Mathematical Methods

Problem Sheet 1: "Normed and inner product vector space"

Andre Lukas, MT 2018

1) (Examples of function vector spaces)
Consider the set $\mathcal{F}(S, V)$ of all functions $S \to V$ from a set S to a vector space V.

(a) Write down the definitions of vector addition and scalar multiplication for $\mathcal{F}(S, V)$ and argue that this makes $\mathcal{F}(S, V)$ a vector space.

(b) Sub-vector spaces of $\mathcal{F}(S, V)$ can be obtained by restricting to functions with certain properties which are preserved under vector addition and scalar multiplication. Based on this observation find three vector spaces of functions.

(c) Discuss possible norms and scalar product which can be introduced on the vector spaces found in part (b).

2) (Polarisation identities)
Let V be a (complex) inner product vector space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$ defined in the usual way as $\| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

(a) Show that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \left(\| \mathbf{v} + \mathbf{w} \|^2 - \| \mathbf{v} - \mathbf{w} \|^2 - i \| \mathbf{v} + i\mathbf{w} \|^2 + i \| \mathbf{v} - i\mathbf{w} \|^2 \right) , \tag{1}$$

for all $\mathbf{v}, \mathbf{w} \in V$. (This identity, also called polarisation identity, implies that the scalar product is completely determined by its associated norm.)

(b) Find the polarisation identity if V is a real inner product space.

(c) Let $T : V \to V$ be a linear operator on the complex inner product space V. Show that the following generalisation of the polarisation identity holds:

$$\begin{aligned} \langle \mathbf{v}, T\mathbf{w} \rangle \;=\; & \frac{1}{4} \left(\langle \mathbf{v} + \mathbf{w}, T(\mathbf{v} + \mathbf{w}) \rangle - \langle \mathbf{v} - \mathbf{w}, T(\mathbf{v} - \mathbf{w}) \rangle \right. \\ & \left. - i \langle \mathbf{v} + i\mathbf{w}, T(\mathbf{v} + i\mathbf{w}) \rangle + i \langle \mathbf{v} - i\mathbf{w}, T(\mathbf{v} - i\mathbf{w}) \rangle \right) . \end{aligned} \tag{2}$$

(d) Suppose $T : V \to V$ is a linear operator on a complex inner product space V with $\langle \mathbf{v}, T\mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. Use the result from part c) to show that $T = 0$.

(e) Show that the linear operator $T : V \to V$ is hermitian if and only if $\langle \mathbf{v}, T\mathbf{v} \rangle \in \mathbb{R}$ for all $\mathbf{v} \in V$. (Hint: Apply the statement of part (d) to $T - T^\dagger$.)

3) (The normed vector space ℓ^p and the parallelogram identity)

(a) Let V be an inner product vector space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Show that the associated norm satisfies the parallelogram identity

$$\| \mathbf{v} + \mathbf{w} \|^2 + \| \mathbf{v} - \mathbf{w} \|^2 = 2 \left(\| \mathbf{v} \|^2 + \| \mathbf{w} \|^2 \right) , \tag{3}$$

for all $\mathbf{v}, \mathbf{w} \in V$.

(b) Consider the normed vector space ℓ^p of sequences $(x_i)_{i=1}^{\infty}$, where $x_i \in \mathbb{C}$, with $(\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ finite. Show that, for $1 < p < \infty$, the expression

$$\| (x_i) \| := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \tag{4}$$

defines a norm on ℓ^p.

(c) For the normed space ℓ^p consider the vectors $\mathbf{v} = (1, 0, 0, \ldots)$ and $\mathbf{w} = (0, 1, 0, 0, \ldots)$ together with the parallelogram identity from part (a) to show that the above norm on ℓ^p is not associated to an inner product for $p \neq 2$.

4) (Recap of Gram-Schmidt procedure)
Consider the Hilbert space $L^2_w(\mathbb{R})$ with weight function $w(x) = e^{-x^2}$ and scalar product

$$\langle f, g \rangle := \int_{\mathbb{R}} dx \, w(x) f(x) g(x) \, . \tag{5}$$

(a) Apply the Gram-Schmidt procedure to the sub-space $V \subset L^2_w(\mathbb{R})$ with basis $1, x, x^2$ to obtain ortho-normal polynomials p_i of degree i, where $i = 0, 1, 2$.

