Problem Set 1. **General Relativity, HT22**

THE SIGN OF FOUR: COORDINATES, 4-VECTORS and 4-TENSORS

"That's its business," said Lambert. "If Balbus says it's the same bulk, why, it's the same bulk, you know."

"Well, I don't believe it," said Hugh.

"You needn't," said Lambert. "Besides, it's dinner-time. Come along." They found Balbus waiting dinner for them, and to him Hugh at once propounded his difficulty.

"Let's get you helped first," said Balbus, briskly cutting away at the joint. "You know the old proverb, 'Mutton first, mechanics afterwards'?"

The boys did not know the proverb, but they accepted it in perfect good faith, as they did every piece of information, however startling, that came from so infallible an authority as their tutor. They ate on steadily in silence, and, when dinner was over, Hugh set out the usual array of pens, ink, and paper, while Balbus repeated to them the problem he had prepared for their afternoon's task.

— Excerpt from *A Tangled Tale*, by Lewis Carroll

NOTE: Problem 5 is optional.

1.) Consider the following thought:

"Special relativity holds for frames moving at constant relative velocity, but of course acceleration requires general relativity because the frames are noninertial."

Wait, what? Nonsense! Special relativity is a formulation encompassing all nongravitational physics, and it certainly doesn't collapse before simple kinematical accleration. On the other hand, acceleration, even just uniform accleration in one dimension, is not without its connections to general relativity. We shall explore some of them here. *For ease of notation, let us set* $c = 1$. In part (d) we'll put c back.

1a.) Let us first ask what we mean by "uniform acceleration." After all, a rocket approaching the speed of light c can't change its velocity at a uniform rate forever without exceeding c at some point. Go into the frame moving instantaneously at velocity v, *with* the rocket relative to the "lab." By definition, in this frame, the instantaneous rocket velocity v' is zero. Now, wait a time dt' later, as measured in this frame. The rocket will now have a velocity dv' in this same frame. What we mean by constant acceleration is that $dv'/dt' \equiv a'$ is constant. By contrast, the acceleration measured in the fixed lab is certainly not constant! The question is, how is the lab acceleration $a = dv/dt$ related to the truly constant a'?

To answer this, let $V = v/\sqrt{1-v^2}$, the spatial part of the 4-vector V^α associated with the ordinary velocity $\boldsymbol{v}$, and the same for V' and v'. Assume, for the moment, that the primed and unprimed frames differ by some arbitrary velocity w. The 4-velocity differentials are then given by:

$$dV' = (dV - w\, dV^0)/\sqrt{1-w^2}$$

where $V^0 = 1/\sqrt{1-v^2}$. Explain.

1b.) Now, set $w = v$. We thereby go into the frame in which $v' = 0$; the rocket is instantaneously at rest. Prove that $dv = dv'(1-v^2)$. (Remember, v and v' are the *ordinary*

velocities.) From here, prove that

$$\frac{dv}{dt} = a'(1-v^2)^{3/2}.$$

1c.) Show that, starting from rest at $t = t' = 0$,

$$v = \frac{a't}{\sqrt{1+a'^2t^2}}, \qquad a't = \sinh(a't'),$$

and hence show that (for $x = 0$ at $t = t' = 0$):

$$v = \tanh(a't'), \qquad x = \frac{1}{a'}[\cosh(a't') - 1]$$

The integrals are not difficult; do them yourselves.

1d.) Let's use these results to construct a full coordinate transformation from the lab frame x, t to the accelerating x', t' frame. A good start is to guess a transform of the form

$$t = A(x')\sinh(a't') + B(x'), \qquad x = A(x')\cosh(a't') + C(x')$$

where A, B, and C depend only upon x'. Then on x' = constant surfaces, $dx/dt = \tanh(a't') = v$, which is indeed what we need.

By definition, constant t' surfaces are constant-time surfaces in the (x', t') frame that move instantaneously with velocity $v = \tanh(a't')$ with respect to the (x, t) frame. On such a surface, $dt'/dx' = 0$. We fix the origin by demanding that as $t' \to 0$, $x \to x'$. We fix our clock by demanding that as $t' \to 0$, $t \to t'$ at the rocket location $x' = 0$. (This must be done locally: since A depends on x', this time agreement can be exact at only one value of x'.) Show that these constraints force B and C to be constant, and that B in particular must vanish.

Finally, put the speed of light c back into the equations, demand that x goes to x' at $t' = 0$, and show that

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh(a't'/c), \qquad x = \left(\frac{c^2}{a'} + x'\right)\cosh(a't'/c) - \frac{c^2}{a'}$$

1e.) Show that the invariant Minkowski line element may be written in x', t' coordinates as:

$$c^2d\tau^2 = c^2dt^2 - dx^2 = \left(1 + \frac{a'x'}{c^2}\right)^2 c^2dt'^2 - dx'^2.$$

Provide a physical interpretation of your result in terms of a gravitational redshift. How do you interpret the region $x' \le -c^2/a'$? (Review the results of 1d.)

