Vectors and Matrices, Problem Set 1

Vectors, vector spaces and geometry

Prof Andre Lukas, MT2013

Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Which of the following sets of vectors are linearly independent? For each linearly dependent set, identify a maximal subset of linearly independent vectors. Provide detailed reasoning in each case.

(a) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (b) $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$ (d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix}$

2. Which of the following sets constitute sub vector spaces of the given vector space? Provide reasoning in each case.

(a) All three-dimensional vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ satisfying $x = y = 2z$.

(b) All two-dimensional vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$.

(c) Within the vector space of real functions $f : \mathbb{R} \to \mathbb{R}$, the subset of even functions, that is, the functions satisfying $f(x) = f(-x)$.

3. Consider the vector space of 3×3 matrices A with real entries A_{ij}. What is the dimension of this vector space?

(a) Show that the subset of symmetric 3×3 matrices, that is, matrices satisfying $A_{ij} = A_{ji}$, forms a sub vector space. Write down an explicit basis for this sub vector space. What is its dimension?

(b) Do the same for the subset of 3×3 anti-symmetric matrices, that is, matrices satisfying $A_{ij} = -A_{ij}$.

(c) Generalize the results from (a) and (b) to $n \times n$ matrices with real entries.

4. Show that the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^3$. Write a general vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of this basis. What are the coordinates of $\mathbf{v}$ relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$?

5.* Consider a real vector space V and two sub vector spaces U and W of V.

(a) Show that the intersection $U \cap W$ is a sub vector space of V.

(b) Show that $U + W$ (the set of all sums $\mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$) is a sub vector space of V.

(c) The dimensions of the above vector spaces are related by $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. Verify this formula for the specific example where $V = \mathbb{R}^3$, U is spanned by $\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j}$, $\mathbf{u}_2 = \mathbf{k}$ and W is spanned by $\mathbf{w}_1 = \mathbf{j} + \mathbf{k}$, $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$.

(If you are ambitious, try to prove the dimension formula in (c) in general. Start by writing down a basis for $U \cap W$ and complete this to a basis of U and W, respectively.)

6. For three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ prove the following relations:

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$,

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$,

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.

7. For three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, prove the following:

(a) $\mathbf{a} \times \mathbf{b} = \mathbf{a} - \mathbf{b}$ implies that $\mathbf{a} = \mathbf{b}$,

(b) $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ implies that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$,

(c) if $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, this implies that $\mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = \pm|\mathbf{c}| \cdot |\mathbf{a} - \mathbf{b}|$,

(d) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})]$.

8. The three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form the sides of a triangle.

(a) Show that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}|$.

(b) Find the area of the triangle with vertices at $P = (1, 3, 2)$, $Q = (2, -1, 1)$ and $R = (-1, 2, 3)$.

9. (a) Prove that the three vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ are coplanar.

(b) For $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, find a vector which is coplanar with $\mathbf{a}$ and $\mathbf{b}$, but perpendicular to $\mathbf{a}$.

10.* The three-dimensional vectors $\mathbf{v}_i$, where $i = 1, 2, 3$ are not co-planar with triple product $V = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$. Define the *reciprocal vectors* $\mathbf{v}'_i$ by $\mathbf{v}'_1 = \frac{1}{V}\mathbf{v}_2 \times \mathbf{v}_3$, $\mathbf{v}'_2 = \frac{1}{V}\mathbf{v}_3 \times \mathbf{v}_1$ and $\mathbf{v}'_3 = \frac{1}{V}\mathbf{v}_1 \times \mathbf{v}_2$.

(a) Show that $\mathbf{v}_i \cdot \mathbf{v}'_j = \delta_{ij}$ and, hence, that the coordinates of a vector $\mathbf{w}$ relative to the basis $\mathbf{v}_i$ are given by $\mathbf{w} \cdot \mathbf{v}'_i$.

(b) Show that the triple product $V' = \mathbf{v}'_1 \cdot (\mathbf{v}'_2 \times \mathbf{v}'_3)$ of the reciprocal vectors equals $1/V$.