(b) Write a general quadratic polynomial $q(x) = \sum_{k=0}^{2} a_k x^k$ as a linear combination of the ortho-normal polynomials in part (a).

(c) Use the result from (b) to work out $\| q \|^2$.

(d) Consider the linear operator $T = \frac{1}{w} \frac{d}{dx} w \frac{d}{dx}$ and show that it is hermitian, relative to the above scalar product. Compute the matrix which describes the restriction of T to V relative to the basis of ortho-normal polynomials from part (a).

5) (Ortho-normal basis)
Let V be a (finite-dimensional) inner product space and $(\boldsymbol{\epsilon}_i)_{i=1}^n$ and $(\boldsymbol{\epsilon}'_i)_{i=1}^n$ be two ortho-normal basis sets.

(a) For any two vectors $\mathbf{v}, \mathbf{w} \in V$ show that $\langle \mathbf{v} | \mathbf{w} \rangle = \sum_i \langle \mathbf{v} | \boldsymbol{\epsilon}_i \rangle \langle \boldsymbol{\epsilon}_i | \mathbf{w} \rangle$.

(b) Show that the matrix P with entries $P_{ij} = \langle \boldsymbol{\epsilon}'_i | \boldsymbol{\epsilon}_j \rangle$ is unitary.

(c) For a linear operator $\hat{T} : V \to V$ consider the matrices T and T' which consist of the entries $T_{ij} = \langle \boldsymbol{\epsilon}_i | \hat{T} | \boldsymbol{\epsilon}_j \rangle$ and $T'_{ij} = \langle \boldsymbol{\epsilon}'_i | \hat{T} | \boldsymbol{\epsilon}'_j \rangle$, respectively. Show that $T' = P T P^\dagger$, where P is the matrix from part (b).

6) (Rotations and unitary matrices)
Consider three-dimensional rotation matrices, that is, 3×3 real matrices R which satisfy $R^T R = \mathbb{1}_3$ and $\det(R) = 1$, as well as two-dimensional special unitary matrices, that is, complex 2×2 matrices U which satisfy $U^\dagger U = \mathbb{1}_2$ and $\det(U) = 1$.

(a) Write $R = \mathbb{1}_3 + iT + \cdots$ (where T is purely imaginary) and show that, to linear order in T, R satisfies the defining relation for rotations iff T is anti-symmetric. Such matrices T can be seen as "infinitesimal rotations".

(b) Convince yourself that the space of infinitesimal rotations is three-dimensional and that the matrices $\tilde{T}_i$ with entries $(\tilde{T}_i)_{jk} = -i\epsilon_{ijk}$ form a basis. Show that $[\tilde{T}_i, \tilde{T}_j] = i\epsilon_{ijk}\tilde{T}_k$, where the square bracket denotes the commutator.

(c) Find the set of matrices S such that $U = \mathbb{1}_2 + iS + \cdots$ satisfies the special unitary constraints to linear order and show that these are precisely the 2×2 hermitian matrices with vanishing trace.

(d) Convince yourself that the matrices $\tau_i := \frac{\sigma_i}{2}$, where σ_i are the Pauli matrices, form a basis of the space of 2×2 hermitian traceless matrices and that $[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_k$.

(e) Given the above results, which statements can you make about the linear map r which maps anti-symmetric 3×3 matrices into hermitian, traceless 2×2 matrices and is defined by $r(\tilde{T}_i) := \tau_i$?

7) (Convergence and completeness)
Let V be the inner product space of polynomials in x, where $x \in [0, a]$ and $0 < a < 1$, with the standard inner product defined by an integral.

(a) Show that the sequence $(x^k)_{k=1}^\infty$ is a Cauchy sequence and show that it converges to the zero function.

(b) Consider the sequence $(s_k)_{k=0}^\infty$, where $s_k = \sum_{i=0}^k x^i$. Show that this is a Cauchy sequence.

(c) Argue from the result in (b) that V is not complete and, hence, that it is not a Hilbert space.

