BASIC GEOPHYSICAL FLUID DYNAMICS

Tim Woollings, HT 2022

(Thanks to David Anderson, David Andrews and Peter Read for earlier versions)

Topics to be covered (6 lectures):

- Introduction and overview. Dynamics and motion of Earth's atmosphere: differential heating, buoyant response, heat and angular momentum transport.
- Basic equations of motion, mass conservation, equations of state.
- Rotating frame of reference, Centrifugal and Coriolis forces, Local Cartesian coordinates, Scale analysis.
- Geostrophic balance, hydrostatic balance, thermal wind.

- Pressure coordinates, geopotential.
- Thermodynamic energy equation.
- Shallow water and reduced gravity models, f and β-planes, potential vorticity.
- Inertia-gravity waves, dispersion relation, phase and group velocity.
- Rossby number, equations for nearly geostrophic motion, Rossby waves, Kelvin waves.

Further topics on dynamics in oceans and on other planets will be covered in the Advanced GFD next term.

Suggested Books

• Vallis, *Essentials of Atmospheric and Oceanic Dynamics* (CUP 2019)

• Hoskins & James. *Fluid Dynamics of the Mid-Latitude Atmosphere* (Wiley-Blackwell, 2014)

• Andrews, *An Introduction to Atmospheric Physics*, Chapter 4

[Either 1st edition (2000) or 2nd edition (2010) will do.]

• Holton, *An Introduction to Dynamic Meteorology*, Chapters 1–3

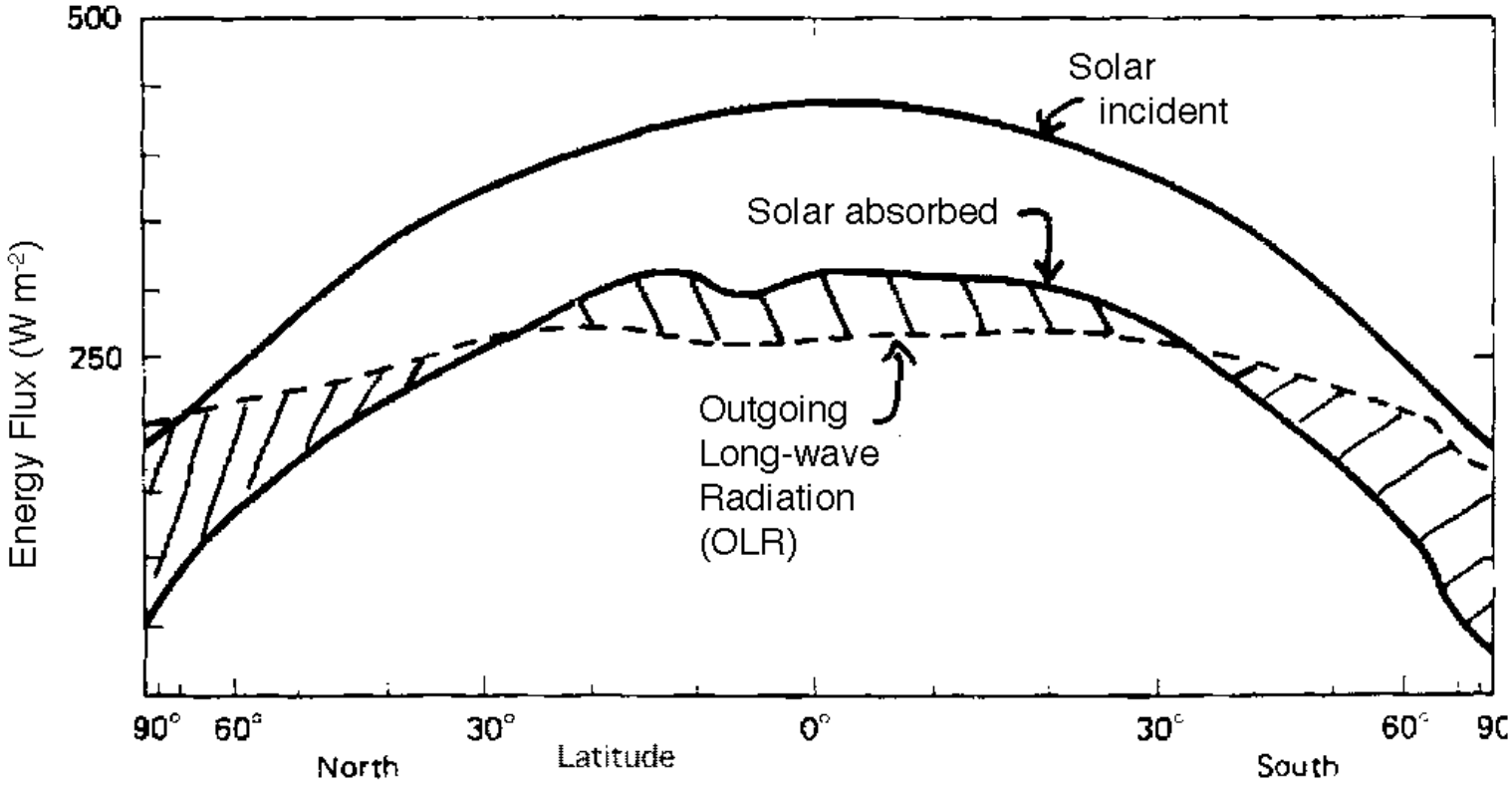
More advanced:

• Vallis, *Atmospheric and Oceanic Fluid Dynamics* (CUP 2006)

• Marshall and Plumb, *Atmosphere, Ocean and Climate Dynamics* (Academic Press 2008)

1. Introduction to Atmosphere-Ocean Dynamics

- Why does the atmosphere move to form winds?



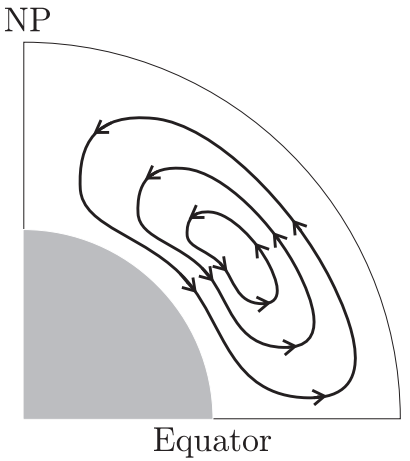
[After Gill (1978).]

Heat balance between equator and poles

- Observations show that the equator absorbs more heat than it loses via outgoing long-wave radiation (OLR) and the poles lose more heat via OLR than gained from insolation.
- So a balance between latitudes must be achieved by *fluid motion* (in atmosphere and ocean) carrying heat from equator to poles.
- The *energy* source for the motion is differential heating/cooling.
- Leads to temperature contrasts $\partial T/\partial y$.
- Density depends on temperature, hence leads to *buoyancy contrasts*, forces and **motion**.

2. Transport of Heat and Angular Momentum in the Atmosphere

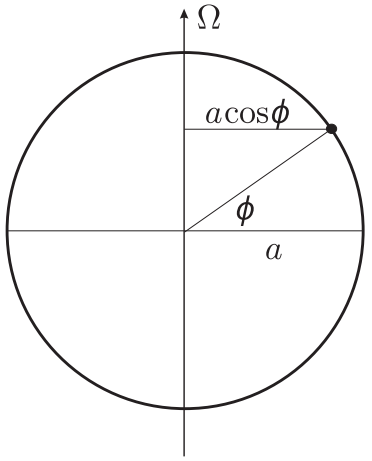
• You might think the simplest response to differential radiative heating would be a simple zonally-symmetric equator-pole convection cell (Hadley 1735):



• Hadley circulation transports heat upwards and polewards, **but** Earth is rotating, so also need to consider conservation (or otherwise) of angular momentum of air 'parcel' with zonal speed u relative to rotating Earth.

• Angular momentum/unit mass about Earth's axis: $m = a\cos\phi\,(\Omega a\cos\phi + u)$, where u is the eastward velocity component w.r.t the Earth.

($a = 6.37 \times 10^6$ m, $\Omega = 7.29 \times 10^{-5}$ rad s^{-1})

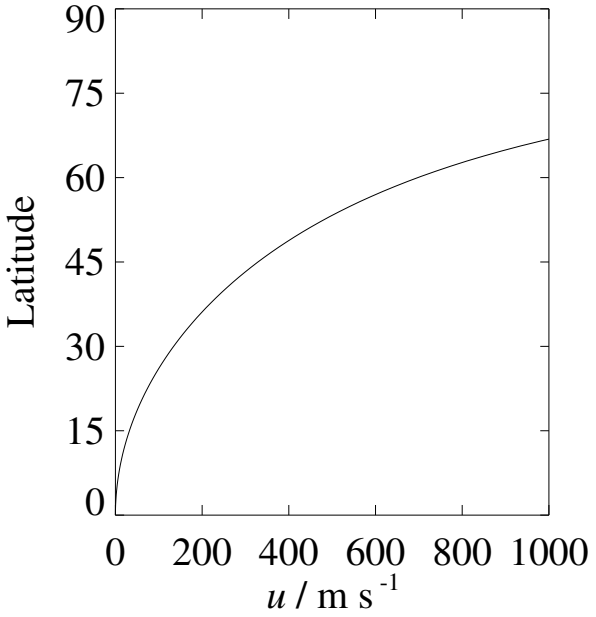


• For simplicity, consider a parcel starting at rest on the equator (i.e. $u = 0$) so initial angular momentum $m = a^2\Omega$.

• **If the air parcel conserves its angular momentum** as it moves polewards in the upper part of the Hadley cell, then

$$a\cos\phi\,[u(\phi) + \Omega a\cos\phi] = a^2\Omega \quad \Rightarrow \quad u = u_M \equiv a\Omega\left(\frac{1}{\cos\phi} - \cos\phi\right) \qquad (1)$$

• For the Earth, $a\Omega \sim 465$ m s^{-1}. So if angular momentum of parcel were to be conserved, its variation of zonal velocity with latitude would be:



• **But** winds $\gtrsim 150$ m s^{-1} are not observed: the simple Hadley cell can't extend far from equator; large shears, etc, lead to instability and complicated 3D eddies/waves (at least outside tropics). [More on this later: Held & Hou model.]

3. Ocean circulation: thermohaline and wind-driven

Buoyancy forces are also important for the circulation of the oceans, but things are more complicated than for the atmosphere...

- Oceans are broken up into bounded domains or 'basins'.
- Buoyancy variations are due to both differential heating **and** salinity (freshwater fluxes etc).
- Direct differential heating is restricted to near the surface - role of deep convection?
- Part of the circulation is driven by surface wind stresses.

Much more detail in the advanced GFD lectures...

4. Basic equations of atmosphere-ocean dynamics

The atmosphere and oceans are *fluids* – we need to apply laws for *bulk* motion of fluids. To recap, see B1 Flows, Fluctuations and Complexity lecture notes.

These laws are:

1. Newton's Second Law ('Navier-Stokes equation')
2. Mass conservation law ('continuity equation')
3. Equation of State (atmosphere or ocean)
4. Thermodynamic Energy equation (from First Law of Thermodynamics)

4.1 Equations of motion

Based on Newton's second law:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\boldsymbol{k} + \frac{\mu}{\rho}\nabla^2\boldsymbol{u} \quad \left[+ \frac{\mu}{3\rho}\nabla\left(\nabla\cdot\boldsymbol{u}\right) \right] \tag{2}$$

– the *Navier-Stokes equation* of fluid flow, in an **inertial reference frame.**

Here

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\ .$$

The term on the far right of equation (2) is present when the fluid is compressible ($\nabla\cdot\boldsymbol{u} \neq 0$.)

4.2 Mass conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) \quad = \quad \frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} \quad = \quad 0\ . \tag{3}$$

4.3 Thermodynamic energy equation

$$c_p \frac{DT}{Dt} - \frac{1}{\rho}\left(\frac{Dp}{Dt}\right) = Q \tag{4}$$

where c_p is the specific heat capacity at constant pressure and Q the diabatic heating rate. [See equation (19), page 46.]

4.4 Equation of state

$$\rho = \rho(p, T, S...) \qquad [S = \text{Salinity}, \text{etc...}] \tag{5}$$

Equation of state: Atmosphere

- Assume an ideal (perfect) gas:

$$pV = RT \qquad \text{for 1 mole}$$

where R = molar gas const = 8.3 J mol^{-1} K^{-1}.

If M = mass of 1 mole,

$$\rho = M/V \ , \qquad \text{so} \qquad \rho = Mp/RT$$

Gas constant for unit mass of dry air:

$$R_{\rm a} = R/M = 287\,{\rm J\,K^{-1}\,kg^{-1}}$$

So equation of state is

$$p = R_{\rm a} T \rho \ . \tag{6}$$

Warning: many Meteorology books use the symbol R for $R_{\rm a}$!

Equation of state: Oceans

$$\rho = \rho(p, T, S, \text{etc...}) \quad \text{where } S \text{ is the salinity.}$$

For small departures from a mean state ρ_0, T_0, S_0,

$$\rho \approx \rho_0 \Big[1 - \alpha(T - T_0) + \beta(S - S_0) \Big]$$

where $(\rho_0, T_0, S_0) \simeq (10^3 \text{ kg m}^{-3}, 10°\text{C}, 35 \text{ parts per thousand})$ respectively,

$$\alpha = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_{p,S}$$

is the thermal expansion coefficient, ~ 1–$3 \times 10^{-4} \text{ K}^{-1}$,

$$\beta = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial S} \right)_{p,T}$$

is the salinity coefficient $\sim 8 \times 10^{-4} \text{ (ppt)}^{-1}$.

Thus far, we have...

- Three equations of motion – components of NS equation (2).
- Equation of mass conservation (3).
- Equation of state (5) (for atmosphere or ocean).
- Thermodynamic energy equation (4) - to be studied later, page 46.

$\Rightarrow$ 5 or 6 equations in unknowns u, v, w, p, ρ [and T].

This is good enough in principle for most purposes (apart from moisture or salinity transport...), but not very convenient in an **inertial reference frame.**

Also, equations are too complicated to solve in practice - **need to simplify!**

5. Rotating reference frames

• The rotation of the Earth is significant for large-scale dynamics of the atmosphere/ocean

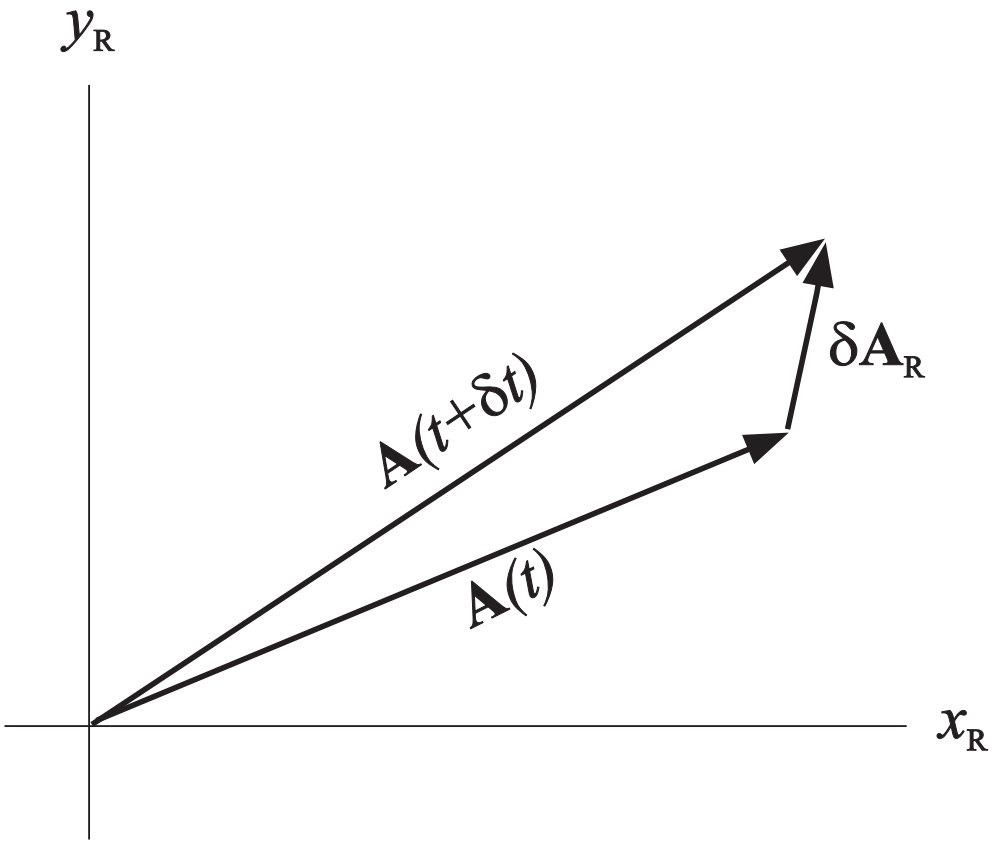
• Use rotating axes, fixed w.r.t. Earth –

but time-derivatives of vectors must be modified:

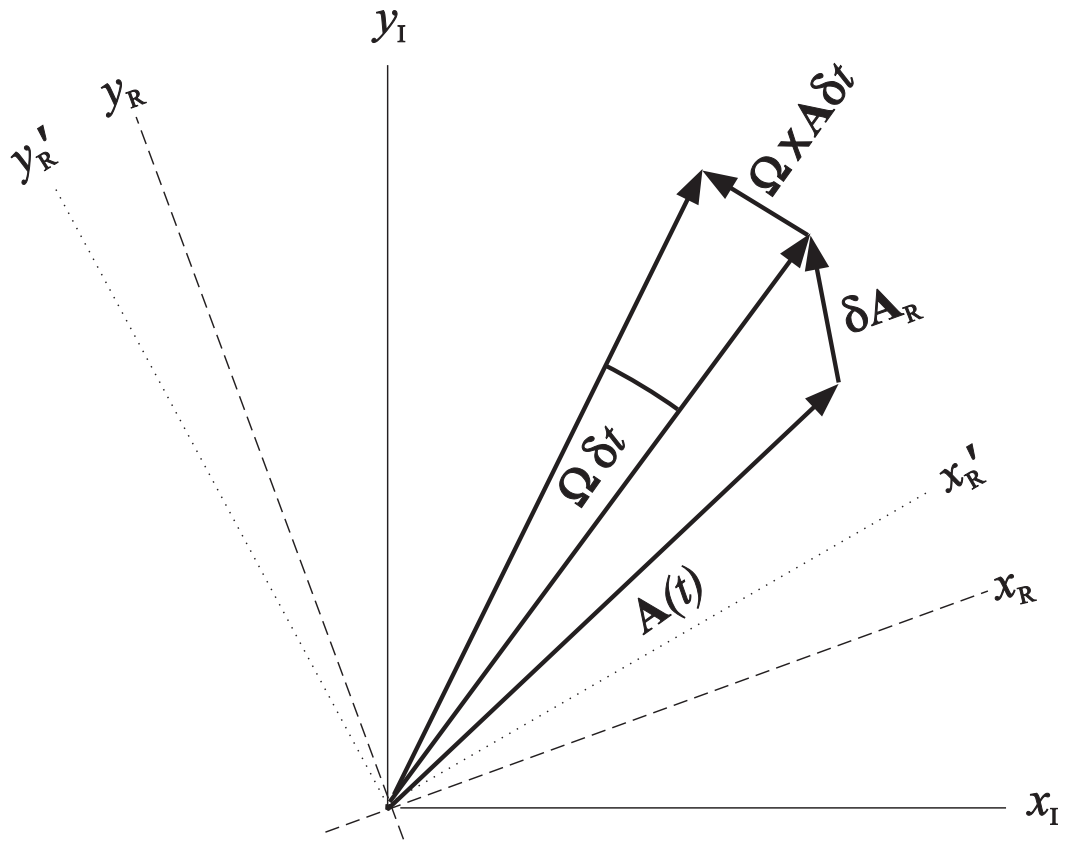
Suppose frame R rotates at constant angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$ w.r.t. an inertial frame I.

Consider vector $\boldsymbol{A}(t)$ – suppose for simplicity that it lies in the $xy-$plane of R and I.

As viewed in R, in time interval δt, $\boldsymbol{A} \to \boldsymbol{A}(t+\delta t) \approx \boldsymbol{A}(t) + \delta \boldsymbol{A}_{\rm R}$, where $\delta \boldsymbol{A}_{\rm R} = \left(\dfrac{d\boldsymbol{A}}{dt}\right)\delta t$:



But if same $\boldsymbol{A}$ is viewed in I, we must allow for the rotation of frame R w.r.t frame I in time δt; this gives an extra bit of $\left(\dfrac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}}$:



Extra bit $= \boldsymbol{\Omega} \times \boldsymbol{A}\delta t$. (The same works if $\boldsymbol{A}$ is not in the $xy-$plane.)

So as $\delta t \to 0$,

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{A} \; .$$

Repeat this procedure:

$$\left(\frac{d^2\boldsymbol{A}}{dt^2}\right)_{\mathrm{I}} = \left(\frac{d^2\boldsymbol{A}}{dt^2}\right)_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{A}) \; .$$

In particular, if $\boldsymbol{A} = \boldsymbol{r}(t)$, the position vector,

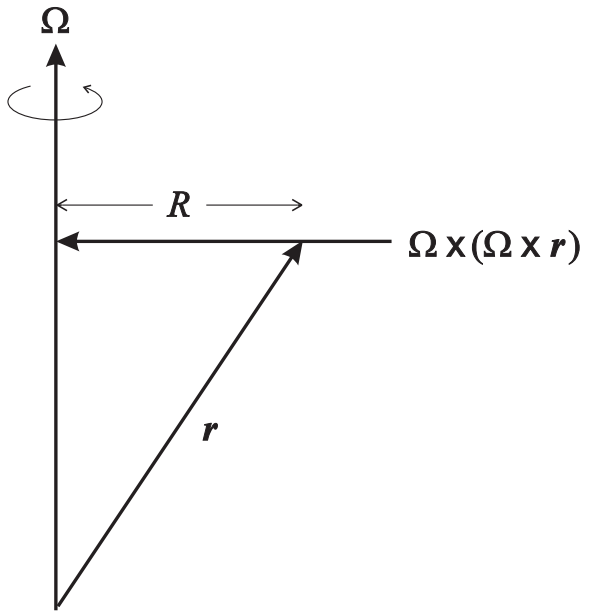
$$\boldsymbol{a}_{\mathrm{I}} = \boldsymbol{a}_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \; .$$

NB: with $\boldsymbol{A} = \boldsymbol{r}(t)$, the position vector of a moving fluid parcel, we are talking about the *Lagrangian* description, so

$$\frac{d}{dt} \quad \text{should be replaced by} \quad \frac{D}{Dt} \; .$$

• $2\boldsymbol{\Omega} \times \boldsymbol{u}_{\mathrm{R}}$ is the *Coriolis acceleration*: it is perpendicular to the velocity $\boldsymbol{u}_{\mathrm{R}}$ in the rotating frame, and to $\boldsymbol{\Omega}$.

• $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ is the *centripetal acceleration*: it has magnitude $\Omega^2 R$, and points perpendicularly towards the rotation axis:



[**Note errors** on p. 107 of Andrews, *An Introduction to Atmospheric Physics*, 1st edition (2000)!]

• Now replace $\boldsymbol{a}$ in Newton 2 by $\boldsymbol{a}_{\mathrm{I}}$, drop subscript R, to get the equation of motion *with respect to the rotating frame:*

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p \ \underline{-2\boldsymbol{\Omega}\times\boldsymbol{u} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})} - g\boldsymbol{k} + \text{friction} \tag{7}$$

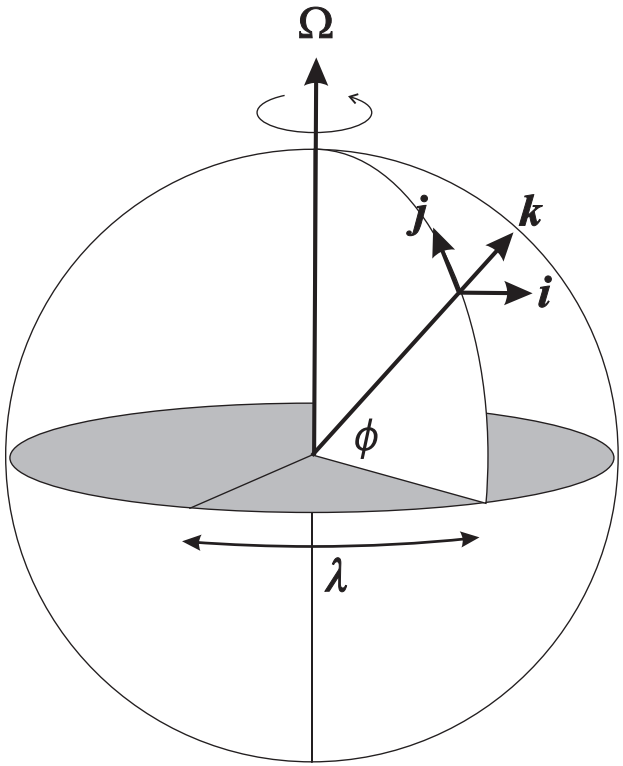
where $\boldsymbol{u}$ is now the velocity measured in the rotating frame R.

• The underlined terms are 'fictitious forces': the *Coriolis force* and the *centrifugal force*, respectively.

• We often put $\boldsymbol{g}' = -g\boldsymbol{k} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$, the *effective gravity*. The extra term is $\ll g$ in magnitude, and in a different direction.

6. Equations in local Cartesian coordinates

• We can work in spherical coords, but slightly easier to use local Cartesians:



• $\boldsymbol{i}$ points eastwards ($x-$direction), $\boldsymbol{j}$ northwards ($y-$direction), $\boldsymbol{k}$ upwards ($z-$direction) at latitude ϕ.

• Note that $\boldsymbol{\Omega} = \Omega(\boldsymbol{j}\cos\phi + \boldsymbol{k}\sin\phi)$

• Velocity components: $\boldsymbol{u} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$. But note that $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ change with time, following the motion. So get

$$\frac{D\boldsymbol{u}}{Dt} = \frac{Du}{Dt}\boldsymbol{i} + \frac{Dv}{Dt}\boldsymbol{j} + \frac{Dw}{Dt}\boldsymbol{k} + u\frac{D\boldsymbol{i}}{Dt} + v\frac{D\boldsymbol{j}}{Dt} + w\frac{D\boldsymbol{k}}{Dt} \tag{8a}$$

$$\nabla p = \frac{\partial p}{\partial x}\,\boldsymbol{i} + \frac{\partial p}{\partial y}\,\boldsymbol{j} + \frac{\partial p}{\partial z}\,\boldsymbol{k} \tag{8b}$$

$$2\boldsymbol{\Omega} \times \boldsymbol{u} = 2\Omega(w\cos\phi - v\sin\phi)\,\boldsymbol{i} + 2\Omega u\sin\phi\,\boldsymbol{j} - 2\Omega u\cos\phi\,\boldsymbol{k} \tag{8c}$$

Neglecting the centripetal acceleration, (7), (8) $\Rightarrow$

$$\text{Eastward :} \qquad \frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho}\frac{\partial p}{\partial x} = F^{(x)} \tag{9a}$$

$$\text{Northward :} \qquad \frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{1}{\rho}\frac{\partial p}{\partial y} = F^{(y)} \tag{9b}$$

$$\text{Upward :} \qquad \frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\Omega u\cos\phi + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = F^{(x)} \tag{9c}$$

where $F^{(x)}$, $F^{(y)}$ and $F^{(z)}$ are components of the frictional force and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

[See e.g. Andrews, Appendix B for more details.]

7. Simplifications

Eqs (9) are complicated, but we can simplify them:

• Atmosphere/ocean are *shallow* compared with Earth's radius ($a \sim 6400$ km).

So replace r by a in (9).

• Other terms in (9) can be neglected for motions with relevant time and space scales: e.g. for atmospheric weather systems:

Scale	*Symbol*	*Typical magnitude*
Horizontal scale	L	1000 km $= 10^6$ m
Vertical scale	H	10 km $= 10^4$ m
Horizontal velocity	U	10 m s^{-1}
Vertical velocity	W	10^{-2} m s^{-1}
Time scale	T	1 day $\sim 10^5$ s
Density	ρ	1 kg m^{-3}
Earth's radius	a	6.4×10^6 m
$2 \times$ rotation rate	2Ω	10^{-4} s^{-1}
Acceleration of gravity	g	10 m s^{-2}

Size of terms in vertical momentum eqn (9c):

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$\sim \frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} \sim \frac{10^{-2}}{10^5} + \frac{10^{-1}}{10^6} + \frac{10^{-4}}{10^4} \sim 10^{-7}$$

$$\frac{u^2 + v^2}{r} \sim \frac{10^2}{6 \times 10^6} < 10^{-4}$$

$$2\Omega u \cos\phi \sim 10 \times 10^{-4} = 10^{-3}$$

$$g \sim 10^1 \,, \qquad F^{(z)} \text{ negligible.}$$

(All in m s^{-2}.)

The remaining term is $\frac{1}{\rho}\frac{\partial p}{\partial z}$: this is the only term that can possibly balance the large g term, so $(9c)$ must become, to a good approximation,

$$\frac{\partial p}{\partial z} = -g\rho \tag{10}$$

This is the *hydrostatic approximation*.

Now consider the horizontal momentum equation (9a): [(9b) is similar.]

$$\partial u/\partial t \sim 10^{-4}\,, \qquad u\partial u/\partial x \sim 10^{-4}$$

$$w\partial u/\partial z \sim 10^{-5}\,, \qquad uw/r \sim 10^{-8}$$

$$2\Omega v \sin\phi \sim 10^{-3}\,, \qquad 2\Omega w \cos\phi \sim 10^{-6}$$

and $F^{(x)}$ is negligible.

Here $2\Omega v \sin\phi$ is the biggest term (just), so must be balanced by the remaining term $\rho^{-1}\partial p/\partial x$, giving the *geostrophic approximation*:

$$fv = \frac{1}{\rho}\frac{\partial p}{\partial x}\,, \qquad -fu = \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{11}$$

where $f = 2\Omega \sin\phi$ is the *Coriolis parameter*. (Local vertical component of $2\boldsymbol{\Omega}$).

Note: $f > 0$ in Northern Hemisphere, $f < 0$ in Southern Hemisphere.

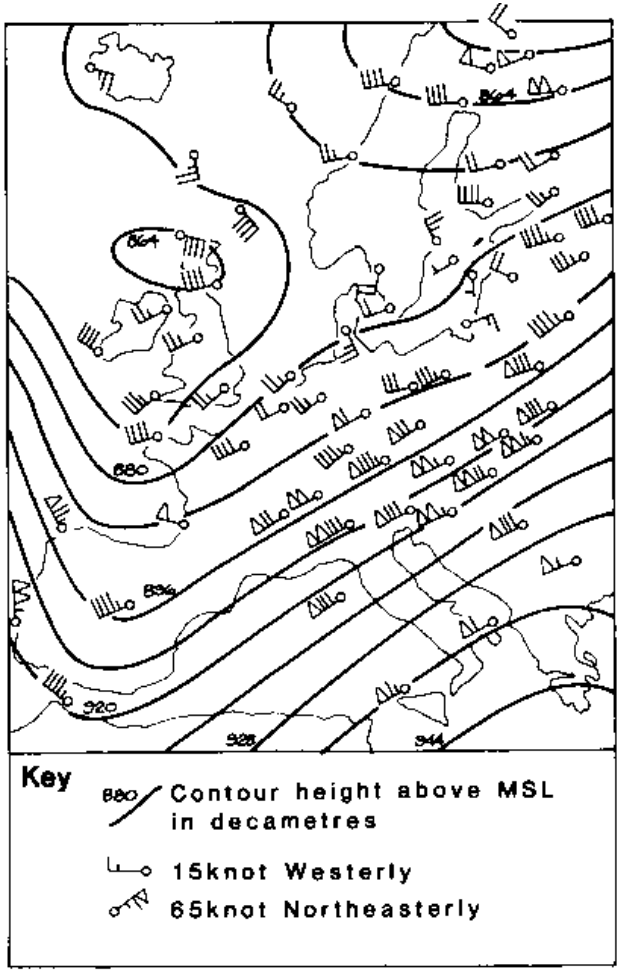
• Geostrophic balance means that horizontal pressure gradients are largely balanced by Coriolis forces associated with the horizontal winds.

• Note that $(u, v, 0)\cdot\nabla p = 0$ so, at a given height, the horizontal wind *blows along the isobars* (constant-pressure surfaces), under the geostrophic approximation.

• Note also that it is twice the *vertical* component of the rotation vector, $2\Omega \sin\phi$, that enters.

• Geostrophic balance is a **diagnostic** relationship - no time derivatives, hence no *prediction.*

Example of geostrophic wind balance: 300 hPa flow



NB: contours close together ⇒ high winds, contours far apart ⇒ weak winds.

• The ratio of the 'horizontal advection' term to the 'Coriolis term' in $(9a)$ is approximately

$$\frac{u\partial u/\partial x}{fv} \sim \frac{U^2/L}{fU} = \frac{U}{fL} \equiv \mathrm{Ro} \tag{12}$$

Ro is a dimensionless number, the *Rossby Number*.

• If $\mathrm{Ro} \gg 1$, the Coriolis term is comparatively small.

• If $\mathrm{Ro} \ll 1$, the Coriolis term is comparatively large, and the geostrophic approximation is usually valid.

8. The 'thermal wind'

Recall the ideal gas law (6):

$$p = R_{\mathrm{a}} T \rho \tag{13}$$

where $R_{\mathrm{a}} = R/M$, the gas constant for dry air.

