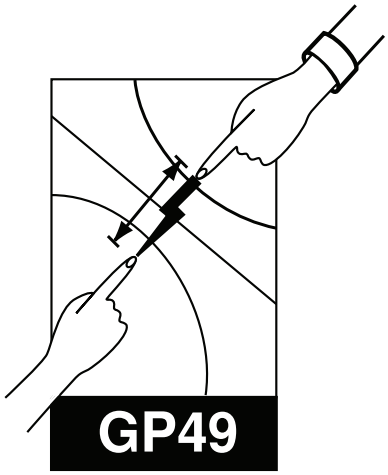
Thermal Waves in a Bar



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GP49

1 Abstract

The aim of this experiment is to study the transport of heat along a copper bar. The heat input at one end of the bar can be varied with time and the temperature profile along the bar recorded both as a function of position and of time. Solutions of the heat conduction equation appropriate to this case give predictions for the dispersion and thus the group and phase velocities.

2 Preliminary data collection

The first thing that you should do for this experiment is to perform the initial calibration (which takes approximately 30 minutes) and collect some preliminary data (which should take at least an hour). This is explained in the Measurements section below. Once your data is being logged, check through the Introduction and Theory sections again.



Remember that you are responsible for your own safety when working in the laboratory. Always talk to a demonstrator or technician if you are concerned that anything may be unsafe.

3 Introduction

Thermocouple #	Distance (mm)	Thermocouple #	Distance (mm)
1	21	8	121
2	31	9	141
3	41	10	161
4	51	11	181
5	61	12	221
6	81	13	261
7	101	-	-

Table 1: Distances to thermocouple sensors from base of apparatus

In this experiment, the variation in temperature at fixed points along a copper rod are studied (see figure 1 and table 1). The temperature is recorded with a set of thermocouples and the heat input at one end of the rod is varied by controlling the electrical power to a Peltier heat pump. By switching the power on and off (a square pulse, figure 2) the propagation of the composite Fourier components of the temperature wave can be investigated. Each frequency component travels with its own characteristic velocity and with a coefficient of attenuation which is also depends on the frequency. The higher frequencies are attenuated more rapidly so that near the end with cooling fins only the fundamental frequency remains; the temperature variation here therefore has a simple harmonic form.

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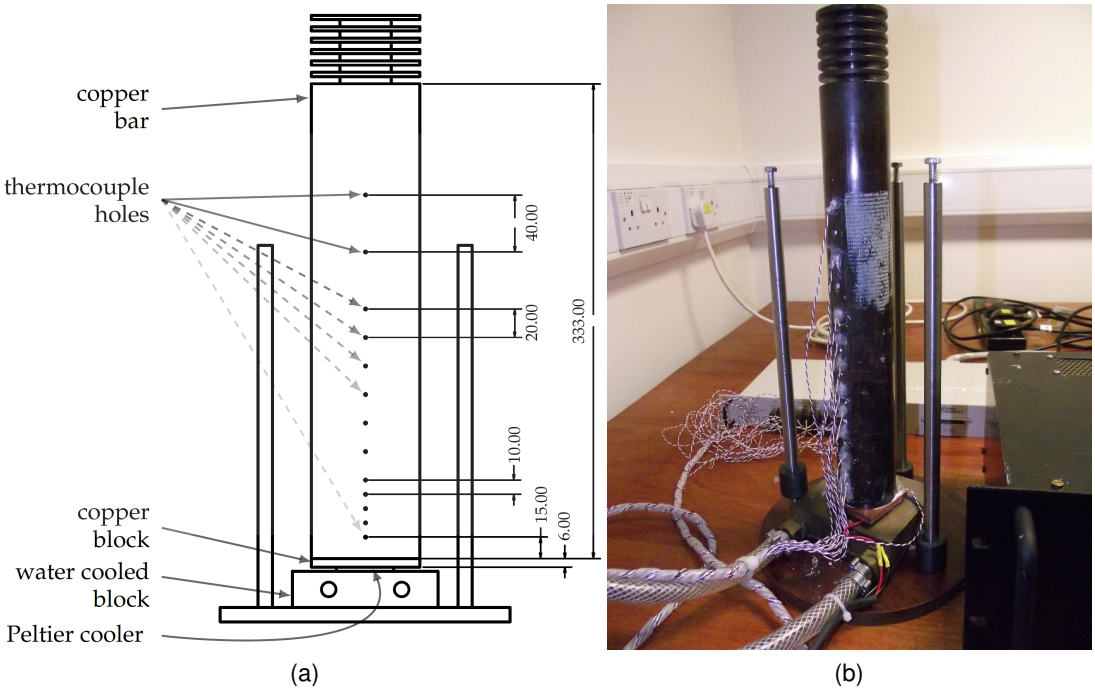


Figure 1: (a) Schematic of the apparatus. The Peltier cooler is sandwiched between the copper and water cooled blocks. Thermocouple positions and spacings are indicated in millimetres. (b) A photograph of the same.