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Problem Sheet 2: "Fourier series and Fourier integrals"

Andre Lukas, MT 2018

1) (Fourier series)

a) Find the Fourier series for the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \leq x < 0 \\ \sin x & \text{for} \quad 0 \leq x \leq \pi \end{cases} \tag{1}$$

(b) Find the Fourier series for the functions $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = x^2$.

(c) Use the result from part (b) to sum the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$.

2) (Sine and cosine Fourier series)

(a) For the functions $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x \sin x$ find the cosine Fourier series.

(b) For the function from part (a), find the sine Fourier series.

(c) For the function $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x$ compute the cosine Fourier series.

(d) Compute the sine Fourier series for the function from part (c). Comment on the convergence properties of this series and its cosine Fourier series counterpart. Sum the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

3) (Legendre polynomials as an orthogonal basis)
Denote by $(P_n)_{n=0}^{\infty}$ the Legendre polynomials in $L^2([[-1, 1])$, normalised as $\| P_n \|^2 = \frac{2}{2n+1}$.

(a) Show that, for any polynomial $p \in L^2([-1, 1])$ with degree less than n, we have $\langle P_n, p \rangle = 0$.

(b) Find explicitly the polynomials $P_0, \ldots, P_4$.

(c) Write the function $f \in L^2([-1, 1])$ with $f(x) = x^4$ as a linear combination of Legendre polynomials.

(d) Write the function $f \in L^2([-1, 1])$ defined by $f(x) = \frac{2}{\sqrt{5-4x}}$ as an expansion in terms of Legendre polynomials. (Hint: Think about the generating function for Legendre polynomials.)

4) (Examples of Fourier transforms)
Recall that the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined as

$$\hat{f}(k) = \mathcal{F}(f)(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, f(x) e^{-ikx} \, . \tag{2}$$

(a) Work out the Fourier transform of the function

$$\chi(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases} \tag{3}$$

(b) Show that the Fourier transform $\hat{\chi}$ from part (a) is not in $L^1(\mathbb{R})$.

(c) Work out the function $f = \chi \star \chi$, where χ is the function from part (a) and the star denotes the convolution.

(d) Find the Fourier transform of the function f from part (c).

(e) Compute the integral $\int_{\mathbb{R}} dx \, \frac{\sin^4 x}{x^4}$. (Hint: Use the fact that the Fourier transform is unitary.)

5) (Some properties of Fourier transforms)
For a constant $a \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{C}$, define the dilation operator $\mathcal{D}_a$, the translation operator T_a, the modulation operator E_a and the multiplication operator M_g (for a function g) by

$$\mathcal{D}_a(f)(x) := f(ax) \, , \quad T_a(f)(x) := f(x-a) \, , \quad E_a(f)(x) := e^{iax} f(x) \, , \quad M_g(f)(x) := g(x)f(x) \, .$$

Also, denote the Fourier transform by $\mathcal{F}$ and $D_x = \frac{d}{dx}$, $D_k = \frac{d}{dk}$.

(a) Show that $\mathcal{F} \circ T_a = E_{-a} \circ \mathcal{F}$ and $\mathcal{F} \circ E_a = T_a \circ \mathcal{F}$.

(b) Show that $\mathcal{F} \circ \mathcal{D}_a = \frac{1}{|a|} \mathcal{D}_{1/a} \circ \mathcal{F}$.

(c) Show that $\mathcal{F} \circ D_x = M_{ik} \circ \mathcal{F}$ and $\mathcal{F} \circ M_x = iD_k \circ \mathcal{F}$.

6) (More Fourier transforms)

(a) Show that the Gaussian $f(x) = e^{-\frac{x^2}{2}}$ is invariant under Fourier transformation.

(b) Work out the Fourier transform of a Gaussian $f_a(x) = e^{-\frac{x^2}{2a^2}}$ with width a.

(c) Work out the Fourier transform of a Gaussian $f_{a,c}(x) = e^{-\frac{(x-c)^2}{2a^2}}$ with width a and peaked at $x = c$.

7) (Hermite polynomials and Fourier transform)
Denote by $(H_n)_{n=0}^{\infty}$ the standard Hermite polynomials and by $h_n(x) = e^{-\frac{x^2}{2}} H_n(x)$ the Hermite functions which form an orthogonal basis of $L^2(\mathbb{R})$.