From (11a) and (13) we have

$$fv = \frac{R_{\mathrm{a}} T}{p} \frac{\partial p}{\partial x} = R_{\mathrm{a}} T \frac{\partial \ln p}{\partial x}$$

while from (10) and (13) we have

$$-\frac{g}{R_{\mathrm{a}} T} = \frac{\partial \ln p}{\partial z} \, .$$

Cross-differentiation of these (to eliminate p) gives

$$\begin{aligned} f\frac{\partial v}{\partial z} &= R_{\mathrm{a}}\left(\frac{\partial T}{\partial z}\right)\left(\frac{\partial \ln p}{\partial x}\right) + R_{\mathrm{a}}T\frac{\partial}{\partial z}\left(\frac{\partial \ln p}{\partial x}\right) \\ &= \frac{1}{T}\left(fv\frac{\partial T}{\partial z} + g\frac{\partial T}{\partial x}\right) \end{aligned}$$

Typical scales:

$$\frac{\partial T}{\partial x} \sim \frac{10\ \mathrm{K}}{1000\ \mathrm{km}} \quad \Rightarrow \quad g\frac{\partial T}{\partial x} \sim 10^{-4} \quad \mathrm{K\,s^{-2}}\,,$$

$$\frac{\partial T}{\partial z} \sim 6\ \mathrm{K\ km^{-1}} \quad \Rightarrow \quad fv\frac{\partial T}{\partial z} \sim 10^{-6} \quad \mathrm{K\,s^{-2}}\,.$$

Hence, neglecting vertical (but not horizontal) variations in T,

$$f\frac{\partial v}{\partial z} \approx \frac{g}{T}\frac{\partial T}{\partial x} \tag{14a}$$

And similarly from (11b)

$$f\frac{\partial u}{\partial z} \approx -\frac{g}{T}\frac{\partial T}{\partial y} \tag{14b}$$

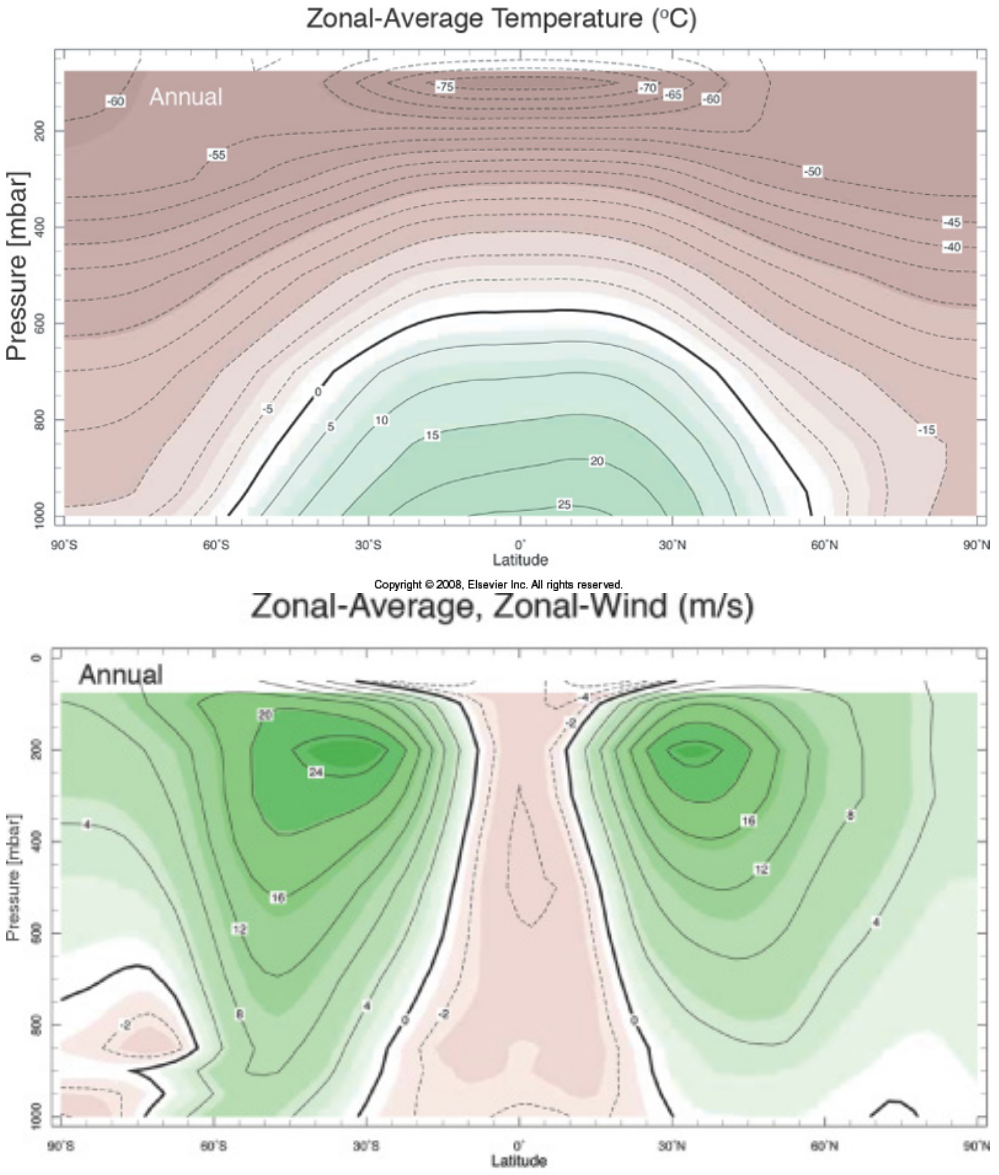
These are called the *thermal wind equations*. (More correctly the thermal wind *shear* equations.)

• NB: Vertical variations of T need not be neglected if we use pressure p, rather than height z, as a vertical coordinate. See §10.

Example

The zonally-averaged (i.e. longitudinally-averaged) zonal (i.e. eastward) winds tend to be nearly in thermal-wind balance with the zonally-averaged temperature.

NB: this may break down near the equator, where $f = 2\Omega \sin\phi \approx 0$.

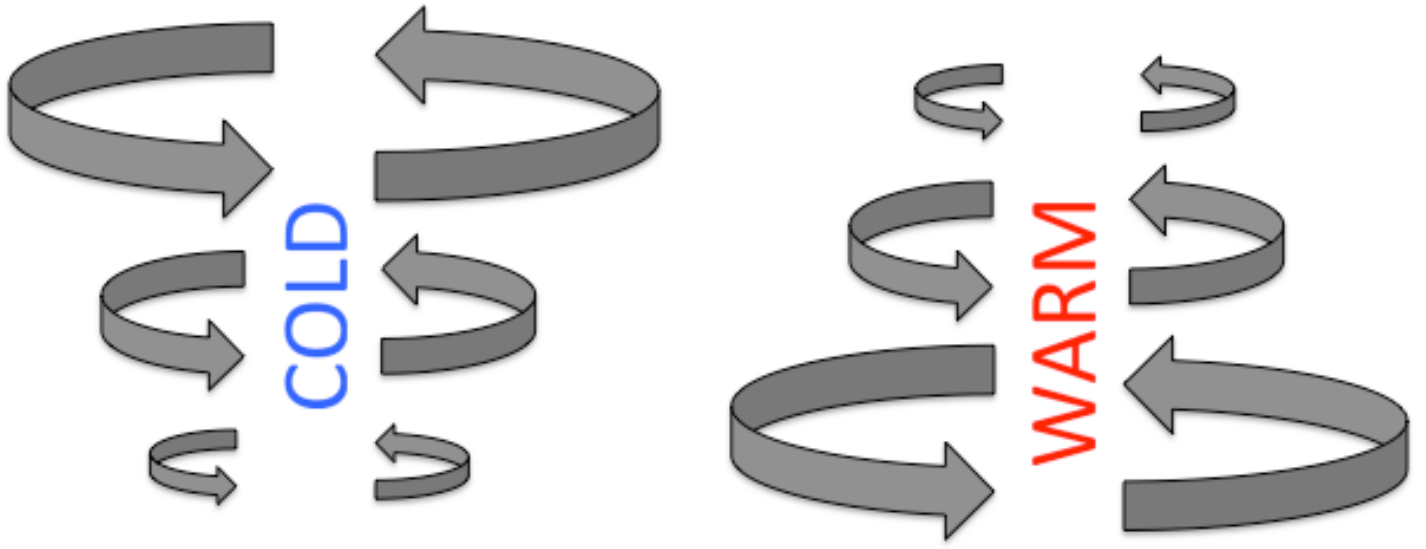


Marshall and Plumb 2008

Another example: Cold and Warm core cyclones

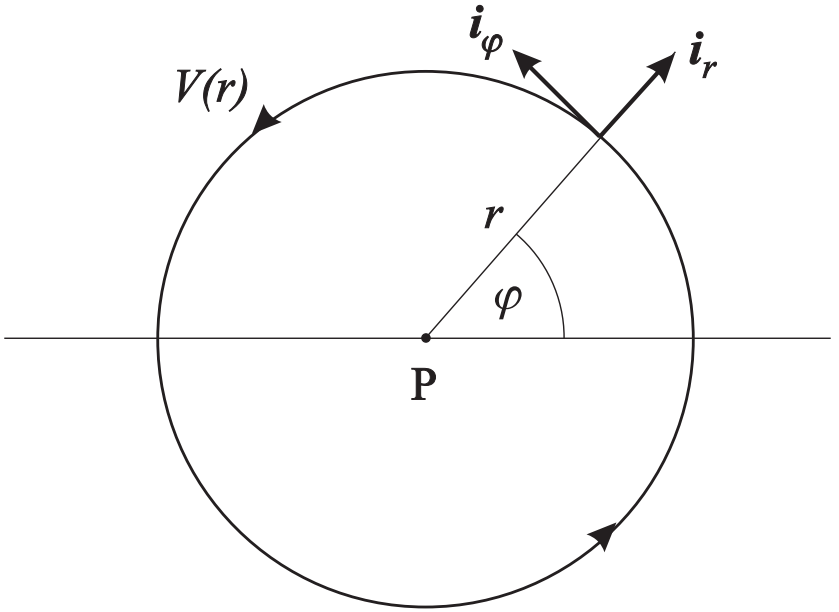
Forming the vorticity from equation 14 gives:

$$\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\approx\frac{g}{fT}\nabla^2 T \tag{15}$$



9. Circular vortex: gradient-wind balance

• Consider a steady vortex with circular streamlines, centred at point P, near latitude ϕ. Assume flow is horizontal. Use polar coordinates r, φ, z, centred at P:



Velocity $\boldsymbol{u} = V(r)\,\boldsymbol{i}_\varphi$. Coriolis force $= -\rho f \boldsymbol{k} \times \boldsymbol{u} = \rho f V \boldsymbol{i}_r$.

Also include centrifugal force $(\rho V^2/r)\boldsymbol{i}_r$ (associated with circular vortex, **not** with Earth's rotation).

These have to be balanced by a radial pressure gradient, so

$$\frac{V^2}{r} + fV = \frac{1}{\rho}\frac{dp}{dr} \equiv G \ , \ \text{say.}$$

This expresses *gradient wind balance* and is a better approximation than geostrophic balance (which neglects the V^2/r term). It is a quadratic for V, if G is given, with solutions:

$$V(r) = -\frac{rf}{2} \pm \sqrt{\frac{r^2 f^2}{4} + Gr}$$

Note that real solutions are only possible if $G \geq -rf^2/4$, so the pressure can't drop too rapidly with radius.

- When fV is negligible compared with V^2/r we have *cyclostrophic balance.*

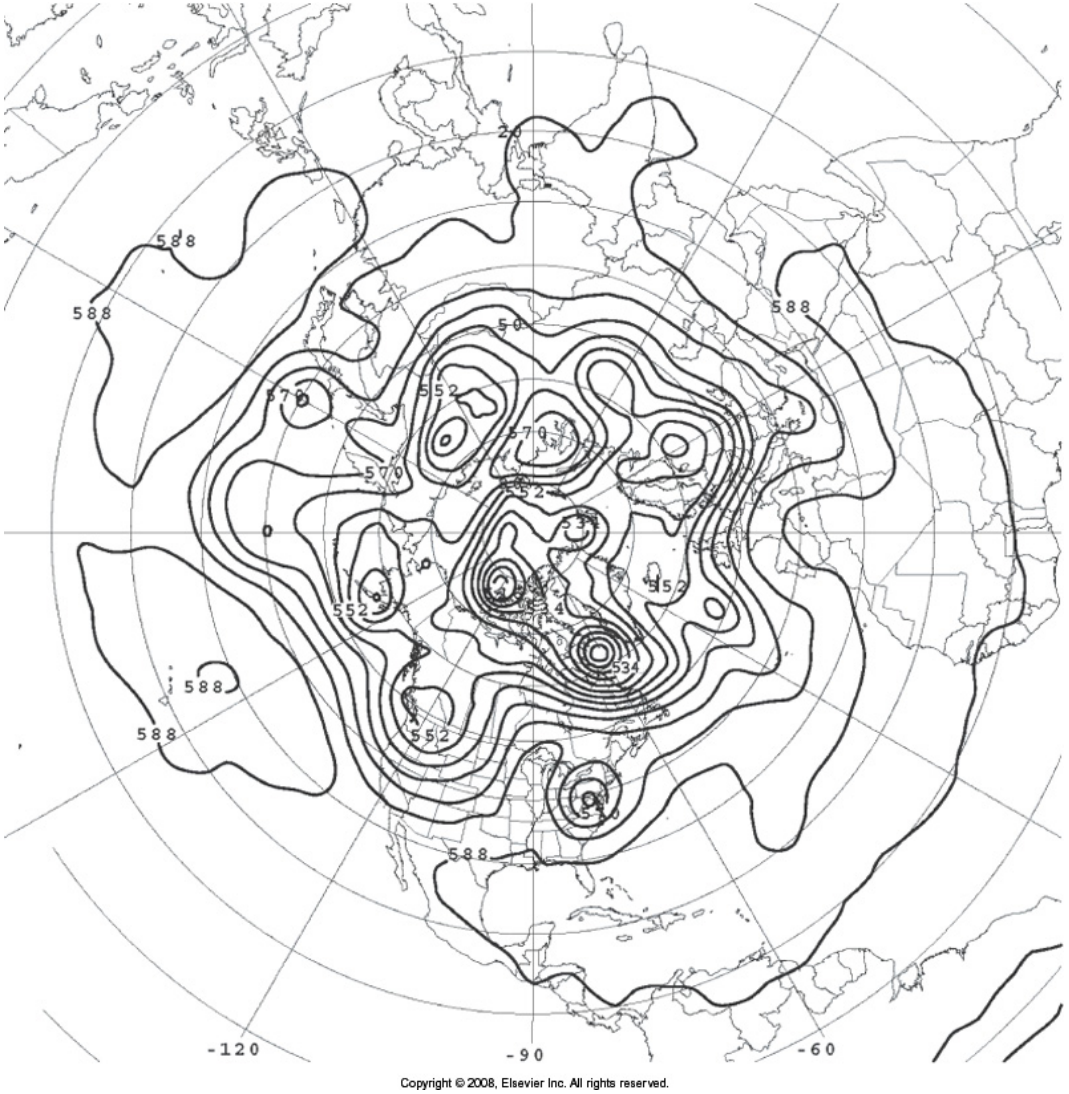
10. Pressure coordinates and geopotential

• Meteorologists often use p, rather than z, as a vertical coordinate. We need to be careful about *which* variables are held constant in partial differentiation!

• For a small column of air, $dp = -g\rho\, dz$ from hydrostatic balance (10), so the pressure decreases monotonically with height.

• We introduce the *geopotential* $\Phi = gz$.

(We often rescale Φ to have units of height by defining the *geopotential height* $h = \Phi/g$, which is just the height of a pressure surface.)



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Typical 500mb height analysis; Marshall and Plumb 2008.

• We can turn (10) upside down, and use (13), to get

$$g\left(\frac{\partial z}{\partial p}\right)_{x,y,t} = -\left(\frac{1}{\rho}\right) = -\frac{R_{\rm a}T}{p}$$

Using Φ, the hydrostatic equation becomes

$$\frac{\partial \Phi}{\partial p} = -\frac{R_{\rm a}T}{p} \tag{16}$$

in pressure coordinates.

We can derive the pressure-coordinate versions of geostrophic balance by eliminating ρ from (10), (11a):

$$fv = -g\left(\frac{\partial p}{\partial x}\right) \Big/ \left(\frac{\partial p}{\partial z}\right) = -g\left(\frac{\partial p}{\partial x}\right)\left(\frac{\partial z}{\partial p}\right)$$

There is a well-known theorem (used e.g. in thermodynamics):

$$\left(\frac{\partial\alpha}{\partial\beta}\right)_\gamma \left(\frac{\partial\beta}{\partial\gamma}\right)_\alpha \left(\frac{\partial\gamma}{\partial\alpha}\right)_\beta = -1$$

Applying this here gives

$$fv = g\left(\frac{\partial z}{\partial x}\right)_{y,p,t} = \left(\frac{\partial\Phi}{\partial x}\right)_{y,p,t} \tag{17a}$$

and similarly

$$fu = -\left(\frac{\partial\Phi}{\partial y}\right)_{x,p,t} \tag{17b}$$

[Exercise: derive the $p-$coordinate version of the thermal wind equations, without assuming that T is independent of z: GFD Problem Set.]