(c) Show that taking the reciprocal of the reciprocal leads back to the original vectors, that is, show that $\frac{1}{V'}\mathbf{v}'_2 \times \mathbf{v}'_3 = \mathbf{v}_1$, $\frac{1}{V'}\mathbf{v}'_3 \times \mathbf{v}'_1 = \mathbf{v}_2$ and $\frac{1}{V'}\mathbf{v}'_1 \times \mathbf{v}'_2 = \mathbf{v}_3$.

11. What is the shortest distance of $\mathbf{p} = (2, 3, 4)$ from the x-axis?

12. Write down the vector equation of the line

$$\frac{(x-2)}{4} = \frac{(y-1)}{3} = \frac{(z-5)}{2}$$

and find the minimum distance of this line from the origin.

13. Derive an expression for the shortest distance between the two lines $\mathbf{r}_i = \mathbf{q}_i + \lambda_i\mathbf{m}_i$, where $i = 1, 2$. Hence find the shortest distance between the lines

$$\frac{(x-2)}{2} = (y-3) = \frac{(z+1)}{2} \quad \text{and} \quad (x+2) = \frac{(y+1)}{2} = (z-1)\,.$$

14. Find the Cartesian equation for the plane passing through $P_1 = (2, -1, 1)$, $P_2 = (3, 2, -1)$ and $P_3 = (-1, 3, 2)$.

15. Find the Cartesian and vector description of the plane which contains the three position vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

16. A line goes through the origin and the point $P = (1, 1, 1)$; a plane goes through points $A = (-1, 1, -2)$, $B = (1, 5, -5)$, $C = (0, 2, -3)$. Find the intersection point of the plane and the line.

Vectors and Matrices, Problem Set 2

Matrices, linear maps and linear equations

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Un-starred questions indicate standard problems students should be familiar with.
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1. For matrices A, B, C of compatible sizes and scalars α show that
(a) matrix multiplication is associate, so $A(BC) = (AB)C$,
(b) $(A+B)^T = A^T + B^T$, $(\alpha A)^T = \alpha A^T$ and $(AB)^T = B^T A^T$,
(c) $(A+B)^\dagger = A^\dagger + B^\dagger$, $(\alpha A)^\dagger = \alpha^* A^\dagger$ and $(AB)^\dagger = B^\dagger A^\dagger$,
(d) $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^T)^{-1} = (A^{-1})^T$ provided A, B are invertible.

2. Using row operations, work out the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 3 & 2 & 0 & -4 \\ 1 & -2 & a & 0 \end{pmatrix},$$

where a is a real parameter. (Keep in mind that the rank of A depends on the value of a so a case distinction is required.)

3. Using row operations, find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & -3 & 0 \end{pmatrix}.$$

Check your result!

4. A linear map in $\mathbb{R}^2$ is given by the action of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(a) Work out the matrix A' which represents this linear map in the basis

$$\mathbf{e}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}'_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) Find a 2×2 matrix P such that $A' = PAP^{-1}$.
(c) What is the interpretation of the matrix P?

5.* For a fixed unit vector $\mathbf{n}$ in $\mathbb{R}^3$ define the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\mathbf{v}) = \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n}$.
(a) Show that f is linear and that $f(f(\mathbf{v})) = \mathbf{v}$ for all vectors $\mathbf{v}$.
(b) Work out the matrix A describing f relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of standard unit vectors and show that $A^2 = \mathbf{1}$.
(c) Work out the matrix $\hat{A}$ describing f relative to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$, where $\mathbf{u}_a$, $a = 1, 2$, are two linearly independent vectors perpendicular to $\mathbf{n}$, so $\mathbf{u}_a \cdot \mathbf{n} = 0$.
(d) Discuss your results. What is the geometrical interpretation of f?

6. For a quadractic matrix A, the trace is defined as the sum of its diagonal elements, so $\mathrm{tr}(A) = \sum_i A_{ii}$.

(a) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Show that $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A)$. Discuss this result in relation to basis change of matrices.

(c)* What is the trace of the linear map f defined in the previous question?