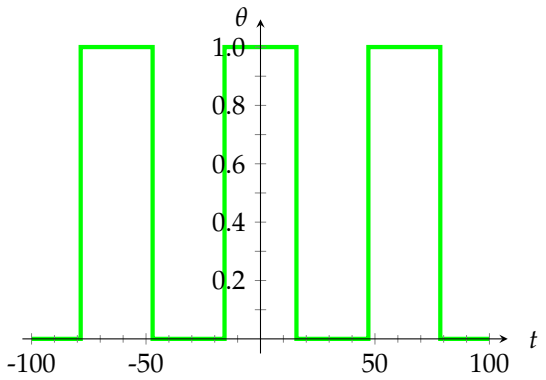


Figure 2: The variation in power applied to the copper bar.

▶ Check you have identified all the parts of the apparatus and that you understand how they connect to the logging computer and the heater driver.

4 Theoretical analysis

We start by considering an element of the rod which has its long dimension along the x-axis and which is lagged to prevent heat loss in the transverse directions (y- and z-). The heat conduction equation states that the rate of flow of heat (power) through a section is proportional to the temperature differential, $\frac{\partial T}{\partial x}$,

and to the cross-sectional area, A, i.e.

$$P(x) = -\kappa A \left(\frac{\partial T}{\partial x}\right) \tag{1}$$

where the constant of proportionality κ is the thermal conductivity. For copper this conductivity is about $401\,\mathrm{W\,m^{-1}\,K^{-1}}$. For the elemental disc then the net flow of heat is given by the heat entering one face (at say distance x from the Peltier) minus that leaving the second face (at $x + \delta x$). Thus, since

$$P(x + \delta x) = P(x) + \frac{\partial P}{\partial x}\,\delta x \tag{2}$$

we have by differentiating equation 1:

$$\frac{\partial P}{\partial x} = -\kappa A\,\frac{\partial^2 T}{\partial x^2} \tag{3}$$

This represents the heat leaving the element. With no heat loss along the lagged sides of the bar the heat balance equation then gives us:

$$C\rho A\delta x \left(\frac{\partial T}{\partial t}\right) = -\delta P = \kappa A\,\frac{\partial^2 T}{\partial x^2}\,\delta x \tag{4}$$

where ρ is the density of the metal and C its specific heat capacity. The heat flow equation is therefore finally given by:

$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial x^2} \tag{5}$$

where D, the thermal diffusivity of the metal is given by $\kappa/\rho C$.

For copper at 25 °C the heat capacity is $385.0 \pm 1.0\,\mathrm{J\,kg^{-1}\,K^{-1}}$ and the density is $8960.0 \pm 10.0\,\mathrm{kg\,m^{-3}}$.

▶ **Evaluate D and determine its units.**

4.1 Solutions to the heat flow equation

If the temperature is caused to vary sinusoidally at angular frequency ω we will need to seek a solution of equation 5 which is of the form:

$$T(x,t) = T_0 e^{-\alpha x} e^{i(\omega t - kx + \phi)} \tag{6}$$

This represents a *temperature wave* travelling down the bar in the +ve x direction with velocity ω/k. As the wave propagates down the bar its amplitude decreases exponentially. The phase angle ϕ is an arbitrary phase which can be chosen to satisfy the initial conditions. When equation 6 is differentiated twice with respect to x and once with respect to t and substituted back into equation 5 we obtain

$$i\omega T(x,t) = K(\alpha + ik)^2 T(x,t) \tag{7}$$

so that

$$\alpha^2 = k^2 \tag{8}$$

$$\alpha k = \omega/(2D) \tag{9}$$

and therefore

$$\alpha = k = \left(\frac{\omega}{2D}\right)^{1/2} \tag{10}$$

The experiment is also concerned with the velocity of propagation of the wave. Any identifiable point on the wave form (e.g. the maximum amplitude point) corresponds to a constant value of the third term on the RHS of equation 6, i.e.:

$$\omega t - kx + \phi = \text{constant} \tag{11}$$

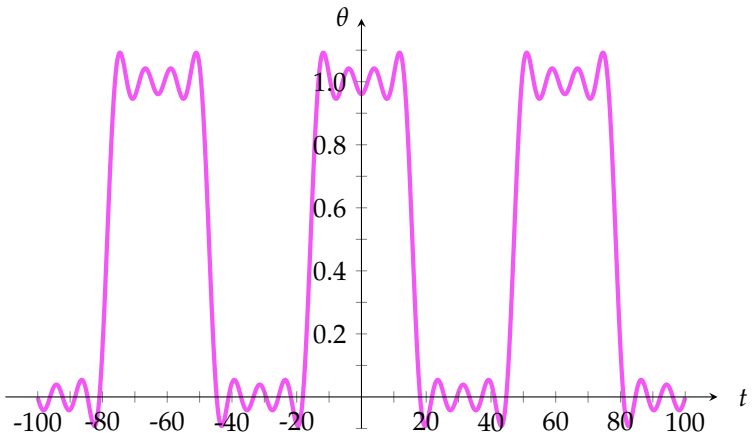


Figure 3: Fourier sum up to 7th harmonic.

When this is differentiated with respect to time we obtain:

$$\omega - k\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{12}$$

which shows that the identified point moves with velocity:

$$v = \omega/k \tag{13}$$

▶ How does the propagation velocity and the attenuation coefficient vary with frequency? The power to the Peltier is varied sinusoidally with a period of $1\,\mathrm{min}$. Find an expression for the wavelength of the temperature wave. Using this expression, show the distance over which the amplitude of the wave falls by 1/e.

4.2 Non-sinusoidal waves

If the Peltier is simply switched on and off in a regular way (i.e. a pulsed pattern) then the temperature variation at different points along the bar represents the progression of the Fourier components on the initial wave form. Thus the square wave pattern at $x = 0$ changes shape further down the bar. The appropriate solution is now a Fourier sum of harmonics, i.e.:

$$T(x,t) = \sum_{n=1}^{\infty} T_0^{(n)} e^{-\alpha_n x} e^{i(n\omega_n t - k_n x + \phi_n)} \tag{14}$$

Figure 3 illustrates this Fourier series for a square pulse using frequencies up to the seventh harmonic.

The coefficients (amplitudes) $T_0^{(n)}$ are determined by the temperature variation recorded at a particular position along the bar. The attenuation (α) is greater for the higher harmonics so that after a certain distance down the bar the waveform is almost completely sinusoidal with fundamental frequency ω.

At a distance of $0.02\,\mathrm{m}$ along the bar from the Peltier the waveform is already showing a change as the higher harmonics are progressively attenuated. At $0.1\,\mathrm{m}$ the waveform is dominated by the fundamental component, as in figure 5. Expressed as a function of both variables, the 3D plot looks like figure 6.

4.3 Effect of heat losses

Suppose that there is now a heat loss from the sides of the bar to its surroundings which is proportional to the excess temperature of the bar above its surroundings, i.e. define the surrounding temperature

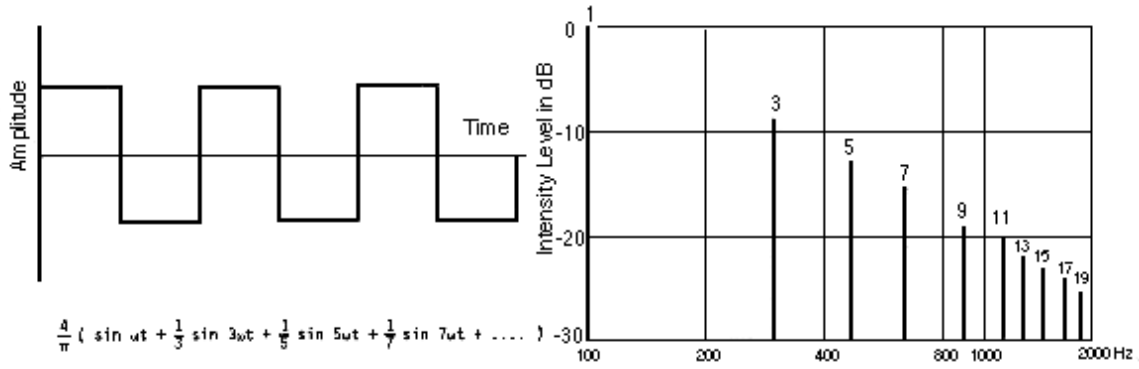


Figure 4: A square wave and its characteristic equation, along with its Fourier decomposition.

