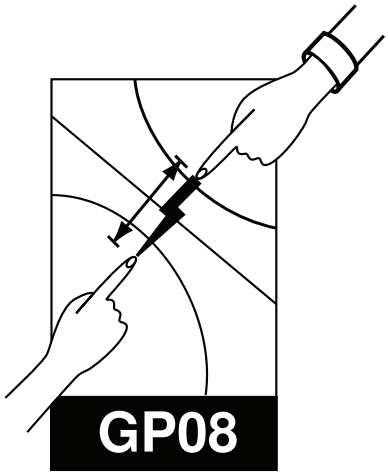
Rotational dynamics



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GP08

1 Introduction

The dynamics of rotating bodies is analogous in many respects to that of linearly translating bodies, though with certain important differences.

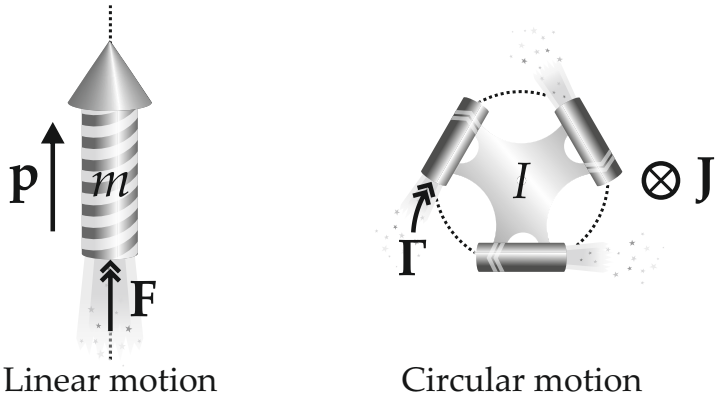


Figure 1: Left: a firework of mass m, velocity $\mathbf{v}$, and linear momentum $\mathbf{p} = m\mathbf{v}$ undergoing linear motion, subjected to net force $\mathbf{F} = d\mathbf{p}/dt$. Right: a rotationally symmetric Catherine wheel with moment of inertia I, angular velocity $\boldsymbol{\omega}$, and angular momentum $\mathbf{J} = I\boldsymbol{\omega}$ undergoing circular motion, subjected to net torque $\boldsymbol{\Gamma} = d\mathbf{J}/dt$. Note that both $\mathbf{J}$ and $\boldsymbol{\Gamma}$ point into the page (right-hand rule).

You will already be well-acquainted with the variables pertinent to linear motion (including mass m, momentum $\mathbf{p}$, and force $\mathbf{F}$) that are tied together via Newton's second law of motion. It transpires that the rotation of a rigid body can be described by an analogous set of variables and equations: mass m is replaced by moment of inertia I, linear momentum $\mathbf{p}$ by angular momentum $\mathbf{J}$, and so on. The theory describing rotational dynamics is an essential part of any physicist's toolkit; this practical will help you gain fluency in some of the relevant equations and test their validity.

The aim of this experiment is to investigate a number of aspects of rotational dynamics using an apparatus consisting of a set of rotating masses, a low-friction bearing with an optical encoder, and a pulley system to accelerate the rotating masses with known torques. The experiment is divided into three sections, which take you through a sequence of investigations of increasing complexity. The early sections entail measuring the moment of inertia of a metal disc, and comparing the theoretical value with that experimentally observed. The later sections involve investigation of the conservation of angular momentum and energy, and the factors which contribute to the moment of inertia.

As for all experiments, you should record your results in your logbook in the form of a table of measurements with appropriate notes. Computer output should be printed and securely attached into your notebook. Estimates of errors should be given and sources of error discussed.

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2 Theory

2.1 Rotational dynamics of a rigid body

Consider a rigid body consisting of a large number N of small mass elements of mass m_i whose relative positions are fixed and whose absolute positions measured with respect to an origin fixed in the laboratory are $\mathbf{r}_i$ (see [1]). If we consider the origin to lie somewhere along the axis of rotation, the angular momentum $\mathbf{J}$ is defined as

$$\mathbf{J} = \sum_{i=1}^{N} m_i(\mathbf{r}_i \times \mathbf{v}_i) \tag{1}$$

where $\mathbf{v}_i$ is the velocity of material element i. By differentiating Eq. (1), we find immediately that the rate of change of $\mathbf{J}$ is

$$\frac{d\mathbf{J}}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i \equiv \boldsymbol{\Gamma} \tag{2}$$

where $\boldsymbol{\Gamma}$ is the *torque* resulting from the applied forces $\mathbf{F}_i$. Eq. (2) is the rotational equivalent of Newton's second law. Note that $\mathbf{J}$ is conserved in the absence of external torques.

