $P_2(x_0) = 0$, the line $x = x_0$ is an integral curve, and so is the line $y = y_0$ if $Q_1(y_0) = 0$. There can be a number of such lines, one for each of the zeros of P_2 or Q_1.

If the integral curves of (2.17) touch (i.e., intersect and have the same slope at the point of intersection[9] as) the lines $x = x_0$ and $y = y_0$, one can construct *composite solutions* by gluing together at the points where they touch bits of those integral curves and the lines $x = x_0$ and $y = y_0$.

Example. Let us consider (1.12) again:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{2/3}. \tag{2.18}$$

First, $y = 0$ is a solution. Assuming $y \neq 0$, we can separate variables and integrate our equation:

$$\int \frac{\mathrm{d}y}{y^{2/3}} = \int \mathrm{d}x \quad \Rightarrow \quad y = \left(\frac{x}{3} + C\right)^3. \tag{2.19}$$

These integral curves touch the line $y = 0$ at a multitude of points $x = -3C$ and have the same slope $y' = 0$ there. Any number of composite solutions can now be constructed by gluing together cubics and bits of the x axis.

The moral of the story is: do not lose these extra solutions when manipulating ODEs (**example:** Q1.4).

Example. Consider

$$xy' + y = y^2 \quad \Rightarrow \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y(y-1) \quad \Rightarrow \quad x\mathrm{d}y = y(y-1)\mathrm{d}x. \tag{2.20}$$

This equation is easily separable and, therefore, integrable:

$$\begin{aligned} \int \frac{\mathrm{d}y}{y(y-1)} = \int \frac{\mathrm{d}x}{x} \quad &\Rightarrow \quad \ln|x| + C = \int \frac{\mathrm{d}y}{y-1} - \int \frac{\mathrm{d}y}{y} = \ln\left|\frac{y-1}{y}\right| \\ &\Rightarrow \quad \frac{y-1}{y} = Cx \quad \Rightarrow \quad y = \frac{1}{1+Cx}. \end{aligned} \tag{2.21}$$

[9] They must always have the same slope at the point of intersection because integral curves by definition are tangent to $(-Q(x,y), P(x,y))$ at every point $(x,y) \in \mathscr{D}$ [see (1.18)] and so there is only one slope available at each point (but see §2.7.1).

Note that, here and henceforth, I change the integration constant at each step in some opportune way, but keep calling it C.

Is (2.21) it, have I solved (2.20)? Not quite: to separate variables, I used the integrating factor $1/xy(y-1)$, so should worry about dividing by 0.

First consider $x = 0$. This is a solution of the last equation in (2.20), but not of the first. Indeed, the last step in (2.20), taking our equation from the form resolved with respect to derivative, (1.9), to the symmetric form (1.14), made our equation a little more general: the line $x = 0$ was not a solution of it in its original form but is a solution of the symmetric version (cf. the discussion at the end of §1.4). Thus, $x = 0$ is not a legitimate solution of our original equation.

Now consider $y = 0$ and $y = 1$. Both of these lines are solutions of our equation. One of them, $y = 1$, is, in fact, already covered by (2.21) with $C = 0$, whereas $y = 0$ is new. Thus, the full set of solutions is

$$y = \frac{1}{1+Cx}, \quad y = 0. \tag{2.22}$$

2.2.2. *Reduction to Separable Form by Linear Change of Variables*

[Literature: Tikhonov *et al.* (1985, §2.1.1), Yeomans (2014, §II.1'), Binney (2002, §5.4)]

Another simple case of an ODE that can be made separable is

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax+by)}, \tag{2.23}$$

where a and b are constant coefficients. The change of (dependent) variable

$$z(x) = ax + by(x) \tag{2.24}$$

does the trick:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x} = a + bf(z) \quad \Rightarrow \quad \int \frac{\mathrm{d}z}{a+bf(z)} = \int \mathrm{d}x = x + C. \tag{2.25}$$

Here again keep in mind that $\forall z_0$ such that $a + bf(z_0) = 0$, $z = z_0$ is a solution and so, besides the integral curves (2.25), (2.23) has solutions (integral curves) that are the lines

$$y(x) = -\frac{a}{b}x + \frac{z_0}{b}. \tag{2.26}$$

Example. Q1.2(d).

Example. Consider

$$(x+2y)y' = 1 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+2y} \equiv f(x+2y). \tag{2.27}$$

Clearly, we must assume $x + 2y \neq 0$ in order for the equation to make sense. Following (2.24), let $z = x + 2y$, whence

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{2}{z}. \tag{2.28}$$

This is a separable equation:

$$\begin{aligned} \int \frac{\mathrm{d}z}{1+2/z} = \int \mathrm{d}x \quad &\Rightarrow \quad \cancel{+} + C = \int \frac{(z+2-2)}{2+z}\mathrm{d}z = \int \mathrm{d}z - 2\int \frac{\mathrm{d}z}{2+z} \\ &= z - 2\ln|2+z| = \cancel{+} + 2y - 2\ln|2+x+2y| \\ &\Rightarrow \quad 2 + x + 2y = Ce^y. \end{aligned} \tag{2.29}$$

This may not be the only solution because we have to consider separately the case when an attempt to separate variables would have led to dividing by 0, viz.,

$$1+\frac{2}{z}=0 \quad \Rightarrow \quad z=-2 \quad \Rightarrow \quad x+2y=-2. \tag{2.30}$$

This solution is, in fact, already captured by (2.29) with $C=0$. The point of the last argument is to show that $C=0$ is allowed in (2.29).

2.3. *Homogeneous Equations*

[Literature: Pontryagin (1962, §2.2), Tikhonov *et al.* (1985, §2.1.1), Arnold (2006, §6.3), Yeomans (2014, §II.2), Binney (2002, §5.1), Tenenbaum & Pollard (1986, §7)]

An ODE in the symmetric form (1.14) is called *homogeneous*[10] if $\forall\lambda\neq 0$,

$$\frac{P(\lambda x,\lambda y)}{P(x,y)}=\frac{Q(\lambda x,\lambda y)}{Q(x,y)}, \tag{2.31}$$

i.e., P and Q scale in the same way when their arguments are scaled.[11] Equivalently, (1.14) does not change under the rescaling $x\to\lambda x$, $y\to\lambda y$.

Since (2.31) works for $\forall\lambda$, it should work, e.g., for $\lambda=1/x$. Assuming $Q\neq 0$ and $x\neq 0$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{P(x,y)}{Q(x,y)}=-\left.\frac{P(\lambda x,\lambda y)}{Q(\lambda x,\lambda y)}\right|_{\lambda=1/x}=-\frac{P(1,y/x)}{Q(1,y/x)}. \tag{2.32}$$

Thus, a homogeneous ODE can be reduced (with all the usual caveats about dividing by zero) to

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=f\left(\frac{y}{x}\right)}, \tag{2.33}$$

which is sometimes quoted as the definition of a homogeneous first-order equation.

This equation is solved by the ansatz

$$y(x)=xz(x). \tag{2.34}$$

This gives us

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}=z+x\frac{\mathrm{d}z}{\mathrm{d}x}=f(z) \quad &\Rightarrow \quad x\,\mathrm{d}z=[f(z)-z]\,\mathrm{d}x \\ &\Rightarrow \quad \int\frac{\mathrm{d}z}{f(z)-z}=\int\frac{\mathrm{d}x}{x}=\ln|x|+C,\end{aligned} \tag{2.35}$$

assuming $f(z)-z\neq 0$. If the equation $f(z_0)=z_0$ has any roots, these give rise to additional solutions, which are the lines

$$y(x)=z_0x. \tag{2.36}$$

Example. Q1.2(e). A slight generalisation of this scheme is explored in Q1.7.

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[10] With respect to x and y simultaneously, so not quite the same as the homogeneity of linear equations introduced after (1.5), but you can, hopefully, see the connection.

[11] This is sometimes written so: $\forall\lambda\neq 0\ \exists\alpha$ such that $P(\lambda x,\lambda y)=\lambda^\alpha P(x,y)$ and $Q(\lambda x,\lambda y)=\lambda^\alpha Q(x,y)$, with the same α for P and for Q.

Example. Consider

$$xy' = \sqrt{x^2 - y^2} + y. \tag{2.37}$$

Notice that this equation stays the same under the rescaling $x \to \lambda x$, $y \to \lambda y$ and so is homogeneous. It is rendered into the form (2.33) if it is divided through by x (assuming, of course, $x \neq 0$):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - \frac{y^2}{x^2}} + \frac{y}{x} \equiv f\left(\frac{y}{x}\right). \tag{2.38}$$

Making the ansatz (2.34), we get a separable equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cancel{z} + x\frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{1 - z^2} + \cancel{z} \quad \Rightarrow \quad \int \frac{\mathrm{d}z}{\sqrt{1 - z^2}} = \int \frac{\mathrm{d}x}{x}$$
$$\Rightarrow \quad \arcsin z = \ln|x| + C \quad \Rightarrow \quad y = x \sin \ln|Cx|, \tag{2.39}$$

assuming $1 - z^2 \neq 0$. As usual, follow mathematical hygiene by considering separately the case

$$1 - z^2 = 0 \quad \Rightarrow \quad z = \pm 1 \quad \Rightarrow \quad y = \pm x. \tag{2.40}$$

These too are solutions, additional to (2.39), and we now have a complete set.

2.3.1. *Reduction to Homogeneous Form by Linear Change of Variables*

[Literature: Yeomans (2014, §§II.2′-2″), Tenenbaum & Pollard (1986, §8)]

As usual, there are classes of ODEs that can be reduced to (2.33) via simple manipulations. Consider, for example,

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)}. \tag{2.41}$$

We can reduce this to (2.33) if we can somewhow get rid of the constants c_1 and c_2. Let us do it by changing variables:

$$x = x_0 + \xi, \quad y = y_0 + \eta, \tag{2.42}$$

where x_0 and y_0 are some constants. Our aim will be accomplished if these constants satisfy

$$\begin{cases} a_1 x_0 + b_1 y_0 + c_1 = 0, \\ a_2 x_0 + b_2 y_0 + c_2 = 0. \end{cases} \tag{2.43}$$

These are two lines on the $\mathbb{R}^2$ plane of points (x_0, y_0). If they intersect, i.e., if $a_1 b_2 \neq a_2 b_1$, then there is a solution (x_0, y_0) and (2.41) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = f\left(\frac{a_1\xi + b_1\eta}{a_2\xi + b_2\eta}\right) = f\left(\frac{a_1 + b_1\eta/\xi}{a_2 + b_2\eta/\xi}\right) \equiv g\left(\frac{\eta}{\xi}\right), \tag{2.44}$$

assuming $\xi \neq 0$. We are back to (2.33).

Example. Q1.2(f).

If $a_1 b_2 = a_2 b_1$, the two lines (2.43) are parallel and there is no solution (x_0, y_0) for which we can pull this trick. However, if the two lines are parallel, then $\exists k$ such that $\forall (x, y)$, $a_2 x + b_2 y = k(a_1 x + b_1 y)$, so (2.41) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_1 x + b_1 y + c_1}{k(a_1 x + b_1 y) + c_2}\right) \equiv g(a_1 x + b_1 y). \tag{2.45}$$

This is (2.23), which we already know how to solve. Victory.

Example. Consider

$$y' = \frac{y+2}{2x+y-4}. \tag{2.46}$$

Chaging variables according to (2.42), we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{y_0+2+\eta}{2x_0+y_0-4+2\xi+\eta} = \frac{\eta}{2\xi+\eta} = \frac{\eta/\xi}{2+\eta/\xi} \equiv g\left(\frac{\eta}{\xi}\right), \tag{2.47}$$

provided $y_0+2=0$ and $2x_0+y_0-4=0$, or $x_0=3$ and $y_0=-2$. We now have a homogeneous equation, which is solved in the usual way. You can check that you now know how to do it: the full set of solutions is

$$(y+2)^2 = C(x+y-1), \quad y = 1-x. \tag{2.48}$$

What if instead of (2.46), we are dealing with

$$y' = \frac{2x+y+2}{2x+y-4}? \tag{2.49}$$

To change it into a homogeneous equation via (2.42), we would require $2x_0+y_0+2=0$ and $2x_0+y_0-4=0$, which obviously cannot be satisfied simultaneously (the two lines are parallel). This is the case where (2.45) works: the equation is of the form $y'=g(2x+y)$ and is solved using the method described in §2.2.2. Check your answer:

$$(2x+y-2)^2 = Ce^{y-x}. \tag{2.50}$$

2.4. *Linear Equations*

[Literature: Tikhonov *et al.* (1985, §2.1.2), Arnold (2006, §3), Tenenbaum & Pollard (1986, §11), Coddington (1990, Ch. 1)]

Linear equations (1.5) have special status because, in the vicinity of any given point, any equation can be expanded and turned into a linear one ("linearised"; see §3.5.3). I will spend quite a lot of time on their general properties in subsequent chapters (§§4–6), but let us have a preview here by considering the Cauchy problem for the simplest, first-order linear ODE:

$$\boxed{\dot{y} = a(t)y + f(t)}, \quad y(0) = y_0. \tag{2.51}$$

I am setting the initial condition at $t_0=0$ without loss of generality as I can always declare the time at which the initial condition is set to be zero by definition.

The Cauchy problem (2.51) can be solved completely in two equivalent ways, which are both quite instructive, so I will go through both.

2.4.1. *Solution via Integrating Factor*

[Literature: Yeomans (2014, §II.3), Binney (2002, §2.3)]

Let us recast (2.51) in the symmetric form (1.14),

$$\mathrm{d}y - [a(t)y + f(t)]\,\mathrm{d}t = 0, \tag{2.52}$$

and look for an integrating factor. In general, it must satisfy (2.9), but we shall look for (and easily find) one that depends just on t: $\Lambda = \Lambda(t)$:

$$\dot{\Lambda} = -a(t)\Lambda \quad \Rightarrow \quad \int \frac{\mathrm{d}\Lambda}{\Lambda} = -\int \mathrm{d}t\, a(t) + C \quad \Rightarrow \quad \Lambda(t) = \exp\left[-\int_0^t \mathrm{d}t'\, a(t')\right]. \tag{2.53}$$



FIGURE 8. Joseph-Louis Lagrange (1736-1813), born Giuseppe Lodovico Lagrangia, great Italian-French mathematician, astronomer, Euler's (Fig. 22) protegé, Fourier's teacher, senator and count of the Napoleonic Empire.

The integration is from zero for future convenience—since I only need a solution, I can integrate from wherever I like. Now multiplying (2.51) by $\Lambda(t)$, we see that

$$\Lambda\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Lambda y - y\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Lambda y + a(t)\Lambda y \tag{2.54}$$

and so our general linear equation (2.4) reduces to something that we can integrate directly:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t)y = \Lambda(t)f(t) \quad \Rightarrow \quad y(t) = \frac{1}{\Lambda(t)}\left[C + \int_0^t \mathrm{d}t'\Lambda(t')f(t')\right]. \tag{2.55}$$

Since $y(0) = C$, the solution of the Cauchy problem (2.51) is achieved by setting $C = y_0$. So, finally,

$$\boxed{y(t) = y_0\, e^{\int_0^t \mathrm{d}t' a(t')} + \int_0^t \mathrm{d}t'\, e^{\int_{t'}^t \mathrm{d}t'' a(t'')} f(t')}\,. \tag{2.56}$$

2.4.2. *Solution via Variation of Constant*

[Literature: Pontryagin (1962, §2B)]

Here is another way to obtain this result. Consider first the homogeneous version of (2.51):

$$\dot{y} = a(t)y. \tag{2.57}$$

It is separable:

$$\int \frac{\mathrm{d}y}{y} = \int \mathrm{d}t\, a(t) \quad \Rightarrow \quad y(t) = C\exp\left[\int_0^t \mathrm{d}t' a(t')\right] \equiv C\mathfrak{E}(t), \tag{2.58}$$

where C is the integration constant and I have introduced the notation $\mathfrak{E}(t)$ for the exponential function to make further algebra look more compact[12] (also, I rather like gothic letters). Once we know this solution of the associated homogeneous equation, there is a nifty trick called the *method of variation of constants* (invented, apparently, by Lagrange [Fig. 8]). It consists in turning C into a function $\psi(t)$ and looking for the solution of the inhomogeneous equation (2.51) in the form

$$y(t) = \psi(t)\mathfrak{E}(t). \tag{2.59}$$

[12] Obviously, $\mathfrak{E}(t) = 1/\Lambda(t)$ from (2.53).

Substituting this into (2.51), we get

$$\dot{y} = \dot{\psi}\,\mathfrak{E}(t) + \cancel{\psi\,a(t)\mathfrak{E}(t)} = \cancel{a(t)\psi\,\mathfrak{E}(t)} + f(t) \quad \Rightarrow \quad \dot{\psi} = \frac{f(t)}{\mathfrak{E}(t)}. \tag{2.60}$$

Integrating this and then substituting into (2.59) gives us

$$y(t) = \mathfrak{E}(t)\left[C + \int_0^t \mathrm{d}t'\,\frac{f(t')}{\mathfrak{E}(t')}\right]. \tag{2.61}$$

The initial condition is enforced by $C = y_0$ and we again have our solution (2.56).

2.4.3. *CF + PI*

When we study the linear equations of arbitrary order in §4.4, this will be a rather obvious particular case. Still, let me flag a key insight here. The solution (2.61) consists of the solution of the associated homogeneous equation (2.57) plus a certain particular solution of the inhomogeneous equation (2.51). In fact, I claim that if I somehow found (deduced, dreamed, intuited, guessed, stole) a particular solution of (2.51)—let us call it $y_{\mathrm{PI}}(t)$, where "PI" = "particular integral"—then the *general* solution of (2.51) is

$$y(t) = y_{\mathrm{CF}}(t) + y_{\mathrm{PI}}(t), \tag{2.62}$$

where $y_{\mathrm{CF}}(t)$, known as the "complementary function" (hence "CF"), is the general solution of the associated homogeneous equation (2.57). That (2.62) is the general solution follows from the fact that (2.62) is clearly *a* solution, but $y_{\mathrm{CF}}(t)$ already has a constant in it, which can be used to accommodate any initial condition—and so the resulting solution is *the* solution because the solution of the Cauchy problem is unique. Note, by the way, that we have thus explicitly shown that, for a linear equation, the existence and uniqueness is global, rather than local—indeed, our solution is valid at all t, not just in the vicinity of $t = 0$. This property, as well as the prescription (2.62), will turn out to carry over to linear ODEs and systems of ODEs of any order (see §4).

Example. Q1.2(g).

Example. Consider

$$xy' + (x+1)y = 3x^2 e^{-x}. \tag{2.63}$$

The associated homogeneous equation is (assuming $x \neq 0$)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x+1}{x}\,y \quad \Rightarrow \quad \int \frac{\mathrm{d}y}{y} = -\int \mathrm{d}x \left(1 + \frac{1}{x}\right) \quad \Rightarrow \quad y = C\frac{e^{-x}}{x}. \tag{2.64}$$

To find the solution of the inhomogeneous equation (2.63), weaponise the constant: $C \to \psi(x)$. Then

$$y' = \psi'\frac{e^{-x}}{x} - \cancel{\psi\frac{e^{-x}}{x}} - \cancel{\psi\frac{e^{-x}}{x^2}} = -\cancel{\frac{x+1}{x}}\,y + 3xe^{-x} \quad \Rightarrow \quad \psi' = 3x^2 \quad \Rightarrow \quad \psi = x^3 + C. \tag{2.65}$$

Therefore, the solution of (2.63) is

$$y = (x^3 + C)\frac{e^{-x}}{x}. \tag{2.66}$$

2.5. *Bernoulli Equations*

[Literature: Yeomans (2014, §II.4), Tenenbaum & Pollard (1986, §11D), Bender & Orszag (1999, §1.6)]



(a) Bernoulli



(b) Riccati

FIGURE 9. (a) Jacob Bernoulli (1654-1705), Swiss mathematician, one of eight Bernoulli mathematicians (all related to each other), Leibniz's (Fig. 1) student and ally against Newton (Fig. 11), discoverer of e and of the Law of Large Numbers. His younger brother Johann was Euler's (Fig. 22) adviser (as well as l'Hôpital's). (b) Jacopo Francesco Riccati (1676-1754), Venetian mathematician and lawyer, known for little else besides equation (2.70).

A few further tricks exist. Consider *Bernoulli's equation* (Fig. 9a)

$$\boxed{y' + a(x)y = b(x)y^n}\,, \tag{2.67}$$

where $n \neq 0, 1$, so the equation is nonlinear. If $n > 1$, $y(x) = 0$ is clearly a solution. Putting it to one side, divide (2.67) by y^n:

$$b(x) - a(x)y^{1-n} = \frac{y'}{y^n} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{y^{1-n}}{1-n}. \tag{2.68}$$

Letting $z = y^{1-n}$, we turn this into

$$z' + (1-n)a(x)z = (1-n)b(x). \tag{2.69}$$

This is a linear equation: follow standard operating procedure (§2.4).

Example. Q1.2(h).

2.6. *Riccati Equations*

[Literature: Bender & Orszag (1999, §1.6)]

Now consider *Riccati's equation* (Fig. 9b)

$$\boxed{y' = a(x)y^2 + b(x)y + c(x)}\,. \tag{2.70}$$

Equations of this form are generally impossible to solve in quadratures. There is, however, one loophole. If you can guess some particular solution of (2.70), let us call it $y_0(x)$, then the ansatz

$$y(x) = z(x) + y_0(x) \tag{2.71}$$

turns your Riccati equation into a Bernoulli equation (2.67) for $z(x)$. Watch:

$$y' = z' + \cancel{y_0'} = az^2 + 2azy_0 + \cancel{ay_0^2} + bz + \cancel{by_0} + \cancel{c} \quad \Rightarrow \quad z' = [2a(x)y_0(x) + b(x)]\,z + a(x)z^2. \tag{2.72}$$

You have an opportunity to try your hand at this method in Q1.9 (**example**). In §4.6, it will turn into something very useful indeed.

2.7. *Equations Unresolved With Respect to Derivative*

[Literature: Arnold (2006, §8.5)]

Let us go back to the most general form of ODE (1.1) and consider its first-order version:

$$\boxed{F(x, y, y') = 0}\,. \tag{2.73}$$

What can you do about this equation? Well, obviously, if you can resolve (2.73) with respect to y', this gets you back to familar territory (1.9):

$$y' = f(x, y). \tag{2.74}$$

Note, however, that (2.73) can have more than one solution of the form (2.74)—e.g., if (2.73) is quadratic in y'—so solving (2.73) may amount to solving several ODEs of the form (2.74).

2.7.1. *Cauchy Problem*

[Literature: Tikhonov *et al.* (1985, §§2.3.1, 2.3.3)]

What, you might wonder, about existence, uniqueness and all that? Do integral curves of (2.73) intersect? Yes, they generally do intersect—basically because (2.73) generally corresponds to more than one equation of the form (2.74). A generalisation of the Cauchy problem for this case is to specify

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = p_0. \tag{2.75}$$

Obvioulsy, not every set (x_0, y_0, p_0) is legitimate as these numbers must satisfy $F(x_0, y_0, p_0) = 0$. Since, for any given (x_0, y_0), this equation generally has more than one root p_0, the point of (2.75) is to specify which root we are interested in. In other words, we ask for an integral curve of (2.73) that passes through (x_0, y_0) *and* has a slope p_0 there. Existence and uniqueness of the solution of (2.73) with initial condition (2.75) turn out to be guaranteed, analogously to Theorem 1, provided F, $\partial F/\partial y$, $\partial F/\partial y'$ are continuous in some neighbourhood of (x_0, y_0, p_0) and

$$\frac{\partial F}{\partial y'}(x_0, y_0, p_0) \neq 0. \tag{2.76}$$

The necessity of the last condition is intuitively obvious: in the close vicinity of the point (x_0, y_0, p_0), our equation (2.73) is

$$\frac{\partial F}{\partial x}(x_0, y_0, p_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, p_0)(y - y_0) + \frac{\partial F}{\partial y'}(x_0, y_0, p_0)(y' - p_0) = 0. \tag{2.77}$$

Unless (2.76) holds true, the above cannot be resolved for y'.

Example. Consider

$$xy'(xy' + y) = 2y^2. \tag{2.78}$$

This is quadratic in xy'. Solving it gives us two equations:

$$xy' = y \quad \text{or} \quad xy' = -2y. \tag{2.79}$$

Their solutions are

$$y = Cx \quad \text{and} \quad y = \frac{C}{x^2}, \tag{2.80}$$

respectively. If we ask for a solution with $y(1) = 1$, we get $C = 1$ in both cases, so both solutions pass through the point $(1, 1)$. To decide which of them to pick, we must specify $y'(1) = 1$ or $y'(1) = -2$.

Note that places where the condition (2.76) is broken are interesting. A curve in the (x, y) plane that satisfies

$$F(x, y, p) = 0, \quad \frac{\partial F}{\partial p}(x, y, p) = 0 \tag{2.81}$$

is called the *p-discriminant curve* of (2.73). If this curve is also a solution (an integral curve) of

this equation, then it is a rather special solution, in every point of which uniqueness is violated. This means that at its every point, this solution will be tangent to another integral curve of the equation. You will find a cute **example** of this in Q1.10 and others in Q1.11.

2.7.2. *Solution via Introduction of Parameter*

[Literature: Tikhonov *et al.* (1985, §2.3.2)]

How do we solve equations like (2.73) if we cannot resolve them with respect to y'—or get something horrible/intractable if we do? Some useful advice on this is as follows.

First suppose that you can resolve (2.73) with respect to y, rather than y':

$$\boxed{y = f(x, y')}\,. \tag{2.82}$$

Then let $p = y'$ and rewrite (2.82) as

$$\mathrm{d}y = p\,\mathrm{d}x \quad \text{and} \quad y = f(x, p). \tag{2.83}$$

This implies

$$p\,\mathrm{d}x = \mathrm{d}y = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial p}\,\mathrm{d}p \quad \Rightarrow \quad \left(\frac{\partial f}{\partial x} - p\right)\mathrm{d}x + \frac{\partial f}{\partial p}\,\mathrm{d}p = 0. \tag{2.84}$$

The resulting ODE has the symmetric form (1.14). Suppose we can integrate it and get the solution in the form

$$p = p(x, C), \tag{2.85}$$

where C is an integration constant. Then the solution of (2.82) is

$$y = f(x, p(x, C)). \tag{2.86}$$

This scheme is called the *method of introduction of parameter*.

Example. A useful and pretty class of ODEs that are solvable by this method is introduced in Q1.10; see also Q1.11.

Example. Consider

$$y = xy' - (y')^2 = xp - p^2, \tag{2.87}$$

where, as per the algorithm laid out above, a parameter was introduced: $p = y'$. Then

$$\cancel{p\mathrm{d}x} = \mathrm{d}y = x\mathrm{d}p + \cancel{p\mathrm{d}x} - 2p\mathrm{d}p \quad \Rightarrow \quad (x - 2p)\mathrm{d}p = 0. \tag{2.88}$$

There are two ways in which this can be satisfied:

$$p = \frac{x}{2} \quad \Rightarrow \quad y = \frac{x^2}{4}, \tag{2.89}$$

$$\mathrm{d}p = 0 \quad \Rightarrow \quad p = C \quad \Rightarrow \quad y = C(x - C). \tag{2.90}$$

In both cases, to get to the answer, the solution for p was substituted into (2.87). Note that (2.89) is the p-discriminant curve of our equation and all the other integral curves (2.90) are lines tangent to it.

Exercise 2.1. Work out how to solve (2.73) if it can be resolved with respect to x, i.e., how to solve an ODE that has the form

$$\boxed{x = f(y, y')}\,. \tag{2.91}$$

Example. Q1.11(iii).

A more general/abstract version of the above is as follows. Consider the general first-order ODE (2.73). The equation

$$F(x, y, p) = 0 \tag{2.92}$$

defines a 2D surface in the 3D space $\mathbb{R}^3$ of points (x, y, p). Suppose the function F is such that

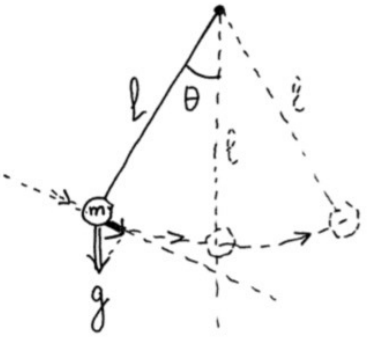


FIGURE 10. Mathematical pendulum.

this surface can be smoothly parametrised, i.e., it is described by some continuously differentiable functions

$$x = \xi(u, v), \quad y = \eta(u, v), \quad p = \zeta(u, v), \tag{2.93}$$

where (u, v) are auxiliary variables parametrising the surface (finding those and working out in what ranges of their values they describe which bits of the surface is the key nontrivial task here).[13] Then

$$\mathrm{d}y = p\,\mathrm{d}x \quad \Leftrightarrow \quad \frac{\partial \eta}{\partial u}\,\mathrm{d}u + \frac{\partial \eta}{\partial v}\,\mathrm{d}v = \zeta\left(\frac{\partial \xi}{\partial y}\,\mathrm{d}u + \frac{\partial \xi}{\partial v}\,\mathrm{d}v\right), \tag{2.94}$$

leading to an ODE in the symmetric form (1.14):

$$\left(\frac{\partial \eta}{\partial u} - \zeta\,\frac{\partial \xi}{\partial u}\right)\mathrm{d}u + \left(\frac{\partial \eta}{\partial v} - \zeta\,\frac{\partial \xi}{\partial v}\right)\mathrm{d}v = 0. \tag{2.95}$$

Suppose we have found its solution

$$v = v(u, C). \tag{2.96}$$

Then the original ODE (2.73) has the parametric solution

$$x(u) = \xi(u, v(u, C)), \quad y(u) = \eta(u, v(u, C)). \tag{2.97}$$

Obviously, the devil is in the detail—you have an opportunity to practice this detail in Q1.12 (**example**).

3. The Language of the Game: Going to Higher Order

Hopefully you now feel sufficiently empowered actually to solve things that you might tolerate a resumption of conceptual/generalist discussion. It is time for us to consider how to handle ODEs of order higher than first. Let me start with an example of a second-order ODE, which is perhaps the second most important [after (1.4)] ODE in the world—and possibly *the* most important one in physics.

3.1. *Mathematical Pendulum*

Consider a pendulum of length l, with a point mass m at the end, subject to acceleration of gravity g (Fig. 10). The mass executes a trajectory that is parametrised by the

[13] In order for this approach to work, at every point (and its neighbourhood) at least one of the mappings $(u, v) \to (x, y)$, $(u, v) \to (y, p)$ or $(u, v) \to (x, p)$ must be one-to-one. Formally, this is expressed as the requirement that the Jacobians of these mappings do not all vanish simultaneously: $[\partial(x, y)/\partial(u, v)]^2 + [\partial(y, p)/\partial(u, v)]^2 + [\partial(x, p)/\partial(u, v)]^2 > 0$.



FIGURE 11. Sir Isaac Newton FRS (1643-1727), who needs no introduction.

angle θ and follows the arc of the circle of radius l. The velocity, always tangent to this circle, is $l\dot{\theta}$ and the acceleration is $l\ddot{\theta}$. Newton's (Fig. 11) Second Law reads

$$ml\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \quad \boxed{\ddot{\theta} = -\omega_0^2 \sin\theta}\,, \quad \omega_0 = \sqrt{\frac{g}{l}}. \tag{3.1}$$

This nonlinear, second-order ODE is the *equation of mathematical pendulum* ("mathematical" because the rod is massless, the mass at the end of it is a point mass, and there is no friction of any kind). It describes the pendulum's oscillations around the equilibrium $\theta = 0$. If we further assume that these oscillations are small, $\theta \ll 1$, then we can approximate $\sin\theta \approx \theta$ and turn our ODE into a linear one:

$$\boxed{\ddot{\theta} = -\omega_0^2\,\theta}\,. \tag{3.2}$$

This is the famous equation of a *simple harmonic oscillator (SHO)*, which describes not just a gently peturbed pendulum but also countless other physical systems (or bits of physical systems) ranging from a mass attached to a spring to an elementary excitation in an (imaginary) box filled with radiation. Note that the linear ODE emerged as a result of the nonlinear ODE being expanded in the vicinity of its equilibrium point (the steady solution $\theta = 0$). This is a sign of things to come, but first let us occupy ourselves with our new toy (3.2).

The solution of (3.1) is (verified by substitution)[14]

$$\theta(t) = A\cos(\omega_0 t + \phi), \tag{3.3}$$

where A and ϕ are arbitrary constants (now there are two of them, as there must be, for a second-order ODE—we shall shortly see why). We can fix these constants by setting two initial conditions:

$$\theta(0) = \theta_0 \quad \text{and} \quad \dot{\theta}(0) = \Omega_0. \tag{3.4}$$

the intial angular displacement and the initial angular velocity, respectively.

3.2. *Laws of Motion*

The equation of the pendulum is, of course, a particular case of the much more general differential equation that expresses Newton's Second Law and, therefore, describes

[14]Cosine is a familiar elementary function, but again, in fact, you might think of it as, by definition, the function that is the solution of (3.2), rendered familiar and, therefore, "elementary", by heavy use. You get to "derive" the cosine from a position of pretend ignorance in Q2.9(c).

everything in the (classical) world:

$$\boxed{\ddot{\boldsymbol{r}} = \boldsymbol{f}(t, \boldsymbol{r}, \dot{\boldsymbol{r}})}\,. \tag{3.5}$$

This is why ODEs are so dominant in physics: the laws of motion are ODEs. Since they are second-order ODEs, we better learn how to handle those.

Perhaps you already know from mechanics that, in order to solve (3.5) and thus predict the future of the world, you need to know the initial position $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and the initial velocity $\dot{\boldsymbol{r}}(0) = \boldsymbol{v}_0$. It would appear that there must be some way of formulating a Cauchy-style problem for second- (and, presumably, higher-) order ODEs and one must also hope that an existence and uniqueness theorem is available. This better be true, or what will become of the determinism of the natural world and of our ability to predict its fate?

3.3. *All ODEs Are (Systems of) First-Order ODEs*

[Literature: Pontryagin (1962, §4), Tikhonov *et al.* (1985, §1.1), Arnold (2006, §§1.11-1.14, 8.1), Bender & Orszag (1999, §1.1)]

First of all, let me point out that all ODEs are, in fact, first-order ODEs, or systems thereof. Indeed, (3.5) is nothing but

$$\begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v}, \\ \dot{\boldsymbol{v}} = \boldsymbol{f}(t, \boldsymbol{r}, \boldsymbol{v}). \end{cases} \tag{3.6}$$

Thus, instead of having 1 or 3 or n second-order equations for the n-dimensional vector $\boldsymbol{r}$, we have 2 or 6 or $2n$ first-order equations for the $2n$-dimensional vector $(\boldsymbol{r}, \boldsymbol{v})$ (a second order ODE that lives in $\mathbb{R}^{n+1}$ is a first-order ODE that lives in $\mathbb{R}^{2n+1}$, the $+1$ being time). For example, (3.2) is

$$\begin{cases} \dot{\theta} = \Omega, \\ \dot{\Omega} = -\omega_0^2 \theta \end{cases} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \theta \\ \Omega \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \theta \\ \Omega \end{pmatrix}, \tag{3.7}$$

an example of a *system* of coupled, linear, homogeneous, first-order ODEs. More generally,

$$F(t, y, \dot{y}, \ddot{y}, \ldots, y^{(n)}) = 0 \tag{3.8}$$

can be recast as this system of n first-order ODEs:

$$\begin{cases} F(t, y, p_1, \ldots, p_{n-1}, \dot{p}_{n-1}) = 0, \\ \dot{y} = p_1, \\ \dot{p}_1 = p_2, \\ \ldots \\ \dot{p}_{n-2} = p_{n-1} \end{cases} \tag{3.9}$$

Thus, we can talk of n as the order of the original ODE or as the order of the corresponding system of first-order ODEs (n = the number of these ODEs).

If we restrict our attention to ODEs that are resolved with respect to the highest derivative,

$$y^{(n)} = f(t, y, \dot{y}, \ldots, y^{(n-1)}), \tag{3.10}$$

we see that both (3.10) and many other systems of ODEs that do not necessarily reduce to a single higher-order equation can be written in the general form

$$\boxed{\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y})}\,, \tag{3.11}$$



(a) Lipschitz



(b) Picard

FIGURE 12. (a) Rudolf Otto Sigismund Lipschitz (1832-1903), German mathematician, student of Dirichlet. (b) No, this is not Hercule Poirot, this is Charles Émile Picard FRS (1856-1941), a brilliant French mathematician who proved the existence and uniqueness theorem with the Lipschitz continuity condition (3.13). He had a French academic career *par excellence* (ENS, chair at Sorbonne, Académie des sciences, seat 1 in Académie française, etc.), was a great lecturer (according to Hadamard), and married Hermite's daughter Marie, with whom he had three children, all killed in World War I.

where $\boldsymbol{y}(t)$ is an n-dimensional vector function mapping some subset of $\mathbb{R}$ to some subset of $\mathbb{R}^n$ and $\boldsymbol{f}$ is an n-dimensional "velocity field"—it is called the *phase velocity*, while $\boldsymbol{y}$ is the "phase" of the physical system described by (3.11) (as in "phases of the moon", i.e., states of the system). Note that since $\boldsymbol{y}$ includes physical velocities, as in (3.6), $\boldsymbol{f}$ includes the physical forces providing acceleration.

Naturally, the Cauchy (initial-value) problem for (3.11) is the problem of finding the solution that satisfies

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0. \tag{3.12}$$

3.4. *Existence and Uniqueness*

[Literature: Pontryagin (1962, §§3, 21), Tikhonov *et al.* (1985, §§2.4, 2.6-2.7), Tenenbaum & Pollard (1986, §62), Coddington (1990, Ch. 6)]

It turns out, thank God, that the 1D existence and uniqueness Theorem 1 can be generalised to the n-dimensional case.[15]

Theorem 2. *Let all $f_i(t, \boldsymbol{y})$ and $\partial f_i/\partial y_j$ $(i, j = 1, \ldots, n)$ exist and be continuous on some open domain $\mathscr{D} \subset \mathbb{R}^{n+1}$. Then*

(a) $\forall (t_0, \boldsymbol{y}_0) \in \mathscr{D}$, $\exists \Delta t$ such that the Cauchy problem (3.11–3.12) has a solution in the interval $\mathscr{I} = [t_0 - \Delta t, t_0 + \Delta t]$.

(b) This solution is unique, i.e., if $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are solutions of (3.11–3.12) on the intervals $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively, then $\boldsymbol{y}_1(t) = \boldsymbol{y}_2(t)$ $\forall t \in \mathscr{I}_1 \cap \mathscr{I}_2$.

In fact, it turns out that this theorem can be made somewhat more general (meaning that its assumptions can be weakened): $\boldsymbol{f}$ need not have all these derivatives but must only satisfy the so-called *Lipschitz continuity condition* (Fig. 12): $\exists L \in \mathbb{R}$ such that

[15] Note that this theorem officially confirms that the general solution of an n-th-order ODE must have n integration constants because n initial conditions are needed to pin down the solution uniquely [cf. the argument around (4.31)].

$\forall(t, \boldsymbol{y}_1)$ and $(t, \boldsymbol{y}_2) \in \mathscr{D}$,

$$|\boldsymbol{f}(t, \boldsymbol{y}_1) - \boldsymbol{f}(t, \boldsymbol{y}_2)| \leqslant L|\boldsymbol{y}_1 - \boldsymbol{y}_2| \tag{3.13}$$

(any $\boldsymbol{f}$ that has all the first derivatives is Lipschitz-continuous, but not vice versa).

I shall not go any further into these matters, but this does not mean that you should not: in fact, I urge those of you who wish to be mathematically competent to read about the existence and uniqueness theorems and, indeed, work your way through their proofs (Pontryagin 1962 is a good undergraduate textbook on this subject; Arnold 2006 takes you to the next level of enlightenment). There are things that every physicist ought to have done at least once (although possibly not more than once) in her life—and this is one of them.

Thus, to sum up, ODEs in the form (3.11) are equations that express the rate of change of a vector quantity $\boldsymbol{y}(t)$ (the "phase" of the system) as a function of t and $\boldsymbol{y}$ itself—and they have unique solutions if the "phase" $\boldsymbol{y}_0$ where the system finds itself at some time t_0 is specified.

You can hopefully now see why this is an important subject: evolution of systems with given initial conditions is what we study in physics.

The rest of this chapter is an introduction to the ideas that we shall explore further in §8. If you are impatient to learn how to solve higher-order linear equations, jump to §4 now and come back later.

3.5. *Phase Space and Phase Portrait*

Integral curves live in an $(n+1)$-dimensional space, so, for (systems of) ODEs of order $n > 1$, drawing these curves becomes challenging, pending the invention of a 3D iPad. It also turns out that, unless the equations are linear and have constant coefficients, finding their solutions in elemetary functions or even quadratures is usually impossible.

Example. Consider the linear, homogeneous, second-order equation, which looks like a rather simple ODE:

$$y'' + a(x)y' + b(x)y = 0. \tag{3.14}$$

Let $z = y'/y$. Then $y' = zy$, $y'' = zy' + z'y$, and, for $y \neq 0$, the equation becomes

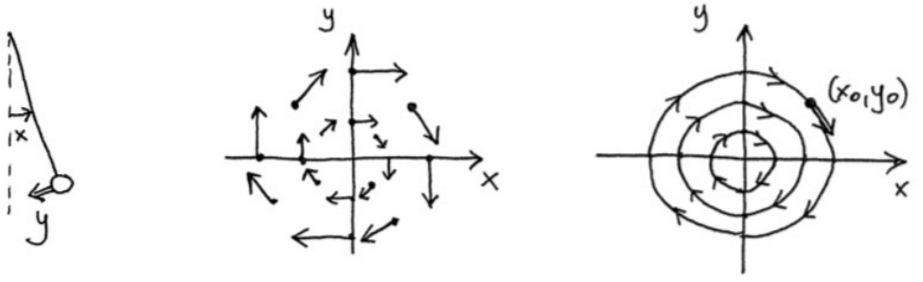
$$z' + z^2 + a(x)z + b(x) = 0. \tag{3.15}$$

This is a Riccati equation (2.70) and so, unless we can guess one of its solutions, it is unsolvable in quadratures. The case in which (3.14) is solvable will be discussed in §4.6. It is basically the same story as in §2.6.

These days, one's reflex in such situations is to stick the recalcitrant equation into a computer and ask for solutions, but it tends to be the case that, if you do not understand what you are doing—and, specifically, if you do not have a good idea as to what to expect—you will get rubbish and be none the wiser. Therefore, what one needs to study is various methods of solving ODEs apprximately and/or of analysing their behaviour qualitatively, i.e., methods for establishing certain key properties of their solutions without having to find these solutions precisely and/or in their entirety.

The objective of such an analysis is to understand how a physical system described by (3.11) might travel around its *phase space*, i.e., the space that contains all values of $\boldsymbol{y}$ accessible to the system (it is usually a subspace of $\mathbb{R}^n$). What parts of the phase space the system visits how often and where it ends up long-term generally must depend on where it started from ($\boldsymbol{y}_0$ at $t = t_0$), but in fact, often does not—or does in such a way that various initial conditions can be classified into broad categories with qualitatively similar evolution.

Let us specialise from (3.11) to a class of ODEs called *autonomous*. These are the systems



(a) (b) (c)

FIGURE 13. Oscillations of a pendulum around the $\theta = 0$ equilibrium (a): (b) the direction field, (c) the phase portrait.

whose phase velocity depends only on their position in phase space:[16]

$$\boxed{\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})}\,. \tag{3.16}$$

Physically, these are systems in which forces depend only on positions and velocities (or analogous phase-space variables), but not explicitly on time—this can be interpreted as absence of external meddling with the system (hence "autonomous"). We may then set ourselves the task of sketching all possible trajectories of the system in phase space. Such a sketch is called the *phase portrait* of the system. Note that this is *not* the same thing as sketching the integral curves: the latter are curves in $\mathbb{R}^{n+1}$ (including time) whereas the *phase trajectories* are curves in $\mathbb{R}^n$ parametrised by t (they are projections of the integral curves onto the phase space). In view of (3.16), these trajectories are everywhere tangent to the system's phase velocity $\boldsymbol{f}$. Obviously, the actual sketching is only a (relatively) easy task when $n = 2$. Note that the task is not thankless for an autonomous system because the phase portrait is independent of time for it, and so only needs to be produced once.

3.5.1. *Linear Pendulum*

[Literature: Arnold (2006, §1.15)]

Let me illustrate what I mean by all this, and why such an approach is useful, on the example of the SHO (3.7). For simplicity of exposition, I will rescale time and angular velocity to get rid of ω_0 and rename variables: $\omega_0 t \to t$, $y = \Omega/\omega_0$, $x = \theta$. Then the system that we are dealing with is just

$$\boxed{\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases}} \tag{3.17}$$

The right-hand side of this equation, $(y, -x)$, defines a velocity field in phase space $\mathbb{R}^2$. It is sketched in Fig. 13(b). The phase trajectories are tangent to this field. These trajectories (Fig. 13c) are circles—this is obvious if one observes that the velocity $(y, -x)$ is always perpendicular to the radius vector (x, y) or, indeed, simply recalls the solution (3.3):

$$x = A\cos(t+\phi), \quad y = \dot{x} = -A\sin(t+\phi) \quad \Rightarrow \quad x^2 + y^2 = A^2. \tag{3.18}$$

Thus, the phase portrait of the SHO is a family of concentric circles parametrised by the integration constant A (Fig. 13c). If you pick a point on the (x, y) plane, you fix A and ϕ, i.e.,

[16] In purely formalistic terms, all systems of ODEs can be reduced to autonomous systems. Indeed, consider a generic non-autonomous system (3.11) with $\boldsymbol{y} \in \mathbb{R}^n$. Define $\boldsymbol{\eta} \in \mathbb{R}^{n+1}$ to have the components $\eta_i = y_i$ for $i = 1, \ldots, n$ and $\eta_{n+1} = t$. Then (3.11) can be written as $\dot{\eta}_i = f_i(\boldsymbol{y}, \eta_{n+1}) \equiv g_i(\boldsymbol{\eta})$ for $i = 1, \ldots, n$ and $\dot{\eta}_{n+1} = 1 \equiv g_{n+1}(\boldsymbol{\eta})$, or $\dot{\boldsymbol{\eta}} = \boldsymbol{g}(\boldsymbol{\eta})$, an autonomous system in $\mathbb{R}^{n+1}$. In practice, this is not terribly helpful, because increasing the number of dimensions, even from 2 to 3, makes things vastly more complicated.

which circle you are on and where on it. The phase portrait now tells you where you would go from there: around the circle, clockwise.

There is another way to see this, which is very illuminating physically: multiply the first equation in (3.17) by x, the second by y, and add them:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^2}{2}+\frac{y^2}{2}\right)=0 \quad \Rightarrow \quad \mathscr{E}(x,y)=\frac{x^2}{2}+\frac{y^2}{2}=C. \tag{3.19}$$

This integral is the *energy* (potential + kinetic), which, unsurprisingly, is conserved. The solutions lie on circles because they are constrained by the conservation of $\mathscr{E}$.

3.5.2. *Nonlinear Pendulum*

[Literature: Arnold (2006, §§1.16-1.17, 12), Strogatz (1994, §6.7)]

This is all rather elementary, but becomes less so if we raise our ambitions and attempt the phase portrait of the more general, nonlinear pendulum, described by (3.1). In our rescaled variables, the system is

$$\boxed{\begin{cases} \dot{x}=y, \\ \dot{y}=-\sin x. \end{cases}} \tag{3.20}$$

The easiest way to draw the phase portrait is by the same method (which, however, does *not* necessarily work for all systems) as we used to obtain (3.19). Indeed, the system (3.20) also has a conserved quantity and so can be integrated: multiply the first equation by $\sin x$, the second by y and add them, to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\cos x+\frac{y^2}{2}\right)=0 \quad \Rightarrow \quad \mathscr{E}(x,y)=1-\cos x+\frac{y^2}{2}=C. \tag{3.21}$$

I have adjusted the constants in such a way as to make sure that $\mathscr{E}(x,y)$ returns to the old expression (3.19) when $x \ll 1$.

The phase trajectories of the system are level sets of $\mathscr{E}(x,y)$. You already know from your Calculus course how to sketch them. The idea is first to find "stationary points" (extrema) of $\mathscr{E}$, then Taylor-expand $\mathscr{E}$ around them to determine their nature, and then "fill in the blanks", i.e., use common sense to work out how these local phase portraits fit together.

Stationary points are not called "stationary points" by accident. Indeed, the system (3.20) can be written equivalently as (cf. §I.6)

$$\dot{x}=\frac{\partial \mathscr{E}}{\partial y}, \quad \dot{y}=-\frac{\partial \mathscr{E}}{\partial x}. \tag{3.22}$$

Thus the extrema of $\mathscr{E}$ are indeed the stationary points (or *fixed points*, or *equilibria*) of the pendulum. They are

$$y=0 \quad \text{and} \quad \sin x=0 \quad \Rightarrow \quad x=\pi n, \quad n\in\mathbb{Z}. \tag{3.23}$$

For $n=2k$, this is the obvious equilibrium at $x=0$ and its identical copies at $x=2\pi k$.[17] Clearly, $(x,y)=(0,0)$ is a minimum of $\mathscr{E}$:

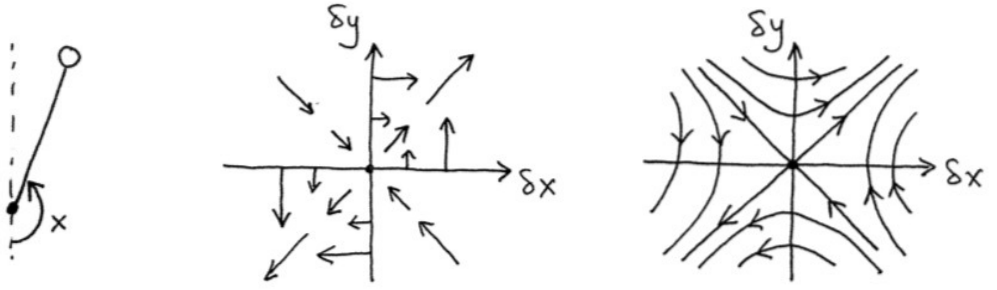
$$\left.\frac{\partial^2 \mathscr{E}}{\partial x^2}\right|_{(0,0)}=1, \quad \left.\frac{\partial^2 \mathscr{E}}{\partial y^2}\right|_{(0,0)}=1. \tag{3.24}$$

In the vicinity of $(0,0)$, the phase portrait is the family of concentric circles already found in §3.5.1 and depicted in Fig. 13(c).

