S31 Numerical Methods

1. Use Euler's method to approximate the solution to

$$\frac{dy}{dt} = y - t$$

at time $t = 1$ given that $y = \frac{1}{2}$ at $t = 0$. Do this in 3 steps. Redo the estimate, this time with a fully implicit scheme. Is one scheme more accurate than the other? Note: the analytic solution to this problem is $y(t) = t + 1 - \frac{1}{2}\exp(t)$, so that the true solution at t = 1 is y = 0.640859.

2. Consider the second order ordinary differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

If you solve it numerically with an explicit forward (Euler) scheme, are there limitations on the updating step? If so what are the limitations?

3. (a) Consider the finite difference approximation to the second derivative of a function $u(x, y)$ on an evenly spaced grid with spacing Δx and Δy.

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

The subscripts i and j index the grid cells in the x and y directions. Determine the accuracy of the approximation and give an explicit expression for the truncation error.

(b) Given a two-dimensional function $f(x, y)$, find a second-order accurate (i.e. $O((\Delta x)^2, (\Delta y)^2)$) finite difference expression for the mixed derivative

$$\frac{\partial^2 f}{\partial x \partial y}.$$

4. The question of estimating the heat transfer to a wall

$$q = \kappa \frac{\partial T}{\partial z}$$

raises the problem of how to compute an approximation to a derivative at a boundary. Here κ is the thermal conductivity (assumed to be constant) and z is the direction perpendicular to the wall. Let grid point 1 be on the wall with points 2 and 3 a distance Δz and $2\Delta z$ above the wall respectively.

(i) Assuming that T can be expressed by a second order polynomial $T = a + bz + cz^2$, find a finite difference approximation for $\frac{\partial T}{\partial z}$ at the wall in terms of the temperature at grids points 1, 2, and 3.

(ii) Use a Taylor series expansion to show its order of accuracy.

5. By representing $\frac{\partial T}{\partial t}$ with a forward difference in time and $\frac{\partial^2 T}{\partial x^2}$ with a central second order difference, the one dimensional heat diffusion equation with constant heat conduction coefficient α

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

becomes

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}.$$

Another version of this finite difference equations is the Crank-Nicholson scheme. It is obtained by rewriting each of the temperature terms on the right hand side of the above finite difference equation in terms of the average temperatures occurring at time levels n and $n+1$.

$$T_i^n \longrightarrow \frac{1}{2}(T_i^{n+1} + T_i^n).$$

(i) Write the finite difference equation for the Crank-Nicholson scheme.

(ii) Assuming an evenly spaced spatial grid with grid cells indexed by i ranging from $i = 1$ to $i = I$ and assuming that the temperature keeps fixed values at the boundaries $T_1 = T_{\text{left}}$ and $T_I = T_{\text{right}}$, express the Crank-Nicholson scheme in matrix form

$$\mathbf{M} \begin{pmatrix} T_2^{n+1} \\ \vdots \\ \vdots \\ T_{I-1}^{n+1} \end{pmatrix} = \begin{pmatrix} K_2^n \\ \vdots \\ \vdots \\ K_{I-1}^n \end{pmatrix}$$

giving explicit expressions for the non-zero elements of the matrix $\mathbf{M}$ and expressions for elements K_i^n in the vector on the right hand side.

6. Use von-Neumann stability analysis to assess the stability of the following approximation to the linear advection equation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

where a is a positive constant, the j subscript indexes the evenly spaced grid in x and the n superscript indexes the timestep. Change the scheme to a fully implicit one, and determine whether the stability of the scheme changes.

Relevant past exam questions:
S17 2008 Q3
S17 2007 Q2
S17 2005 Q1 (the 2nd part for 9 points)

① Using Euler's method to approximate the solution to

$$\frac{dy}{dt} = y - t$$

(ANALYTIC SOLUTION: $y(t) = t + 1 - \frac{1}{2}e^t$, so $y = 0.640859$ at $t = 1$)

at time $t = 1$ given that $y = \frac{1}{2}$ at $t = 0$.

