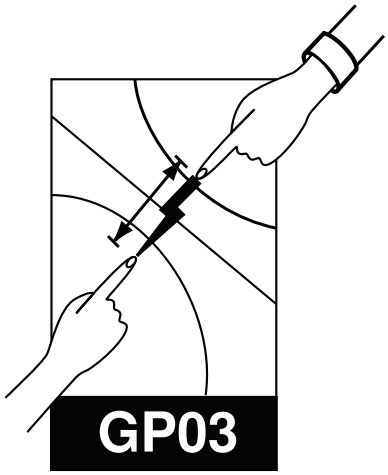
Resonances and harmonics



PGR, IK, PH July 2020

GP03

1 Introduction

The physics of standing waves on a wire is an essential part of the syllabus. This system demonstrates features of wave motion and allows accurate measurements of physical quantities by exploiting the ability to 'tune' the device accurately.

As with all experiments, record your findings in your log book (see below). Take care to describe, not what you believe you ought to see, but what you *do* actually see. The distinction may be important. You may find that after an initial round of measurements some observed effects need to be followed up, and some assumptions challenged. For each part of the experiment, you should do the following:

(a) Consider the theory and what you expect of each section.

(b) Carry out the investigation noting in your log book

  (i) your comments on what you observe,

  (ii) the raw readings for each measurement you make, and

  (iii) your answer to each question asked.

(c) In your log book analyse the raw readings, tabulate and/or plot the results, calculate and justify the errors on your result. Understanding the errors is arguably one of the most important parts of the work. Evaluating and analysing the errors will help you to understand how good the data you have collected are. Show which sources of error are important and which are not and explain why.

After finishing the experiment, write a brief summary of the experiment aims, method and results in your log book.

2 Theory

2.1 The physics of a taut string

The amplitude $A(x,t)$ of a simple wave travelling along the x-axis on a taut wire can be described by

$$A(x,t) = A_0 \sin(kx - \omega t) \tag{1}$$

This sine-wave repeats with a wavelength along x of $\lambda = 2\pi/k$ and a period in time of $\tau = 2\pi/\omega$. Its frequency is therefore $f = \omega/2\pi$. The peaks and troughs of the wave move with a speed $v = \lambda f = \omega/k$ in the positive x-direction. The expression in brackets $kx - \omega t = \phi$ is the phase of the wave.

If the string is fixed at one end ($x = 0$), the above wave will be reflected. The reflected wave, which is given by equation 2 with time reversed in the expression for the phase, indicating the reverse direction of propagation of the wave:

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$$B(x,t) = A_0 \sin(kx + \omega t) \tag{2}$$

Its constant A_0 is determined by the need for the resulting total wave $C(x,t)$ to have a node at $x = 0$. This means that it is equal to zero there at all times. Combining eq. 2 with eq. 1 we obtain:

$$C(x,t) = A_0(\sin(kx + \omega t) + \sin(kx - \omega t)) \tag{3}$$

which by trigonometry reduces to

$$C(x,t) = 2A_0 \sin(kx)\cos(\omega t) \tag{4}$$

This is a standing wave. It has nodes, not only at $x = 0$, but also at $x = n\pi/k$ for any integer value of n. Now, if the wire is fixed at a second position, $x = L$, then there must be a node there. Thus $L = n\pi/k$ and k is restricted to the values $n\pi/L$. Only at these values of k can a standing wave solutions exist. We say that the system has a *discrete spectrum of eigenmodes* — a characteristic shared by all bound systems. Re-expressed in terms of λ, this 'quantisation' condition has a particularly simple interpretation, $L = n\lambda/2$. In other words, the length of the string L must be a multiple of $\lambda/2$.

For a taut string made of an elastic medium, the velocity of propagation of these waves is given by the formula $v = \sqrt{T/\rho}$, where T is the tension of the string (in N) and ρ is the *mass per unit length* of the string (in Kg). From this, one can deduce that the string can vibrate at frequencies $f_n = nf_1$, where f_n is the frequency in Hz and $f_1 = v/2L$ is the *fundamental frequency*. For a given n, the standing wave will have $n-1$ *nodes* at positions $x_m = Lm/n$, with m an integer in the interval $1 \ldots n-1$.