For $n=2k+1$, (3.23) is the upright equilibrium of the pendulum ($x=\pi$, Fig. 14a) and its periodic copies. This equilibrium, $(x,y)=(\pi,0)$, is a saddle point:

$$\left.\frac{\partial^2 \mathscr{E}}{\partial x^2}\right|_{(\pi,0)}=-1, \quad \left.\frac{\partial^2 \mathscr{E}}{\partial y^2}\right|_{(\pi,0)}=1. \tag{3.25}$$

[17] Remember that $x=\theta$, the angle, and so the phase space of the pendulum is, in fact, not really $\mathbb{R}^2$ but the surface of a cylinder: $-\infty<y<\infty$, $0\leqslant x<2\pi$, with the ends, 0 and 2π, glued together.



(a) (b) (c)

FIGURE 14. Oscillations of a pendulum around the $\theta = \pi$ equilibrium (a): (b) the direction field, (c) the phase portrait.

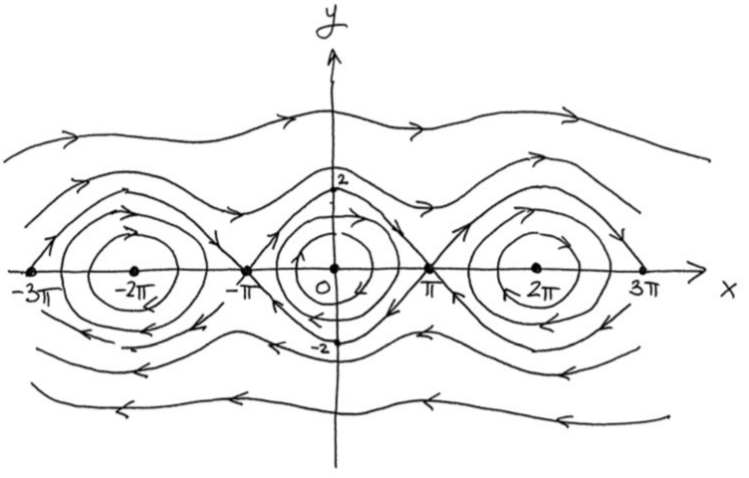


FIGURE 15. Phase portrait of an undamped pendulum.

Let us look at the phase portrait in its vicinity. To do that, we expand (3.20) around this equilibrium point:

$$x = \pi + \delta x, \quad y = 0 + \delta y \quad \Rightarrow \quad \begin{cases} \delta\dot{x} = \delta y, \\ \delta\dot{y} = \delta x. \end{cases} \tag{3.26}$$

The phase velocity field $(\delta y, \delta x)$ is sketched in Fig. 14(b) and the resulting phase portrait in Fig. 14(c). The phase trajectories are hyperbolae, which is easily confirmed by expanding the energy (3.21) around $(\pi, 0)$:

$$\mathscr{E} \approx 2 - \frac{\delta x^2}{2} + \frac{\delta y^2}{2} = C. \tag{3.27}$$

Finally, this is all pieced together in Fig. 15. The phase portrait consists of

—closed trajectories around the minima of energy—they have $\mathscr{E} < 2$ and describe a pendulum oscillating around its usual downward equilibrium,

—open trajectories—they have $\mathscr{E} > 2$ and describe a pendulum executing full 360° rotations,

—the *separatrices* at $\mathscr{E} = 2$.

Picking a point (x_0, y_0) and following the phase trajectory on which it lies will tell you what the pendulum will do if it starts at that point: it will oscillate if you start with $\mathscr{E} < 2$ and rotate if you start with $\mathscr{E} > 2$.

Thus, we have developed a pretty good picture of the behaviour of our system even though we did not solve the nonlinear equations (3.20) explicitly, i.e., obtained $x = x(t)$, $y = y(t)$.[18] We have also been able to classify all possible initial conditions into broad groups that had different subsequent *qualitative* behaviour.

Of course, we were greatly helped by the existence of $\mathscr{E}(x, y)$ and our ability to write an explicit expression (3.21) for it. In practice, however, all we really needed in order to sketch the global phase portrait were the positions of the fixed points and the phase portraits in their close vicinity. In §8.1, I will classify all possible fixed points for second-order ODEs.

This was the most elementary example of a qualitative solution of a system of coupled ODEs.

3.5.3. *Local Linear Analysis*

[Literature: Arnold (2006, §13.1)]

Let me spell out a key general lesson that you are supposed to learn from this example. In order to analyse qualitatively an autonomous system of ODEs (3.16), first find its fixed points:

$$\boldsymbol{f}(\boldsymbol{y}_0) = 0. \tag{3.28}$$

Then expand the system around them:

$$\boldsymbol{y} = \boldsymbol{y}_0 + \delta\boldsymbol{y} \quad \Rightarrow \quad \delta\dot{y}_i = f_i(\boldsymbol{y}_0 + \delta\boldsymbol{y}) \approx \cancel{f_i(\boldsymbol{y}_0)} + \left.\frac{\partial f_i}{\partial y_j}\right|_{\boldsymbol{y}_0} \delta y_j. \tag{3.29}$$

This gives you a linear, homogeneous system of ODEs with constant coefficients that describes the behaviour of the original system in the vicinity of each fixed point:

$$\boxed{\delta\dot{\boldsymbol{y}} = \mathsf{A} \cdot \delta\boldsymbol{y}}, \quad A_{ij} = \left.\frac{\partial f_i}{\partial y_j}\right|_{\boldsymbol{y}_0}. \tag{3.30}$$

After you learn how to handle such linear systems (§6), you will be able to describe life in the vicinity of fixed points and hence piece together global phase portraits for nonlinear systems of which they are local approximations (§8).

3.5.4. *Damped Pendulum (Introduction to Dissipative Systems)*

[Literature: Arnold (2006, §20.5), Strogatz (1994, §6.7)]

Before I wrap up this extended introduction, let me show you what sort of thing happens when energy is *not* conserved and so we cannot obtain the phase portrait of the system by plotting level sets $\mathscr{E}(x, y) = C$.

Let us add friction to the range of experiences available to our idealised pendulum. This is a force that opposes the pendulum's motion and is proportional to its velocity (there is no friction on things standing still), so (3.1) is now written so:

$$\boxed{\ddot{\theta} = -\omega_0^2 \sin\theta - \gamma\dot{\theta}}. \tag{3.31}$$

Recasting it as a system of first-order ODEs in dimensionless variables, analogous to (3.20), gives

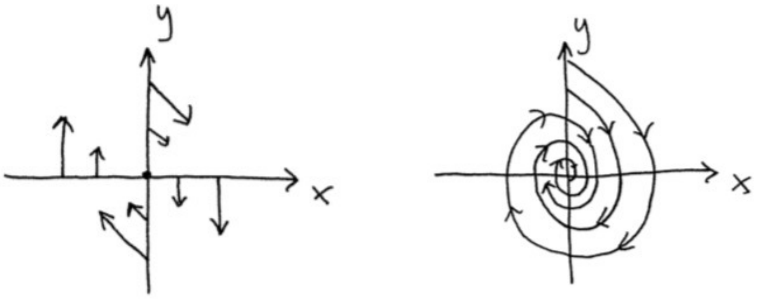
$$\boxed{\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x - \gamma y. \end{cases}} \tag{3.32}$$

Let us see how the energy (3.21) evolves for such a system:

$$\dot{\mathscr{E}} = \frac{\partial\mathscr{E}}{\partial x}\,\dot{x} + \frac{\partial\mathscr{E}}{\partial y}\,\dot{y} = -\gamma y^2 \leqslant 0. \tag{3.33}$$

Thus, the pendulum's energy declines with time (naturally, since there is damping and no forcing). This means that the system is constantly slipping off the conservative trajectories (3.21) towards smaller values of $\mathscr{E}$ (the pendulum's oscillations are losing amplitude).

[18] It is, in this case, in fact, possible to do it in special functions, called elliptic functions.



(a) (b)

FIGURE 16. Motion of a damped pendulum around the $\theta = 0$ equilibrium: (a) the direction field, (b) the phase portrait.

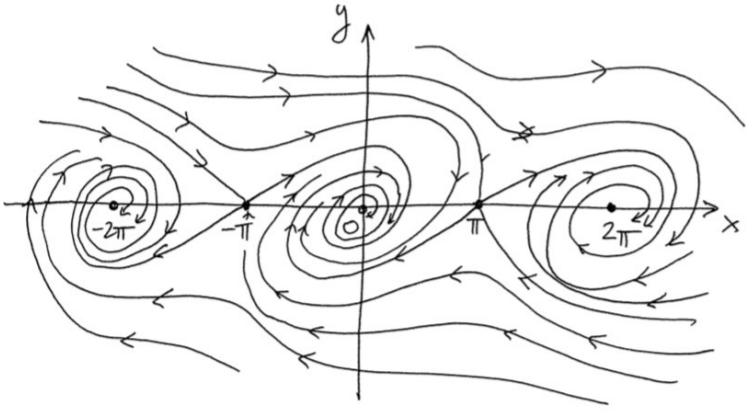


FIGURE 17. Phase portrait of a damped pendulum.

For example, in the vicinity of the fixed point $(x_0, y_0) = (0, 0)$ (which is still a fixed point),

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \gamma y. \end{cases} \tag{3.34}$$

The phase velocity of this system is not perendicular to the radius vector: indeed,

$$\begin{pmatrix} y \\ -x - \gamma y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -\gamma y^2 < 0, \tag{3.35}$$

so the trajectories are ever tending inwards (Fig. 16a). The resulting phase portrait is a clockwise spiral wrapping onto the origin (Fig. 16b). We shall learn how to investigate stability and categorise the system's behavior in such situations when we pick up this story again in §8. For now, noticing that the phase portrait near $(x_0, y_0) = (\pi, 0)$ is distorted but not qualitatively changed (still a saddle), we can adjust the phase portrait of Fig. 15 to include the effect of damping, as shown in Fig. 17: the effect is that all trajectories eventually wrap onto the still-life equilibrium of a downward hanging, energetically exhausted, unmoving pendulum.

As before, without obtaining an explicit solution of the full nonlinear problem, we have been able to deduce a qualitative phase portrait that gives us a pretty good idea of the system's behaviour. The eventual fate of a damped pendulum is independent of initial conditions, although the details of getting there do depend on them:

—if the pendulum starts with $\mathscr{E} \leqslant 2$, it spirals straight into the $(0, 0)$ equilibrium;

—if it starts with $\mathscr{E} > 2$, it rotates a few times, getting slower as it loses energy, and then spirals in.

The key ingredient in developing this understanding was again the local analysis (§3.5.3) of the linearised system around the fixed points.

The damped pendulum is an example of a *dissipative system*, which, when left to its own devices, decays into a boring, quiescent state. This opens the door to a class of questions as to what will happen if we "drive" the system, i.e., continuously put energy into it. This can be done by introducing a force, making the system inhomogeneous—and, if the force is time-dependent, also non-autonomous. We shall study the forced oscillator in §5.3.

Example. Q5.1 offers you an opportunity to investigate another type of nonlinear oscillator, along similar lines.

Let me observe, in anticipation of the developments in §8, that a fixed point is not the only possible type of *attractor*—a set towards which a dissipative system's trajectories converge regardless of initial conditions. Other possibilities are a *limit cycle* (§8.3) and a *strange attractor*—harder beasts to handle but not outside our intellectual range!

Hopefully this extended introduction has given you an idea of the general narrative arc of this subject. We shall now return to learning some practical methods for solving (linear) ODEs.

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4. Linear ODEs: General Principles

[Literature: Pontryagin (1962, §17), Binney (2002, §2), Coddington (1990, Ch. 3)]

There is little one can do about integrating higher-order (higher than first) *nonlinear* ODEs explicitly. Most useful methods here are qualitative (§8). So we are now going to spend some time with *linear* ODEs, because we can achieve something with them and because this is rather useful—both by itself (there are many physical systems described by them) and as a door to qualitative analysis of nonlinear equations, a key component of which is local linear analysis (§3.5.3).

As I have explained above (§3.3), all ODEs are (systems of) first-order ODEs. Thus, in our treatment of linear ODEs, it is formally sufficient to consider systems of the form

$$\boxed{\dot{\boldsymbol{y}} = \mathsf{A}(t) \cdot \boldsymbol{y} + \boldsymbol{f}(t)}\,, \tag{4.1}$$

where $\boldsymbol{y}$ and $\boldsymbol{f}$ are vectors that live in $\mathbb{R}^n$ (which can be extended to $\mathbb{C}^n$ without any complications) and A is an $n \times n$ matrix, in general time-dependent and with elements in $\mathbb{R}$ (or $\mathbb{C}$). In most books, the general theory of (4.1) is presented separately from the general theory of n-th-order linear ODEs (1.5). While this leads to a number of useful practical prescriptions, conceptually it is mostly a waste of time to deal with the latter type of ODEs separately. You will, however, find it instructive to work out in detail how what I am about to do here applies specifically to (1.5) (and, in particular, to the case with with constant coefficients: see §§5.1.3, 5.2.2, and 6). Also, because of its multitudinous applications and relative lack of algebraic cumbersomeness, the second-order linear ODE,

$$y'' + a(x)y' + b(x)y = f(x), \tag{4.2}$$

is particularly important and popular. In many cases below, (4.2) will serve me as a default example of the application of general principles (and in §5, I will study to death its version with constant coefficients).

4.1. *Existence and Uniqueness Theorem for Linear Equations*

[Literature: Arnold (2006, §27.2), Tenenbaum & Pollard (1986, §§62C, 65)]

A radical strengthening of Theorem 2 is possible for linear ODEs: the existence and uniqueness are guaranteed not just *locally* (in a neighbourhood of the point where initial conditions are set), but also *globally*.

Theorem 3. *If all $A_{ij}(t)$ and $f_i(t)$ $(i,j=1,\dots,n)$ are continuous functions on the interval $\mathscr{I}=(t_1,t_2)$, then $\forall t_0\in\mathscr{I}$ and $\forall \boldsymbol{y}_0\in\mathbb{R}^n$, the Cauchy problem*

$$\dot{\boldsymbol{y}}=\mathsf{A}(t)\cdot\boldsymbol{y}+\boldsymbol{f}(t),\quad \boldsymbol{y}(t_0)=\boldsymbol{y}_0 \tag{4.3}$$

has a unique solution on the entire interval $\mathscr{I}$.

For linear ODEs, existence will be ascertained explicitly in what follows—by constructing a solution. Uniqueness will be needed to give us certainty that *a* solution that we construct for any given Cauchy problem (4.3) is *the* solution of it.

The rest of this chapter is a general theory of how to solve linear equations, both homegeneous and inhomogeneous. If you do not feel up to suffering generality for a long, continuous stretch, you can take an alternative route through these notes. First learn how to handle homogeneous equations: read §4.2 (but not yet §4.2.1), then work through the example of a homogeneous, 2nd-order, linear, non-degenerate ODE with constant coefficients (§§5.1–5.1.1), then return to the general exposition of §4.3, followed by the practical recipe for finding solutions of 2nd-order linear ODEs in §4.6 (where, at this first reading, you can set $f=0$ everywhere)—the simplest example of how to apply this is the degenerate 2nd-order ODE with constant coefficients (§5.1.2), and some further examples are found in §4.6.1. Once you are thus an expert on homogeneous equations (an expertise that you can further practice in Problem Set 2), move on to inhomogeneous ones: read §4.2.1, then §4.4, and then again §4.6 but this time allow $f\neq 0$. Finally, move on to the inhomogeneous, 2nd-order equations with constant coefficients: §§5.2 and 5.3. Practice using all this in Problem Set 3.

4.2. *Superposition Principle*

[Literature: Arnold (2006, §27.3), Yeomans (2014, §III.0)]

Consider first the homogeneous version of (4.1):

$$\boxed{\dot{\boldsymbol{y}}=\mathsf{A}(t)\cdot\boldsymbol{y}}\,. \tag{4.4}$$

It is very easy to prove, but momentously significant, that if $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are solutions of (4.4), then $\forall$ constants C_1 and C_2,

$$\boldsymbol{y}(t)=C_1\boldsymbol{y}_1(t)+C_2\boldsymbol{y}_2(t) \tag{4.5}$$

is also a solution of (4.4)—this can be verified by direct substitution. This result is called the *superposition principle*—the fundamental property that makes linear ODEs special.

Its mathematical meaning is quite transparent. Stated more generally, it says that any linear combination of solutions of (4.4) is also a solution of (4.4)—the consequence of the linearity of the operator $\mathrm{d}/\mathrm{d}t-\mathsf{A}(t)$. In the language of linear algebra, this means that solutions of (4.4) form a linear vector subspace of the space of $\mathbb{R}^n$-valued functions of a single variable. Note that extending all this to the case of $\mathsf{A}(t)$, $\boldsymbol{f}(t)$ and $\boldsymbol{y}(t)$ being complex presents no difficulty (because a complex number is a kind of linear combination of real numbers, once i has been properly defined).

Example. §5.1.

4.2.1. *Superposition Principle for Inhomogeneous Equations*

[Literature: Tikhonov *et al.* (1985, §3.6.2), Yeomans (2014, §III.3)]

The superposition principle for homogeneous ODEs has two immediate corollaries that apply to inhomogeneous ODEs.

1) If $\boldsymbol{y}_1(t)$ is a solution of

$$\dot{\boldsymbol{y}} = \mathsf{A}(t) \cdot \boldsymbol{y} + \boldsymbol{f}_1(t) \tag{4.6}$$

and $\boldsymbol{y}_2(t)$ is a solution of

$$\dot{\boldsymbol{y}} = \mathsf{A}(t) \cdot \boldsymbol{y} + \boldsymbol{f}_2(t), \tag{4.7}$$

then, $\forall C_1$ and C_2,

$$\boldsymbol{y}(t) = C_1 \boldsymbol{y}_1(t) + C_2 \boldsymbol{y}_2(t) \tag{4.8}$$

is a solution of

$$\dot{\boldsymbol{y}} = \mathsf{A}(t) \cdot \boldsymbol{y} + C_1 \boldsymbol{f}_1(t) + C_2 \boldsymbol{f}_2(t). \tag{4.9}$$

This property, instantly verifiable by direct substitution, gives us a strategy for solving inhomogeneous ODEs whose $\boldsymbol{f}$'s are linear superpositions of more elementary/easier-to-handle functions.

2) A corollary of the above (for $\boldsymbol{f}_1 = 0$, $\boldsymbol{f}_2 = \boldsymbol{f}$) is the statement that if $\boldsymbol{y}_{\mathrm{PI}}(t)$ is some *particular* solution of (4.1) (a "particular integral", PI) and $\boldsymbol{y}_{\mathrm{CF}}(t)$ is the *general* solution of the associated homogeneous equation (4.4) (the "complementary function", CF), then

$$\boldsymbol{y}(t) = \boldsymbol{y}_{\mathrm{CF}}(t) + \boldsymbol{y}_{\mathrm{PI}}(t) \tag{4.10}$$

is the *general* solution of the inhomogeneous equation (4.1).[19] The proof is simply the observation that, having found some $\boldsymbol{y}_{\mathrm{PI}}(t)$, we can make the ansatz (4.10) and discover that $\boldsymbol{y}_{\mathrm{CF}}(t)$ satisfies (4.4), i.e., such an ansatz reduces an inhomogeneous equation to a homogeneous one. It is perfectly alright that $\boldsymbol{y}_{\mathrm{PI}}(t)$ can be chosen in many different ways because the difference between any two particular solutions of the inhomogeneous equation (4.1) is a solution of its homogeneous counterpart (4.4) and so is included in $\boldsymbol{y}_{\mathrm{CF}}(t)$.

Thus, if we learn how to find general solutions to homogeneous linear ODEs, we might have a fighting chance of solving inhomogeneous ODEs as well.

Example. §5.2.

4.3. *General Solution of Homogeneous Equations*

[Literature: Pontryagin (1962, §17), Tikhonov *et al.* (1985, §3.6.3)]

Since solutions of (4.4) form a linear vector space, we can capture them all if we are able to find a basis in their space. This basis will turn out to be a set of n linearly independent solutions

$$\{\boldsymbol{y}_1(t), \ldots, \boldsymbol{y}_n(t)\}, \tag{4.11}$$

where n is the order of the ODE. Such a set is called the *fundamental system* of solutions of (4.4). I will often use the shorter notation $\{\boldsymbol{y}_i(t)\}$ to designate the set (4.11) (and similarly for other sets of objects indexed by $i = 1, \ldots, n$). Solutions of an ODE are

[19] We have already met this principle in the context of first-order linear equations: see (2.62).

linearly *dependent* if $\exists C_1, \ldots, C_n$, not all of which are zero, such that[20]

$$\forall t \in \mathscr{I}, \quad C_1 \boldsymbol{y}_1(t) + \cdots + C_n \boldsymbol{y}_n(t) \equiv C_i \boldsymbol{y}_i(t) = 0 \tag{4.12}$$

(I will use Einstein's convention of summation over repeated indices). The solutions $\{\boldsymbol{y}_i(t)\}$ are *independent* otherwise, i.e., if (4.12) implies that all $C_i = 0$.

I am now going to demonstrate

1) that a fundamental system can always be constructed (i.e., it exists);

2) how, given a fundamental system, any Cauchy problem (4.3) can be solved in practice.

4.3.1. *(In)dependent Somewhere—(In)dependent Everywhere*

I will need the following two lemmas.

Lemma 1. *If $\exists t_0 \in \mathscr{I}$ such that the set of vectors $\{\boldsymbol{y}_i(t_0)\}$ is linearly independent, then the set of solutions $\{\boldsymbol{y}_i(t)\}$ is linearly independent.*

Proof. Suppose the solutions $\{\boldsymbol{y}_i(t)\}$ are linearly dependent. Then $\exists\{C_i\}$ such that (4.12) is satisfied. Since it is satisfied for all t, it is satisfied for t_0 and so $\{\boldsymbol{y}_i(t_0)\}$ are linearly dependent. Contradiction. Q.E.D.

Lemma 2. *If $\exists t_0 \in \mathscr{I}$ such that the set of vectors $\{\boldsymbol{y}_i(t_0)\}$ is linearly dependent, then the set of solutions $\{\boldsymbol{y}_i(t)\}$ is linearly dependent.*

Proof. If $\{\boldsymbol{y}_i(t_0)\}$ are linearly dependent, then $\exists\{C_i\}$ such that

$$C_i \boldsymbol{y}_i(t_0) = 0. \tag{4.13}$$

Consider the function $\boldsymbol{y}(t) = C_i \boldsymbol{y}_i(t)$. Since $\boldsymbol{y}_i(t)$ are all solutions of (4.4), so is $\boldsymbol{y}(t)$. In view of (4.13), $\boldsymbol{y}(t_0) = 0$. But identical 0 is a solution of (4.4) satisfying this initial condition. Since the solution of any Cauchy problem for (4.4) is unique, this implies $\boldsymbol{y}(t) = 0$ identically. Therefore, $C_i \boldsymbol{y}_i(t) = 0\ \forall t$. The solutions are dependent. Q.E.D.

Thus, if the vectors $\{\boldsymbol{y}_i(t)\}$ are (in)dependent for any $t = t_0$, they are (in)dependent for all t and so are the solutions that they represent.

4.3.2. *The Fundamental Matrix and the Wronskian*

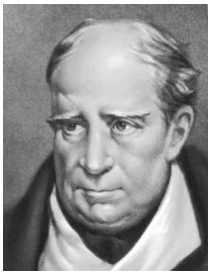
Let me introduce some notation that will make everything slightly more compact and also push us further into speaking of linear ODEs in the language of linear algebra. If I denote $y_i^j(t)$ the j-th vector component of the solution $\boldsymbol{y}_i(t)$, then the fundamental system $\{\boldsymbol{y}_i(t)\}$ can be packed into a single matrix with components $y_i^j(t)$, i.e., with the solution $\boldsymbol{y}_i(t)$ serving as the i-th column of the matrix:

$$\mathsf{Y}(t) = (\boldsymbol{y}_1(t) \ldots \boldsymbol{y}_n(t)), \quad \text{or} \quad Y_{ji}(t) = y_i^j(t). \tag{4.14}$$

[20] This says that $\{\boldsymbol{y}_i(t)\}$ are linearly dependent *as functions*, i.e., *the same* set of constants $\{C_i\}$ gets them all to sum up to zero at every t. This is *not* the same thing as *vectors* $\{\boldsymbol{y}_i(t)\}$ being linearly dependent at every t, because for the latter, it is enough to have a different set of constants $\{C_i(t)\}$ at every t. This distinction is important for understanding why the two Lemmas that are coming up are both true and non-trivial. In principle, the *vectors* $\{\boldsymbol{y}_i(t)\}$ being linearly dependent at some particular t does not automatically imply that the *functions* $\{\boldsymbol{y}_i(t)\}$ are also dependent in the sense of the definition (4.12), but Lemma 2 will say that if these functions are solutions of (4.4), then they are.



(a) Liouville



(b) Wroński

FIGURE 18. (a) Joseph Liouville FRS (1809-1882), great French mathematician and unsuccessful politician. He was a student of Poisson (who was a student of Lagrange; Fig. 8) and the adviser of Hermite (who was the father-in-law of Picard; Fig. 12b). Everyone who was anyone knew everyone else in those days. As they do now. Intellectual life is a small-group activity. Do jump in while you have the chance. (b) Józef Maria Hoene-Wroński (1776-1853), Polish messianic philosopher, mathematician, fierce critic of Lagrange's iniquitous use of infinite series and self-proclaimed author of the general theory of everything. While he was largely ignored as a crackpot during his life (apparently, not entirely unjustly), it did later turn out that he had invented some useful mathematical concepts and so his name lives forever as a determinant.

I shall call this the *fundamental matrix*. In view of the two lemmas proven in §4.3.1, the determinant of this matrix,

$$W(t) = \det \mathsf{Y}(t), \tag{4.15}$$

called the *Wronskian* (Fig. 18b), does not vanish anywhere if $\{\boldsymbol{y}_i(t)\}$ is a fundamental system. Conversely, if $\{\boldsymbol{y}_i(t)\}$ is just some set of n solutions of (4.4), checking whether it is a fundamental system amounts to checking that $W(t) \neq 0$ at some t.

The Wronskian turns out to be easy to compute:

$$W(t) = W(t_0) \exp\left[\int_{t_0}^{t} \mathrm{d}t' \operatorname{tr} \mathsf{A}(t')\right], \tag{4.16}$$

where $\operatorname{tr} \mathsf{A} = A_{ii}$, the trace of the matrix. This is *Liouville's formula* (Fig. 18a). Its proof is based on showing that $\dot{W} = \operatorname{tr} \mathsf{A}(t) W$. I leave it to you as an **exercise**, which you can complete either by figuring out on your own how to differentiate a determinant or by looking it up in a book, e.g., Pontryagin (1962, §17) or Arnold (2006, §§16.4, 27.6).

Note that the formula (4.16) makes it explicit that if $W(t)$ is (or is not) zero at any $t = t_0$, then it is (or is not) zero at all t and so linear (in)dependence of a set of solutions is equivalent to linear (in)dependence of these solutions at any one point in time.

4.3.3. *How to Construct a Fundamental System*

We are now ready for the following result.

Corollary 1. *A fundamental system of solutions of (4.4), i.e., a set of n linearly independent solutions (4.11) with n equal to the order of the ODE, always exists.*

Proof. Take any basis set of vectors $\{\boldsymbol{v}_i\}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$). By Theorem 3, there is a unique solution $\boldsymbol{y}_i(t)$ for each of n Cauchy problems, set at some t_0,

$$\dot{\boldsymbol{y}}_i = \mathsf{A}(t) \cdot \boldsymbol{y}_i, \quad \boldsymbol{y}_i(t_0) = \boldsymbol{v}_i, \quad i = 1, \ldots, n. \tag{4.17}$$

Since the vectors $\{\boldsymbol{y}_i(t_0)\}$ are linearly independent, so are the solutions $\{\boldsymbol{y}_i(t)\}$, by Lemma 1. There will always be n of them because any smaller system can be amended

with more linearly independent solutions until we have n, while no larger system can be independent because you cannot have more than n linearly independent vectors in $\mathbb{R}^n$ and if vectors $\{\boldsymbol{y}_i(t)\}$ are linearly dependent for any t, the corresponding set of solutions is also dependent (by Lemma 2).

4.3.4. *How to Construct the Solution*

Now let me show that the fundamental system (4.11) truly is a basis in the space of solutions.

Corollary 2. *If $\{\boldsymbol{y}_i(t)\}$ is a fundamental system of solutions of (4.4), then, for any Cauchy problem*

$$\dot{\boldsymbol{y}} = \mathsf{A}(t)\cdot\boldsymbol{y}, \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \tag{4.18}$$

$\exists\{C_i\}$ such that

$$\boldsymbol{y}(t) = C_i\boldsymbol{y}_i(t) \tag{4.19}$$

is the solution.

Proof. I will prove this by showing explicitly how to find the set of constants $\{C_i\}$ such that any given initial condition is satisfied (this solution is then the solution by the uniqueness Theorem 3). The linear combination (4.19) is, using the notation (4.14),

$$\boldsymbol{y}(t) = \mathsf{Y}(t)\cdot\boldsymbol{C}, \quad \text{where} \quad \boldsymbol{C} = \begin{pmatrix} C_1 \\ \dots \\ C_n \end{pmatrix}. \tag{4.20}$$

In order to have $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, we must have

$$\boldsymbol{C} = \mathsf{Y}^{-1}(t_0)\cdot\boldsymbol{y}_0. \tag{4.21}$$

The inverse $\mathsf{Y}^{-1}(t_0)$ exists $\forall t_0$ because the columns of the matrix are linearly independent and so the Wronskian $\det\mathsf{Y} \neq 0$. Finally, the (unique) solution of the Cauchy problem (4.18) is

$$\boxed{\boldsymbol{y}(t) = \mathsf{Y}(t)\cdot\mathsf{Y}^{-1}(t_0)\cdot\boldsymbol{y}_0}\,. \tag{4.22}$$

The interpretation of this formula is that the initial condition $\boldsymbol{y}_0$ gets decomposed—by multiplication by $\mathsf{Y}^{-1}(t_0)$—into bits corresponding to each member of the fundamental system ("projected" onto the basis "vectors" of the space of solutions) and then—by multiplication by $\mathsf{Y}(t)$—each of these bits is forwarded in time along its respective basis "vector" $\boldsymbol{y}_i(t)$.

Example. Let me apply all this to the homogeneous version of the second-order linear ODE (4.2):

$$y'' + a(x)y' + b(x)y = 0. \tag{4.23}$$

Formally, this reduces to (4.4) as follows:

$$\begin{cases} y' = p, \\ p' = -a(x)p - b(x)y, \end{cases} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} y \\ p \end{pmatrix} = \mathsf{A}(x)\cdot\begin{pmatrix} y \\ p \end{pmatrix}, \quad \text{where} \quad \mathsf{A}(x) = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, \tag{4.24}$$

but we do not actually need to write it like this explicitly. We know from Corollary 1 that the fundamental system for (4.23) must consist of two ($n = 2$) linearly independent solutions: $\{y_1(x), y_2(x)\}$. To make sure they are linearly independent, we must check that their Wronskian does not vanish anywhere (see §4.3.2), viz.,

$$\mathsf{Y}(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, \quad W(x) = \det\mathsf{Y}(x) = y_1y_2' - y_2y_1' \neq 0. \tag{4.25}$$

If this is true, then the general solution of (4.23) is (4.19):

$$y(x) = C_1 y_1(x) + C_2 y_2(x). \tag{4.26}$$

The Cauchy problem (4.18) is to find a solution that satisfies

$$y(x_0) = y_0, \quad y'(x_0) = p_0 \tag{4.27}$$

for some given y_0 and p_0. In order for that to happen, the two constants C_1 and C_2 must satisfy

$$\begin{cases} C_1 y_1(x_0) + C_2 y_2(x_0) = y_0, \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) = p_0 \end{cases}, \quad \text{or} \quad \mathsf{Y}(x_0) \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}. \tag{4.28}$$

The desired solution is then obtained by resolving these two simultaneous equations for C_1 and C_2: this is (4.21). It is possible to do this because $\det \mathsf{Y} \neq 0$, so the inverse of $\mathsf{Y}(x_0)$ exists:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{W(x_0)} \begin{pmatrix} y_2'(x_0) & -y_2(x_0) \\ -y_1'(x_0) & y_1(x_0) \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}. \tag{4.29}$$

Exercise 4.1. Sturm's Separation Theorem. Show that the zeros of two linearly independent solutions of (4.23) always alternate. In other words, show that if x_1 and x_2 are successive zeros of $y_1(x)$, then $y_2(x)$ has exactly one zero in the interval (x_1, x_2). *Hint.* Use the fact that the Wronskian $W(x) = y_1 y_2' - y_2 y_1'$ is never zero.

It should now be obvious to you how to deal with the n-th-order linear ODE

$$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = 0. \tag{4.30}$$

Here is an elegant argument (which I learned from Binney 2002, §2.1.3) that concisely illustrates the whole story. Consider the n-th-order linear ODE (4.30). Let us imagine that we know $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$. Then from (4.30) we can calculate $y^{(n)}(x_0)$. Taking a derivative of (4.30), we can then calculate $y^{(n+1)}(x_0)$. If we keep going, we can determine all derivatives of y at x_0, provided all $\{a_i(x)\}$ are as differentiable as they need to be. Knowing all the derivatives $y^{(k)}(x_0)$, we can reconstruct the whole solution as a Taylor series (cf. Q2.9):

$$y(x) = y(x_0) + \sum_{k=1}^{\infty} (x - x_0)^k \frac{y^{(k)}(x_0)}{k!}. \tag{4.31}$$

Thus, if we have n initial conditions that fix the values of $y(x_0), \ldots, y^{(n-1)}(x_0)$, we can find the (unique) solution (4.31) to (4.30). Clearly, to fix the values of these derivatives, we need to have n relationships between them (which do not have to be explicitly their values), amounting to having n constants $\{C_i\}$, on which the general solution of (4.30) then depends. If we now let $\{y_i(x)\}$ be n particular solutions of (4.30) such that for the i-th solution all constants except the i-th are zero, then the general solution is $y(x) = C_i y_i(x)$. In the dark ages before Lukas (2017), when Oxford physics students did not know linear algebra, this was all one could tell them.

4.4. *General Solution of Inhomogeneous Equations*

[Literature: Pontryagin (1962, §17), Tikhonov *et al.* (1985, §3.6.4), Arnold (2006, §29), Bender & Orszag (1999, §1.4)]

Now that we know how to find all solutions of the homogeneous linear ODE (4.4), we are ready to go back to the inhomogeneous one (4.1). According to the general principle expressed by (4.10), once we have the fundamental system of solutions $\mathsf{Y}(t)$ for (4.4), all we need is a particular solution of (4.1), $\boldsymbol{y}_{\mathrm{PI}}(t)$, and then its general solution is

$$\boldsymbol{y}(t) = \mathsf{Y}(t) \cdot \boldsymbol{C} + \boldsymbol{y}_{\mathrm{PI}}(t). \tag{4.32}$$

The Cauchy problem $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$ is then solved as in (4.21): the constants are

$$\boldsymbol{C} = \mathsf{Y}^{-1}(t_0) \cdot [\boldsymbol{y}_0 - \boldsymbol{y}_{\rm PI}(t_0)]\,. \tag{4.33}$$

This is great, but how do we find $\boldsymbol{y}_{\rm PI}(t)$ if we cannot just guess it? Here is a rather clever method, called the *method of variation of constants*, which we have already encountered in the context of a single first-order linear ODE (§2.4.2).

The idea is to weaponise the constants by turning them into functions, $C_i \to \psi_i(t)$ (same trick as in §2.4.2), and look for the solution of (4.1) in the form

$$\boldsymbol{y}(t) = \psi_i(t)\boldsymbol{y}_i(t) = \mathsf{Y}(t) \cdot \boldsymbol{\psi}(t), \quad \text{where} \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \dots \\ \psi_n \end{pmatrix}. \tag{4.34}$$

Substituting this into (4.1), we get

$$\dot{\boldsymbol{y}} = \cancel{\dot{\mathsf{Y}} \cdot \boldsymbol{\psi}} + \mathsf{Y} \cdot \dot{\boldsymbol{\psi}} = \cancel{\mathsf{A} \cdot \mathsf{Y} \cdot \boldsymbol{\psi}} + \boldsymbol{f} \quad \Rightarrow \quad \mathsf{Y} \cdot \dot{\boldsymbol{\psi}} = \boldsymbol{f}, \tag{4.35}$$

because, Y being a matrix composed of solutions of (4.4), $\dot{\mathsf{Y}} = \mathsf{A} \cdot \mathsf{Y}$. Since $\det \mathsf{Y} \neq 0$, (4.35) is easy to turn into an explicit expression for $\dot{\boldsymbol{\psi}}$, which we can then promptly integrate:

$$\dot{\boldsymbol{\psi}} = \mathsf{Y}^{-1}(t) \cdot \boldsymbol{f}(t) \quad \Rightarrow \quad \boldsymbol{\psi}(t) = \boldsymbol{C} + \int_{t_0}^{t} \mathrm{d}t'\, \mathsf{Y}^{-1}(t') \cdot \boldsymbol{f}(t'), \tag{4.36}$$

where $\boldsymbol{C}$ are integration constants, as usual. Putting this back into (4.34), we get the desired general solution of (4.1):

$$\boldsymbol{y}(t) = \mathsf{Y}(t) \cdot \left[\boldsymbol{C} + \int_{t_0}^{t} \mathrm{d}t'\, \mathsf{Y}^{-1}(t') \cdot \boldsymbol{f}(t')\right]. \tag{4.37}$$

The first term is $\boldsymbol{y}_{\rm CF}$ and the second is $\boldsymbol{y}_{\rm PI}$, which we have now found explicitly and in general (assuming that we know the fundamental system). Since I have picked the limits of integration in such a way as to make $\boldsymbol{y}_{\rm PI}(t_0) = 0$, the initial condition $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$ is enforced by (4.21) and so the solution of the Cauchy problem (4.3) is

$$\boxed{\boldsymbol{y}(t) = \mathsf{Y}(t) \cdot \left[\mathsf{Y}^{-1}(t_0) \cdot \boldsymbol{y}_0 + \int_{t_0}^{t} \mathrm{d}t'\, \mathsf{Y}^{-1}(t') \cdot \boldsymbol{f}(t')\right]}, \tag{4.38}$$

which is the generalisation of (4.22) to inhomogeneous ODEs. Recalling our interpretation of (4.22), it is easy to see the meaning of (4.38): the "force" $\boldsymbol{f}(t')$ can be thought of as a sequence of kicks to the system, each of these kicks gets decomposed into basis "vectors" $\boldsymbol{y}_i(t')$ of the space of solutions—by multiplication by $\mathsf{Y}^{-1}(t')$,—and then they all get forwarded in time along these basis "vectors" by multiplication by $\mathsf{Y}(t)$.

4.5. *Green's Function*

[Literature: Binney (2002, §6)]

This is a good place for the following consequential digression.

Let us define a matrix $\mathsf{G}(t, t_0)$, called *Green's function* (Fig. 19a), to be the solution of the following Cauchy problem

$$\frac{\partial}{\partial t}\mathsf{G}(t, t_0) = \mathsf{A}(t) \cdot \mathsf{G}(t, t_0), \quad \mathsf{G}(t_0, t_0) = \mathsf{I}. \tag{4.39}$$


(a) Green


(b) Dirac (in 1907)


(c) Heaviside

FIGURE 19. (a) George Green (1793-1841), an English baker and miller, who, in a triumph of mind (and inherited wealth) over matter, extricated himself and became a celebrated mathematical physicist and a Cambridge fellow. (b) Paul Adrien Maurice Dirac FRS (1902-1984), great English physicist, one of the founding fathers of quantum mechanics, protagonist of many stories of strange behaviour and author of many famous *bons mots*, allegedly unpremediated. (c) Oliver Heaviside FRS (1850-1925), English mathematician, electrical engineer and autodidact, Maxwell's apostle, bicycling enthusiast, inventor of a lot of very usefully non-rigorous (at the time) notation, including differential operators and vector calculus.

Then the solution of the Cauchy problem (4.3) is

$$\boldsymbol{y}(t) = \mathsf{G}(t, t_0) \cdot \boldsymbol{y}_0 + \int_{t_0}^{t} \mathrm{d}t' \, \mathsf{G}(t, t') \cdot \boldsymbol{f}(t'). \tag{4.40}$$

Indeed, checking this directly, we get $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$ and

$$\dot{\boldsymbol{y}}(t) = \mathsf{A}(t) \cdot \mathsf{G}(t, t_0) \cdot \boldsymbol{y}_0 + \mathsf{G}(t, t) \cdot \boldsymbol{f}(t) + \int_{t_0}^{t} \mathrm{d}t' \, \mathsf{A}(t) \cdot \mathsf{G}(t, t') \cdot \boldsymbol{f}(t') = \mathsf{A}(t) \cdot \boldsymbol{y}(t) + \boldsymbol{f}(t), \quad \text{Q.E.D.} \tag{4.41}$$

The meaning and purpose of Green's function (which is also sometimes called the *response function*) is to quantify the response of the linear system to initial ($\boldsymbol{y}_0$) and/or external ($\boldsymbol{f}$) disturbances.

In fact, it is sometimes convenient (and conceptually appealing) to treat these two types of externally prescribed influences on equal grounds, interpreting the initial condition as a kind of instantaneous burst at $t = t_0$ and the external force as a protracted sequence of bursts, or kicks,[21] at later (or, if you wish, also earlier) times—all this occurring within the overall history of evolution of $\boldsymbol{y}$ that stretches from $t = -\infty$ to $t = +\infty$. One then defines Green's function as the solution of the ODE

$$\boxed{\frac{\partial}{\partial t}\mathsf{G}(t, t_0) = \mathsf{A}(t) \cdot \mathsf{G}(t, t_0) + \mathsf{I}\,\delta(t - t_0)}\,, \tag{4.42}$$

where $\delta(t - t_0)$ is *Dirac's delta function* (Fig. 19b), a spiky "function" (in a generalised sense that is made precise in the theory of distributions) defined to be zero everywhere except at $t = t_0$ and having the property that

$$\forall f(t), \quad \int_{-\infty}^{+\infty} \mathrm{d}t \, \delta(t - t_0) f(t) = f(t_0). \tag{4.43}$$

Such a function can be formally constructed as the limit of a sequence of functions peaked at t_0,

[21] See discussion after formula (4.38).

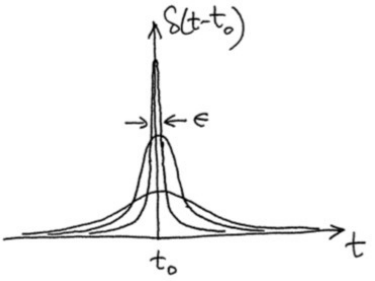


FIGURE 20. Dirac's delta function = limit of a sequence of collapsing functions with constant area underneath.

with ever decreasing width, increasing height and constant unit area underneath, e.g.,[22]

$$\delta(t-t_0) = \lim_{\epsilon\to+0} \frac{e^{-(t-t_0)^2/\epsilon^2}}{\sqrt{\pi}\epsilon} = \lim_{\epsilon\to+0} \frac{e^{-|t-t_0|/\epsilon}}{2\epsilon} = \lim_{\epsilon\to+0} \frac{\epsilon/\pi}{(t-t_0)^2+\epsilon^2}. \tag{4.44}$$

To (4.42), we attach a further stipulation that $\mathsf{G}(-\infty, t_0) = 0$. This implies that $\mathsf{G}(t, t_0) = 0$ for all $t < t_0$ because nothing happens in (4.42) until the unit burst at $t = t_0$, so zero is a perfectly good—and unique—solution until that moment. With Green's function defined in this new way, the solution (4.40) becomes

$$\boldsymbol{y}(t) = \int_{-\infty}^{+\infty} \mathrm{d}t'\, \mathsf{G}(t,t') \cdot \left[\boldsymbol{f}(t')H(t'-t_0) + \delta(t'-t_0)\boldsymbol{y}_0\right]. \tag{4.45}$$

Here $H(t-t_0)$ is *Heaviside's step function* (Fig. 19c), defined to be[23]

$$H(t-t_0) = \begin{cases} 1 & \text{if} \quad t > t_0, \\ 0 & \text{if} \quad t < t_0 \end{cases} \tag{4.46}$$

and used in (4.45) to restrict the application of the external force only to $t > t_0$. You might find it amusing to convince yourself that, for all practical purposes, $H'(t-t_0) = \delta(t-t_0)$.

It is not hard to check that (4.45) is indeed the solution of the Cauchy problem (4.3). Thus, this solution is now expressed as a convolution over the entire timeline between external disturbances and the system's response, the latter encoded in $\mathsf{G}(t, t_0)$ and entirely independent of those external disturbances. Obviously, we can easily read off from (4.38) an explicit expression for the Green's function in terms of the fundamental matrix:

$$\boxed{\mathsf{G}(t,t_0) = \mathsf{Y}(t) \cdot \mathsf{Y}^{-1}(t_0) H(t-t_0)}. \tag{4.47}$$

While these ideas might appear somewhat superfluous at this stage, you will encounter them many times again, e.g., in the Mathematical Methods course (Lukas 2019, §§5-8), when you deal with boundary-value problems for ODEs or with partial differential equations (and indeed already when you study the wave equation next term, there will be some tiptoeing around these notions).

Exercise 4.2. What is the Green's function for the linear, first-order ODE that we solved in §2.4?

Thus, if we can find the fundamental system $\{\boldsymbol{y}_i(t)\}$ [equivalently, the fundamental matrix $\mathsf{Y}(t)$ or the Green's function $\mathsf{G}(t, t_0)$] for the homogeneous linear ODE (4.4), we now know how to solve any Cauchy problem for any inhomogeneous ODE (4.1). But how do we find $\{\boldsymbol{y}_i(t)\}$? For a general linear ODE with time-dependent coefficients, this rather depends on what those coefficients are. When they are constant in time, everything can

[22] Another example is (5.54).

[23] What it is at $t = t_0$ is a moot point.

be solved completely generally—I shall do that in §6 (and, for a simple sub-case, in §5). But some progress can also be made with the case of time-dependent coefficients if we focus on our favorite example (4.2).

4.6. *Buy One Get One Free*

[Literature: Binney (2002, §2.4)]

So, consider the Cauchy problem for the second-order linear ODE (4.2):

$$\boxed{y'' + a(x)y' + b(x)y = f(x)}\,, \quad y(x_0) = y_0, \quad y'(x_0) = p_0. \tag{4.48}$$

Suppose that, by some feat of luck or inspiration, we have found one particular solution of the associated homogeneous equation (4.23). Let us call this solution $y_1(x)$. It turns out that a kind of "buy one get one free" deal is available: with $y_1(x)$ in hand, we can easily find another, linearly independent, solution $y_2(x)$ of (4.23) and, therefore, the general solution of (4.48).

This is again done by the method of variation of constants, viz., by looking for a general solution in the form[24]

$$y(x) = \psi(x)y_1(x). \tag{4.49}$$

Let us see what this ansatz does for us. Substituting (4.49),

$$y' = \psi' y_1 + \psi y_1' \quad \text{and} \quad y'' = \psi'' y_1 + 2\psi' y_1' + \psi y_1'' \tag{4.50}$$

into (4.48), we get

$$\psi'' y_1 + 2\psi' y_1' + \cancel{\psi y_1''} + a\psi' y_1 + \cancel{a\psi y_1'} + \cancel{b\psi y_1} = f, \tag{4.51}$$

where the cancellations are due to y_1 being a solution of the homogeneous equation (4.23). As a result, we are left with a first-order ODE for ψ':

$$\psi'' + \left(2\,\frac{y_1'}{y_1} + a\right)\psi' = \frac{f}{y_1} \tag{4.52}$$

(assuming $y_1 \neq 0$), i.e., the highly beneficial effect of the ansatz (4.49) is to lower the order of our original equation (4.48). We already know how to solve any first-order linear ODE, of which (4.52) is a perfectly ordinary example: using (2.61), we get

$$\psi'(x) = \mathfrak{E}(x)\left[C_1 + \int_{x_0}^{x} \mathrm{d}x'\,\frac{f(x')}{y_1(x')\mathfrak{E}(x')}\right], \tag{4.53}$$

where

$$\mathfrak{E}(x) = \exp\left\{-\int_{x_0}^{x} \mathrm{d}x'\left[2\,\frac{y_1'(x')}{y_1(x')} + a(x')\right]\right\} = \left[\frac{y_1(x_0)}{y_1(x)}\right]^2 \exp\left[-\int_{x_0}^{x} \mathrm{d}x' a(x')\right]. \tag{4.54}$$

Integrating (4.53) and putting it back into (4.49), we arrive at the general solution of our inhomogeneous ODE, complete with two constants:

$$\boxed{y(x) = y_1(x)\left[C_2 + C_1\int_{x_0}^{x} \mathrm{d}x'\mathfrak{E}(x') + \int_{x_0}^{x} \mathrm{d}x'\,\mathfrak{E}(x')\int_{x_0}^{x'} \mathrm{d}x''\,\frac{f(x'')}{y_1(x'')\mathfrak{E}(x'')}\right]}\,. \tag{4.55}$$

[24] The reason this is going to work is that multiplying $y_1(x)$ by a non-constant function $\psi(x)$ is guaranteed to give you a function that is linearly independent of $y_1(x)$—it has the effect of restricting our search to the (1D) subspace of the space of solutions of our ODE that is not covered by $y_1(x)$.