EXPLICIT SCHEME

$$\frac{y_{n+1} - y_n}{\Delta t} = y_n - t_n \qquad \text{let } t_n = n\Delta t$$

If want to reach $t = 1$ in 3 steps, then $\Delta t = \frac{1}{3}$

$$y_{n+1} = y_n + \Delta t\,(y_n - n\Delta t)$$

$n = 0$ $\quad y_1 = y_0 + \frac{1}{3}(y_0 - 0) = \frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right) = \frac{2}{3}$

$n = 1$ $\quad y_2 = y_1 + \frac{1}{3}\left(y_1 - \frac{1}{3}\right) = \frac{2}{3} + \frac{1}{3}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

$n = 2$ $\quad y_3 = y_2 + \frac{1}{3}\left(y_2 - \frac{2}{3}\right) = \frac{7}{9} + \frac{1}{3}\left(\frac{7}{9} - \frac{2}{3}\right) = \frac{7}{9} + \frac{1}{27} = \frac{22}{27} = 0.814$

IMPLICIT SCHEME

$$\frac{y_{n+1} - y_n}{\Delta t} = y_{n+1} - t_{n+1}$$

$$y_{n+1} - y_n = \Delta t\, y_{n+1} - \Delta t^2 (n+1)$$

$$y_{n+1} = \frac{1}{(1 - \Delta t)}\left[y_n - (\Delta t)^2 (n+1)\right]$$

$\Delta t = \frac{1}{3}$, $\frac{1}{1-\Delta t} = \frac{3}{2}$

$n = 0$ $\quad y_1 = \frac{3}{2}\left[y_0 - \left(\frac{1}{9}\right)\right] = \frac{3}{2}\left[\frac{1}{2} - \frac{1}{9}\right] = \frac{3}{2}\left(\frac{7}{18}\right) = \frac{7}{12}$ (6)

$n = 1$ $\quad y_2 = \frac{3}{2}\left[y_1 - \frac{1}{9}2\right] = \frac{3}{2}\left[\frac{7}{12} - \frac{2}{9}\right] = \frac{3}{2}\left[\frac{21}{36} - \frac{8}{36}\right] = \frac{3}{2}\left[\frac{13}{36}\right] = \frac{13}{24}$ (12)

$n = 2$ $\quad y_3 = \frac{3}{2}\left[y_2 - \frac{1}{9}(3)\right] = \frac{3}{2}\left[\frac{13}{24} - \frac{1}{3}\right] = \frac{3}{2}\left[\frac{5}{24}\right] = \frac{5}{16} = .3125$ (8)

WE ILLUSTRATED THE TWO SCHEMES WITH A VERY LARGE TIME STEP FOR WHICH THE EXPLICIT SCHEME WAS MORE ACCURATE.

(2) Given second order ODE

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

to solve numerically with explicit forward Euler scheme, first reduce to set of first order ODEs

$$y_1 = y$$

$$y_2 = \frac{dy_1}{dx} = -3y_1 - 2y$$

$$\frac{d}{dx}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{A}\begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\frac{d\vec{y}}{dx} = A\vec{y}$$

If solving by explicit Euler method

$$\frac{y_{n+1} - y_n}{\Delta x} = A y_n \Rightarrow \vec{y}_{n+1} = \vec{y}_n + \Delta x A \vec{y}_n$$

$$\vec{y}_{n+1} = (1 + \Delta x A)\vec{y}_n$$

$$\vec{y}_n = (1 + \Delta x A)^n \vec{y}_0$$

Expand $\vec{y}_0$ in terms of eigenvectors of A

$$\vec{y}_0 = \sum \alpha_i \vec{v}_i$$

$$\vec{y}_n = \sum (1 + \Delta x A)^n \alpha_i v_i = \sum \alpha_i (1 + \Delta x \lambda_i)^n v_i$$

Require $|(1 + \Delta x \lambda_i)| < 1$ FOR STABILITY

To find eigenvalues

$$\det(A - \lambda I) = 0 \Rightarrow \det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = 0$$

$$-\lambda(-3-\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{9-8}}{2}$$

$$= -1 \text{ or } -2$$

If $\lambda_1 = -1$ $\quad 1 - \Delta x > -1 \Rightarrow \Delta x < 2$

If $\lambda_2 = -2$ $\quad 1 - 2\Delta x > -1 \Rightarrow \Delta x < 1$

For stability have to satisfy both conditions so the most stringent one in this case is $\Delta x < 1$

③ (a) GIVEN THE FOLLOWING FINITE DIFFERENCE APPROXIMATION TO THE SECOND DERIVATIVE ON AN EVENLY SPACED GRID

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

TO DETERMINE ACCURACY WE TAYLOR EXPAND ABOUT (i,j)

$$u_{i+1,j} = u_{i,j} + \left.\frac{\partial u}{\partial x}\right|_{i,j} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} (\Delta x)^2 + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{i,j} (\Delta x)^3 + \frac{1}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} (\Delta x)^4 + \dots$$