• The full equations of motion (9) can be derived in pressure coordinates (not needed here!). Note that instead of w we have a vertical 'velocity'

$$\omega \equiv \frac{Dp}{Dt}$$

which can be shown to be

$$\omega \approx -\rho g w.$$

• Then the equation of continuity (3) takes the simple form

$$\left(\frac{\partial u}{\partial x}\right)_{y,p,t} + \left(\frac{\partial v}{\partial y}\right)_{x,p,t} + \left(\frac{\partial \omega}{\partial p}\right)_{x,y,t} = 0 \tag{18}$$

with no time-derivative (or density ρ) appearing.

[Derivation: see e.g. Andrews, §4.9.]

11. Thermodynamic energy equation

First Law of Thermodynamics for a moving parcel of fluid of unit mass, undergoing a change δS of entropy, δU of internal energy and δV of volume in time δt:

$$T\delta S = \delta U + p\,\delta V$$

For unit mass of ideal gas, $U = c_v\,T$ (c_v = specific heat capacity at constant volume) and $V = 1/\rho = R_{\rm a}T/p$. So

$$T\delta S = c_v\,\delta T + p\,\delta\left(\frac{R_{\rm a}T}{p}\right) = (c_v + R_{\rm a})\,\delta T - \frac{R_{\rm a}\,T}{p}\,\delta p$$

$$= c_p\,\delta T - \frac{1}{\rho}\,\delta p$$

where $c_p = c_v + R_{\rm a}$, specific heat capacity at const p.

• This is equivalent to the standard formula for the entropy of 1 mole of ideal gas:

$$S = C_p \ln T - R \ln p + \text{const}$$

where C_p is molar heat capacity and R is molar gas constant. (Check: divide by molar mass M to convert to entropy per unit mass, differentiate to get δS.)

• Dividing by δt, letting $\delta t \to 0$, and remembering that we are considering a moving (Lagrangian) mass, so that the time derivative is D/Dt, not $\partial/\partial t$,

$$Q \equiv T\frac{DS}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} \tag{19}$$

• Q is called the *diabatic heating rate*. For a parcel in adiabatic (adiathermal, reversible) motion, $Q = 0$.

• The main physical processes contributing to Q in the lower atmosphere are *latent heating/cooling* (from condensation/evaporation of water vapour – see Composition lectures) and *radiative heating/cooling* (from absorption/emission of e.m. radiation – see Radiation lectures).

• Note that if $Q = 0$,

$$\frac{DT}{Dt} = \frac{1}{\rho c_p} \frac{Dp}{Dt}$$

so in the absence of diabatic heating, the temperature of a moving parcel of air will increase if it *descends* (moves to higher pressure) and will decrease if it *ascends*.

• Another way of showing this: if the parcel moves *adiabatically*, it conserves its *potential temperature*, θ (recall thermodynamics lectures):,

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} \quad \text{where} \quad \kappa = \frac{\gamma - 1}{\gamma} = \frac{R_\mathrm{a}}{c_p}$$

As p increases, T has to increase as well, to keep θ constant.

(NB: Molar entropy $S = C_p \ln\theta + \text{const.}$)

• Exercise: prove

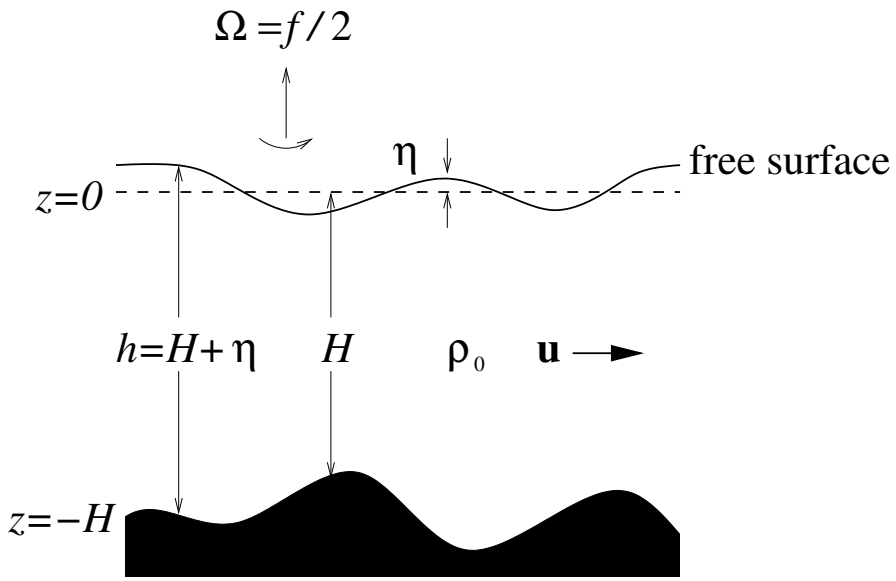
$$\frac{D\theta}{Dt} = \frac{Q}{c_p}\left(\frac{p}{p_0}\right)^{-\kappa} \tag{20}$$

• Numerical values: for dry air, $c_p = 1004\,\mathrm{J\,K^{-1}\,kg^{-1}}$, $c_v = 717\,\mathrm{J\,K^{-1}\,kg^{-1}}$ and $R_\mathrm{a} = 287\,\mathrm{J\,K^{-1}\,kg^{-1}}$.

12. Shallow-water theory

12.1 Equations of motion

Consider a *shallow* layer of incompressible fluid rotating about the z-axis. The elevation of the free surface from its reference level ($z = 0$) is η and the depth of the lower boundary is $z = -H(x, y)$.



The momentum equations can be derived from eq. (7) by taking $\rho = \rho_0$ (constant), $\boldsymbol{\Omega} = \Omega \boldsymbol{k}$ and neglecting the centripetal term and friction.

The horizontal momentum equations are:

$$\frac{Du}{Dt} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0, \tag{21}$$

$$\frac{Dv}{Dt} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = 0. \tag{22}$$

Since the layer is shallow ($H/L \ll 1$), the vertical momentum equation is well approximated by *hydrostatic balance*:

$$g + \frac{1}{\rho_0}\frac{\partial p}{\partial z} = 0. \tag{23}$$

The mass conservation (continuity) equation (3) reduces to incompressibility:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\ . \tag{24}$$

Now integrating hydrostatic balance in the vertical, the pressure is

$$p = p_a + \rho_0 g(\eta - z)$$

where p_a is the atmospheric pressure (assumed constant).

Thus the horizontal pressure gradient is independent of depth:

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} = g\frac{\partial \eta}{\partial x}.$$

If the horizontal velocities are initially depth-independent, then they remain so for all future times.

However, the vertical velocity *does* vary with depth. Integrating (24) wrt z gives:

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w_{\text{top}} - w_{\text{bottom}} = 0.$$

But

$$w_{\text{top}} = \left(\frac{Dz}{Dt}\right)_{\text{top}} = \frac{D\eta}{Dt}, \;\; w_{\text{bottom}} = \left(\frac{Dz}{Dt}\right)_{\text{bottom}} = -\frac{DH}{Dt} \;,$$

and hence the continuity equation (24) can be rewritten [exercise]:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{25}$$

Finally (21) and (22) can be written:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial \eta}{\partial x} \;=\; 0, \tag{26}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial \eta}{\partial y} \;=\; 0. \tag{27}$$

Equations (25) – (27) are the *free surface equations* (or shallow-water equations).

Note that they are *nonlinear*.

12.2 Conservation of energy

The scalar product of $\rho_0 h\boldsymbol{u}$ and the momentum equations (26, 27) $\Rightarrow$

$$h\frac{\partial}{\partial t}\left(\rho_0\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)+h\boldsymbol{u}\cdot\nabla\left(\rho_0\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)+h\boldsymbol{u}\cdot\nabla(\rho_0 g\eta)=0.$$

Adding $\frac{1}{2}\rho_0\boldsymbol{u}\cdot\boldsymbol{u}\times$ (25) (and noting $\partial\eta/\partial t=\partial h/\partial t$) gives the kinetic energy equation:

$$\frac{\partial}{\partial t}\left(\rho_0 h\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)+\nabla\cdot\left(h\boldsymbol{u}\rho_0\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)+h\boldsymbol{u}\cdot\nabla(\rho_0 g\eta)=0.$$

The potential energy equation is formed by taking $\rho_0 g\eta\times$ (25) $\Rightarrow$

$$\frac{\partial}{\partial t}\left(\rho_0\frac{g\eta^2}{2}\right)+\rho_0 g\eta\nabla\cdot(h\boldsymbol{u})=0.$$

Adding the above, we obtain the energy conservation equation:

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{S} = 0 \,, \tag{28}$$

where

$$E = \rho_0 h \, \frac{|\boldsymbol{u}|^2}{2} + \rho_0 \, \frac{g\eta^2}{2}$$

is the energy density per unit horizontal area and

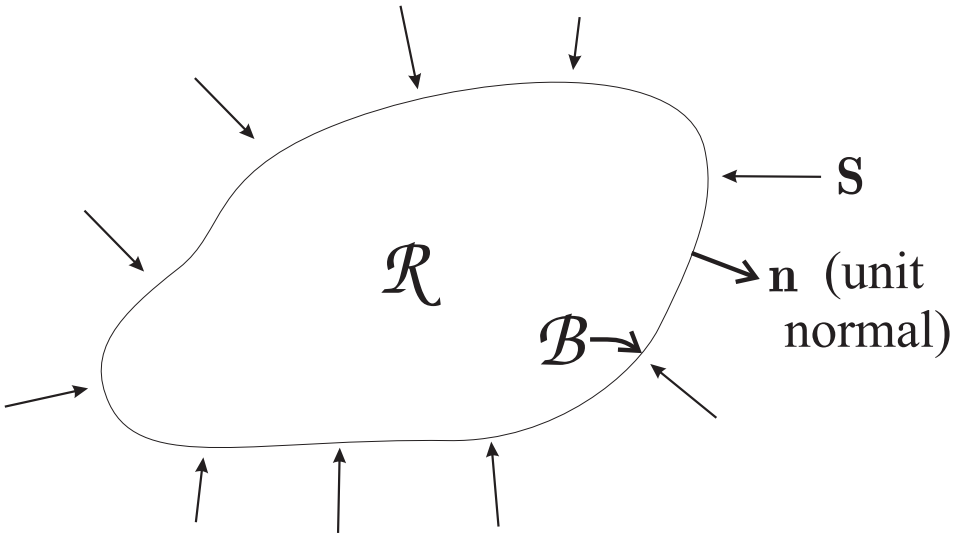
$$\mathbf{S} = h\boldsymbol{u}\rho_0 \left(\frac{|\boldsymbol{u}|^2}{2} + g\eta \right)$$

is the energy flux vector (cf. Poynting vector in Electromagnetism).

When integrated over a horizontal area $\mathcal{R}$ with boundary $\mathcal{B}$, (28) gives

$$\frac{\partial}{\partial t} \int_{\mathcal{R}} E \, dx \, dy = - \oint_{\mathcal{B}} \mathbf{S} \cdot \mathbf{n} \, dl.$$

The rate of change of energy within $\mathcal{R}$ equals the flux of energy into $\mathcal{R}$:



Note that $\int\int$(PE density) $dx\,dy$ is the *difference* between the potential energy of the disturbed and resting states and is termed the *available potential energy*.

[Exercise: Check that $\frac{1}{2}\rho_0 g\eta^2$ *is* the PE density in this sense.]

12.3 Material conservation of potential vorticity

In a rotating frame, the *absolute vorticity*, $\boldsymbol{\omega}_\mathbf{a}$ is the sum of the *relative vorticity*, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and the *planetary vorticity*, $2\boldsymbol{\Omega}$, due to the rotation of the reference frame.

Here flow is in the (x, y)-plane (neglecting w) and is independent of z

$$\Rightarrow \boldsymbol{\omega} = (0, 0, \xi) \quad \text{where} \quad \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\text{and} \quad \boldsymbol{\omega}_\mathbf{a} = (0, 0, f + \xi).$$

Using the nonlinear free surface equations (25) – (27), we can show:

$$\frac{D}{Dt}\left(\frac{f + \xi}{h}\right) = 0. \tag{29}$$

Thus the *potential vorticity*, $Q = (f + \xi)/h$ is materially conserved by moving fluid columns — a *very* useful fact!

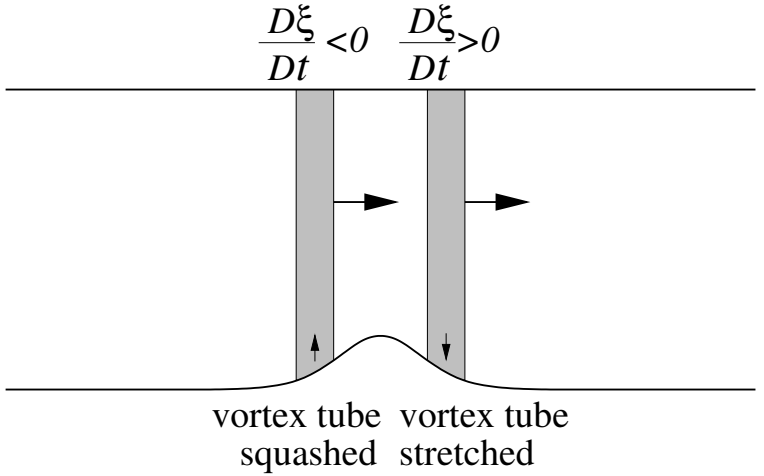
Physical interpretation: as columns of fluid stretch (i.e., h increases), the vertical component of absolute vorticity ($f + \xi$) also increases due to the vortex stretching. These two effects cancel each other in the potential vorticity.

Note the difference between *material* conservation of potential vorticity following a fluid column and *global* conservation of energy.

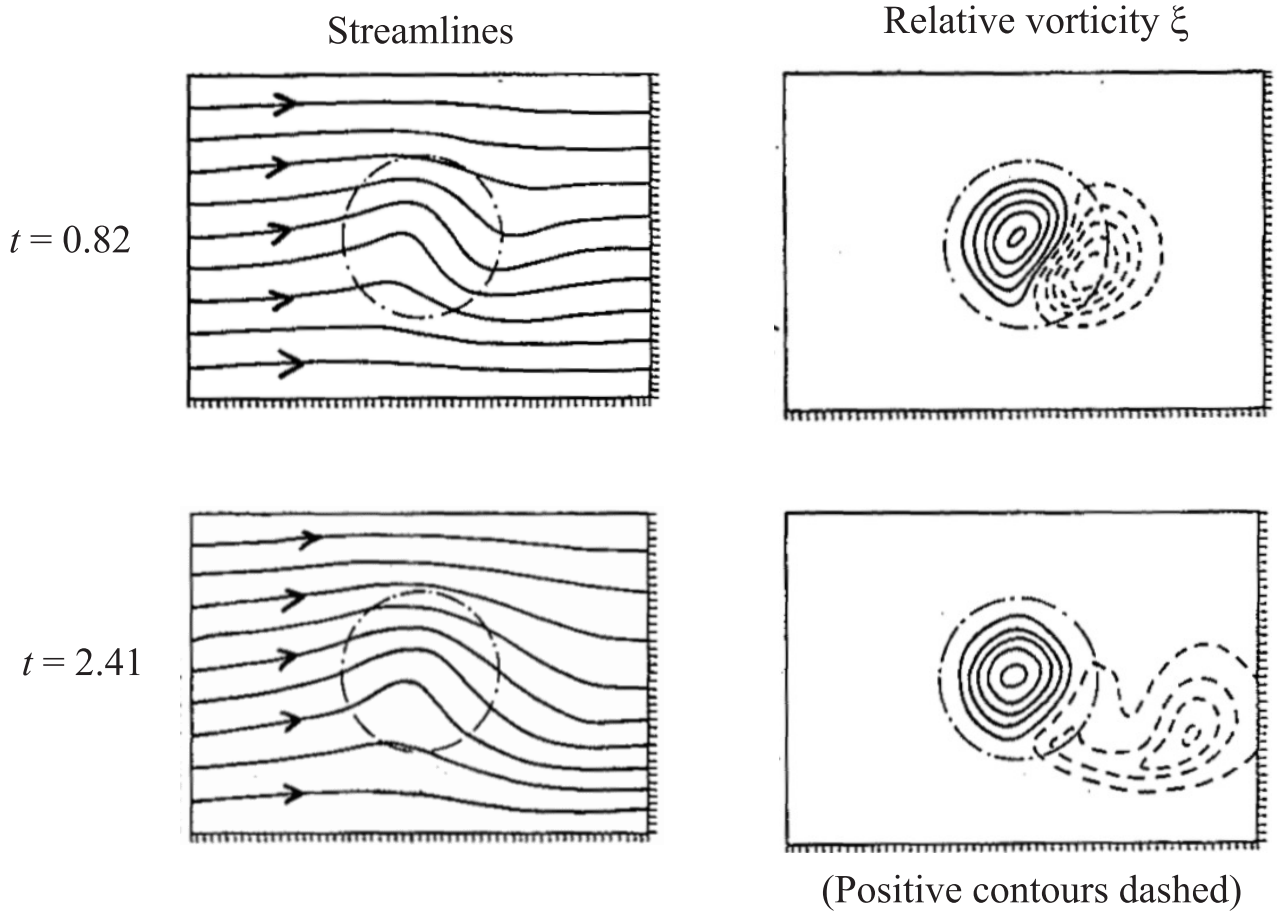
Under appropriate conditions, the PV can be *inverted* to obtain all of the dynamical fields. The two properties of invertibility and conservation make the PV a very useful dynamical quantity.