(a) Show that the Fourier transform of the function $g(x) = e^{-\frac{x^2}{2}+2xz-z^2}$ is given by $\hat{g}(k) = e^{-\frac{k^2}{2}-2kiz+z^2}$.

(b) Show that the Hermite functions h_n are eigenfunctions of the Fourier transform $\mathcal{F}$, so $\mathcal{F}h_n = \lambda_n h_n$, and compute the eigenvalues λ_n. (Hint: Think about the generating function of the Hermite polynomials and use part (a).)

(c) Every function $f \in L^2(\mathbb{R}^n)$ has an expansion $f = \sum_{n=0}^{\infty} a_n h_n$ in terms of Hermite functions. Write down the expansion of the Fourier transform $\hat{f}$ in terms of Hermite functions and in terms of the coefficients a_n .

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Problem Sheet 3: "Ordinary differential equations and special functions"

Andre Lukas, MT 2018

1) (Green function)
Consider the differential operator $T = \frac{d^2}{dx^2} + 2c\frac{d}{dx} + 1$, where $0 < c < 1$.

a) Determine the value of ω for which $y_1(x) = e^{-cx}\cos(\omega x)$ and $y_2(x) = e^{-cx}\sin(\omega x)$ solve the homogeneous equation $Ty = 0$.

(b) Show that the solutions y_1 and y_2 are linearly independent by computing the Wronski determinant $W = y_1 y_2' - y_1' y_2$.

(c) Find the Green function $G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$ for the operator T.

(d) Write down the general solution for the inhomogeneous equation $Ty = f$, where f is a given function, in terms of the Green function.

2) (Hermite polynomials again)
Consider the space $L^2_w(\mathbb{R})$ with weight function $w(x) = e^{-x^2}$. The Hermite polynomials $H_n \in L^2_w(\mathbb{R})$ can be defined by the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \ . \tag{1}$$

(a) Show that the H_n form an orthogonal system on $L^2_w(\mathbb{R})$ with $\langle H_n, H_m\rangle = \sqrt{\pi} 2^n n!\, \delta_{nm}$.

(b) Show that the Hermite polynomials satisfy the recursion relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ and $H_n' = 2nH_{n-1}$.

(c) Show that H_n satisfies the differential equation $y'' - 2xy' + 2ny = 0$.

(d) Solve the differential equation from part (c) with a power series Ansatz.

(e) Using reduction of order, show that the second solution to the Hermite differential equation for $n = 1$ (in addition to $H_1(x) = 2x$) is given by

$$\tilde{y}(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2(2k-1)k!} \ . \tag{2}$$

Verify that the above solution $\tilde{y}$ is consistent with the recursion relation for the coefficients obtained in part (d).

3) (Hermitian and unitary operators)
Let V be the vector space of infinitely many times differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with $x^n f(x) \to 0$ for $x \to \pm\infty$ and any $n \geq 0$. On V we have the usual scalar product

$$\langle f, g\rangle = \int_{\mathbb{R}} dx\, f(x)^* g(x) \ . \tag{3}$$

(a) Let $T : V \to V$ and $S : V \to V$ be two hermitian operators. Show that $T \circ S$ is hermitian iff $[T, S] = 0$.

(b) For $T : V \to V$ anti-hermitian, show that iT is hermitian. Show that every linear operator $T : V \to V$ can be written as a sum of a hermitian and an anti-hermitian operator.

(c) Which of the following operators are hermitian? (i) $i\frac{d}{dx}$, (ii) x^k, (iii) $\frac{d}{dx} + x^2$, (iv) $i\frac{d^3}{dx^3}$, (v) $ix\frac{d}{dx}$. Write the ones which are not hermitian as a sum of a hermitian and an anti-hermitian operator.

(d) Show that eigenvalues λ of a unitary operator $U : V \to V$ must satisfy $|\lambda| = 1$.

(e) Show that the translation operator $T_a : V \to V$ defined by $T_a(f)(x) := f(x-a)$ is unitary.

4) (Quantum harmonic oscillator)
Consider the operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{d\xi^2} + \frac{1}{2}m\omega^2\xi^2 \,, \tag{4}$$

associated to the quantum harmonic oscillator, where $\xi \in \mathbb{R}$. We would like to solve the eigenvalue problem $H\Psi = E\Psi$.