The first term (one with C_2) is our already known solution $y_1(x)$ of the associated homogeneous equation (4.23), the second (with C_1) is another, linearly independent, solution of this equation [cf. (4.26)], and the third (with f) is a particular solution of the inhomogeneous equation (4.48), so the overall solution has the structure (4.32), or (4.37). Finally, the constants are found by enforcing the initial conditions:

$$y(x_0) = y_0 \quad \Rightarrow \quad y_0 = C_2 y_1(x_0) \quad \Rightarrow \quad C_2 = \frac{y_0}{y_1(x_0)}, \tag{4.56}$$

$$y'(x_0) = p_0 \quad \Rightarrow \quad p_0 = C_2 y_1'(x_0) + C_1 y_1(x_0) \quad \Rightarrow \quad C_1 = \frac{p_0 y_1(x_0) - y_0 y_1'(x_0)}{y_1^2(x_0)}. \tag{4.57}$$

The Cauchy problem is gloriously solved.

It is a straightforward but useful **exercise** for you to work out how this maps onto the general scheme laid out in §4.4.

Example. §§5.1.2, 5.2.1.

Exercise 4.3. Use Liouville's formula (4.16) to find the second linearly independent solution of a second-order ODE on the basis of knowing one of its solutions. Convince yourself that this is essentially the same procedure as laid out above.

Exercise 4.4. Work out the Green's function for (4.48).

Exercise 4.5. Show that the "buy one get one free" scheme introduced here is, in fact, the same deal as was available for Riccati equations (§2.6).

You might be wondering how the scheme presented for a second-order ODE generalises to n-th-order ODEs of the type (4.30). The basic answer is that if you know some set of m linearly independent solutions, you can lower the order of the ODE to $n - m$. I leave it to your curiosity to investigate how this is done.

There is, alas, no general method for finding $y_1(x)$—and indeed it is generally impossible to do, as you must have realised if you still remember the discussion around (3.14). Everything becomes rather simple, however, if we specialise to the case of constant coefficients—we shall do that in §5.

4.6.1. *Tips for Guessing Games*

While there is no general method, there are some empirical recommendations that apply to equations whose coefficients are polynomials. Here they are:

- Try

$$y = e^{\lambda x}. \tag{4.58}$$

Plug this into your equation and see if it works for some λ. If it does, hurrah. **Example.** Q3.5(b) (and all of §5!).

- Try

$$y = x^n. \tag{4.59}$$

Plug this into your equation and set to zero the coefficient of the highest power of x. Hence get the value of n. Then check if $y = x^n$ is a solution. **Example.** Q3.5(a) (and §5.1.4).

- If (4.59) is not a solution, but n is a positive integer, try putting the polynomial

$$y = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n \tag{4.60}$$

into the equation and finding the coefficients $\{\alpha_i\}$ such that it is a solution. **Example.** Q2.3.

Example. Consider

$$2x(x+2)y'' + (2-x)y' + y = 0. \tag{4.61}$$

Try $y = x^n$. The term containing the highest power of x, which is x^n, is

$$2x^2 n(n-1)x^{n-2} - xnx^{n-1} + x^n = 0 \quad \Rightarrow \quad (2n-1)(n-1) = 0. \tag{4.62}$$

Let us try $n = 1/2$: success, $y(x) = \sqrt{x}$ is a solution! Let us then look for a general solution in the form (4.49):

$$y(x) = \psi(x)\sqrt{x} \quad \Rightarrow \quad y' = \psi\sqrt{x} + \frac{\psi}{2\sqrt{x}}, \quad y'' = \psi''\sqrt{x} + \frac{\psi'}{\sqrt{x}} - \frac{\psi}{4x^{3/2}}. \tag{4.63}$$

Putting this back into (4.61), we get, after some expected cancellations,

$$2x(x+2)\psi'' + (x+6)\psi' = 0 \quad \Rightarrow \quad \psi' = C_1\left(\frac{1}{2\sqrt{x}} + \frac{1}{x^{3/2}}\right) \quad \Rightarrow \quad \psi = C_1\left(\sqrt{x} - \frac{2}{\sqrt{x}}\right) + C_2. \tag{4.64}$$

The solution of (4.61) is, therefore,

$$y = \psi(x)\sqrt{x} = C_1(x-2) + C_2\sqrt{x}. \tag{4.65}$$

The two solutions are quite obviosuly linearly independent and we have two constants ready to meet our every desire as far as initial conditions are concerned.

What if, in (4.62), we had chosen the root $n = 1$ instead of $n = 1/2$? Trying $y(x) = x$ in (4.61) leads to disappointment: it is not a solution. But we do not despair and try the prescription (4.60): $y(x) = x + \alpha$. Putting this into (4.61) shows that it is a solution if $\alpha = -2$. Of course it is—this is the other solution in (4.65). Pretending you do not know what the second solution is, you can now look for a general solution in the form $y(x) = \psi(x)(x-2)$, follow the same algorithm as above and see if you can get (4.65) again.

Another useful exercise is to turn (4.61) into an inhomogeneous equation by putting some $f(x)$ into its right-hand side and then to solve it by variation of the constant C_1 in (4.64) [or just use (4.53)]. Try, e.g., $f(x) = 1$. It is, of course, ridiculously easy to guess what the particular integral is, but see if you can recover it by the standard method that I have described.

- If the finite sum (4.60) does not work, the next thing to try is an infinite series. How to do this is explained in Q2.9.

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5. Second-Order Linear ODE with Constant Coefficients

[Literature: Pontryagin (1962, §18), Tikhonov *et al.* (1985, §3.1), Coddington (1990, Ch. 2)]

While, in principle, I could have moved straight on to §6, the second-order linear ODE with constant coefficients is both ubiquitous and important all across physics, so I will discuss it in particular detail. This section is one long worked example. One way for you not to get too bored with it might be to do it all entirely on your own and then check back that you have caught all the interesting features.

5.1. *Homogeneous Equation*

[Literature: Pontryagin (1962, §7), Yeomans (2014, §III.1), Binney (2002, §2.5)]

Let us start with the homogeneous equation

$$\boxed{\ddot{y} + a\dot{y} + by = 0}\,, \tag{5.1}$$

where a and b are (real) constants. Let me take a "rabbit out of hat" approach and say that the winning strategy is to seek a solution in the form (4.58):

$$y(t) = C\,e^{\lambda t} \tag{5.2}$$

(you might already have realised that this is the inevitable choice for linear ODEs with

constant coefficients; in §6, we shall arrive at it naturally). Substituting (5.2) into (5.1), we find the condition (known as the *auxiliary equation*) for (5.2) to be a solution:

$$\lambda^2 + a\lambda + b = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{5.3}$$

This is excellent news: unless $a^2 = 4b$, we have two different values of λ and so two different solutions of (5.1). They are very obviously linearly independent, so we have our general solution (4.26):

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \tag{5.4}$$

The constants are determined by the initial conditions

$$y(0) = y_0 \quad \text{and} \quad \dot{y}(0) = v_0, \tag{5.5}$$

as follows:

$$C_1 + C_2 = y_0 \quad \text{and} \quad \lambda_1 C_1 + \lambda_2 C_2 = v_0. \tag{5.6}$$

Since (5.4) with C_1 and C_2 given by (5.6) is clearly a solution of the Cauchy problem (5.1) and (5.5), and since its solution is, by Theorem 3, unique, then this is *the* solution. There are two interesting cases.

$\boxed{a^2 > 4b}$: then λ_1 and λ_2 are both real and the solution is

$$\boxed{y(t) = e^{-at/2}\left(C_1 e^{t\sqrt{a^2/4-b}} + C_2 e^{-t\sqrt{a^2/4-b}}\right)}. \tag{5.7}$$

Thus, our equation (5.1) has growing and/or decaying solutions depending on the signs of λ_1 and λ_2. The initial conditions (5.5) determine what mixture of the two solutions emerges as a result. If we wait long enough, we will be left with the fastest growing (or, failing growth, slowest decaying) solution, with everything else exponentially small in comparison (cf. §8.1.1).

$\boxed{a^2 < 4b}$: in this case, λ_1 and λ_2 are complex:

$$\lambda_{1,2} = -\frac{a}{2} \pm i \underbrace{\sqrt{b - \frac{a^2}{4}}}_{\equiv\, \Omega}. \tag{5.8}$$

The solution is

$$y = e^{-at/2}\left(C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right), \tag{5.9}$$

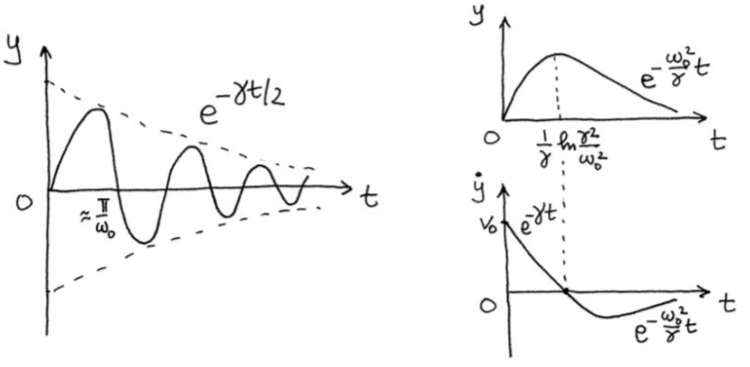
where the constants C_1 and C_2 are, in general, complex. However, one often wants a real solution, representing some real physical quantity. The solution (5.9) can be recast in two equivalent forms in which the constants can, if we wish, be purely real:

$$\boxed{y = e^{-at/2}\left(A\cos\Omega t + B\sin\Omega t\right) = e^{-at/2}\alpha\cos(\Omega t - \phi)}, \tag{5.10}$$

where $A = C_1 + C_2 = \alpha\cos\phi$ and $B = i(C_1 - C_2) = \alpha\sin\phi$ are both real if the initial conditions (5.5) are real.

Exercise 5.1. Find A, B, α and ϕ in terms of y_0 and v_0.

The solution (5.10) describes oscllations of frequency Ω, phase-shifted by ϕ, and with the amplitude $\alpha e^{-at/2}$, which either exponentially grows ($a < 0$) or decays ($a > 0$).



(a) (b)

FIGURE 21. (a) Underdamped and (b) overdamped oscillator. In (b), the upper plot shows the displacement (starting from $y_0 = 0$), the lower one the velocity (starting from v_0) in the limit $\gamma \gg \omega_0$.

5.1.1. *Damped Oscillator*

[Literature: Arnold (2006, §17.2), Yeomans (2014, §III.2), Binney (2002, §3.1), Tenenbaum & Pollard (1986, §§28-29A), Landau & Lifshitz (1976, §25)]

When $a \equiv \gamma > 0$ and $b = \omega_0^2 > 0$, (5.1), or

$$\boxed{\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0}\,, \tag{5.11}$$

describes small oscillations of a damped pendulum (damped SHO) with "own" (or "natural") frequency ω_0[25] and friction coefficient γ—we have encountered its nonlinear version in §3.5.4. The solution (5.10) describes explicitly what we already deduced "graphically" in working out its phase portrait (Fig. 16): oscillations at the frequency

$$\Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{5.12}$$

with superimposed exponential damping at the rate $\gamma/2$ (Fig. 21a). Note that in the physically interesting limit $\gamma \ll \omega_0$, $\Omega \approx \omega_0$.

In general, when $\omega_0 > \gamma/2$, the oscillator is called *underdamped* (cf. §8.1.6). When $\omega_0 < \gamma/2$, it is called *overdamped* and is described by (5.7).

It is interesting to look at the severe-overdamping limit, $\gamma \gg \omega_0$:

$$\begin{aligned} y(t) &= C_1 \exp\left[-\frac{\gamma t}{2}\left(1 - \sqrt{1 - \frac{4\omega_0^2}{\gamma^2}}\right)\right] + C_2 \exp\left[-\frac{\gamma t}{2}\left(1 + \sqrt{1 - \frac{4\omega_0^2}{\gamma^2}}\right)\right] \\ &\approx C_1 e^{-(\omega_0^2/\gamma)t} + C_2 e^{-(\gamma - \omega_0^2/\gamma)t}. \end{aligned} \tag{5.13}$$

Thus, we have two modes: one damped quickly (at the rate $\approx \gamma \gg \omega_0$), the other slowly (at the rate $\approx \omega_0^2/\gamma \ll \omega_0$). What is the physics of these two solutions?

To work out the physics, one usually must look at what kind of perturbations are subject to

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[25] This comes from the restoring force: gravity for a pendulum, Hookean tension for a spring, etc.

which of the damping rates. Let us impose the initial conditions (5.5):

$$C_1 + C_2 = y_0, \tag{5.14}$$

$$-\frac{\omega_0^2}{\gamma} C_1 - \left(\gamma - \frac{\omega_0^2}{\gamma}\right) C_2 = v_0. \tag{5.15}$$

If $v_0 = 0$, i.e., if there is only an initial displacement,

$$C_2 \approx -\frac{\omega_0^2}{\gamma^2} C_1 \ll C_1 \quad \Rightarrow \quad C_1 \approx y_0, \quad y(t) \approx y_0\, e^{-(\omega_0^2/\gamma)t}. \tag{5.16}$$

The solution decays slowly. In contrast, if $y_0 = 0$, i.e., if the oscillator is given an initial push but no displacement,

$$C_1 = -C_2 \quad \Rightarrow \quad C_2 \approx -\frac{v_0}{\gamma}, \quad y(t) \approx \frac{v_0}{\gamma}\left(1 - e^{-\gamma t}\right) e^{-(\omega_0^2/\gamma)t}. \tag{5.17}$$

The initial velocity first decays quickly, $\dot{y}(t) \approx v_0 e^{-\gamma t}$, while the displacement grows until reaching a peak $\approx v_0/\gamma$ at $t \approx \gamma^{-1} \ln(\gamma^2/\omega_0^2)$. After this time, we are back to the regime in which the residual displacement decays at the slow rate.[26] This is all illustrated in Fig. 21(b).

All this makes straightforward sense: in a highly frictional/viscous environment, an initial displacement of a pendulum will relax very slowly, whereas any motion initially imparted to it will stop very quickly.

Another way to see this is as follows. The equation

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0 \tag{5.18}$$

has two possible dominant balances: either between the restoring force and friction,

$$\gamma\dot{y} \approx -\omega_0^2 y, \tag{5.19}$$

giving the slowly damped displacement (5.16), or between friction and inertia,

$$\ddot{y} \approx -\gamma\dot{y}, \tag{5.20}$$

giving the quickly damped velocity (5.17).

All this might appear rather elementary, but it is worth understanding it clearly early on in your career: you will be surprised how relevant these things prove to be later on!

Example. Q6.3(d).

5.1.2. *Homogeneous Equation: Degenerate Case*

[Literature: Pontryagin (1962, §8), Yeomans (2014, §III.1), Binney (2002, §2.5)]

Now consider the special case of $a^2 = 4b$ (an oscillator with $\gamma = 2\omega_0$ is called *critically damped*). In this case, the two solutions of the auxiliary equation (5.3) are *degenerate* (meaning they are the same):

$$\lambda_1 = \lambda_2 = -\frac{a}{2}. \tag{5.21}$$

This means that we only have one solution available to us:

$$y_1(t) = Ce^{-at/2}. \tag{5.22}$$

To find the second one, we must invoke the "buy one get one free" deal of §4.6. Namely, we seek the solution of (5.1) in the form

$$y(t) = \psi(t)e^{-at/2}. \tag{5.23}$$

[26] In systems with several different damping rates, the eventual approach to equilibrium always happens at the slowest of these rates, and along the corresponding eigendirection (cf. §8.1.1).



FIGURE 22. Leonhard Euler (1707-1783) may well have been the greatest mathematician of all time—and was certainly the most prolific. He was also, it appears, the best paid professor in the Russian history (in St. Petersburg)—possibly the best-spent money in the Russian history.

Substituting this back into (5.1) and recalling that $b = a^2/4$, we find

$$\ddot{\psi} = 0 \quad \Rightarrow \quad \psi(t) = C_1 t + C_2 \quad \Rightarrow \quad \boxed{y(t) = (C_1 t + C_2) e^{-at/2}} \,. \tag{5.24}$$

Obviously, you do not need to repeat this calculation every time. If $\lambda_1 = \lambda_2 = \lambda$, just remember that one solution is $e^{\lambda t}$ and the other is $te^{\lambda t}$ (they are, clearly, linearly independent—if this is not obvious, prove it by calculating their Wronskian). In §§5.2.2 and 6.3, we shall see that there is a certain systematicity to the apperance of powers of t in front of exponentials.

5.1.3. *Above and Beyond: n-th-Order Homogeneous Equation*

[Literature: Pontryagin (1962, §§7-8), Tikhonov *et al.* (1985, §§3.1, 3.5), Bender & Orszag (1999, §1.4)]

Exercise 5.2. Consider an n-th-order ODE

$$\boxed{y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = 0} \,, \tag{5.25}$$

where $\{a_i\}$ are (real) constant coefficients. Let $\lambda_1, \ldots, \lambda_n$ be the roots of the auxiliary equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{5.26}$$

What is the general solution of (5.25)? How does one handle the case when some roots are complex? What about the case when some are degenerate? You may well have to resort to reading books in order to do this exercise in full generality.[27]

5.1.4. *Scale-Invariant (Euler's) Equation*

[Literature: Yeomans (2014, §III.4a), Coddington (1990, §4.2), Bender & Orszag (1999, §1.4)]

Having worked out the general method for solving (5.1), we have, in fact, also solved *Euler's equation* (Fig. 22)

$$\boxed{x^2 y'' + axy' + by = 0} \,. \tag{5.27}$$

This equation has the property of *scale invariance:* it does not change under rescaling of x by an arbitrary factor: $x \to \alpha x$. Therefore, it makes sense to seek solutions as power laws (which are scale-free functions):

$$y(x) = Cx^{\lambda}. \tag{5.28}$$

[27] If you wish to race ahead, you might derive some inspiration for this exercise from §6.3, but it is also possible to solve it without matrix algebra.

Substituting this back into (5.27) gives a polynomial auxiliary equation,

$$\lambda(\lambda-1)+a\lambda+b=0, \tag{5.29}$$

which has two solutions λ_1 and λ_2. If $\lambda_1\neq\lambda_2$, they give us the fundamental system for (5.27), so the general solution is

$$y(x)=C_1x^{\lambda_1}+C_2x^{\lambda_2}. \tag{5.30}$$

If $\lambda_1=\lambda_2=\lambda$, this gives us only one solution, x^λ; the second one is $x^\lambda\ln x$ (check).

In fact, we did not need a separate theory for Euler's equation because it is reduced to (5.1) by a simple change of variables:

$$x=e^t \quad\Rightarrow\quad xy'=\dot y, \quad x^2y''=\ddot y-\dot y \quad\Rightarrow\quad \ddot y+(a-1)\dot y+by=0 \tag{5.31}$$

and then the standard operating procedure applies. It is not hard to see that the same change of variables will reduce the n-th-order scale-invariant equation for $y(x)$

$$x^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\cdots+a_1xy'+a_0y=0 \tag{5.32}$$

to an n-th-order equation with constant coefficents for $y(t)$. If you have time on your hands, you can try and work out what those coefficients are.

5.2. *Inhomogeneous Equation*

Inevitably, we must now consider the inhomogeneous equation

$$\boxed{\ddot y+a\dot y+by=f(t)}\,. \tag{5.33}$$

According to the general scheme explained in §4.4, since we already know how to solve the homogeneous version of (5.33), all we need is to work out a particular solution, $y_{\rm PI}(t)$. Then

$$y(t)=C_1e^{\lambda_1t}+C_2e^{\lambda_2t}+y_{\rm PI}(t) \quad\text{or}\quad y(t)=(C_1t+C_2)\,e^{\lambda t}+y_{\rm PI}(t) \tag{5.34}$$

if $\lambda_1\neq\lambda_2$ or $\lambda_1=\lambda_2=\lambda$, respectively. There are two methods for finding $y_{\rm PI}(t)$:

—for certain simple forms of $f(t)$, it is easy to guess;

—for general $f(t)$, it can be determined via the method of variation of constants: the answer is contained in (4.37) or in (4.55).

Exercise 5.3. Given a and b, work out the explicit formula for $y_{\rm PI}(t)$ for general $f(t)$. Identify special cases requiring separate consideration.

Leaving the general consideration to you as an exercise, I will instead go through a few simple special cases of $f(t)$ for which the task of guessing $y_{\rm PI}(t)$ is particularly easy.

5.2.1. *Some Tips for Finding Particular Solutions*

[Literature: Pontryagin (1962, §10), Yeomans (2014, §III.3), Binney (2002, §2.6)]

Polynomial force. Suppose $f(t)$ is a polynomial of degree m:

$$\boxed{f(t)=F_0+F_1t+\cdots+F_mt^m}\,. \tag{5.35}$$

It is then not hard to guess that

$$\boxed{y_{\rm PI}(t)=\alpha_0+\alpha_1t+\cdots+\alpha_mt^m} \tag{5.36}$$

will work. Indeed, substituting this into (5.33) will produce powers of t up to t^m in the

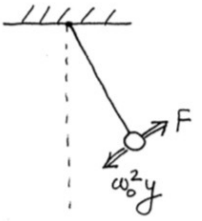


FIGURE 23. Pendulum in balance with a constant force.

left-hand side. Collect the coefficient in front of each power $k = 0, \ldots, m$ and equate it to F_k. This will give you m equations for m unknowns $\{\alpha_i\}$.

In particular, if $m = 0$, i.e., $f(t) = F_0 = \text{const}$, then $y_{\text{PI}} = F_0/b$. Physically, recalling our oscillator (§5.1.1), we interpret this solution as the constant displacement resulting from the balance between the restoring force $by = \omega_0^2 y$ and the constant applied force $f = F_0$ (Fig. 23).

Exponential force. Now consider

$$\boxed{f(t) = Fe^{\mu t}}\,. \tag{5.37}$$

The obvious solution to try is

$$\boxed{y_{\text{PI}}(t) = Ae^{\mu t}}\,. \tag{5.38}$$

Substituting this back into (5.33) gives us a relationship between A and μ:

$$A\left(\mu^2 + a\mu + b\right) = F \quad \Rightarrow \quad \boxed{A = \frac{F}{\mu^2 + a\mu + b}}\,. \tag{5.39}$$

This is fine unless $\mu^2 + a\mu + b = 0$, i.e., unless μ coincides with λ_1 or λ_2, the solutions of the auxiliary equation (5.3). Indeed, in this case (5.38) is one of the solutions of the associated homogeneous equation, so cannot work as a particular solution of the inhomogeneous one! What should we do?—What we always do, try variation of constants: supposing $\mu = \lambda_1$, we seek a solution in the form

$$y(t) = \psi(t)e^{\lambda_1 t}. \tag{5.40}$$

This is obviously a particular case of the calculation that we already went through in §4.6, but let me do this simple case from scratch, lest you complain that I never show you any worked examples: since

$$\dot{y} = (\dot{\psi} + \lambda_1\psi)e^{\lambda_1 t} \quad \text{and} \quad \ddot{y} = (\ddot{\psi} + 2\lambda_1\dot{\psi} + \lambda_1^2\psi)e^{\lambda_1 t}, \tag{5.41}$$

our equation becomes, after we use the fact that $\lambda_1^2 + a\lambda_1 + b = 0$,

$$\ddot{\psi} + (a + 2\lambda_1)\dot{\psi} = F. \tag{5.42}$$

We only need one particular solution. If $a + 2\lambda_1 \neq 0$, this one works:

$$\dot{\psi} = \frac{F}{a + 2\lambda_1} \quad \Rightarrow \quad \psi = \frac{Ft}{a + 2\lambda_1} \quad \Rightarrow \quad \boxed{y_{\text{PI}}(t) = \frac{Fte^{\lambda_1 t}}{a + 2\lambda_1}}\,. \tag{5.43}$$

Just as in §5.1.2, multiplication by t has done the trick.

What if $a + 2\lambda_1 = 0$? This is the degenerate case discussed in §5.1.2, in which $\lambda_1 = \lambda_2 = \lambda = a/2$, so both $e^{\lambda t}$ and $te^{\lambda t}$ are solutions of the homogeneous equation (5.1) and

cannot, therefore, be particular solutions of the inhomogeneous one. In this case, (5.42) is, in fact, even easier to solve:

$$\ddot{\psi} = F \quad \Rightarrow \quad \psi = \frac{F}{2}\,t^2 \quad \Rightarrow \quad \boxed{y_{\rm PI}(t) = \frac{F}{2}\,t^2 e^{\lambda t}}\,. \tag{5.44}$$

Composite force. Obviously, if $f(t)$ is a linear combination of more elementary functions, such as (5.35) or (5.37), we can use the superposition principle (4.8) to construct $y_{\rm PI}(t)$ as the same linear combination of particular solutions of the kind derived above. For example,

$$f(t) = A\cos\omega t = \frac{A}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \tag{5.45}$$

is just a linear combination of two exponentals (5.37), with $\mu = \pm i\omega$. This situation describes the physically important case of an oscillator subjected to a periodic force, so I will analyse it to death in §5.3.

Example. Q3.1, Q3.2.

5.2.2. *Above and Beyond: Quasipolynominals and n-th-Order Inhomogeneous Equation*

[Literature: Pontryagin (1962, §10), Tikhonov *et al.* (1985, §§3.1, 3.5), Arnold (2006, §26)]

Let me mention a general result that summarises and generalises the examples that I showed you in §5.2.1.

Consider the n-th-order ODE

$$\boxed{y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = \left(F_0 + F_1 t + \cdots + F_m t^m\right) e^{\mu t}}\,. \tag{5.46}$$

This equation has a particular solution

$$\boxed{y_{\rm PI}(t) = t^k\left(\alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m\right) e^{\mu t}}\,, \tag{5.47}$$

where $k = 0$ if μ is *not* a root of the auxiliary equation (5.26) and k is equal to the degree of degeneracy of μ if it *is* a root of (5.26) (i.e., k is equal to the number of times that μ occurs as a root). The constants $\{\alpha_i\}$ can found by substituting (5.47) into (5.46) and equating coefficients.

Functions of the form $(\alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m)e^{\mu t}$ are called *quasipolynomials*. The reason (5.47) works is that derivatives of quasipolynomials of degree m are also quasipolynomials of degree m. I leave it to your own researches (or literature searches) to work out the complete proof.

If you have done Exercise 5.2, you now know the general solution of (5.46), an n-th-order inhomogeneous ODE with constant coefficients and a quasipolynomial force.

5.3. *Forced Oscillator*

[Literature: Pontryagin (1962, §12), Arnold (2006, §26.6), Yeomans (2014, §IV), Binney (2002, §3.2), Tenenbaum & Pollard (1986, §29B), Landau & Lifshitz (1976, §26)]

We are now going to spend some time with the equation of the forced oscillator:

$$\boxed{\ddot{y} + \gamma\dot{y} + \omega_0^2 y = F\cos\omega t}\,. \tag{5.48}$$

To reiterate, the first term is inertia, the second friction, the third the Hookean (or gravitational, or whatever) restoring force and the right-hand side is the external periodic driving force.[28] We already know the solution of the associated homogeneous equation

[28] This is an equation of more general interest than it might superficially appear. When you poke

(see §5.1). Let us find the particular solution. As I explained after (5.45), the force in (5.48) is a linear combination of two complex exponentials, so the particular solution will also be some linear combination of these complex exponentials, or, equivalently, of sines and cosines. The unimaginative (and laborious) way of finding it is to write

$$y_{\mathrm{PI}}(t) = A\cos\omega t + B\sin\omega t, \tag{5.49}$$

substitute this into (5.48) and work out the coefficients A and B. There is, however, a much quicker way. Notice that $\cos\omega t = \operatorname{Re} e^{i\omega t}$, the coefficients in (5.48) are real and so the operation of taking the real part commutes with the differential operator in the left-hand side of (5.48). So we "complexify" (which, in this particular case, is a synonym of "simplify") our equation by introducing a complex function $z(t)$ that satisfies

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = Fe^{i\omega t}. \tag{5.50}$$

Then, clearly, $y = \operatorname{Re} z$, so all we need to do is solve (5.50) and then take the real part. As we did for the exponential force (5.37), let us look for a solution in the form

$$z(t) = Ce^{i\omega t}. \tag{5.51}$$

This gives us, upon substitution of (5.51) into (5.50),

$$C = \frac{F}{\omega_0^2 - \omega^2 + i\omega\gamma} = \frac{F\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2} = \frac{F\left(\cos\phi - i\sin\phi\right)}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}, \tag{5.52}$$

whence

$$z(t) = \frac{Fe^{i(\omega t-\phi)}}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}, \quad \text{where} \quad \boxed{\tan\phi = \frac{\omega\gamma}{\omega_0^2-\omega^2}}. \tag{5.53}$$

Taking the real part gives us

$$y_{\mathrm{PI}}(t) = A\cos(\omega t - \phi), \quad \text{where} \quad \boxed{A = \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}}. \tag{5.54}$$

Finally, the general solution of (5.48), with two constants eager to accommodate initial conditions, is

$$\boxed{y(t) = e^{-\gamma t/2}\left[C_1\cos\Omega t + C_2\sin\Omega t\right] + A\cos(\omega t - \phi)}, \tag{5.55}$$

where Ω is given by (5.12).

Exercise 5.4. Let $y(0) = y_0$ and $\dot{y}(0) = v_0$. Work out C_1 and C_2.

The part of the solution (5.55) containing the integration constants is *transient:* it decays away after $t \gg \gamma^{-1}$ and with it all memory of the initial conditions is lost. What does matter in the long-time limit is the interaction between the external forcing and the linear response properties of the system, the outcome of which is represented here by the last term in (5.55), called the *stationary*, or sometimes *steady-state*, solution. This part of the solution endures as long as the force is applied and describes *forced oscillations.* They are independent of the initial state of the oscillator, they have the amplitude A that

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an oscillator with a (perhaps periodic) force of arbitrary functional form $f(t)$, this force is usually representable (via Fourier transform) as a sum of periodic functions of different frequencies. Many complicated physical systems can be described, at least in the linear approximation, as superpositions of SHOs (which may be damped, forced, coupled). Thus, what we are studying here is a fundamental building block of many more complicated situations.

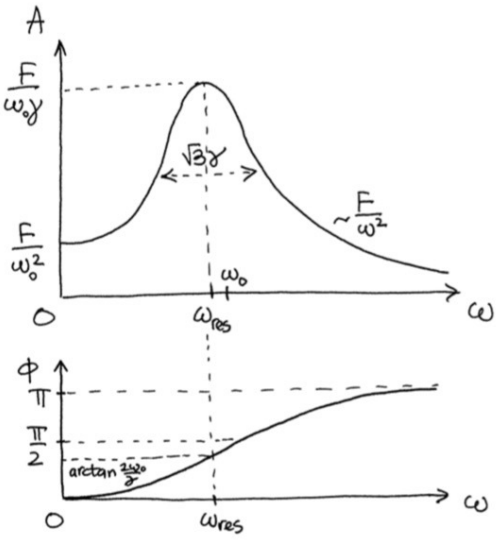


FIGURE 24. Amplitude A [see (5.54) or (5.59)] and phase ϕ [see (5.53)] of forced oscillations as a function of the frequency ω of the driving force. There is a resonance at the frequency (5.56), just below ω_0.

is proportional but not equal to the amplitude of the force and they are phase-shifted by ϕ with respect to it.

This is the simplest example of a generic feature of forced dissipative systems: their long-time behaviour is usually independent of initial conditions because dissipation washes away the system's memory.

5.3.1. *Resonance*

Let us examine how the amplitude of forced oscillations, given in (5.54), depends on the parameters of the problem (Fig. 24). The most interesting feature is that, when $\gamma \ll \omega_0$ (which is a "natural" physical limit for an oscillator—assuming we do want it to oscillate), $A(\omega)$ has a sharp peak around ω_0, i.e., when the frequency of the driver is close to the "own" frequency of the oscillator. This phenomenon is called *resonance*. The precise calculation, achieved by setting $A'(\omega) = 0$ places the peak at

$$\omega_{\rm res} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} \approx \omega_0. \tag{5.56}$$

This *resonant frequency* is just slightly below the own frequency of the undamped oscillator. The peak amplitude is

$$A_{\rm res} \approx A(\omega_0) = \frac{F}{\omega_0 \gamma}. \tag{5.57}$$

The phase at the resonance, (5.53) with $\omega = \omega_{\rm res}$ given by (5.56), satisfies

$$\tan \phi_{\rm res} = \frac{2\omega_0}{\gamma}. \tag{5.58}$$

The width of the peak (or, as one sometimes says, "of the resonance") is $\sim \gamma$. This is manifest in the functional dependence $A(\omega)$, but can be made more quantitative in the following way, popular in experimental circles. When $|\omega - \omega_0| \sim \gamma \ll \omega_0$, $\omega_0^2 - \omega^2 \approx$

$2\omega_0(\omega_0 - \omega)$ and (5.54) becomes

$$A(\omega) \approx \frac{F}{2\omega_0\sqrt{(\omega_0-\omega)^2+\gamma^2/4}}. \tag{5.59}$$

This implies that the amplitude drops to half of its maximum value (5.57) when

$$\frac{A(\omega)}{A_{\rm res}} \approx \frac{\gamma}{2\sqrt{(\omega_0-\omega)^2+\gamma^2/4}} = \frac{1}{2} \quad \Rightarrow \quad |\omega-\omega_0| = \frac{\sqrt{3}}{2}\,\gamma. \tag{5.60}$$

Thus, the so-called *full width at half maximum (FWHM)* is $\sqrt{3}\,\gamma$.

Other interesting features of the $A(\omega)$ dependence are

— the amplitude at zero frequency is

$$A(0) = \frac{F}{\omega_0^2}, \tag{5.61}$$

which is just the force balance between a constant driving force and the restoring force, $\omega_0^2 y = F$ (Fig. 23);

— the amplitude decays at high frequencies,

$$A(\omega) \approx \frac{F}{\omega^2}, \quad \text{when} \quad \omega \gg \omega_0 \gg \gamma, \tag{5.62}$$

because, when the force oscillates too fast, the system has no time to respond, so the force averages out and nothing much happens.[29]

As $\gamma \to +0$, the resonance becomes infinitely sharp, $A_{\rm res} \to \infty$ and $\phi_{\rm res} \to \pi/2$.

It is instructive to examine the case when $\gamma = 0$ literally (which is, of course, a mathematical abstraction) and $\omega = \omega_0$. The calculation of the particular solution in this case changes because $e^{i\omega_0 t}$ is itself one of the solutions of the homogeneous counterpart of (5.50). The answer is (5.43) with $a = 0$ and $\lambda_1 = i\omega_0$:

$$y_{\rm PI}(t) = {\rm Re}\left(\frac{Fte^{i\omega_0 t}}{2i\omega_0}\right) = \frac{tF}{2\omega_0}\,\sin\omega_0 t. \tag{5.63}$$

The full solution of (5.48) is then

$$y(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{tF}{2\omega_0}\,\sin\omega_0 t. \tag{5.64}$$

This solution has no damping away of the initial conditions and no stationary state; the amplitude of forced oscillations grows secularly with time (Fig. 25).

Another instructive exercise is to work out the solution of (5.48) with $\gamma = 0$ and $\omega \neq \omega_0$ and then show how (5.64) is recovered in the limit $\omega \to \omega_0$ (see Q3.3).

5.3.2. *Energy Budget of Forced Oscillator*

In §3.5.4, we already saw how the energy of a (damped) pendulum would behave and how understanding its behaviour could help one work out the system's phase portrait (Fig. 16b). Let us now investigate how the oscillating force enters the energy budget.

Multiplying (5.48) by $\dot{y}$, we find that the energy, kinetic ($\dot{y}^2/2$) plus potential ($\omega_0^2 y^2/2$),

[29] The case of a rapidly oscillating force becomes much more interesting if the force is made to depend on the displacement: see §5.4.

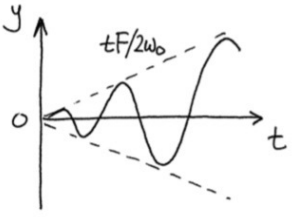


FIGURE 25. Resonance in the case of $\gamma = 0$: oscillations grow secularly.

evolves according to

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}^2}{2} + \omega_0^2\frac{y^2}{2}\right) = -\gamma\dot{y}^2 + \dot{y}F\cos\omega t. \tag{5.65}$$

The first term on the right-hand side is the dissipation (work done per unit time by the oscillator against the friction force), the second term is the power injected by the driving force (work done per unit time by the driver on the oscillator).

At $\gamma t \gg 1$, we have stationary forced oscillations:

$$y = A\cos(\omega t - \phi), \quad \dot{y} = -\omega A\sin(\omega t - \phi). \tag{5.66}$$

Using these expressions, we can find the *stored energy* in the oscillator averaged over the period $T = 2\pi/\omega$, $\langle\mathscr{E}\rangle = (1/T)\int_0^T \mathrm{d}t\,\mathscr{E}(t)$: since the mean square of a cosine or a sine over a period is $1/2$,

$$\langle\mathscr{E}\rangle = \frac{\langle\dot{y}^2\rangle + \omega_0^2\langle y^2\rangle}{2} = \frac{(\omega^2+\omega_0^2)A^2}{4} = \frac{(\omega^2+\omega_0^2)F^2}{4\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]} \tag{5.67}$$
$$\left(= \frac{F^2}{2\gamma^2} \quad \text{when} \quad \omega = \omega_0\right).$$

The dissipated power is

$$\mathscr{D} = \gamma\langle\dot{y}^2\rangle = \frac{\gamma\omega^2A^2}{2} = \frac{\gamma\omega^2F^2}{2\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]} \quad \left(= \frac{F^2}{2\gamma} \quad \text{when} \quad \omega = \omega_0\right). \tag{5.68}$$

The injected power is

$$\mathscr{P} = \langle\dot{y}F\cos\omega t\rangle = -\frac{\omega F^2\langle\cancel{\sin\omega t\cos\omega t}\,\cos\phi - \cos^2\omega t\,\sin\phi\rangle}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$
$$= \frac{\gamma\omega^2F^2}{2\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]} \quad \left(= \frac{F^2}{2\gamma} \quad \text{when} \quad \omega = \omega_0\right) \tag{5.69}$$

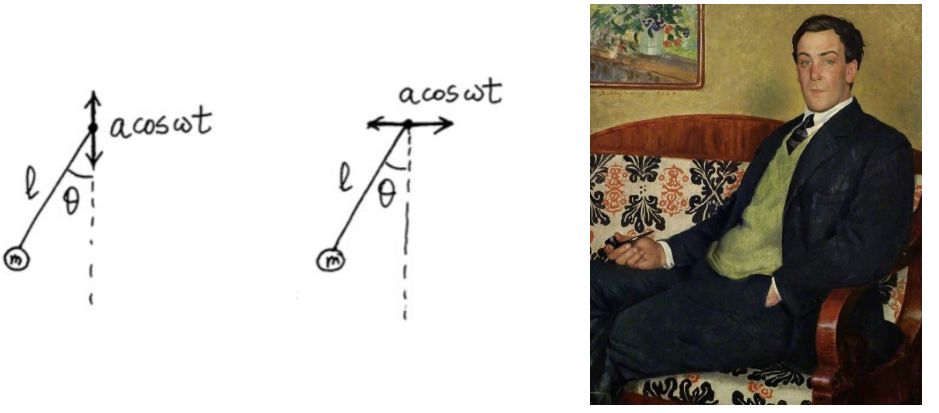
because, from (5.52), $\sin\phi = \omega\gamma/\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$. Thankfully, $\mathscr{P} = \mathscr{D}$, as it ought to be in steady state—that this ought to be the case follows from averaging $\langle(5.65)\rangle$:

$$\mathscr{P} - \mathscr{D} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\mathscr{E}\rangle = 0. \tag{5.70}$$

To measure how well an oscillator can store energy, one sometimes defines the so-called *quality factor* (or "Q-factor"):

$$Q = 2\pi\frac{\text{stored energy}}{\text{energy lost per period}}\bigg|_{\text{at resonance}} = \frac{2\pi\langle\mathscr{E}\rangle}{(2\pi/\omega)\mathscr{D}}\bigg|_{\omega=\omega_0} = \frac{\omega_0}{\gamma}. \tag{5.71}$$

Note that the quality factor can be defined in the same way for a damped oscillator



(a) (b) (c)

FIGURE 26. Pendulum whose point of suspension rapidly oscillates (a) vertically, (b) horizontally (Q3.9). (c) Peter Kapitza FRS (1894-1984) (painted in 1926 by B. Kustodiyev), one of the greatest experimental physicists of the 20th century, who worked with Rutherford in Cambridge, then (in 1934) made the mistake of a lifetime of visiting the Soviet Union, from whence he was not allowed to return, and where he proceeded to become the "godfather" of Soviet physics, both experimental and theoretical (Landau was in charge of the theory department at Kapitza's Institute). The Kapitza Club met informally in 1920-30s, at Trinity College, Cambridge, after dinner (obviously), and was famously a place where young physicists could argue freely, without regard to academic seniority.

(§5.1.1), using the (slowly damped) energy in the homogeneous solution, and it is equal to the same thing (I leave this to you as an exercise; see Q2.2).

Exercise 5.5. Work out the steady-state solution for a forced, overdamped oscillator ($\gamma \gg \omega_0$; see §5.1.1). How long does one have to wait for the initial conditions to be forgotten? Show that $\mathscr{P} \propto \omega^{-2}$ when $\gamma \ll \omega$ and $\mathscr{P} \propto \gamma^{-1}$ when $\gamma \gg \omega$ and explain why this is, physically.

I reckon I have spent quite enough time belabouring the second-order linear ODE with constant coefficients! Let us move on to more sophisticated material.[30]

5.4. *(Nonlinear) Digression: Rapidly Oscillating Force*

[Literature: Kapitza (1951*a*,*b*), Landau & Lifshitz (1976, §30), Arnold (2006, §28.4)]

To take a break from the tedium of linear equations and linear oscillations, let me show you an interesting piece of physics that is applicable to both linear and nonlinear systems. Let us consider a 1D system whose behaviour is described by the following ODE

$$\ddot{x} = -U'(x) + f(x,t). \tag{5.72}$$

Physically, x is some displacement variable, $U(x)$ is the potential energy, and $f(x,t)$ is an additional force that is allowed to depend on time. For the forced linear oscillator (in this case, without damping), $U(x) = \omega_0^2 x^2/2$ and $f(x,t) = F\cos\omega t$. For the nonlinear pendulum considered in §3.5.2, $U(x) = \omega_0^2(1-\cos x)$.

I would like to study a situation in which the force f oscillates extremely rapidly, i.e., its frequency ω is much larger than any other frequencies appearing in the equation (e.g., $\omega \gg \omega_0$). We already saw in §5.3.1 that when the force f had a constant amplitude, this was not a terribly

[30] This course's obsession with oscillators is set to continue though: I will interfere with a single SHO nonlinearly (§5.4) and parametrically (§8.2), couple several of them together (§7), and construct some nonlinear ones (§8.4, Q5.1).

interesting scenario: the amplitude of forced oscillations became very small because the effect of the force largely averaged out [see (5.62)]. Let us complicate matters by assuming that the oscillating force also depends on the displacement x:

$$f(x,t) = F(x)\cos\omega t. \tag{5.73}$$

An **example** of such a situation is *Kapitza's pendulum*—a mathematical pendulum (§3.1) whose point of suspension oscillates vertically as, say, $a\cos\omega t$ (Fig. 26a). This gives rise to an additional (non-inertial) vertical force $-ma\omega^2\cos\omega t$, which, when projected on the direction of the motion of the mass, becomes $-ma\omega^2\cos\omega t\sin\theta$. The equation (3.1) for the pendulum is then modified as follows:

$$ml\ddot\theta = -mg\sin\theta - ma\omega^2\sin\theta\cos\omega t. \tag{5.74}$$

In the notation of this chapter, $x=\theta$ and

$$\ddot x = -\omega_0^2\sin x - \omega_1^2\sin x\cos\omega t, \quad \text{where} \quad \omega_0^2 = \frac{g}{l}, \quad \omega_1^2 = \frac{a}{l}\,\omega^2. \tag{5.75}$$

This is (5.72) with the force (5.73):

$$\boxed{\ddot x = -U'(x) + F(x)\cos\omega t}\,, \quad \text{where} \quad U(x) = \omega_0^2(1-\cos x) \quad \text{and} \quad F(x) = -\omega_1^2\sin x. \tag{5.76}$$

We are assuming $\omega \gg \omega_0, \omega_1$ (the latter means $a \ll l$).

Let us seek the solution to this equation as a sum of some slow avearge motion and a fast oscillation on top of it. Formally, let

$$x(t) = \bar x(t) + \delta x(t), \quad \text{where} \quad \bar x(t) = \langle x(t)\rangle = \frac{1}{T}\int_t^{t+T}\mathrm{d}t' x(t'), \quad T = \frac{2\pi}{\omega} \quad \text{and} \quad \langle \delta x(t)\rangle = 0. \tag{5.77}$$

Assuming $\delta x \ll \bar x$ (as can be verified *a posteriori*), we can expand the right-hand side of our equation in δx:

$$\ddot{\bar x} + \delta\ddot x = -U'(\bar x) - \delta x\, U''(\bar x) + F(\bar x)\cos\omega t + \delta x\, F'(\bar x)\cos\omega t. \tag{5.78}$$

Taking the average of this equation gives us

$$\ddot{\bar x} = -U'(\bar x) + F'(\bar x)\langle \delta x(t)\cos\omega t\rangle. \tag{5.79}$$

Subtracting (5.79) from (5.78) and neglecting all terms that contain δx (which is small) but keeping its time derivatives (which are large), we get[31]

$$\delta\ddot x = F(\bar x)\cos\omega t \quad \Rightarrow \quad \delta x(t) = -\frac{F(\bar x)}{\omega^2}\cos\omega t \quad \Rightarrow \quad \langle \delta x(t)\cos\omega t\rangle = -\frac{F(\bar x)}{2\omega^2}. \tag{5.80}$$

Substituting the last expression into (5.79), we arrive at the following rather neat result:

$$\ddot{\bar x} = -U'(\bar x) - \frac{F'(\bar x)F(\bar x)}{2\omega^2} = -U'_{\rm eff}(\bar x), \quad \text{where} \quad \boxed{U_{\rm eff}(\bar x) = U(\bar x) + \frac{F^2(\bar x)}{4\omega^2}}\,. \tag{5.81}$$

Thus, the effect of a rapidly oscillating force is to change the potential energy of the system to a new, "effective" form $U_{\rm eff}$, which includes what is easily seen to be the average kinetic energy of the rapid oscillations:

$$\frac{F^2(\bar x)}{4\omega^2} = \frac{\langle \delta\dot x^2\rangle}{2}. \tag{5.82}$$

The effect of the additional force is to push the system out of the regions of phase space where the rapid oscillations have larger amplitude.

Applying these general results to our **example** of a pendulum with an oscillating suspension

[31] Check that, with this $\delta x(t)$, all the terms that we have neglected to get it are indeed smaller than the term that we have retained. Note, by the way, that the solution that we have got for $\delta x(t)$ is exactly the same solution as (5.62), but it is made consequential by the fact that F now depends on x.

point [see (5.76)], we find that

$$U_{\text{eff}} = \omega_0^2(1-\cos x) + \frac{\omega_1^4 \sin^2 x}{4\omega^2} = \frac{g}{l}(1-\cos\theta) + \frac{a^2\omega^2}{4l^2}\sin^2\theta. \tag{5.83}$$

Equilibria of the system satisfy

$$U'_{\text{eff}}(\theta_0) = \left(\frac{g}{l} + \frac{a^2\omega^2}{2l^2}\cos\theta_0\right)\sin\theta_0 = 0 \quad \Rightarrow \quad \theta_0 = 0,\ \pi,\ \arccos\left(-\frac{2gl}{a^2\omega^2}\right). \tag{5.84}$$

As in §3.5.2, an equilibrium is stable if it is a minimum of the energy, i.e., if

$$U''_{\text{eff}}(\theta_0) = \left(\frac{g}{l} + \frac{a^2\omega^2}{2l^2}\cos\theta_0\right)\cos\theta_0 - \frac{a^2\omega^2}{2l^2}\sin^2\theta_0 > 0. \tag{5.85}$$

Thus, the downward equlibrium $\theta_0 = 0$ is always stable, but, for a wiggly pendulum, the upright equilibrium $\theta_0 = \pi$ can also be stable, provided $a^2\omega^2 > 2gl$! There is also a "slanted" equilibrium, but it is always unstable.[32]

Exercise 5.6. Armed with the experience of §3.5.2, sketch the "effective" phase portrait [i.e., the phase portrait of the averaged motion $(\bar{x}, \dot{\bar{x}})$] for the wiggly pendulum discussed above. Investigate what effect friction will have.