In this experiment, we will consider a vibrating string that is both *driven* (by a magnetic drive) and *damped* by air friction and internal friction within the material. The physics of a driven and dampened string is closely related to that of a driven and dampened *harmonic oscillator*, which is discussed in detail in the script for experiment GP04 — *Driven Harmonic Motion*[1]. In summary, if a sinusoidally modulated force $F \sin \omega_D t$ is applied at a given position x_0 along the string and the angular frequency of the drive ω_D is in the vicinity of one of the n-th harmonic, using Fourier analysis one finds that the maximum *complex amplitude* A_{ω_D} (occurring at the *antinodal* positions) is given by

$$A_{\omega_D} = \frac{\tilde{F}}{M} \frac{e^{-i\phi}}{\left((n^2\omega_1^2 - \omega_D^2)^2 + \gamma^2\omega_D^2\right)^{\frac{1}{2}}} \tag{5}$$

with phase

$$\tan\phi = \frac{\omega_D \gamma}{n^2\omega_1^2 - \omega_D^2} \tag{6}$$

Here, $\omega_1 = 2\pi f_1$ is the fundamental angular frequency, $\tilde{F} = 2F \sin kx_0$ is the effective driving force amplitude, and $M = L\rho$ is the total mass of the string. In other words, the string behaves like a driven and damped harmonic oscillator with mass M and driving force $2F \sin kx_0 \sin \omega_D t$, with x_0 being the point of application of the force on the string. **The expression for the effective driving force is particularly significant for our experiment**: it tells us that *the drive is completely ineffective when applied at a node (since $kx_0 = 0$), and is maximally effective when applied at an antinode.*

Otherwise, eq. 5 is almost identical to the equation of a simple driven and damped oscillator (eq. 15 in the script for experiment GP04), the main difference being that there are multiple resonances rather than a single one (note also the substitution $\omega_0 \to n\omega_1$). In particular, one can deduce that the characteristic damping time of free oscillations is $\tau = \frac{2}{\gamma}$, that the frequency FWHM of all resonances is γ and the quality factor is $Q = n\omega_1/\gamma$.

[1] `www-teaching.physics.ox.ac.uk/practical_course/scripts/srv/local/rscripts/trunk/General/GP04/GP04.pdf`

2.2 The physics of the magnetic drive

In order to test all aspects of the physics of a vibrating string as described in section 2.1, we would ideally require a sinusoidally modulated driving force $F \sin \omega_D t$ to be applied at a single point x_0 along the string. In our apparatus, the driving force is implemented by a magnetic drive; as we shall see, this only approximates the expression $F \sin \omega_D t$ in the limit of a small driving force. **It is important to take this into account when defining the parameters of the experiment**.

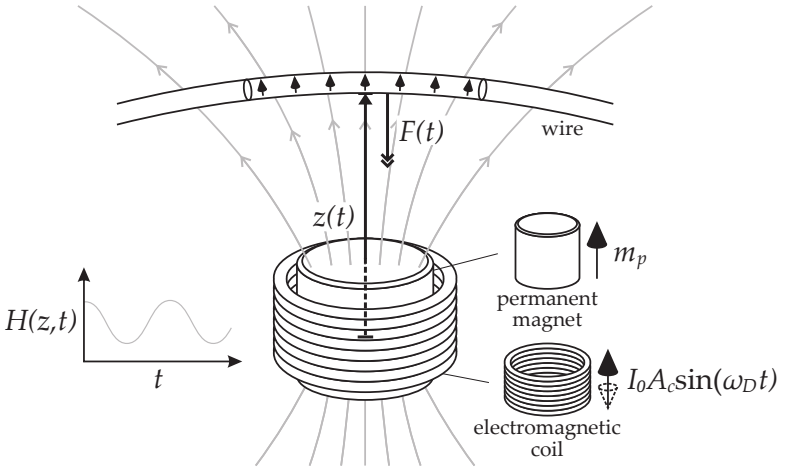


Figure 1: Schematic drawing of magnetic drive system used to excite the string resonances.

The magnetic drive is a modified guitar pick-up (see Figure 1), which consists of a permanent magnet, co-axial with a coil that is connected to a variable-frequency voltage source, which induces a current in the coil[2]. The string is made of a soft magnetic material, which becomes magnetised in the magnetic field of the permanent magnet and coil. **It is important to remember that a soft magnetic material is always attracted by a magnet**. However, the magnitude of the attractive force changes when the current in the coil is sinusoidally modulated. Treating the permanent magnet and the coil as magnetic dipoles, and assuming that the string is at a height z from the centre of the dipole, the magnetic field H at the string position is given by:

$$H = \frac{1}{2\pi} \frac{m_p + IA_c}{z^3} \tag{7}$$

where m_p is the magnetic moment of the permanent magnet, A_c is the area of the coil times the number of wire turns and $I = I_0 \sin \omega_D t$ is the current in a single turn of the coil. The magnetic field does two things: it magnetises the string above it (for simplicity, we can assume that the string magnetisation is proportional to H) and it attracts it with a force that is proportional to the magnetisation of the string times the *gradient of H*. This implies that the force will contain the *square* of the term $m_p + IA_c$. In fact, in the approximations just stated, the force on the string can be written as:

$$F = -\frac{C}{z^7} \left(m_p + IA_c\right)^2 \tag{8}$$

where C is a positive constant. From eq. 8, we can deduce the following important facts:

- As previously stated, the force is always attractive (due to the *minus* sign).

[2]The current is the drive voltage divided by the complex impedance of the circuit, including the coil. Since it is not easy to determine the frequency-dependent impedance with the apparatus at our disposal, we will be unable to test eq. 6 quantitatively. For a determination of the phase evolution through a resonance, see experiment GP04.

- The force varies very rapidly with distance, so we cannot automatically assume in all regimes that the string is at a fixed position z_0 when calculating the force, since the string is oscillating.
- Inserting $I = I_0 \sin \omega_D t$, and taking the limit of small oscillations of the string ($z \approx z_0$) we obtain:

$$\begin{aligned} F &= -\frac{C}{z_0^7}\left(m_p^2 + 2m_p I_0 A_c \sin\omega_D t + I_0^2 A_c^2 \sin^2\omega_D t\right) \\ &= -\frac{C}{z_0^7}\left(m_p^2 + \frac{1}{2}I_0^2 A_c^2 + 2m_p I_0 A_c \sin\omega_D t - \frac{1}{2}I_0^2 A_c^2 \cos 2\omega_D t\right) \end{aligned} \quad (9)$$

As we can see from 9, the driving force is quite different from the simple expression we were hoping to get. It consists of three terms:

- A time-independent term $m_p^2 + \frac{1}{2}I_0^2 A_c^2$, which has simply the effect of deflecting the string slightly and may be ignored as a first approximation (note, however, that this may affect the tension of the string).
- A term proportional to $\sin\omega_D t$, which is exactly what we were hoping to obtain.
- A 'parasitic' term proportional to $\cos 2\omega_D t$, that is, a term that excites the string at *twice* the desired frequency.

We can also see that the ratio between the "$2\omega_D$" and "ω_D" terms is:

$$R_{2\omega_D/\omega_D} = \frac{I_0 A_c}{2m_p} \quad (10)$$

In other words, the drive system approximates a force $\propto \sin\omega_D t$ only in the limit of small currents (i.e., small drive voltages).

We should also briefly consider the fact that $z \neq z_0$. In fact, as the string vibrates near a particular harmonic resonance n,

$$z = z_0(1 + \frac{\tilde{z}}{z_0}\sin n\omega_1 t) \quad (11)$$

Inserting eq. 11 into eq. 8, and performing a Taylor expansion of the denominator under the assumption $\frac{\tilde{z}}{z_0} \ll 1$, one can see that additional time-dependent terms appear in the force. **This further emphasise the importance of keeping the drive voltage (and therefore the amplitude of the string oscillations) as low as possible**.

2.3 The physics of the magnetic detector

The purpose of the magnetic pick-up (detector) is to measure a quantity that is proportional to the amplitude of the string motion at any given point. Naturally, the detector will be most effective near the antinodes of the resonances we are trying to measure. Its design is identical to that of the magnetic drive, but the coil is directly connected to the oscilloscope rather than to the voltage source. Once again, the permanent magnet magnetises the string above it, while the coil senses the time derivative of the magnetic flux Φ (Faraday's law) as the string oscillates. In analogy with the previous section, the electromotive force on the coil can be approximated as:

$$V = -\frac{\partial\Phi}{\partial t} = K\frac{m_p A_c}{z^7}\frac{\partial z}{\partial t} \quad (12)$$

where K is a constant. In the limit $z \approx z_0$, *V is proportional to the vertical velocity of the string*, which is itself proportional to the amplitude of the motion, as desired. As for the magnetic drive, it is important to remember that *the magnetic detector behaves ideally only in the limit of small oscillations of the string*, i.e., when $z \approx z_0$.

3 Getting to know the apparatus



Remember that you are responsible for your own safety when working in the laboratory. Always talk to a demonstrator or technician if you are concerned that anything may be unsafe.