12.4 Application: flow over a circular hill

Physical interpretation (given f = constant):



We can see the effect of the squashing and stretching of vortex tubes in a series of snapshots of the vorticity field and streamlines in an initially resting fluid in which a flow is turned on at time $t = 0$ (James 1980):



Fluid originally over the hill acquires positive relative vorticity due to vortex stretching. However, this positive relative vorticity is advected away by the flow, leaving just the negative relative vorticity over the hill itself.

In the *steady state*, the potential vorticity is constant along streamlines, and since the potential vorticity is uniform upstream of the hill:

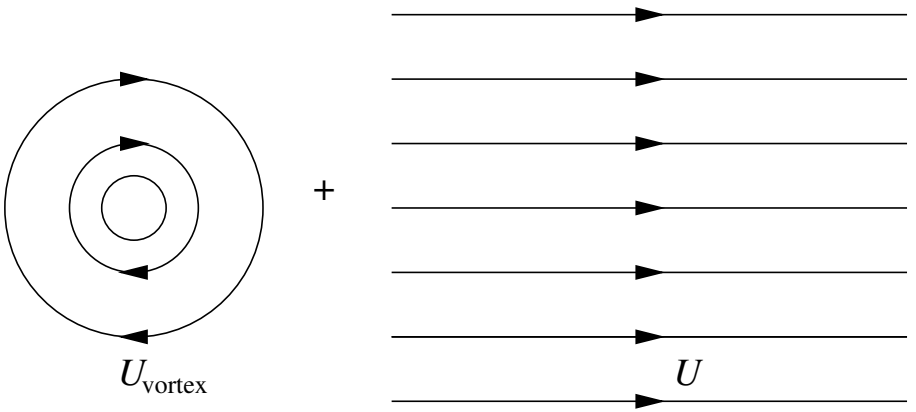
$$\frac{f+\xi}{h} = Q_0 = \frac{f}{h_0}$$

where h_0 is the layer thickness at the inflow

$$\Rightarrow \xi = -f\left(1 - \frac{h}{h_0}\right). \tag{30}$$

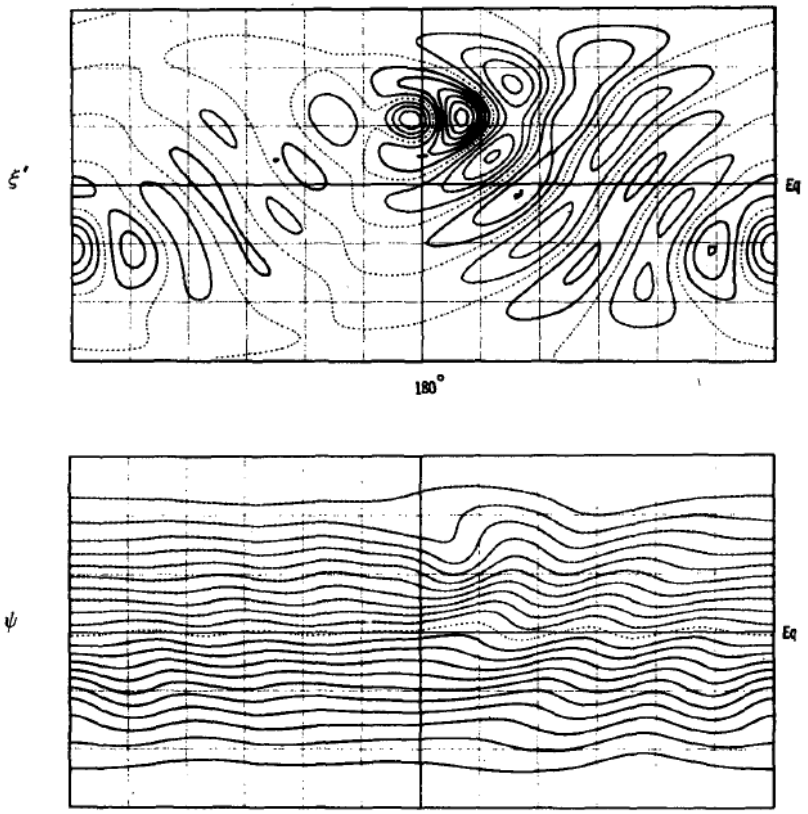
Over the hill, ξ decreases $\Rightarrow$ an anticyclonic patch of relative vorticity $\Rightarrow$ an anticyclonic vortex decaying in strength away from the hill.

Solution = vortex + background zonal flow



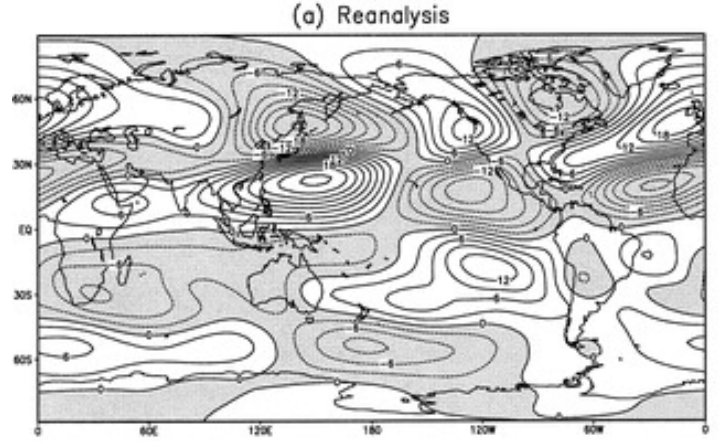
The flow is more complex if we allow the Coriolis parameter f to vary with latitude. This β-*effect* allows *Rossby* waves to form downstream of the hill - we will derive a model for these waves later.

Vorticity anomalies due to flow over a hill on a spherical Earth with β-effect.



Grose and Hoskins 1979

Stationary eddies in the real atmosphere: streamfunction anomalies at 300hPa for January. Negative (cyclonic) anomalies are shaded. The distribution of surface heating as well as orography determines the stationary waves.



Held et al 2002

13. Inertia-gravity waves

13.1 The tangent f-plane

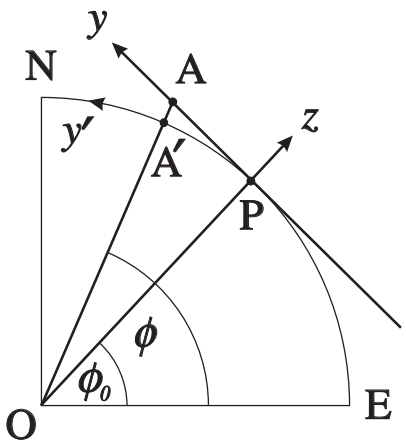
In Section 12 we used a Cartesian geometry and took $\boldsymbol{\Omega}$ in the z-direction.

However, the Earth is (nearly) a rotating sphere. In general, only the radial (local vertical) component

$$\boldsymbol{\Omega} \cdot \boldsymbol{k} = \Omega \sin \phi$$

matters dynamically — this can be confirmed by scale analysis.

When fairly close to reference point P (latitude ϕ_0), we can replace surface A′P on the sphere by AP on the tangent plane:



This simplifies complicated equations in spherical coordinates to simpler equations in Cartesians, with x = eastward distance from P, y = northward distance from P, z = vertical distance from P.

We get the same free surface equations as before, (25) – (27), but now

$$f = 2\Omega \sin \phi = f(y).$$

Sufficiently close to P, we can approximate further:

$$f = 2\Omega \sin \phi_0 = \text{constant}. \tag{31}$$

This is called the *f-plane approximation*.

13.2 Linear shallow-water equations

Now *linearise* the free-surface equations (25) – (27) about a state of rest:

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} = 0, \tag{32}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \eta}{\partial y} = 0, \tag{33}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{34}$$

We will neglect any orographic variations and work on an f-plane, so that (32) – (34) are a set of linear partial differential equations with constant coefficients (much easier to solve!).

The corresponding (linear) potential vorticity equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f}{H}\eta\right) = 0. \tag{35}$$

This is easily obtained by linearising the nonlinear potential vorticity equation (29) or by direct derivation from (32) – (34). Note that there is no contribution from the nonlinear advection term, which is of higher order.

We can also obtain a relation for the relative vorticity $\partial v/\partial x - \partial u/\partial y$ by taking $\partial(32)/\partial x + \partial(33)/\partial y \Rightarrow$ and rearranging the result to give:

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{g}{f}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) - \frac{1}{fH}\frac{\partial^2 \eta}{\partial t^2}. \tag{36}$$

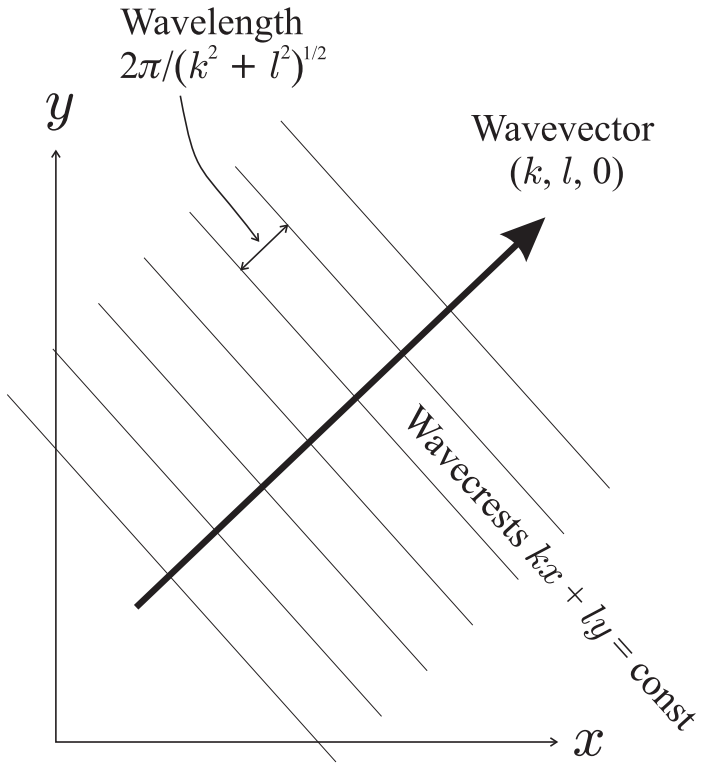
Combining (35) and (36) then gives a single equation for the evolution of the free surface elevation perturbation:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial t^2} + f^2\eta - gH\frac{\partial^2 \eta}{\partial x^2} - gH\frac{\partial^2 \eta}{\partial y^2}\right) = 0. \tag{37}$$

13.3 Wave solutions

We now seek wave solutions of (37) of the form

$\eta = A\, e^{i(kx+ly-\omega t)}$:



Substituting for η in (37) gives the *dispersion relation*:

$$\omega\,[\,\omega^2 - f^2 - gH(k^2 + l^2)\,] = 0\ . \tag{38}$$

• The first root of (38) is $\omega = 0$. This corresponds to steady, geostrophic flow,

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y}\,, \qquad\qquad v = \frac{g}{f}\frac{\partial \eta}{\partial x}\,,$$

consistent with $\partial/\partial t = 0$ in (32) and (33). Note that this geostrophic flow is also nondivergent (since f is constant), consistent with $\partial/\partial t = 0$ in (34).

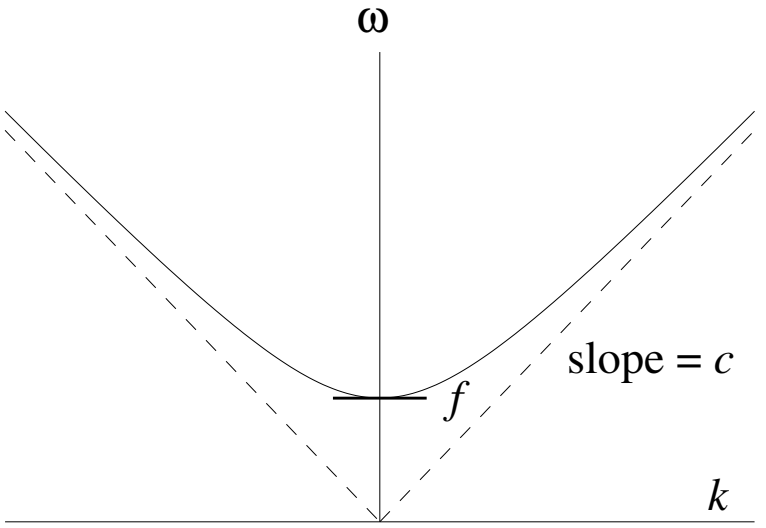
• The remaining two roots of (38) correspond to *inertia-gravity* (or *Poincaré*) *waves* and satisfy the *inertia-gravity wave dispersion relation*:

$$\omega^2 = f^2 + c^2(k^2 + l^2) \tag{39}$$

where $c = (gH)^{1/2}$.

a. *Phase and group velocities*

Without loss of generality, we can set $l = 0$ such that the dispersion relation simplifies to $\omega^2 = f^2 + c^2k^2$:



Note that there is a low-frequency cut-off at the inertial frequency, $\omega \geq |f|$.

The *phase velocity*,

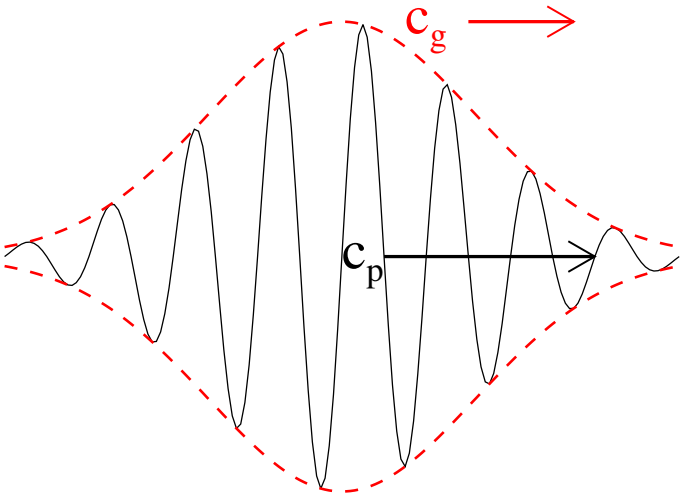
$$c_p = \frac{\omega}{k} = \frac{(f^2 + c^2k^2)^{1/2}}{k} \ ,$$

describes the propagation of peaks and troughs.

The *group velocity*,

$$c_g = \frac{\partial \omega}{\partial k} = \frac{c^2 k}{\omega} = \frac{c^2 k}{(f^2 + c^2 k^2)^{1/2}} \ ,$$

describes the propagation of wave energy or wave packets:



We can also show that

$$\frac{c_g}{c_p} = \frac{c^2 k}{\omega} \Big/ \frac{\omega}{k} = \frac{c^2 k^2}{\omega^2} = \frac{\omega^2 - f^2}{\omega^2} \ .$$

b. *Short and long wave limits*

An important length scale is the *external Rossby deformation radius*,

$$L_d = \frac{c}{|f|} = \frac{(gH)^{1/2}}{f} \ . \tag{40}$$

In the *short-wave limit* ($k \gg L_d^{-1}$),

$$\omega^2 \approx c^2 k^2 \gg f^2.$$

This is the dispersion relation for non-dispersive *surface gravity waves* in shallow water, unaffected by the Earth's rotation. For these short waves,

$$c_p \ \approx \ c_g \approx \ \pm c \ = \ \pm (gH)^{1/2} \ .$$

[Caution: even though these waves are "short", we also require $k \ll H^{-1}$ for the validity of shallow water theory; for this reason you may see c referred to as the "long" wave speed in some books!]

In the *long-wave limit* ($k \ll L_d^{-1}$),

$$\omega^2 \approx f^2$$

and $c_g \approx 0$.

These are *inertial oscillations* with a period $2\pi/f$ (12 hours at the poles).