2.) *Orbits with relativistic kinematics.* The relativistic equations for a test mass with ordinary velocity $\boldsymbol{v}$ in a potential of the form $-\alpha/R$, where α is a constant, are given by:

$$\gamma c^2 - \frac{\alpha}{R} = E$$

$$\gamma R^2 \dot{\phi} = J$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and we use standard R, ϕ cylindrical coordinates in the orbital plane. $\dot{\phi}$ is an ordinary time derivative, i.e. the time is measured in the fixed "lab" frame, and E and J are constants. These equations are in fact the exact classical relativistic equations for an electron in orbit about a positive point charge, but we can solve them as a poor man's substitute for proper general relativity by setting $\alpha = GM$. (To the best of my knowledge, no one even tried this during the years between special and general relativity, when the distinction was less clear.) Later in the course, we can compare these "ersatz orbits" with those of the true theory. In any case, you'll have solved exactly the correct, classical relativistic atomic orbit problem!

2a.) Give a physical interpretation to each of these equations. What do the constants E and J represent physically? Be precise.

2b.) Show that

$$\left(\frac{dR}{d\phi}\right)^2 = \left(\frac{\dot{R}}{\dot{\phi}}\right)^2 = \frac{R^4}{J^2c^2}\left(E^2 - c^4 + \frac{2\alpha E}{R} + \frac{1}{R^2}[\alpha^2 - J^2c^2]\right)$$

(You may wish to begin by first solving for $\dot{\phi}$ in terms of R alone.)

2c.) Set $u = 1/R$ and derive a second order linear differential equation for u:

$$\frac{d^2u}{d\phi^2} + \mu^2 u = \frac{\alpha E}{J^2c^2}$$

where

$$\mu^2 = 1 - \frac{\alpha^2}{J^2c^2}$$

2d.) Finally, show that, up to an unimportant orbital phase constant, the most general solution of this equation takes the form

$$u = \frac{\alpha E}{\mu^2 J^2 c^2}[1 + \epsilon\cos(\mu\phi)]$$

where ϵ is an arbitrary constant of integration. Show that when $\alpha \ll Jc$, this orbit corresponds to a near ellipse, but an ellipse in which the point of closest approach advances by an angle

$$\Delta\phi = \frac{\pi\alpha^2}{J^2c^2}$$

every orbit. We will see that general relativity also predicts such an advance of the *perihelion,* as it is called, but by an amount which is six times larger! The precise match of GR with observations of Mercury's orbit was the theory's first great success.

3.) *Recognising tensors.* One way to prove that something is a vector or tensor is to show explicitly that it satisfies the coordinate transformation laws. This can be a long and arduous procedure if the tensor is complicated with many indices. There is another way, usually much better!

Show that if V_ν is an arbitrary covariant vector and the combination $T^{\mu\nu}V_\nu$ is known to be a contravariant vector (note the free index μ), then

$$\left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}\right)V'_\nu = 0$$

Why does this prove that $T^{\mu\nu}$ is a tensor? Does your proof actually depend on the rank of the tensors involved?

4.) *What about $d^2x_\mu/d\tau^2$?* The geodesic equation in standard form gives us an expresssion for $d^2x^\mu/d\tau^2$ in terms of the affine connection, $\Gamma^\mu_{\nu\lambda}$. For the covariant coordinate x_μ, show that

$$\frac{d^2x_\mu}{d\tau^2} = \frac{1}{2}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\frac{\partial g_{\nu\rho}}{\partial x^\mu}$$

Refer to section 4.7 in the notes if help is needed. Under what conditions is $dx_0/d\tau = V_0 \equiv V_t$ a constant of the motion?

5*.) *What is "the spatial part" of a metric?: An* ***optional*** *problem for further study.* It is easy, even trivial, to get the proper time from a metric. One simply sets all the spatial $dx^i = 0$ in the invariant interval $g_{\mu\nu}dx^\mu dx^\nu$, and reads off a proper time of

$$d\tau = \sqrt{-g_{00}}\,dx^0/c.$$

This is what a local inertial observer reads off on their watch. So to get " the spatial part" of the metric, let's call it dl^2, do we just take whatever is left over from setting $dx^0 = 0$, i.e. $dl^2 = g_{ij}dx^i dx^j$? Not quite; it depends on the application. The point is that one cannot enforce global simultaneity by setting dx^0 everywhere

How does a physical observer actually measure a distance? They take a light ray, bounce it off a mirror some distance dl away, measure their local (proper) time on their watch $d\tau$ for the light to go and come back, and then determine the distance via $dl = cd\tau/2$. Let's go with that.

5a.) Show that for a *diagonal* metric tensor (all $g_{0i} = g_{i0} = 0$), this procedure gives

$$dl^2 = g_{ij}dx^i dx^j,$$

just as we expect. (In interpreting your results, you will find it useful to think of the light ray as leaving at a negative time interval, and returning after a positive time interval.)

5b.) Show that for a general metric tensor $g_{\mu\nu}$, with nonvanishing $g_{0i} = g_{0i}$, this procedure gives

$$dl^2 = \gamma_{ij}dx^i dx^j, \text{ where } \gamma_{ij} = g_{ij} - (g_{0i}g_{0j}/g_{00})$$

The metric tensor of a rotating black hole (the Kerr metric) actually has $g_{0\phi} = g_{\phi 0}$ components, so this formula is very relevant here. We see that the spatial part of the metric may contain mixed time-indexed terms!