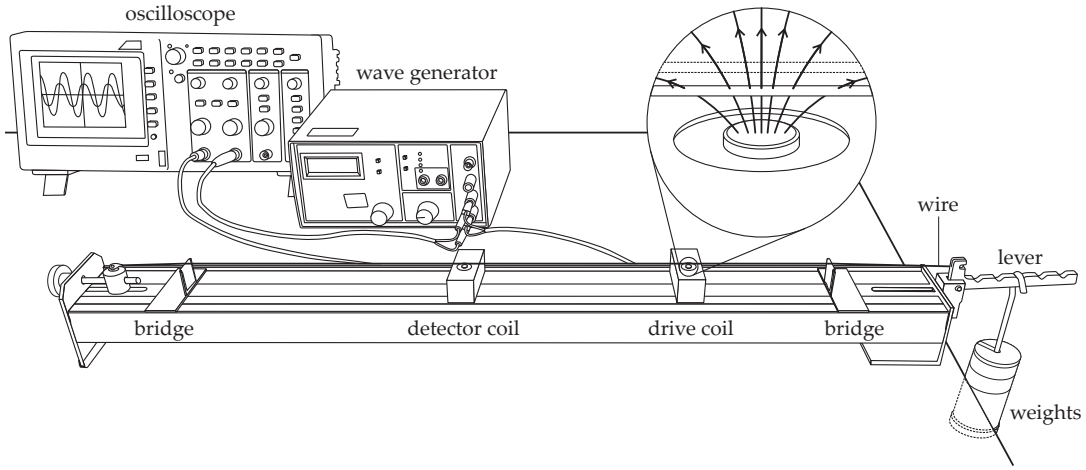


Figure 2: Schematic drawing of the GP03 experimental apparatus.

The basic apparatus is shown in Figure 2, and comprises a set of wires that can be stretched on a bench, a wave generator (oscillator) connected to a 'drive coil' and an oscilloscope connected to a 'detector coil'. Each wire is marked with its nominal diameter (in inches) and is contained in a protective tube. Replace each wire safely back in the tube after use.

The drive coil works as follows: the oscillating current generates a magnetic field according to Ampere's Law. The field acts on the wire with a force given by equation 9. There are two significant terms in the drive equation 9 (see Theory section). One produces an oscillating force at the same angular frequency, ω_D, as the drive coil, while the other produces a force at **twice** the drive frequency $2\omega_D$.

The detector consists of a permanent magnet inside a coil. The magnet induces a constant magnetisation in nearby objects, in this case the wire. If the object then moves, the magnetic field lines move too and the magnetic flux through the coil changes. By Faraday's Law of Electromagnetic Induction this changing magnetic flux generates an electromagnetic field across the ends of the coil, which is detected using the oscilloscope (positioned on the bench near the set-up).

Start with the wire whose nominal diameter is 0.017 inches. A **separate specimen** is provided, 50 cm in length, and with no end tags. This can be found next to the chemical balance, located by the entrance of the lab near experiment GP14. Find the mass per unit length ρ of the 50 cm specimen using a chemical balance.



Do not use the wire provided with your apparatus! Bending these wires will damage them making it difficult for you to get any results.

Set the two bridges over which the wire passes 60 cm apart. This distance will define the length L. Hang 1 kg on the middle notch of the lever.

▶ Calculate the tension in the wire. What determines the error on the tension? How important is it that the lever is horizontal? Are we justified in assuming that the distance between the bridges defines the length L? Is the wire perfectly "fixed" at 0 and L, and can this condition be tested experimentally? Is this condition important and why? Can this condition be affected by large-amplitude oscillations?

Set up the signal generator and oscilloscope with the adjustments shown in the table.

Signal Generator Amplitude		$\simeq 10\%$
Input CH1	Connect to detector coil	2 mV/Div
Input CH2	Connect to wave generator	200 mV/Div
Time base		2.5 ms/Div

Table 1: Basic scope setup

Set the range and frequency of the generator such that two decimal places are displayed for the frequencies. Try turning the frequency knob. Note that its response is velocity-dependent —- the more sharply you turn it, the larger the frequency change.

Initially place the detector coil at the centre of the wire and the drive coil 10 cm from one end. Draw the experimental set-up and the expected waveform for $n = 1$.

4 Experimenting with resonances

Before starting the data collection, you should become familiar with the procedures to estimate and propagate experimental errors, as described, for example in the AD02 script. For derived quantities, it is good practice to calculate values and errors at the same time. Subsequently, only significant digits should be quoted or tabled (one digit only for the error) — for example, 0.2796 ± 0.0017 should be denoted 0.280(2).