7.* Consider the vector space consisting of all quadratic polynomials $p(x) = ax^2 + bx + c$, where a, b, c are arbitrary real coefficients. On this space a map is defined by

$$D(p) = x\frac{d^2p}{dx^2} + (1-x)\frac{dp}{dx} + 2p\,.$$

(a) Why is this map linear?

(b) Work out the matrix, A, which represents D for the standard monomial basis $\{1, x, x^2\}$.

(c) Find the kernel of the matrix A, that is, the vectors $\mathbf{v}$ satisfying $A\mathbf{v} = 0$. Which polynomials p correspond to these vectors?

(d) Show by explicitly applying D that the polynomials p found in part (c) satisfy $D(p) = 0$.

8. Analyze the linear system of equations

$$\begin{aligned} (2-\lambda)x + y + 2z &= 0 \\ x + (4-\lambda)y - z &= 0 \\ 2x - y + (2-\lambda)z &= 0\,, \end{aligned}$$

where λ is an arbitrary real parameter.

(a) What is the rank of the associated coefficient matrix, A, depending on the value of λ?

(b) Based on the rank result in (a) what is your expectation for the qualitative structure of the solution.

(c) Confirm your expectation by an explicit calculation.

9. The linear system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= \eta \\ x + 4y + 10z &= \eta^2 \end{aligned}$$

depends on the real parameter η.

(a) Show that the rank of the coefficient matrix is two. What does this imply for the qualitative structure of the solution?

(b) Explicitly solve the system for the cases where a solution exists.

10. Solve the linear system

$$\begin{aligned} 3x + 2y - z &= 10 \\ 5x - y - 4z &= 17 \\ x + 5y + \alpha z &= \beta\,, \end{aligned}$$

where α, β are arbitrary real parameters using row reduction on the augmented matrix.

11.* A semi-magic square is a 3×3 matrix of (rational) numbers such that all rows and columns sum up to the same total. A simple example is

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} ,$$

where all rows and columns sum up to 6.

(a) Why do the semi-magic squares form a vector space?

(b) Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

form a basis of the semi-magic squares. (Therefore, the dimension of this space is five and all semi-magic squares can be written as linear combinations of the above five matrices.)

12.* Consider an "internet" with n sites, labeled by $i = 1, \ldots, n$. Each site i contains n_i links to some of the other sites and it is linked to by the pages $L_i \subset \{1, \ldots, n\}$. The page rank x_i of each site i is defined by

$$x_i = \sum_{j \in L_i} \frac{x_j}{n_j} \, . \tag{1}$$

(a) Show that Eq. (1) can be written in matrix form as $A\mathbf{x} = 0$, where $\mathbf{x}$ is the vector with components x_i, and determine the components A_{ij} of the matrix A.

(b) Show a non-trivial solution (that is, a solution $\mathbf{x} \neq 0$) always exists.

(c) Analyze a simple example with $n = 4$ and the following structure of links:

$$\begin{array}{ccc} \mathbf{1} & \rightleftarrows & \mathbf{2} \\ \uparrow & \searrow & \downarrow \\ \mathbf{3} & \longleftarrow & \mathbf{4} \end{array}$$

Write down the page rank equations (1) for this case and solve them. Which page is ranked highest?

Vectors and Matrices, Problem Set 3

Scalar products and determinants

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Un-starred questions indicate standard problems students should be familiar with.
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1. Calculate the determinants of the matrices

$$(a)\ A = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \quad (b)\ B = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Are the matrices (i) real, (ii) diagonal, (iii) symmetric, (iv) antisymmetric, (v) singular, (vi) orthogonal, (vii) Hermitian, (viii) anti-Hermitian, (ix) unitary?

2. Use the Gram-Schmidt procedure to find an ortho-normal basis of $\mathbb{R}^3$ (with the standard scalar product), starting with the basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

Check your result.

3. Let A be an $n \times n$ matrix.

(a) Why are the statements "A is invertible" and $\det(A) \neq 0$ equivalent?