Kapitza's original papers on the wiggly pendulum are Kapitza (1951*a*,*b*). If you want to invesigate further, you might find the recent treatment by Butikov (2020) an amusing read.

Another **example** of an effective average force arising due to a fast-oscillating real force is the case of a charged particle subject to rapidly oscillating electric field that has some spatial variation:

$$\ddot{x} = \frac{q}{m}E(x)\cos\omega t \quad \Rightarrow \quad m\ddot{\bar{x}} = -\frac{\mathrm{d}}{\mathrm{d}x}\frac{q^2E^2(\bar{x})}{4m\omega^2}. \tag{5.86}$$

This effective force is called *ponderomotive force*. It pushes charged particles out of regions where the oscillating electric field is stronger.

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6. Systems of Linear ODEs with Constant Coefficients

[Literature: Pontryagin (1962, §14), Tikhonov *et al.* (1985, §3.7), Yeomans (2014, §V), Binney (2002, §4)]

From the point of view of presenting a systematic, general theory, I could have skipped everything from §4.5 onward and jumped right to this point. By the end of §4.4, we learned how to solve the Cauchy problem for any (system of) linear ODEs, provided we knew their fundamental system of solutions, neatly packed into the fundamental matrix $\mathsf{Y}(t)$. I am now going to show you how to find this matrix for the linear ODEs of arbitrary order n but with constant coefficients:

$$\boxed{\dot{\boldsymbol{y}} = \mathsf{A}\cdot\boldsymbol{y} + \boldsymbol{f}(t)}\,. \tag{6.1}$$

This is the point at which this course merges completely with linear algebra.

The basic idea of the method is ridiculously simple but extraordinarily powerful. Suppose we can find n linearly independent eigenvectors $\{\boldsymbol{v}_i\}$ of the matrix A, with n eigenvalues $\{\lambda_i\}$:

$$\mathsf{A}\cdot\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i. \tag{6.2}$$

[32] A slanted equilibrium turns out to be the stable one if the pendulum is wiggled (sufficiently vigorously) in the horizontal direction: see Fig. 26(b) and Q3.9.

If we now consider the homogeneous ODE associated to (6.1),

$$\dot{\boldsymbol{y}} = \mathsf{A} \cdot \boldsymbol{y}, \tag{6.3}$$

and ask for its solutions in the form

$$\boldsymbol{y}_i(t) = \xi_i(t)\boldsymbol{v}_i, \tag{6.4}$$

we find that $\xi_i(t)$ must satisfy

$$\dot{\xi}_i = \lambda_i \xi_i \quad \Rightarrow \quad \xi_i(t) = C_i e^{\lambda_i t}. \tag{6.5}$$

Clearly, $\{\boldsymbol{y}_i(t)\}$ are linearly independent because $\{\boldsymbol{v}_i\}$ are, and (by assumption) there are n of them. This gives us a fundamental system, so any solution of the homogeneous equation (6.3) can be written as[33]

$$\boldsymbol{y}(t) = \sum_{i=1}^{n} C_i \boldsymbol{v}_i e^{\lambda_i t}, \tag{6.6}$$

any solution of the inhomogeneous equation (6.1) can be found by variation of constants (see §4.4) and any Cauchy problem then duly solved by (4.38). The fundamental matrix is

$$\boxed{\mathsf{Y}(t) = \left(\boldsymbol{v}_1 e^{\lambda_1 t} \ldots \boldsymbol{v}_n e^{\lambda_n t}\right)}. \tag{6.7}$$

I am now going to go through the construction of this solution step by step and then also deal with the slighly tricky question of what to do if a basis made of n eigenvectors of A does *not* exist (§6.3).

6.1. *Diagonalisable Systems With No Degenerate Eigenvalues*

[Literature: Arnold (2006, §§17.1, 19.5, 20.6)]

Let us look for *right* eigenvectors[34] of the matrix A:

$$\boxed{\mathsf{A} \cdot \boldsymbol{v} = \lambda \boldsymbol{v}}. \tag{6.8}$$

Let us suppose that the n roots $\lambda_1, \ldots, \lambda_n$ of the n-th-order polynomial equation

$$\boxed{\det(\mathsf{A} - \lambda \mathsf{I}) = 0} \tag{6.9}$$

are all different, i.e., they are non-degenerate right eigenvalues of A. Then their corresponding eigenvectors $\{\boldsymbol{v}_i\}$ form a basis (*not* necessarily orthogonal).

Proof. Let

$$(\mathsf{A} - \lambda_i \mathsf{I}) \cdot \boldsymbol{v}_i = 0, \quad i = 1, \ldots, n. \tag{6.10}$$

Suppose $\{\boldsymbol{v}_i\}$ are not linearly independent. Then it must be possible to write one of them as a linear combination of the rest:

$$\boldsymbol{v}_n = \sum_j \alpha_j \boldsymbol{v}_j, \tag{6.11}$$

where $\alpha_j \neq 0$ are some constants and the summation is over some subset of $\boldsymbol{v}_j$'s that *are*

[33] The solutions of the auxiliary equation (5.3), λ_1 and λ_2, which featured so prominently all through §5, are, of course, the eigenvalues $\{\lambda_i\}$, and the solution (5.4) is a particular case of (6.6), for a second-order ODE.

[34] I assume nothing about the matrix, so, in general, have to allow that its *right* eigenvectors are not the same as its *left* eigenvectors ($\boldsymbol{v} \cdot \mathsf{A} = \lambda \boldsymbol{v}$).

independent (and $j \neq n$). Then, on the one hand,

$$\mathsf{A} \cdot \boldsymbol{v}_n = \lambda_n \boldsymbol{v}_n = \lambda_n \sum_j \alpha_j \boldsymbol{v}_j, \tag{6.12}$$

whereas on the other hand,

$$\mathsf{A} \cdot \boldsymbol{v}_n = \sum_j \alpha_j \mathsf{A} \cdot \boldsymbol{v}_j = \sum_j \alpha_j \lambda_j \boldsymbol{v}_j. \tag{6.13}$$

Therefore,

$$\sum_j \alpha_j (\lambda_j - \lambda_n) \boldsymbol{v}_j = 0 \quad \Rightarrow \quad \alpha_j (\lambda_j - \lambda_n) = 0, \tag{6.14}$$

because $\boldsymbol{v}_j$'s are independent by assumption. Since $\lambda_j \neq \lambda_n$, the above implies $\alpha_j = 0$ for all j. Contradiction. Q.E.D.

Now pack the right eigenvectors into a matrix, as columns, and the eigenvalues into another matrix, as diagonal elements:

$$\mathsf{R} = (\boldsymbol{v}_1 \ldots \boldsymbol{v}_n), \quad \mathsf{L} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ldots & \\ 0 & & \lambda_n \end{pmatrix} \equiv \operatorname{diag}\{\lambda_i\}. \tag{6.15}$$

Then

$$\mathsf{A} \cdot \mathsf{R} = (\lambda_1 \boldsymbol{v}_1 \ldots \lambda_n \boldsymbol{v}_n) = \mathsf{R} \cdot \mathsf{L}. \tag{6.16}$$

Since $\{\boldsymbol{v}_i\}$ are independent, $\det \mathsf{R} \neq 0$, so R has an inverse. Therefore,

$$\boxed{\mathsf{A} = \mathsf{R} \cdot \mathsf{L} \cdot \mathsf{R}^{-1}}. \tag{6.17}$$

Note that this implies

$$\mathsf{R}^{-1} \cdot \mathsf{A} = \mathsf{L} \cdot \mathsf{R}^{-1}, \tag{6.18}$$

so rows of the inverse matrix R^{-1} are *left* eigenvectors of A, whilst its left eigenvalues are the same as its right eigenvalues:[35]

$$\mathsf{R}^{-1} = \begin{pmatrix} \boldsymbol{u}_1 \\ \ldots \\ \boldsymbol{u}_n \end{pmatrix}, \quad \text{where} \quad \boldsymbol{u}_i \cdot \mathsf{A} = \lambda_i \boldsymbol{u}_i. \tag{6.19}$$

Let us now use (6.17) to rewrite our homogeneous equation (6.3) as

$$\dot{\boldsymbol{y}} = \mathsf{A} \cdot \boldsymbol{y} = \mathsf{R} \cdot \mathsf{L} \cdot \mathsf{R}^{-1} \cdot \boldsymbol{y}. \tag{6.20}$$

Dotting this with R^{-1} on the left, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{R}^{-1} \cdot \boldsymbol{y} = \mathsf{L} \cdot \mathsf{R}^{-1} \cdot \boldsymbol{y} \quad \Rightarrow \quad \boxed{\dot{\boldsymbol{\xi}} = \mathsf{L} \cdot \boldsymbol{\xi}}, \quad \text{where} \quad \boldsymbol{\xi} = \mathsf{R}^{-1} \cdot \boldsymbol{y}. \tag{6.21}$$

But L is a diagonal matrix, so our system of ODEs breaks into n decoupled equations:

$$\dot{\xi}_i = \lambda_i \xi_i \quad \Rightarrow \quad \xi_i(t) = C_i e^{\lambda_i t} \tag{6.22}$$

(no summation over i here). What we have done, effectively, is *diagonalised* our system

[35] Note that $\{\boldsymbol{u}_i\}$ that form the rows of R^{-1} are *particular* left eigenvectors of A, i.e., you cannot scale them arbitrarily. If you did scale each $\boldsymbol{u}_i$ by a factor a_i, they would obviously still be left eigenvectors of A, but stacking them together will no longer give you the inverse of R, unless you rescaled the right eigenvectors $\{\boldsymbol{v}_i\}$ that form the columns of R by factors $1/a_i$.

of ODEs by transforming it into coordinates defined by the eigenvectors of the matrix A. Let us transform back into the original coordinates: the solution is

$$\boxed{\boldsymbol{y}(t) = \mathsf{R}\cdot\boldsymbol{\xi}(t) = \sum_i C_i \boldsymbol{v}_i e^{\lambda_i t}}\,, \tag{6.23}$$

just as was promised in (6.6).

Even if A is real, its eigenvalues are under no obligation to be real (unless A is Hermitian; see §6.2). Therefore, the solution (6.23) may well contain complex exponentials, complex eigenvectors and complex constants. There is nothing wrong with this, but it may be a bit unseemly if we are solving a physical system whose solution is supposed to be real. Just like in the case of a linear second-order ODE [see (5.9) and (5.10)], it is quite easy to recast the solution (6.23) in an explicitly real form.

Since $\{\lambda_i\}$ are roots of a polynomial, the complex ones come in conjugate pairs. So suppose λ and λ^* are eigenvalues. Then the corresponding eigevectors are also complex conjugates of each other: indeed, A being real,

$$\mathsf{A}\cdot\boldsymbol{v} = \lambda\boldsymbol{v} \quad\Rightarrow\quad \mathsf{A}\cdot\boldsymbol{v}^* = \lambda^*\boldsymbol{v}^*. \tag{6.24}$$

Then the corresponding bit of the general solution (6.23) is

$$\begin{aligned} C_1\boldsymbol{v}e^{\lambda t} + C_2\boldsymbol{v}^* e^{\lambda^* t} &= A\,\mathrm{Re}(\boldsymbol{v}e^{\lambda t}) + B\,\mathrm{Im}(\boldsymbol{v}e^{\lambda t}) \\ &= e^{\Gamma t}\left[(A\cos\Omega t + B\sin\Omega t)\,\mathrm{Re}(\boldsymbol{v}) + (B\cos\Omega t - A\sin\Omega t)\,\mathrm{Im}(\boldsymbol{v})\right], \end{aligned} \tag{6.25}$$

where $\lambda = \Gamma + i\Omega$ and the constants $A = C_1 + C_2$ and $B = i(C_1 - C_2)$ can be treated as real. The solutions $\mathrm{Re}(\boldsymbol{v}e^{\lambda t})$ and $\mathrm{Im}(\boldsymbol{v}e^{\lambda t})$ are real and linearly independent both of each other and of all the other solutions, so they can be used instead of $\boldsymbol{v}e^{\lambda t}$ and $\boldsymbol{v}^*e^{\lambda^* t}$. You get to practice this trick in Q4.3 (**example**).

For convenience of further manipulations, let me collect all the exponentials into a diagonal matrix:

$$\mathsf{E}(t) = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \cdots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} \equiv \mathrm{diag}\left\{e^{\lambda_i t}\right\}. \tag{6.26}$$

With this notation,

$$\boldsymbol{\xi}(t) = \mathsf{E}(t)\cdot\boldsymbol{C} \quad\Rightarrow\quad \boxed{\boldsymbol{y}(t) = \mathsf{R}\cdot\mathsf{E}(t)\cdot\boldsymbol{C}}\,, \tag{6.27}$$

where, as usual, $\boldsymbol{C}$ is the vector of integration constants. In the language of §4.3, we have found the fundamental matrix to be

$$\boxed{\mathsf{Y}(t) = \mathsf{R}\cdot\mathsf{E}(t)}\,, \tag{6.28}$$

as promised in (6.7).

Thus, we have the general solution of (6.3). Note that we did not need to calculate R^{-1} explicitly in order to obtain (6.23) or (6.27). We *will* need it to find $\boldsymbol{C}$ necessary to enforce any given initial condition.

So, let us now solve the Cauchy problem, $\boldsymbol{y}(0) = \boldsymbol{y}_0$. Since $\mathsf{E}(0) = \mathsf{I}$, we have

$$\mathsf{R}\cdot\boldsymbol{C} = \boldsymbol{y}_0 \quad\Rightarrow\quad \boldsymbol{C} = \mathsf{R}^{-1}\cdot\boldsymbol{y}_0 \quad\Rightarrow\quad \boxed{\boldsymbol{y}(t) = \mathsf{R}\cdot\mathsf{E}(t)\cdot\mathsf{R}^{-1}\cdot\boldsymbol{y}_0}\,, \tag{6.29}$$

which, in view of (6.28), is just (4.22), because $\mathsf{Y}(0) = \mathsf{R}$. Let me reiterate the procedure that the formula (6.29) represents:

—take the initial disturbance $\boldsymbol{y}_0$ and project it ($\mathsf{R}^{-1}\cdot\boldsymbol{y}_0$) onto the eigendirections of the matrix A;

—evolve in time the part of the disturbance corresponding to each eigendirection with the exponential factor corresponding to its eigenvalue ($\mathsf{E}(t) \cdot \mathsf{R}^{-1} \cdot \boldsymbol{y}_0$);

—project the result back onto the original coordinates ($\mathsf{R} \cdot \mathsf{E}(t) \cdot \mathsf{R}^{-1} \cdot \boldsymbol{y}_0$).

The physics of it all is now clear: in a physical system represented by A, initial disturbances along each eigenvector $\boldsymbol{v}_i$ of A grow or decay at the rate λ_i. Any arbitrary initial disturbance $\boldsymbol{y}_0$ can be decomposed into these elementary ones and evolves accordingly.

Exercise 6.1. How does one solve the system

$$\dot{\boldsymbol{y}} = \alpha(t)\, \mathsf{A} \cdot \boldsymbol{y}, \tag{6.30}$$

where $\alpha(t) > 0$ is a known, continuous, scalar function and A a time-independent matrix?

6.1.1. *General Solution of Inhomogeneous Equation*

The solution of the Cauchy problem for the inhomogeneous equation (6.1) is found in the usual way, via variation of constants, etc. In fact, it is just an application of the general result (4.38) to the case where $\mathsf{Y}(t)$ is given by (6.28), $t_0 = 0$ and $\mathsf{Y}(0) = \mathsf{R}$. Noticing that

$$\mathsf{Y}^{-1}(t') = \mathsf{E}^{-1}(t') \cdot \mathsf{R}^{-1} = \mathsf{E}(-t') \cdot \mathsf{R}^{-1} \quad \text{and} \quad \mathsf{Y}(t) \cdot \mathsf{Y}^{-1}(t') = \mathsf{R} \cdot \mathsf{E}(t - t') \cdot \mathsf{R}^{-1} \tag{6.31}$$

because $\forall t_1$ and t_2, $\mathsf{E}(t_1)\mathsf{E}(t_2) = \mathsf{E}(t_1 + t_2)$, we get

$$\boxed{\boldsymbol{y}(t) = \mathsf{R} \cdot \mathsf{E}(t) \cdot \mathsf{R}^{-1} \cdot \boldsymbol{y}_0 + \int_0^t \mathrm{d}t'\, \mathsf{R} \cdot \mathsf{E}(t - t') \cdot \mathsf{R}^{-1} \cdot \boldsymbol{f}(t')}\,. \tag{6.32}$$

Examining (6.32) and comparing it with the solution (2.61) of the 1D problem, you should now be struck by the exhilarating realisation that we have just worked out is how to exponentiate a matrix:

$$e^{\mathsf{A}t} = \mathsf{R} \cdot \mathsf{E}(t) \cdot \mathsf{R}^{-1}. \tag{6.33}$$

The trick was to diagonalise the matrix, exponentiate the eigenvalues, put them on the diagonal and transform it all back to the original coordinates. With this prescription, solving ODEs in n dimensions is no more difficult (conceptually) than solving them in one. Knowledge is power. Now that you have it, use it.

Exercise 6.2. What is the Green's function (§4.5) for a system of linear ODEs with constant coefficients? Does it make sense as a generalisation of the 1D case (Exercise 4.4)?

Example. Consider the vector generalisation of the exponential force (5.37):

$$\boldsymbol{f}(t) = \boldsymbol{F} e^{\mu t}, \tag{6.34}$$

where $\boldsymbol{F}$ is a constant vector. Let us work out the particular solution for this case—the second term in (6.32):

$$\begin{aligned}
\boldsymbol{y}_{\mathrm{PI}}(t) &= \int_0^t \mathrm{d}t'\, \mathsf{R} \cdot \mathsf{E}(t - t') \cdot \mathsf{R}^{-1} \cdot \boldsymbol{F} e^{\mu t'} \\
&= \mathsf{R} \cdot \mathrm{diag} \left\{ e^{\lambda_i t} \int_0^t \mathrm{d}t'\, e^{(\mu - \lambda_i)t'} \right\} \cdot \mathsf{R}^{-1} \cdot \boldsymbol{F} \\
&= \mathsf{R} \cdot \mathrm{diag} \left\{ \frac{e^{\mu t} - e^{\lambda_i t}}{\mu - \lambda_i} \right\} \cdot \mathsf{R}^{-1} \cdot \boldsymbol{F}
\end{aligned} \tag{6.35}$$

This formula works provided $\mu \neq \lambda_i$. If, for some i, $\mu = \lambda_i$, then that element of the diagonal matrix must be replaced by $te^{\lambda_i t}$. This follows either by doing the time integral correctly for this special case or by taking the limit $\mu \to \lambda_i$:

$$\frac{e^{\mu t} - e^{\lambda_i t}}{\mu - \lambda_i} = e^{\lambda_i t} \frac{e^{(\mu-\lambda_i)t} - 1}{\mu - \lambda_i} \to te^{\lambda_i t} \quad \text{as} \quad \mu - \lambda_i \to 0. \tag{6.36}$$

Example. Q4.4.

Sometimes, before solving the Cauchy problem, it is easier to find the general solution first, equipped with n constants:

$$\boldsymbol{y}(t) = \sum_i C_i \boldsymbol{v}_i e^{\lambda_i t} + \boldsymbol{y}_{\rm PI}(t). \tag{6.37}$$

Then, for the force in the form (6.34), one gets the particular solution as follows:

$$\boldsymbol{y}_{\rm PI}(t) = \boldsymbol{u}\, e^{\mu t} \quad \Rightarrow \quad \mu \boldsymbol{u} = \mathsf{A} \cdot \boldsymbol{u} + \boldsymbol{F} \quad \Rightarrow \quad \boldsymbol{u} = -(\mathsf{A} - \mu \mathsf{I})^{-1} \cdot \boldsymbol{F}. \tag{6.38}$$

Again, this only works if μ is not an eigenvalue of A (otherwise the inverse does not exist).

Exercise 6.3. Work out what to do if μ is an eigenvalue of A.

Example. Consider

$$\begin{cases} \dot{x} = 3x + 2y + 4e^{5t}, \\ \dot{y} = x + 2y. \end{cases} \tag{6.39}$$

To solve this, consider the homogeneous equation first. It is in the form (6.3), where the matrix is

$$\mathsf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \det(\mathsf{A} - \lambda \mathsf{I}) = \lambda^2 - 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = 1. \tag{6.40}$$

The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{6.41}$$

Therefore, the general solution of the homegeneous equation is

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\rm CF} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}. \tag{6.42}$$

This is the homogeneous bit of (6.37). Now let us look for the particular solution in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\rm PI} = \begin{pmatrix} A \\ B \end{pmatrix} e^{5t} \quad \Rightarrow \quad A = 3, \quad B = 1. \tag{6.43}$$

The values of the constants are obtained by direct substitution of the trial solution into (6.39). Final answer:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t}. \tag{6.44}$$

If such a simple method of finding $\boldsymbol{y}_{\rm PI}$ does not work, the universal panacea is to vary constants in (6.42): let $C_1 \to \psi_1(t)$, $C_2 \to \psi_2(t)$, substitute into (6.39) and solve for $\psi_{1,2}$.

You can now have some fun generalising some of the other results of §§5.2.1 and 5.2.2 to the case of an n-dimensional system of ODEs. Note, however, that we have not yet learned how to deal with a situation in which not all $\{\lambda_i\}$ are different.

6.2. *Hermitian Systems*

[Literature: Palmer (2019, §2.3)]

Suppose now that our matrix is Hermitian: $\mathsf{A} = \mathsf{A}^\dagger (\equiv \mathsf{A}^{*T})$. If $\{\boldsymbol{v}_i\}$ are its right eigenvectors, then

$$\mathsf{A} \cdot \boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \quad \Rightarrow \quad \lambda_i^* \boldsymbol{v}_i^* = \mathsf{A}^* \cdot \boldsymbol{v}_i^* = \mathsf{A}^T \cdot \boldsymbol{v}_i^* = \boldsymbol{v}_i^* \cdot \mathsf{A}. \tag{6.45}$$

Thus, $\boldsymbol{v}_i^*$ are left eigenvectors of A and, since the left eigenvalues must be the same as the right eigenvalues [see (6.19)], $\lambda_i = \lambda_i^*$ are all real.[36] Another very useful property is that, if $\lambda_i \neq \lambda_j$, then the corresponding eigenvectors are orthogonal:

$$\lambda_i \boldsymbol{v}_i \cdot \boldsymbol{v}_j^* = \boldsymbol{v}_j^* \cdot \mathsf{A} \cdot \boldsymbol{v}_i = \lambda_j \boldsymbol{v}_j^* \cdot \boldsymbol{v}_i \quad \Rightarrow \quad (\lambda_i - \lambda_j) \boldsymbol{v}_i \cdot \boldsymbol{v}_j^* = 0 \quad \Rightarrow \quad \boldsymbol{v}_i \cdot \boldsymbol{v}_j^* = 0. \tag{6.46}$$

If all $\{\lambda_i\}$ are different, we are still in the situation described in §6.1, when the matrix A is diagonalisable and the solution of (6.1) is (6.32), but now there is an important added simplification: if we normalise our eigenvectors in such a way that $|\boldsymbol{v}_i| = 1$, then

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j^* = \delta_{ij} \quad \Rightarrow \quad \boxed{\mathsf{R}^{-1} = \mathsf{R}^\dagger}\,, \tag{6.47}$$

i.e., R is a *unitary* matrix. Note that if A is real, then Hermitian means symmetric, $\mathsf{A} = \mathsf{A}^T$, and $\boldsymbol{v}_i^* = \boldsymbol{v}_i$, so $\mathsf{R}^{-1} = \mathsf{R}^T$ is an *orthogonal* matrix, describing rotations in $\mathbb{R}^n$.

What if not all the eigenvalues of A are different (i.e., some are degenerate)? For a Hermitian matrix, it is still guaranteed that there is an orthogonal basis $\{\boldsymbol{v}_i\}$ in which A is diagonal.

Proof. Formally, this is done by induction.

First, when the dimension of A is $n = 1$, A is automatically diagonal.

Now consider an $n \times n$ matrix A with $n > 1$ and assume that, in fewer dimensions than n, Hermitian matrices are diagonalisable. Suppose λ is one of the eigenvalues of A. Consider all vectors $\boldsymbol{w}$ that satisfy

$$\mathsf{A} \cdot \boldsymbol{w} = \lambda \boldsymbol{w}. \tag{6.48}$$

They form a linear subspace, W. All vectors $\boldsymbol{v}$ orthogonal to W (i.e., all $\boldsymbol{v}$ such that $\forall \boldsymbol{w} \in W$, $\boldsymbol{w}^* \cdot \boldsymbol{v} = 0$) also form a linear subspace, V, with $\dim V < n$. Then, since A is Hermitian,

$$\boldsymbol{w}^* \cdot (\mathsf{A} \cdot \boldsymbol{v}) = (\mathsf{A} \cdot \boldsymbol{w})^* \cdot \boldsymbol{v} = \lambda \boldsymbol{w}^* \cdot \boldsymbol{v} = 0 \quad \Rightarrow \quad \mathsf{A} \cdot \boldsymbol{v} \in V. \tag{6.49}$$

Thus, V is invariant under A, and, since $\dim V < n$ and A is Hermitian, A restricted to V is diagonalisable. But A restricted to W is also diagonal, by assumption (6.48). Thus, A is diagonalisable over all space $W + V$.

In practical terms, what the above argument tells us is as follows. For each eigenvalue λ_i of the matrix, we can construct a linear subspace W_i, defined as in (6.48). Vectors belonging to subspaces that correspond to different eigenvalues ($\lambda_i \neq \lambda_j$) will certainly be orthogonal to each other, by (6.46), and, by (6.49), all these subspaces are invariant under A. Thus, to diagonalise A, you just need to pick an orthogonal basis in each W_i. The induction guarantees that if λ_i's degree of degeneracy (i.e., the number of times it is repeated) is m, then $\dim W_i = m$, not less,[37] so the right number of orthogonal vectors in it will always be available (their choice is not unique, but that is perfectly fine).

[36] This also follows directly from $\lambda_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i^* = \boldsymbol{v}_i^* \cdot \mathsf{A} \cdot \boldsymbol{v}_i = \lambda_i^* \boldsymbol{v}_i^* \cdot \boldsymbol{v}_i$.

[37] It cannot be more because then it would not be possible to accommodate the remaining eigenvalues.

The diagonalisability of A means that the fundamental system of solutions of (6.3) still consists of the functions $\boldsymbol{v}_i e^{\lambda_i t}$. Even if $\lambda_i = \lambda_j$ for some $i \neq j$, the corresponding solutions are linearly idependent because the basis vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal. In other words, the solution of (6.1) is still (6.32). The "physical" difference with the non-degenerate case is that initial disturbances along the directions corresponding to a degenerate eigenvalue grow or decay at the same rate.

Note that the property (6.47) that the diagonalising transformation is a unitary one (or, for real A, is just a rotation) continues to hold, regardless of whether the eigenvalues are degenerate or not.

In §7, I will go through some important physical **examples** of Hermitian systems.

6.3. *Non-Hermitian Systems*

What if the matrix is non-Hermitian? As long as it is diagonalisable (which it definitely is if it has n different eigenvalues and which it might be if they are not all different), i.e., if there is a basis consisting of n linearly independent (but, not necessarily orthogonal) eigenvectors, then everything is fine and you can follow the procedure described in §6.1. But diagonalisability of non-Hermitian matrices is not guaranteed and situations when one must deal with non-diagonalisable matrices are quite common.

Here is a consequential **example**. Consider the n-th order linear ODE with constant coeffients

$$\boxed{y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = f(t)}\,. \tag{6.50}$$

As I have explained before (§3.3), any such equation can be rebranded as a system of n coupled first-order ODEs. Indeed, letting $p_i = y^{(i)}$ for $i = 1, \ldots, n-1$, we can rewrite (6.50) as follows

$$\left\{\begin{array}{rcl} \dot{p}_{n-1} &=& -a_{n-1} p_{n-1} - \cdots - a_1 p_1 - a_0 y + f(t), \\ \dot{p}_{n-2} &=& p_{n-1}, \\ \cdots & & \\ \dot{p}_1 &=& p_2, \\ \dot{y} &=& p_1, \end{array}\right. \tag{6.51}$$

or, in matrix form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_{n-1} \\ p_{n-2} \\ \cdots \\ y \end{pmatrix} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_{n-1} \\ p_{n-2} \\ \cdots \\ y \end{pmatrix} + \begin{pmatrix} f \\ 0 \\ \cdots \\ 0 \end{pmatrix}. \tag{6.52}$$

The matrix here is very obviously *not* Hermitian, so nothing is guaranteed. It is not a hard exercise to show that the equation for its eigenvalues is simply the auxiliary equation (5.26):

$$\det(\mathsf{A} - \lambda \mathsf{I}) = 0 \quad \Leftrightarrow \quad \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{6.53}$$

If this has n distinct roots, then A is diagonalisable by the scheme explained in §6.1, the general solution of the homogeneous equation associated to (6.50) is

$$y(t) = C_1 e^{\lambda_1 t} + \cdots + C_n e^{\lambda_n t} \tag{6.54}$$

(cf. Exercise 5.2), and the solution of the inhomogeneous equation is obtained via variation of constants in the usual manner. If, on the other hand, any of the roots are degenerate, the matrix is not diagonalisable.

There are, of course, many other systems, not necessarily reducible to (6.50), in which A is non-diagonalisable. This happens when for some eigenvalue λ that occurs m times, there are fewer than m independent vectors that satisfy $\mathsf{A} \cdot \boldsymbol{w} = \lambda \boldsymbol{w}$, i.e., the subspace W of vectors that satisfy this equation has $\dim W < m$. It turns out that when one encounters such systems, one need not despair.

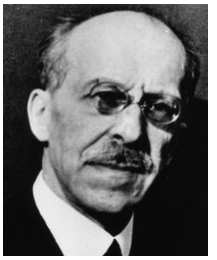


FIGURE 27. Issai Schur (1875-1941), Russian-Jewish mathematician, German professor, and refugee from the Nazis to Palestine. He was a student of Frobenius (see Q2.9) and adviser of Hopf (see footnote 61).

6.3.1. *Solution by Triangulation*

The reason there is hope is *Schur's triangulation theorem* (Fig. 27).

Theorem 4. *For any matrix* A*, there is a unitary transformation that converts it into an (upper) triangular form, viz.,* $\forall \mathsf{A}\ \exists \mathsf{U}$ *such that* $\mathsf{U}^{-1} = \mathsf{U}^\dagger$ *and*

$$\mathsf{U}^\dagger \cdot \mathsf{A} \cdot \mathsf{U} = \mathsf{T} = \begin{pmatrix} \lambda_1 & & \text{stuff} \\ & \dots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{6.55}$$

The triangular matrix T has stuff on and above the diagonal and zeros below. It is very easy to see that, if Schur's theorem holds, the diagonal elements of T are its eigenvalues and that they are also the eigenvalues of A:

$$\det(\mathsf{A}-\lambda\,\mathsf{I}) = \det(\mathsf{U}\cdot\mathsf{T}\cdot\mathsf{U}^\dagger - \lambda\mathsf{U}\cdot\mathsf{U}^\dagger) = \det[\mathsf{U}\cdot(\mathsf{T}-\lambda\,\mathsf{I})\cdot\mathsf{U}^\dagger] = \det(\mathsf{T}-\lambda\,\mathsf{I}) = (\lambda_1-\lambda)\dots(\lambda_n-\lambda). \tag{6.56}$$

I will prove Schur's theorem in §6.3.2—and in the process show you how to find U,[38]—but first let me show you how to solve an arbitrary system of ODEs with constant coefficients once one has the triangulating unitary transformation in hand.

Consider the homogeneous equation (6.3) and assume that we have U that satisfies (6.55), i.e.,

$$\dot{\boldsymbol{y}} = \mathsf{A}\cdot\boldsymbol{y} = \mathsf{U}\cdot\mathsf{T}\cdot\mathsf{U}^\dagger\cdot\boldsymbol{y}. \tag{6.57}$$

Analogously to (6.21), we dot this with $\mathsf{U}^\dagger$ on the left and get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{U}^\dagger\cdot\boldsymbol{y} = \mathsf{T}\cdot\mathsf{U}^\dagger\cdot\boldsymbol{y} \quad\Rightarrow\quad \boxed{\dot{\boldsymbol{\xi}} = \mathsf{T}\cdot\boldsymbol{\xi}}\,, \quad \text{where} \quad \boldsymbol{\xi} = \mathsf{U}^\dagger\cdot\boldsymbol{y}. \tag{6.58}$$

Let us call T_{ij} the elements of T above the diagonal and rewrite our system explicitly as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi_1\\ \dots\\ \xi_n\end{pmatrix} = \begin{pmatrix}\lambda_1 & T_{12} & \dots\\ & \dots & \\ 0 & & \lambda_n\end{pmatrix}\cdot\begin{pmatrix}\xi_1\\ \dots\\ \xi_n\end{pmatrix}. \tag{6.59}$$

We can start solving this system from the bottom up: the n-th line of (6.59) is

$$\dot{\xi}_n = \lambda_n\xi_n \quad\Rightarrow\quad \xi_n(t) = C_n e^{\lambda_n t}. \tag{6.60}$$

Then the $(n-1)$-st line of (6.59) is

$$\dot{\xi}_{n-1} = \lambda_{n-1}\xi_{n-1} + T_{n-1,n}\xi_n(t). \tag{6.61}$$

This is a first-order, inhomogeneous linear ODE and it is solved in the usual manner (§2.4.2),

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[38] In fact, you sort of already know how to do this, operationally: effectively, you have done this many times, when you applied the Gaussian elimination procedure to bring a matrix into an "upper echelon" form (Lukas 2017, §3.3).

by variation of constants:

$$\xi_{n-1}(t) = \psi(t) e^{\lambda_{n-1} t} \quad \Rightarrow \quad \dot{\psi} = T_{n-1,n} C_n e^{(\lambda_n - \lambda_{n-1})t}$$
$$\Rightarrow \quad \psi(t) = C_{n-1} + C_n T_{n-1,n} \begin{cases} \dfrac{e^{(\lambda_n - \lambda_{n-1})t}}{\lambda_n - \lambda_{n-1}} & \text{if} \quad \lambda_{n-1} \neq \lambda_n, \\ t \quad \text{if} \quad \lambda_{n-1} = \lambda_n. \end{cases} \tag{6.62}$$

Thus,

$$\xi_{n-1}(t) = C_{n-1} e^{\lambda_{n-1} t} + C_n T_{n-1,n} e^{\lambda_n t} \begin{cases} \dfrac{1}{\lambda_n - \lambda_{n-1}} & \text{if} \quad \lambda_{n-1} \neq \lambda_n, \\ t \quad \text{if} \quad \lambda_{n-1} = \lambda_n. \end{cases} \tag{6.63}$$

Moving upwards and onwards, the $(n-2)$-nd line of (6.59) is

$$\dot{\xi}_{n-2} = \lambda_{n-2} \xi_{n-2} + T_{n-2,n-1} \xi_{n-1}(t) + T_{n-2,n} \xi_n(t). \tag{6.64}$$

Again we have a first-order linear ODE with a known inhomogeneous part. Etc., etc. Once all ξ_i's have been found in this manner, we can recover the solution of (6.3) by reversing the triangulating transformation,

$$\boxed{\boldsymbol{y}(t) = \mathsf{U} \cdot \boldsymbol{\xi}(t)}\,. \tag{6.65}$$

Observing how powers of t pop up in front of exponentials every time a degenerate eigenvalue is repeated, you can probably see now how to handle the last part of Exercise 5.2 (how did you handle it then? was that equivalent to what we are doing here?).

Finally, extending the above procedure to the general inhomogeneous equation (6.1) presents no conceptual difficulty, so I leave you to your own devices with that (cf. §5.2.2).

6.3.2. *Proof of Schur's Triangulation Theorem*

The proof of Schur's triangulation theorem gives one some idea as to how to construct the unitary transformation U.

Start by finding just one eigenvector of our matrix:

$$\mathsf{A} \cdot \boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1. \tag{6.66}$$

Consider an orthonormal basis $\{\boldsymbol{w}_i\}$ where $\boldsymbol{w}_1 = \boldsymbol{v}_1$ and all the other vectors are chosen so as to satisfy $\boldsymbol{w}_i^* \cdot \boldsymbol{w}_j = \delta_{ij}$. Then the $n \times n$ matrix

$$\mathsf{R} = (\boldsymbol{v}_1 \ \boldsymbol{w}_2 \ \ldots \ \boldsymbol{w}_n) \tag{6.67}$$

in which $\boldsymbol{w}_i$'s serve as columns is a unitary transformation, viz., $\mathsf{R}^{-1} = \mathsf{R}^\dagger$. Apply this transformation to A:

$$\mathsf{R}^\dagger \cdot \mathsf{A} \cdot \mathsf{R} = \begin{pmatrix} \boldsymbol{v}_1^* \\ \boldsymbol{w}_2^* \\ \ldots \\ \boldsymbol{w}_n^* \end{pmatrix} \cdot \Big(\lambda_1 \boldsymbol{v}_1 \ \mathsf{A} \cdot \boldsymbol{w}_2 \ \ldots \ \mathsf{A} \cdot \boldsymbol{w}_n \Big) = \begin{pmatrix} \lambda_1 & \boldsymbol{v}_1^* \cdot \mathsf{A} \cdot \boldsymbol{w}_2 & \ldots & \boldsymbol{v}_1^* \cdot \mathsf{A} \cdot \boldsymbol{w}_n \\ 0 & & & \\ \ldots & & \mathsf{A}_{n-1} & \\ 0 & & & \end{pmatrix}, \tag{6.68}$$

where the $(n-1) \times (n-1)$ matrix A_{n-1} has elements $\boldsymbol{w}_i^* \cdot \mathsf{A} \cdot \boldsymbol{w}_j$ with $i, j = 2, \ldots, n$. In words, we have found a unitary transformation whereby all sub-diagonal elements in the first column of our transformed matrix are now zero.

For $n = 2$, this completes the process of triangulation. For $n > 2$, subject the matrix A_{n-1} to the same treatment by finding one of its eigenvectors and hence a unitary transformation R_{n-1} that clears up the subdiagonal elements in the first column of A_{n-1}, etc. Keep going until the matrix has been turned into a triangular one.

Let me argue formally by induction that this works. We have seen that for $n = 2$, any matrix can be triangulated by a unitary transformation. Now suppose any $(n-1) \times (n-1)$ matrix can be so triangulated. Then, in (6.68), $\exists \mathsf{U}_{n-1}$, a unitary transformation such that

$$\mathsf{A}_{n-1} = \mathsf{U}_{n-1} \cdot \mathsf{T}_{n-1} \cdot \mathsf{U}_{n-1}^\dagger, \tag{6.69}$$

where T_{n-1} is an $(n-1)\times(n-1)$ triangular matrix. Therefore, (6.68) becomes

$$\mathsf{A} = \mathsf{R}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathsf{U}_{n-1}\end{pmatrix}\cdot\begin{pmatrix}\lambda_1 & \text{stuff}\\ 0 & \mathsf{T}_{n-1}\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathsf{U}^\dagger_{n-1}\end{pmatrix}\cdot\mathsf{R}^\dagger = \mathsf{U}\cdot\mathsf{T}\cdot\mathsf{U}^\dagger, \tag{6.70}$$

where

$$\mathsf{T} = \begin{pmatrix}\lambda_1 & \text{stuff}\\ 0 & \mathsf{T}_{n-1}\end{pmatrix} \quad\text{and}\quad \mathsf{U} = \mathsf{R}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathsf{U}_{n-1}\end{pmatrix} \tag{6.71}$$

are, respectively, the desired $n\times n$ triangular matrix and the unitary transformation that turns A into it. Q.E.D.

Note that

$$\mathsf{U}_{n-1} = \mathsf{R}_{n-1}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathsf{U}_{n-2}\end{pmatrix}, \quad \mathsf{U}_{n-2} = \mathsf{R}_{n-2}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathsf{U}_{n-3}\end{pmatrix}, \quad \dots, \tag{6.72}$$

where $\mathsf{R}_{n-1}, \mathsf{R}_{n-2}, \dots$ are unitary transformations constructed analogously to (6.67), but for the $(n-1), (n-2), \dots$-dimensional matrices that are still in need of triangulation at the corresponding steps of this procedure.

Note, finally, that the technique of solution by triangulation is perfectly applicable to systems whose eigenvalues are not degenerate and that are, in fact, diagonalisable. Obviously, the diagonalisation (6.17) is also a triangulation, because a diagonal matrix is triangular. However, there is no requirement that any particular triangulation be unique, and the (in general, non-unitary) diagonalising transformation R in (6.17) is under no obligation to be the same as the unitary triangulating transformation U. Obviously, in solving a diagonalisable system of ODEs, if you go down the triangulation route, while the intermediate steps will be different, you will still get the same answer—to see this, just assume that no two eigenvalues are the same and examine the output of the scheme laid out in §6.3.1.

Example. §8.1.5, the case of degenerate nodes: see (8.19).

Exercise 6.4. Prove that if the matrix A is Hermitian, then any triangulating transformation is the diagonalising transformation, i.e., the tringular matrix T is unique and equal to the diagonal matrix L of the eigenvalues of A.

Exercise 6.5. Consider the second-order homogeneous equation (5.1), turn it into a system of first-order equations, and solve it by the triangulation method. Identify A, U and T. For what value of b is the triangular matrix diagonal? Consider specially the case $b = a^2/4$ and check that you have recovered the solution (5.24).

6.3.3. *Solution via Jordan Form*

[Literature: Arnold (2006, §§25.1-25.2)]

There is, in fact, an (operationally) simpler method for solving ODEs with non-diagonalisable matrices. It is based on another magic result from linear algebra: *Jordan's theorem* (Fig. 28).

Let me first introduce some new objects. Consider a degenerate eigenvalue λ_1 that repeats m times. Suppose it has at least one eigenvector $\boldsymbol{v}_1$. Then the set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$, where $k \leqslant m$, is called a *Jordan chain* of λ_1 if they satisfy

$$\begin{aligned}
&\mathsf{A}\cdot\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1,\\
&\mathsf{A}\cdot\boldsymbol{v}_2 = \lambda_1\boldsymbol{v}_2 + \boldsymbol{v}_1,\\
&\dots\\
&\mathsf{A}\cdot\boldsymbol{v}_k = \lambda_1\boldsymbol{v}_k + \boldsymbol{v}_{k-1}.
\end{aligned} \tag{6.73}$$

The eigenvector $\boldsymbol{v}_1$ is called the *generator* of the chain, and the rest of the vectors *associated vectors*, or *generalised eigenvectors* of A.

Exercise 6.6. Prove that the vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ given by (6.73) are linearly independent.

If λ_1 is non-degenerate ($m = 1$), then no associated vectors exist. If $m > 1$ and λ_1 has $l < m$



FIGURE 28. Marie Ennemond Camille Jordan (1838-1922), French mathematician; the only mildly curious factoids about him that I have been able to find are that he was (i) a nephew of Puvis de Chavannes, (ii) famous for eccentric notation in his lectures at Collège de France (where he was the successor of Liouville; see Fig. 18a), (iii) not to be confused with Pascual Jordan, who worked with Born and Heisenberg on matrix quantum mechanics and was a Nazi storm trooper.

independent eigenvectors, then it will have l different Jordan chains[39] generated by them and the total length of these chains will be $k_1 + \cdots + k_l = m$.

This means that, given n eigenvalues $\{\lambda_i\}$, some of which may be degenerate, one can always construct a basis consisting of their eigenvectors and their associated vectors. If, as usual, we pack these basis vectors together as columns of a matrix R, the latter matrix will transform A into the following *Jordan form:*

$$\boxed{\mathsf{R}^{-1}\cdot\mathsf{A}\cdot\mathsf{R}=\mathsf{J}=\begin{pmatrix}\mathsf{J}_1 & & 0\\ & \mathsf{J}_2 & \\ 0 & & \cdots\end{pmatrix}}, \tag{6.74}$$

where to each Jordan chain of eigenvalue λ_i within the basis corresponds the *Jordan block*

$$\mathsf{J}_i=\begin{pmatrix}\lambda_i & 1 & & 0\\ & \lambda_i & 1 & \\ & & \cdots & 1\\ 0 & & & \lambda_i\end{pmatrix} \tag{6.75}$$

(λ_i on the diagonal, 1's above the diagonal, 0's everywhere else). That all of this can always be done is guaranteed by the following *Jordan's theorem.*

Theorem 5. *For any matrix* A*, there is always a basis in* $\mathbb{C}^n$ *consisting of Jordan chains, or,* $\forall\mathsf{A}\ \exists\mathsf{R}$ *such that* $\mathsf{R}^{-1}\cdot\mathsf{A}\cdot\mathsf{R}=\mathsf{J}$.

While Jordan's theorem is much harder to prove than Schur's, it provides a much easier algorithm for actually solving ODEs with degenerate eigenvalues. Indeed, as usual, we have

$$\dot{\boldsymbol{y}}=\mathsf{A}\cdot\boldsymbol{y}=\mathsf{R}\cdot\mathsf{J}\cdot\mathsf{R}^{-1}\cdot\boldsymbol{y}\quad\Rightarrow\quad\boxed{\dot{\boldsymbol{\xi}}=\mathsf{J}\cdot\boldsymbol{\xi}},\quad\text{where}\quad\boldsymbol{\xi}=\mathsf{R}^{-1}\cdot\boldsymbol{y}. \tag{6.76}$$

Let us consider any one Jordan block in J, say J_1. It generates the following set of equations:

$$\begin{cases}\dot\xi_1=\lambda_1\xi_1+\xi_2,\\ \quad\cdots\\ \dot\xi_{k-1}=\lambda_1\xi_{k-1}+\xi_k,\\ \dot\xi_k=\lambda_1\xi_k.\end{cases} \tag{6.77}$$

[39] Some of which can be trivial 1×1 "matrices" containing just λ_1 as their sole element.

Solving them starting from the last, in the manner similar to what we did in §6.3.1, we get

$$\begin{aligned}
\xi_k(t) &= C_k e^{\lambda_1 t},\\
\xi_{k-1}(t) &= (C_{k-1} + C_k t)\, e^{\lambda_1 t},\\
\xi_{k-2}(t) &= \left(C_{k-2} + C_{k-1} t + C_k \frac{t^2}{2}\right) e^{\lambda_1 t},\\
&\dots\\
\xi_1(t) &= \left(C_1 + C_2 t + \dots + C_k \frac{t^{k-1}}{(k-1)!}\right) e^{\lambda_1 t}.
\end{aligned} \tag{6.78}$$

Since $\boldsymbol{y} = \mathsf{R}\cdot\boldsymbol{\xi}$, the corresponding part of the solution of our original ODE is, in terms of the vectors of the Jordan chain $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$,

$$\begin{aligned}
\boldsymbol{y}(t) &= \xi_1(t)\boldsymbol{v}_1 + \dots + \xi_k(t)\boldsymbol{v}_k\\
&= e^{\lambda_1 t}\left[C_1\boldsymbol{v}_1 + C_2(t\boldsymbol{v}_1 + \boldsymbol{v}_2) + C_3\left(\frac{t^2}{2}\,\boldsymbol{v}_1 + t\boldsymbol{v}_2 + \boldsymbol{v}_3\right) + \dots\right.\\
&\qquad\left. + C_k\left(\frac{t^{k-1}}{(k-1)!}\,\boldsymbol{v}_1 + \dots + t\boldsymbol{v}_{k-1} + \boldsymbol{v}_k\right)\right]\\
&\equiv C_1\boldsymbol{y}_1(t) + \dots + C_k\boldsymbol{y}_k(t).
\end{aligned} \tag{6.79}$$

Doing this for each Jordan block and then adding up all the solutions, we get the general solution of our original system. Indeed, since the number of linearly independent solutions $\boldsymbol{y}_i(t)$ and, accordingly, integration constants C_i, in (6.79) is equal to the length of the Jordan chain (k), the sum of the lengths of the Jordan chains corresponding to each degenerate eigenvalue is equal to its degree of degeneracy (m) and the sum of the degrees of degeneracy of all eigenvalues of A is equal to n, the resulting solution will have n constants multiplying n independent solutions.

This is easy to use without having to remember these derivations: to solve the general homogeneous ODE (6.3),

—find all eigenvalues $\{\lambda_i\}$ of the matrix A;

—find all eigenvectors corresponding to each λ_i;

—if λ_i is repeated m times but only has $l < m$ eigenvectors, find $m - l$ associated vectors by constructing a Jordan chain (6.73) generated by each eigenvector;[40]

—for each Jordan chain, write a piece of the general solution in the form (6.79);

—add them all up—this is the general solution of (6.3);

—enforce initial conditions via the n available constants.

If you are faced with an inhomogeneous equation (6.1), you will need to find a particular solution. In principle, our general formula (4.38) tells you how to do this if you know the fundamental matrix $\mathsf{Y}(t)$. You do know this matrix: the linearly independent solutions $\boldsymbol{y}_i(t)$ that are its columns are just $e^{\lambda_i t}$ times the vector polynominal expressions multiplying the integration constants in (6.79). For a quasipolynomial force, i.e., for a force consisting of products of polynomials and exponentials, you can work out the answer explicitly (cf. §5.2.2). I shall leave you to it.

Example. Q4.7.

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[40] Note that because $\det(\mathsf{A} - \lambda_1\mathsf{I}) = 0$, each successive equation in (6.73) is under no obligation to have a solution, indeed you might think it is a stroke of luck if it does. Jordan's theorem guarantees that your luck will not run out until you get $m - l$ associated vectors, a marvellous piece of mathematics.