(a) Introduce the new coordinate $x = \sqrt{\frac{m\omega}{\hbar}}\xi$ and $\epsilon = \frac{E}{\hbar\omega}$ and show that the equation $H\Psi = E\Psi$ can be re-written as

$$\mathcal{H}\psi = \epsilon\psi \,, \qquad \mathcal{H} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) \,. \tag{5}$$

Comment on the practical and physical significance of this re-writing.

(b) Write $\psi(x) = y(x)e^{-x^2/2}$. Show that ψ satisfies the differential equation (5) iff y satisfies the Hermite differential equation $y'' - 2xy' + (2\epsilon - 1)y = 0$.

(c) Show that the solutions to equation (5) in $L^2(\mathbb{R})$ are given by the hermite functions $h_n(x) = H_n(x)e^{-x^2/2}/A_n$ with $\epsilon = n + \frac{1}{2}$. (Here, $A_n = \pi^{1/4}2^{n/2}\sqrt{n!}$ is a normalisation factor such that $\| h_n \| = 1$.)

(d) Define the operators $a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$, $a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$ and $N = a^\dagger a$ and show that $\mathcal{H} = N + \frac{1}{2}$.

(e) Define $|n\rangle := |h_n\rangle$ and show that $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $a|n\rangle = \sqrt{n}|n-1\rangle$ and $N|n\rangle = n|n\rangle$. (Hint: Use the relations for Hermite polynomials from question 2(b).) Hence, verify that $\mathcal{H}|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle$.

5) (Sturm-Liouville operators)

(a) Show that every second order differential operator

$$T = \alpha_2(x)\frac{d^2}{dx^2} + \alpha_1(x)\frac{d}{dx} + \alpha_0(x) \tag{6}$$

where $\alpha_2(x) \neq 0$ can be written in Sturm-Liouville form.

(b) Write the Legendre, Laguerre and Hermite differential equations as a Sturm-Liouville eigenvalue problem, $T_{\mathrm{SL}}y = \lambda y$, and find the explicit form of T_{SL} in each case.

(c) For the cases in part (b), explain what the relevant inner product vector spaces are and show that the Sturm-Liouville operators T_{SL} are hermitian.

6) (Bessel functions)
The Bessel differential equation is given by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0\ . \tag{7}$$

(a) Solve this equation with a power series Ansatz of the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}$, where $\alpha = \pm\nu$. Show that all a_k with k odd must vanish and find a recursion relation for the a_k with k even.

(b) Solve the recursion relation to find a formula for a_{2k} in terms of a_0 and write down the complete series solutions $J_{\pm\nu}$ for $\alpha = \pm\nu$, choosing $a_0 = (2^{\alpha}\Gamma(\alpha+1))^{-1}$.

(c) Show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$.

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Problem Sheet 4: "Partial differential equations"

Andre Lukas, MT 2018

1) (Laplace equation in two dimensions)

(a) Consider complex coordinates $z = x + iy$ and the 45 degree "cake slice" $\mathcal{V} = \{z \in \mathbb{C} \,|\, \arg(z) \in [0, \pi/4]\,,\ |z| \in [0, a]\}$ in the complex plane. Using complex methods, find solutions ϕ to the two-dimensional Laplace equation on $\mathcal{V}$ which satisfy the boundary conditions $\phi|_{\arg(z)=0} = \phi|_{\arg(z)=\pi/4} = 0$.

(b) From the solutions found in part (a), select the one which, in addition, satisfies the boundary condition $\phi_{|z|=a} = h$, where $h : [0, \pi/4] \to \mathbb{R}$ is a given function.

(c) Show that the most general solution to the two-dimensional Laplace equation in polar coordinates can be written as

$$\phi(r, \varphi) = \frac{a_0}{2} + \frac{\tilde{a}_0}{2} \ln r + \sum_{k=1}^{\infty} (a_k r^k + \tilde{a}_k r^{-k}) \cos(k\varphi) + \sum_{k=1}^{\infty} (b_k r^k + \tilde{b}_k r^{-k}) \sin(k\varphi)\ . \quad (1)$$

(d) Consider the annulus $\mathcal{V} = \{(r, \varphi) \,|\, a_- \leq r \leq a_+\}$ in the two-dimensional plane and find the solution ϕ to the two-dimensional Laplace equation with boundary conditions $\phi|_{r=a_\pm} = g_\pm$ where $a_+ > a_- > 0$ and $g_\pm : [0, 2\pi] \to \mathbb{R}$ are two given functions.