• Some numerical values:

Consider an ocean of depth 4 km at mid-latitudes. Then $c \sim 200\,\mathrm{m\,s^{-1}}$, $f \sim 10^{-4}\,\mathrm{s^{-1}}$ and $L_d \sim 2000$ km. So horizontal wavelengths less than about 1000 km (but much greater than 4 km) count as short in the present context. Long waves will be essentially global in scale!

c. *Fluid parcel trajectories*

From the linear free surface equation (32) – (34), the real part of the wave solution (with $l = 0$) is:

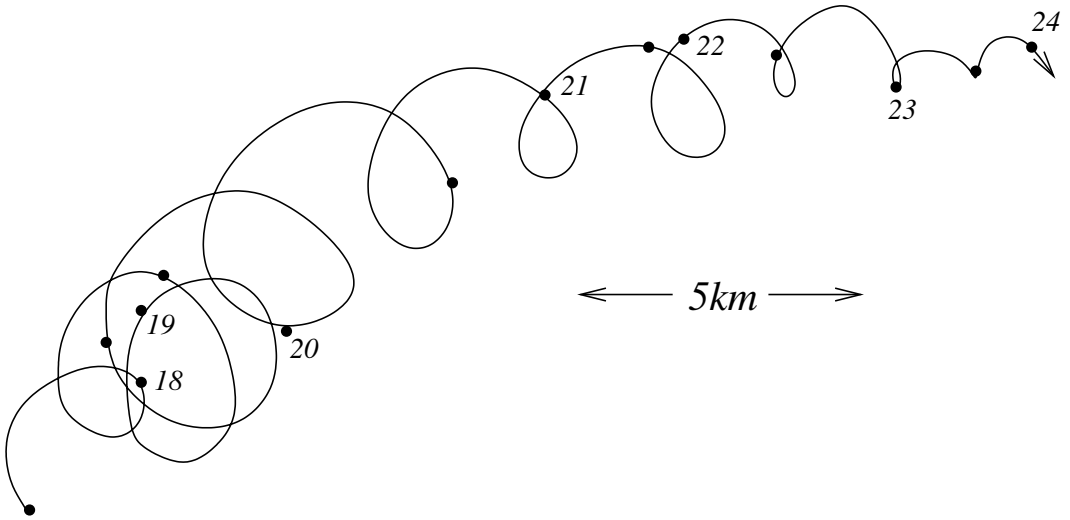
$$\eta = A\cos(kx - \omega t), \quad u = \frac{\omega A}{kH}\cos(kx - \omega t), \quad v = \frac{fA}{kH}\sin(kx - \omega t) \ .$$

Thus the horizontal velocity vector traces an ellipse with time, given by

$$\frac{u^2}{\omega^2} + \frac{v^2}{f^2} = \left(\frac{A}{kH}\right)^2 \ .$$

In the short wave (gravity wave) limit $\omega^2 \gg f^2$, this ellipse becomes elongated and horizontal motion is confined the direction of wave propagation (i.e., $v^2 \ll u^2$).

In the long wave (inertial oscillation) limit, the horizontal velocities and fluid trajectories trace a circular orbit.



The trajectory of a fluid parcel in the Baltic Sea (58°N) in August 1933. The markers are separated by 12 hours, and the numbers indicate the day of the month. The theoretical period of an inertial oscillation at this latitude is 14.1 hours. Note the length scale is more consistent with the reduced gravity case (later).

d. *Energetics*

Averaging the potential energy and (linearised) kinetic energy densities over a wave period, we find:

$$\overline{PE} = \frac{\rho_0 g \overline{\eta^2}}{2} = \frac{\rho_0 g A^2}{4}\,, \quad \overline{KE} = \frac{\rho_0 H(\overline{u^2+v^2})}{2} = \frac{\rho_0 g A^2(\omega^2+f^2)}{4c^2k^2}$$

$$\Rightarrow \overline{E} = \overline{PE} + \overline{KE} = \frac{\rho_0 g A^2}{2}\,\frac{\omega^2}{(\omega^2-f^2)}$$

and

$$\Rightarrow \frac{\overline{KE}}{\overline{PE}} = \frac{\omega^2+f^2}{c^2k^2} = \frac{\omega^2+f^2}{\omega^2-f^2}\,.$$

$\omega \gg f$ (short waves, rotation unimportant) $\Rightarrow \overline{KE} \approx \overline{PE}$: equipartition.

$\omega \to f$ (long waves, rotation important) $\Rightarrow \overline{KE} \gg \overline{PE}$. [GFD Problem set]

Moreover, the linearised energy flux vector (page 54) is

$$\mathbf{S} = H\boldsymbol{u}\rho_0 g\eta = \rho_0 c^2(u\eta\,,\, v\eta).$$

Averaging over a wave period, this becomes [exercise]

$$\overline{\mathbf{S}} = \left(\frac{\rho_0 g A^2}{2} \frac{\omega}{k} \, , \, 0 \right) = (\, \overline{E} c_g \, , \, 0 \,) = \overline{E} \mathbf{c}_g. \tag{41}$$

The result that the wave energy flux is the product of the wave energy and group velocity is general and not specific to inertia-gravity waves.

e. *Example: Tsunami waves*

Tsunamis are surface gravity waves generated by seismic activity or landslides on the sea floor. A typical open-ocean tsunami has a horizontal wavelength of 200 km, i.e., is a short wave in the present context.

Thus the phase and group velocities are $c_p \approx c_g \approx c \approx 200 \, \mathrm{m \, s^{-1}}$ assuming a depth of 4 km (following the calculation on page 72).

The wave energy flux is given by

$$\overline{S} = \overline{E} c_g \approx \frac{\rho_0 g A^2}{2} (gH)^{1/2} \propto A^2 H^{1/2}.$$

Even though we assumed H was constant in the derivation of the dispersion equation, we can now use this energy flux and the energy conservation equation to determine the *slow* variations of the wave amplitude in space and time.
For example, for 1-dimensional waves in equilibrium, the wave energy flux must be constant, giving

$$A \propto H^{-1/4}.$$

If the wave amplitude is 1 m in the open ocean where $H = 4\,\text{km}$, then this increases to 4.5 m when the water depth is 10 m.

In practice, the wave amplitude will decay due to diffraction, but will also be enhanced at certain locations along the coastline due to refraction.

13.4 Reduced gravity model and internal waves

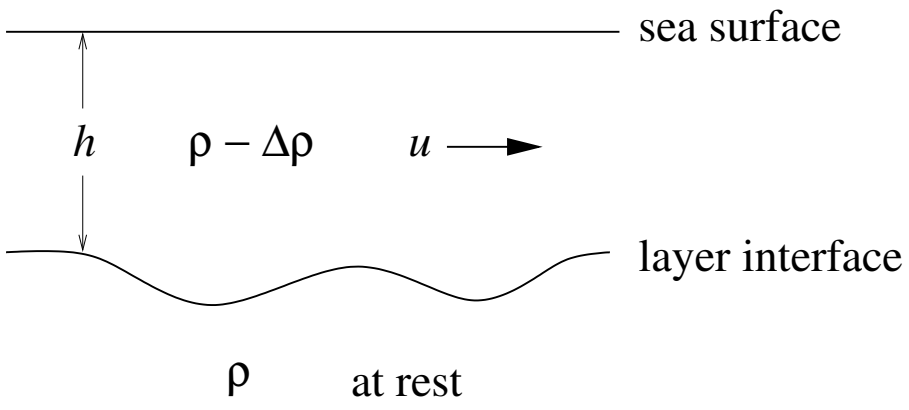
So far, we have been considering *surface* inertia-gravity waves.

However, in a stratified fluid, waves can also arise from perturbations to the depth of density interfaces within the fluid — these are known as *internal inertia-gravity waves*.

Because the restoring force is the buoyancy force rather than the full gravitational force, internal waves evolve more slowly than surface waves.

a. *Reduced gravity model*

Consider a *thin* layer of fluid, density $\rho_0 - \Delta\rho$ and thickness h, overlying a deep, motionless abyssal layer of density ρ_0, and with a rigid lid. Within the surface layer, we assume that the horizontal fluid velocity is independent of depth.



(deep)

The equations of motion are now:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g'\frac{\partial h}{\partial x} = 0, \tag{42}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g'\frac{\partial h}{\partial y} = 0, \tag{43}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{44}$$

These are identical to the shallow-water equations (25) – (27), except that gravity, g, is replaced by the *reduced gravity*,

$$g' = g\frac{\Delta\rho}{\rho_0}.$$

[Exercise: confirm the above using hydrostatic balance and by assuming the horizontal pressure gradients vanish in the abyssal layer.]

b. *Internal inertia-gravity waves*

The equations for internal inertia-gravity waves are identical to those presented in sections 14.2 and 14.3 for surface inertia-gravity waves, except that we replace g by the reduced gravity, g', and H by the mean thickness of the upper layer, h_0. This gives the dispersion equation for internal inertia-gravity waves:

$$\omega^2 = f^2 + g' h_0 (k^2 + l^2) \tag{45}$$

In the short-wave limit, the phase and group speed of internal gravity waves are given by

$$c_p \approx c_g \approx \pm (g' h_0)^{1/2}.$$

In the ocean, $g' \sim 10^{-2} \mathrm{m\,s^{-1}}$ and $h_0 \sim 500\,\mathrm{m}$ gives $c_p \sim 2\,\mathrm{m\,s^{-1}}$.

We can also define an *internal Rossby deformation radius*,

$$L_d = \frac{(g'h_0)^{1/2}}{f} \approx 30\,\mathrm{km} \tag{46}$$

at mid-latitudes.

Thus internal gravity waves have extremely short wavelengths in the ocean.

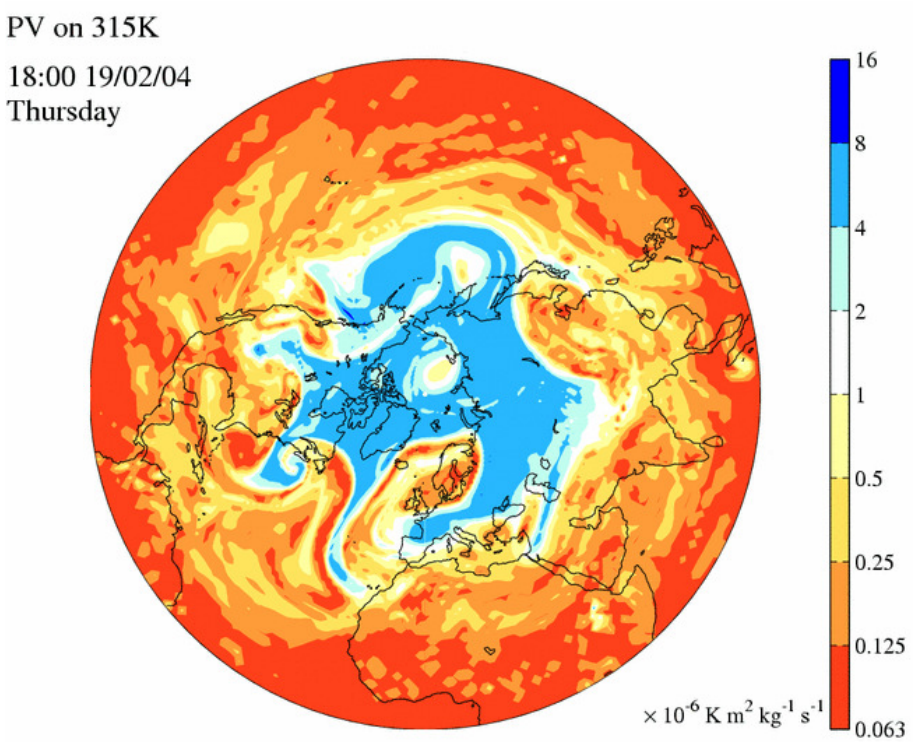
The internal Rossby deformation radius is of such importance in atmospheric and oceanic dynamics that it is often simply referred to as "the" deformation radius. It defines, for example, the characteristic length scales of atmospheric weather systems and ocean eddies, and it will often appear in later GFD lectures.

14. Rossby and Kelvin Waves

The large-scale circulation of the atmosphere and ocean is dominated by wave motions which feel the effects of the Earth's rotation. These are known as *Rossby waves* (or *planetary waves*).

Along the equator and along coastlines, where the Coriolis acceleration vanishes, a different class of waves, known as *Kelvin waves*, is found.

Example of planetary Rossby waves in the atmosphere:

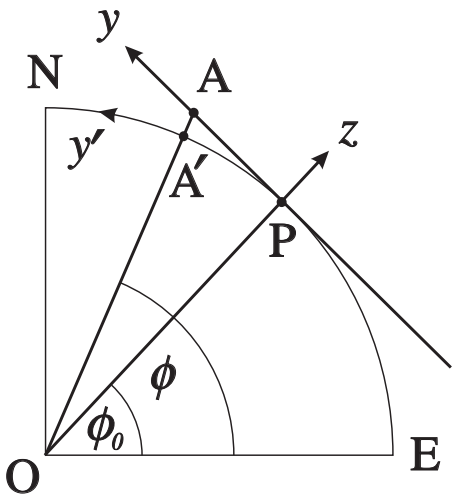


[Potential vorticity on θ=315K]

14.1 The β-plane approximation

For many flows in the atmosphere and ocean, the f-plane approximation, $f \approx 2\Omega \sin \phi_0$, is inadequate.

These include Rossby waves, for which the meridional gradient in the Coriolis parameter is essential.



Instead, we can use a Taylor expansion around $\phi = \phi_0$,

$$f(\phi) = 2\Omega \sin \phi = 2\Omega\{\sin \phi_0 + (\phi - \phi_0) \cos \phi_0 + \ldots\} \ .$$

On the tangent plane, AP,

$$f(y) \approx f_0 + \beta \underbrace{(y - y_0)}_{\approx a(\phi - \phi_0)} + \ldots \tag{47}$$

$$\text{where} \quad f_0 = 2\Omega \sin\phi_0 = f(y = y_0) \ ,$$

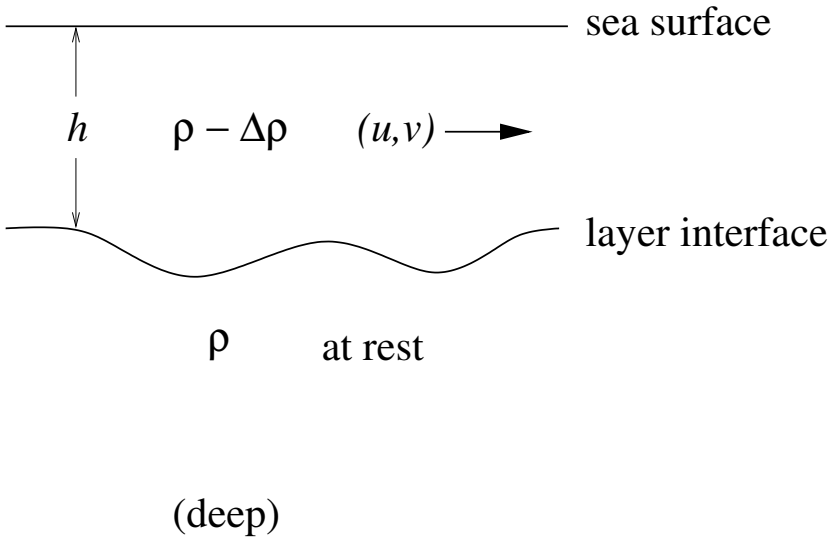
$$\text{and} \quad \beta = \frac{2\Omega \cos\phi_0}{a} = \left(\frac{df}{dy}\right)_{y=y_0} \ .$$

Equation (47) is called the *β-plane approximation.*

The β-plane approximation captures the main effect of the Earth's spherical geometry on large-scale motions in the atmosphere and ocean. While the formal requirement for its validity is $|y - y_0| \ll a$, the β-plane approximation is often a useful starting point for motions with scales comparable with the Earth's radius because of the simplicity of the resultant equations in Cartesians compared with spherical coordinates.

14.2 Linearised reduced-gravity model

In this lecture we will use the reduced-gravity model (section 13.4) consisting of a moving layer of fluid, thickness h and horizontal velocity (u, v), overlying a motionless abyssal layer:



This is the simplest model capable of including the effects of both rotation and (simple) stratification.

We will linearise about a state of rest and mean layer thickness h_0.

The linearised equations of motion, including rotation, are thus:

$$\frac{\partial u}{\partial t} - fv + g' \frac{\partial h}{\partial x} = 0, \tag{48}$$

$$\frac{\partial v}{\partial t} + fu + g' \frac{\partial h}{\partial y} = 0, \tag{49}$$

$$\frac{\partial h}{\partial t} + h_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \tag{50}$$

14.3 Rossby waves

a. *Nearly geostrophic flow*

In the atmosphere and ocean, the flow is *nearly* in geostrophic balance.

We can rewrite the momentum equations (48) and (49) as:

$$u = -\frac{g'}{f}\frac{\partial h}{\partial y} - \frac{1}{f}\frac{\partial v}{\partial t},$$
$$v = \frac{g'}{f}\frac{\partial h}{\partial x} + \frac{1}{f}\frac{\partial u}{\partial t}.$$

large *small*

The second terms on the right-hand sides of these equations represent small departures from geostrophic balance.