EXPRESSION FOR $u_{i-1,j}$ IS THE SAME WITH $\Delta x \longrightarrow -\Delta x$

$$u_{i-1,j} = u_{i,j} - \left.\frac{\partial u}{\partial x}\right|_{i,j} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} (\Delta x)^2 - \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{i,j} (\Delta x)^3 + \frac{1}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} (\Delta x)^4 + \dots$$

SO $$u_{i+1,j} + u_{i-1,j} = 2u_{i,j} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} (\Delta x)^2 + \frac{2}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} (\Delta x)^4 + \dots$$

$$\Rightarrow \quad u_{i+1} - 2u_{i,j} + u_{i-1,j} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} (\Delta x)^2 + \frac{1}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} (\Delta x)^4 + \dots$$

so when divide by $(\Delta x)^2$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u_{i+1} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} - \underbrace{\frac{1}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} (\Delta x)^2}$$

SECOND ORDER TRUNCATION ERROR

(b) Finding a second-order accurate $(O((\Delta x)^2, (\Delta y)^2)$ finite difference expression for the mixed derivative.

$$\frac{\partial^2 f}{\partial x \partial y} \qquad \text{where} \qquad f \equiv f(x,y)$$

$$f_{i+1,j} = f_{i,j} + \left(\frac{\partial f}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 f}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots$$

differentiate w.r.t. y

[1] $$\left(\frac{\partial f}{\partial y}\right)_{i+1,j} = \left(\frac{\partial f}{\partial y}\right)_{i,j} + \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j} \Delta x + \left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{i,j} \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^4 f}{\partial x^3 \partial y}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots$$

also $$f_{i-1,j} = f_{i,j} - \left(\frac{\partial f}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 f}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots$$

differentiate w.r.t. y

[2] $$\left(\frac{\partial f}{\partial y}\right)_{i-1,j} = \left(\frac{\partial f}{\partial y}\right)_{i,j} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j} \Delta x + \left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{i,j} \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^4 f}{\partial x^3 \partial y}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots$$

THEN [1] − [2]

$$\left(\frac{\partial f}{\partial y}\right)_{i+1,j} - \left(\frac{\partial f}{\partial y}\right)_{i-1,j} = 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j} \Delta x + 2\left(\frac{\partial^4 f}{\partial x^3 \partial y}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \ldots$$

solve for $\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j}$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j} = \frac{\left(\frac{\partial f}{\partial y}\right)_{i+1,j} - \left(\frac{\partial f}{\partial y}\right)_{i-1,j}}{2\Delta x} - \left(\frac{\partial^4 f}{\partial x^3 \partial y}\right)_{i,j} \frac{(\Delta x)^2}{3!} + \ldots$$

NOW RECALL SECOND ORDER CENTRAL FINITE DIFFERENCE EXPRESS FOR 1st ORDER DERIVATIVE

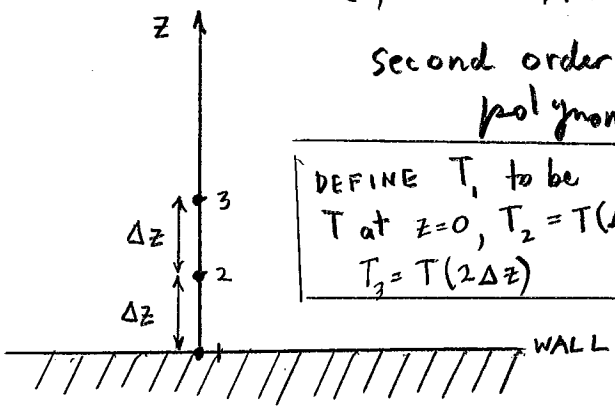
$$\left(\frac{\partial f}{\partial y}\right)_{i+1,j} = \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta y} + O(\Delta y^2)$$

$$\left(\frac{\partial f}{\partial y}\right)_{i-1,j} = \frac{f_{i-1,j+1} - f_{i-1,j-1}}{2\Delta y} + O(\Delta y^2)$$

so $\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j} = \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4 \Delta x \Delta y} + O\left((\Delta x)^2, (\Delta y)^2\right)$

④

(i) Assume T can be expressed as a second order polynomial : $T = a + bz + cz^2$



DEFINE T_1 to be T at $z=0$, $T_2 = T(\Delta z)$, $T_3 = T(2\Delta z)$

At grid point 1

$z = 0$

$\Rightarrow T_1 = a$

At grid point 2, $z = \Delta z$

$\rightarrow T_2 = a + b\Delta z + c(\Delta z)^2$

At grid point 3, $z = 2\Delta z$

$$T_3 = a + b(2\Delta z) + c(2\Delta z)^2$$

so have 3 equations for 3 unknowns (a, b, c).

