NCP1: Numerical problems for CP1: Classical Mechanics and Special Relativity

The full solutions to these problems require the use of numerical methods. The problems have been designed with `Mathematica` in mind, however, they can be solver with any other numerical framework/package or even by coding everything up from scratch using a general-purpose programming language.

You are encouraged to use a symbolic solver/framework even for calculations that are entirely analytic.

1. Large-amplitude pendulum: *This illustrates how the results of Question 7 for Problem set 3 could be obtained on the grounds of numerical results and physical reasoning alone.*

The equation of motion for the deflection angle $\theta(t)$ of a simple pendulum that consists of a massless rod of length l and a mass at its end, is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2 \sin\theta = 0, \tag{1}$$

where $\omega^2 = g/l$ and the gravitational acceleration is $g = 9.8\,\mathrm{m\,s^{-2}}$.

(a) Consider a pendulum with $l = 10\,\mathrm{cm}$ released from rest at $\theta_0 = 30°$. Plot the solutions $\theta(t)$ and the simple-harmonic approximation $\theta_{\mathrm{SHO}}(t) = \theta_0 cos(\omega t)$ in the range $t \in [0s, 20s]$. Convince yourself that the SHO approximation is unreliable in this case.

(b) Which solution oscillates faster? Does this make physical sense? By plotting $\theta(t) + \theta_{\mathrm{SHO}}(t)$, show that the two solutions first become out of phase at $t \approx 18.5s$. How many seconds will a pendulum clock with $l = 10\,\mathrm{cm}$, $\theta_0 = 30°$ lose/gain in a day if we used the SHO approximation to design it?

(c) Convince yourself that even though the solution $\theta(t)$ does not coincide with $\theta_{\mathrm{SHO}}(t)$, it still resembles a cosine wave. Let us conjecture that $\theta(t) \approx \theta_0 \cos[(\omega + \omega_1)t]$, where ω_1 is the correction to the SHO frequency $\omega = \sqrt{g/l}$. Argue that $\omega_1 = F(\theta_0)\omega$.

(d) Suppose that $F(\theta_0)$ can be Taylor-expanded around $\theta_0 = 0$ as

$$F(\theta_0) = a_0 + a_1\theta_0 + a_2\theta_0^2 + \ldots$$

Argue that $a_0 = 1$, $a_1 = 0$, and $a_2 < 0$ (it may be useful to think of the symmetry of the pendulum's motion).

(e) Using

$$\cos[(\omega + \omega_1)t] + \cos(\omega t) = 2\cos\left(\frac{\omega_1}{2}\right)\cos\left(\omega + \frac{\omega_1}{2}\right),$$

show that $a_2 \approx -1/16$. Does this agree with the result of Question 7(c) from Problem set 3?

The surprising thing here is that using some intuition about what F should be and the solution to (1) for a particular set of parameters, we were able to deduce a general result about the lowest-order correction to the SHO frequency.

2. Lagrange points: Consider a star S and a planet P with masses $M_{\rm S}$ and $M_{\rm P}$, respectively, in circular orbits in the (x, y) plane around their centre of mass.

(a) If R is the radius of the orbit of P, show that the orbital angular frequency Ω is given by

$$\Omega^2 = \frac{GM_{\rm S}}{(M_{\rm P}/M_{\rm S}+1)^2 R^3}.$$

(b) We would like to work out the equations of motion for an asteroid A of negligibly small mass in the noninertial frame rotating with S and P. Let $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ be the vectors

$$\hat{\boldsymbol{e}}_1 = \cos(\Omega t)\hat{\boldsymbol{x}} + \sin(\Omega t)\hat{\boldsymbol{y}}, \tag{2}$$

$$\hat{\boldsymbol{e}}_2 = -\sin(\Omega t)\hat{\boldsymbol{x}} + \cos(\Omega t)\hat{\boldsymbol{y}}, \tag{3}$$

such that the locations $\boldsymbol{r}_{\rm S}$ and $\boldsymbol{r}_{\rm P}$ of S and P are given by

$$\boldsymbol{r}_{\rm S} = -\frac{M_{\rm P}}{M_{\rm S}} R\hat{\boldsymbol{e}}_1, \quad \boldsymbol{r}_{\rm P} = R\hat{\boldsymbol{e}}_1,$$

respectively. If $\boldsymbol{r} = (r_1, r_2, r_3)$ is the position of A in the basis $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{z}}\}$, show that the acceleration of A is given by

$$\frac{{\rm d}^2\boldsymbol{r}}{{\rm d}t^2} = \ddot{\boldsymbol{r}} - 2\Omega\hat{\boldsymbol{z}} \times \dot{\boldsymbol{r}} + \Omega^2\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{r}), \tag{4}$$

where the dots represent the time derivatives of the components r_1, r_2, and r_3. What do the second and third terms on the right-hand side of (4) represent?