Example. Consider

$$\begin{cases} \dot{x} = -x + y - 2z, \\ \dot{y} = 4x + y, \\ \dot{z} = 2x + y - z. \end{cases} \tag{6.80}$$

First, find the eigenvalues:

$$\mathsf{A} = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \quad \Rightarrow \quad \det(\mathsf{A} - \lambda \mathsf{I}) = (1+\lambda)^2(1-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 1. \tag{6.81}$$

The eigenvector corresponding to the non-degenerate eigenvalue $\lambda_3 = 1$ satisfies

$$\begin{pmatrix} -2 & 1 & -2 \\ 4 & 0 & 0 \\ 2 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \Rightarrow \quad x = 0, \ y = 2z \quad \Rightarrow \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}. \tag{6.82}$$

We now expect to find two eigenvectors or an eigenvector and an associated vector corresponding to $\lambda_1 = -1$:

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \Rightarrow \quad y - 2z = 0, \ 2x + y = 0 \quad \Rightarrow \quad \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}. \tag{6.83}$$

No other eigenvector can be found, so we test our confidence in Jordan's theorem by seeking an associated vector according to (6.73):

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad y - 2z = -1, \ 2x + y = 1 \quad \Rightarrow \quad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}. \tag{6.84}$$

Success, we have our Jordan basis! Jordan's theorem tells us that we should not be able to continue this Jordan chain. Let us verify this:

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \quad \Rightarrow \quad y - 2z = -1, \ 4x + 2y = 3, \ 2x + y = 2. \tag{6.85}$$

Indeed, this has no solution. Now with full confidence that the method works, let us write the solution according to (6.79):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left\{ C_1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + C_2 \left[\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right] \right\} e^{-t} + C_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{t}. \tag{6.86}$$

Three constants, three linearly independent solutions—this is the desired general solution of (6.80). It all works hitchlessly, like an app.

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7. Normal Modes of Linear Oscillatory Systems

[Literature: Jarvis (2016, Ch. 2), Palmer (2019), Tenenbaum & Pollard (1986, §33A), Landau & Lifshitz (1976, §23), Arnold (2006, §§24, 25.6), Arnold (1997, §23)]

Like §5, most of this section is a series of worked examples of the application of the methods developed in §6 (in fact, more specifically, in §6.2) to Hermitian systems. We study them in generous detail because they are of special consequence physically: they describe systems of coupled oscillators and, under diagonalisation, split into linearly independent collective oscillations, known as *normal modes*. This is the first step to many tenets of the physics canon: wave motion (Parra 2021), (second) quantisation and quasiparticles (a "gas" of normal modes), and everything that follows thence.

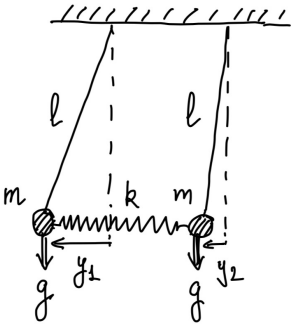


FIGURE 29. Two coupled pendula.

7.1. *Coupled Identical Oscillators*

[Literature: Jarvis (2016, §2.1), Palmer (2019, §3)]

Let us take two pendula and link them with a spring (Fig. 29). Each of them is now anything but free. We shall assume that the equilibrium point—one where the spring is neither stretched not squeezed—is when the two pendula are hanging vertically downward. The dynamical variables of the system will then be deviations from this equilibrium, which I will call y_1 and y_2. We can think of these as displacements along arcs followed by the pendula, but, as long as they are small, there is no difference between those and the horizonal displacements. The equations that they satisfy are

$$\begin{cases} m\ddot{y}_1 = -m\omega_0^2 y_1 - k(y_1 - y_2), \\ m\ddot{y}_2 = -m\omega_0^2 y_2 - k(y_2 - y_1), \end{cases} \tag{7.1}$$

assuming the masses and the lengths of the pendula (m and l, respectively, with $\omega_0^2 = g/l$) are the same and the Hookean spring constant of the spring is k.

7.1.1. *General Method*

[Literature: Lukas (2017, §7.6.1)]

In matrix form,

$$\boxed{\ddot{\boldsymbol{y}} = -\mathsf{A} \cdot \boldsymbol{y}}\,, \qquad \mathsf{A} = \begin{pmatrix} \omega_0^2 + k/m & -k/m \\ -k/m & \omega_0^2 + k/m \end{pmatrix}. \tag{7.2}$$

We are dealing with a system of ODEs—formally, 4th-order (two variables, two time derivatives on each). We could, of course, follow the general prescription of §6: turn this into a system of first-order ODEs with a 4×4 matrix, then solve it via standard operating procedures. However, this is a cluncky, formalistic way to work—and physicists are nimble opportunists. The matrix A is symmetric (I will explain in §7.2.2 why it always is, in such problems), so according to §6.2, we are guaranteed to be able to diagonalise it just by rotating the vector $\boldsymbol{y}$, viz., a change to new variables

$$\boldsymbol{\xi}(t) = \mathsf{R}^T \cdot \boldsymbol{y}(t) \tag{7.3}$$

is guaranteed to exist, where R is the (orthogonal) matrix whose columns are the eigenvectors of A, orthogonal to each other and normalised to unity [see (6.47)]. The equation for $\boldsymbol{\xi}$ will then be just be a set of decoupled SHOs. Indeed:

$$\ddot{\boldsymbol{\xi}} = \mathsf{R}^T \cdot \ddot{\boldsymbol{y}} = -\mathsf{R}^T \cdot \mathsf{A} \cdot \boldsymbol{y} = -\mathsf{L} \cdot \mathsf{R}^T \cdot \boldsymbol{y} = -\mathsf{L} \cdot \boldsymbol{\xi}, \tag{7.4}$$

where $\mathsf{L} = \mathrm{diag}\,\{\lambda_i\}$ is the diagonal matrix of the eigenvalues of A. The above equation is just

$$\ddot{\xi}_i = -\lambda_i \xi_i \quad \Rightarrow \quad \boxed{\xi_i(t) = A_i \cos\bigl(\sqrt{\lambda_i}\, t\bigr) + B_i \sin\bigl(\sqrt{\lambda_i}\, t\bigr)}\,, \tag{7.5}$$

where A_i and B_i are constants of integration and $i = 1, 2$ for the example at hand, although nothing in the procedure that I have described is specific to $n = 2$. Note that $\{\lambda_i\}$ are guaranteed to be real, because A is a symmetric matrix (see §6.2), but they are not necessarily guaranteed to be positive. However, if any $\lambda_i < 0$, the equilibrium around which we are considering our small perturbations is clearly going to be unstable, so not an equilibrium in the vicinity of which we should expect our system to stay for a long time.

Returning to the original variables, we get

$$\boxed{\boldsymbol{y}(t) = \mathsf{R} \cdot \boldsymbol{\xi}(t) = \sum_i \boldsymbol{v}_i \xi_i(t)}\,, \tag{7.6}$$

where $\{\boldsymbol{v}_i\}$ are the eigenvectors of A. These eigenvectors are guaranteed to be orthogonal to each other, so (7.6) is a sum of linearly independent solutions, and it has the right number of constants—four of them for a 2×2 system, $2n$ for an $n \times n$ one—thus, it is the general solution of our system. As usual, the constants are there to accommodate initial conditions: from (7.5),

$$A_i = \xi_i(0), \qquad B_i = \frac{\dot{\xi}_i(0)}{\sqrt{\lambda_i}}, \tag{7.7}$$

where $\boldsymbol{\xi}(0)$ and $\dot{\boldsymbol{\xi}}(0)$ are related to the initial conditions $\boldsymbol{x}(0)$ and $\dot{\boldsymbol{x}}(0)$ via the transformation (7.3). Note that, in order for (7.3) to hold, the eigenvectors must be normalised to $|\boldsymbol{v}_i| = 1$ [see (6.47)]. In general, in (7.6), it is not, in fact, necessary to normalise them, as any normalisation factors can be absorbed into the integration constants, as usual. However, if the columns of R are unnormalised eigenvectors, then R^T in (7.3) must be replaced by $\mathsf{R}^{-1} \neq \mathsf{R}^T$. Then the most straightforward way to recover $\boldsymbol{\xi}$ from $\boldsymbol{y}$ is just to dot (7.6) with $\boldsymbol{v}_j$ on both sides and use the orthogonality of the eigenvectors ($\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ if $i \neq j$):

$$\xi_j(t) = \frac{\boldsymbol{v}_j \cdot \boldsymbol{y}(t)}{|\boldsymbol{v}_j|^2}. \tag{7.8}$$

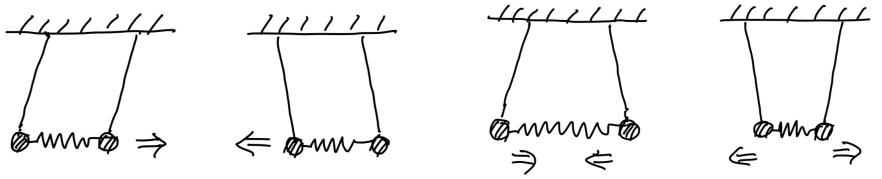
In other words, R^{-1} is a matrix whose rows are the vectors $\boldsymbol{v}_j/|\boldsymbol{v}_j|^2$.

The individual "eigensolutions" $\boldsymbol{v}_i \xi_i(t)$ are, in the context of linear oscillatory systems, called *normal modes*. That the system can be decomposed into them reflects the fact that it is not really coupled, it was just written in the "wrong" variables—the two pendula in Fig. 29 can still be described as two independent SHOs, but those SHOs are not the pendula, but certain collective motions of them. Let us solve the system (7.2) according to the general scheme developed above and see what these collective motions are, physically.

7.1.2. *Solution for Two Coupled Pendula*

The eigenvalues of the matrix A satisfy

$$\left(\omega_0^2 + \frac{k}{m} - \lambda\right)^2 - \frac{k^2}{m^2} = 0 \quad \Rightarrow \quad \lambda_1 = \omega_0^2 \equiv \Omega_1^2, \qquad \lambda_2 = \omega_0^2 + \frac{2k}{m} \equiv \Omega_2^2. \tag{7.9}$$



(a) (b)

FIGURE 30. (a) Normal mode 1: oscillations of the centre of mass. (b) Normal mode 2: "internal" oscillations.

The eigenvalues are real, as expected, and positive, so the equilibrium is a good one (we knew that!). Now let us find the corresponding eigenvectors:

$$\begin{pmatrix} k/m & -k/m \\ -k/m & k/m \end{pmatrix} \cdot \boldsymbol{v}_1 = 0 \quad \Rightarrow \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{7.10}$$

$$\begin{pmatrix} -k/m & -k/m \\ -k/m & -k/m \end{pmatrix} \cdot \boldsymbol{v}_2 = 0 \quad \Rightarrow \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{7.11}$$

Thus, the first normal mode, which, according to (7.9), oscillates at the frequency ω_0 and does not "know" about the connecting spring, consists of the two pendula swinging back and forth together as a rigid body—indeed, without perturbing the spring (Fig. 30a). According to (7.8), the variable that describes this is

$$\xi_1(t) = \frac{y_1(t) + y_2(t)}{2}, \tag{7.12}$$

i.e., the coordinate of the centre of mass of the system. The second normal mode is

$$\xi_2(t) = \frac{y_1(t) - y_2(t)}{2}. \tag{7.13}$$

This is a collective motion in which the centre of mass of the two pendula stays put at zero, while the pendula swing in concert towards, or away from, each other (Fig. 30b). The frequency $\Omega_2 = \sqrt{\omega_0^2 + 2k/m}$ of this SHO is greater than ω_0 because the restoring force is not just gravity but also the Hookean tension in the spring.

In retrospect, it might appear obvious that the system (7.1) could have been decoupled simply by adding and subtracting the two equations, leading to two independent SHOs, one for $y_1 + y_2$, the other for $y_1 - y_2$, so why bother with all the linear algebra? For this particular example, there was indeed little need to do this, but there will be less simple cases for us to deal with, where guessing the right linear combinations of the original equations is as laborious (or, indeed, more laborious) a task as finding the eigenvectors, entirely equivalent to the latter but thoroughly unenlightened. Another way to avoid enlightenment would have been to use the "method of elimination": express y_2 in terms of y_1 from the first equation in (7.1) and substitute into the second; this gives you a single 4th-order ODE for y_1, whose characteristic equation is biquadratic and can be solved, etc. Do not waste your time on such silliness and use linear algebra instead!

Example. Q6.2(a,b), Q6.3(a,b), Q6.6(a,b).

7.1.3. *Examples of Initial Conditions*

If we set the two pendula in motion with such an initial condition that only one of the two normal modes is excited, for **example**, the first one,

$$y_1(0) = y_2(0) = a, \qquad \dot{y}_1(0) = \dot{y}_2(0) = 0, \tag{7.14}$$

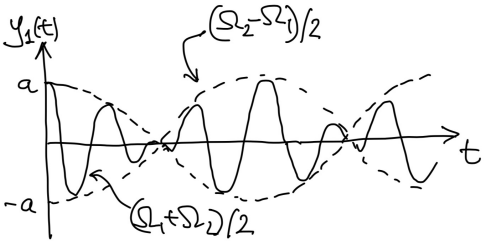


FIGURE 31. Beats: oscillations (7.23) with two frequencies (a superposition of two normal modes).

then that will be the only oscillation that the system will perform forever. Indeed, according to (7.7) and (7.8),

$$A_1 = a, \qquad B_1 = 0, \qquad A_2 = 0, \qquad B_2 = 0, \tag{7.15}$$

so the solution (7.6) is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \Omega_1 t. \tag{7.16}$$

To belabour the point with another **example**, letting

$$y_1(0) = y_2(0) = 0, \qquad \dot{y}_1(0) = -\dot{y}_2(0) = v \tag{7.17}$$

gives us

$$A_1 = 0, \qquad B_1 = 0, \qquad A_2 = 0, \qquad B_2 = \frac{v}{\Omega_2}, \tag{7.18}$$

so the solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = v \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{\sin \Omega_2 t}{\Omega_2}. \tag{7.19}$$

Things become a bit more interesting if we give the system an initial kick without tailoring it to a particular normal mode: for **example**,

$$y_1(0) = a, \qquad y_2(0) = 0, \qquad \dot{y}_1(0) = \dot{y}_2(0) = 0. \tag{7.20}$$

The constants that accommodate this are

$$A_1 = A_2 = \frac{a}{2}, \qquad B_1 = B_2 = 0. \tag{7.21}$$

Hence the solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{a}{2} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \Omega_1 t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos \Omega_2 t \right], \tag{7.22}$$

or, trigonometry tells us,

$$y_1(t) = a \cos \left(\frac{\Omega_1 + \Omega_2}{2}\, t \right) \cos \left(\frac{\Omega_2 - \Omega_1}{2}\, t \right), \tag{7.23}$$

$$y_2(t) = a \sin \left(\frac{\Omega_1 + \Omega_2}{2}\, t \right) \sin \left(\frac{\Omega_2 - \Omega_1}{2}\, t \right). \tag{7.24}$$

The reason this last rewriting is enlightening is that $\Omega_2 - \Omega_1 < \Omega_1 + \Omega_2$ and so these formulae represent an oscillation at the larger frequency encased inside an envelope oscillating at the smaller frequency (Fig. 31). This gives you a feel for what superpositions of normal modes look like in time. The picture of a fast oscillation *modulated* by a slowly

oscillating amplitude is particularly useful when $\Omega_2 - \Omega_1 \ll \Omega_1 + \Omega_2$, i.e., when the coupling spring is weak:

$$\frac{k}{m} \ll \omega_0^2 \quad \Rightarrow \quad \Omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}} \approx \omega_0 \left(1 + \frac{k}{m\omega_0^2}\right). \tag{7.25}$$

The two pendula then oscillate at approximately their natural frequency, but with slight modulation due to their interaction via the wimpy spring:

$$y_1(t) \approx a \cos \Delta t \cos \bar{\Omega} t, \qquad y_2(t) \approx a \sin \Delta t \sin \bar{\Omega} t. \tag{7.26}$$

where $\Delta = k/2m\omega_0$ and $\bar{\Omega} = \omega_0 + \Delta \gg \Delta$. These modulations are called *beats*.

Example. Q6.1, Q6.2(c), §7.3.5.

7.2. *Energetics of Coupled Oscillators*

[Literature: Jarvis (2016, §2.1.4), Palmer (2019, §4)]

What "follow the money" is to politics, "follow the energy" is to physics. In §I.6, this will become the starting point for the whole construction of mechanics (as it is also for quantum mechanics), but for now, let us investigate the energetics of the oscillatory dynamics that we have so far deduced from Newton's equations of motion.

7.2.1. *Energy of a System of Coupled Oscillators*

Going back to the example of two coupled identical pendula (7.1), it does not take very much physics education to write down the energy of the system: it is kinetic ($m\dot{y}_{1,2}^2/2$) plus potential ($m\omega_0^2 y_{1,2}^2/2$) energy of the two oscillators plus the potential energy associated with the extensions or contractions of the spring:

$$\mathscr{E} = \frac{m\dot{y}_1^2}{2} + \frac{m\dot{y}_2^2}{2} + \frac{m\omega_0^2 y_1^2}{2} + \frac{m\omega_0^2 y_2^2}{2} + \frac{k(y_1 - y_2)^2}{2}. \tag{7.27}$$

A few lines of trivial algebra confirm that the system (7.1) does indeed conserve this quantity.

For the purposes of application to other problems, and to extract some enlightening structual features, let me consider how to derive the energy function for a general system in the matrix form (7.2),

$$\ddot{\boldsymbol{y}} = -\mathsf{A} \cdot \boldsymbol{y}, \tag{7.28}$$

the only restriction on A being that it is a symmetric matrix. Multiplying both sides of (7.28) by $\dot{\boldsymbol{y}}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\dot{\boldsymbol{y}}|^2}{2} = -\dot{\boldsymbol{y}} \cdot \mathsf{A} \cdot \boldsymbol{y} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{y} \cdot \mathsf{A} \cdot \boldsymbol{y}}{2}, \tag{7.29}$$

the last step enabled by the symmetry of the matrix. Thus, (7.28) conserves

$$\boxed{\mathscr{E} = \frac{|\dot{\boldsymbol{y}}|^2}{2} + \frac{\boldsymbol{y} \cdot \mathsf{A} \cdot \boldsymbol{y}}{2} \equiv K + U}\,, \tag{7.30}$$

the first term being kinetic (K), and the second potential (U), energy of the system. Unpacking (7.30) for the specific matrix (7.2) that describes the case of two coupled pendula, we get

$$\mathscr{E} = \frac{\dot{y}_1^2 + \dot{y}_2^2}{2} + \left(\omega_0^2 + \frac{k}{m}\right)\frac{y_1^2 + y_2^2}{2} - \frac{k}{m} y_1 y_2, \tag{7.31}$$

which is the same as (7.27), up to the (here irrelevant) mass prefactor.

7.2.2. *Restoring Forces Are Potential*

Note that the equation of motion (7.28) is

$$\ddot{y}_i = -\frac{\partial U}{\partial y_i}, \tag{7.32}$$

so the restoring forces are all potential (a property, and with it energy conservation, broken by damping: see §7.3.3). This is actually true for such systems also nonlinearly (see §3.5.2), with (7.28) being the result of linearising (7.32) around the equilibrium point $\boldsymbol{y} = 0$ (cf. §3.5.3).

This sheds some light on why the matrix A that emerges from such a linearisation should always be symmetric: since $\partial U/\partial y_i = 0$ at $\boldsymbol{y} = 0$, in the vicinity of this point, we have

$$\ddot{y}_i \approx -\left.\frac{\partial^2 U}{\partial y_i \partial y_j}\right|_{\boldsymbol{y}=0} y_j \equiv -A_{ij} y_j. \tag{7.33}$$

Note that not all oscillators in nature are limited to potential forces—in §8.4, I will introduce some examples of (nonlinear) oscillators that are neither harmonic nor conservative, and indeed constantly extract and dissipate energy as they oscillate (your grandfather's clock belongs to that class).

7.2.3. *Energy in Terms of Normal Modes*

The conserved energy of a system of coupled oscillators is manifestly *not* just the sum of the energies of the individual oscillators—the terms associated with the off-diagonal elements of A are coupling terms, the last term in (7.31) illustrating the point. It is clear (and can be easily checked) that the energies of the individual oscillators are not conserved, and so energy flows back and forth between them. However, the appearance that the system is an interacting one is, in fact, an optical illusion: since we know that it can be decomposed into uncoupled SHOs (normal modes), we expect that its energy must simply be the sum of the energies of these SHOs. Let us check this directly.

The decomposition is (7.6), $\boldsymbol{y} = \mathsf{R}\cdot\boldsymbol{\xi}$, so let us just substitute this into (7.30):

$$\mathscr{E} = \frac{\dot{\boldsymbol{\xi}}\cdot\mathsf{R}^T\cdot\mathsf{R}\cdot\dot{\boldsymbol{\xi}}}{2} + \frac{\boldsymbol{\xi}\cdot\mathsf{R}^T\cdot\mathsf{A}\cdot\mathsf{R}\cdot\boldsymbol{\xi}}{2} = \frac{|\dot{\boldsymbol{\xi}}|^2}{2} + \frac{\boldsymbol{\xi}\cdot\mathsf{L}\cdot\boldsymbol{\xi}}{2} = \sum_i\left(\frac{\dot{\xi}_i^2}{2} + \frac{\lambda_i\xi_i^2}{2}\right), \tag{7.34}$$

where R has been assumed to contain as its columns the normalised eigenvectors of A, so $\mathsf{R}^T\cdot\mathsf{R} = \mathsf{I}$ (it is a rotation matrix). Thus, indeed, the total energy of our system is the sum of the energies of the decoupled oscillators (7.5) (normal modes)—obviously, each of these energies is individually conserved (and so determined purely by how much of each normal mode was present in the initial condition).

Example. Q6.2(d), Q6.4(c).

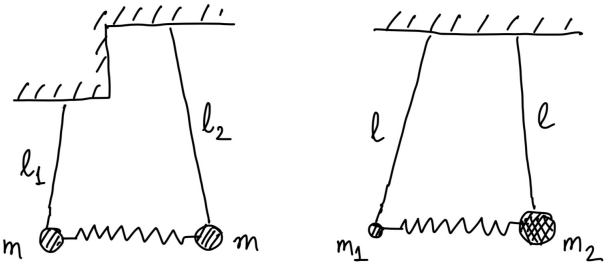
7.3. *Generalisations and Complications*

The general scheme laid out in §7.1.1 is not difficult to make even more general. First, as I already mentioned, there is nothing there that restricts its applicability to $n = 2$ oscillators—and I will consider the highly instructive case of N oscillators in §7.3.6.

7.3.1. *Pendula of Unequal Length*

[Literature: Jarvis (2016, §2.2)]

Secondly, the pendula need not be identical. The simplest generalisation is to let them



(a) (b)

FIGURE 32. Unequal (but equally entitled to be solved!) pendula: (a) different lengths (§7.3.1); (b) different masses (§7.3.2).

have different lengths $l_{1,2}$ (Fig. 32a), which would result in two different frequencies, $\omega_{1,2} = \sqrt{g/l_{1,2}}$. The matrix A will still be symmetric (as this only changes its diagonal elements) and the general scheme descrived in §7.1.1 works, with some obvious adjustments to the specific calculation of §7.1.2.

Exercise 7.1. Make these obvious adjustments and calculate the general solution for two coupled pendula of lengths $l_{1,2}$.

Example. §7.3.5.

7.3.2. *Pendula of Unequal Mass*

[Literature: Palmer (2019, §6)]

Giving the two pendula different masses (Fig. 32b) produces a formal wrinkle: the matrix A is no longer symmetric and you might worry that all the nice things that came with that might be imperilled. Let us iron this wrinkle out. Let me rewrite the new system in the following matrix form:

$$\boxed{\mathsf{M}\cdot\ddot{\boldsymbol{y}} = -\mathsf{A}\cdot\boldsymbol{y}}\,, \qquad \mathsf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \qquad \mathsf{A} = \begin{pmatrix} m_1\omega_0^2 + k & -k \\ -k & m_2\omega_0^2 + k \end{pmatrix}, \tag{7.35}$$

where I have redefined the matrix A so that it is still symmetric, even if $\mathsf{M}^{-1}\cdot\mathsf{A}$ is not.

Let us now solve a generalised eigenvalue problem:

$$\boxed{\mathsf{A}\cdot\boldsymbol{v}_i = \lambda_i \mathsf{M}\cdot\boldsymbol{v}_i}\,. \tag{7.36}$$

The equation for the eigenvalues defined in this way is

$$\boxed{\det(\mathsf{A} - \lambda_i \mathsf{M}) = 0}\,, \tag{7.37}$$

an n-th-order polynomial equation with n roots ($n = 2$ in our current example, but everything that is about to happen works for general n). The nice properties of the eigenvalues and eigenvectors of a symmetric matrix proven in §6.2 all generalise to this new construction:

- The eigenvalues are real:

$$\lambda_i \boldsymbol{v}_i^*\cdot\mathsf{M}\cdot\boldsymbol{v}_i = \boldsymbol{v}_i^*\cdot\mathsf{A}\cdot\boldsymbol{v}_i = \boldsymbol{v}_i\cdot(\mathsf{A}\cdot\boldsymbol{v}_i)^* = \lambda_i^*\boldsymbol{v}_i\cdot\mathsf{M}\cdot\boldsymbol{v}_i^* = \lambda_i^*\boldsymbol{v}_i^*\cdot\mathsf{M}\cdot\boldsymbol{v}_i \quad \Rightarrow \quad \lambda_i = \lambda_i^*. \tag{7.38}$$

- The eigenvectors are orthogonal, in the following sense: if $\lambda_i \neq \lambda_j$,

$$\lambda_j \boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{v}_j = \boldsymbol{v}_i \cdot \mathsf{A} \cdot \boldsymbol{v}_j = \lambda_i \boldsymbol{v}_j \cdot \mathsf{M} \cdot \boldsymbol{v}_i \quad \Rightarrow \quad (\lambda_i - \lambda_j) \boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{v}_j = 0 \quad \Rightarrow \quad \boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{v}_j = 0, \tag{7.39}$$

i.e., they are orthogonal with respect to a scalar product "weighted" by the matrix M—this matrix is a *metric*.

- For any degenerate eigenvalue, we can always find the right number of orthogonal [in the sense (7.39)] eigenvectors—the proof is entirely analogous to that given in §6.2.

- Finally, normalising the eigenvectors with respect to our M-weighted scalar product, we get, for the matrix R that contains $\{\boldsymbol{v}_i\}$ as columns:

$$\boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{v}_j = \delta_{ij} \quad \Rightarrow \quad \mathsf{R}^T \cdot \mathsf{M} \cdot \mathsf{R} = \mathsf{I} \quad \Rightarrow \quad \boxed{\mathsf{R}^{-1} = \mathsf{R}^T \cdot \mathsf{M}}\,. \tag{7.40}$$

With these properties in hand, let $\boldsymbol{y} = \mathsf{R} \cdot \boldsymbol{\xi}$ in (7.35):

$$\mathsf{M} \cdot \mathsf{R} \cdot \ddot{\boldsymbol{\xi}} = -\mathsf{A} \cdot \mathsf{R} \cdot \boldsymbol{\xi} = -\mathsf{M} \cdot \mathsf{R} \cdot \mathsf{L} \cdot \boldsymbol{\xi} \quad \Rightarrow \quad \boxed{\ddot{\boldsymbol{\xi}} = -\mathsf{L} \cdot \boldsymbol{\xi}}\,, \tag{7.41}$$

where $\mathsf{L} = \operatorname{diag}\{\lambda_i\}$. The system has been duly diagonalised. The solution for ξ_i is again (7.5) and the general solution is again reassembled according to (7.6). The only difference is in how we extract $\boldsymbol{\xi}$ from $\boldsymbol{y}$, to work out the normal modes and to calculate the integration constants from the initial conditions: instead of (7.8), we have

$$\xi_j(t) = \frac{\boldsymbol{v}_j \cdot \mathsf{M} \cdot \boldsymbol{y}(t)}{\boldsymbol{v}_j \cdot \mathsf{M} \cdot \boldsymbol{v}_j}. \tag{7.42}$$

If the eigenvectors are normalised as in (7.40), then the denominator in (7.42) is unity.

To belabour the point, the solution is

$$\boldsymbol{y}(t) = \sum_i \boldsymbol{v}_i \left(A_i \cos \Omega_i t + B_i \sin \Omega_i t\right), \tag{7.43}$$

where $\Omega_i = \sqrt{\lambda_i}$ and the integration constants are determined from initial conditions according to

$$A_i = \frac{\boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{y}(0)}{\boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{v}_i}, \qquad B_i = \Omega_i^{-1} \frac{\boldsymbol{v}_i \cdot \mathsf{M} \cdot \dot{\boldsymbol{y}}(0)}{\boldsymbol{v}_i \cdot \mathsf{M} \cdot \boldsymbol{v}_i}. \tag{7.44}$$

Exercise 7.2. (a) Implement this scheme specifically for the example (7.35) with two coupled pendula of unequal mass: find the general solution, the normal modes, their frequencies, etc. You can check your algebra by making sure that its outtcome reduces to the solution obtained in §7.1.2 when $m_1 = m_2$.

(b) Work out the energy of the system (7.35) and show that it is equal to the sum of the energies of the decoupled oscillators (7.41).

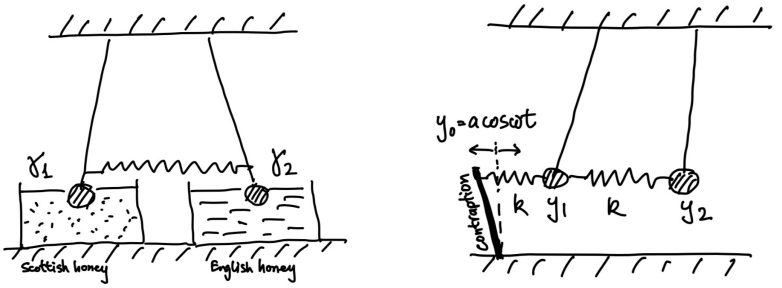
(c) What is the physical nature of the two normal modes of this system in the limit $m_1 \ll m_2$? Note that you can answer this question by pure physical intuition, without doing the maths in (a).

Example. §7.3.5, Q6.4(a,b), Q6.5.

7.3.3. *Damped Oscillators*

[Literature: Jarvis (2016, §2.7)]

Now allow the pendula to be subject to friction against air (or put the whole contrap-



(a) (b)

FIGURE 33. (a) Coupled pendula with damping (§7.3.3). (b) Coupled pendula with periodic forcing (§7.3.4). Obviously, the two springs do not have to have the same spring constant, but post-Brexit, we only have one type of spring available (this makes no difference to how we solve the system).

tion into a jar of honey). Then

$$\boxed{\ddot{\boldsymbol{y}} = -\gamma\dot{\boldsymbol{y}} - \mathsf{A}\cdot\boldsymbol{y}}\,, \tag{7.45}$$

where γ is the friction rate. This system is easily reduced to the old one. Indeed, let

$$\boldsymbol{y}(t) = \boldsymbol{z}(t)\, e^{-\gamma t/2} \quad \Rightarrow \quad \ddot{\boldsymbol{z}} = -\left(\mathsf{A} - \frac{\gamma^2}{4}\,\mathsf{I}\right)\cdot\boldsymbol{z} \tag{7.46}$$

(**exercise:** check that this works). All friction does is reduce the frequency of each normal mode,

$$\Omega_i^2 \to \Omega_i^2 - \frac{\gamma^2}{4}, \tag{7.47}$$

and introduce an overall damping factor (just as it did in §5.1.1). It is not hard to see that the eigenvectors, i.e., the physical nature of the normal modes, do not change.

Finally, it stands to reason that energy is no longer conserved: dotting (7.45) with $\dot{\boldsymbol{y}}$ on both sides, like we did in (7.29), we get

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = -\gamma|\dot{\boldsymbol{y}}|^2 < 0. \tag{7.48}$$

Thus, friction dissipates energy, of course it does.

Example. Q6.3(c,d).

Exercise 7.3. Work out how to deal with the system where each oscillator has its own friction coefficient (each pendulum has its own jar of different honey; Fig. 33a), i.e.,

$$\ddot{y}_i = -\gamma_i\dot{y}_i - A_{ij}y_j \tag{7.49}$$

(no summation in the first term on the right-hand side).

7.3.4. *Forced Oscillators and Resonances*

[Literature: Jarvis (2016, §§2.6-2.7), Palmer (2019, §5.2)]

Forced oscillator systems are just the inhomogeneous versions of the unforced ones, and all the principles that we learned for other linear systems apply here:

- The general solution is the sum of the general solution of the associated homogeneous

equation (the CF) and any particular solution of the inhomogeneous one (the PI)—see (4.10).

- For a system with damping, the CF will take care of the initial conditions and then damp away, while the PI will stick around, representing forced oscillations—as happened for a single oscillator in (5.55).
- Forcing at (or close to) any of the frequencies of the system's normal modes is special, producing a sharp peak in the amplitude of the forced oscillations of the normal mode with which the force is in resonance (cf. §5.3.1).

Let us work all this out. The system that we shall look at is the forced version of (7.45):

$$\boxed{\ddot{\boldsymbol{y}} = -\gamma\dot{\boldsymbol{y}} - \mathsf{A}\cdot\boldsymbol{y} + \boldsymbol{f}(t)}\,, \tag{7.50}$$

where $\boldsymbol{f}(t)$ is a time-periodic force. In general, it can consist of many "harmonics" containing sines and cosines with different frequencies. The system is linear, so, by the superposition principle (4.8), its solution will be the sum of solutions corresponding to individual additive parts of $\boldsymbol{f}(t)$. Thus, we will learn everything that needs learning by just considering

$$\boldsymbol{f}(t) = \boldsymbol{F}\cos\omega t, \tag{7.51}$$

where $\boldsymbol{F}$ is a constant vector. A physical situation that might bring about such a system can be envisioned by adding to our workhorse example of two coupled (damped) pendula another spring connecting the first pendulum to some mechanical contraption oscillating horizontally around its equilibrium point according to $y_0(t) = a\cos\omega t$ (Fig. 33b). Then the equation of motion for the first oscillator is

$$m\ddot{y}_1 = -m\gamma\dot{y}_1 - m\omega_0^2 y_1 - k(y_1 - y_2) - k\,[y_1 - y_0(t)]\,. \tag{7.52}$$

This gives us a system of the form (7.50) with a force (7.51), where

$$\boldsymbol{F} = \begin{pmatrix} ka/m \\ 0 \end{pmatrix}, \qquad \mathsf{A} = \begin{pmatrix} \omega_0^2 + 2k/m & -k/m \\ -k/m & \omega_0^2 + k/m \end{pmatrix} \tag{7.53}$$

(two different frequencies now, but, as we saw in §7.3.1, this is not a problem).

In order to find a particular solution of (7.50), we do what we always do (§5.3): complexify our variable, $\boldsymbol{y} = \mathrm{Re}\,\boldsymbol{z}$, consider

$$\ddot{\boldsymbol{z}} + \gamma\dot{\boldsymbol{z}} + \mathsf{A}\cdot\boldsymbol{z} = \boldsymbol{F}e^{i\omega t}, \tag{7.54}$$

and look for a particular solution in the form

$$\boldsymbol{z}(t) = \boldsymbol{C}e^{i\omega t}. \tag{7.55}$$

This produces the following equation for the complex amplitude:

$$\left[\mathsf{A} + (i\gamma\omega - \omega^2)\mathsf{I}\right]\cdot\boldsymbol{C} = \boldsymbol{F}. \tag{7.56}$$

If $\gamma \neq 0$, the matrix in the left-hand side is always invertible because the eigenvalues of A are real, so $\det\left[\mathsf{A} + (i\gamma\omega - \omega^2)\mathsf{I}\right] \neq 0$.[41] Therefore,

$$\boldsymbol{C} = \left[\mathsf{A} + (i\gamma\omega - \omega^2)\mathsf{I}\right]^{-1}\cdot\boldsymbol{F}. \tag{7.57}$$

[41] If $\gamma = 0$ and ω happens to be equal to the frequency of any of the system's normal modes, special measures are needed—you know enough at this point to figure out on your own what they are; reread the end of §5.3.1 for inspiration.

Finally, the forced oscillations are

$$\boxed{\boldsymbol{y}_{\rm PI}(t) = \mathrm{Re}\left\{e^{i\omega t}\left[\mathsf{A} + (i\gamma\omega - \omega^2)\mathsf{I}\right]^{-1} \cdot \boldsymbol{F}\right\}}\,. \tag{7.58}$$

Working out the inverse of the matrix, calculating all the relevant phase factors and extracting the real part entails considerable tedium, so I leave it to you to prove to yourselves that you have the stamina to endure it and get the full answer for the specific example (7.53), or any other example that you care to work through (one such **example** is done in Jarvis 2016, §2.7).

It is, however, also possible to make some progress in a non-tedious way, if one asks a physically useful question. We shall aim to find out what happens when ω is close to the frequency of one of the normal modes of the system—we expect a resonance (cf. §5.3.1).

Let us first use the fact that $\mathsf{A} = \mathsf{R}\cdot\mathsf{L}\cdot\mathsf{R}^T$ and $\mathsf{I} = \mathsf{R}\cdot\mathsf{R}^T$, where R is the rotation matrix made up of A's normalised, orthogonal eigenvectors. Then (7.57) is

$$\boldsymbol{C} = \left\{\mathsf{R}\cdot\left[\mathsf{L} + (i\gamma\omega - \omega^2)\mathsf{I}\right]\cdot\mathsf{R}^T\right\}^{-1}\cdot\boldsymbol{F} = \mathsf{R}\cdot\mathsf{G}\cdot\mathsf{R}^T\cdot\boldsymbol{F}, \tag{7.59}$$

where $\mathsf{G} = \left[\mathsf{L} + (i\gamma\omega - \omega^2)\mathsf{I}\right]^{-1} = \mathrm{diag}\left\{1/(\lambda_i - \omega^2 + i\gamma\omega)\right\}$. Here $\lambda_i = \Omega_i^2$ are the eigenvalues of A, and Ω_i are the frequencies of the corresponding normal modes. The above formula simply says that the forcing gets decomposed into parts "aligned" with the normal modes of the system ($\mathsf{R}^T\cdot\boldsymbol{F}$), then each of these gives rise to independent forced oscillations of its corresponding normal mode, with an amplitude determined in exactly the same way as in (5.52) ($\mathsf{G}\cdot\mathsf{R}^T\cdot\boldsymbol{F}$).

Now, if the forcing frequency is close to one of the normal-mode frequencies, say, $\omega \approx \Omega_1$, and γ is small compared to it, then the largest element of the diagonal matrix G will be the one with $i = 1$. Ignoring all others, we get

$$\boldsymbol{C} \approx \boldsymbol{v}_1\frac{\boldsymbol{v}_1\cdot\boldsymbol{F}}{i\gamma\Omega_1} \quad\Rightarrow\quad \boldsymbol{y}_{\rm PI} = \mathrm{Re}\left(\boldsymbol{C}e^{i\omega t}\right) \approx \boldsymbol{v}_1\frac{\boldsymbol{v}_1\cdot\boldsymbol{F}}{\gamma\Omega_1}\sin\Omega_1 t, \tag{7.60}$$

where $\boldsymbol{v}_1$ is the normalised ($|\boldsymbol{v}_1| = 1$) eigenvector of the resonant normal mode. Thus, if you force an oscillating system at one of its normal modes' frequencies, the forced oscillations in the stationary state will be dominated by the corresponding normal mode and have an amplitude that is just the same as the amplitude (5.57) at resonance for a single weakly damped oscillator, but with only the projection of the forcing amplitude onto the corresponding eigenvector ($\boldsymbol{v}_1\cdot\boldsymbol{F}$) contributing. The phase shift between the force and the forced oscillations is approximately $\pi/2$, as always at resonance (see §5.3.1).

This all makes perfect sense (I hope). It should be reassuring to be able to prove in such a simple manner what intuition tells us must happen *volens nolens*.

Example. Q6.3(e), §7.3.5.

Exercise 7.4. Use (7.59) to calculate $\boldsymbol{y}_{\rm PI}(t)$ in terms of $\boldsymbol{F}$, ω, γ, eigenvectors $\{\boldsymbol{v}_i\}$ of A and the corresponding normal-mode frequencies $\{\Omega_i\}$ without making any assumptions about any of these quantities.

Exercise 7.5. Imagine that a weakly damped ($\gamma \ll \Omega_i$ for all i's) system is forced in a resonance, $\omega = \Omega_1$, but the force is applied nearly perpendicularly to the eigenvector of the corresponding normal mode, i.e., $\boldsymbol{v}_1\cdot\boldsymbol{F} \ll |\boldsymbol{F}|$. How small does $\boldsymbol{v}_1\cdot\boldsymbol{F}$ have to get in order for the approximation (7.60) to stop being valid?

Exercise 7.6. Work out the energy budget for a system of weakly damped, coupled oscillators (7.50), viz., show that the injected power is equal to the dissipated power (cf. §5.3.2).

7.3.5. *A Worked Example: Can Coupled Pendula Be in Resonance With Each Other?*

I want to show you a long worked **example** that nicely encapsulates several important features and ideas introduced in the preceding sections. I got the idea of this calculation from Karlov & Kirichenko (2003).

We know from §5.3.1 (and again from §7.3.4) that if we force an oscillatory system at its natural frequency, we can get it to develop oscillations with very large amplitude—in the absence of damping, with amplitude that grows $\propto t$ [see (5.64)]. But when we couple two oscillators, we could consider one of them to be a kind of forcing of the other. Is there some arrangement in which this would lead to large amplitudes, i.e., can two coupled oscillators be, in some sense, in a resonance with each other?

To explore this question, let us consider two coupled pendula unequal in every respect that we have so far considered: let them have different lengths (and hence different natural frequencies $\omega_{1,2}$; see §7.3.1) and different masses (§7.3.2). The equations for this system are

$$\begin{cases} \ddot{y}_1 = -\left(\omega_1^2 + \dfrac{k}{m_1}\right) y_1 + \dfrac{k}{m_1} y_2, \\ \ddot{y}_2 = -\left(\omega_2^2 + \dfrac{k}{m_2}\right) y_2 + \dfrac{k}{m_2} y_1. \end{cases} \tag{7.61}$$

I have written them in this way to emphasise the fact that the effect of having different masses, besides modifying the natural frequencies by different amounts, is also to make the coupling coefficients between the oscillators different. We can see immediately that in the limit $m_2 \to \infty$, the second oscillator no longer cares about the first (why would it, if it is infinitely more massive!) and so its effect on the first is mathematically the same as that of a sinusoidal forcing with frequency ω_2. If we adjusted l_2 so that

$$\omega_2^2 = \omega_1^2 + \frac{k}{m_1}, \tag{7.62}$$

we should have a resonance: the amplitude of y_1 would grow as in (5.64).

With this insight in hand, let us explore what happens when m_2 is not infinite, with a view to making it much larger than m_1. From §7.3.2, we know that we are guaranteed to have real eigenvalues $\lambda_{1,2}$ and orthogonal (when weighted by the matrix of the masses) normal modes oscillating with frequencies $\Omega_{1,2} = \sqrt{\lambda_{1,2}}$. Let us work out these frequencies:

$$\det \begin{pmatrix} \omega_1^2 + \dfrac{k}{m_1} - \Omega^2 & -\dfrac{k}{m_1} \\ -\dfrac{k}{m_2} & \omega_2^2 + \dfrac{k}{m_2} - \Omega^2 \end{pmatrix} =$$

$$\Omega^4 - \left(\omega_1^2 + \frac{k}{m_1} + \omega_2^2 + \frac{k}{m_2}\right)\Omega^2 + \left(\omega_1^2 + \frac{k}{m_1}\right)\left(\omega_2^2 + \frac{k}{m_2}\right) - \frac{k^2}{m_1 m_2} = 0. \tag{7.63}$$

The discriminant of this equation is

$$D = \left(\omega_1^2 + \frac{k}{m_1} - \omega_2^2 - \frac{k}{m_2}\right)^2 + \frac{4k^2}{m_1 m_2} > 0, \tag{7.64}$$

always positive as long as $k \neq 0$ and the masses are finite. Therefore, the two normal-mode frequencies are always different, $\Omega_1 \neq \Omega_2$, and there are no resonances. However, if we take $m_2 \gg m_1$, $k/m_2 \ll \omega_2^2$, and arrange for the "resonance condition" (7.62) to hold, we can make D very small and the two frequencies very close to each other:

$$D \approx \frac{4k^2}{m_1 m_2} \quad \Rightarrow \quad \Omega_{1,2} \approx \sqrt{\omega_2^2 \pm \frac{k}{\sqrt{m_1 m_2}}} \approx \omega_2 \pm \frac{k}{2\omega_2\sqrt{m_1 m_2}} \quad \Rightarrow \quad \boldsymbol{v}_{1,2} = \begin{pmatrix} \mp\sqrt{m_2/m_1} \\ 1 \end{pmatrix}. \tag{7.65}$$

The eigenvectors corresponding to two normal modes feature the first pendulum oscillating at a much greater amplitude than the second—so there is a kind of resonance effect.

Since the two frequencies are close, we expect beats (§7.1.3). The general solution is

$$\boldsymbol{y}(t) = \begin{pmatrix} -\sqrt{m_2/m_1} \\ 1 \end{pmatrix} (A_1 \cos\Omega_1 t + B_1 \sin\Omega_1 t) + \begin{pmatrix} \sqrt{m_2/m_1} \\ 1 \end{pmatrix} (A_2 \cos\Omega_2 t + B_1 \sin\Omega_2 t), \tag{7.66}$$

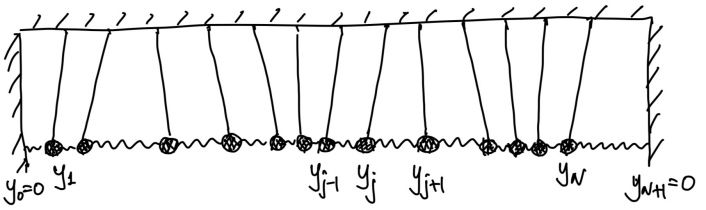


FIGURE 34. Beats: oscillations (7.23) with two frequencies (a superposition of two normal modes).

and, setting a finite displacement of the second oscillator as the initial condition, we get

$$y_1(0) = 0 = \sqrt{\frac{m_2}{m_1}}\,(-A_1 + A_2)\,, \quad y_2(0) = a = A_1 + A_2 \quad \Rightarrow \quad A_1 = A_2 = \frac{a}{2}, \tag{7.67}$$

$$\dot{y}_1(0) = \dot{y}_2(0) = 0 \quad \Rightarrow \quad B_1 = B_2 = 0. \tag{7.68}$$

Assembling the solution, we get

$$y_1(t) = a\sqrt{\frac{m_2}{m_1}}\,\sin\left(\frac{kt}{2\omega_2\sqrt{m_1 m_2}}\right)\sin\omega_2 t, \tag{7.69}$$

$$y_2(t) = a\cos\left(\frac{kt}{2\omega_2\sqrt{m_1 m_2}}\right)\cos\omega_2 t. \tag{7.70}$$

Both pendula oscillate at the natural frequency of the second one, ω_2, with a low-frequency modulation and a very large amplitude ratio—the first pendulum is in "resonance" with the second. In energy terms, by setting the more massive pendulum in motion, we have created a large energy reservoir, which the other pendulum can tap to oscillate wildly.

Finally, if we consider times $t \ll 2\omega_2\sqrt{m_1 m_2}/k$ (which allows $t \gg \omega_2^{-1}$) and expand the modulating sine and cosine in small argument, we recover precisely the familiar resonant solution (5.64):

$$y_1(t) \approx \frac{akt}{2\omega_2 m_1}\sin\omega_2 t, \qquad y_2(t) \approx a\cos\omega_2 t. \tag{7.71}$$

This limit is, of course, equivalent to taking $m_2 \to \infty$, so this is exactly the solution we are supposed to get.

7.3.6. *N Coupled Oscillators*

[Literature: Jarvis (2016, §3.1)]

So, we know how to handle one pendulum and two pendula—the next step is, naturally, to solve N pendula. Let us, therefore, consider a row of N identical pendula (Fig. 34), all connected by identical springs to their nearest neighbours and the 1-st and the N-th also tethered by the same springs to an ajacent wall (this is for simplicity of algebra down the line). Then the equation for the j-th pendulum is

$$m\ddot{y}_j = -m\omega_0^2 y_j - k(y_j - y_{j+1}) - k(y_j - y_{j-1}), \tag{7.72}$$

where $j = 1, \ldots, N$ and, to represent the walls, we set $y_0 = y_{N+1} = 0$ (what should we do if there are no walls and the 1-st and the N-th pendula are free on one side?).

Rearranging,

$$\ddot{y}_j = -\left(\omega_0^2 + \frac{2k}{m}\right) y_j + \frac{k}{m}\,(y_{j+1} + y_{j-1})\,, \tag{7.73}$$

we see that the resulting matrix A is tridiagonal (and also has identical elements on each diagonal—such matrices are called *Toeplitz*). Finding its eigenvalues λ and eigenvectors $\boldsymbol{v}$ turns out to be quite an easy task. We are looking for solutions to

$$\left(\omega_0^2 + \frac{2k}{m}\right) v_j - \frac{k}{m}\,(v_{j+1} + v_{j-1}) = \lambda v_j \quad \Rightarrow \quad \frac{v_{j+1} + v_{j-1}}{v_j} = \frac{\omega_0^2 + 2k/m - \lambda}{k/m}. \tag{7.74}$$

The winning trick is to look for a solution in the form $v_j = \sin j\theta$, which gives

$$\frac{\omega_0^2 + 2k/m - \lambda}{k/m} = \frac{\sin(j+1)\theta + \sin(j-1)\theta}{\sin j\theta} = 2\cos\theta. \tag{7.75}$$

To figure out what θ is, recall the requirement $v_0 = 0$ and $v_{N+1} = 0$. The first of these is satisfied automatically, whereas the second requires

$$\sin(N+1)\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi p}{N+1}, \tag{7.76}$$

where p is an arbitrary integer. It can only take N interesting values: $p = 1, \ldots, N$ ($p = 0$ and $p = N+1$ correspond to $y_j = 0$, so no displacement of any pendula, and for all other p, the values of $\cos\theta$ repeat).