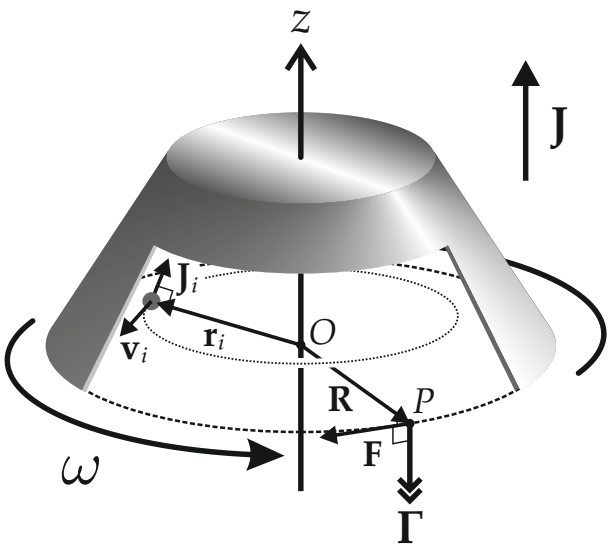


Figure 2: A conical frustum rotating about the z axis at angular frequency ω. Its total angular momentum $\mathbf{J}$ is obtained by summing over contributions from its constituent material elements with individual momenta $\mathbf{J}_i = m_i(\mathbf{r}_i \times \mathbf{v}_i)$ (see cutaway). Due to the frustum's rotational symmetry, $\mathbf{J} \parallel \hat{\mathbf{z}}$. An external force $\mathbf{F}$ applied at point P with position vector $\mathbf{R}$ generates torque $\boldsymbol{\Gamma} = \mathbf{R} \times \mathbf{F}$. Note the sense of the torque is such as to decrease $|\mathbf{J}|$, i.e. the frustum's angular velocity ω will decrease.

As it stands, Eq. (1) is difficult to interpret, but, just as we can express the total linear momentum of an extended body in terms of its total mass and its centre-of-mass velocity, we can simplify $\mathbf{J}$ if the body is rigid and it is rotating about a fixed axis. If we define this rotation axis as the z-direction, the coordinates of each individual material element may be expressed in the form $\mathbf{r}_i = (R_i \cos\phi_i,\ R_i \sin\phi_i,\ z_i)$, where both R_i and z_i are fixed, and where $d\phi_i/dt = \omega$, the angular velocity of the body. Expressed in this way, the total angular momentum defined by Eq. (1) becomes

$$\mathbf{J} = \sum_i m_i(-z_i R_i \omega \cos\phi_i,\ -z_i R_i \omega \sin\phi_i,\ R_i^2 \omega). \tag{3}$$

Now, in general, the components of the angular momentum that are orthogonal to the rotation axis, J_x and J_y, are nonzero, and must still be evaluated explicitly via an unpleasant summation. *However*, if the object in question is rotating about one of its axes of symmetry – as will be the case throughout the

majority of this practical – J_x and J_y vanish identically, and the only component of $\mathbf{J}$ that survives is the one parallel to the rotation axis:

$$\mathbf{J} = \sum_i m_i R_i^2 \omega \, \hat{\mathbf{z}}. \tag{4}$$

Note that we can express J_z – which, again, is conserved in the absence of external torque – in the form

$$J_z = I_z \omega \tag{5}$$

where the constant I_z is defined as

$$I_z \equiv \sum_{i=1}^{N} m_i R_i^2. \tag{6}$$

This property is referred to as the *moment of inertia* of the body about the z-axis. Just as the inertial mass m characterises how much a body resists changes to its linear velocity, I_z expresses the resistance of the body to having its angular velocity ω changed. Note that it depends not only upon the total mass of the body, but upon how that mass is distributed: the contribution of any given element of material to I_z scales with the *square* of how far it is from the rotation axis. Note also the directly analogous mathematical form of Eq. (5) with that for the linear momentum, $p_z = mv_z$.