2) (Laplace equation in three dimensions)

(a) Consider the region $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| \geq b\}$ outside a ball of radius b and a charge distribution ρ in $\mathcal{V}$ given by a point charge q located at $\mathbf{y} \in \mathcal{V}$. Using the method of mirror charge, solve the Laplace equation $\Delta\phi = -4\pi\rho$ in $\mathcal{V}$, subject to the boundary condition $\phi_{|\mathbf{x}|=b} = 0$. Generalise to the case where ρ corresponds to a number of point charges $q_1, \ldots, q_n$ located at $\mathbf{y}_1, \ldots \mathbf{y}_n \in \mathcal{V}$.

(b) Using properties of the spherical harmonics Y_{lm}, show that the most general solution to the Laplace equation $\Delta\phi = 0$ in spherical coordinates can be written as

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta, \varphi)\ . \quad (2)$$

For cases where ϕ is independent of φ, write the above expression in terms of Legendre polynomials.

(c) Find the solution to the homogeneous Laplace equation inside a ball $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| \leq a\}$ of radius a, given that the boundary condition is $\phi(a, \theta, \varphi) = \phi_0(1 + \cos\theta)$, where ϕ_0 is a constant.

(d) For the region $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x} \geq b\}$ outside a ball of radius b find the solution to the homogeneous Laplace equation which satisfies $\phi(b, \theta, \varphi) = \phi_0 \sin^2\theta$ and approaches zero as $r \to \infty$.

3) (Multipole expansion)

(a) Explain why the solution to $\Delta\phi = -4\pi\rho$ which goes to zero at infinity can be written as

$$\phi(\mathbf{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm}}{2l+1} \frac{Y_l^m(\mathbf{n})}{r^{l+1}} \,, \qquad q_{lm} = \int_{\mathbb{R}^3} Y_l^m(\mathbf{n}')^* (r')^l \rho(\mathbf{r}')\, d^3r' \,. \tag{3}$$

(b) Find explicitly the spherical harmonics Y_0^0, Y_1^0 and $Y_1^{\pm 1}$.

(c) Use the results from (a) and (b) to write the two lowest terms in the above expansion (that is the monopole term for $l = 0$ and the dipole term for $l = 1$) in terms of Cartesian coordinates. To do this, introduce the total charge $Q = \int_{\mathbb{R}^3} \rho(\mathbf{r}') d^3r'$ and the dipole moment $\mathbf{p} = \int_{\mathbb{R}^3} \mathbf{r}' \rho(\mathbf{r}') d^3r'$.

4) (Strings)
Consider the wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \psi = 0 \,, \tag{4}$$

for the string with $x \in [0, a]$ and Dirichlet boundary conditions $\psi(t, 0) = \psi(t, a) = 0$.

(a) Write down the most general solution for this Dirichlet string using the sine Fourier series.

(b) Find the solution with initial condition $\psi(0, x) = \psi_0 \sin\left(\frac{\pi x}{a}\right)$ and $\dot{\psi}(0, x) = 0$, where ψ_0 is a constant.

(c) Find the solution with initial condition $\psi(0, x) = 0$ and $\dot{\psi}(0, x) = \psi_0 \sin\left(\frac{2\pi x}{a}\right)$.

(d) Find the solution with initial condition $\dot{\psi}(0, x) = 0$ and

$$\psi(0, x) = \begin{cases} \frac{hx}{b} & \text{for} \quad 0 \leq x \leq b \\ \frac{h(a-x)}{a-b} & \text{for} \quad b < x \leq a \end{cases} \tag{5}$$

where b and h are constants. Plot the amplitude b_k/h of the k^{th} eigenmode as a function of $b/a \in [0.1]$ for $k = 1, 2, 3, 4$ and discuss.