Thus, we have found N distinct eigenvalues λ_p and their corresponding eigenvectors $\boldsymbol{v}^{(p)}$:

$$\lambda_p = \omega_0^2 + \frac{2k}{m}\left(1 - \cos\frac{\pi p}{N+1}\right) = \omega_0^2 + \frac{4k}{m}\sin^2\frac{\pi p}{2(N+1)} \equiv \Omega_p^2, \qquad v_j^{(p)} = \sin\frac{\pi p j}{N+1}. \tag{7.77}$$

These are the normal modes of the system. It general solution is

$$y_j(t) = \sum_{p=1}^{N} \sin\frac{\pi p j}{N+1}\left(A_p\cos\Omega_p t + B_p \sin\Omega_p t\right), \tag{7.78}$$

with $2N$ integration constants $\{A_p, B_p\}$. This formula is begging us to go to the "continuous limit": rather than indexing our pendula by j, we can do so by their spatial coordinate (in equilibrium) $x = ja$, where a is the spacing between them (and between the marginal pendula and walls). Then the solution is

$$\boxed{y(t,x) = \sum_{p} \sin\frac{\pi p x}{L}\left(A_p\cos\Omega_p t + B_p\sin\Omega_p t\right)}, \tag{7.79}$$

where $L = (N+1)a$ is the overall length of the system. This tells us that the normal modes are "standing waves" for which p determines the number of maxima and minima—our pendula show an impressive propensity to organise themselve into coherent collective activity.

Obviously, the most interesting case is now $N \gg 1$, and (7.79) suggests that we should take this limit while holding $L = (N+1)a$ fixed. For $p \ll N$, the frequencies can be simplified:[42]

$$\Omega_p^2 \approx \omega_0^2 + \frac{ka^2}{m}\left(\frac{\pi p}{L}\right)^2. \tag{7.80}$$

This suggests that we ought to hold ka^2/m fixed, but that makes perfect sense: $m/a = \rho$ is the linear density of the row of masses and $ka = T$ is the tension force along the combined string—both of these quantities can be reasonably held fixed and so the system has a characteristic speed $c = \sqrt{T/\rho}$ (physically, this is the speed at which perturbations propagate through the system).

We could have taken the $N \to \infty$ limit earlier. Going back to (7.73), it is not hard to notice that it can be rewritten as

$$\ddot{y}_j = -\omega_0^2 y_j + \frac{ka^2}{m}\frac{y_{j+1} - 2y_j + y_{j-1}}{a^2} \quad \Rightarrow \quad \boxed{\frac{\partial^2 y}{\partial t^2} = -\omega_0^2 y + c^2\frac{\partial^2 y}{\partial x^2}}. \tag{7.81}$$

With $\omega_0 = 0$ (i.e., with the masses oscillating just due to their interaction via springs),[43] this is the *wave equation*—the topic of sequel to this course. Parra (2021) will pick up this thread where

[42] In the opposite (less interesting) limit $p \to N$, the frequencies tend to a constant $\Omega_N = (\omega_0^2 + 4k/m)^{1/2}$ and $v_j^{(N)} = -(-1)^j \sin[\pi j/(N+1)]$—this describes adjacent pendula with sign-alternating displacements.

[43] The wave equation (7.81) with $\omega_0 \neq 0$ describes propagation of waves through certain *dispersive media* that are also capable of oscillating at their own frequency ω_0—an example is propagation of light through plasma (see, e.g., Schekochihin 2020*a*).

I am leaving it off. You will see that it indeed can be solved via decomposition into (an infinite number of) normal modes, known as *Fourier modes* (if you did your research for Q2.7c, you are already familiar with them). The solution (7.79) (with $N = \infty$) is what such a decomposition looks like for the case of zero boundary conditions $y(t, 0) = y(t, L) = 0$.

Example. Q6.6(c,d).

Exercise 7.7. Show that for the system of N coupled oscillators considered in §7.3.6, the energy is

$$\mathscr{E} = \frac{1}{2}\sum_{j=1}^{N}\left(\dot{y}_j^2 + \omega_0^2 y_j^2\right) + \frac{k}{2m}\sum_{j=0}^{N}(y_{j+1} - y_j)^2 \approx \frac{N}{2L}\int_0^L \mathrm{d}x\left(\dot{y}^2 + \omega_0^2 y^2 + c^2 y'^2\right), \tag{7.82}$$

the first expression being exact and the second aproximate in the limit $N \gg 1$. Work out what the energy is in terms of the normal-mode amplitudes $\{A_p, B_p\}$.

7.4. *(Nonlinear) Digression: Coupled Oscillators and Hamiltonian Chaos*

Coming soon... In the meanwhile, read Zaslavsky *et al.* (1991) (not an easy read, but great fun).

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8. Qualitative Solution of Systems of Autonomous Nonlinear ODEs

I promised at the start of this course that, once we have got through all the necessary standard material, we could have some fun and learn how to solve complicated systems in an uncomplicated (=qualitative) way. One might call it an enlightened theory of sketching—fittingly, it has an element of art to it, as well as of science. What we shall be sketching will be *phase portraits*—the basic lay out of *phase trajectories* for autonomous ODEs

$$\boxed{\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})} \tag{8.1}$$

(the salient feature being the absence of explicit time dependence in the right-hand side).

Let me remind you that phase trajectories are the projections of the integral curves (§1.2) onto the phase space, which is the space where the solutions $\boldsymbol{y}(t)$ live (typically, it is all or part of $\mathbb{R}^n$). In other words, they are the "field lines" of the phase velocity $\boldsymbol{f}(\boldsymbol{y})$, i.e., they are the curves whose tangent vector at $\boldsymbol{y}$ is $\boldsymbol{f}(\boldsymbol{y})$. The usefulness of a phase portrait is that it tells you where the system will go from any given initial location in phase space—regardless of exactly *when* it might find itself at that location. It only makes sense to draw phase portraits for autonomous systems because if the phase velocity $\boldsymbol{f}$ changed with time, any one-time snapshot of its field lines would be transient and we could not trace the system's time evolution by following those field lines.[44]

Mathematically, this narrative amounts to saying that

1) if $\boldsymbol{y}(t)$ is a solution of (8.1), then $\forall t_0$ $\boldsymbol{y}(t + t_0)$ is also a solution, i.e., *a time-shifted phase trajectory is the same trajectory;*

2) if $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are phase trajectories and $\boldsymbol{y}_1(t_1) = \boldsymbol{y}_2(t_2)$ for some t_1 and t_2, then $\boldsymbol{y}_2(t) = \boldsymbol{y}_1(t + t_1 - t_2)$, i.e., *any two phase trajectories either do not intersect or coincide.*

Exercise 8.1. These statements are perhaps obvious, but do practice your mathematical reasoning by proving them formally. Think also about how they fail for non-autonomous systems.

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[44] Sometimes, for non-autonomous systems, one can do this in some average sense: usually, when the rate of non-autonomous change of $\boldsymbol{f}$ is either very large (§5.4) or very small (§8.2).

All phase trajectories can be classed into three types:

1) trajectories without self-intersections;

2) closed loops, or *cycles*, i.e., trajectories that bite their own tail;

3) *fixed points* (or *stationary points*, or *equilibria*)—these are phase trajectories consisting of a single point such that the system starting exactly at that point remains there forever. They are simply solutions of

$$\boldsymbol{f}(\boldsymbol{y}_0) = 0. \tag{8.2}$$

Note that fixed points do not have to be isolated—in principle, there can be whole continuous sets of them (see, e.g., §8.1.3).

Exercise 8.2. This classification is a theorem. Prove it. What you have to prove is that if a phase trajectory is not a solution of (8.2) and intersects itself, then it is closed loop, periodic forever.

Clearly, fixed points are special.

Mathematically, their intuitive specialness can be given precise meaning by the statement that, in the vicinity of any point in phase space that is not a fixed point, the phase trajectories can be mapped onto a set of straight-line segments, whereas near a fixed point, they cannot be [this should be obvious from (8.3)]. This is the basis for the intuitive expectation that one might be able to sketch the phase portrait of a system if one works out what happens near all of its fixed points and then "connects" these local mini-portraits in some sensible way. This is not always quite as straightforward as it sounds, but it is the basic idea.

Physically, one is also obviously interested in the fixed points because they are the equilibria of the system. If a fixed point proves to be stable, i.e., if one can show that, starting in some vicinity of it, the system eventually ends up arbitrarily close to it,[45] then such a fixed point represents long-term fate of the system, at least for some range of initial conditions (this is called the *basin of attraction*). Long-time behaviour is what one is often after, so this is clearly useful.

Since we are interested in the system's behaviour near a particular point $\boldsymbol{y}_0$, let us expand (8.1) in the vicinity of $\boldsymbol{y}_0$, as promised in §3.5.3:

$$\delta\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}_0 + \delta\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y}_0) + \mathsf{A}\cdot\delta\boldsymbol{y} + \boldsymbol{f}_{\rm nl}(\boldsymbol{y}_0, \delta\boldsymbol{y}), \quad A_{ij} = \left.\frac{\partial f_i}{\partial y_j}\right|_{\boldsymbol{y}=\boldsymbol{y}_0}, \tag{8.3}$$

where $\boldsymbol{f}_{\rm nl}(\boldsymbol{y}_0, \delta\boldsymbol{y})$ represents the nonlinear terms and is assumed to vanish quicker than

[45] In technical language, such a fixed point is called *attracting*, or an *attractor*. There is another type of stability, called *Lyapunov stability*, whereby if the system starts in some vicinity of a fixed point, it stays within some vicinity of it *at all times*. This does not mean that the point is necessarily attracting, but if it is, then it is called *asymptotically stable*. The theory of (Lyapunov) stability is an important branch of the theory of ODEs. You will find an introduction to it in Pontryagin (1962, §26) or Arnold (2006, §23). Alexander Lyapunov (1857-1918) was a prominent Russian mathematician who essentially started the subject. Word has it that he first reported some of his findings in his lectures to undergraduates at the University of Kharkov, where he was employed as a lecturer, but found teaching standard material rather tedious (he was, however, reputed to produce rather good lecture notes).

linearly with $\delta\boldsymbol{y}$:

$$\lim_{|\boldsymbol{y}|\to 0} \frac{\boldsymbol{f}_{\rm nl}(\boldsymbol{y}_0, \delta\boldsymbol{y})}{|\delta\boldsymbol{y}|} = 0. \tag{8.4}$$

If $\boldsymbol{y}_0$ is not a fixed point, then (8.3) just describes locally constant motion [$\delta\dot{\boldsymbol{y}} \approx \boldsymbol{f}(\boldsymbol{y}_0) = \text{const}$], but if it is, $\boldsymbol{f}(\boldsymbol{y}_0) = 0$ and we have an approximately linear system in $\delta\boldsymbol{y}$. This is what we are now going to study, while returning from time to time to the question of whether our qualitative conclusions are likely to withstand the effect of small $\boldsymbol{f}_{\rm nl}$.

Life in n dimensions is complicated and sketching it on a 2D sheet of paper requires more artistic skill and geometrical imagination than we can perhaps muster within the confines of this course. I will, therefore, start—and stay—with 2D systems. I hasten to say that 2D is rather special: basically because in 2D, a trajectory constrained never to cross itself is very constrained indeed.[46] However, a physicist's intuition is not built in a day, so let us see what a 2D world can teach us.

8.1. *Classification of 2D Equilibria*

[Literature: Pontryagin (1962, §30), Tikhonov *et al.* (1985, §5.4), Arnold (2006, §20.4), Strogatz (1994, §5-6), Glendinning (1994, §5.1-5.4), Bender & Orszag (1999, §4.4)]

You can view this section as a preparation for something bigger (not, I am afraid, to be fully consummated within the time limitations of this course) or just as a set of (very simple) examples of how one uses the analytical machinery developed in §6 (this time, unlike in §7, not necessarily for Hermitian oscillatory systems).

Entirely without loss of generality, $\boldsymbol{y}_0 = 0$ and $\delta\boldsymbol{y} = \boldsymbol{y}$ in (8.3). We are specialising to the 2D system

$$\dot{\boldsymbol{y}} = \mathsf{A}\cdot\boldsymbol{y} \iff \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}}, \quad \text{where} \quad \boldsymbol{y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{8.5}$$

You know the drill: the first order of business is to find eigenvalues:

$$\det(\mathsf{A} - \lambda\mathsf{I}) = 0 \quad \Rightarrow \quad \lambda^2 - \underbrace{(a+d)}_{=T}\lambda + \underbrace{ad - bc}_{=D} = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}}, \tag{8.6}$$

where $T = \operatorname{tr}\mathsf{A}$ and $D = \det\mathsf{A}$. There are a number of possibilities, each of which has a name and a distinctive phase portrait.

8.1.1. *Nodes:* $T^2 > 4D > 0$

If we assume also $T > 0$, this corresponds to

$$\boxed{\lambda_1 > \lambda_2 > 0}. \tag{8.7}$$

With two different eigenvalues available, we are guaranteed that there will be two linearly independent eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, which, when packed into $\mathsf{R} = (\boldsymbol{v}_1\,\boldsymbol{v}_2)$, will transform our system into a diagonal form (see §6.1):

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathsf{R}\cdot\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi\boldsymbol{v}_1 + \eta\boldsymbol{v}_2 \quad \Rightarrow \quad \begin{cases} \dot{\xi} = \lambda_1\xi, \\ \dot{\eta} = \lambda_2\eta \end{cases} \quad \Rightarrow \quad \begin{cases} \xi = \xi_0 e^{\lambda_1 t}, \\ \eta = \eta_0 e^{\lambda_2 t} \end{cases}. \tag{8.8}$$

[46] For example, in 2D, if there is a closed trajectory, any unlucky trajectory starting with initial conditions inside it is trapped forever. In 3D, it would barely notice.

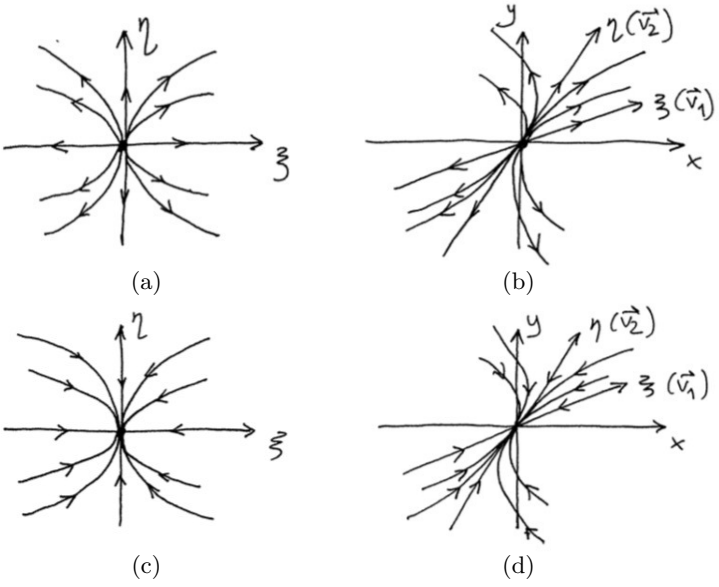


FIGURE 35. (a) An unstable node in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ where A is diagonal ($\lambda_1 > \lambda_2 > 0$). (b) An unstable node in the original Cartesian basis. (c,d) Same as (a,b), but the node is stable ($\lambda_1 < \lambda_2 < 0$).

The phase trajectories in the plane (ξ, η) are, therefore, the curves[47]

$$\frac{\xi}{\xi_0} = \left(\frac{\eta}{\eta_0}\right)^{\lambda_1/\lambda_2}. \tag{8.9}$$

Since $\lambda_1/\lambda_2 > 1$, these are "generalised parabolae", i.e., they are tangent at zero to the "slow" direction $\boldsymbol{v}_2$. Note that $\xi = 0$ and $\eta = 0$ are also phase trajectories, corresponding to the initial conditions $\xi_0 = 0$ and $\eta_0 = 0$, respectively. The corresponding phase portrait is sketched in Fig. 35(a)—the arrows, indicating the directions of the system's travel, are based on the fact that both ξ and η must increase with time. This type of equlibrium is called an *unstable node*.

Exercise 8.3. The phase trajectories seem to intersect at zero. Is this a problem?

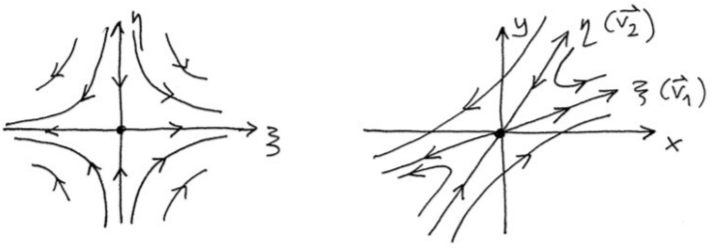
Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not necessarily orthogonal, this picture gets somewhat distorted when transformed back into the (x, y) plane (Fig. 35b), but its basic feature remains the same: trajectories run away from the fixed point, hugging the slow eigendirection.

If we let $T < 0$ and rearrange the eigenvalues to be

$$\boxed{\lambda_1 < \lambda_2 < 0}, \tag{8.10}$$

we get the same story, but with direction of time reversed, i.e., the trajectories now all rush into the fixed point, again hugging the slow eigendirection (Figs. 35c,d). This is called a *stable node*. An **example** of this type of behaviour was the overdamped oscillator, where indeed it was the slow damping that dominated long-time dynamics (see §5.1.1, where $T = \gamma$ and $D = \omega_0^2$).

[47] You can obtain (8.9) also by noticing that $\mathrm{d}\xi/\mathrm{d}\eta = \dot{\xi}/\dot{\eta} = (\lambda_1/\lambda_2)\xi/\eta$ and integrating this equation.



(a) (b)

FIGURE 36. (a) A saddle point in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ where A is diagonal ($\lambda_1 > 0 > \lambda_2$). (b) A saddle point in the original Cartesian basis.

8.1.2. *Saddles:* $T^2 > 0 > 4D$

In this case, the two eigenvalues have opposite signs:

$$\boxed{\lambda_1 > 0 > \lambda_2}\,. \tag{8.11}$$

The same analysis applies as in §8.1.1, but (8.9) now becomes an equation for "generalised hyperbolae":

$$\frac{\xi}{\xi_0} = \left(\frac{\eta}{\eta_0}\right)^{-\lambda_1/|\lambda_2|}. \tag{8.12}$$

The trajectories approach the fixed point along the stable eigendirection ($\boldsymbol{v}_2$) and then veer off and away along the unstable one ($\boldsymbol{v}_1$): see Fig. 36. This type of equlibrium is called a *saddle*. We saw an **example** of it when we studied the "upright" equilibrium of the mathematical pendulum (§3.5.2, Fig. 14).

The two axes, $\xi = 0$ and $\eta = 0$, are also trajectories, the former leading the system straight into the fixed point, the latter out of it. These are called *separatrices*. The first separatrix, the η axis, also known in the litterati circles as the *stable manifold*, is quite special in the sense that for a system starting off close to it, a small perturbation in the initial conditions that puts the system on the other side of the separatrix has the effect of dramatically altering subsequent evolution: the system will end up at $\xi = -\infty$ as opposed to $\xi = +\infty$, or vice versa. This is your first inkling of the phenomenon of *sensitive dependence on initial conditions*, which, transferred to more dimensions and more complicated systems, makes the world chaotic, but also opens the door to its statistical description (see Schekochihin 2020*b*, §1).

8.1.3. *Pathologies:* $D = 0$

In this case, at least one of the eigenvalues is zero:

$$\boxed{\lambda_1 > \lambda_2 = 0} \quad \text{or} \quad \boxed{\lambda_1 < \lambda_2 = 0}\,. \tag{8.13}$$

This means that

$$\dot{\eta} = 0, \tag{8.14}$$

so the straight lines $\eta = \text{const}$ are phase trajectories, which all run away from the $\xi = 0$ axis if $\lambda_1 > 0$ or towards it if $\lambda_1 < 0$ (Fig. 37). The entire line $\xi = 0$ consists of (non-isolated) fixed points. I will call these cases *stripes* (to match the stars in §8.1.5).

Obviously, if $\lambda_1 = 0$ as well, then every point in the plane is a fixed point of our linear system, but this is an intensely uninteresting case—or, depending on your perspective, an intensely interesting one, because the nonlinear system of which it is the linearisation is "fully nonlinear", i.e., its dynamics are fully determined by the nonlinearity (see §8.1.4).

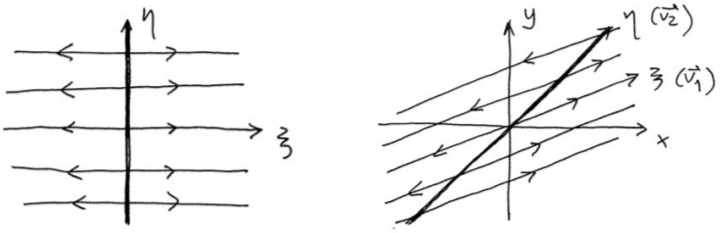


FIGURE 37. (a) Unstable stripes in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ where A is diagonal ($\lambda_1 > \lambda_2 = 0$). The η axis is a line of fixed points. (b) Unstable stripes in the original Cartesian basis. Stable stripes ($\lambda_1 < \lambda_2 = 0$) look the same but with arrows reversed.

8.1.4. *Effect of Nonlinearity*

I have called the above case a pathology because it requires some rather special arrangements ($D = 0$). It is also a bit of an irrelevance because what, in fact, a zero eigenvalue implies is that the linearisation of the system in the corresponding eigendirection has proved trivial and so the dynamics in that direction is fully decided by the nonlinearity, which has so far been lurking invisibly on the sidelines. For example, if, in (8.3), $\boldsymbol{f}_{\rm nl}$ projected on the $\xi = 0$ axis proves to be a positive contribution at $|\eta| > 0$, say η^2, so

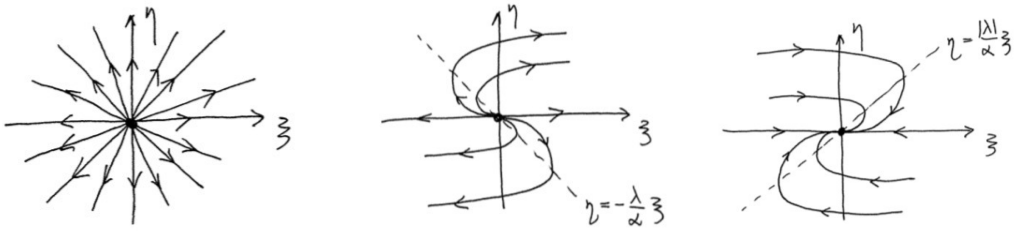
$$\dot{\eta} \approx \eta^2, \tag{8.15}$$

then we again only have a single fixed point (at zero) and this fixed point is unstable in the η direction—a nonlinearly unstable saddle (if $\lambda_1 < 0$) or node (if $\lambda_1 > 0$), with the departing trajectories hugging the slow (nonlinear) direction even more tightly than in the linear case.[48]

This brings in the broader question: to what extent are the local phase portraits that we construct on the basis of linear theory likely to be good qualitative descriptions of what happens nonlinearly, even in the vicinity of the fixed point? The basic qualitative answer is that linear conclusions are fine as long as they do not depend on the eigenvalues being precisely equal to some special value: thus, nodes and saddles survive because a small perturbation of the trajectories by the nonlinear terms cannot knock them off the basic trends of exponential growth or decay set by robustly non-zero, non-equal eigenvalues—unlike in the pathological case of §8.1.3, where a zero eigenvalue gives the nonlinearity license to take over.[49] Mathematically, it is possible to prove qualitative robustness of the linear phase portraits by placing careful upper bounds on the nonlinearity's contribution. All this can be made much more precise, but I hope this qualitative discussion makes enough sense to be believable at the first pass. If you want to see for yourself how to investigate the matter quantitatively, and, in particular, how to prove some of the statements about the effect of the nonlinearities that I have made above or will make below, a good undergraduate read is Glendinning (1994, §5.2), while the original source of biblical proportions is Andronov *et al.* (1973).

[48] An **example** of such a situation is the $x = y = 0$ equilibrium in Q5.4.

[49] A fixed point with all $\mathrm{Re}\,\lambda_i \neq 0$ is called *hyperbolic* (just a name, nothing to do with saddles, necessarily). One can show that local phase portraits around hyperbolic fixed points are robust. Mathematically, "robust" means *topologically equivalent*, which means that the phase portraits of the nonlinear and the linearised systems can be mapped onto each other by a continuous one-to-one transformation.



(a) (b) (c)

FIGURE 38. (a) An unstable star in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ where A is diagonal ($\lambda_1 = \lambda_2 > 0$). In the original Cartesian basis, the picture is the same, with trajectories still approaching the fixed point at all angles. A stable case ($\lambda_1 = \lambda_2 < 0$) looks the same but with arrows reversed. (b) An unstable degenerate node in the triangulating basis $\{\boldsymbol{v}, \boldsymbol{w}\}$. (c) A stable degenerate node in the same basis—it is a mirror image of (b).

8.1.5. *Pathologies: $T^2 = 4D$*

This is another pathological case: the eigenvales are equal,

$$\boxed{\lambda_1 = \lambda_2 \equiv \lambda \neq 0}\,, \tag{8.16}$$

and can be either positive or negative (we are not interested in the trivial case of them being zero).

In this case, it matters whether the matrix A has two linearly independent eigenvectors or only one. Suppose it has two. Then they span the plane and, in fact, any vector is an eigenvector, i.e., the system can hurtle towards ($\lambda < 0$) or away from ($\lambda > 0$) the fixed point along any straight line originating at zero (Fig. 38a). Indeed, the phase trajectories satisfy (8.9) with $\lambda_1/\lambda_2 = 1$:

$$\frac{\xi}{\xi_0} = \frac{\eta}{\eta_0}, \tag{8.17}$$

which describes straight lines. Such a node is called a *star* (stable or unstable depending on the sign of λ).

Let us work out precisely for what kind of matrix A this scenario occurs. All vectors must be eigenvectors, so we must have

$$\forall \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad b = c = 0, \quad a = d = \lambda \quad \Rightarrow \quad \mathsf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \tag{8.18}$$

Thus, stars are a bit boring. Are they also irrelevant, requiring $b = c = 0$ and thus potentially sensitive to nonlinear perturbations?

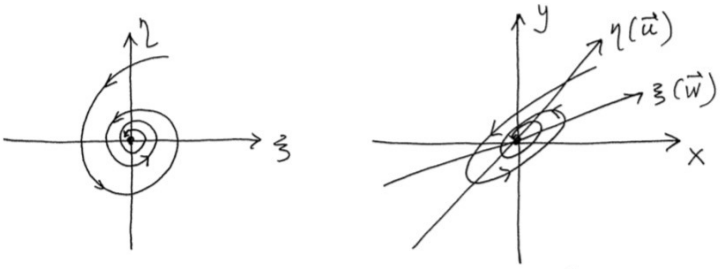
It turns out that it is possible to find classes of $\boldsymbol{f}_{\text{nl}}$ such that a star will stay a star (i.e., trajectories will still approach the fixed point at all angles) or will look more like a node (with trajectories hugging one direction) or a focus (§8.1.6).[50] In particular, a star stays a star if the nonlinearity is quadratic or higher-order.

Let us see now what happens in the arguably more generic situation when $bc \neq 0$, i.e., when A is allowed to have off-diagonal terms. It is then not diagonalisable, but, as we know from Schur's theorem (§6.3.1), it can be turned into a triangular form by a unitary transformation. The matrix of this transformation is $\mathsf{U} = (\boldsymbol{v}\,\boldsymbol{w})$, where $\boldsymbol{v}$ is the sole available eigenvector and $\boldsymbol{w}$ is any vector orthogonal to it (see §6.3.2). Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathsf{U} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi \boldsymbol{v} + \eta \boldsymbol{w} \quad \Rightarrow \quad \begin{cases} \dot{\xi} = \lambda \xi + \alpha \eta, \\ \dot{\eta} = \lambda \eta, \end{cases} \tag{8.19}$$

where we can always have $\alpha > 0$ (by reversing the sign of $\boldsymbol{w}$ if not). In order to sketch the phase

[50] Q5.2(ii) is an **example** of the latter scenario.



(a) (b)

FIGURE 39. (a) A stable focus ($\lambda_{1,2} = \Gamma \pm i\Omega$, $\Gamma < 0$) in the basis $\{\boldsymbol{v}, \boldsymbol{w}\}$, where A is transformed according to (8.22). (b) A stable focus in the original Cartesian basis. An unstable focus ($\Gamma > 0$) looks the same but with arrows reversed.

portrait of this system, let us observe that

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \left(\frac{\xi}{\eta} + \frac{\alpha}{\lambda}\right)^{-1} \to \begin{cases} 0, & \text{when} \quad \eta = 0, \ \xi \text{ finite}, \\ \dfrac{\lambda}{\alpha}, & \text{when} \quad \xi = 0, \ \eta \text{ finite}, \\ \infty, & \text{when} \quad \eta = -\dfrac{\lambda}{\alpha}\,\xi. \end{cases} \tag{8.20}$$

This is enough to piece together a phase portrait: when $\lambda > 0$, phase trajectories run away from the fixed point along the one available eigendirection before executing a partial U-turn (at $\eta/\xi = -\lambda/\alpha$, where $\mathrm{d}\eta/\mathrm{d}\xi = \infty$) and taking off to infinity (Fig. 38b); when $\lambda < 0$, they do the same thing in reverse and in mirror reflection (Fig. 38c). These are unstable and stable *degenerate (or improper) nodes*. An **example** of this kind of behaviour was the critically damped oscillator (§5.1.2).

Like the stars, the degenerate nodes qualitatively survive the effect of quadratic or higher-order nonlinear terms, but for gentler nonlinearities, they are liable to change, roughly speaking in the direction of looking more like spirals converging into or coming out of foci (§8.1.6)—not, perhaps, surprising as they are a case that sits at the boundary in the (T, D) space separating nodes and foci.

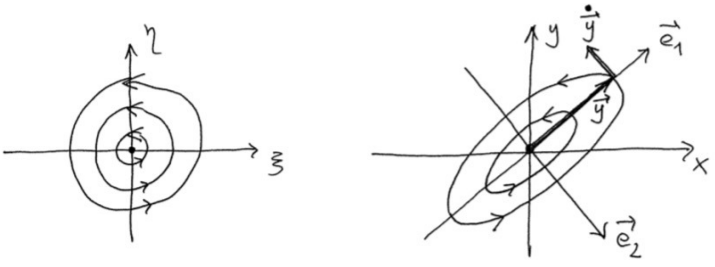
Note that the method that I have used to figure out the phase portraits in Figs. 38(b,c) is representative of the sort of thing you might generally have to resort to when in doubt: work out the phase-velocity field $(\dot{\xi}, \dot{\eta})$ [or $(\dot{x}, \dot{y})$] on some special curves in phase space and try to make the phase portrait consistent with these scraps of information. The most obvious special curves are the axes of whatever coordinate system you choose to work in (here $\xi = 0$ and $\eta = 0$) and the *null-clines*, which are the curves where $\dot{\xi} = 0$ or $\dot{\eta} = 0$ (equivalently, $\mathrm{d}\eta/\mathrm{d}\xi = \infty$ or 0), i.e., where the "phase flow" is either vertical or horizontal.

8.1.6. *Foci:* $4D > T^2 > 0$

In this case, we have two different complex eigenvalues that are conjugates of each other:

$$\boxed{\lambda_{1,2} = \Gamma \pm i\Omega}, \quad \text{where} \quad \Gamma = \frac{T}{2}, \quad \Omega = \sqrt{D - \frac{T^2}{4}}. \tag{8.21}$$

An **example** of such a system is the underdamped oscillator (5.10), or, equivalently, the $\theta = 0$ equilibrium of the damped mathematical pendulum (§3.5.4, Fig. 16). It is instantly clear what will happen qualitatively: the trajectories will wind around the fixed point with the period $2\pi/\Omega$ (this is the 2D representation of an oscillation) and spiral either in or out of it, depending on the sign of Γ (Fig. 39). Fixed points of this type are called stable or unstable *foci (or spirals)*. They turn out to be robust to any nonlinear meddling: close enough to the fixed point, a spiral stays a spiral, regardless of what the nonlinearity does (this makes intuitive sense).



(a) (b)

FIGURE 40. (a) A centre ($\lambda_{1,2} = \pm i\Omega$) in the basis $\{\boldsymbol{v}, \boldsymbol{w}\}$, where A is transformed according to (8.22). (b) A centre in the original Cartesian basis. The basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is one where the energy has the form (8.34); the basis vectors determine the directions of the semi-major and semi-minor axes of the elliptic trajectories.

In the interest of show-casing some nice mathematical tricks, let me demonstrate explicitly that phase trajectories are indeed spirals. Since the two eigenvalues are different, there will be two linearly independent eigenvectors, complex conjugates of each other: $\boldsymbol{v}$ and $\boldsymbol{v}^*$ [see (6.24)]. If we denote $\boldsymbol{v} = \boldsymbol{u} + i\boldsymbol{w}$, where $\boldsymbol{u}$ and $\boldsymbol{w}$ are both real, we can make a matrix $\mathsf{S} = (\boldsymbol{w}\,\boldsymbol{u})$ and ask what it will do to A: since A is real,

$$\begin{aligned}\mathsf{A}\cdot\mathsf{S} &= (\mathsf{A}\cdot\boldsymbol{w}\ \mathsf{A}\cdot\boldsymbol{u}) = (\mathrm{Im}(\mathsf{A}\cdot\boldsymbol{v})\ \mathrm{Re}(\mathsf{A}\cdot\boldsymbol{v})) = (\mathrm{Im}(\lambda\boldsymbol{v})\ \mathrm{Re}(\lambda\boldsymbol{v}))\\ &= (\Gamma\boldsymbol{w} + \Omega\boldsymbol{u}\ \ \Gamma\boldsymbol{u} - \Omega\boldsymbol{w}) = \mathsf{S}\cdot\begin{pmatrix}\Gamma & -\Omega\\ \Omega & \Gamma\end{pmatrix} \quad\Rightarrow\quad \mathsf{A} = \mathsf{S}\cdot\begin{pmatrix}\Gamma & -\Omega\\ \Omega & \Gamma\end{pmatrix}\cdot\mathsf{S}^{-1}. \end{aligned} \tag{8.22}$$

If we apply this decomposition to our equation (8.5), we get

$$\dot{\boldsymbol{y}} = \mathsf{S}\cdot\begin{pmatrix}\Gamma & -\Omega\\ \Omega & \Gamma\end{pmatrix}\cdot\mathsf{S}^{-1}\cdot\boldsymbol{y} \quad\Rightarrow\quad \dot{\boldsymbol{\xi}} = \begin{pmatrix}\Gamma & -\Omega\\ \Omega & \Gamma\end{pmatrix}\cdot\boldsymbol{\xi}, \quad \text{where} \quad \boldsymbol{\xi} = \mathsf{S}^{-1}\cdot\boldsymbol{y} \equiv \begin{pmatrix}\xi\\ \eta\end{pmatrix}. \tag{8.23}$$

Now let $\zeta = \xi + i\eta$. Then

$$\dot{\zeta} = \dot{\xi} + i\dot{\eta} = \Gamma\xi - \Omega\eta + i\Omega\xi + i\Gamma\eta = (\Gamma + i\Omega)\zeta \quad\Rightarrow\quad \zeta = r_0 e^{\Gamma t + i(\Omega t + \phi_0)}, \tag{8.24}$$

where $r_0 = \sqrt{\xi_0^2 + \eta_0^2}$ and ϕ_0 are the radius and phase of the initial condition. In the (ξ, η) plane, this is an anticlockwise logarithmic spiral with period $2\pi/\Omega$ and radius $r = r_0 e^{\Gamma t} = r_0 e^{(\Gamma/\Omega)(\phi - \phi_0)}$. When transformed back into the (x, y), this is still a spiral, if possibly somewhat squashed ($\boldsymbol{u}$ and $\boldsymbol{w}$ are not necessarily orthogonal). Whether it is still anticlockwise depends on whether the $\{\boldsymbol{u}, \boldsymbol{w}\}$ basis has the same handedness as the Cartesian one.

Note that we have discovered the usefulness of the transformation to polar coordinates: for $(\xi, \eta) \to (r, \phi)$, our original equation (8.5) becomes just

$$\begin{cases} \dot{r} = \Gamma r, \\ \dot{\phi} = \Omega. \end{cases} \tag{8.25}$$

It is often a good idea to manipulate even your nonlinear system into these coordinates and think of the 2D dynamics in terms of distance from the origin and phase and of the nonlinearities as making the effective growth/damping rate Γ and frequency Ω depend on r and ϕ (see, e.g., Glendinning 1994, §5.2).

8.1.7. *Centres:* $4D > T^2 = 0$

In this case, the eigenvalues are purely imaginary:

$$\boxed{\lambda = \pm i\Omega}\,. \tag{8.26}$$

From the above analysis, it is instantly clear that phase trajectories will circle around the fixed point (Fig. 40), rather than spiralling in or out. These trajectories are ellipses,

as I will prove in §8.1.8. A fixed point of this type is called a *centre*. An **example** is the $\theta = 0$ equlibrium of an undamped pendulum (§3.5.1, Fig. 13).

In principle, this is again a pathological case and you might imagine (and would be correct in many cases) that a small nonlinearity that provides a damping or a growth of perturbations will turn a centre into a focus.

8.1.8. *Conservative Systems*

However, the case of centres is rather important because it describes a situation in which there is a *conservation law:* the closed trajectories are level sets of some function that, throughout the system's evolution, stays the same as it was initially. We shall call this function *energy*. In many physical settings, energy conservation is a fundamental principle that governs both linear and nonlinear dynamics—in which case, the energy of the linearised system is an approximate version of the energy of the nonlinear system, so the nonlinearity conspires in making the system *conservative*. The phase trajectories must then stay on surfaces of constant energy. In such situations, a centre remains resilient to nonlinear perturbations.

We have seen an **example** of this in the nonlinear, undamped pendulum (§3.5.2, Fig. 15; a host of other such systems was studied in §7). A slight generalisation of this example is as follows. Consider a 1D system subject to a potential force:

$$\ddot{x} = -U'(x), \quad \text{or} \quad \begin{cases} \dot{x} = y \\ \dot{y} = -U'(x). \end{cases} \tag{8.27}$$

This conserves energy:

$$\mathscr{E} = \frac{y^2}{2} + U(x) \quad \Rightarrow \quad \dot{\mathscr{E}} = 0. \tag{8.28}$$

At the fixed point, assumed to be at the origin, $U'(0) = 0$, so, in its vicinity,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -U''(0)x \end{cases} \quad \text{and} \quad \mathscr{E} = \frac{y^2}{2} + U(0) + U''(0)\frac{x^2}{2}. \tag{8.29}$$

If $U''(0) > 0$, the trajectories are ellipses $y^2 + U''(0)x^2 = \text{const}$, and the energy conserved by the linear system is just the true energy (8.28) approximated near the fixed point. The phase trajectories of the full nonlinear system are level sets of $\mathscr{E} = \text{const}$ and so will also be closed curves, at least as long as $\mathscr{E}$ is sufficiently close to $U(0)$. Thus, the qualitative nature of the centre is not destroyed by nonlinearity.

It is not hard to figure out what the conserved energy for the linearised system (8.5) is:[51] it is clear that it should be quadratic in x and y and so we may let

$$\mathscr{E} = \alpha x^2 + 2\beta xy + \gamma y^2. \tag{8.30}$$

Since, $\forall x$ and y, we must have

$$\dot{\mathscr{E}} = 2\alpha x\dot{x} + 2\beta(\dot{x}y + x\dot{y}) + \gamma y\dot{y} = 2(\alpha a + \beta c)x^2 + 2(\alpha b + \beta a + \beta d + \gamma c)xy + 2(\beta b + \gamma d)y^2 = 0, \tag{8.31}$$

setting to zero the coefficients of x^2, y^2 and xy gives us three linear equations for α, β and γ:

$$\begin{pmatrix} a & c & 0 \\ b & a+d & c \\ 0 & b & d \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0. \tag{8.32}$$

These equations are readily seen to be solvable when $T = a + d = 0$, as indeed we assumed when

[51] This somewhat more laborious procedure than that described in §7.2.1 is needed because here we are dealing with with a matrix that is not necessarily symmetric.

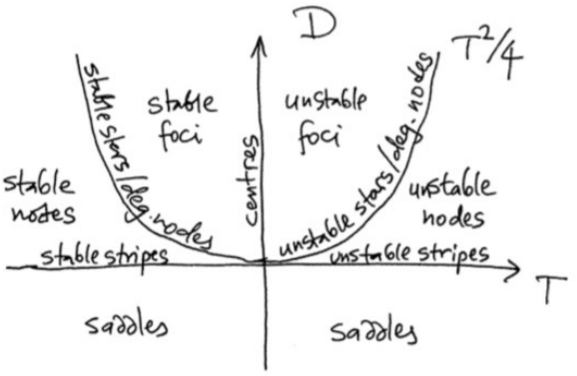


FIGURE 41. Classification of 2D equilibria based on the values of $T = \operatorname{tr}\mathsf{A}$ and $D = \det\mathsf{A}$.

we embarked on studying the case of two imaginary eigenvalues. This gives us[52]

$$\mathscr{E} = -\frac{1}{2}\,cx^2 + axy + \frac{1}{2}\,by^2 = \boldsymbol{y}\cdot\mathsf{B}\cdot\boldsymbol{y}, \quad \text{where} \quad \mathsf{B} = \frac{1}{2}\begin{pmatrix} -c & a \\ a & b \end{pmatrix}. \tag{8.33}$$

Since B is a symmetric matrix, it has two real eigenvalues Λ_1 and Λ_2 and two orthonormal eigenvectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, which, once combined into the orthogonal matrix $\mathsf{R} = (\boldsymbol{e}_1\,\boldsymbol{e}_2)$, rotate our plane into the coordinates in which B is diagonal:

$$\mathscr{E} = \boldsymbol{y}\cdot\mathsf{R}\cdot\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}\cdot\mathsf{R}^T\cdot\boldsymbol{y} = \Lambda_1 X^2 + \Lambda_2 Y^2, \quad \text{where} \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \mathsf{R}^T\cdot\begin{pmatrix} x \\ y \end{pmatrix}. \tag{8.34}$$

Since $\Lambda_1\Lambda_2 = \det\mathsf{B} = -(bc + a^2)/4 = D/4 > 0$ by the founding assumption of this section, the eigenvalues have the same sign, which can be assumed positive (otherwise, redefine $\mathscr{E} \to -\mathscr{E}$). Thus, the integral curves of our equation (8.5) are ellipses $\mathscr{E} = \text{const}$ with semi-major and semi-minor axes $(\mathscr{E}/\Lambda_1)^{1/2}$ and $(\mathscr{E}/\Lambda_2)^{1/2}$ (assuming $\Lambda_1 \leqslant \Lambda_2$). Their orientation is given by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively.[53]

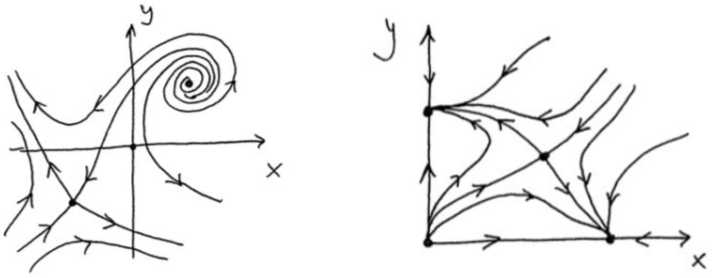
This is it: we have covered all possible 2D equilibria. This is sometimes called the *Poincaré classification*. In Fig. 41, all the different cases (12 of them: saddles, stable/unstable nodes, degenerate nodes, stars, stripes, foci, and centres) are placed in their applicability regions of the (T, D) plane.

The way forward now is to identify the fixed points of nonlinear systems, work out local phase portraits and then "connect the dots" (Fig. 42). This works surprisingly well for many (simple) systems,[54] especially if one is willing to be assisted by some additional tricks like looking at the null-clines (see discussion at the end of §8.1.5). However, before we can do this with a modicum of confidence, we have to learn about one other possible

[52] Another way to get this result is to notice that, in view of (8.24), $\mathscr{E} = \xi^2 + \eta^2 = \boldsymbol{y}\cdot(\mathsf{S}^{-1})^T\cdot\mathsf{S}^{-1}\cdot\boldsymbol{y}$, so $\mathsf{B} = (\mathsf{S}^{-1})^T\cdot\mathsf{S}^{-1}$ (up to an arbitrary multiplicative constant because if $\mathscr{E}$ is conserved, so is any multiple of it). Working out the eigenvector $\boldsymbol{v} = \boldsymbol{u} + i\boldsymbol{w}$, hence $\mathsf{S} = (\boldsymbol{w}\,\boldsymbol{u})$, and hence B, should get you (8.33). Yet another, and the quickest, way of getting (8.33) is to find $\mathscr{E} = H(x, y)$ from (8.35).

[53] A somewhat quicker, but less satisfying to linear-algebra aficionados, way to pin down these ellipses is to notice that the phase-space points at the ends of the major and minor axes must satisfy $\boldsymbol{y}\cdot\dot{\boldsymbol{y}} = 0$ (the phase velocity at these points is perpendicular to the radius vector; see Fig. 40b). This gives $\boldsymbol{y}\cdot\mathsf{A}\cdot\boldsymbol{y} = 0$, whence one gets a quadratic equation for y/x, its two solutions giving the orientations of the axes of the ellipses. Their lengths are then given by $(x^2 + y^2)^{1/2}$.

[54] A nice, carefully worked **example** (Fig. 42b) can be found in Strogatz (1994, §6.4) or in Glendinning (1994, §5.3, the same example).



(a) (b)

FIGURE 42. Simple **examples** of phase portraits. (a) An abstract system with an unstable focus and a saddle: $\dot{x} = x - y$, $\dot{y} = x^2 - 4$ (as an **exercise**, check that this is indeed the phase portrait). (b) A Lotka–Volterra population model ("rabbits and sheep" competing for the same grass) with three nodes and a saddle: $\dot{x} = x(3 - x - 2y)$, $\dot{y} = y(2 - x - y)$; see Strogatz (1994, §6.4) or Glendinning (1994, §5.3). The one interesting qualitative feature here is that, depending on which side of the stable manifold (the separatrix going into the saddle; see §8.1.2) you start, you will end up in a different long-term equilibrium.

feature of 2D systems that can also describe their long-time behaviour and that certainly affects the look of their phase portraits: limit cycles.

Exercise 8.4. Strogatz's Classification of Love Affairs. It might amuse you to recast the classification of 2D equilibria in terms of the dynamics of a relationship between two lovers whose feelings for one another depend on the other's response: see Strogatz (1994, §5.3 and its accompanying exercises). Thinking about romantic interpretations and implifications of different parameter regimes might help you master the material. #oxlove

Exercise 8.5. Classify the 3D equilibria of linear ODEs (Arnold 2006, §21.1 lets you have a glimpse of the answer but leaves some space for creativity).

8.2. *(Nonlinear) Digression: Hamiltonian Systems and Adiabatic Invariants*

[Literature: Landau & Lifshitz (1976, §49)]

This topic does not strictly belong here, but there is nothing wrong with taking a quick peek at more advanced material when the context is begging for it.[55]

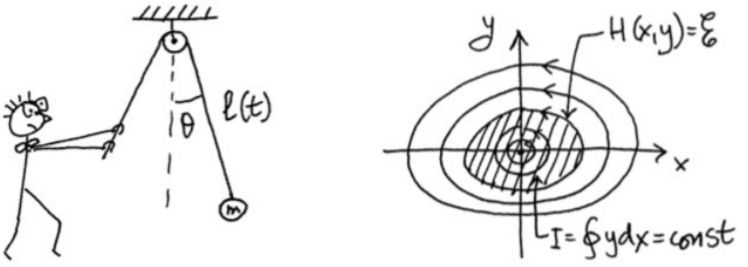
The context in question is that of conservative systems, whose phase portraits locally look like centres and remain robust to nonlinearity because there is a conserved energy—the centres are the minima of this energy and its level sets are the phase trajectories of the system. Such systems of ODEs in mechanics are known as *Hamiltonian systems* (§I.6). They have the general form

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y}, \\ \dot{y} = -\dfrac{\partial H}{\partial x}, \end{cases} \tag{8.35}$$

where $H(x, y)$ is a function, known as the *Hamiltonian*, which turns out to be the energy of the system: indeed, defining

$$\mathscr{E}(t) = H(x(t), y(t)), \tag{8.36}$$

[55] I am grateful to Radek Grabarczyk for getting me to include this material.



(a) (b)

FIGURE 43. (a) Pendulum with slowly changing length. (b) Conserved adibatic invariant (8.48) is the area of phase space enclosed by a closed phase trajectory (which is a level set of the Hamiltonian).

where $(x(t), y(t))$ is the phase trajectory, we find immediately, using (8.35), that[56]

$$\dot{\mathscr{E}} = \frac{\partial H}{\partial x}\,\dot{x} + \frac{\partial H}{\partial y}\,\dot{y} = 0. \tag{8.37}$$

Going back to §3.5.2, you might recall that equations of motion for the undamped pendulum could indeed be written in the form (8.35) [see (3.22)]. The same is very obviously true for the slightly more general case (8.27) and also for the general 2D linearised system (8.5) with $d = -a$ and the energy (8.33).