In continuous solid objects of mass density ρ, the sums in Eqs (1–6) become integrals, and the formula for I_z can then be written thus:

$$I_z = \iiint dx\, dy\, dz\, \rho(x, y, z)(x^2 + y^2), \tag{7}$$

where the volume integral extends over the entire object. For a uniform disk (i.e., one whose mass density ρ is a constant), these integrals yield $I_z = \frac{1}{2}Ma^2$, where M is the total mass of the disk and a its radius. In general, for objects of regular shape,

$$I = M(\lambda_1 b^2 + \lambda_2 c^2) \tag{8}$$

where b and c are dimensions of the object along axes perpendicular to the rotation axis, and λ_1 and λ_2 are constants that depend on the cross-section: $\lambda_i = \frac{1}{3}$ for 'rectangles', $\frac{1}{4}$ for 'ellipses', and $\frac{1}{5}$ for 'ellipsoids' (*Routh's rules*, see [2]). More generally, I will also have additional components I_x and I_y about axes perpendicular to z which are typically different from I_z – this possibility is addressed in the Optional portion of this practical.

2.2 Rotational kinetic energy

Rotational equivalents of kinetic and potential energy may also be derived. The body's total kinetic energy T is given by the sum

$$T = \sum_i^N \tfrac{1}{2} m_i R_i^2 \omega^2 = \tfrac{1}{2} I \omega^2. \tag{9}$$

Note again the similarity between the mathematical form of T and that of the translational kinetic energy, $\frac{1}{2}mv^2$. Furthermore, if all external forces $\mathbf{F}$ are conservative, one can define a potential energy $V = V(\mathbf{F})$ and show (*e.g.* see [1]) that

$$T + V = \text{ constant}$$

so that total mechanical energy is conserved.

▶ How do figure skaters vary their moment of inertia, and why? Show that, if the angular momentum of the figure skater is conserved, varying the moment of inertia requires changing their kinetic energy. Where does this energy come from?

2.3 The parallel axis theorem

Suppose we have calculated the moment of inertia I_{cm} of an object of mass M about an axis z that passes through the body's centre of mass. The parallel axis theorem (also known as the *Huygens–Steiner theorem*, or just as *Steiner's theorem*) enables us to calculate the moment of inertia about any axis z' that is parallel to z. If the perpendicular distance between z and z' is d, the theorem states that

$$I_{z'} = I_{cm} + Md^2. \tag{10}$$

One can see that $I_{z'}$ is always *greater* than I_{cm}. You will test the validity of this theorem in Section 5.

3 The apparatus

The apparatus consists of a rotational motion sensor mounted on a retort stand and connected to a PC runing 'PASCO Capstone' software, together with a set of torque pulleys cast from clear plastic. There are three masses available for the experiment: a thin aluminium plate, a heavy black aluminium ring, and a pair of sliding brass masses mounted on a long thin rod. There is a hanger and a set of masses provided in a blue box that allow for accelerating the mechanism with a constant torque, using a fine cotton thread.

Make sure you have identified the following:

1. The blue rotational motion sensor.
2. The clear plastic pulley set.
3. The black low-friction pulley.
4. The thin aluminium plate.
5. The heavy black aluminium ring.
6. The pair of brass weights on a long thin rod.
7. The hanger mass set.
8. The cotton thread.

The plastic parts of the apparatus are fragile and only fit with the metal parts in a particular way. It is easy to break the clear plastic pulley by trying to force it onto the metal pin of the rotational motor sensor. Please be careful!



3.1 Introduction to the data acquisition program

Log in to the PC[1] and complete the following steps:

1. Load the PASCO Capstone software by double-clicking the icon on the desktop.
2. Choose the Table & Graph option presented on the first screen.

[1] If you have forgotten your password, find a computer already logged in and navigate to:
`http://www2.physics.ox.ac.uk/students/undergraduates/teaching-lab-password`

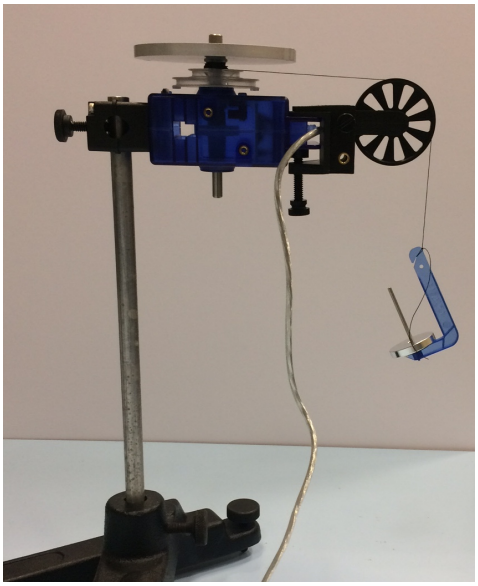


Figure 3: The apparatus assembled to measure the moment of inertia of the thin aluminium plate.