5c.) Using the $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$ relations, show that

$$g^{ij}\gamma_{jk} = \delta^i_k$$

and that, defining γ^{ij} by raising indices via $\gamma^{ij} \equiv g^{ik} g^{jm} \gamma_{km}$, leads to

$$\gamma^{ij} = g^{ij},$$

the "pure spatial part" of $g^{\mu\nu}$. γ^{ij} defined this way is indeed the inverse of γ_{ij}. Hence, we are justified in regarding $\gamma_{ij} dx^i dx^j$ as the invariant interval in its own three-dimensional space, with inverse γ^{ij}, within the more encompassing four-dimensional $g_{\mu\nu}$ spacetime. (Note that this also shows that the indices on γ_{ij} may be raised with γ^{ij}.)

5d.) Show that $\det g_{\mu\nu} = g_{00} \det \gamma_{ij}$, which is consistent with identifying γ_{ij} as the spatial metric. You may find it useful to recall that the determinant of a matrix is unchanged when a multiple of one row is added to another.

Problem Set 2. **General Relativity, HT22**

EQUILIBRIUM, FLOWS, AND ORBITS IN GENERAL RELATIVITY

1.) *Hydrostatic Equilibrium in GR.* Model a neutron star atmosphere with a simple equation of state: $P = K\rho^\gamma$, where P is pressure, ρ is mass density, γ is the adiabatic index and K is a constant. Assume that $g_{00} = -(1 - 2GM/rc^2)$, where M is the mass of the star and r is radius. If $\rho = \rho_0$ at the surface $r = r_0$, solve the equation of hydrostatic equilibrium to show that

$$\frac{1 + K\rho^{\gamma-1}/c^2}{1 + K\rho_0^{\gamma-1}/c^2} = \left(\frac{1 - R_S/r_0}{1 - R_S/r}\right)^\alpha$$

where $R_S = 2GM/c^2$ is the so-called Schwarzschild radius, and $2\alpha\gamma = \gamma - 1$. (Hint: See §4.6 of the notes.) What is the Newtonian limit of the above equation? Express your answer in terms of the speed of sound a, $a^2 = \gamma P/\rho$ and the potential $\Phi(r) = -GM/r$. (OPTIONAL: For those who have studied fluids, what quantity is being conserved in the Newtonian limit?)

2.) *Bondi Accretion: go with the flow.* To get some practise working with the equations of GR as well as some insight into relativistic dynamics in a practical problem in astrophysics, consider what is known as (relativistic) Bondi Accretion, the spherical flow of gas into a black hole. (The original Bondi accretion problem was Newtonian accretion onto an ordinary star.) We assume a Schwarzschild metric in the usual spherical coordinates:

$$g_{00} = -(1 - 2GM/rc^2),\ g_{rr} = (1 - 2GM/rc^2)^{-1},\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2\sin^2\theta.$$

2a.) First, let us assume that particles are neither created or destroyed. So particle number is conserved. If n is the particle number density in the local rest frame of the flow, then the particle flux is $J^\mu = nU^\mu$, where U^μ is the flow 4-velocity. Justify this statement, and using §4.5 in the notes, show that particle number conservation implies:

$$J^\mu{}_{;\mu} = 0.$$

If nothing depends upon time, show that this integrates to

$$nU^r|g'|^{1/2} = \text{constant},$$

where g' is the determinant of $g_{\mu\nu}$ divided by $\sin^2\theta$, and U^r is...well, you tell me what U^r is.

2b.) We move on to energy conservation, $T^{t\nu}{}_{;\nu} = 0$. (Refer to §4.6 in the notes.) Show that the only nonvanishing affine connection that we need to use is

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{\partial \ln|g_{tt}|}{\partial r}$$

Derive and solve the energy equation. Show that its solution may be written

$$(P + \rho c^2)U^r U_t|g'|^{1/2} = \text{constant}$$

where $U_t = g_{t\mu}U^\mu$, and ρ is the total energy density of the fluid in its rest frame, including any thermal energy.

2c.) We next define

$$\varpi = \mu n,$$

where μ is the rest mass per particle and ϖ is a Newtonian density. This is not to be confused with ρ, the true relativistic energy density divided by c^2. P and ϖ are assumed to be related by a simple power law relationship,

$$P = K\varpi^\gamma$$

where K is a constant, and γ is called the adiabatic index. This is not an entirely artificial problem: it is valid for cold classical particles ($\gamma = 5/3$) or hot relativistic particles ($\gamma = 4/3$). The first law of thermodynamics then tells us that the thermal energy per unit volume is

$$\epsilon = \frac{P}{\gamma - 1}$$

(You needn't derive that here, just use it!) Show that this implies:

$$\rho = \varpi + \frac{P}{c^2(\gamma - 1)}.$$

2d.) Verify that

$$|g'| = r^4$$

and using $g_{\mu\nu}U^\mu U^\nu = -c^2$, show that

$$U_t = \left[c^2 - \frac{2GM}{r} + (U^r)^2\right]^{1/2}$$

(Take care to distinguish U^t and U_t.)

2e.) With

$$a^2 = \gamma P/\varpi,$$

(this is the speed of sound in a nonrelativistic gas), combine our mass and energy conservation equations to show that

$$\left(\mathbf{c^2} + \frac{\mathbf{a^2}}{\boldsymbol{\gamma - 1}}\right)^{\mathbf{2}} \left(\mathbf{c^2} + \mathbf{U^2} - \frac{\mathbf{2GM}}{\mathbf{r}}\right) = \textbf{constant}.$$

We have dropped the superscript r on U^r for greater clarity. How does a^2 depend upon ϖ? The other equation we shall use is just that of mass conservation itself. Show that this may be written as

$$4\pi\varpi r^2 U = \dot{m},$$

which defines the net, constant mass accretion rate $\dot{m} < 0$. With a^2 depending entirely on ϖ, and $\varpi = \dot{m}/(4\pi r^2 U)$, the equation in **boldface** becomes a single algebraic equation for U as a function of r, and the formal solution to our problem.