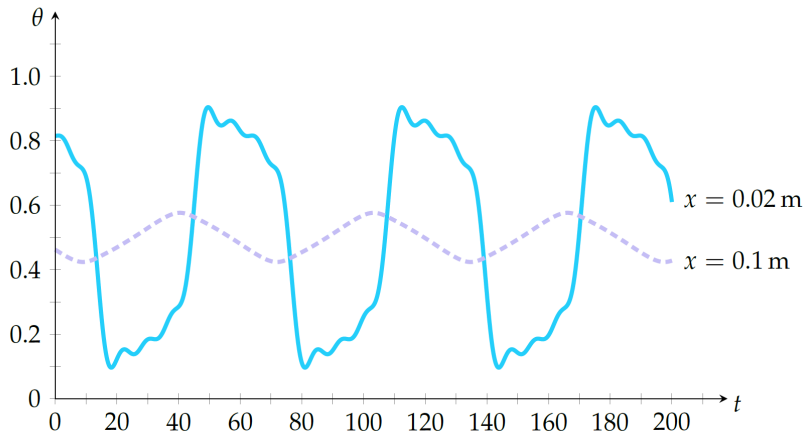


Figure 5: Waveform with inclusion of attenuation and phase shift at $x = 0.02\,\mathrm{m}$ and at $x = 0.1\,\mathrm{m}$ from heater.

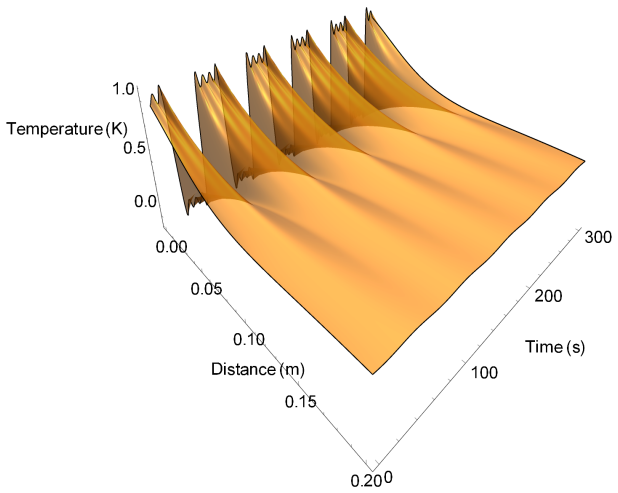


Figure 6: Temperature as a function of x and t (surface colour for illustrative purposes only; waveforms at x =0.02 m and 0.1 m illustrated).

as T_s and assume the heat loss is from a small element of the bar at temperature T is proportional to

$(T - T_s)$, then equation 5 takes the amended form:

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - L(T - T_s) \tag{15}$$

where L is a constant characterising the heat loss. We use the variable $\theta = (T - T_s)$ and try a solution of the form:

$$\theta = \theta_0 e^{-\alpha x} e^{i(\omega t - kx)} \tag{16}$$

When differentiated and substituted back into equation 15 we obtain:

$$(\alpha^2 - k^2)D = L \text{ and } \omega - 2D\alpha k = 0 \tag{17}$$

or equivalently:

$$(\alpha^2 - k^2) = \frac{L}{D} \quad \text{and} \quad \alpha k = \frac{\omega}{2D} \tag{18}$$

If $L = 0$ the equations above reduce to the values of α and k previously obtained. If $L \neq 0$ then:

$$\alpha^2 = \frac{1}{2D}\left(L \pm \sqrt{[L^2 + \omega^2]}\right) \tag{19}$$

Since α^2 is positive we must take the plus sign in the bracket. The effect of a heat loss therefore is to increase the attenuation coefficient α and to decrease k. Thus if the heat losses are ignored the value of the diffusivity calculated will be too small, and from the velocity, too large. It follows from the relationship $\alpha k = \omega/2D$ that the best value is obtained by taking the geometric mean of the two values.

▶ Derive expressions for both the phase and group velocity of the temperature wave. Which is greater? What is the product of these velocities for a light wave? Compare this with that for the temperature wave.

▶ At what depth should a trench for an underground water pipe be dug to ensure that the water never freezes in this country? Give reasons and values for the important parameters in your estimate.