3. On the right-hand side of the screen, click the button on the graph y-axis and select Angular velocity (rad/s) from the menu presented. The x-axis will automatically default to time.

4. On the left-hand side of the screen, click the table columns to select time and Angular Velocity (rad/s).

5. Prepare the apparatus by mounting the thin aluminium plate on the rotary motion sensor as shown in Figure 3. You may need to flip the pulley set over to mount the plate correctly.

Now briefly experiment with the apparatus itself, to explore how to record and fit data:

6. With the apparatus at rest, click the Record button on the lower left-hand side of the screen, then set the aluminium plate spinning by hand for a few seconds, then stop the aluminum plate, and finally click Stop to end the recording, as shown in Figure 4. You should obtain a dataset similar to the one shown in Figure 5.

7. Use the selection tool to highlight a range of data collected whilst the disc was spinning, and use the linear fit tool to extract the equation of the line through this data, see figure 6.

When data acquisition is complete, you can save the recording either through choosing Save ⟩ Experiment or Export ⟩ Data on the file menu to produce a text file suitable for importing into other analysis software, such as Python or Excel. It's also possible to select and copy/paste data directly from the table on the left-hand side of the PASCO Capstone screen into the analysis program of your choice.

When saving data, make sure you choose an informative filename for future reference. For example, ⟨`data1`⟩ is a poor choice, whereas ⟨`angular acceleration - aluminium plate - 100g - 1`⟩ is better,

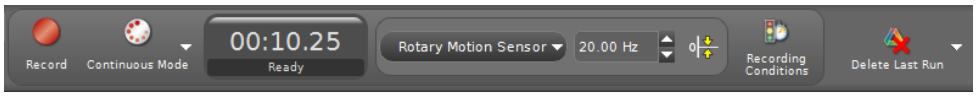


Figure 4: The 'Record' and 'Delete Last Run' controls in the PASCO Capstone software.