Hint: use (2) and (3) to express the time derivatives of the basis vectors and then differentiate $\boldsymbol{r} = r_1\hat{\boldsymbol{e}}_1 + r_2\hat{\boldsymbol{e}}_2 + r_3\hat{\boldsymbol{z}}$ with respect to t twice.

(c) Show that, in the rotating frame, A's equations of motion can be written as

$$\ddot{\boldsymbol{r}} - 2\Omega\hat{\boldsymbol{z}} \times \dot{\boldsymbol{r}} = -\boldsymbol{\nabla} V_{\rm eff}, \tag{5}$$

where the effective potential $V_{\rm eff}$ is given by

$$V_{\rm eff} = -\Omega^2 R^2 \left[\frac{r_1^2 + r_2^2}{2R^2} + \left(\frac{M_{\rm P}}{M_{\rm S}} + 1\right)^2 \frac{R}{|\boldsymbol{r} - \boldsymbol{r}_{\rm S}|} + \left(\frac{M_{\rm P}}{M_{\rm S}} + 1\right)^2 \frac{M_{\rm P}}{M_{\rm S}} \frac{R}{|\boldsymbol{r} - \boldsymbol{r}_{\rm P}|} \right].$$

For the rest of this question, work in units $R = 1$ and $\Omega = 1$.

(d) Using a contour plot of $V_{\rm eff}$ for $r_1 \in (-1.5, 1.5)$, $r_2 \in (-1.5, 1.5)$, $r_3 = 0$, and $M_{\rm P}/M_{\rm S} = 0.1$, convince yourself that the effective potential has three saddle points along the $r_2 = 0$ line and two local maxima at $(r_1, r_2) \approx (0.45, \pm 0.95)$.

These five extrema of $V_{\rm eff}$ are commonly known as the Lagrange points, with the saddles labelled as L1, L2, and L3 and the maxima labelled as L4 and L5. In the limit $M_P/M_S \to 0$, the points {L4, S, P} and {L5, S, P} form equilateral triangles.

(e) Surprisingly, even though L4 and L5 are maxima of $V_{\rm eff}$, they can be dynamically stable in the sense that, for small enough $M_{\rm P}/M_{\rm S}$, there are bounded orbits in their vicinity. To see this, solve (5) numerically for $M_{\rm P}/M_{\rm S} = 10^{-2}$ with initial conditions $r_1(0) = 0.495$, $r_2(0) = 0.874$, $r_3(0) = 0$ and $\dot{r}_i(0) = 0$. Show that the orbits are bounded within a distance of ~ 0.02 of the starting point for $0 < t < 10000$. What is the shape of the orbits? Are they closed?

(f) What part of (5) is responsible for the existence of the bounded orbits near L4 and L5? Does that term remind you of something you have seen elsewhere?

3. Catenary: Consider a piece of inelastic string of length L and uniform density per unit length ρ that is hung from the points $(-l/2, 0)$ and $(l/2, 0)$ in the (x, y) plane. Let $y = f(x)$ describe the shape of the string in equilibrium under uniform gravitational acceleration g.

The first three parts are analytical derivations that may prove challenging. If that is the case, feel free to jump straight to (d) assuming that (7) is the correct ODE to solve.

(a) Show that the length of the string is given by

$$L = \int_{-l/2}^{l/2} dx \sqrt{1 + f'(x)^2}.$$

(b) Argue that the potential energy of the string is given by

$$V = \rho g \int_{-l/2}^{l/2} dx f(x) \sqrt{1 + f'(x)^2},$$

and hence that the shape $f(x)$ of the string in equilibrium must minimise

$$S = V - \rho g \lambda \left(\int_{-l/2}^{l/2} dx \sqrt{1 + f'(x)^2} - L \right) \tag{6}$$

with respect to $f(x)$ and the scalar parameter λ. What is the role of λ?

(c) Consider

$$S = \int_{-l/2}^{l/2} \mathcal{L}(f(x), f'(x)),$$

where $\mathcal{L}$ does not have explicit x dependence, i.e., it depends on x only through $f(x)$. Using the Euler-Lagrange equations, show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\mathcal{L} - f' \frac{\partial \mathcal{L}}{\partial f'} \right) = 0$$

for the function $f(x)$ that extremises S. Hence or otherwise, show that the shape $f(x)$ of the string that minimises (6) satisfies

$$\frac{f(x) - \lambda}{\sqrt{1 + f'(x)^2}} = a, \tag{7}$$

where a is a constant.