We can now approximate the small terms by substituting for u and v using the geostrophic velocity and replacing f by its mean value f_0 (since $f \approx f_0$ to leading order):

$$\begin{aligned} -\frac{1}{f}\frac{\partial v}{\partial t} &\approx -\frac{1}{f_0}\frac{\partial}{\partial t}\left(\frac{g'}{f_0}\frac{\partial h}{\partial x}\right), \\ \frac{1}{f}\frac{\partial u}{\partial t} &\approx \frac{1}{f_0}\frac{\partial}{\partial t}\left(-\frac{g'}{f_0}\frac{\partial h}{\partial y}\right). \end{aligned}$$

This gives:

$$\begin{aligned} u &\approx -\frac{g'}{f}\frac{\partial h}{\partial y} - \frac{g'}{f_0^2}\frac{\partial^2 h}{\partial t \partial x}, \\ v &\approx \frac{g'}{f}\frac{\partial h}{\partial x} - \frac{g'}{f_0^2}\frac{\partial^2 h}{\partial t \partial y}. \end{aligned}$$

[Exercise: using velocity scale U, time scale T and length scale $L = UT$, show that the ratio of the small terms to the large terms in the above equations is of order the Rossby number $\mathrm{Ro} = U/f_0 L$: cf. eq. (12).]

b. *Rossby wave equation and dispersion relation*

Substituting the above into the continuity equation (50) gives the Rossby wave equation [exercise]:

$$\frac{\partial}{\partial t}\left(\nabla^2 h - \frac{h}{L_d^2}\right) + \beta\frac{\partial h}{\partial x} = 0 \tag{51}$$

where

$$L_d = \frac{\sqrt{g'h_0}}{f_0}$$

is the internal Rossby deformation radius.

We now seek a wave solution of the form:

$$h = Ae^{i(kx+ly-\omega t)}.$$

Substituting this trial solution into the wave equation (51) gives:

$$-i\omega\left(-k^2 - l^2 - \frac{1}{L_d^2}\right) + i\beta k = 0.$$

$$\Rightarrow \quad \omega = -\frac{\beta k}{k^2 + l^2 + 1/L_d^2}. \tag{52}$$

This is the *Rossby wave dispersion relation.*

Note that if $\beta = 0$, then $\omega = 0$. Thus on the f-plane this reduces to the solution of steady geostrophic flow that we found in Section 13.3. On the β-plane, however, this solution is not stationary but propagates as a Rossby wave due to the meridional gradient in the Coriolis parameter.

c. *Zonal phase and group speed*

The zonal phase speed is:

$$c_p^{(x)} = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2 + 1/L_d^2}. \tag{53}$$

This is always negative, implying *westward* phase propagation.

Physical mechanism?

• As fluid parcels are displaced north/south, they conserved their *total* (potential) vorticity.

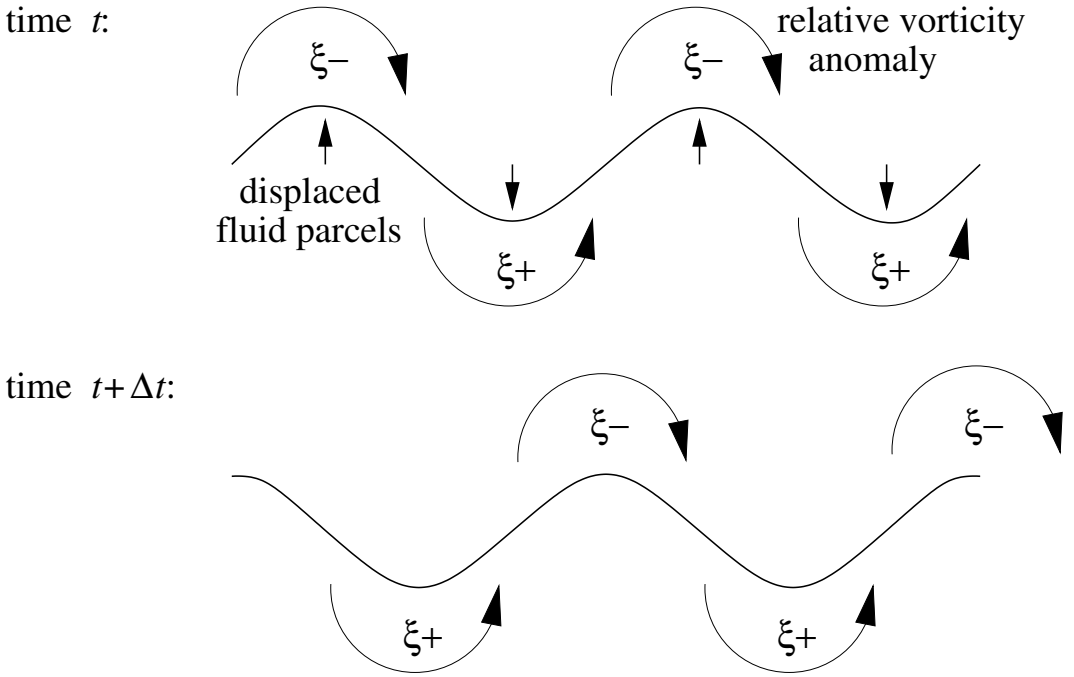
• But the *background* (potential) vorticity $\propto f_0 + \beta y$ increases northwards

$\Rightarrow$ parcels' *relative* vorticity ξ (or layer thickness) anomalies are generated:

ξ *decreases* for parcels moving northwards, ξ *increases* for parcels moving southwards.

These induce circulation anomalies which tend to 'push' neighbouring parcels northwards or southwards.

Thus the wavy pattern tends to move westwards:



The zonal group speed is:

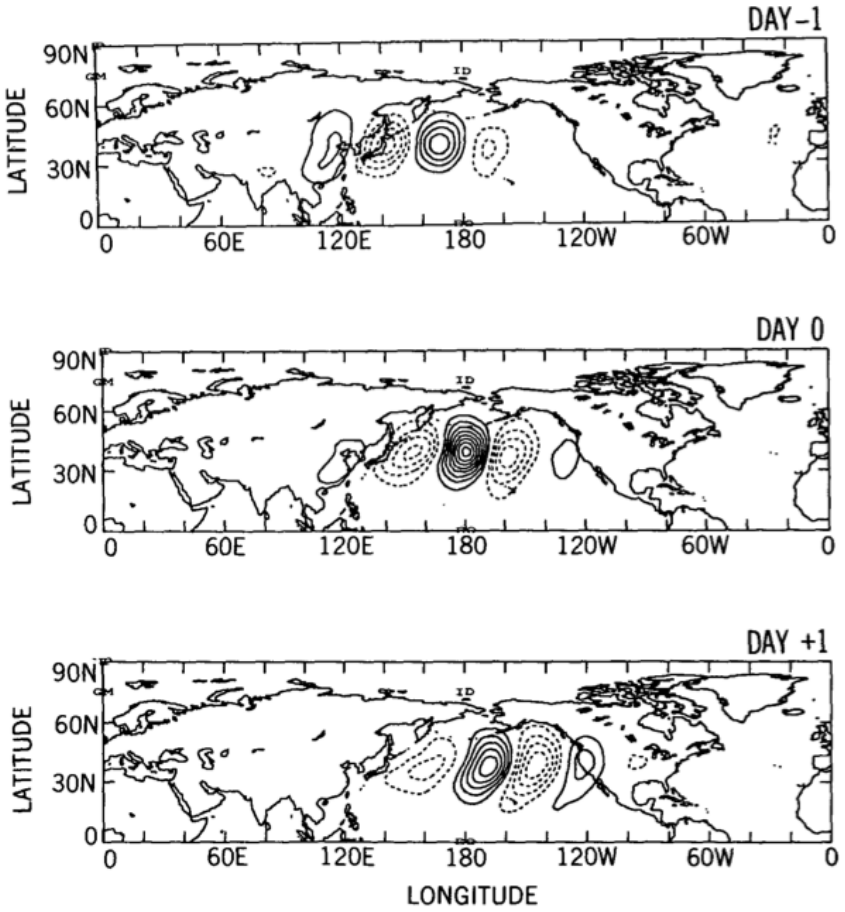
$$c_g^{(x)} = \frac{\partial \omega}{\partial k} = \beta \frac{(k^2 - l^2 - 1/L_d^2)}{(k^2 + l^2 + 1/L_d^2)^2}. \tag{54}$$

This is positive for short waves in the x direction (large k^2), but negative for long waves in the x direction (small k^2).

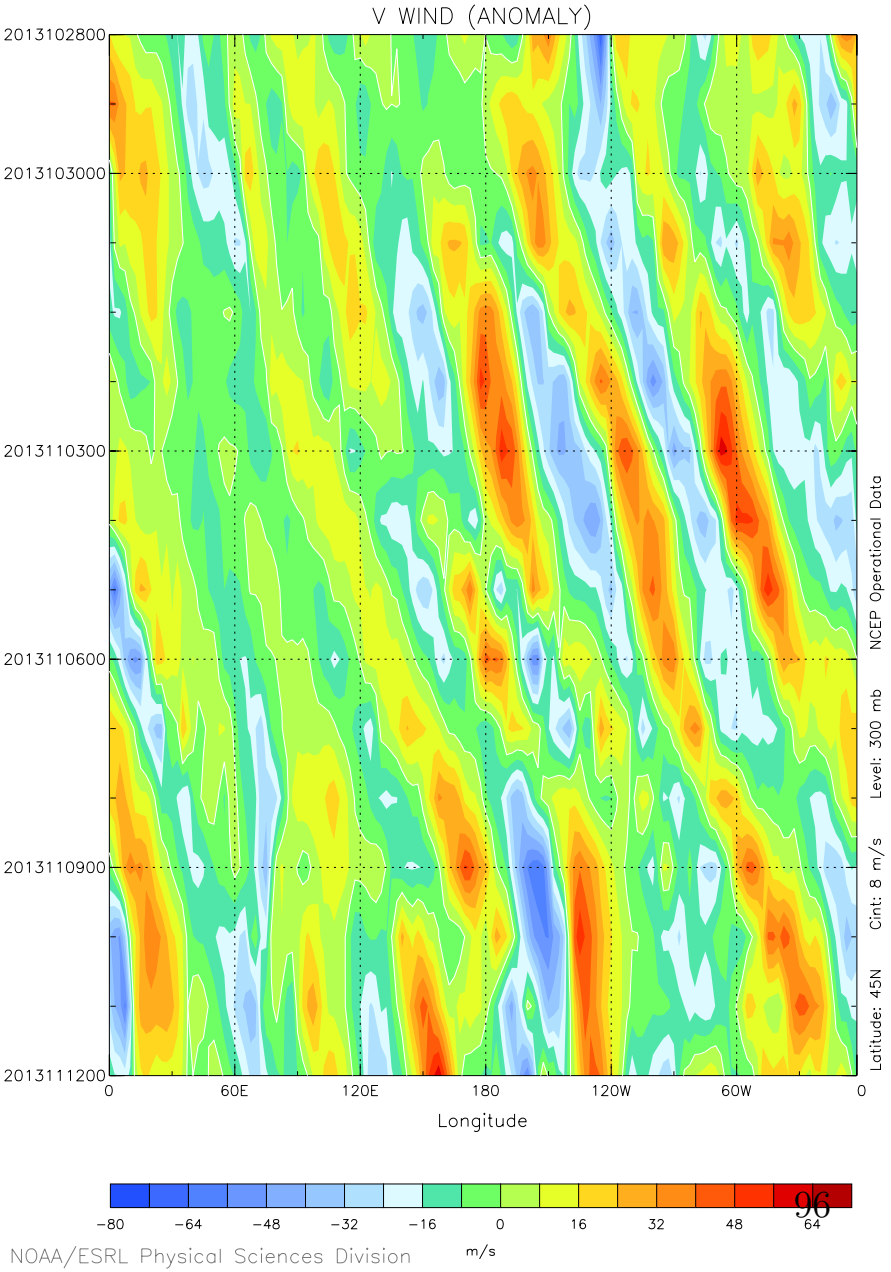
Thus packets of *long waves move westward*, but packets of *short waves move eastward*.

Wave packets are *dispersive*: the wave energy moves at a different speed from individual highs and lows. In reality (with background westerly winds present) this means that new cyclones often develop downstream (i.e. east) of existing cyclones.

Dispersion of wave packets derived from regression analysis (Chang 1993).



Example meridional wind *Hovmöller*.

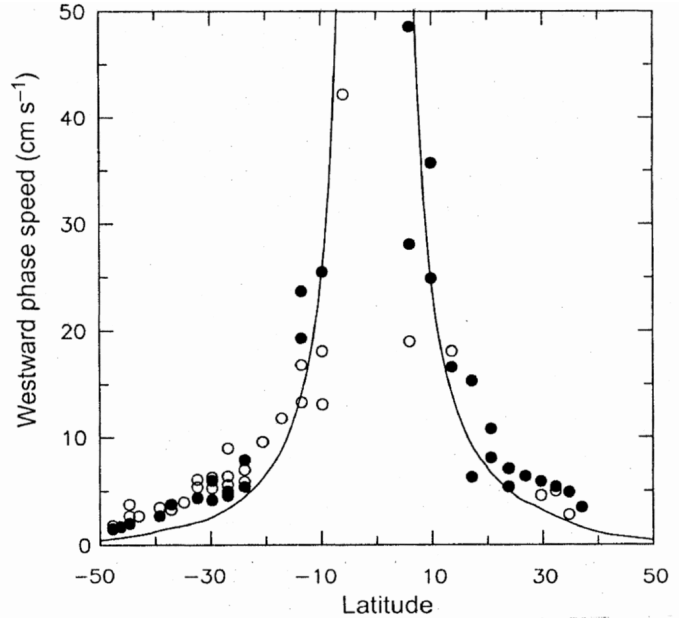


In the ocean, L_d is small $\Rightarrow$ we can generally assume $1/L_d^2 \gg k^2,\, l^2$.

In this limit, both the phase speed and group speed simplify to

$$c_p^{(x)} \approx c_g^{(x)} \approx -\beta L_d^2 = -\frac{\beta g' h_0}{f^2} \tag{55}$$

so the waves are non-dispersive.



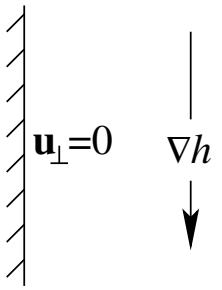
Estimated speed of westward propagation from altimeter data (• = Pacific, ∘ = Atlantic); solid line is the prediction from (55). (Chelton and Schlax 1986)

14.4 Kelvin waves

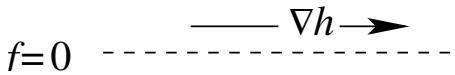
In a Rossby wave, the flow is in geostrophic balance to first approximation.

However, there are two circumstances under which a pressure gradient *cannot* be balanced by a Coriolis force:

- along a coastline:



- along the equator:



Instead, such pressure gradients result in *Kelvin waves* which propagate along coastlines and the equator.

a. *Coastal Kelvin waves*

Close to a meridional coastline, we can anticipate that $u \approx 0$, giving:

$$-f_0 v + g' \frac{\partial h}{\partial x} = 0, \tag{56}$$

$$\frac{\partial v}{\partial t} + g' \frac{\partial h}{\partial y} = 0, \tag{57}$$

$$\frac{\partial h}{\partial t} + h_0 \frac{\partial v}{\partial y} = 0. \tag{58}$$

Note that we have assumed $f = f_0$ is constant.

Firstly taking

$$\frac{\partial}{\partial t}(58) - h_0 \frac{\partial}{\partial y}(57)$$

gives:

$$\frac{\partial^2 h}{\partial t^2} - g' h_0 \frac{\partial^2 h}{\partial y^2} = 0. \tag{59}$$

Secondly taking

$$\frac{1}{g'}\frac{\partial}{\partial t}(56) + \frac{f_0}{g'}(57)$$

gives:

$$\frac{\partial^2 h}{\partial t \partial x} + f_0 \frac{\partial h}{\partial y} = 0. \tag{60}$$

We now seek a solution of the form:

$$h = A(x)\, e^{i(ly - \omega t)} \ .$$

Substituting this solution into (59) gives the phase speed

$$\frac{\omega}{l} = \pm\sqrt{g' h_0} \ .$$

Thus, along the coast, these waves move at the internal gravity wave speed [note: even when $\omega \ll f$ — how?]

Substituting this solution into (60) gives

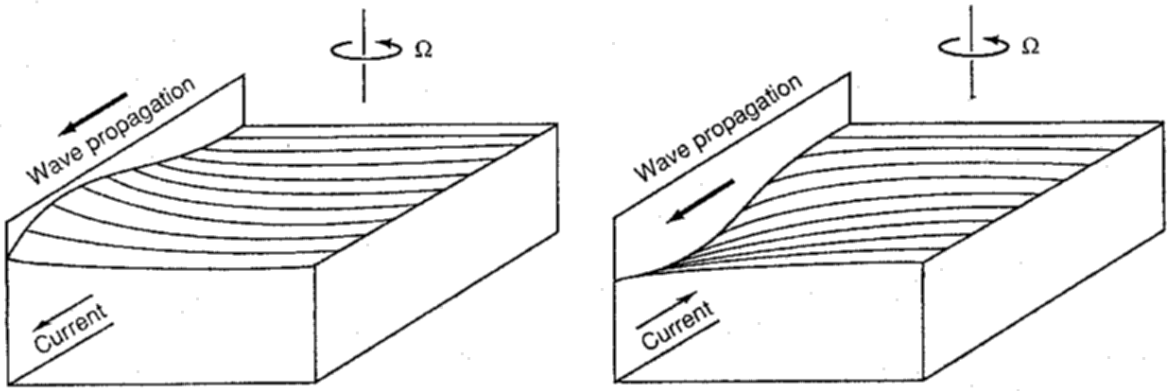
$$\frac{dA}{dx} = \pm\frac{A}{L_d}.$$

$$\Rightarrow \qquad A = A_0\, e^{\pm x/L_d}.$$

Only the wave that decays away from the boundary is physical

$\Rightarrow$ the wave travels with the coast to its right in the northern hemisphere, and with the coast to its left in the southern hemisphere.