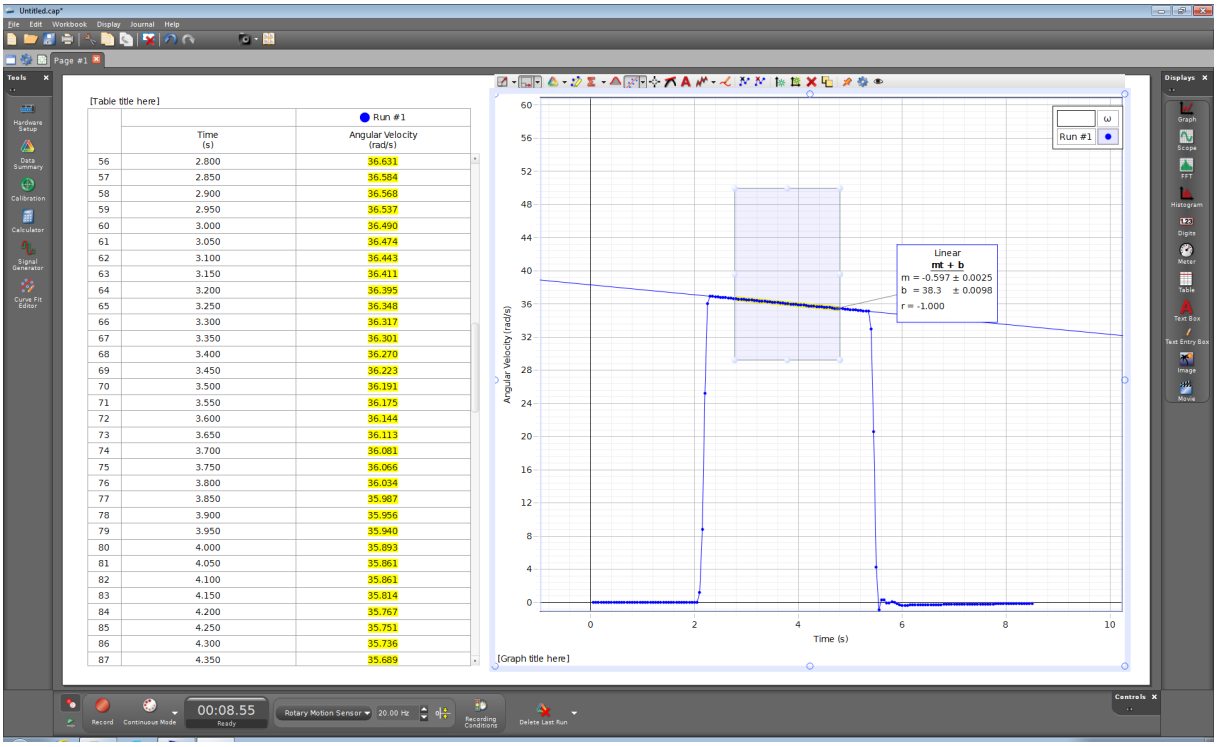


Figure 5: Example dataset recorded from the freely rotating thin aluminium disc, showing gradual deceleration due to friction. The highlighting tools in figure 6 are used to select a region of interest in the data.

indicating what you were measuring and what the conditions for the experiment were, and the repeat number used to gain an estimate of measurement errors. If you don't want to save the data (for example, if something went wrong during data acquisition) click Delete Last Run at the bottom of the screen.

4 Measurements using the PC for data acquisition

4.1 Angular acceleration and moment of inertia

The clear plastic torque pulley has three diameters available (approximately 1.0 cm, 2.5 cm and 5.0 cm) for use with the hanger and mass set.

> Obtain precise values for the diameters of the torque pulleys by measuring them using the calipers supplied. Check the extent to which the pulleys are circular and use an average radius. An approximately constant torque can be applied by attaching the torque pulley to the hanger set using the fine cotton thread supplied. Suspend the hanging mass over the edge of the bench using the low friction pulley. Make sure the thread is long enough to take the hanger and weights down to just short of the floor. Raise the hanging mass to just below the motion sensor by rotating the thin aluminium plate, wrapping the thread around the thinnest diameter on the clear plastic torque pulley.



Figure 6: Location of tools to select and fit to data on the PASCO Capstone software.

On releasing the hanger, a constant torque of

$$\Gamma \approx mgr \tag{11}$$

is applied to the apparatus, where m is the total mass of the hanger ($\sim 5\,\mathrm{g}$) and any extra masses suspended, r is your measured value of the radius of the torque pulley, and g the acceleration due to gravity.

▶ Why is Equation (11) an approximation to the torque applied?[2] Can you propose a better alternative to Eq. (11)? Does the 'corrected' equation give a better prediction within the errors?

Find the mass of the hanger and each mass you add using the balances near the entrance to the lab. Use these measured masses along with your measured diameter to determine the applied torque.

Using the apparatus you are given, consider how you can determine most efficiently and accurately the angular acceleration of the thin aluminium plate under acceleration by a constant torque. How can you manage your data in order to minimize the experimental error?

When projected onto the axis of rotation, Eq. (2) for the rotational motion (corresponding to Newton's second law) states that

$$\Gamma = I\alpha = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{12}$$

where Γ is the applied torque, α is the angular acceleration, ω is the angular velocity, θ is the angular displacement of the disk about its rotation axis, and I is the moment of inertia — the rotational equivalent of mass. Determine experimentally the moment of inertia for the thin aluminium plate using Eq. (12) and your measurements of angular acceleration, together with the applied torque found from measuring the hanger masses. Tabulate your results.

The theoretical expression for the moment of inertia of a uniform disk of mass M, calculated about the axis of symmetry, is

$$I_z = \tfrac{1}{2}Ma^2 \tag{13}$$

where a is the radius of the disk. Calculate I using measured values of M and a (found with the electronic balance and calipers, respectively) and then compare your measured and calculated values of I. How accurately can you determine I in each case? Try repeating the measurements using several (perhaps five) different hanger masses m (to change Γ) and plot a graph of $\frac{\mathrm{d}\omega}{\mathrm{d}t}$ against Γ. Investigate whether the diameter of the pulley you use affects the results (e.g., changing Γ by changing r whilst keeping F constant). To what extent can you verify Newton's second law of rotational motion?

4.2 Moment of inertia of a ring

Identify the black aluminium ring and place it on the thin aluminium plate. Use the pins to guide the ring into position as in Figure 7.

Repeat some of the measurements in the previous section with the combined ring and plate assembly in order to measure the combined moment of inertia:

$$I_{\mathrm{total}} = I_{\mathrm{ring}} + I_{\mathrm{plate}}. \tag{14}$$

Subtract the moment of inertia due to the thin aluminium plate that you found in Section 4.1 to find the moment of inertia of the ring itself.