solve for b

$$b = \frac{-3T_1 + 4T_2 - T_3}{2\Delta z}$$

We want a finite difference approximation for for $\frac{\partial T}{\partial z}$ at the wall. Differentiating the polynomial:

$$\frac{\partial T}{\partial z} = b + 2cz$$

Evaluating $\frac{\partial T}{\partial z}$ at the wall $(z=0)$

$$\frac{\partial T}{\partial z} = b$$

so finite difference approximation for $\frac{\partial T}{\partial z}$ at the wall in terms of temperature at grid points 1, 2, 3 is:

$$\frac{\partial T}{\partial z} = \frac{-3T_1 + 4T_2 - T_3}{2\Delta z}$$

(ii) Using a Taylor series expansion to show its order of accuracy...

A Taylor series expansion about point 1 on the wall gives

$$T(z) = T_1 + \left(\frac{\partial T}{\partial z}\right)_1 z + \left(\frac{\partial^2 T}{\partial z^2}\right)_1 \frac{z^2}{2!} + \left(\frac{\partial^3 T}{\partial z^3}\right)_1 \frac{z^3}{3!} + \dots$$

Our assumed polynomial in (i) $T = a + bz + cz^2$ is the same as using the first 3 terms in the Taylor series. Hence the polynomial is $O(\Delta z)^3$

T_1, T_2 and T_3 in the numerator of the expression for $\left(\frac{\partial T}{\partial z}\right)_1$ can each be expressed as the polynomial $T = a + bz + cz^2$ so the numerator is $O(\Delta z)^3$.
Now to get the derivative, we divided by Δz so $\left(\frac{\partial T}{\partial z}\right)_1$ is $O(\Delta z)^2$, i.e. SECOND ORDER ACCURATE

$$\left(\frac{\partial T}{\partial z}\right)_1 = \frac{-3T_1 + 4T_2 - T_3}{2\Delta z} + O(\Delta z)^2$$

⑤ ONE-DIMENSIONAL HEAT DIFFUSION EQ $\quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}$$

FORWARD IN TIME — CENTRAL SECOND ORDER DIFFERENCE

(i) CRANK-NICHOLSON $\quad T_i^n \longrightarrow \frac{1}{2}\left(T_i^{n+1} + T_i^n\right)$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{\frac{1}{2}\left(T_{i+1}^{n+1} + T_{i+1}^n\right) + \frac{1}{2}\left(-2T_i^{n+1} - 2T_i^n\right) + \frac{1}{2}\left(T_{i-1}^{n+1} + T_{i-1}^n\right)}{(\Delta x)^2}$$

(ii) REARRANGE TO PUT ALL UNKNOWNS $\left(T_{i+1}^{n+1}, T_i^{n+1}, T_{i-1}^{n+1}\right)$ ON THE LEFT AND ALL KNOWNS ON THE RIGHT

$$\underbrace{\frac{\alpha \Delta t}{2(\Delta x)^2}}_{A} T_{i-1}^{n+1} - \underbrace{\left(1 + \frac{\alpha \Delta t}{(\Delta x)^2}\right)}_{B} T_i^{n+1} + \underbrace{\frac{\alpha \Delta t}{2(\Delta x)^2}}_{A} T_{i+1}^{n+1} = \underbrace{-T_i^n - \frac{\alpha \Delta t}{2(\Delta x)^2}\left(T_{i-1}^n - 2T_i^n + T_{i+1}^n\right)}_{K_i^n}$$

BOUNDARY CONDITIONS $\quad T_1 = T_{left}$

$$T_I = T_{right}$$

AT GRID POINT 2 $(i = 2)$

$$A T_1 - B T_2^{n+1} + A T_3^{n+1} = K_2^n$$

PUTTING ALL KNOWN QUANTITIES TO THE RIGHT

$$-B T_2^{n+1} + A T_3^{n+1} = K_2^n - A T_1$$

$$= K_2^n - A T_{left}$$

SIMILARLY FOR GRID POINT I-1 $\quad (i = I-1)$

$$A T_{I-2}^{n+1} - B T_{I-1}^{n+1} + A T_I = K_{I-1}^n$$

$$A T_{I-2}^{n+1} - B T_{I-1}^{n+1} = K_{I-1}^n - A T_I = K_{I-1}^n - A T_{right}$$

$$\begin{pmatrix} -B & A & 0 & 0 & 0 & \cdots & & & & \\ A & -B & A & 0 & 0 & \cdots & & & & \\ 0 & A & -B & A & 0 & \cdots & & & & \\ 0 & 0 & A & -B & A & \cdots & & & & \\ & & & & & & & & & \\ \cdots & \cdots & 0 & 0 & 0 & 0 & A & -B & A & 0 \\ \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & A & -B & A \\ \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & A & -B \end{pmatrix} \begin{pmatrix} T_2^{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T_{I-1}^{n+1} \end{pmatrix} = \begin{pmatrix} K_2^n - AT_{left} \\ K_3^n \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ K_{I-2}^n \\ K_{I-1}^n - AT_{right} \end{pmatrix}$$