Now let me venture slightly outside my declared remit of studying only autonomous systems and show you how sometimes one can get somewhere with certain non-autonomous ones. Imagine that some parameter of our system, which I will call $\Lambda(t)$, is changing very slowly.[57] "Slowly" means that, for a system following closed trajectories around a centre, the parameter changes little over one period T of the motion: formally,

$$T\dot{\Lambda} \ll \Lambda. \tag{8.38}$$

In the new, non-autonomous system that such a situation engenders, x and y still satisfy (8.35), but the Hamiltonian function now depends on time via Λ, viz., $H = H(x, y, \Lambda(t))$. A good **example** of such a situation is again our pendulum, whose length is now slowly changing: e.g., the rope on which the mass hangs could be slowly pulled up (Fig. 43a), i.e., the slow-changing parameter is $\Lambda = l(t)$. Then

$$\left\{\begin{array}{l} \dot{x} = y, \\ \dot{y} = -\omega_0^2(t)x \end{array}\right. \quad \Rightarrow \quad H = \frac{y^2}{2} + \omega_0^2(t)\frac{x^2}{2}, \quad \text{where} \quad \omega_0(t) = \sqrt{\frac{g}{l(t)}}. \tag{8.39}$$

The energy is no longer conserved:

$$\dot{\mathscr{E}} = \frac{\partial H}{\partial x}\,\dot{x} + \frac{\partial H}{\partial y}\,\dot{y} + \frac{\partial H}{\partial \Lambda}\,\dot{\Lambda} = \frac{\partial H}{\partial \Lambda}\,\dot{\Lambda}. \tag{8.40}$$

It turns out, however, that there is another quantity that *is* conserved—more robustly than energy, even if only approximately. This quantity is called the *adiabatic invariant* and is equal to the area of the region of the phase space enclosed by the trajectory followed by the system in each oscillation. I am going to derive this result now.

Let us average (8.40) over the period T of oscillations:

$$\langle\dot{\mathscr{E}}\rangle = \dot{\Lambda}\left\langle\frac{\partial H}{\partial \Lambda}\right\rangle = \dot{\Lambda}\,\frac{1}{T}\int_t^{t+T} \mathrm{d}t'\,\frac{\partial H}{\partial \Lambda}(x(t'), y(t'), \Lambda). \tag{8.41}$$

[56] Recalling the definition (1.17) of a parametric solution of the first-order ODE (1.14), let us observe that the phase trajectories of (8.35) are the integral curves of (1.14) with $P = \partial H/\partial x$ and $Q = \partial H/\partial y$, which is then just $\mathrm{d}H = 0$, an equation in full differentials (cf. §2.1). The Hamiltonian system (8.35) is *integrable*.

[57] I dealt with the opposite situation, that of rapid change, in §5.4.

Here $\dot{\Lambda}$ has been pulled outside the averaging because, by the assumption (8.38), $\Lambda(t)$ changes little over one period. Let us change the integration variable in the time integral:

$$\int_t^{t+T} \mathrm{d}t' \, \frac{\partial H}{\partial \Lambda} = \oint \mathrm{d}x \, \frac{\mathrm{d}t'}{\mathrm{d}x} \frac{\partial H}{\partial \Lambda} = \oint \mathrm{d}x \left(\frac{\partial H}{\partial y}\right)^{-1} \frac{\partial H}{\partial \Lambda}, \tag{8.42}$$

where the first of the equations (8.35) has been used for $\mathrm{d}x/\mathrm{d}t'$. The integral is now over a closed phase trajectory described by (8.35), or, equivalently, over a level set of the Hamiltonian with a given fixed energy:

$$H(x, y, \Lambda) = \mathscr{E} = \text{const}. \tag{8.43}$$

If we treat x as the independent variable (over which we integrate), then $y = y(x, \Lambda, \mathscr{E})$. Let us differentiate (8.43) with respect to Λ:

$$\frac{\partial H}{\partial \Lambda} + \frac{\partial H}{\partial y} \frac{\partial y}{\partial \Lambda} = 0. \tag{8.44}$$

From here, we can substitute $\partial H/\partial \Lambda$ into (8.42):

$$\int_t^{t+T} \mathrm{d}t' \, \frac{\partial H}{\partial \Lambda} = -\oint \mathrm{d}x \, \frac{\partial y}{\partial \Lambda}. \tag{8.45}$$

This goes into (8.41), which also contains the period of the motion T. We can calculate it by the same change-of-integration-variable trick as in (8.42):

$$T = \int_t^{t+T} \mathrm{d}t' = \int_{t'=t}^{t'=t+T} \frac{\mathrm{d}t'}{\mathrm{d}x} \mathrm{d}x = \oint \mathrm{d}x \left(\frac{\partial H}{\partial y}\right)^{-1} = \oint \mathrm{d}x \, \frac{\partial y}{\partial \mathscr{E}}. \tag{8.46}$$

Using (8.45) and (8.46), we now deduce from (8.41):

$$\langle \dot{\mathscr{E}} \rangle = -\dot{\Lambda} \frac{\displaystyle\oint \mathrm{d}x \, \frac{\partial y}{\partial \Lambda}}{\displaystyle\oint \mathrm{d}x \, \frac{\partial y}{\partial \mathscr{E}}} \quad \Rightarrow \quad 0 = \oint \mathrm{d}x \left(\frac{\partial y}{\partial \mathscr{E}} \langle \dot{\mathscr{E}} \rangle + \frac{\partial y}{\partial \Lambda} \dot{\Lambda} \right) = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \oint y \, \mathrm{d}x \right\rangle. \tag{8.47}$$

Thus, we have a quantity, the *adiabatic invariant*, that does not change on average, even though both $\mathscr{E}$ and Λ do (i.e., it changes even more slowly than they do):

$$\boxed{\langle \dot{I} \rangle = 0, \quad \text{where} \quad I = \oint y \, \mathrm{d}x}. \tag{8.48}$$

But I is nothing but the area inside the closed phase trajectory (Fig. 43b), as I promised.

Let us return to our **example** (8.39) of a linear oscillator with a slowly changing length, and ask how the amplitude of the oscillations will change as the mass is slowly pulled up. For any given value of energy, the phase trajectory is an ellipse:

$$\frac{y^2}{2} + \omega_0^2 \frac{x^2}{2} = \mathscr{E}, \tag{8.49}$$

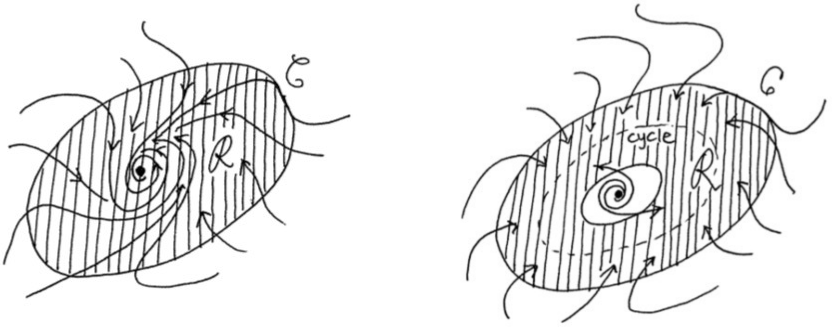
with semi-axes $\sqrt{2\mathscr{E}}$ and $\sqrt{2\mathscr{E}}/\omega_0$. The adiabatic invariant (8.48) is the area of this ellipse:

$$I = \frac{2\pi \mathscr{E}}{\omega_0}. \tag{8.50}$$

Since this must stay constant as the frequency increases (as l gets smaller, ω_0 gets larger), the energy of the oscillations will increase proportionally to it.

Thus, armed with adiabatic invariance, we can trace the time evolution of the phase portrait of a non-autonomous system and thus predict the system's behaviour.

Exercise 8.6. What will happen to the mean square amplitude of the oscillations and to their mean square velocity? Sketch how the phase portrait of the pendulum will change with time.



(a) (b)

FIGURE 44. Trapping regions: (a) with a fixed point inside, (b) with no fixed points and a limit cycle inside.

8.3. *Limit Cycles*

[Literature: Strogatz (1994, §7), Glendinning (1994, §5.8)]

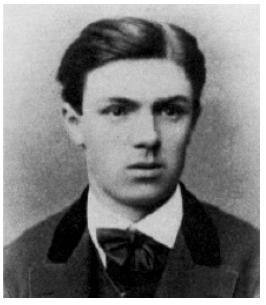
Apart from fixed points, limit cycles are the only other non-boring type of phase trajectories in 2D (see the classification before Exercise 8.2)—the closed trajectories.[58] Physically, they represent a nonlinear phenomenon known as *auto-oscillations*, which are instances of periodic (but not usually sinusoidal) behaviour that can happen in dissipative systems *at finite amplitude*[59] and are an interesting example of a natural tendency for things to go back and forth.

As I noted before, the 2D world is quite restrictive for phase trajectories because they must wonder on a plane without ever intersecting each other or themselves. Imagine, e.g., that we can identify a region $\mathscr{R}$ of phase space, called a *trapping region*, that trajectories can enter but not exit: this can be done by tracing a closed curve $\mathscr{C}$ in the (x, y) plane and showing that everywhere on $\mathscr{C}$ the phase-velocity feld $(\dot{x}, \dot{y})$ either points inwards or is tangent to $\mathscr{C}$.[60] What kind of future does a trajectory have if it enters such a region? One obvious possibility is that there is a stable fixed point inside it, perhaps a node (§8.1.1) or a focus (§8.1.6), into which all trajectories eventually converge (Fig. 44a). What, however, if there are no fixed points there or if there are but they are unstable? It is not hard to imagine how to construct the latter situation. Let us say that we have a nonlinear system whose linearisation around a fixed point identifies that fixed point as a stable node or focus and whose global nonlinear behaviour is such that phase trajectories from far away drift towards that fixed point (i.e., there is a trapping region $\mathscr{R}$ enclosing the fixed point). Now imagine that we change some parameter of the system in such a way as to render the fixed point unstable (say, turn it into an unstable focus) without necessarily changing very much the far-away nonlinear behaviour of phase trajectories, so the trapping region still exists. Then we can easily draw a small circle enclosing our

[58] I mean isolated closed trajectories, i.e., ones in whose immediate vicinity the phase trajectories are not closed. In conservative systems, the whole phase space can be foliated into level sets of a conserved energy, many of which are closed trajectories (§8.1.7), but, having spent quite a lot of time on them (see §§3.5.2, 8.1.7, 8.2, Q3.9, Q5.1), I am not interested in them anymore.

[59] Unlike the good old linear oscillations at infinitesimal amplitude, studied in exhausting detail in §§5.1.1, 5.3 and 7.

[60] The latter case corresponds to a situation in which some part of $\mathscr{C}$ coincides with one of the phase trajectories. This sometimes proves to be a useful trick in identifying uncrossable borders in phase space (try this in Q5.4).



(a) Poincaré



(b) Bendixson

FIGURE 45. (a) Jules Henri Poincaré (1854-1912), great French mathematician and polymath, responsible for too many foundational results and insights to give him justice in a caption. Neither modern mathematics nor physics would have been the same without him. A case in point for Landau's observation that ability in science is measured on a log scale. (b) Ivar Otto Bendixson (1861-1935), Swedish mathematician, who was also a politician interested in the plight of poor students. He is famous mostly for dotting the *i*'s and crossing the *t*'s on Poincaré's theorem.

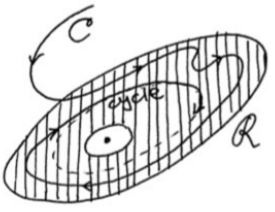
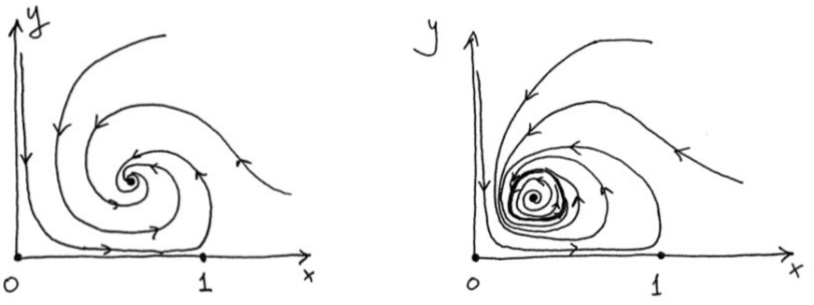


FIGURE 46. Poincaré–Bendixson theorem: trajectory $\mathscr{C}$ wonders into trapping region $\mathscr{R}$, never leaves, ends up on the limit cycle.

unstable fixed point in such a way that all phase trajectories cross this circle in the outward direction. Thus, we now have an annular trapping region (Fig. 44b), with no fixed points inside and with phase trajectories rushing in from both outside and inside the annulus. It is perhaps intuitive that they will not be able to wonder around there aimlessly forever but instead eventually converge onto a closed trajectory, a *limit cycle*.

8.3.1. *Poincaré–Bendixson Theorem*

This narrative is formally confirmed by the following *Poincaré–Bendixson theorem* (Fig. 45).



(a) (b)

FIGURE 47. Phase portraits for Odell's model of population dynamics (Q5.4): (a) when there is a stable focus, (b) when there is an unstable focus and a limit cycle (the bold curve).

Theorem 6. *Suppose that*

(a) the phase velocity field of the ODE

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) \tag{8.51}$$

is continuously differentiable in some open domain $\mathscr{D} \subset \mathbb{R}^2$*;*

(b) there exists a phase trajectory $\mathscr{C}$ *of (8.51) that enters and never leaves some closed, bounded region* $\mathscr{R} \subset \mathscr{D}$*, which contains no fixed points of (8.51).*

Then $\mathscr{R}$ *contains at least one closed, periodic phase trajectory of (8.51) and the trajectory* $\mathscr{C}$ *converges to it as* $t \to \infty$ *(Fig. 46).*

You will find a readable proof of this result in, e.g., Glendinning (1994, §5.8).

While, formally speaking, you only need one phase trajectory trapped in $\mathscr{R}$, the practical strategy for applying the Poincaré–Bendixson theorem is to go for overkill and find $\mathscr{R}$, a trapping region, such that no trajectories can leave and some definitely enter. Practical tips for constructing $\mathscr{R}$ are:

—look for curves in phase space that phase trajectories can cross in one direction only;

—where opportune, complete the boundary of $\mathscr{R}$ with pieces of phase trajectories (which other phase trajectories cannot cross);

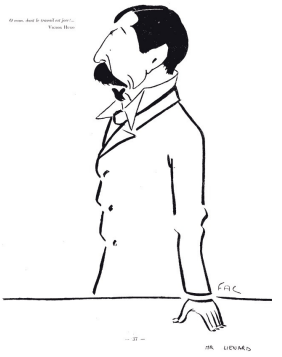
—more often than not, limit cycles enclose unstable fixed points,[61] as envisioned above, in which case $\mathscr{R}$ will have "holes" surrounding those fixed points.

An excellent **example** of all this is Q5.4 (Fig. 47), dealing with a model of *population dynamics*.

[61] That is because they are often "born" as some parameter is changed and a stable fixed point becomes unstable—this is called a *Hopf bifurcation* (see Strogatz 1994, §8.2 or, for a somewhat more detailed treatment, Glendinning 1994, §8.8). This happens, e.g., in Q5.4: Fig. 47 shows the phase portraits of the system considered there with a stable focus (a) and with an unstable focus and a limit cycle (b); a Hopf bifurcation happened between (a) and (b). Eberhard Hopf (1902-1983) was an Austrian-German-American mathematician, student of Schur (Fig. 27), a founding father of the bifurcation theory and the ergodic theory. In a feat of bad judgment (or even worse politics), he managed to move *to* Nazi Germany from the US in 1936, where he proceeded to have a successful professorial career all through the lifetime of the Third Reich—and ended up back in the US after the war!



(a) van der Pol



(b) Liénard

FIGURE 48. (a) Balthasar van der Pol (1889-1959), a Dutch radio engineer, famous for his oscillator, which was a model of an electrical circuit—do investigate further if our department's pedagogical efforts have made you a fan of analogue electronics. (b) Alfred-Marie Liénard (1869-1958), French mining and electrical engineer, inventor of the Liénard–Wiechert retarded potentials for a moving point charge—and of Theorem 7.

Models of population dynamics, sometimes referred to as *Lotka–Volterra*[62] *equations*, are a prolific field of application of the theory of dynamical systems. They typically involve two species that multiply and die at some rates related to their number and to the availability of food, and whose numbers are coupled to each other either because they compete for the same food supply (e.g., Fig. 42b) or because one of them *is* the food supply of the other (in which case we are dealing with a *predator-prey model*; e.g., Fig. 47).

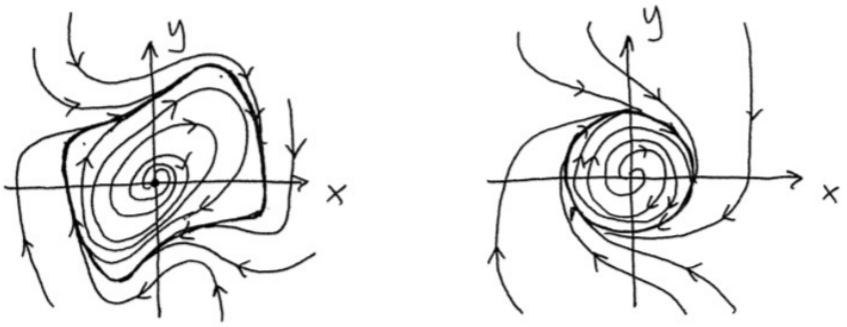
If you followed the advice of footnote 54, you have already encountered one such model. In that example, there were a number of fixed points but no limit cycles. In fact, limit cycles are an endemic feature of predator-prey models. This makes sense: what they describe is a basic sequence of events in which the predators eat the prey, get happy and numerous, eat ever more prey, drive it near extinction, the situation becomes unsustainable, food shortages ensue, predator numbers start dwindling, prey gets a respite, starts procreating with renewed vigour, its numbers recover, at which point predators perk up at the increased food supply and the cycle repeats itself. Regrettable perhaps, but such is the (limit) cycle of life.

8.4. *Auto-Oscillations*

Since fixation on oscillators has been a hallmark of our course, let me discuss the circumstances in which limit cycles make an appearance in this context. Physically, they represent a situation in which a system undergoes sustained oscillations with a fixed period and finite amplitude (so they are a *nonlinear* phenomenon). This comes about when there is an *energy source* and an *energy sink* and the system finds a regime in which it can periodically draw energy from the source, then dissipate it, then draw more, dissipate again, etc. Your heart is a nonlinear oscillator of this kind, as is your grandfather's clock (if you thought that its pendulum was an example of the boring, linear oscillations that we discussed in §§3.5 and §5, do investigate the matter more thoroughly!).

The simplest, best known **example** of such a system is *van der Pol's oscillator*

[62] Alfred James Lotka (1880-1949), American chemist and statistician. Vito Volterra (1860-1940), Italian mathematician best known for his theory of integral equations. He was a senator of the Italian Kingdom, an enthusiastic participant of World War I, and an opponent of Mussolini, who kicked him out of his job and, in the end, Italy (Volterra was philosophical about it: *"Muoiono gl'imperi, ma i teoremi d'Euclide conservano eterna giovinezza"*).



(a) (b)

FIGURE 49. Typical phase portraits of (a) van der Pol's oscillator (8.52), (b) the easy case (8.53). The limit cycles are shown as bold curves.

(Fig. 48a):

$$\boxed{\ddot{x} + \gamma(x^2 - 1)\dot{x} + x = 0}\,, \quad \gamma > 0. \tag{8.52}$$

When $x \ll 1$, this looks like the usual linear oscillator (5.11) (time rescaled to make $\omega_0 = 1$), but with the sign of friction reversed: now γ represents an *instability*, i.e., any small oscillation initially triggered in the system will grow exponentially at the rate γ. This is the *energy source* in the system. Since small initial perturbations grow, they must eventually stop being small, x reaching order-unity amplitude. At that point the $\gamma x^2 \dot{x}$ term switches on, representing *nonlinear friction:* finite displacements are dissipated by the system—this is the *energy sink*.

Intuitively, it is perhaps clear that this should lead to a limit cycle: the instability amplifies perturbations, they get to a finite amplitude, overshoot ($x > 1$), nonlinear friction kicks in, perturbations decay, x drops below unity, instability turns on again, etc. Mathematically, it is not an elementary task either to prove that there is a limit cycle or to work out its shape—there is, in fact, a sizeable body of literature dedicated solely to van der Pol's oscillator, which has proved an inspiration to nonlinear dynamicists.

In order to show you a solvable **example** of a limit cycle (there will be another in §8.4.2), let me consider a modified version of van der Pol's equation (Anosov 2016), which, as far as I know, does not describe anything real,[63] but is easy to analyse:

$$\ddot{x} + \gamma(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0. \tag{8.53}$$

Recasting (8.53) as a 2D dynamical system, we make it into

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \gamma(1 - x^2 - y^2)y. \end{cases} \tag{8.54}$$

Let us go to polar coordinates, $x = r\cos\phi$, $y = r\sin\phi$, or, equivalently, to make calculations even more compact, turn our $\mathbb{R}^2$ plane into $\mathbb{C}$: $z = x + iy = re^{i\phi}$. Then

$$\dot{z} = \dot{r}e^{i\phi} + i\dot{\phi}re^{i\phi} = \underbrace{y - ix}_{\substack{= -iz \\ = -ire^{i\phi}}} + i\gamma(1 - r^2)r\sin\phi. \tag{8.55}$$

[63] Although it is not necessarily unphysical: there is nonlinear friction here that is due to finite displacement *and* finite velocity. Why not! The reason everyone prefers van der Pol's equation (8.52) is that it is much harder and wonderfully nontrivial, with a whimsically shaped limit cycle (Fig. 49a; see also §8.4.2).

Dividing through by $e^{i\phi}$ and requiring the real and imaginary parts of the resulting equation to be satisfied separately, we get

$$\begin{cases} \dot{r} = \gamma r(1-r^2)\sin^2\phi, \\ \dot{\phi} = -1 + \gamma r(1-r^2)\sin\phi\cos\phi. \end{cases} \tag{8.56}$$

There is an unstable focus at the origin and a stable cycle at $r = 1$, which phase trajectories approach clockwise both from inside and outside (Fig. 49b). There is, of course, a trapping region: this can be, e.g., any annulus $r_1 < r < r_2$ with $r_1 < 1$ and $r_2 > 1$. All trajectories cross these circular boundaries inwards except on the x axis ($\sin\phi = 0$), but that is OK: they slip up or down the circle tangentially and then turn inwards anyway.

8.4.1. *Liénard's Theorem*

[Literature: Strogatz (1994, §7.4)]

There is a rigorous mathematical result that guarantees a limit cycle for van der Pol's equation as well as for a family of similar systems:

$$\boxed{\ddot{x} + f(x)\dot{x} + g(x) = 0}\,, \tag{8.57}$$

where $f(x)$ and $g(x)$ are continuously differentiable functions. For such systems, we have the following *Liénard's theorem* (Fig. 48b).

Theorem 7. *Suppose that*

(a) $g(x)$ is an odd function, $g(x) > 0$ for $x > 0$;

(b) $f(x)$ is an even function;

(c) the function $F(x) = \int_0^x \mathrm{d}x' f(x')$, which is odd, has a single zero at $x > 0$, viz., $F(x_0) = 0$, $F(x) < 0$ at $0 < x < x_0$, $F(x) > 0$ and $F'(x) \geqslant 0$ at $x > x_0$, and $F(x \to \infty) \to \infty$.

Then the system (8.57) has a unique, stable limit cycle enclosing the origin.

I am not going to prove this result, but perhaps I can convince you that it makes sense. Observe that $\dot{F} = f(x)\dot{x}$. Let us define new phase-space coordinates (known as *Liénard's coordinates*) to be x and $y = \dot{x} + F(x)$. In these coordinates, (8.57) becomes

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = \ddot{x} + f(x)\dot{x} = -g(x). \end{cases} \tag{8.58}$$

The condition (a) of Theorem 7 implies that $g(x)$ is the usual kind of restoring force accelerating the system in the direction opposing displacement. The conditions (c) on $F(x)$ imply that the nonlinear friction is negative (pushes x to larger values) at low x and positive (tries to reduce x) at high x. This is a general description of the basic physical situation that I posited in the context of van der Pol's equation (8.52).

Using the latter as our working **example**, we see that for it,

$$g(x) = x, \quad f(x) = \gamma(x^2 - 1), \quad F(x) = \gamma x\left(\frac{x^2}{3} - 1\right), \tag{8.59}$$

so Theorem 7 applies: there is a unique, stable limit cycle (Fig. 49a). Hurrah.

8.4.2. *Relaxation Oscillations*

[Literature: Strogatz (1994, §7.5)]

There is a parameter regime in which it is relatively easy to see that van der Pol's

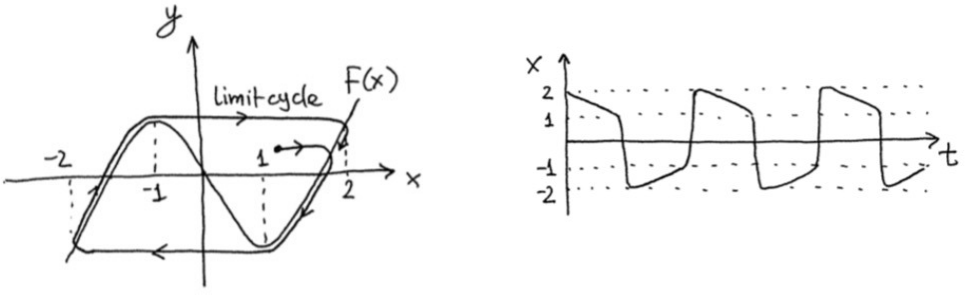


FIGURE 50. Overdamped van der Pol's oscillator: (8.52) with $\gamma \gg 1$. (a) A typical trajectory's journey onto the limit cycle in the Liénard's phase plane [see (8.60)]. (b) Relaxation oscillations: x vs. t. The long, slow bits taking x from ± 2 to ± 1 correspond to the phase trajectory crawling along the cubic $y = F(y)$; they last for a time $\sim \gamma \gg 1$. The quick transitions are the mad dashes from the extrema of the cubic to its opposite brach; they take a time $\sim \gamma^{-1} \ll 1$.

equation (8.52) exhibits a limit cycle—and that limit cycle has quite pleasingly nontrivial and physically interesting behaviour. This is the case of $\gamma \gg 1$, a kind of "overdamped" nonlinear oscillator (cf. §5.1.1).

Let us use Liénard's variables [see (8.58)] for (8.52), except also rescale everything with γ:

$$y = \gamma^{-1}\dot{x} + F(x), \quad \text{where} \quad F(x) = x\left(\frac{x^2}{3} - 1\right) \quad \Rightarrow \quad \begin{cases} \dot{x} = \gamma[y - F(x)], \\ \dot{y} = -\gamma^{-1}x. \end{cases} \tag{8.60}$$

The salient property of this system (not unlike a regular overdamped oscillator in that respect; see §5.1.1) is the presence of two disparate time scales: generally speaking, x changes very fast, at the rate $\dot{x} \sim \gamma \gg 1$, while y changes slowly, at the rate $\dot{y} \sim \gamma^{-1} \ll 1$.[64] The presence of a large coefficient γ in the equation for $\dot{x}$ means that, in any generic place on the phase plane, the system will have a very large horizontal velocity towards a state in which the expression multiplied by γ is small, i.e., towards the null-cline where $\dot{x} = 0$. This is the cubic curve $y = F(x)$: see Fig. 50(a). When $y > F(x)$, $\dot{x} > 0$ pushes the system right, towards the upward-pointing branch of the cubic; when $y < F(x)$, $\dot{x} < 0$ and the system moves left, towards the downward-pointing branch. Once the system's trajectory hits this cubic, it is forced to stay in its vicinity, because were it to try and run away, it would be pushed right back by increased $\dot{x}$. In the vicinity of the cubic, $y - F(x) \sim \gamma^{-2}$, $\dot{x}$ and $\dot{y}$ are comparable and slow ($\sim \gamma^{-1}$). At $x > 0$ ($x < 0$), we have $\dot{y} < 0$ ($\dot{y} > 0$), so the system will crawl slowly down (up) along the cubic, until it reaches its minimum (maximum), can dash across to the other branch of the cubic and resume the crawling (Fig. 50a). This describes a cyclic auto-oscillation, consisting of slow crawls and mad dashes (or, if you like, gradual build-ups and fast releases).

It is possible to calculate approximately the period of these oscillations. Clearly, most of the time, the system is crawling (Fig. 50b), so all we need to know is the duration of a crawl. Since the crawl happens along the null-cline,

$$y \approx F(x) \quad \Rightarrow \quad \dot{y} \approx F'(x)\dot{x}. \tag{8.61}$$

[64] Since the frequency of the oscillator is 1, this means time scales much shorter and much longer, respectively, than the period of the oscillator.

But, since, also $\dot{y} = -\gamma^{-1}x$, we find

$$F'(x)\dot{x} \approx -\gamma^{-1}x \quad \Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\gamma F'(x)}{x} = -\gamma\,\frac{x^2-1}{x}. \tag{8.62}$$

The crawl occurs between the maximum (or minumum) of the cubic at $x = \pm 1$ and the point that is the final destination of each dash, where $F(x) = F(\mp 1) = \pm 2/3$, which is $x = \pm 2$. Therefore, the period is

$$T = 2\gamma \int_1^2 \mathrm{d}x\,\frac{x^2-1}{x} = (3 - 2\ln 2)\gamma. \tag{8.63}$$

It is always nice to be able to solve something.

One can solve a lot more than this. For example, the case of weak damping, $\gamma \ll 1$, is begging to be analysed by some suitable approximate method (see Strogatz 1994, §7.6). One could also ask what happens if you force these kinds of oscillators (they can go chaotic!). You are now fully (or, at least, somewhat) equipped both to ask such questions and to work through the answers to them contained in the literature. Enjoy.

Exercise 8.7. Sketch roughly the phase portrait of van der Pol's oscillator (8.52) with $\gamma \gg 1$ in the usual phase plane $(x, \dot{x})$.

8.5. *Outlook: Attractors, Dissipative Chaos, and Many Degrees of Freedom*

Let me say a few words about where it all leads from here—as usual, all the really exciting stuff is for the future!

[to be finished up...]



FIGURE 51. Sir William Rowan Hamilton (1805-65), a great Irish mathematician. He became Professor of Astronomy at Trinity College Dublin while still an undergraduate there. The sky is the limit to a clever undergraduate's career prospects!

APPENDIX: **Introduction to Lagrangian and Hamiltonian Mechanics**

This is not a course on Theoretical Mechanics, but we brush so close against the subject in so many places that it seems opportune to explain some of it, to get it over with.

I.1. *Hamilton's Action Principle*

[Literature: Landau & Lifshitz (1976, §2), Arnold (1997, §§12-13)]

Have you asked yourselves what physics starts with and why? In the standard school curriculum, it starts, effectively, with Newton's Second Law, relating the acceleration of point masses to something called forces, acting on those masses. Because one is taught all that at an unquestioning age, one comes to regard this starting point as "natural".

Theoretical physics (as codified by, obviously, Landau & Lifshitz 1976, and the subsequent volumes) comes as a culture shock to many because that is not where it starts at all. It starts with the notion that "systems", whatever they are, live in "phase space", parametrised by some generalised coordinates $\boldsymbol{q}$ and their time derivatives $\dot{\boldsymbol{q}}$. These can be the displacements of the kind that I have, in the above, called $\boldsymbol{y}$ (or sometimes x), or they can be the "eigencoordinates" $\boldsymbol{\xi}$, or something else—at this point in the argument, it does not matter which, as long as they have the right number of dimensions to capture all of the system's *degrees of freedom*. As time (whatever that is) advances, systems move along trajectories in their phase space, $\boldsymbol{q}(t)$. Doing physics is about predicting these trajectories.

The starting point for that is *Hamilton's action principle* (Fig. 51): the postulate that systems always follow trajectories that minimise something called *action*, which is an integral over something else called a *Lagrangian*, $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, a scalar function of t, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$:

$$\boxed{S[t, \boldsymbol{q}] = \int_0^t \mathrm{d}t' L(t', \boldsymbol{q}(t'), \dot{\boldsymbol{q}}(t')) \to \min} \,. \tag{I.1}$$

Action is a *functional*, meaning that it depends on the entire phase-space trajectory $\boldsymbol{q}(t')$ from $t' = 0$ to $t' = t$. To minimise action associated with getting from some place $\boldsymbol{q}(0)$ to some other place $\boldsymbol{q}(t)$, one allows small perturbations of the phase-space trajectory connecting these points, $\boldsymbol{q}(t) \to \boldsymbol{q}(t) + \delta\boldsymbol{q}(t)$, holding the start and end points of the evolution fixed, i.e., $\delta\boldsymbol{q}(0) = \delta\boldsymbol{q}(t) = 0$. To find the minimum (or, to be precise, the extremum) point of S, one sets the *variation* of the action to zero:

$$\begin{aligned} 0 = \delta S \equiv S[t, \boldsymbol{q} + \delta\boldsymbol{q}] - S[t, \boldsymbol{q}] &\approx \int_0^t \mathrm{d}t' \left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta\dot{\boldsymbol{q}} \right)(t') \\ &= \int_0^t \mathrm{d}t' \delta\boldsymbol{q}(t') \cdot \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t'} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)(t'). \end{aligned} \tag{I.2}$$

The last step involves integration by parts; boundary terms vanish because $\delta\boldsymbol{q}(0) = \delta\boldsymbol{q}(t) = 0$. As δS has to vanish for any (infinitesimal) variation $\delta\boldsymbol{q}(t')$ of the phase-space trajectory $\boldsymbol{q}(t')$, it must be the case for the optimal trajectory that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0}\,. \tag{I.3}$$

This is called the *Euler–Lagrange equation* (Figs. 22 and 8), a differential equation connecting $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and t that describes the system's peregrinations in phase space, but is not very useful until we know what L is.

To work out what it is, we do not have much to fall back on apart from some very basic principles.

I.2. *Lagrangian of a Point Mass*

[Literature: Landau & Lifshitz (1976, §4)]

Consider a point mass in an otherwise empty space. Both space and time are homogeneous and isotropic, i.e., there are no special points or directions in either. Therefore, if $\boldsymbol{q} = \boldsymbol{r}$, the point mass's position, and $\dot{\boldsymbol{q}} = \boldsymbol{v}$ its velocity, its Lagrangian can only depend on v^2:

$$L = L(v^2) \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} = 0 \quad\Rightarrow\quad \frac{\partial L}{\partial \boldsymbol{v}} = \text{const} \quad\Rightarrow\quad \boxed{\boldsymbol{v} = \text{const}}\,, \tag{I.4}$$

the last step following from the fact that $\partial L/\partial \boldsymbol{v} = 2\boldsymbol{v}\partial L/\partial v^2$ is a function only of $\boldsymbol{v}$. This is called the *principle of inertia*—free point masses move at constant velocities.

Let us now postulate *Galilean invariance* of the laws of physics: action must not change under a Galilean transformation,

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}t, \qquad t' = t, \tag{I.5}$$

where $\boldsymbol{u}$ is a constant. Let us consider the particular case of $\boldsymbol{u}$ being infinitesimal and work out what happens to the Lagrangian of a free point mass under this transformation:

$$L(v'^2) = L(v^2 + 2\boldsymbol{v}\cdot\boldsymbol{u} + u^2) \approx L(v^2) + \frac{\partial L}{\partial v^2}\,2\boldsymbol{v}\cdot\boldsymbol{u}. \tag{I.6}$$

Now we notice that, as far as action is concerned, you can add to the Lagrangian a full time derivative of any function $f(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ without affecting the value of the integral (I.1) between two fixed phase-space points. Therefore, it must be the case that, for some f,

$$L(v'^2) - L(v^2) = \frac{\partial L}{\partial v^2}\,2\,\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{u} = \frac{\mathrm{d}f}{\mathrm{d}t} \quad\Rightarrow\quad \frac{\partial L}{\partial v^2} = \text{const} \equiv \frac{m}{2} \quad\Rightarrow\quad \boxed{L = \frac{mv^2}{2}}\,, \tag{I.7}$$

where *mass*, by definition, is (twice) the constant of proportionality between v^2 and L. Mass must be positive, because otherwise the action associated with a free particle would be negative and minimised at infinite velocities.

Exercise I.1. Show that, with the Lagrangian (I.7), action is invariant with respect to the Galilean transformation (I.5) with *finite* $\boldsymbol{u}$.

I.3. *Lagrangian of a System of Point Masses*

[Literature: Landau & Lifshitz (1976, §5)]

It is not much fun for a point mass to be alone in the world, so let us give it company. The Lagrangian of a system of point masses with positions and velocities $\{\boldsymbol{r}_i, \boldsymbol{v}_i\}$ is

$$\boxed{L = \sum_i \frac{m_i v_i^2}{2} - U(\boldsymbol{r}_1, \boldsymbol{r}_2, \dots)}\,, \tag{I.8}$$

where U is some function of positions, called *potential energy*. The first term is called *kinetic energy*—I will call it K. The rationale for (I.8) is as follows.

First, if we want to cook up a Lagrangian for a number of particles or systems that do not interact with each other, it needs to be such that the Euler–Lagrange equation (I.3) for each of them does not contain any quantities associated with the rest. It then makes sense that their combined Lagrangian just be a sum of the Lagrangians of each individual system. Hence the sum of single-particle Lagrangians in (I.8).

Secondly, we shall simply postulate that if these particles *interact* in some way, either with some "external" system(s)[65] or with other particles in the same population, those interactions shall be encoded by some function $-U$ of the particles' spatial coordinates, which is to be added to the Lagrangian. Then the Euler–Lagrange equation (I.3) for a system of particles is

$$\boxed{m_i \dot{\boldsymbol{v}}_i = -\frac{\partial U}{\partial \boldsymbol{r}_i}} \,. \tag{I.9}$$

The gradients of $-U$ with respect to particle positions that appear in the right-hand side of this equation are called *forces*. This is a "derivation" of Newton's (Fig. 11) second law for any potential (conservative) forces in a nonrelativistic setting. What used to be the starting point has now become a corollary of the Lagrangian formulation of mechanics.

I.4. *Momentum*

[Literature: Landau & Lifshitz (1976, §7)]

Now consider a *closed* system of particles (no external forces). Such a system must be *translationally invariant*, in particular, invariant with respect to a constant infinitesimal shift of all spatial coordinates $\boldsymbol{r}_i \to \boldsymbol{r}_i + \delta\boldsymbol{r}$. Therefore, no such transformation can be allowed to change the Lagrangian:

$$\delta L = \sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} \cdot \delta\boldsymbol{r} = 0 \;\forall \delta\boldsymbol{r} \quad \Rightarrow \quad \sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} = 0. \tag{I.10}$$

The Euler–Lagrange equation (I.3) then implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i \frac{\partial L}{\partial \boldsymbol{v}_i} = \sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} = 0. \tag{I.11}$$

This is the law of conservation of *momentum*, which leads to the general definition of momentum associated with the phase-space coordinate $\boldsymbol{q}$:

$$\boxed{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}} \,. \tag{I.12}$$

For a point particle ($\boldsymbol{q} = \boldsymbol{r}$, $\dot{\boldsymbol{q}} = \boldsymbol{v}$), we get what we expect to get based on the familiar, more mundane definition of momentum: $\boldsymbol{p} = m\boldsymbol{v}$.

I.5. *Energy*

[Literature: Landau & Lifshitz (1976, §6)]

[65] An "external" system is a system whose dynamics are assumed to be known (i.e., when we minimise action to find the motion of the system under scrutiny, we do not vary the phase trajectories of the systems external to it). Then, in general, if we have a system described by some phase-space variables $\boldsymbol{q}$, to be determined, and "external" system(s) described by $\boldsymbol{q}_{\mathrm{ext}}(t)$, which are known functions of time, then the combined Lagrangian is $L = K_{\mathrm{sys}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + K_{\mathrm{ext}}(\boldsymbol{q}_{\mathrm{ext}}(t), \dot{\boldsymbol{q}}_{\mathrm{ext}}(t)) - U(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{ext}}(t))$. Since the kinetic energy K_{ext} of the external system(s) is a known, fixed function only of time, it is a full time derivative of some other function, and so can be dropped from the Lagrangian. We are left with the Lagrangian $L_{\mathrm{sys}} = K_{\mathrm{sys}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U_{\mathrm{sys}}(\boldsymbol{q}, t)$, where $U_{\mathrm{sys}}(\boldsymbol{q}, t) = U(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{ext}}(t))$, i.e., the same as (I.8), but allowing U to depend on time explicitly [if it does not, our system is *autonomous*; see (8.1)]. An example of an external system is planet Earth whose known dynamics is to sit there and exert a constant gravitational force on us. But of course as far as Earth itself is concerned, it moves under the influence of another external system, the Sun, slaved to the latter's gravitational pull. And so on.

In a similar vein, *homogeneity of time* implies that the Lagrangian of a closed system must not depend explicitly on time:

$$L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \quad \Rightarrow \quad \frac{\mathrm{d}L}{\mathrm{d}t} = \dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \boldsymbol{q}} + \ddot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \dot{\boldsymbol{q}} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right) + \ddot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)$$
$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L \right) = 0. \tag{I.13}$$

The Euler–Lagrange equation (I.3) was used in the first line above. Thus, we have found another conserved quantity associated with the Lagrangian, which is called *energy*:

$$\boxed{\mathscr{E} = \dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L}\,. \tag{I.14}$$

It is immediately obvious that, for the Lagrangian (I.8), $\mathscr{E} = K + U$. Note that, here and in what follows, we can think of the vector $\boldsymbol{q}$ as having the number of components equal to the number of coordinates for all the particles in the system, i.e., the number of particles N times 3, e.g., $\boldsymbol{q} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$.

I.6. *Hamilton's Equations of Motion*

[Literature: Landau & Lifshitz (1976, §40), Arnold (1997, §15)]

The emergence of momentum (I.12) and energy (I.14) as interesting and, since conserved, physically relevant quantities, motivates us to change our phase-space variables $(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \to (t, \boldsymbol{q}, \boldsymbol{p})$ and introduce a new function, called the *Hamiltonian*:

$$H(t, \boldsymbol{p}, \boldsymbol{q}) = \dot{\boldsymbol{q}}(t, \boldsymbol{p}, \boldsymbol{q}) \cdot \boldsymbol{p} - L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}(t, \boldsymbol{p}, \boldsymbol{q})). \tag{I.15}$$

Let us observe that

$$\frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}} + \cancel{\frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{p}} \cdot \boldsymbol{p}} - \cancel{\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{p}}} = \dot{\boldsymbol{q}}, \qquad \frac{\partial H}{\partial \boldsymbol{q}} = \cancel{\frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{q}} \cdot \boldsymbol{p}} - \frac{\partial L}{\partial \boldsymbol{q}} - \cancel{\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{q}}} = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = -\dot{\boldsymbol{p}}, \tag{I.16}$$

where the Euler–Lagrange equation (I.3) was used at the penultimate equality. Thus, we can now discard the Lagrangian and declare that if we have a Hamiltonian for a system, as a function of its momenta and coordinates, then the system's equations of motion are

$$\boxed{\begin{cases} \dot{\boldsymbol{p}} = -\dfrac{\partial H}{\partial \boldsymbol{q}}, \\[2ex] \dot{\boldsymbol{q}} = \dfrac{\partial H}{\partial \boldsymbol{p}}. \end{cases}} \tag{I.17}$$

The energy of the system is just $\mathscr{E}(t) = H(t, \boldsymbol{p}(t), \boldsymbol{q}(t))$, so

$$\dot{\mathscr{E}} = \frac{\partial H}{\partial t} + \cancel{\frac{\partial H}{\partial \boldsymbol{p}} \cdot \dot{\boldsymbol{p}}} + \cancel{\frac{\partial H}{\partial \boldsymbol{q}} \cdot \dot{\boldsymbol{q}}} = \frac{\partial H}{\partial t}, \tag{I.18}$$

where cancellation is achieved by means of Hamilton's equations (I.17). If a Hamiltonian system is autonomous, i.e., $H = H(\boldsymbol{p}, \boldsymbol{q})$ does not depend explicitly on time, then the energy is conserved.

This is the starting point of the *Hamiltonian mechanics*—an alternative formulation both to Newton's and to Hamilton's own that was based on the action principle. This new formulation is the ultimate expression of the "follow the energy" principle: start with some physically and/or mathematically motivated expression for a system's energy (the Hamiltonian) and hence derive the phase-space dynamics via (I.17).[66] This is, manifestly, a system of first-order ODEs in a

[66] **Example.** Under this scheme, for the two coupled pendula that were our workhorse example of §7, you would start by writing the system's energy as (7.27), then turning it into a Hamiltonian (with $p_i = m\dot{y}_i$, $q_i = y_i$) and using that to derive the governing equations (7.1) via (I.17).

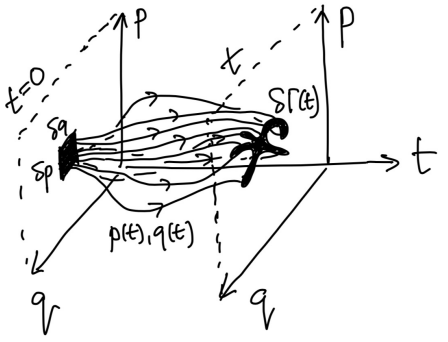


FIGURE 52. Evolution of a phase volume with time.

phase space whose dimensionality is the number of particles N times 2 times the dimension of position space (3), so, in general, $6N$.

Because autonomous Hamiltonian systems have a conserved energy, their trajectories always lie on hypersurfaces in phase space described by $H(\boldsymbol{p},\boldsymbol{q}) = \mathscr{E} = \text{const}$, i.e., on the level sets of the Hamiltonian. If we are dealing with a 1D space of displacements for a single particles and, therefore, a 2D phase space, this hypersurfaces is actually just a curve, so $H(p,q) = \mathscr{E}$ is the actual trajectory—the corresponding system of ODEs is fully integrable. Our first and main example of this was the mathematical pendulum, linear and nonlinear (§3.5.2; see also §8.1.8 and Q5.1).

I.7. *Liouville's Theorem*

[Literature: Landau & Lifshitz (1976, §46), Arnold (1997, §16)]

The fact that phase-space trajectories of Hamiltonian systems have to stick to surfaces of constant H gives them a certain "incompressible" quality. Here is what I mean by this.

Consider a generic autonomous ODE,

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}). \tag{I.19}$$

From each point $\boldsymbol{y}$ in its phase space springs a trajectory—the solution of the Cauchy problem for this ODE with that point as the initial condition. We can think of these trajectories as paths along which a kind of "phase fluid" flows, with $\boldsymbol{f}(\boldsymbol{y})$ representing its velocity field at every point in phase space. Let us now take some volume in phase space and evolve it in time with this flow, i.e., let a trajectory start at time zero from every point in the volume and then consider the phase-space volume formed by the set of points to which these trajectories arrive at time t. It turns out that, for a Hamiltonian system, in the process of evolution defined in this way, the phase volume stays constant, even if its shape can change. In other words, the "phase fluid" is incompressible—a statement known as *Liouville's Theorem* (Fig. 18a).

Here is an elementary proof. Consider an infinitesimal cube in phase space,

$$\delta\Gamma = \prod_{i=1}^{3N} \delta p_i \delta q_i, \tag{I.20}$$

where δp_i and δq_i are increments of the phase-space coordinates of a Hamiltonian system (I.17). Let us now evolve this volume with time, i.e., follow the system's trajectories originating from it (Fig. 52):

$$\frac{\mathrm{d}\delta\Gamma}{\mathrm{d}t} = \sum_{i=1}^{3N} \left(\frac{\delta\dot{p}_i}{\delta p_i} + \frac{\delta\dot{q}_i}{\delta q_i} \right) \delta\Gamma = \sum_{i=1}^{3N} \left(\frac{\partial\dot{p}_i}{\partial p_i} + \frac{\partial\dot{q}_i}{\partial q_i} \right) \delta\Gamma = \sum_{i=1}^{3N} \left(-\frac{\partial H}{\partial p_i \partial q_i} + \frac{\partial H}{\partial q_i \partial p_i} \right) \delta\Gamma = 0. \tag{I.21}$$

Finite volumes can be constructed out of infinitesimal ones, so phase-space volumes are conserved, q.e.d.

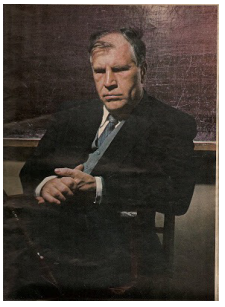
Note that, by the same token, for a general ODE (I.19), the phase volume $\delta\Gamma = \prod_i \delta y_i$ evolves

according to

$$\frac{\mathrm{d}\delta\Gamma}{\mathrm{d}t} = \sum_i \frac{\delta\dot{y}_i}{\delta y_i}\,\delta\Gamma = \sum_i \frac{\partial\dot{y}_i}{\partial y_i}\,\delta\Gamma = (\boldsymbol{\nabla}\cdot\boldsymbol{f})\,\delta\Gamma, \tag{I.22}$$

i.e., the compressibility of phase volumes is related to the divergence of the phase velocity $\boldsymbol{f}$—in exactly the same way as, in fluid dynamics, the compressibility of a fluid is related to the divergence of its velocity field. The Hamiltonian phase velocity $(-\partial H/\partial\boldsymbol{q}, \partial H/\partial\boldsymbol{p})$ is divergence-free. In contrast, for dissipative systems, it is intuitive that phase volumes should generally tend to be compressed with time, as the system's trajectories converge towards an equilibrium point (see, e.g., §3.5.4), a limit cycle (§8.3), or, in three or more dimensions, to a strange attractor (§8.5)—all objects whose phase volume is zero.