2f.) Three final simple tasks for now:

i) Show that the constant on the right of the **bold** equation of problem (2e) is

$$c^2\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2$$

where a_∞ is the sound speed at infinite distance from the black hole, if the gas starts accreting from rest.

ii) Show that the Newtonian limit of the equation is

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}$$

where v is the ordinary velocity, not the 4-velocity. This is a statement that a quantity known as enthapy (energy plus the work done by pressure) is conserved. This is the original nonrelativistic Bondi 1952 solution for accretion onto a star.

iii) Show that as r approaches the Schwarzschild radius $R_S = 2GM/c^2$, then if $a \ll c$ everywhere, then dr/dt satisfies the condition of a "null geodesic," a fancy way to say the inflow follows the equation of light:

$$\frac{dr}{dt} = -c(1 - R_S/r).$$

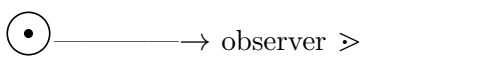
Like stalled photons, from the point of view of a distant observer, the flow never crosses R_S.

3a.) *Kinematic and gravitational redshifts.* One of the most important observational black hole diagnostics is a calculation of the radiation spectrum from the surrounding disc. In particular we are interested in how the frequency of a photon is shifted due to space-time distortions and relativistic kinematics. Show that:

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R) V^\mu(R)}{p_\mu(E) V^\mu(E)}$$

where R denotes the received the photon and E the emitted photon, ν is a frequency (not an index here!), p_μ a covariant photon 4-momentum, and V^μ is the normalised 4-velocity in the form $(dt/d\tau, d\mathbf{x}/cd\tau)$ for the emitted material (E) or the distant observer at rest (R).

3b.) In the problem at hand, the observer views the disc edge-on, in the plane of the disc. The gas moves in circular orbits. Here is the disc spherical midplane ($\theta = \pi/2$), viewed from above:



Show that in t, r, θ, ϕ coordinates for the $0, 1, 2, 3$ components,

$$V^\mu(R) = (1, 0, 0, 0), \quad V^\mu(E) = V_E^0(1, 0, 0, d\phi/cdt), \text{ with } V_E^0 = dt/d\tau$$

Then, using $g_{\mu\rho} V^\mu V^\rho = -1$, conclude that

$$V_E^0 = (1 - 3GM/rc^2)^{-1/2}$$

You may use the result from problem (5c) below that $\Omega^2 = (d\phi/dt)^2 = GM/r^3$. (You will prove it later!)

3c.) Finally, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \left(1 + \frac{\Omega p_\phi(E)}{cp_0(E)}\right)^{-1}, \quad \Omega^2 = GM/r^3.$$

A result from problem (4) of Problem Set 1 may be useful.

From disk material moving at right angles across the line of sight, ν_R/ν_E reduces to

$$(1 - 3GM/rc^2)^{1/2}.$$

Why? From disk material moving precisely along the line of sight, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - 3GM/rc^2\right)^{1/2} / \left(1 \pm (rc^2/GM - 2)^{-1/2}\right)$$

(Hint: $g^{\nu\rho}p_\nu p_\rho = 0$. You may ignore deflection of the photon trajectories and assume they move in straight lines.) Interpret the $\pm$ sign. In general, the photon paths must be calculated from the dynamical equations to determine the $p(E)$ ratio.

4a.) *The perihelion advance of Mercury.* In the notes we found that the differential equation for $u = 1/r$ for Mercury's orbit could be written as follows. $u = u_N + \delta u$ with the Newtonian solution u_N given by

$$u_N = (GM/J^2)(1 + \epsilon\cos\phi)$$

and the differential equation for δu is

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2J^4}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi).$$

Show that this is equivalent to solving the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2)$$

where $a = 3(GM)^3/(c^2J^4)$ and $b = 1 + \epsilon^2/2$.

To solve this, try a solution of the form

$$\delta u = A_0 + A_1\phi e^{i\phi} + A_2 e^{2i\phi}$$

where the A's are constants. Why do we need an additional factor of ϕ in the A_1 term?

4b.) Show that the solution for $u = u_N + \delta u$ is

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi + \epsilon a\phi\sin\phi$$

Since a is very small, show that this equivalent to

$$u = ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}[1 + \epsilon(\cos\phi(1 - \alpha))]$$

where

$$\alpha = aJ^2/GM = 3(GM/Jc)^2$$

4c.) In the equation for u, the first two terms in a cause tiny (and unmeasurable) distortions in the shape of the ellipse, but do not affect the 2π perodicity in ϕ of the orbit. Show however that the final term, proportional to GM/J^2, results in a periastron advance of

$$\Delta\phi = 6\pi \left(\frac{GM}{cJ}\right)^2$$

each orbit. This is the classic Einstein result.