5) (Membranes)
Consider the membrane

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2} \right) \psi = 0 \,, \tag{6}$$

an the rectangular spatial patch $\mathcal{V} = [0, a] \times [0, b]$ with Dirichlet boundary conditions $\psi(t, 0, y) = \psi(t, a, y) = \psi(t, x, 0) = \psi(t, x, b) = 0$.

(a) Write down the most general solution for the rectangular membrane and find its spectrum of frequencies. What is the ratio of the two lowest eigenfrequencies for the square membrane?

(b) Find the most general solution to a triangular membrane with Dirichlet boundary conditions along all sides, by constraining the general solution of the square membrane along the diagonal $x = y$. What is the spectrum of the triangular membrane?

6) (Eigenfunctions of the Laplacian)
Consider the eigenvalue problem

$$-\Delta\psi = E\psi \tag{7}$$

in three dimensions and with boundary condition $\psi|_{|\mathbf{x}|=a} = 0$. (In quantum mechanics, this correspond to finding the energy eigenstates of a particle in an infinite spherical well.)

(a) Write the three-dimensional Laplacian in terms of a radial coordinate r and the Laplacian Δ_{S^2} on the two-sphere and start by expanding $\psi(r, \theta, \varphi) = \sum_{l.m} R_{l,m}(r)Y_{lm}(\theta, \varphi)$, where R_{lm} are functions to be determined. Show that the R_{lm} must satisfy the differential equation $r^2y'' + 2ry' + (Er^2 - l(l + 1))y = 0$.

(b) Introduce a new coordinate $\rho = \sqrt{E}r$ and define $\tilde{y} = \sqrt{\rho}y$. Show that, in terms of these new quantities, the differential equation from part (a) becomes a Bessel differential equation $\rho^2\tilde{y}'' + \rho\tilde{y}' + (\rho^2 - (l+1/2)^2)\tilde{y} = 0$ (where the prime now denotes a ρ derivative).

(c) Find the solutions for R_{lm} and impose the boundary condition $R_{lm}(a) = 0$. Which eigenvalues E do you find?

7 (Heat equation)
Consider the heat equation

$$\kappa\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi}{\partial t}\ , \tag{8}$$

where κ is a positive constant, $x \in [0, a]$ and ψ is real-valued. For $t > 0$, we demand Dirichlet boundary conditions $\psi(t, 0) = \psi_0$ at $x = 0$ (where ψ_0 is a constant) and von Neumann boundary conditions $\frac{\partial\psi}{\partial x}(t, a) = 0$ at $x = a$. The initial condition is $\psi(0, x) = 0$.

(a) As a preparation, we introduce the functions $g_k : [0, a] \to \mathbb{R}$ defined by

$$g_k(x) = \sqrt{\frac{2}{a}}\sin(q_k x)\ , \qquad q_k = \frac{\pi}{a}\left(k + \frac{1}{2}\right) \tag{9}$$

where $k = 0, 1, \ldots$. Show that these functions satisfy the boundary conditions $g_k(0) = 0$ and $g_k'(a) = 0$, that they form an ortho-normal system relative to the standard scalar product on $L^2([0, a])$ and that they are eigenfunctions of $\frac{d^2}{dx^2}$ with

$$\frac{d^2}{dx^2}g_k = -q_k^2 g_k\ . \tag{10}$$

(b) Argue that the functions g_k from part (a) form, in fact, an ortho-normal basis of the space $L^2_b([0,a])$, of square integrable functions f on $[0,a]$ with a Dirichlet boundary condition $f(0)=0$ at $x=0$ and a von Neumann condition $f'(a)=0$ at $x=a$. (Hint: Think about the sine Fourier series on $[0,2a]$.)

(c) Based on the results in (a) and (b) argue that the most general ψ with the correct boundary conditions can be written as $\psi(t,x)=\psi_0+\sum_{k=0}^{\infty} T_k(t)g_k(x)$. Find the solutions for the functions T_k.

(d) Fix the remaining constants in your solution by imposing the initial condition. Compute the average value $\bar{\psi}(t)$ of $\psi(x,t)$ by averaging over $x\in[0,a]$ and find an approximate equation for the time as a function of $r:=(\psi_0-\bar{\psi}(t))/\psi_0$.