(b) For which values of the parameters a, b is the matrix

$$A = \begin{pmatrix} a & 1 & a \\ 1 & b & -1 \\ 0 & -1 & a \end{pmatrix}$$

not invertible?

4.* In $\mathbb{R}^n$, we have $n-1$ linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$. Define the vector $\mathbf{w}$ with components $w_i = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{e}_i)$, where $\mathbf{e}_i$ are the standard unit vectors.

(a) Show that $\mathbf{w}$ is perpendicular (with respect to the standard scalar product in $\mathbb{R}^n$) to all vectors $\mathbf{v}_a$, where $a = 1, \ldots, n-1$.

(b) Show that $|\mathbf{w}| = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{n})$, where $\mathbf{n} = \mathbf{w}/|\mathbf{w}|$.

(c) Work out the vector $\mathbf{w}$ if $\mathbf{v}_a = \mathbf{e}_a$, for $a = 1, \ldots, n-1$.

(d) Show that in three dimensions $\mathbf{w}$ can be written in terms of a cross product.

5. Solve the following system of linear equations

$$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 4y + 5z &= 4 \\ x + 3y + 4z &= 6 \end{aligned}$$

by

(a) calculating the matrix inverse

(b) Cramer's method

(c) row reduction on the augmented matrix.

If you had to write a computer program solving systems of linear equations (of arbitrary and possibly large size) which of the above methods would you base it on?

6.* On the vector space of polynomials $f : \mathbb{R} \to \mathbb{R}$ we define $\langle f, \mathrm{g}\rangle = \int_{-\infty}^{\infty} dx\, e^{-x^2} f(x)g(x)$.

(a) Why does this define a scalar product?

(b) Consider the polynomials $p_0(x) = 1/n_0$, $p_1(x) = 2x/n_1$ and $p_2(x) = (4x^2 - 2)/n_2$, where the n_a are real numbers. Show that these polynomials are orthogonal under the above scalar product.

(c) Determine the numbers n_a such that the polynomials p_a are normalized to one, so $\langle p_a, p_a\rangle = 1$.

(Hint: You can use that $\int_{-\infty}^{\infty} dx\, x^n e^{-x^2} = 0$ for n odd (why?) and $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$, $\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2} = \sqrt{\pi}/2$, $\int_{-\infty}^{\infty} dx\, x^4 e^{-x^2} = 3\sqrt{\pi}/4$.)

7. The vector space V is equipped with a hermitian scalar product $\langle \cdot, \cdot \rangle$ and an ortho-normal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

(a) Show that non-zero and pairwise orthogonal vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linearly independent.

(b) Show that the coordinates v_i of a vector $\mathbf{v}$, relative to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, are given by $v_i = \langle \mathbf{e}_i, \mathbf{v}\rangle$.

(c) Show that the scalar product of two vectors $\mathbf{u}$, $\mathbf{v}$ can be written as $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{e}_i\rangle\langle \mathbf{e}_i, \mathbf{v}\rangle$.

(d) A second ortho-normal basis $\mathbf{e}'_1, \ldots, \mathbf{e}'_n$ is related to the first one by $\mathbf{e}'_j = \sum_i U_{ij}\mathbf{e}_i$, where U_{ij} are complex numbers. Show that $U_{ij} = \langle \mathbf{e}_i, \mathbf{e}'_j\rangle$ and that the matrix U with entries U_{ij} is unitary.

8. Consider $\mathbb{R}^n$ with the standard scalar product $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T\mathbf{w}$ and orthogonal matrices R, that is, matrices satisfying $\langle R\mathbf{v}, R\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all vectors $\mathbf{v}$, $\mathbf{w}$.

(a) Show that orthogonal matrices can, alternatively, also be characterized by the equation $R^T R = \mathbf{1}$ and that $\det(R) = \pm 1$.

(b) Focus on $n = 2$ and show that two-dimensional orthogonal matrices can be written in the form

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \mp\sin\varphi \\ \sin\varphi & \pm\cos\varphi \end{pmatrix} .$$

Which of these matrices correspond to two-dimensional rotations? What is the interpretation of the other matrices?