Physically, Kelvin waves are internal gravity waves in the direction $||$ to the coast, *but* are in geostrophic balance in the direction $\perp$ to the coast.



(Cushman Roisin, 1994)

b. Equatorial Kelvin waves

At the equator, we can anticipate that $v \approx 0$, giving:

$$\begin{aligned} \frac{\partial u}{\partial t} + g' \frac{\partial h}{\partial x} &= 0, \\ \beta y u + g' \frac{\partial h}{\partial y} &= 0, \\ \frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} &= 0, \end{aligned}$$

where we use the β-plane approximation.

Following the same approach as above, the solution for an equatorial Kelvin wave is [GFD Problem set]:

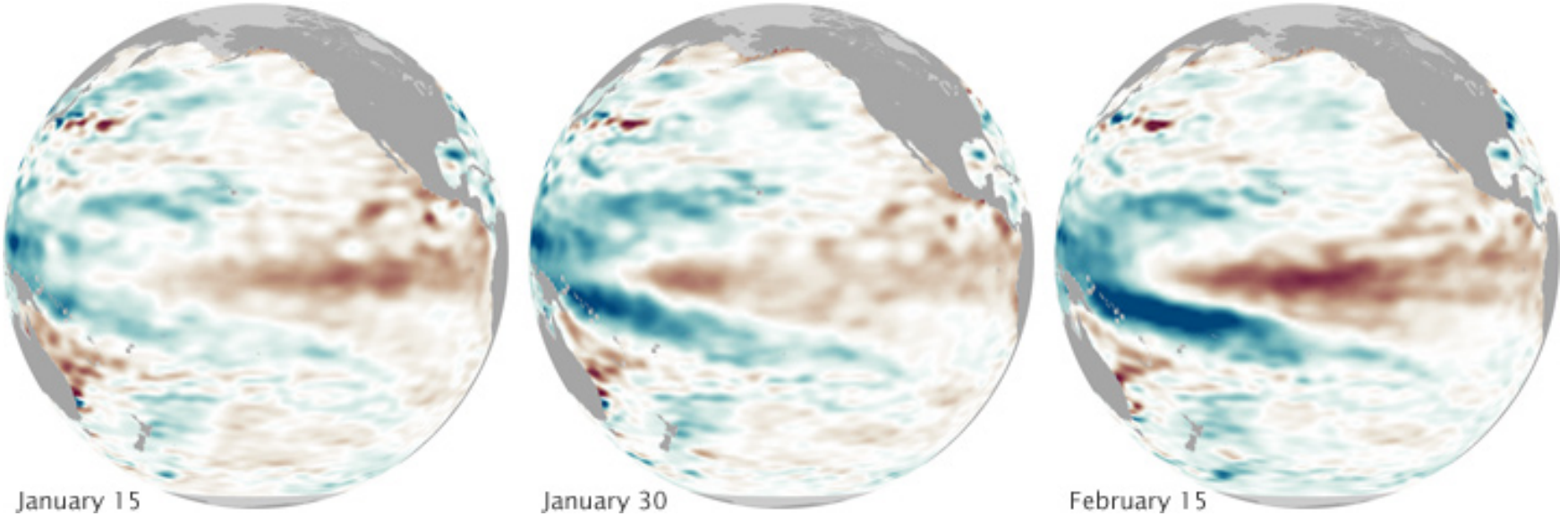
$$h = A_0\, e^{-y^2/L_{eq}^2}\, e^{i(kx-\omega t)}\ , \tag{61}$$

where $\omega = +(g'h_0)^{1/2}k$, so that these waves propagate *eastwards* along the equator with phase (and group) speed $(g'h_0)^{1/2}$, and decay away from the equator on a length scale given by the *equatorial deformation radius*

$$L_{eq} = \left(\frac{2(g'h_0)^{1/2}}{\beta}\right)^{1/2}\ .$$

One can think of an equatorial Kelvin wave as two coastal Kelvin waves propped up against each other, with the Equator playing the role of 'coast'.

They are observed both in the atmosphere (e.g., the equatorial stratosphere) and the ocean, in particular during the development and decay of an El Niño event.



NASA Earth Observatory

Evolution of sea surface height (red = high) during the 2010 El Niño. In mid-January the El Niño was weakening, but a new Kelvin wave emerges in late January (high heights in west Pacific) and propagates east to re-enforce the anomalies.

C5 Problem Set 5 (HT2022): Geophysical Fluid Dynamics

1. **Coriolis**

   If a ball is thrown a horizontal distance of 100 m at 30° latitude in 4 seconds, what is its sideways deflection due to the Coriolis force? **[Total: 5]**

2. **Scale analysis**

   Perform a scale analysis for the horizontal and vertical momentum equations for oceanic motion of Atlantic ocean scale:

   $$U \sim 10\,\mathrm{cm\,s^{-1}}, \quad W \sim 1\,\mathrm{mm\,s^{-1}}, \quad H \sim 4\,\mathrm{km}, \quad T \sim 1\,\text{month}.$$

   (Choose an appropriate horizontal length scale, and justify your choice; neglect friction. Note that the sizes of horizontal and vertical pressure gradients should be inferred from the other terms.) [6]

   Estimate the Rossby number for this flow.

   What U or L scales would be required for the flow to be significantly non-geostrophic? [4]

   [Total: 10]

3. **Rossby number**

   Estimate the Rossby numbers for the following flows, and comment:

   (a) A hurricane at 20°N, with windspeed $\sim 70\,\mathrm{m\,s^{-1}}$ and horizontal lengthscale ~ 100 km.

   (b) A tornado in the American Midwest, with windspeed $\sim 100\,\mathrm{m\,s^{-1}}$ and horizontal lengthscale ~ 100 m.

   (c) Flow in a bathtub vortex.

   (d) A "Polar Low" at 60°N, with windspeed $\sim 15\,\mathrm{m\,s^{-1}}$ and horizontal lengthscale ~ 500 km.

   [Total: 5]

4. **Thermal wind**

   Discuss briefly why the geostrophic balance assumption is important in the initialization of numerical weather forecasts. Show that, if the flow is geostrophic,

   $$f\frac{\partial u}{\partial \ln p} = R\left(\frac{\partial T}{\partial y}\right)_p,$$

   $$f\frac{\partial v}{\partial \ln p} = -R\left(\frac{\partial T}{\partial x}\right)_p,$$

   where p is pressure, u and v are the zonal and northward velocity components of the thermal wind and f is the Coriolis parameter. [6]

The *bearing* of the geostrophic wind is defined as the angle ϕ, measured clockwise from North, to the direction from which the wind blows. Obtain an expression relating the turning of the geostrophic wind with pressure $(\partial\phi/\partial \ln p)$ to the horizontal advection of temperature $(\mathbf{u}\cdot\nabla T)_p$ on an isobaric surface. Hence show that *veering* (in which ϕ increases with height) is associated with warm advection, and *backing* (in which ϕ decreases with height) with cold advection. [4]

A radiosonde measures the wind speed between 1000 hPa and 500 hPa to remain constant with height at 15 ms^{-1}. The wind is observed to be northerly between 1000 hPa and 700 hPa but then backs in direction from $\phi = 0°$ at 700 hPa to a bearing of $\phi = 300°$ at 500 hPa. Estimate the mean rate of change of temperature in K day^{-1} within the layer of the atmosphere between 1000 hPa and 500 hPa over the site of the radiosonde station, clearly stating any assumptions or approximations you make.

Deduce what you can about the synoptic situation implied by these measurements, and possible significant weather to be expected during the following few hours. [10]

[Total: 20]

5. **Inertial oscillations**

Pure inertial oscillations can be modelled by

$$\frac{\partial u}{\partial t} - fv = 0 \,, \qquad \frac{\partial v}{\partial t} + fu = 0 \,,$$

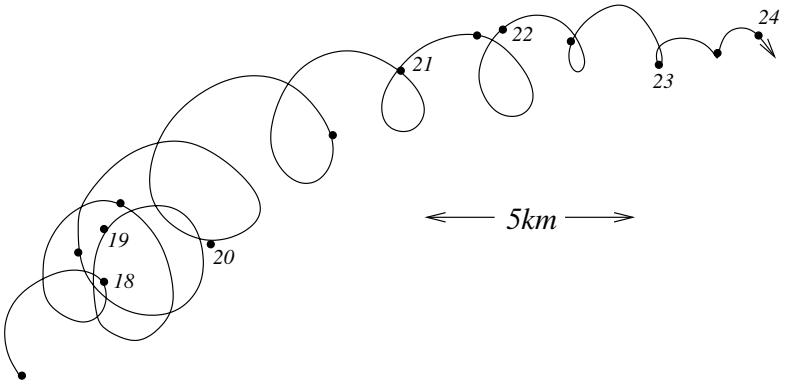
where f is constant and horizontal pressure gradients are neglected.

Solve for (u, v), given that $(u, v) = (u_0, 0)$ at $t = 0$. [3]

Integrate again to find particle displacements, and show that the particle trajectories for these oscillations are circular.

Calculate the radius of the circle at 45°N if the particle speed is (a) 1 $\mathrm{m\,s}^{-1}$, (b) 10 $\mathrm{m\,s}^{-1}$. [5]

The following figure shows the trajectory of a fluid parcel in the Baltic Sea (58°N) in August 1933. The markers are separated by 12 hours, and the numbers indicate the day of the month. Calculate the theoretical period of an inertial oscillation at this location. How does this compare with the observed oscillation period?



[2]

[Total: 10]

6. **Inertia-gravity waves**

Starting with the linear free surface equations with $f =$ constant, derive the linearised potential vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f\eta}{H}\right) = 0,$$

and hence show that

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2 \eta}{\partial t^2} - c^2\nabla_h^2\eta + f^2\eta\right\} = 0$$

where $c^2 = gH$. [6]

Look for solutions of the form $\eta = \eta_0 \cos(kx - \omega t)$ and show that either

$$\text{(i)} \quad \omega = 0\ , \qquad \text{or} \qquad \text{(ii)} \quad \omega^2 = f^2 + c^2k^2\ .$$

[3]

Show that (i) represents a steady geostrophic current, and find the corresponding u and v. [3]

Equation (ii) is the dispersion relation for inertia-gravity (or Poincaré) waves; in this case verify (e.g., by direct substitution) that

$$(u, v) = \frac{\eta_0 g}{c^2 k}(\omega\cos\theta\,,\ f\sin\theta)$$

where $\theta = kx - \omega t$. In both cases ignore any superimposed inertial oscillations. [3]

[Total: 15]

7. **Energetics**

Show that the available potential energy per unit horizontal area of a homogeneous fluid with free surface elevation η is $P = \rho g\eta^2/2$, where the available potential energy is defined as the potential energy relative to to the reference level $z = 0$, i.e. the undisturbed free-surface level. [2]

Let $K = \rho H(u^2+v^2)/2$, the kinetic energy per unit horizontal area. Find the ratio $\bar{P}/\bar{K}$ for the geostrophic disturbances (i) and the inertia-gravity waves (ii) of Question 6, where $\bar{}$ denotes a time or space average. In each case express this ratio in terms of the quantity $\alpha = \lambda/(2\pi L_d)$ (where λ is the total horizontal wavelength and $L_d = c/|f|$ is the Rossby deformation radius) and comment on the relative magnitudes of $\bar{P}$ and $\bar{K}$ when λ is (a) much greater than $2\pi L_d$ and (b) much less than $2\pi L_d$. [8]

[Total: 10]

8. **Equatorial Kelvin waves**

The reduced-gravity equations for oceanic motion close to the equator can be written

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \beta y v + g'\frac{\partial h}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \beta y u + g'\frac{\partial h}{\partial y} &= 0, \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}$$

Linearise these equations about a state of rest. Show that there exists an equatorial Kelvin wave solution to the linearised equations for which v is identically zero and h takes the form

$$h = h_0 + A(y)F(x - ct) \ ,$$

where F is an arbitrary function. [7]

Find the functional form of $A(y)$ and show that the amplitude decays to the north and south of the equator over a scale set by the equatorial deformation radius,

$$L_{eq} = \left(\frac{2(g'h_0)^{1/2}}{\beta}\right)^{1/2} .$$

Show that equatorial Kelvin waves propagate eastward at a speed $c = (g'h_0)^{1/2}$.

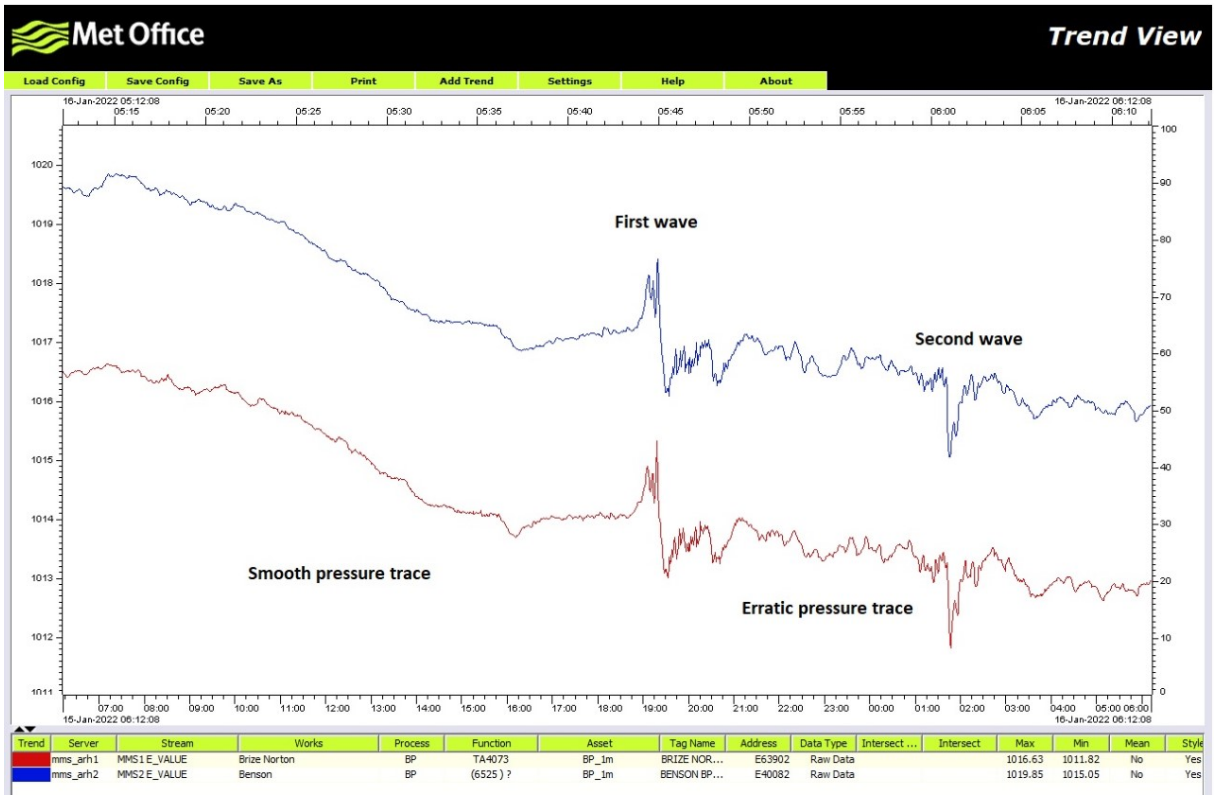
Are these waves dispersive? [9]

During an El Niño event, westerly wind anomalies excite equatorial Kelvin waves over the central Pacific. Assuming $h_0 \approx 200\,\mathrm{m}$ and $g' \approx 0.01\,\mathrm{m\,s^{-1}}$, estimate the time taken for the Kelvin wave to reach the coastline of Central America and estimate the equatorial deformation radius. [4]

[Total: 20]

9. **External Gravity Waves**

The volcanic eruption of Hunga Tonga (4:15 UTC, 15 January 2022) sent atmospheric waves around the planet, as clearly recorded in surface pressure measurements in Oxfordshire (see Figure, which shows two waves associated with the two great circle pathways). Test whether this data is consistent with the theory of external inertia-gravity waves in the shallow water system, by treating the wave as a perturbation in the height of the tropopause. Consider (i) the time of arrival of the waves and (ii) whether inertia-gravity wave dispersion can account for the difference in structure between the two signals. **[Total: 5]**



[Total for Problem Set: 100]

[AD 20220124]