5 Measurements

5.1 Initial measurement

Software is used to control the power supply for the heater and record data being read from the thermocouples.

The heater software allows you to program a variety of waveforms by modifying the shape, amplitude, period, offset, and number of cycles. It is a good idea to set the ambient temperature to be equal to the mean temperature of your waveform.

The reader software displays the current temperature being read by each thermocouple, and the thermocouples are listed according to their distance along the copper bar (for a list of these distances, see the schematic at the end of this document). You may see that they do not all read the same temperature; this is most likely due to slightly different amounts of glue fixing them to the bar. To account for this, the "Set reference" button will normalise the readings across all of the temperature sensors. The save button writes the current data to a file you select.

In addition to controlling the heater, the box of electronics attached to the power supply also monitors the flow of the coolant water and the temperature of the bar. If the water is not flowing or the bar

becomes too hot (above 35 °C), no power will be supplied to the heater. If either of these protections trip, the 'reset' button on the electronics box must be pressed before power will be supplied again to the heater. You will need to press this reset button after initially turning the water on.

First, switch the power supply on (it must be switched on before opening the 'heater' program). Now open both the 'heater' and 'reader' programs.

For your first measurement, program a square waveform with a period of 3 min, and a number of cycles that will bring the measurement time up to approximately 1 hour. Later you can investigate longer periods, and will see why this is necessary. Once the waveform is set and the measurement software is running, start the waveform - once the heat pulses have finished you must save your data to a file.

▶ What can you conclude about thermal relaxation in this system from your first set of measurements?

To take a set of measurements that you can analyse effectively, you should set up an overnight run. This allows you to measure many cycles with a period sufficient enough to allow thermal equilibration between pulses. It is important to measure many cycles in order to obtain a clean fourier transform.

5.2 Analysis

Once your measurement is over and the data is saved, you will see that the file contains 15 columns representing each of the thermocouple temperatures as a function of time. Plot the results and observe the change in the waveform shape along the bar. To determine the initial time constant γ, fit a function with the form $(1 - e^{-\gamma t})$ to the data for the first thermocouple in the region where it is increasing in temperature from ambient to the maximum. Next, use data analysis software to perform a Fourier decomposition of each of the thermocouple waveforms, examining the presence of harmonics in the waveforms. You can use any software package, but if you want to perform a Fourier transform in Origin, select the data, and choose the menu options "analysis" $\rightarrow$ "signal processing" $\rightarrow$ "FFT" which stands for Fast Fourier Transform. You will need to adjust the scales substantially in order to look at different parts of your spectrum, and you should note that the zero-frequency component is simply a measure of the static noise in your signal.

▶ After what distance have most higher harmonics decayed?

5.3 Attenuation of the wave

Use your Fourier transformed data from the thermocouples to determine the decay of several different frequency components of the wave. Make a log plot of the amplitude of the Fourier components of the wave versus position. Use this to determine the thermal diffusivity for each different frequency, and compare these different values.

▶ Use values for C and ρ given in section 4 to find the thermal conductivity. Is there a frequency dependence to your results, and if you were to expect one, where would you look for it?

5.4 Velocity of a single pulse

Make a second measurement of the velocity of the wave by sending a single square heat pulse down the bar. This pulse should be as close to a delta function as possible, so match the period of the wave to the rise time of the first thermocouple. Measure the time taken for a particular part of the wave (e.g. the peak) to travel a certain distance down the bar. Plotting the position of this component

versus time will allow you to determine the velocity of the thermal wave, and another value of the diffusivity D and the thermal conductivity κ. Additionally, since this pulse will approximate a gaussian wave-packet, it will spread as it travels down the bar. You can obtain a measure of the diffusivity by measuring the width of the pulse versus time (i.e. versus thermocouple). The width of the wave should vary as

$$\Delta_x \propto \sqrt{Dt} \tag{20}$$

Thus, a plot of pulse-width versus time$^{\frac{1}{2}}$ will yield a gradient of D$^{\frac{1}{2}}$

▶ Compare your results and use the analysis given earlier to comment on your final results. Additionally, reassure yourself that a Gaussian function is indeed a solution to equation 5.

6 Appendix

You may find it helpful to use the ansatz:

$$T = \frac{T_0}{\sqrt{\alpha t}} \exp\left(-\frac{x^2}{\beta t}\right) \tag{21}$$

to solve the partial differential equation:

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \tag{22}$$