(c) For two-dimensional rotations, show that $R(\varphi_1)R(\varphi_2) = R(\varphi_1 + \varphi_2)$.

(d) The vectors $\mathbf{x} = (x, y)^T$ and $\mathbf{x}' = (x', y')^T$ are related by a rotation, so $\mathbf{x}' = R(\varphi)\mathbf{x}$. What is the relation between the two associated complex numbers $z = x + iy$ and $z' = x' + iy'$?

9.* (a) A three-dimensional rotation, R_3, around the z-axis can be constructed from the previous two-dimensional rotations by setting

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and analogously for three-dimensional rotations R_1 and R_2 around the x and y axis. Construct a general three-dimensional rotation by combining three such rotations, that is, work out $R = R_1(\alpha_1)R_2(-\alpha_2)R_3(\alpha_3)$ for three angles α_1, α_2 and α_3.

(b) Show that for small angles α_i the three-dimensional rotations from part (a) can be written as $R = \mathbf{1}_3 + \sum_{i=1}^{3} \alpha_i T_i + \cdots$, where the dots stand for terms of quadratic or higher order in the angles, and determine the (angle-independent) matrices T_1, T_2, T_3.

(c) The change of a vector $\mathbf{x}$ under a rotation R is given by $\delta\mathbf{x} = R\mathbf{x} - \mathbf{x}$. Show that for small-angle rotations this can be written as $\delta\mathbf{x} = \boldsymbol{\alpha} \times \mathbf{x} + \cdots$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$ and the dots stand for quadratic or higher order terms in the angles.

10.* A bi-linear form on $\mathbb{R}^2$ is defined by $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \eta \mathbf{w}$, where $\eta = \mathrm{diag}(-1, 1)$.

(a) Why is this bi-linear form not a scalar product?

(b) Consider the 2×2 matrices Λ which leave the above bi-linear form invariant, that is, which satisfy $\langle \Lambda\mathbf{v}, \Lambda\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all vectors $\mathbf{v}$, $\mathbf{w}$. Show that these matrices can also be characterized by the equation $\Lambda^T \eta \Lambda = \eta$ and that $\det(\Lambda) = \pm 1$.

(c) Show that the matrices Λ with $\det(\Lambda) = 1$ and $\Lambda_{11} > 0$ can be written in the form

$$\Lambda(\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} ,$$

where ξ is a real parameter. Re-write Λ in terms of the parameter β, defined as $\beta = \tanh\xi$.

(d) Verify that $\Lambda(\xi_1)\Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2)$. What does this rule imply for the parameter β, that is, if $\Lambda(\beta_1)\Lambda(\beta_2) = \Lambda(\beta)$, how does β depend on β_1 and β_2?

Vectors and Matrices, Problem Set 4

Eigenvectors, eigenvalues and diagonalization

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Un-starred questions indicate standard problems students should be familiar with.
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1. Find the eigenvalues and a set of normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Verify that its eigenvectors are mutually orthogonal and construct an orthogonal matrix R such that $R^T M R$ is diagonal.

2. Find the eigenvalues and normalized eigenvectors of the Hermitian matrix

$$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}.$$

and construct a unitary matrix U such that $U^\dagger H U$ is diagonal. (Remember to use the standard *hermitian* scalar product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\dagger \mathbf{w}$.)

3. A curve in two-dimensional space is defined by all $\mathbf{x} = (x, y)^T$ which solve the equation $x^2 + 3y^2 - 2xy = 1$.

(a) Show that this equation can be written in the form $\mathbf{x}^T A \mathbf{x} = 1$ and determine the matrix A.

(b) By diagonalizing the matrix A, show that this curve is an ellipse and determine the length of its two axis.

4. (a) Diagonalize the two-dimensional rotation

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

over the complex numbers.

(b) Why can the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

not be diagonalized?

5. A real, symmetric $n \times n$ matrix M is called positive definite if $\mathbf{v}^T M \mathbf{v} > 0$ for all vectors $\mathbf{v}$, negative definite if $-M$ is positive definite and indefinite otherwise.