4.1 Locating the fundamental frequency

The simplest theory of resonance of a vibrating wire held at each end is given in the Theory section.

Scan for resonances *slowly and systematically*. Set the drive amplitude at 500 mV (so 1 V peak-to-peak).

The resonances are quite narrow (approximately 0.5 Hz FWHM), so one must turn the frequency knob *very slowly*.

▶ Why should the amplitude knob be kept as low as possible?

Scan the drive frequency over the range $f_D = 120$ to 140 Hz in steps of 0.5 Hz, **dwelling at each step for 5-10 seconds** to permit the response to develop. These settings are approximate and should not be followed to the letter. If you cannot find the resonance, feel free to explore the values outside the ranges stated here. Note the range of frequencies over which the response is large, and the exact value of f_D where the response is maximum. Lower the drive amplitude to 100 mV and repeat the scan in steps of 0.1 Hz within this restricted frequency range. Look for a response in the form of a steady sine wave of amplitude less than 5 mV. You may need to repeat with a reduced drive amplitude at the beginning to keep the response small and sinusoidal.

4.2 Deviation from sinusoidal behaviour

▶ In some cases, the wave observed on the oscilloscope is not a sine wave. Can you explain why? Remember that any waveform can be described as a superposition of pure sine waves. The full set of these waves (the 'harmonics') is known as the 'harmonic series' and their analysis is known as 'harmonic analysis'.

Investigate systematically any deviations from sinusoidal behaviour by a) increasing the drive amplitude at fixed frequency and b) moving the drive and detector away from the original position.

▶ Can you understand what is happening based on the discussion in the Theory section?

▶ Comment on any instability. (There will be an opportunity to investigate it in section 6.2 below.)

4.3 Scanning the resonance peak

Tabulate the amplitude as a function of frequency in the immediate neighbourhood of the resonance. Find the value of f, the resonant frequency, to within 0.1 Hz. Plot the shape of the resonance (amplitude versus frequency) and determine its width. What is the Q factor of this resonance?

Record the change in the phase difference between drive and response waveforms when the frequency is moved from just below resonance to just above.

▶ Explain this change based on the theory of the driven and damped harmonic oscillator.

4.4 Confirming that the resonance you found is the fundamental

▶ You should now ascertain that the resonance you found corresponds to the fundamental frequency ($n = 1$).

This can be done in two ways (try both).

- After carefully removing the drive unit away from the string, gently pluck the string near its centre. This excites predominantly the fundamental frequency, which has an antinode at the centre. Verify on the scope that this is indeed the same as the previous drive frequency f_D. *Do not disconnect the driver*, as this provides the signal to which the scope is locked.
- Change the drive frequency to $f_D/2$, $f_D/3$ etc., and show that, at these frequencies, you cannot excite the string *with the same frequency as the drive* (note down any parasitic excitations for later).

4.5 Inferring the wave velocity and the mass density ρ

▶ Determine the velocity of waves on the wire and its error. What happens if the frequency is shifted away from the exact resonance?

Using the formula $v = \sqrt{T/\rho}$ for the velocity in terms of the tension T and mass per unit length ρ, determine the value of ρ for the wire with its error. Compare your new inferred value of ρ with your earlier direct measurement.

▶ Are they consistent? Which is more accurate? What is the dominant systematic uncertainty in this calculation?

5 Further investigations

5.1 Harmonics and nodes

Predict the frequency of the first, second and third harmonics. Also, calculate the exact position of the nodes and antinodes for each harmonic. Consider carefully their pattern of nodes and find the optimal position of the drive and detector for each resonance. Remember that your goal is to drive each resonance with the drive component f_D, *not* with the 'parasitic' component $2f_D$. You also want to minimise the excitation of higher harmonics with the parasitic component of the force.

For each harmonic, set up the drive and the detector appropriately and scan for these harmonics in the same systematic way as section 4 and determine their frequencies. Remember to keep the amplitude of the drive as low as possible.

▶ Do the frequencies match your predictions within errors? Comment on any discrepancies. Is the wire a truly isolated system?

(OPTIONAL) You can measure the wave profile by scanning the position of the detector, while the position of the driver remains fixed at an antinode. In order to get an accurate profile, detector and drive must be as far apart as possible to avoid interferences, so *it is strongly advised to attempt this method on the first harmonic rather than on the fundamental.* Every time the detector is moved, the resonance frequency changes slightly and it needs to be retuned. You should use the MEASURE function on the scope to obtain an accurate reading of the peak-to-peak amplitudes.