5a.) *Black hole orbits.* In Newtonian theory, the energy equation for a test particle in orbit around a point mass is

$$\frac{v^2}{2} + \frac{l^2}{2r^2} - \frac{GM}{r} = \mathcal{E}$$

where r is radius, v is the radial velocity, l the angular momentum per unit mass, $\mathcal{E}$ the constant energy per unit mass, and $-GM/r$ is of course the potential energy. For the Schwarzschild solution show that the integrated geodesic equation may also be written in the form

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathcal{E}_S$$

where r is the standard radial coordinate, l_S and $\mathcal{E}_S$ are constants, $\Phi_S(r)$ a modified potential function, and $v_S = dr/d\tau$. Determine l_S, Φ_S and $\mathcal{E}_S$ in terms of the fundamental angular momentum and energy constants J and γ_∞ from lecture (or the notes) and GM or c as needed. The form of l_S, $\mathcal{E}_S$, and Φ_S should be chosen to go over to their Newtonian counterparts in the appropriate limit.

5b.) Sketch the effective potential $l_S^2/2r^2 + \Phi_S(r)$. Prove that there is always a potential minimum in Newtonian theory, but that this is not the case in general relativity. What is the mathematical condition for the existence of a potential minimum for the effective potential, and what does it mean physically if it does not exist?

When there are no extrema there are no stable orbits, since a stable orbit corresponds to radial oscillations about the potential minimum (of zero amplitude in the case of a circle).

5c.) Show that for the Schwarzschild metric, circular orbits satisfy

$$\Omega^2 = \frac{GM}{r^3},$$

exactly the Newtonian form. Here $\Omega \equiv d\phi/dt$ at the coordinate location r, where dt is the proper time interval at infinity. Derive expressions for E and J in terms of GM, c^2 and r.

5d.) Below what value of r does the effective potential not have any local extrema? (Answer: $6GM/c^2$.)

Problem Set 3. **General Relativity, HT22**

Birkhoff Theorem and Neutron Stars

Affine connection for diagonal $g_{\mu\nu}$:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right) \quad \text{NO SUM OVER } \lambda$$

Ricci Tensor:

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \frac{\Gamma^{\eta}_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^{\eta}} \quad \text{FULL SUMMATION}$$

N.B.: In this problem, we will set $c = 1$.

1a.) Birkhoff's theorem states that outside of a spherical distribution of matter, the metric tensor must be independent of time and equal to the Schwarzschild metric—even if the matter distribution is changing (keeping spherical symmetry) with time. A corollary is that within the hollow of an external spherical distribution of matter, the metric tensor must be Minkowski spacetime, even if there are external, spherical changes. These are the precise relativistic analogues of the Newtonian results of a point mass $1/r$ potential outside any spherical distribution of matter, and the vanishing of the gravitational field inside a cavity with a spherical external distribution of matter. Birkhoff's theorem is critical to formulating modern cosmology.

To prove the theorem is straightforward but painful, because we need to calculate the Ricci tensor $R_{\mu\kappa}$, and that is always a nuisance. Both because Birkhoff's theorem is important, as well as to get practice working with the Ricci tensor, we will explicitly evaluate the key R_{tr} component here, a critical step in the proof[1] . Recall that the R_{tr} component vanishes identically for the static Schwarzschild metric.

Consider the line element for a general time-dependent spherical system,

$$-d\tau^2 = -B(r,t)dt^2 + A(r,t)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \equiv g_{\mu\nu}dx^{\mu}dx^{\nu}$$

The nonvanishing affine connection components Γ^{a}_{bc} from this metric tensor are the same nonvanishing set we found for the Schwarzschild metric, plus three others that used to be zero. In particular, show that

$$\Gamma^{r}_{rt} = \frac{\dot{A}}{2A}, \qquad \Gamma^{t}_{rr} = \frac{\dot{A}}{2B} \qquad \Gamma^{t}_{tt} = \frac{\dot{B}}{2B}$$

where we will use notation $\dot{A}$ for a time derivative, and A' for an r derivative. A warm welcome to our new three new affine connection club members!

1b.) We will now show that $R_{tr} = -\dot{A}/rA$ precisely, which sure looks simple, but in fact involves a large cancellation. The point now is that since all the $R_{\mu\kappa}$ terms must vanish in a vacuum, $\dot{A} = 0$, and A cannot depend on time. The other components of the Ricci tensor then all revert back to their Schwarzschild forms. (We won't show this explicitly, only

[1] See Weinberg for a complete proof.

because it is a long and dull exercise, but it is not particularly difficult). A reduction to the Schwarzschild problem means that any possible time dependence in B can appear only as an overall multiplicative factor $f(t)$, which can then be completely eliminated by a simple time coordinate transformation $dt' = f\,dt$. The static metric is then identical to Schwarzschild. This is Birkoff's theorem.

Using the Ricci tensor above, show that the first two groupings

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda}$$

cancel one another out precisely. This is progress. (g is the determinant of $g_{\mu\nu}$.)

1c.) Show next that for $R_{\mu\kappa} = R_{tr}$,

$$\Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^\eta} = -\frac{\dot{A}}{rA}$$

(Use §6.1 from the notes for any Γ's you need.) You will find that everything cancels once again, except for one final term in the $\ln|g|$ derivative, shown on the right. With $\dot{A} = 0$, Birkhoff's theorem follows relatively easily, as the remaining $R_{\mu\kappa} = 0$ equations reduce to the Schwarzschild problem.

2.) In this problem we will show that the TOV equation must be obeyed if we demand that the total energy (including the gravitational contribution) is minimised when we vary $\rho(r)$ throughout a star of uniform entropy per particle, subject to the constraint that the total number of particles the same.