(d) Argue that the equilibrium shape of the string $f(x)$ must be an even function of x. Using a symbolic solver or otherwise, show that the appropriate solution to (7) is

$$f(x) = \lambda + a \cosh \frac{x}{a}, \tag{8}$$

where λ and a are related to L and l by

$$L = 2a \sinh \left(\frac{l}{2a} \right)$$

$$\lambda = -\sqrt{a^2 + \frac{L^2}{4}}.$$

(e) Plot (8) for $l = 2$ and $L = 4$ in the range $x \in (-1, 1)$ and verify that it behaves as expected. What's the height of the lowest point of the string?

4. Rotating catenary*: *The setup for this problem is similar to the previous one. However, this time we end up with a differential equation whose solution cannot be written in terms of known function. As a consequence, some additional steps are needed.*

Consider a piece of inelastic string of length L and uniform density per unit length ρ attached to two points a distance $l = 2$ from each other. Suppose that the string is spinning with a constant angular velocity Ω about the axis that joins the two points where it is attached. Assume that the string remains in a single rotating plane.

(a) Transform to the rotating coordinate system where the axis of rotation is the $\hat{\boldsymbol{x}}$ axis and the string lies in the (x, y) plane. Let the string be attached at $x = \pm 1$, $y = 0$. Using the results of Problem 2, argue that a stationary string experiences the centrifugal force $\boldsymbol{F} = -\boldsymbol{\nabla} V$, where

$$V = -\frac{1}{2}\Omega^2 y^2.$$

(b) Following a similar derivation to parts (a)–(c) of Problem 3, show that the equilibrium shape $f(x)$ of the string satisfies

$$\frac{f(x)^2 - \lambda}{\sqrt{1 + f'(x)^2}} = -a, \tag{9}$$

where λ and a are constants. Without loss of generality, let's assume that $f(x) > 0$.

Equation (9) is difficult to solve because the constants a and λ need to be determined self-consistently via the string's end points at $x = \pm 1$ and length L.

(c) By considering $x = \pm 1$, argue that $\lambda/a > 1$. Then, by considering $x = 0$, argue that $f(0) = \sqrt{\lambda - a}$ and $\lambda > 0$, $a > 0$.

(d) Using

$$f'(x) = \frac{1}{a}\sqrt{\left[\lambda - f(x)^2\right]^2 - a^2}.$$

and the substitution $y = f(x)$, deduce that

$$1 = \int_{-1}^{0} \mathrm{d}x = a \int_{0}^{\sqrt{\lambda - a}} \frac{\mathrm{d}y}{\sqrt{\left[\lambda - y^2\right]^2 - a^2}}.$$

Finally, letting $u = (\lambda - y^2)/a$, show that

$$\frac{2}{\sqrt{a}} = F(\lambda/a) \equiv \int_1^{\lambda/a} \frac{\mathrm{d}u}{\sqrt{(\lambda/a - u)(u^2 - 1)}} \tag{10}$$

(e) Similarly, show that the length of the string L can be expressed as

$$\frac{L}{2} = \int_{-1}^{0} \mathrm{d}x \, \frac{\lambda - f(x)^2}{a} = \int_0^{\sqrt{\lambda - a}} \mathrm{d}y \frac{\lambda - y^2}{\sqrt{(\lambda - y^2)^2 - a^2}}.$$

Again, using $u = (\lambda - y^2)/a$, show that

$$\frac{L}{\sqrt{a}} = G(\lambda/a) \equiv \int_1^{\lambda/a} \frac{u \, \mathrm{d}u}{\sqrt{(\lambda/a - u)(u^2 - 1)}}. \tag{11}$$

You might be wondering why this helps. The point of these manipulations is that the functions F and G depend only on the ratio λ/a and that $G/F = L/2$ is independent of the unknown constants λ and a.

(f) Let $L = 3$. Using (10) and (11), first determine λ/a and then λ and a individually. Plot the solution $f(x)$ for $-1 < x < 0$. Compare the value $f(0)$ with that from Problem 3.

You might find that your numerical ODE solver cannot integrate past the point $x = 0$. Why could that be?

5. Visualising the headlight effect: Consider a spherically symmetric light source in its rest frame S'. Let F' be a set of uniformly spaced light rays in the (x', y') plane in S' originating from the light source at $x' = y' = 0$. Let S be the lab frame in which S' is moving at a constant speed v in the x direction.

(a) Construct the Lorentz transformation from S' to S as a numerical 4×4 matrix.

(b) Construct F' as a list/array of uniformly spaced null vectors. Visualise them in the (x', y') plane.

(c) Apply the Lorentz transformation from part (a) and obtain the set F of the light rays as seen in S. Visualise them (it is best to do it in an interactive way where v can be varied). What can you deduce about the intensity of light as observed in S?

(d*) *Bonus part:* Initialise F' with null vectors that correspond to a wavelength in the middle of the visible spectrum, say, 565 nm. Colour the visualisation in part (c) with the correct Doppler-shifted colour for each ray.