(a) Show that M is positive definite iff all its eigenvalues are positive and that it is negative definite iff all its eigenvalues are negative.

(b) Focus on the two-dimensional case, $n = 2$, and denote the two eigenvalues of M by λ_1 and λ_2. Why is $\mathrm{tr}(M) = \lambda_1 + \lambda_2$ and $\det(M) = \lambda_1 \lambda_2$?

(c) For $n = 2$, formulate criteria for positive definiteness/negative definiteness/indefiniteness of M in terms of $\mathrm{tr}(M)$ and $\det(M)$.

(d) Are the following matrices positive definite, negative definite or indefinite?

$$M_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} , \quad M_2 = \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix} , \quad M_3 = \begin{pmatrix} -4 & 3 \\ 3 & -5 \end{pmatrix} .$$

6. (a) Show that the characteristic polynomial of an arbitrary 2×2 matrix A can be written as

$$\chi_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) .$$

(b) Do the same for an arbitrary 3×3 matrix A and show that

$$\chi_A(\lambda) = -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))\lambda + \det(A) .$$

(c)* For a general $n \times n$ matrix A, what are the coefficients of λ^n, λ^{n-1} and the constant term in the characteristic polynomial $\chi_A(\lambda)$.

7. Consider real, symmetric $n \times n$ matrices P which satisfy $P^2 = P$. (Such matrices are also referred to as projectors.)

(a) Show that the possible eigenvalues of P are either 0 or 1. What does the diagonalized form of P look like?

(b) Explain why such matrices are called projectors and why the dimension of the space projected onto is given by $\mathrm{tr}(P)$.

(c) For an n-dimensional unit vector $\mathbf{n}$ with components n_i let Q be the matrix with components $Q_{ij} = n_i n_j$. Show that Q is a projector and that it projects to a one-dimensional space.

(d) Show that the matrix $P = \mathbf{1} - Q$ with Q defined as in part (c) is also a projector. What is the dimension of the space it projects onto?

8.* (a) Unitary matrices U are characterized by $U^\dagger U = \mathbf{1}$. Show that eigenvalues λ of unitary matrices must be phases, that is, $|\lambda| = 1$.

(b) Consider three-dimensional rotations R. Intuitively, it is clear that R has an eigenvector, $\mathbf{n}$, with eigenvalue 1, which corresponds to the axis of rotation. Construct an argument which proves this fact, that is, show that a three-dimensional real matrix R with $R^T R = \mathbf{1}$ and $\det(R) = 1$ always has an eigenvalue 1.

(c) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$ be an ortho-normal basis of $\mathbb{R}^3$, where $\mathbf{n}$ is the eigenvector with eigenvalue 1 of a rotation R. Work out the matrix which represents the rotation in this basis.

(d) Show that the angle of rotation, φ, of a three-dimensional rotation R satisfies $\cos\varphi = (\mathrm{tr}(R) - 1)/2$.

(e) Show that the matrix

$$R = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & 0 & 3 \\ -1 & -4 & 1 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$$

is a rotation. Compute the characteristic polynomial of R and verify that 1 is an eigenvalue. Compute the axis of rotation, $\mathbf{n}$, and $\cos\varphi$, where φ is the angle of rotation.

9.* Consider the system of second order differential equations

$$\frac{d^2\mathbf{x}}{dt^2} = -M\mathbf{x}\,,$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}\,, \quad \frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \vdots \\ \frac{d^2x_n}{dt^2} \end{pmatrix}\,,$$

and M is a real, symmetric matrix.

(a) Explain how this system of differential equations can be solved by diagonalizing M.

(b) Discuss the relation between the eigenvalues of M and the qualitative behaviour of the solutions.

(c) Focus on the two-dimensional system

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -kx_1 - lx_2 \\ \frac{d^2x_2}{dt^2} &= -lx_1 - kx_2 \end{aligned}$$

where k and l are arbitrary real constants. Depending on the values of k and l, discuss the qualitative behaviour of the solutions of this system.