2a.) Use the method of Lagrange multipliers to show that the above statement translates into the condition that variations of the quantity

$$M - \lambda N \equiv \int_0^\infty 4\pi r^2 \rho\, dr - \lambda \int_0^\infty 4\pi n r^2 A^{1/2}\, dr$$

must be zero. Of course ρ and n vanish outside the star, so the integration limits are formal. Notation: ρ is the energy density (divided by c^2), n the number density of baryons, and A is g_{rr} from the interior stellar metric of §8.2 in the notes:

$$A = (1 - 2G\mathcal{M}(r)/rc^2)^{-1}, \quad \mathcal{M}(r) = \int_0^r 4\pi\rho r'^2 dr'.$$

Finally, λ is the (constant) Lagrange multiplier.

2b.) Prove that the first order variation δ of this equation gives:

$$\delta M - \lambda \delta N = \int_0^\infty 4\pi r^2 \left[\delta\rho - \lambda A^{1/2}\delta n - \lambda \frac{GA^{3/2}}{rc^2} n\, \delta\mathcal{M}\right] dr$$

2c.) For the constant entropy (adiabatic) perturbations, the first law of thermodynamics is $dE = -PdV$ where E is the energy within some volume V, P is the pressure and dV is

a small volume change. If particle number is conserved so that nV also remains constant, show that δn and $\delta\rho$ are related by:

$$\delta n = \frac{n\,\delta\rho}{\rho + P/c^2}.$$

2d.) Put all these results together and show that

$$\delta M - \lambda\delta N = \int_0^\infty 4\pi r^2 \delta\rho \left[1 - \frac{\lambda n A^{1/2}}{\rho + P/c^2} - \frac{\lambda G}{c^2}\int_r^\infty (4\pi r' n A^{3/2} dr')\right] dr = 0$$

Hint: In the expression for $\delta\mathcal{M}$, you will need to invert the order of integration for r and r'. Also, they are just dummy variables, so you are allowed to interchange their names at the end!

2e.) If the result of 2d) is to hold for any $\delta\rho$, show that $1/\lambda = F(r)$, where F is a function of r, which you should determine. But remember that λ must be constant, so that in fact $dF/dr = 0$! Show that this leads to the TOV equation,

$$\frac{dP}{dr} = -\frac{GA}{r^2}(\mathcal{M} + 4\pi r^3 P/c^2)(\rho + P/c^2).$$

(Hint: in a constant entropy star, the equation of part 2c) also holds with δn and $\delta\rho$ replaced by dn/dr and $d\rho/dr$. Why is that?)

We have thus shown that when the TOV is satisfied, the total energy is minimised for a star of constant entropy. The TOV equation itself is valid independently of any thermodynamics: it is a direct consequence of the field equations of gravity. But the constant entropy case allows a variational formulation for it. Notice as well that we reach the Newtonian limit in our final equation by setting $A = 1$ and letting $c \to \infty$. Could we set $A = 1$ at the start of our derivation and reach the Newtonian limit?

3a.) *Rotating, relativistic stars.* Do the following Exercise from §4.6 in the notes. Starting with the formal stress tensor conservation equation for an ideal fluid,

$$0 = \frac{\partial P}{\partial x^\nu} + \frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|g|^{1/2}(\rho + P/c^2)U^\mu U_\nu\right] - \Gamma^\mu_{\nu\lambda}(\rho + P/c^2)U_\mu U^\lambda$$

use the general expression for $\Gamma^\mu_{\nu\lambda}$ to show that this may be written (more simply) as:

$$0 = \frac{\partial P}{\partial x^\nu} + \frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|g|^{1/2}(\rho + P/c^2)U^\mu U_\nu\right] + (\rho + P/c^2)U_\mu\frac{\partial U^\mu}{\partial x^\nu}$$

3b.) Next, consider the case of a rotating star. The uniform rotation rate $\Omega = d\phi/dt$ is assumed to be constant. On the surface (and in the interior) of the star, the 4-velocity component $U^\phi = d\phi/d\tau = \Omega U^0$, where as usual $U^0 = dt/d\tau$. There are no other 4-velocity U^μ components, and no t or ϕ dependence of the stellar structure. Show that, under these conditions, our equation becomes

$$0 = \frac{\partial P}{\partial x^\nu} - (\rho c^2 + P)\frac{\partial \ln U^0}{\partial x^\nu}$$

3c.) Work now with spatial coordinates, i, j, k for ν. Recall that if A_i is a covariant vector, the curl operator $\boldsymbol{\nabla}\times\boldsymbol{A} = \partial_j A_i - \partial_i A_j$ is also a vector, as the affine connection terms from the covariant derivatives cancel. Show that if surfaces of constant of ρ and constant P(or any two functions at all) coincide, then $(\partial_i P)(\partial_j \rho) = (\partial_j P)(\partial_i \rho)$.

3d.) Using 3c), show that for a rotating star, surfaces of constant ρ, P, and U^0 all coincide. Viewed by an observer at infinity, do surface clocks on a rotating neutron star run faster at the equator or the poles? Does spherical symmetry matter?