[2] Hint: Draw a free-body diagram of the falling hanger and masses at the end of the cotton thread. In what limit can the acceleration of the weight be ignored?

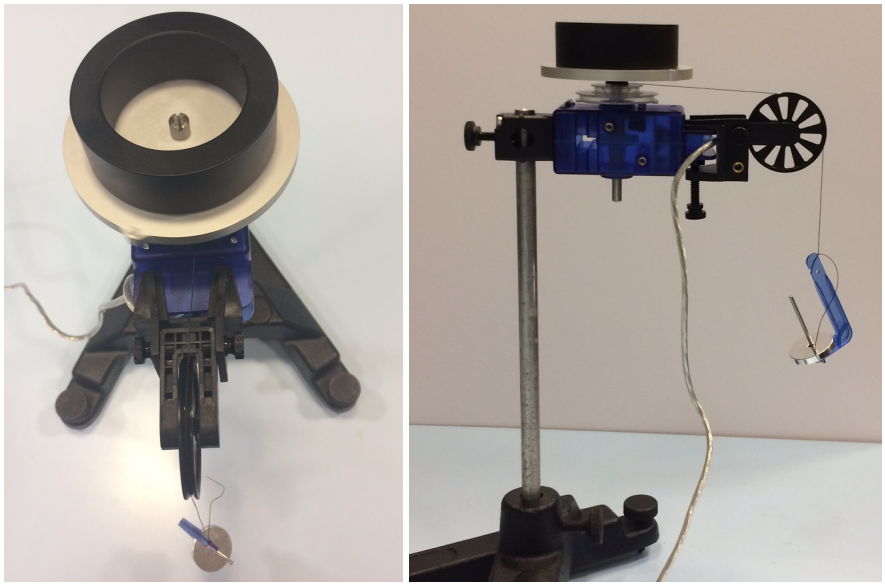


Figure 7: Ring and plate assembled for moment of inertia measurements.

The theoretical moment of inertia of a ring is given by:

$$I_{\text{ring}} = \tfrac{1}{2} M_{\text{ring}} \left(R_{\text{inner}}^2 + R_{\text{outer}}^2 \right) \tag{15}$$

where M_{ring} is the ring mass, and R_{inner} and R_{outer} are the inner and outer radii of the ring. Compare the moment of inertia found from your angular acceleration measurements with that derived from Eq. (15). How accurately can you derive each quantity, and do the values agree to within experimental error?

4.3 Conservation of angular momentum: inelastic collisions

The *conservation of angular momentum* is a fundamental principle that can be nicely illustrated and verified by the inelastic collision/coalescence of two rotating objects. To explore this, you will drop the non-rotating heavy black aluminium ring onto the spinning thin aluminium plate, recording the angular velocities of the plate before and after the collision.

To investigate this, make use of the facts that:

- The angular momentum of a spinning object is given by $J = I\omega$.
- The rotational kinetic energy is given by $T = \tfrac{1}{2} I\omega^2$.

To perform this experiment:

1. Remove the cotton thread and hanger set, so the aluminium plate can rotate freely.
2. Start taking data by clicking Record .
3. Start the thin aluminium plate spinning by hand.
4. Hold the black aluminium ring centred over the spinning aluminium plate.
5. Drop the black aluminium ring onto the spinning plate.
6. Record a few seconds of data after the collision before clicking Stop .

As we will see shortly, to assess the conservation of (rotational) energy and momentum, we will need the moments of inertia of the disk and of the ring, both calculated *about the axis of rotation*. In order for us to be able to use Eq. (15), we would have to drop the black aluminium ring so that it lands *exactly centred* on the aluminium plate — a task that is harder than you might think. You may need to practise and repeat the procedure a few times to obtain satisfactory results. If the ring is dropped off-centre onto the plate, then the moment of inertia of the final state will be greater than suggested by Eq. (15).

▶ Could you use the parallel axis theorem (Section 2.3) to calculate the moment of inertia of the black disk when it lands off-axis? Propose a measurement procedure.

If the plate is not level, you may observe oscillations in the angular velocity of the combined plate-ring system. Try to level the plate carefully to minimise this effect.