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(a) Pontryagin



(b) Arnold

FIGURE 53. (a) Lev Pontryagin (1908-1988), eminent Soviet mathematician with an impressive story of achievement in the face of adversity (he was blind); he was not, however, a nice guy and allegedly took some opportunities to behave disgracefully presented by the Soviet regime. (b) Vladimir Arnold (1937-2010), great Soviet/Russian mathematician, unbeloved by the Soviet government, student of Kolmogorov, author of cult texts, educator and superhero. Depending on how hard-core a mathematician you prove to be, you might continue from his ODE book to the sequels: Arnold (1997), Arnold (2004) and Arnold (1989), although perhaps not in your first year.

Sources and Further Reading

In preparing this course, I have drawn inspiration from a number of sources. The main ones were

—the previous set of Oxford Physics' ODE lectures, for which very concise, prescription-style notes are available (Yeomans 2014)—this may well be all you need to pass your exam; another worthwhile read is the earlier version of the Oxford course by Binney (2002), written in his signature lucid and idiosyncratic style;[67]

—for the material of §7, which formally belongs to a different Oxford course, "Wave Motion", the *status quo ante* is codified in the very detailed notes by Jarvis (2016) and a handout by Palmer (2019);

—the textbook by Romanko (2001), with an attendant gigantic collection of problems (Romanko *et al.* 2011; Ipatova *et al.* 2012), reflecting many decades of experience of teaching ODEs at MIPT (I took this course as an undergraduate there in the early 1990s, when it was taught by A. A. Abramov, whose lectures, as recorded by me then, have proved useful again, after all these years);[68]

—the celebrated (if slightly dated) and very lucid lectures by the famous blind mathematician Pontryagin (1962, in English) (Fig. 53a)—this is a traditional exposition of the subject *par excellence*; assuming you do not read Russian and assuming also that you are interested in learning the subject rather than in obtaining only the minimum training required for your exam, this is the book that I recommend to you as a more thorough undergraduate course than I can offer within the existing time constraints;

—the classic cult text by the great Arnold (2006) (Fig. 53b), which exists in English

[67] Both sets of notes have many examples of how to solve standard Oxford homework/collection/exam-style questions.

[68] Other, somewhat older, Russian books that I have consulted were Fedoryuk (1985), an MIPT classic, and the rival couse from MSU, Tikhonov *et al.* (1985, in English), which, I hate to admit, is also quite good, and rather more concise.

and which I recommend as further reading to those of you who have a taste for intellectual challenge; Arnold will not teach you how to solve ODEs, but he will teach you how to think about them and much else besides.

Two old, affordable (in Dover editions) and, on first inspection, entirely adequate undergraduate textbooks from this side of the Iron Curtain are Tenenbaum & Pollard (1986) (written in a cheerfully spoon-feeding style, with a huge number of worked examples, but of fairly limited scope overall) and Coddington (1990) (a more formal, although still undergraduate, text, focusing primarily on a thorough treatment of linear equations and the existence theorems; looks dull but solid to me; its "adult" version is Coddington & Levinson 1955). Oxford's own official textbook (from the Mathematical Institute) appears to be Collins (2006), although it does not belong to the affordable category and, at any rate, seems to view ODEs as a trivial footnote to PDEs. Hirsch *et al.* (2013) is (I am told) a famous, classic textbook worshipped by many generations of American undergraduates; it covers most of what I cover (including the extracurricular topics on dynamical systems), and much more, in great spoon-feeding detail, which can get a bit irritating, but is also likely to prove very helpful to beginners. Another American textbook on both ODEs and dynamical systems, with careful explanations and lots of examples, but perhaps a bit more mathsy, is Perko (2001). Yet another recent (possibly the most recent) American textbook is Schaeffer & Cain (2016), which looks quite civilised to me—it quickly gets to much more advanced topics than my course ever could, but is written in a friendly style and may be suitable as further reading on the modern mathematical theory of ODEs.

Another direction for further reading is asymptotic methods. These are for those who like *doing* stuff. There are elements here of dark arts and black magic. Much of it has gone from the world, but it is still possible to learn from the masters of old: I have given several references in the text (see, e.g., §1.1 and Q2.9), but if you have to pick one, pick Bender & Orszag (1999), which is a wild, exuberant ride. It is a graduate-level book,[69] supposedly, but, after you have done Complex Analysis (and maybe some Mathematical Methods), you will know everything you need to know to attack all of it—and quite a bit of it you can, in fact, attack right away (this said, I cannot promise you definite success as it depends on whether the Force is strong with you).

Finally, §8 is the tip of a large iceberg that is the theory of dynamical systems, chaos, and nonlinear oscillations. Some good books on these topics, in the order of increasing difficulty, are Strogatz (1994), Glendinning (1994), and Guckenheimer & Holmes (1983).[70] The last of these is a classic, serious, graduate-level text that is not always easy to read even for graduates, whereas the other two are undergraduate lecture courses—both are good but in rather different ways: the first, American one (from MIT and Cornell), is cheerful, friendly, spoon-feeding (especially at first), eclectic, effusive, and mindful of a student audience of potentially variable mathematical ability; the second,

[69] In fact, its §1 is a pretty good introduction to the main methods for solving ODEs and can be read as a short (33 pages), to-the-point, example-rich text for the present course.

[70] Those who read Russian might also like Karlov & Kirichenko (2003) (which also covers oscillations and waves extremely well, in an engaging, lecture-notes style) and, on Hamiltonian chaos, Zaslavsky & Sagdeev (1988) (or, in English, a shorter, somewhat narrower in scope, but, in my view, better written little book by Zaslavsky *et al.* 1991). Note that both Perko (2001) and Schaeffer & Cain (2016), already mentioned above, also deal with the theory of dynamical systems—but, being mathematicians, they think of it more as a brunch of the general theory of ODEs than of nonlinear physics. In contrast, Hirsch *et al.* (2013), while also mathematicians (and famous ones, too), are very keen on all manner of applications.

British (from Cambridge and Warwick), is more relaxed about stretching those who have not been properly introduced (to basic maths), but is unfailingly nice, polite, and gracefully informal in its style of exposition.

Acknowledgments

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Problem Sets

For each topic, a "Minimal Set" of exercises is offered first, followed by "Supplementary Questions". The latter are not necessarily any more difficult than the former: "supplementary" just means that you can get by without them if you choose to do as little as possible. If you are struggling with time, you might want to do some of these during the vacation. Questions that I consider slightly more challenging are marked with a star. Finally, there are "Extracurricular Questions", which are extracurricular (but not necessarily hard or surplus to requirements for a proper education). Some of these will require independent study of material that may or may not be covered in small font in my lecture notes.

Most of these exercises came from the problem sets that were passed on to me by Julia Yeomans (see Yeomans 2014), via Michael Barnes, both of whom lectured this course before me. Those are marked [JY]; [JY+] means that I have amended the question in some (usually minor) way. Quite a few of these questions appear to originate from Binney (2002). Those are marked [JB]. Similarly, the questions about "masses on springs" are adapted from what I inherited from Christopher Palmer and marked [CP+] (I did tinker with most of them). There are some questions that I have inherited from past tutors, at Merton and elsewhere; who invented them can no longer be traced. Those are marked [Ox]. There are some questions that I borrowed from elsewhere but left unattributed because the original source contains solutions.

All "§" references below are to my lecture notes, see `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/ODE/2018/ODELectureNotes.pdf`.

Problem Set 1: First-Order ODEs

Minimal Set

1.1. [JY+] Determine the order of the following differential equations and whether they are linear or nonlinear. Rewrite them as systems of 1st-order ODEs.

$$\text{(i)}\quad y'' + k^2 y = f(x), \tag{1.1}$$

$$\text{(ii)}\quad y''' + 2yy' = \sin x, \tag{1.2}$$

$$\text{(iii)}\quad y' + y^2 = yx. \tag{1.3}$$

1.2. [JB/JY+] Solve the following differential equations using the method stated.

(a) Full differential (§2.1):

$$y' = \frac{(3x^2 + 2xy + y^2)\sin x - 2(3x + y)\cos x}{2(x + y)\cos x}. \tag{1.4}$$

(b) Integrating factor (§2.1.1):

$$y^2 \mathrm{d}x - (xy + x^3)\mathrm{d}y = 0. \tag{1.5}$$

(c) Separable (§2.2):

$$\text{(i)}\quad y' = \frac{xe^y}{1 + x^2}, \quad y = 0 \text{ at } x = 0, \tag{1.6}$$

$$\text{(ii)}\quad y' = \frac{2xy^2 + x}{x^2 y - y}. \tag{1.7}$$

(d) Reducible to separable by change of variables (§2.2.2):

$$y' = 2(2x + y)^2. \tag{1.8}$$

(e) Homogeneous (§2.3):

$$2y' = \frac{xy + y^2}{x^2}. \tag{1.9}$$

(f) Reducible to homogeneous by change of variables (§2.3.1):

$$y' = \frac{x + y - 1}{x - y - 2}. \tag{1.10}$$

(g) Linear (§2.4):

$$\text{(i)}\quad y' + \frac{y}{x} = 3, \quad x = 0 \text{ at } y = 0, \tag{1.11}$$

$$\text{(ii)}\quad y' + y\cos x = \sin 2x. \tag{1.12}$$

(h) Bernoulli (§2.5):

$$y' + y = xy^{2/3}. \tag{1.13}$$

1.3. [JB/JY+] Solve the following 1st-order differential equations:

$$\text{(i)}\quad y' = \frac{x - y\cos x}{\sin x}, \tag{1.14}$$

$$\text{(ii)}\quad (3x + x^2)y' = 5y - 8, \tag{1.15}$$

$$\text{(iii)}\quad y' + \frac{2x}{y} = 3, \tag{1.16}$$

$$\text{(iv)}\quad y' + \frac{y}{x} = 2x^{3/2}y^{1/2}, \tag{1.17}$$

$$\text{(v)}\quad 2y' = \frac{y}{x} + \frac{y^3}{x^3}, \tag{1.18}$$

$$\text{(vi)}\quad xyy' - y^2 = (x + y)^2 e^{-y/x}, \tag{1.19}$$

$$\text{(vii)}\quad x(x - 1)y' + y = x(x - 1)^2, \tag{1.20}$$

$$\text{(viii)}\quad 2xy' - y = x^2, \tag{1.21}$$

$$\text{(ix)}\quad y' = \cos(y + x), \quad y = \frac{\pi}{2} \text{ at } x = 0, \tag{1.22}$$

$$\text{(x)}\quad y' = \frac{x - y}{x - y + 1}, \tag{1.23}$$

$$\text{(xi)}\quad y' = \cos 2x - y\cot x, \quad y = \frac{1}{2} \text{ at } x = \frac{\pi}{2}, \tag{1.24}$$

$$\text{(xii)}\quad y' + ky = y^n \sin x, \quad n \neq 1. \tag{1.25}$$

Supplementary Questions

1.4. [JB+] By introducing a new variable $z = 4y - x$, or otherwise, find all solutions of the ODE

$$y' - 16y^2 + 8xy = x^2. \tag{1.26}$$

You should find this solution:

$$y = \frac{x}{4} - \frac{1}{8}\tanh(2x + C). \tag{1.27}$$

Are there any others? Hint: if you divide by zero too recklessly, you may lose some solutions.

1.5. Full Differentials. Solve the following equations, which can be reduced to equations in full differentials.

(a)

$$2\left(x - y^4\right) \mathrm{d}y = y \,\mathrm{d}x, \tag{1.28}$$

Hint: look for an integrating factor (§2.1.1) in the form $\Lambda(y)$.

(b*)

$$y\left(2y \,\mathrm{d}x - x \,\mathrm{d}y\right) + x^2\left(y \,\mathrm{d}x + 2x \,\mathrm{d}y\right) = 0. \tag{1.29}$$

Hint: identify within the equation combinations that are full differentials of some functions of x and y and then use those functions to introduce a change of variables that will allow you to separate (the new) variables and integrate.

1.6. Sometimes an ODE can be turned into a homogeneous one (§2.3) by the change of variables $y = z^n$. Solve the following system by this method:

$$y \,\mathrm{d}x + x\left(2xy + 1\right) \mathrm{d}y = 0. \tag{1.30}$$

Hint: you will need to find n such that the equation does become homogeneous; once you have done this, remember that sometimes it is more convenient to look for a solution in the form $x = x(y)$, rather than $y = y(x)$.

1.7. Quasi-homogeneous Equations. (a) An ODE $y' = f(x, y)$ is called *quasi-homogeneous* if $\forall\lambda$ and some $\alpha \neq 0$, $\beta \neq 0$,

$$f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta-\alpha} f(x, y). \tag{1.31}$$

Show that the change of variables $y = x^{\beta/\alpha} z$ reduces a quasi-homogeneous equation to a separable one.

(b) Solve the resulting equation in quadratures. Show that if $\exists z_0$ satisfying $f(1, z_0) = \beta z_0/\alpha$, then $y = z_0 x^{\beta/\alpha}$ is also a solution of the equation. Under what condition are composite solutions (§2.2.1) possible?

You can learn more about quasi-homogeneous equations from Arnold (2006, §§6.4-6.5).

Extracurricular Questions

1.8. Integral Curves and Orthogonal Curves. (a) If a famlily of curves on the plane (x, y) is specified in the form

$$f(x, y, C) = 0, \tag{1.32}$$

where C is a parameter, then one can find a first-order ODE for which these curves are integral curves by taking the differential of (1.32) and then substituting $C = C(x, y)$ extracted from (1.32).

By this method, find the ODE whose integral curves are

$$y = \tan\left[\ln(Cx)\right]. \tag{1.33}$$

(b) Show that the curves that are orthogonal to the integral curves of the ODE

$$F(x, y, y') = 0 \tag{1.34}$$

are integral curves of the ODE

$$F\left(x, y, -\frac{1}{y'}\right) = 0. \tag{1.35}$$



FIGURE 54. Alexis Claude Clairaut FRS (1713-1765), French mathematician, astronomer, Arctic explorer and womaniser.

The explicit expression for these *orthogonal curves* can then be found by integrating (1.35).

(c) Find the curves orthogonal to the family (1.33). Check your answer:

$$3x^2 + 2y^3 + 6y = C. \tag{1.36}$$

1.9. Riccati equations. (a) By the method of §2.6, solve the following Riccati equation

$$y' = y^2 - 2e^x y + e^{2x} + e^x. \tag{1.37}$$

(b) The equation in Q1.4 is also a Riccati equation. Solve it again, but now using the method of §2.6.

1.10* Lagrange's and Clairaut's Equations. (a) *Lagrange's equation* (Fig. 8) is

$$y = a(y')x + b(y'), \tag{1.38}$$

where $a(p)$ and $b(p)$ are some given continuously differentiable functions. As suggested in §2.7.2, introduce $p = y'$ and rewrite this equation as a first-order ODE with respect to p and x. Find the solution in quadratures, in the form $x(p)$, assuming $a(p) \neq p$.

(b) What if $\exists p_0$ such that $a(p_0) = p_0$? Can you find a special solution of (1.38) that exists in this case?

(c) Now suppose that $a(p) = p\ \forall p$. Then (1.38) is called *Clairaut's equation* (Fig. 54):

$$y = xy' + b(y'). \tag{1.39}$$

Find all possible solutions of this equation, commenting on the number of continuous derivatives that $b(p)$ must have in order for these solutions to be valid.

(d) Sketch the integral curves of (1.39) if $b(p) = p^2$. They will turn out to be a family of lines hugging a certain envelope. Find the curve that describes this envelope.

(e) Does the solution of (1.39) with $b(p) = p^2$ that passes through the point (x_0, y_0) exist $\forall (x_0, y_0) \in \mathbb{R}^2$? If not, for which (x_0, y_0) does it exist? Given some (x_0, y_0) for which

it does exist, is it unique? If not, is the solution unique in the vicinity of (x_0, y_0) if we also specify $y'(x_0) = p_0$? How many legitimate options for p_0 are there at each point? How far can such a solution be continued uniquely? Show graphically using the result of (d) how an infinite number of composite solutions (§2.2.1) can be constructed.

If you are intrigued by Clairaut's equation, you will find a number of interesting facts, examples and applications in Arnold (2006, §8.5) and Tenenbaum & Pollard (1986, §61).

1.11. Equations unresolved with respect to derivative. Solve the following ODEs via introduction of parameter (§2.7.2):

$$\text{(i)}\quad (y')^2 - 2xy' = x^2 - 4y, \tag{1.40}$$

$$\text{(ii)}\quad y' = e^{xy'/y}, \tag{1.41}$$

$$\text{(iii}^*\text{)}\quad y' = y\,e^{-xy'/y}, \tag{1.42}$$

$$\text{(iv}^*\text{)}\quad y' = y^2 e^{-xy'/y}. \tag{1.43}$$

Are p-discriminant curves [see (2.81) in §2.7.1] of these equations also their integral curves? What does this imply?

1.12*. Parametric Solutions. Consider an equation formed by y' and x lying on the "folium of Descartes":

$$(y')^3 + x^3 = xy'. \tag{1.44}$$

Try $y' = p = xu$, work out a parametrisation of the curve (x, y'), and then find a parametric solution of (1.44) in the form $x = x(u)$ and $y = y(u)$. In what range of values of u (and, therefore, of x and y) is your solution valid?

Problem Set 2: Second-Order ODEs, Part I

Minimal Set

2.1. [JY] **Homogeneous ODEs.** Solve these equations:

$$\text{(i)}\ \ y'' + 2y' - 15y = 0, \tag{2.1}$$

$$\text{(ii)}\ \ y'' - 6y' + 9y = 0, \quad y = 0,\ y' = 1 \text{ at } x = 0, \tag{2.2}$$

$$\text{(iii)}\ \ y'' - 4y' + 13y = 0, \tag{2.3}$$

$$\text{(iv)}\ \ y''' + 7y'' + 7y' - 15y = 0. \tag{2.4}$$

In (iii), write the solution both in terms of complex exponentials *and* in terms of sines and cosines.

2.2. [JY] **Damped Oscillator.** A damped harmonic oscillator is displaced by a distance y_0 and released at time $t = 0$. Show that the subsequent motion is described by the differential equation

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = 0 \quad \text{with } y = y_0,\ \dot{y} = 0 \text{ at } t = 0, \tag{2.5}$$

explaining the physical meaning of the parameters m, γ and ω_0.

(a) Find and sketch solutions for (i) overdamping, (ii) critical damping, and (iii) underdamping. (iv) What happens for $\gamma = 0$?

(b) For a lightly damped oscillator ($\gamma \ll \omega_0$), the quality factor, or Q-factor, is defined as

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per period of oscillation}}. \tag{2.6}$$

Show that $Q = \omega_0/\gamma$.

2.3. [JB] Verify that $y = x + 1$ is a solution of

$$(x^2 - 1)y'' + (x + 1)y' - y = 0. \tag{2.7}$$

Hence find the general solution of this equation. Check your answer:

$$y = C_1(x + 1) + C_2\left(\frac{x+1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\right). \tag{2.8}$$

2.4. Nonlinear ODEs. All of this problem set so far has been a drill in solving linear, second-order ODEs. There are few general methods for solving nonlinear ones (and most of them cannot be integrated in quadratures anyway). Still, there are a few tricks, which you now have an opportunity to practice. All of them, naturally, are based on methods for lowering the order of the equation down to first.

(a)[JY] If bits of the equation can be manipulated into full derivatives of some expressions, the equation's order can sometimes be lowered by direct integration. Practice this by solving the ODE

$$1 + yy'' + (y')^2 = 0. \tag{2.9}$$

(b)[Ox] If the equation does not contain y, only its derivatives, the order is lowered by treating the lowest-order derivative that does appear as the new function. Practice this by solving the ODE

$$xy'' = y' + (y')^3. \tag{2.10}$$

(c) If the equation does not contain x, you can lower its order by letting $p = y'$ and looking for solutions in the form $p = p(y)$. Note that then, by the chain rule, $y'' = pp'$. Solve by this method (plus other appropriate changes of variable) the ODE

$$y''(y-1) + y'(y-1)^2 = (y')^2. \tag{2.11}$$

In Q2.6, you will learn two other methods for solving nonlinear ODEs.

Supplementary Questions

2.5* Consider the differential equation

$$9xy'' + (6+x)y' + \lambda y = 0. \tag{2.12}$$

There are several values of λ for which this can be solved via reduction to a 1st-order equation. For at least one of them, it is possible to find a solution that

(i) satisfies $y(x \to \pm\infty) \to 0$, but is not zero everywhere,

(ii) is continuous at $x = 0$.

Experiment with solutions corresponding to various tractable values of λ and find one for which the above two properties are satisfied. The solution you find may, but does not have to, be

$$y = C\, e^{-x/9} \int_{-\infty}^{x} dz\, \frac{e^{z/9}}{|z|^{2/3}}. \tag{2.13}$$

2.6. Homogeneous Nonlinear ODEs. Continuing from Q2.4, here are some more methods for lowering the order of nonlinear ODEs.

(a) An equation is called *homogeneous* with respect to y if it does not change under the rescaling $y \to \lambda y\ \forall \lambda \neq 0$ (i.e., every term contains the same number of powers of y or its derivatives). For such an equation, one can lower the order by introducing a new function $z(x)$ via $y' = yz$. Then $y'' = y(z' + z^2)$, so the second derivative is expressible in terms of the first; powers of y will cancel because the equation is homogeneous.

Solve by this method the following ODE:

$$xyy'' + x(y')^2 = 3yy'. \tag{2.14}$$

Check your answer: $y^2 = Ax^4 + B$.

(b*) This equation can also be solved by the method that you practiced in Q2.4(a), if you can spot the full derivatives. Do it.

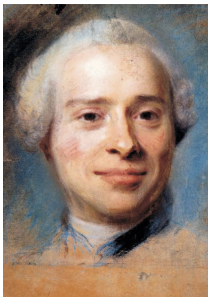
(c) A generalised version of (a) is an equation that does not change under the rescaling $x \to \lambda x$, $y \to \lambda^\alpha y\ \forall \lambda \neq 0$ and for some α (the linear version of this with $\alpha = 0$ is Euler's equation, §5.1.4; you encountered the first-order case in Q1.7). This is solved by letting $x = e^t$ when $x > 0$, $x = -e^t$ when $x < 0$, and $y = z(t)e^{\alpha t}$. The result will be an equation for z in which t does not appear explicitly, so you can use the method practiced in Q2.4(c).

Consider the following ODE:

$$x^2(y'' + 2yy') + 2xy^2 - 2y = 0, \quad x > 0 \tag{2.15}$$

and establish for what value of α it has the invariance property described above. Then solve the equation by the proposed method.


(a) Airy


(b) D'Alembert


(c) Frobenius

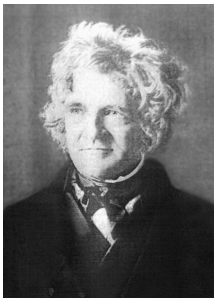

(d) Bessel

FIGURE 55. (a) Sir George Biddell Airy KCB FRS (1801-1892) was Astronomer Royal and put the Prime Meridian at Greenwich. (b) Jean-Baptiste le Rond d'Alembert (1717-1783), great French mathematician, physicist, philosopher, musical theorist, Diderot's fellow encyclopedist and full-time admirer of Mme de Lespinasse. (c) Ferdinand Georg Frobenius (1849-1917), German mathematician, student of Weierstrass. (d) Friedrich Wilhelm Bessel (1784-1846), a fairly boring German mathematician and astronomer. Bessel's functions were in fact discovered by Daniel Bernoulli, nephew of Jacob Bernoulli of Bernoulli's equation (Fig. 9).

Extracurricular Questions

2.7. [JY+] **Boundary-Value Problem**. (a) Solve the ODE

$$y'' + k^2 y = 0. \tag{2.16}$$

(b) Consider the above equation on the interval $x \in [0, L]$. Are there solutions that are not identically zero everywhere but have $y(0) = y(L) = 0$? For which values of k do such solutions exist? Find these solutions and explain in terms of linear algebra the meaning of what you have done.

(c*) Can any function that is defined in the interval $[0, L]$ and vanishes at its ends be represented as a linear combination of solutions of (2.16) with different k's? Work out how to do it.

The full theory of boundary-value problems is not covered in this course. Read up on the topic or wait till the Mathematical Methods course in your 2nd year (see Eßler 2009, §25 and Lukas 2019, §5)—although you will, in fact, need to solve (2.16) many times in CP4 (Parra 2021).

2.8. Higher-Order Linear ODEs with Constant Coefficients. Solve the following ODEs

$$\text{(i)}\quad y'''' - 6y''' + 8y'' + 6y' - 9y = 0, \tag{2.17}$$

$$\text{(ii)}\quad y'''' + 4y'' + 4y = 0. \tag{2.18}$$

Hint: in (i), you will need to guess some roots of a 4th-order polynomial and then factorise it.

2.9*. Series Solution of Linear ODEs. ODEs (usually second order) that cannot be solved in quadratures can sometimes be solved in terms of series. One can then give these solutions names and establish all their properties, thus expanding the library of functions that we know how to handle. Such functions are called *special functions* (a classic textbook on them is Lebedev 1972).

Consider *Airy's equation* (Fig. 55a)

$$y'' + xy = 0. \tag{2.19}$$

This equation cannot, alas, be solved in quadratures. Seek its solution in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{2.20}$$

(a) Without worrying about convergence of this series or about the legitimacy of differentiating it term by term, work out what equations the coefficients c_n must satisfy in order for (2.20) to be a solution of (2.19).

(b) Find two linearly independent such series solutions (remember that if they are linearly independent at one point, they are linearly independent everywhere; see §4.3). Use d'Alembert's Ratio Test (Fig. 55b) to show that both series converge. What is the general solution of (2.19)?

If you are having a difficulty with these derivations, you will find the solution in, e.g., Tikhonov *et al.* (1985, §3.8).

Certain linear combinations of the solutions that you have found are called *Airy functions*, an example of special functions. Entire books are written about them (Vallée & Soares 2004).

(c) Find the general solution of (2.16) of Q2.7 (with $k = 1$ for simplicity) by the same method and convince yourself that the series that you have obtained are sines and cosines. These functions too were once special.

Note that the method presented above is a generalisation of (4.60) in §4.6.1, where the series was able to be truncated at the n-th coefficient. But even the Taylor series (2.20) is not a universal panacea. It is not always possible to find solutions in this form. For example, the equation

$$x^2 y'' + xp(x)y' + q(x)y = 0, \tag{2.21}$$

where $p(x)$ and $q(x)$ are representable as convergent Taylor series, is only guaranteed to have one solution in the form of a *Frobenius series* (Fig. 55c)

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} c_n x^n, \tag{2.22}$$

where α is not, in general, a (positive) integer; you can always get the second solution via the "buy one get one free" scheme (§4.6). You will encounter the Frobenius method in the Mathematical Methods course (see Eßler 2009, §23 or Lukas 2019, §5.3 for some examples; if you want to learn the general method, see, e.g., White 2010, §4 or Bender & Orszag 1999, §3; see also Coddington 1990, Ch. 4). However, no one stops you from getting intrigued now and at least convincing yourself that (2.22) works for (2.21) and that (2.20) in general does not. A good example of (2.21) to play with is *Bessel's equation* (Fig. 55d):

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \tag{2.23}$$

where ν is a real number (interesting things happen depending on whether it is an integer). On Bessel functions too, there are entire books in the library (short and elementary: Bowman 2003; long and comprehensive: Watson 1944).

Problem Set 3: Second-Order ODEs, Part II

Minimal Set

3.1. [JY] **Inhomogeneous ODEs.** Consider the equation

$$y'' - 3y' + 2y = f(x). \tag{3.1}$$

What is the general form (in terms of functions and constants) of its particular solution for

$$\begin{aligned} f(x) = \quad &\text{(i)} \ x^2, && (3.2)\\ &\text{(ii)} \ e^{4x}, && (3.3)\\ &\text{(iii)} \ e^{x}, && (3.4)\\ &\text{(iv)} \ \sinh x, && (3.5)\\ &\text{(v)} \ \sin x, && (3.6)\\ &\text{(vi)} \ x\sin x, && (3.7)\\ &\text{(vii)} \ e^{2x} + \cos^2 x. && (3.8) \end{aligned}$$

If you dislike unfinished business, work out the constants.

3.2. [JB/JY+] **Inhomogeneous ODEs.** Solve these equations:

$$\begin{aligned} &\text{(i)} \ 5y'' + 2y' + y = 2x + 3, \quad y = -1,\ y' = 0 \text{ at } x = 0, && (3.9)\\ &\text{(ii)} \ y'' - y' - 2y = e^{2x}, && (3.10)\\ &\text{(iii)} \ 4y'' - 4y' + y = 8e^{x/2}, \quad y = 0,\ y' = 1 \text{ at } x = 0, && (3.11)\\ &\text{(iv)} \ y'' + 3y' + 2y = xe^{-x}, && (3.12)\\ &\text{(v)} \ y'' - 4y' + 3y = 10\cos x, && (3.13)\\ &\text{(vi)} \ y'' + 4y = x + \cos 2x, \quad y = 0 \text{ when } x = 0, && (3.14)\\ &\text{(vii)} \ y'' - 2y' + 2y = e^x(1 + \sin x), \quad y = 0 \text{ at } x = 0 \text{ and at } x = \frac{\pi}{2}, && (3.15)\\ &\text{(viii)} \ y'' + 2y' + y = 2e^{-x} + x^3 + 2\cos x, && (3.16)\\ &\text{(ix)} \ y'' - 2y' + y = 3e^x, \quad y = 3,\ y' = 0 \text{ at } x = 0, && (3.17)\\ &\text{(x)} \ x^2y'' + xy' + y = x. && (3.18) \end{aligned}$$

3.3. [JB] **Forced Oscillator.** When a varying couple $I\cos\omega t$ is applied to a torsional pendulum with natural period $2\pi/\omega_0$ and moment of inertia I, the angle of the pendulum obeys the equation of motion

$$\ddot{\theta} + \omega_0^2\theta = \cos\omega t. \tag{3.19}$$

The couple is first applied at time $t = 0$ when the pendulum is at rest in equilibrium.

(a) Show that, in the subsequent motion, the root-mean-square angular displacement is

$$\langle\theta^2\rangle^{1/2} = \frac{1}{|\omega_0^2 - \omega^2|}, \tag{3.20}$$

where the average is taken over a time large compared to $1/|\omega_0 - \omega|$.

(b) Discuss the motion as $|\omega_0 - \omega| \to 0$.

3.4. [JY] **Forced and Damped Oscillator.** Consider the damped oscillator of Q2.2

subject to an oscillatory driving force:

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = F\cos\omega t. \tag{3.21}$$

(i) Explain what is meant by the stationary solution of this equation, and calculate this solution for the displacement $y(t)$ and the velocity $\dot{y}(t)$.

(ii) Sketch the amplitude and phase of $y(t)$ and $\dot{y}(t)$ as a function of ω.

(iii) Determine the resonant frequency for both the displacement and the velocity.

(iv) Defining $\Delta\omega$ as the full width at half maximum of the resonance peak calculate $\Delta\omega/\omega_0$ to leading order in γ/ω_0.

(v) For a lightly damped, driven oscillator near resonance, calculate the energy stored and the power supplied to the system. Hence confirm that $Q = \omega_0/\gamma$ as in Q2.2. How is Q related to the width of the resonance peak?

3.5. [JY+] Consider the differential equation

$$x(x+1)y'' + (2-x^2)y' - (2+x)y = (x+1)^2. \tag{3.22}$$

(a) One of its homogeneous solutions is $y_1(x) = 1/x$. Find the general solution.

(b) Now pretend that you do not know that $1/x$ is a homogeneous solution, but know the second homogeneous solution, $y_2(x)$, that you found in (a) (in fact, if you stare at the equation for a few seconds, or minutes, you will see that you could have guessed that solution). Use the knowledge of $y_2(x)$ to find both $y_1(x)$ and the general solution of the equation.

Supplementary Questions

3.6. [JB] Solve the differential equation

$$y'' - 2y' + (k^2+1)y = e^x \sin^2 x \tag{3.23}$$

for general values of the real parameter k. Identify any special values of k for which your solution fails and solve the equation also for those values.

3.7. [Ox] Find the continuous solution with continuous first derivative of the equation

$$y'' + 2y' + 2y = \sin x + f(x), \quad \text{where} \quad f(x) = \begin{cases} 0, & x \leqslant 0, \\ x^2, & x > 0, \end{cases} \tag{3.24}$$

subject to $y(-\pi/2) = y(\pi) = 0$. Hint: obtain a general solution for each of the cases $x < 0$ and $x > 0$ and then determine any loose constants by making these solutions agree at $x = 0$.

3.8. [JB+] **Oscillator with Modulated Force.** A mass m is constrained to move in a straight line and is attached to a spring of strength $m\omega_0^2$ and a dashpot which produces a retarding force $m\gamma v$, where v is the velocity of the mass and $\gamma \ll \omega_0$. An amplitude-modulated periodic force $mA\cos\sigma t\sin\omega t$ with $\sigma \ll \omega$ and $\omega = \omega_0$ is applied to the mass. Show that, in the long-time limit, the displacement is an amplitude-modulated wave

$$y = -\frac{A\sin(\sigma t + \phi)\cos\omega t}{2\omega\sqrt{\sigma^2 + \gamma^2/4}}, \quad \tan\phi = \frac{\gamma}{2\sigma}. \tag{3.25}$$

Extracurricular Questions

3.9. Wiggly Pendulum. Consider a nonlinear pendulum whose point of supension rapidly oscillates in the horizontal direction as $a\cos\omega t$ (see Fig. 26b of §5.4), where $\omega \gg \sqrt{g/l}$ and $a \ll l$ (l is the length of the pendulum, g is the acceleration of gravity). Determine its equilibria and the conditions under which they are stable. Sketch the phase portrait for the pendulum's motion averaged over the rapid oscillations.

Problem Set 4: Systems of Linear ODEs

Minimal Set

4.1. [JY] Solve the coupled differential equations

$$\begin{cases} \dot{x} + ax - by = f, \\ \dot{y} + ay - bx = 0, \end{cases} \tag{4.1}$$

where a, b, and f are constants.

4.2. [JB] Solve the coupled differential equations

$$\begin{cases} y' + 2z' + 4y + 10z - 2 = 0, \\ y' + z' + y - z + 3 = 0, \end{cases} \tag{4.2}$$

where $y = 0$ and $z = -2$ at $x = 0$.

4.3. Find the general, *real* solutions of the following homogeneous systems of ODEs

$$\text{(i)} \quad \begin{cases} \dot{x} = -2y + 2z, \\ \dot{y} = x - y + z, \\ \dot{z} = y - z, \end{cases} \tag{4.3}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = 4x - y - z, \\ \dot{y} = x + 2y - z, \\ \dot{z} = x - y + 2z. \end{cases} \tag{4.4}$$

4.4. Find the general, *real* solutions of the following inhomogeneous systems of ODEs

$$\text{(i)} \quad \begin{cases} \dot{x} = 4x + 3y - 3z, \\ \dot{y} = -3x - 2y + 3z, \\ \dot{z} = 3x + 3y - 2z + 2e^{-t}, \end{cases} \tag{4.5}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = -5x + y - 2z + \cosh t, \\ \dot{y} = -x - y + 2\sinh t + \cosh t, \\ \dot{z} = 6x - 2y + 2z - 2\cosh t. \end{cases} \tag{4.6}$$

Supplementary Questions

4.5. [JB] Solve the coupled differential equations

$$\begin{cases} 2y'' - 3y' + 2z' + 3y + z &= e^{2x}, \\ y'' - 3y' + z' + 2y - z &= 0. \end{cases} \tag{4.7}$$

Is it possible to have a solution to these equations for which $y = z = 0$ at $x = 0$? What is wrong with this system?

4.6. Charged Particle in Electromagnetic Field. A particle of mass m and charge q is placed, initially at rest, in straight, constant electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields, which are neither perpendicular nor parallel to each other. It will experience the Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right), \tag{4.8}$$

where $\boldsymbol{v}$ is the particle's velocity. Find $\boldsymbol{v}(t)$ and sketch the particle's motion.

EXTRACURRICULAR QUESTIONS

4.7. Non-diagonalisable systems. Find the general solutions of the following homogeneous systems of ODEs

$$\text{(i)} \quad \begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 4y - x, \end{cases} \tag{4.9}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = x - 2y, \\ \dot{y} = -x - y - 2z, \\ \dot{z} = y + z, \end{cases} \tag{4.10}$$

$$\text{(iii)} \quad \begin{cases} \dot{x} = 2x - 5y - 8z, \\ \dot{y} = 7x - 11y - 17z, \\ \dot{z} = -3x + 4y + 6z. \end{cases} \tag{4.11}$$

You can find solutions by triangulation (§6.3.1) or via Jordan's basis (§6.3.3), whichever you prefer.

Problem Set 5: Systems of Nonlinear ODEs

Extracurricular Questions

In the following questions, especially Q5.1 and Q5.4, you might gain interesting insight (and a useful preview of the answer) by plotting phase portraits on the computer (using, e.g., `Mathematica`*,* `Python` *or something of the sort).*

5.1. Duffing's[71] Oscillator. Consider the following nonlinear oscillator:

$$\ddot{x} + \gamma\dot{x} + x + \alpha x^2 + \beta x^3 = 0. \tag{5.1}$$

With $\alpha = 0$, this describes a (damped) oscillator whose restoring force strengthens ($\beta > 0$) or weakens ($\beta < 0$) with amplitude—these are called the cases of *hardening spring* or *softening spring*. When $\alpha \neq 0$, there is another amplitude-dependent force, which pushes/pulls at the oscillator differently depending on the direction of the displacement.

Sketch the phase portraits of this system for the following cases:

$$\text{(i)}\quad \gamma = 0, \quad \alpha = 0, \quad \beta > 0, \tag{5.2}$$

$$\text{(ii)}\quad \gamma = 0, \quad \alpha = 0, \quad \beta < 0, \tag{5.3}$$

$$\text{(iii)}\quad \gamma = 0, \quad 0 < \alpha \ll \sqrt{|\beta|}, \quad \beta < 0. \tag{5.4}$$

Having done that, deduce what happens in each of these cases when $0 < \gamma \ll 1$. The name of the game is to get a qualitatively adequate sketch with as little work as possible.

Sketch the functional form of the potential energy. If you think of (5.1) as describing a particle moving in this potential, what do your phase portraits say about this particle's motion?

If you liked this so much that you want to play a bit more, look at the case when the sign of the x term is reversed (i.e., linearly, instead of a restoring force, there is an instability).

You can read more about this kind of nonlinear oscillators in Strogatz (1994, §§7.6, 12.5), Landau & Lifshitz (1976, §29) and Glendinning (1994, §7.2).

5.2. How Nonlinearity Takes Over. Consider the nonlinear systems

$$\text{(i)}\quad \begin{cases} \dot{x} = -y + \alpha x(x^2 + y^2), \\ \dot{y} = x + \alpha y(x^2 + y^2), \end{cases} \tag{5.5}$$

$$\text{(ii)}\quad \begin{cases} \dot{x} = -x - \dfrac{\alpha y}{\ln(x^2 + y^2)}, \\ \dot{y} = -y + \dfrac{\alpha x}{\ln(x^2 + y^2)}, \end{cases} \tag{5.6}$$

where α is a (real) parameter. Determine the nature of the fixed point at $(x, y) = 0$. Does the answer survive the restoration of the nonlinearity? Solve the nonlinear equations exactly and determine the nature of the fixed point depending on the value of α.

Hint. Going to the complex plane in the vein of (8.25) (§8.1.6) and (8.56) (§8.4) might help shorten calculations.

[71] Georg Duffing (1861-1944), a German engineer, famous for his oscillator, which, with the sign of the x term reversed and with a periodic external force added, is one of the simplest known systems that can exhibit chaotic behaviour.

5.3. Limit Cycles. Consider the nonlinear systems

$$\text{(i)} \quad \begin{cases} \dot{x} = x - (x-y)\sqrt{x^2+y^2}, \\ \dot{y} = y - (x+y)\sqrt{x^2+y^2}, \end{cases} \tag{5.7}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = y + \dfrac{1}{4}\, x(1-2x^2-2y^2), \\ \dot{y} = -x + \dfrac{1}{2}\, y(1-x^2-y^2). \end{cases} \tag{5.8}$$

Sketch their phase portraits. Show, in particular, that there is a stable limit cycle in both cases. In (i), you should be able to derive the cycle explicitly. In (ii), you will need to construct a trapping region.

5.4. [Ox] **Odell's Predator-Prey Model.** Consider the following system of nonlinear, coupled ODEs:

$$\begin{cases} \dot{x} = x[x(1-x) - y], \\ \dot{y} = y(x-a), \end{cases} \tag{5.9}$$

where $a > 0$ is a parameter (Odell 1980). This model describes a vegetarian species x, who are omnigamous (birth rate $\propto x^2$), limited by the availability of food (death rate $\propto x^3$), and are eaten by a predator species y at the rate proportional to the latter's population size; the predators are monogamous but procreate conditionally on the availability of prey, at a rate proportional to x, and have a death rate controlled by the parameter a. The population sizes x and y are normalised in some appropriate way (x to its value corresponding to the equilibrium in the absence of predator).

Determine the equilibria (fixed points) of the system and their nature depending on the value of the parameter a. Sketch the phase portrait for each qualitatively different parameter regime. What is the condition for a limit cycle to exist? Interpret your qualitative solutions in terms of population dynamics.

A preview of some of the answers is Fig. 47 (§8.3.1). Before doing this problem, you may wish to practice sketching phase portraits by reproducing ones given in Fig. 42.

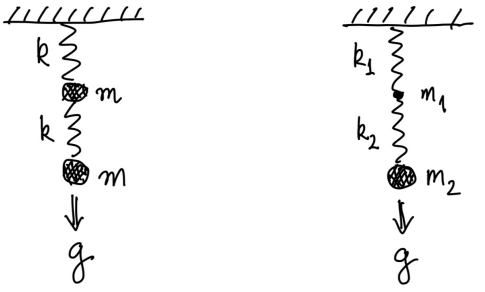


FIGURE 56. Hanging masses: (a) equal, on identical strings (Q6.2); (b) unequal, on different springs (Q6.5).

Problem Set 6: Masses on Springs and Other Things

MINIMAL SET

6.1. [CP+] **Coupled Pendula.** Consider the two-pendula system in Fig. 29. Assume that at $t = 0$, both pendula are at their equilibrium positions, the first is stationary, while the second is instantaneously imparted an initial velocity v.

(a) Calculate their displacements $y_1(t)$ and $y_2(t)$ at all $t > 0$ [the notation is the same as in §7.1].

(b) Taking $k/m = 2\epsilon g/l$, where $\epsilon \ll 1$, sketch these displacements as functions of time. Do they show a precise beat pattern, i.e., one where a higher-frequency oscillation is modulated by a sinuisoidal envelope with frequency $\Delta = (\Omega_2 - \Omega_1)/2$?

(c*) If not, is there such a pattern approximately? If so, show that the minimal amplitude of the oscillations is $O(\epsilon)$. Every how many high-frequency oscillations do the pendula return to this minimal amplitude?

6.2. [CP+] **Hanging Masses.** Two equal masses m are connected with each other and the ceiling with two idential springs, whose stiffness is k (Fig. 56a).

(a) Considering only vertical motion of the masses, obtain the system of equations that govern this motion.

(b) What are the normal modes and their frequencies for this system? Why do they not depend on the acceleration of gravity?

(c) At $t = 0$, the lower mass is pulled down a distance $2a$ and then released. Show that this corresponds to the intitial displacements from equilibrium position $(y_1, y_2) = (a, 2a)$ and find (y_1, y_2) at $t > 0$.

(d) Show that the total energy of the system that has been set in motion by the initial displacement described in (c) is ka^2 (obviously!—why obviously?) and that it is divided between the two normal modes of the system in the ratio approximately 95 : 5 (but do calculate it exactly).

6.3. Sliding Masses. Two equal masses m are connected to each other by a spring of stiffness k_1 and to the walls on either side by two springs of stiffness k_0 and able to slide horizontally left and right, without friction (Fig. 57a).

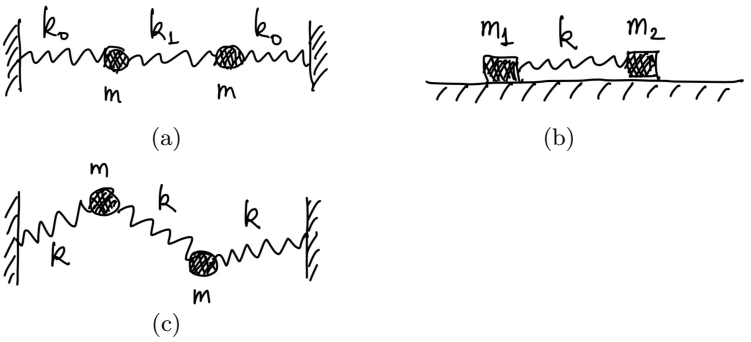


FIGURE 57. Masses sliding (a) horizontally (Q6.3); (b) unboundedly (Q6.4); (c) transversely (Q6.6).

(a) [Ox+] If the mass on the right is clamped, the other mass oscillates with frequency Ω_0. When both masses are free to oscillate, the frequency of the lower-frequency normal mode is Ω_1. Imagining you have measured, and thus know, Ω_0 and Ω_1, but nothing else, determine the ratio k_1/k_0 and the frequency Ω_2 of the higher-frequency normal mode.

(b) [Ox] What is the nature of the two normal-mode oscillations? Explain qualitatively the difference between their frequencies.

(c) Now immerse the whole contraption into some viscous medium and let the moving masses be resisted by it with the damping coefficient γ. Derive the equations that govern this dynamics and their general solution.

(d*) Consider the unusual limit $\sqrt{k_0/m} \ll \gamma \ll \sqrt{k_1/m}$. If at $t = 0$, the first mass is displaced by a distance a, while the other mass is at its equilibrium position, and neither is moving, what will be the subsequent evolution of the system? Sketch it and discuss why it is that, physically.

(e) In the case with damping, assume now $\gamma \ll \sqrt{k_0/m}$ and imagine that one of the walls (say, the one the right) oscillates horizontally as $y_0 = a \sin \Omega_2 t$. Determine the long-time behaviour of the displacements of both masses.

6.4. Sliding Masses Unbound. Consider two masses m_1 and m_2, not necessarily equal,[72] able to slide frictionlessly along a 1D horizontal track and connected to each by a spring of stiffness k (Fig. 57b).

(a) [CP+] Determine the general form of the solution of this system, i.e., the masses' coordinates as functions of time. What are the normal modes and why? How many oscillatory degrees of freedom does this system have?

(b) At $t = 0$, the first (left) mass is set in motion with velocity v, while the other is at rest. Determine the subsequent evolution of the system.

(c) What is the energy of the motion obtained in (b)? How is it distributed between the two normal modes?

[72] Although it is useful throughout this question to keep checking what happens when $m_1 = m_2$ and convincing yourself that your answers make sense in this case.

Supplementary Questions

6.5.* **Unequal Hanging Masses.** Consider again the system of two hanging masses but assume now that the two springs, still of equal length, have different stiffnesses, k_1 (upper spring) and k_2 (lower spring), and the masses are unequal (m_1 upper, m_2 lower): see Fig. 56(b).

(a) Determine the frequencies and normal modes of this system when $m_1 \ll m_2$ (check that they are orthogonal in the appropriate sense). [NB: be careful with your expansions in small m_1/m_2, don't take zero for an answer!]

(b) Explain the physical nature of each of the normal modes. Comment on the limit of $m_1 = 0$, i.e., when there is just one mass m_2 hanging on two different springs tied together.

(c) What will be the subsequent motion of the system if you pull down the second (lower) mass a distance a, then let go?

(d) Show that if you instead hold the second mass in place but displace the first (upper) mass by a distance a, the subsequent displacement of the second (lower) mass will be

$$y_2(t) \approx a\,\frac{m_1 k_2}{m_2(k_1+k_2)}\left[\cos\left(\sqrt{\frac{k_1 k_2}{m_2(k_1+k_2)}}\,t\right) - \cos\left(\sqrt{\frac{k_1+k_2}{m_1}}\,t\right)\right]. \tag{6.1}$$

Show also that this prediction is strictly valid only for $t \ll m_2(k_1+k_2)^{3/2}/\sqrt{m_1}k_2^2$.

6.6. [CP+] **Transversely Sliding Masses.** Consider again the system of two equal masses m connected to each other and the side walls by three identical springs of stiffness k and length l. Assume the natural length of the springs to be $l_0 < l$, i.e., in equilibrium, the springs have finite tension $T = k(l - l_0) > 0$. Consider transverse sliding motions of the masses (Fig. 57c).

(a) Derive a linear system of evolution equations for small transverse displacements of the masses. What do the displacements need to be small compared to, and why, in order for these equations to be valid? What is the nature of the restoring force and does it depend on the fact that displacements lead to springs being stretched longer than l?

(b) Find the normal modes and their frequencies for this system. Describe these motions physically.

(c) Now generalise to the case of N masses and $N+1$ springs (all identical), connected to the side walls at the ends. Find the normal modes and frequencies of this system.

(d) Describe what happens in the limit $N \to \infty$. What quantities is it sensible to hold fixed when taking this limit? What physical system does such a limit describe?