4a.) *Explicit degenerate ideal gas neutron star equations.* Show that the differential equations for a star neutron star with a fully degenerate ideal gas equation of state are (refer to and use §8.5 of the notes for complete background and definitions):

$$\frac{dt}{dy} = -\frac{4m}{y^2(1-2m/y)}\left(\frac{\sinh t - 2\sinh(t/2)}{\cosh t - 4\cosh(t/2) + 3}\right)\left(1 + \frac{\pi y^3}{8m}[\sinh t - 8\sinh(t/2) + 3t]\right)$$

$$\frac{dm}{dy} = \frac{3\pi y^2}{8}(\sinh t - t)$$

The dimensionless variables m and y are defined in terms of radius r and $\mathcal{M}$ (mass within r) by:

$$\rho_{char} = 8\pi m_n^4 c^3/3h^3, \quad r \equiv c(G\rho_{char})^{-1/2} y, \quad \mathcal{M} \equiv c^3 G^{-3/2} \rho_{char}^{-1/2} m$$

Recall the equation of state parameterisation from the notes:

$$\rho(t) = \frac{3\rho_{char}}{32}(\sinh t - t), \quad P(t) = \frac{\rho_{char} c^2}{32}[\sinh t - 8\sinh(t/2) + 3t]$$

A neutron star model consists of picking some $t = t_0$ at $y = 0$ to fix the central density $\rho_0 = (3/32)\rho_{char}(\sinh t_0 - t_0)$, setting $m = \pi(\sinh t_0 - t_0)y^3/8$ at small y (justify!), and integrating the equations until $t = 0$ at some finite value of y. This defines the outer edge of the star. A value of $t_0 \simeq 3$ yields the maximum mass of $0.7 M_\odot$.

4b.) Take the limit $t \to \infty$ and recover the extreme relativistic TOV equation :

$$\frac{d\rho}{dr} = -\frac{4G\mathcal{M}(r)\rho}{r^2 c^2}\left(1 + \frac{4\pi\rho r^3}{3\mathcal{M}(r)}\right)\left(1 - \frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1}$$

Problem Set 4. **General Relativity, HT22**

Gravitational Radiation

1a.) *The Burke-Thorne Potential.* Consider the following unusual Newtonian potential, due to Burke and Thorne:

$$\Phi = \frac{G J^{(5)}_{ij}}{5c^7} x^i x^j$$

where $J^{(5)}_{ij}$ is the traceless (energy) moment of inertia tensor, differentiated five times with respect to time:

$$J^{(5)}_{ij} = c^2 \frac{d^5}{dt^5}\left[\int \rho(x_i x_j - \delta_{ij} r^2/3)\, dV\right] = \frac{d^5}{dt^5}\left[I_{ij} - \delta_{ij} I_{kk}/3\right].$$

Here, $I_{ij} = c^2 \int \rho x_i x_j \, dV$ is the standard moment of inertia tensor. The indices i and j represent spatial Cartesian coordinates, and we use the Minkowski metric, so spatial index placement is unimportant. The radius $r^2 = x^i x^i$. Show that this potential gives rise to a force, $-\partial_i \Phi$, which is exactly analogous to the "radiation reaction force" in electromagnetism. In other words, show that we recover Einstein's gravitational energy loss formula,

$$\frac{dE}{dt} = \left\langle - \int \rho v_i \partial_i \Phi \, dV \right\rangle = -\frac{G}{5c^9} \langle \dddot{J}_{ij} \dddot{J}_{ij} \rangle.$$

This equation states that the work done by the force, averaged over time (this is the meaning of the angle brackets $\langle\rangle$) equals the rate at which energy is lost from the system. This also works for angular momentum loss as well. Show that:

$$\frac{dL}{dt} = -\left\langle \int \epsilon^{ijk} \rho x_i \partial_j \Phi \, dV \right\rangle = -\frac{2G}{5c^9} \left\langle \epsilon^{imk} \ddot{J}_{mn} \dddot{J}_{in} \right\rangle,$$

which states that the effect of "$\mathbf{r} \times \mathbf{F}$" torque, averaged over time, equals the angular momentum loss.

Here are some hints:

i.) When in doubt, integrate by parts, either in time or in space. The system is assumed to be spatially finite so that "integrals at infinity" may be ignored. Time averages over exact time derivatives may also be neglected. (Assumes periodic or strictly bounded motion.)

ii.) The equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x^i} = 0$$

is used for the energy loss formula derivation. (You don't have to take the time here to prove this, though seriously, you ought to be familiar with mass conservation by now.)

iii.) You should find that for the angular momentum loss formula, the result holds either for the traceless moment of inertia J_{ij} or for I_{ij}.

2.) *Desert island GR.* Here we will construct a linear, weak field theory gravity from scratch. Then we will construct GR from scratch! (Well, practically.)

Imagine that it is 1912. Minkowski has formulated the concept of his spacetime geometry (1908), but has departed the scene. Einstein has had his happy (1907) Equivalence Principle thought, and has at last understood that gravity is a Riemannian geometric theory of a distorted Minkowski spacetime, and that the name of the game is to relate the derivatives of $g_{\mu\nu}$ to $T_{\mu\nu}$. But he knows nothing more. Let's help him out.

2a.) Our weak gravity field equation will need, on the left side, a sum of second derivatives of $g_{\mu\nu}$. More conveniently, we use derivatives of the small quantity $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Not only is the background spacetime geometry flat Minkowski, our coordinates are very close to Cartesian. So, with $h \equiv h^{\rho}{}_{\rho}$, there are but five combinations that could possibly appear:

$$\Box h_{\mu\nu},\ \partial_\mu\partial_\nu h,\ (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu),\ \eta_{\mu\nu}\Box h,\ \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda}$$

($\Box \equiv \partial^\rho\partial_\rho$. We use the handy notation $\partial^\mu = \partial/\partial x_\mu$, $\partial_\mu = \partial/\partial x^\mu$, and raise and lower indices on $h_{\mu\nu}$ with $\eta^{\rho\mu}$.) Justify this statement and explain fully.