Calculate the combined angular momentum of the ring and plate before and after the collision, using the formulae:

$$J_{\text{ring}}^{\text{before}} = 0 \qquad J_{\text{ring}}^{\text{after}} = I_{\text{ring}}\omega_{\text{after}} \tag{16}$$

$$J_{\text{plate}}^{\text{before}} = I_{\text{plate}}\omega_{\text{before}} \qquad J_{\text{plate}}^{\text{after}} = I_{\text{plate}}\omega_{\text{after}} \tag{17}$$

$$J^{\text{before}} = J_{\text{ring}}^{\text{before}} + J_{\text{plate}}^{\text{before}} \qquad J^{\text{after}} = J_{\text{ring}}^{\text{after}} + J_{\text{plate}}^{\text{after}} \tag{18}$$

Also, calculate the kinetic energy prior and subsequent to the collision.

$$T_{\text{ring}}^{\text{before}} = 0 \qquad T_{\text{ring}}^{\text{after}} = \tfrac{1}{2} I_{\text{ring}}\omega_{\text{after}}^2 \tag{19}$$

$$T_{\text{plate}}^{\text{before}} = \tfrac{1}{2} I_{\text{plate}}\omega_{\text{before}}^2 \qquad T_{\text{plate}}^{\text{after}} = \tfrac{1}{2} I_{\text{plate}}\omega_{\text{after}}^2 \tag{20}$$

$$T^{\text{before}} = T_{\text{ring}}^{\text{before}} + T_{\text{plate}}^{\text{before}} \qquad T^{\text{after}} = T_{\text{ring}}^{\text{after}} + T_{\text{plate}}^{\text{after}} \tag{21}$$

Calculate the experimental uncertainties on these quantities.

Is the total angular momentum conserved within your experimental error? Is kinetic energy likewise conserved and if not, why not? Try repeating this experiment using different values of ω_{before}.

5 Moments of inertia and geometry

The following experiment investigates how the geometry of an object controls its moment of inertia.

5.1 Variable-radius mass (VRM)

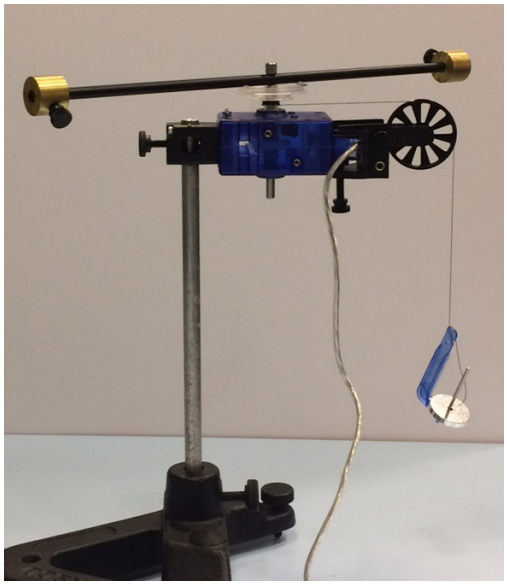


Figure 8: The variable-radius mass apparatus.

The variable-radius mass (VRM) assembly shown in Figure 8 represents an rigid body whose moment of inertia you can vary continuously by changing the way its mass is distributed around its axis of rotation.

First, locate one of the two cylindrical brass masses pictured in Figure 8. Let I_c be the moment of inertia of this mass about a line through its centre of mass and *perpendicular to* (not parallel to) its axis of rotational symmetry. Locate the adaptor with threads on both ends, and use it to mount the brass cylinder onto the top of the clear plastic torque pulley. Measure I_c in the usual way and then compare your value with that given by the theoretical expression for a hollow cylinder on its side:

$$I_c = M_c \left[\frac{1}{4}(b_{out}^2 + b_{in}^2) + \frac{1}{12} l^2 \right] \tag{22}$$

where M_c is its mass, l is its length, and b_{out} and b_{in} are, respectively, its outer and inner radii.

Next, build the VRM assembly on the rotational motion encoder as shown in Figure 8. You will need to flip the clear plastic torque pulleys over so that you can sit the rod in the two grooves on the back of the largest pulley.

Now replace the sliding masses as shown in Figure 8, their centres of mass being an equal distance r from the rotational axis. Then measure the moment of inertia I of the whole system over the full range of r permitted by the apparatus.

▶ Confirm that your measured values are consistent with a relation of the form:

$$I_c = A + Br^2 \tag{23}$$

where A and B are constants. Find values of A and B and comment on them with reference to the parallel axis theorem (see Section 2.3).

▶ If you do not wish to do the optional work below, hand-write a brief summary of the experiment in your logbook. State the aims of the experiment, the method used, quantitative results, and your conclusions, in not more than about half a page.

▶ If stopping here, discuss your results with a demonstrator as part of your assessment.