5.2 Reproducibility

Remove the 1 kg weight, replace it a few times, observe the resonance, record the observations and explain.

▶ Does the value of the resonant frequency of the wire change? Why? In the light of these observations how might the apparatus be redesigned?

5.3 Testing the formula for wave velocity

The formula used above for the velocity of the waves on the string was $v = \sqrt{T/\rho}$, and you have verified this formula for the wire with diameter 0.017 inches. In this section we test the predicted dependence on T and ρ.

Using the wire having diameter 0.017 inches, measure the resonant frequency f for a range of different values of T by varying the suspended mass M.

The frequency formula predicts a dependence of the form

$$f = \sqrt{KM} \qquad \text{or} \qquad f^2 = KM.$$

▶ Deduce an expression for K.

Plot the data and fit to a formula of this form. There are a several ways of doing this, using linear or quadratic fits, and plotting f or f^2, and M or $\sqrt{M}$. Remember that you should plot the data with the smaller fractional error as x, as the fitting routine will assign all the error to the y-data. Is the formula consistent with your data and their independently assessed errors? (The answer to this question, for large, carefully taken data sets is usually no, or only marginally so. You should consider whether your data can shed any light on the question.)

▶ Is the fitted value of K consistent with the value you calculate from your expression? What is the dominant error in the comparison?

Now use the same T that you have applied in your original investigation, and calculate the fundamental frequencies for wires of nominal diameters 0.020 and 0.022 inches. The appropriate values of ρ can be obtained either by weighing the 50 cm sample wires in the chemical balance or by scaling the linear mass density with the diameter of the wire (use both methods, if possible).

Now take data near the predicted resonances for different wires and at fixed T. Fit the data to a formula of the form $f^2\rho = G$, using the values of ρ determined by the two methods.

▶ Are the data consistent with the formula? Is the fitted value of G consistent with your expression? What is the dominant error in the comparison?

Optional

6 Optional section: Further investigations

6.1 Role of damping in the behaviour of resonances

Set the wire oscillating at resonance as in section 4. Change the oscilloscope to trigger on the detected waveform. Using a stopwatch, measure the time taken for the amplitude of the wire vibration to fall by a factor 1/e after disconnection of the drive coil from the generator.

Estimate τ the time for the *energy* of the resonance to fall to 1/e. Then determine the Q of the resonance, $Q = \omega\tau$, see the experiment GP04 — *Driven harmonic motion*[3], or [2]. Is this value of Q consistent with the value you previously obtained by measuring the width of the resonance?

▶ Discuss two distinct ways in which the wire can lose energy.

6.2 Precise position of nodes and coupling to vibrations of the bench

With the wire vibrating at resonance, place the end of your finger on the bridge at the end of the wire. (You may need to turn up the amplitude to detect anything.) Is this a true node?

Set the wire oscillating at resonance with low amplitude as in section 4. Place your hand firmly on the metal bench. Observe and discuss whether the bench itself (through its mass, tension and elastic properties) is part of the normal vibration mode, just as the resonance of a violin or guitar string is coupled to the vibration of the bridge and body.

6.3 Frequency multiplication and nonlinear effects.

It is possible to drive the *first* harmonic of the wire with frequency f_2 ($n = 2$) by setting $f_D = f_1$ (the *fundamental* frequency). In doing this, one exploits the 'parasitic' component of the drive, which is increased at high drive amplitude (See Theory section for a detailed explanation). To observe this effect, you should set up both drive and detector so as to maximise the amplitude and detection of the first harmonic. Discuss the optimal configuration of the apparatus.

At larger wire displacements (large drive amplitude), much higher harmonics of the drive frequency can be observed. Can you find them?

▶ Discuss how this effect relates to the Taylor expansion of the force on the wire and of the electromotive force at the detector coil about the equilibrium position z_0 of the string (see Theory section). What are the optimal conditions to observe this effect?

▶ Now hand-write a brief summary of the experiment in your logbook. State the aims of the experiment, the method used, quantitative results, and your conclusions, in not more than about half a page.

▶ Discuss your results with a demonstrator as part of your assessment.

Bibliography

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[2] A. P. French, *Vibrations and Waves*, W. W. Norton & Company; 1st edition (1971)

[3] W. J. Duffin, *Electricity and Magnetism*, McGraw-Hill Publishing Co., 3rd Edition (1980), paragraph 8.6