M

(TRIDIAGONAL)

⑥ Using von Neumann stability analysis to assess stability of the following approximation to LINEAR ADVECTION EQUATION

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

where "a" is a positive constant

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left\{u_{j+1}^n - u_{j-1}^n\right\} \qquad \text{let } \mu = \frac{a\Delta t}{\Delta x}$$

$$u_j^{n+1} = u_j^n - \frac{\mu}{2}\left\{u_{j+1}^n - u_{j-1}^n\right\}$$

Find amplification factor by going to Fourier space

$$u_j^n = U_k^n e^{ikx_j} \qquad u_j^{n+1} = U_k^{n+1} e^{ikx_j}$$

$$u_{j+1}^n = U_k^n e^{ikx_j + ik\Delta x}$$

$$u_{j-1}^n = U_k^n e^{ikx_j - ik\Delta x}$$

substitute into FDA

$$U_k^{n+1} e^{ikx_j} = U_k^n e^{ikx_j} - \frac{\mu}{2}\left\{U_k^n e^{ikx_j + ik\Delta x} - U_k^n e^{ikx_j - ik\Delta x}\right\}$$

$$U_k^{n+1} = U_k^n - \frac{\mu}{2} U_k^n \left\{e^{ik\Delta x} - e^{-ik\Delta x}\right\}$$

$$= U_k^n - U_k^n \frac{\mu}{2} 2i\sin k\Delta x$$

so $\lambda_{FDA}(k) = \dfrac{U_k^{n+1}}{U_k^n} = 1 - i\mu\sin k\Delta x$

(λ_{FDA}: AMPLIFICATION FACTOR)

$$|\lambda_{FDA}(k)| = \sqrt{(1 - i\mu\sin k\Delta x)(1 + i\mu\sin k\Delta x)}$$

$$= 1 + \mu^2 \sin^2 k\Delta x$$

so $|\lambda_{FDA}(k)| > 1$ FOR ALL WAVENUMERS k AND ALL VALUES OF Δt

(UNCONDITIONALLY UNSTABLE)

CHANGING THE SCHEME TO A FULLY IMPLICIT ONE

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}$$

$$u_j^{n+1} - u_j^n = -\frac{a\Delta t}{2\Delta x} u_{j+1}^{n+1} + \frac{a\Delta t}{2\Delta x} u_{j-1}^{n+1}$$

$$u_j^{n+1} + \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n$$

$$U_k^{n+1} e^{ikx_j} + \frac{a\Delta t}{2\Delta x}\left(U_k^{n+1} e^{ikx_j} e^{ik\Delta x} - U_k^{n+1} e^{ikx_j} e^{-ik\Delta x}\right) = U_k^n e^{ikx_j}$$

let $\mu = \frac{a\Delta t}{\Delta x}$

$$U_k^{n+1} + \frac{\mu}{2} U_k^{n+1}\left(2i \sin k\Delta x\right) = U_k^n$$

$$U_k^{n+1} = \frac{1}{1 + i\mu \sin k\Delta x} U_k^n$$

so $$\lambda_{FDA} = \frac{U_k^{n+1}}{U_k^n} = \frac{1}{1 + i\mu \sin k\Delta x}$$

$$\lambda_{FDA} = \frac{1 - i\mu \sin k\Delta x}{(1 + i\mu \sin k\Delta x)(1 - i\mu \sin k\Delta x)}$$

$$= \frac{1}{(1 + \mu^2 \sin^2 k\Delta x)} - i \frac{\mu \sin k\Delta x}{(1 + \mu^2 \sin^2 k\Delta x)}$$

$$|\lambda_{FDA}| = \sqrt{\frac{1}{(1+\mu^2 \sin^2 k\Delta x)^2} + \frac{\mu^2 \sin^2 k\Delta x}{(1+\mu^2 \sin^2 k\Delta x)^2}} = \sqrt{\frac{1}{1+\mu^2 \sin^2 k\Delta x}} < 1$$

FOR ALL k

SO STABLE