Optional

5.2 Principal axes of triangular mass

Rotating objects are not always symmetrical about their axis of rotation. This can complicate the analysis of their motion, since the angular momentum vector $\mathbf{J}$ is no longer parallel to the axis of rotation $\hat{\boldsymbol{\omega}}$. In fact, while the component of $\mathbf{J}$ *parallel* to $\hat{\boldsymbol{\omega}}$ is given by Eq. (5), there also exists a component *perpendicular* to the rotation axis, given by

$$\mathbf{J}_{\perp} = -\omega \sum_i m_i z_i \mathbf{r}_{i\perp} \tag{24}$$

where $\mathbf{r}_{i\perp}$ is the projection of the position vector $\mathbf{r}_i$ *perpendicular* to $\hat{\boldsymbol{\omega}}$, while z_i is its projection *along* $\hat{\boldsymbol{\omega}}$.

▶ Why should this be significant? What would happen to Eq. (2) if angular momentum was conserved (i.e., absence of torque) but $\hat{\omega}$ was not parallel to $\mathbf{J}$?

The analysis may be simplified, however, because, for any object of any shape, there exist three mutually perpendicular axes through the centre of mass with the following properties:

1. If the object rotates about one of these axes, $\mathbf{J}$ is parallel to $\hat{\boldsymbol{\omega}}$;
2. The moment of inertia I about one of these three axes (I_1) is the largest I for any axis through the centre of mass;
3. I about another of these axes (I_2) takes the smallest possible value;
4. I about the third axis (I_3) is intermediate in size so that $I_1 \geq I_3 \geq I_2$.

This problem is of considerable technical importance in many practical situations since, if an irregular object rotates about an axis that is not a principal axis, the angular momentum vector is not constant in space and couples must act on its support or bearings. These couples are apt to wear the support and make the object wobble. That is why, for instance, it is necessary to 'balance' wheels on a car such that the axis of the wheel is a principal axis of its moment of inertia.

In this part of the experiment, you use the methods of measurement described above to locate the principal axes of an asymmetrical (right-angle triangular) mass.

> Identify the variable-angle triangular mass (VATM) shown in Figure 9 and attach the mounting post to the rotary motion sensor and torque pulleys using the adapter supplied. Remove the triangular mass itself and determine the moment of inertia I_0 of the combined mass of the rotary motion sensor, torque pulley, adapter and mounting post of the VATM (use the PC method as in Section 4.1). Replace the VATM in its mount and adjust so that the hypotenuse is approximately vertical. Measure the new moment of inertia of the entire assembly and infer the moment of inertia I due to the VATM alone.

> Repeat this sequence of measurements, rotating the VATM each time in increments in orientation angle Θ of (say) 15°, and determine I. From a plot of I against Θ, determine the directions of the principal axes and indicate their location relative to the triangular mass.

▶ How would the orientation change if the triangle were isosceles or equilateral?

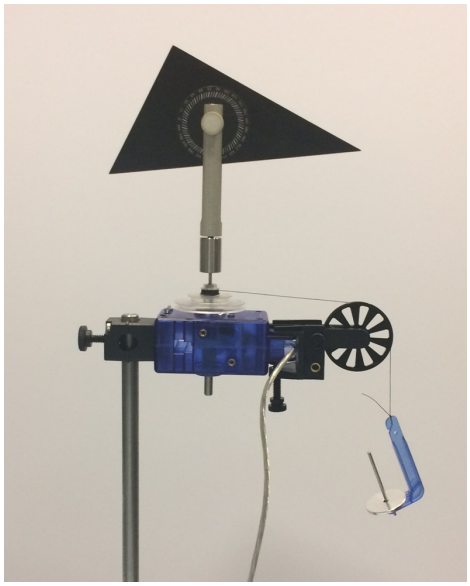


Figure 9: The variable-angle mass.

▶ If you have completed the optional section, now is the time to hand-write a brief summary of the experiment in your logbook. State the aims of the experiment, the method used, quantitative results, and your conclusions, in not more than about half a page. Show this to a demonstrator as part of your assessment.

Bibliography

[1] A. P. French & M. G. Ebison, *Introduction to Classical Mechanics*, Chapman & Hall; March 1987, Chapter 10. doi: 10.1007/978-94-009-4119-9

[2] T. W. B. Kibble, *Classical Mechanics*, Imperial College Press, 5th Edition, 2004. doi: 10.1142/9781860945199˙fmatter