2b.) We accordingly search for an equation of the form:

$$\Box h_{\mu\nu} + \alpha(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h + \eta_{\mu\nu}(\gamma\Box h + \delta\partial_\rho\partial_\lambda h^{\rho\lambda}) = CT_{\mu\nu}$$

where $\alpha, \beta, \gamma, \delta$ and C are constants to be determined. You remember, of course, the stress tensor $T_{\mu\nu}$, now in Newtonian guise. We demand that $\partial^\mu T_{\mu\nu} = 0$ as an identity. What is the reason for this? Show that $\alpha = -1, \delta = 1, \gamma = -\beta$ follow:

$$\Box h_{\mu\nu} - (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h - \eta_{\mu\nu}(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}) = CT_{\mu\nu}$$

2c.) By taking the trace of this last equation and using $T_{00} \gg T_{ii}$ (valid in the Newtonian limit — why?), show that

$$\partial_\rho\partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2}\Box h - \frac{CT_{00}}{2}$$

Be careful with signs and up-down indices.

2d.) Taking the static Newtonian limit of the (2b) final equation, show that

$$\nabla^2 h_{00} + \frac{1-\beta}{2}\nabla^2 h = \frac{C}{2}T_{00}$$

where ∇^2 is the usual Laplacian operator. Explain why this implies $\beta = 1$ and $C = -16\pi G$:

$$\Box h_{\mu\nu} - (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}) = -16\pi G T_{\mu\nu}$$

Compare this with section (9.1) in the notes and comment.

2e.) Given that the Ricci tensor $R_{\mu\nu}$ and $g_{\mu\nu}R^{\rho}{}_{\rho}$ are the only second rank tensors that are linear in the second derivatives of the metric tensor $g_{\mu\nu}$ when the curvature is weak, explain why the general field equations must take the form

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = -8\pi G T_{\mu\nu}$$

where $R \equiv R^\rho_\rho$. Notice: not a Bianchi identity in sight. If Einstein could only have seen this in 1912.

Since $R_{\mu\nu}$ and $g_{\mu\nu}R^{\rho}{}_{\rho}$ are the only possible choices, only one linear combination reduces to our weak field equation, and the combination is indeed the correct field equation! We've derived the full field equations from energy conservation and Newtonian theory alone.

3a.) *Coordinate sinuosities, the speed of gravitational radiation, and the harmonic gauge.* Recall the linear fully covariant curvature tensor:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right).$$

For a plane wave of the form $h_{\mu\nu} = A_{\mu\nu}\exp(ik_\rho x^\rho)$ travelling in vacuum, show that

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(-k_\kappa k_\mu h_{\lambda\nu} + k_\kappa k_\lambda h_{\mu\nu} + k_\mu k_\nu h_{\lambda\kappa} - k_\nu k_\lambda h_{\mu\kappa}\right)$$

and that the linear vacuum field equation is

$$k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$ and $k^2 = k^\rho k_\rho$. We do *not* yet assume that $k^2 = 0$, but shall try to deduce this.

3b.) Show that if $k^2 \neq 0$ then $R_{\lambda\mu\nu\kappa} = 0$. Yikes! No curvature. *A mere coordinate sinuosity propagating at the speed of thought.*

3c.) Finally, show that if we consider only disturbances propagating at the speed of light, then we must have $k^\rho \bar{h}_{\rho\sigma} = 0$. In other words, the harmonic gauge condition *must* be satisfied. You want gravitational radiation to travel at the speed of light and to actually produce curvature? No choice: use a harmonic gauge.

4.) *Radiation from a parabolic fly by.* The Peters—Mathews formula for the time-averaged gravitational wave luminosity of a binary system in an elliptical orbit (with semi-major axis a, masses m_1 and m_2, $M \equiv m_1 + m_2$, eccentricity ϵ) is given by (c is now back in the equation):

$$\langle L_{GW}\rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^5}\left[\frac{1 + (73/24)\epsilon^2 + (37/96)\epsilon^4}{(1-\epsilon^2)^{7/2}}\right]$$

It's derivation is outlined in the notes (§9.6), or you may take it on perfect good faith from your humble instructor, however startling it may seem. Using this result, show that the total gravitational wave energy emitted by a single parabolic encounter between two bodies is

$$E_{GW} = \frac{85\pi\sqrt{2}}{24}\frac{G^{7/2}M^{1/2}m_1^2 m_2^2}{c^5 b^{7/2}}$$

where b is radius of closest approach. Recall that for a parabolic orbit, the radius r and aximuth ϕ are related by $r(1+\cos\phi) = L$, where $L = a(1-\epsilon^2)$ is the "semi-latus rectum," a constant. A parabola corresponds to the $\epsilon \to 1$ limit, with $a(1-\epsilon^2) = L$ finite. You may find the material in §6.8.1 useful.

Problem Set 5. **General Relativity, HT22**

Working with the FRW metric

RECALL:

Dynamical evolution equation:

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2E \text{ (Energy Form)} = -\frac{c^2}{a^2} \text{ (Curvature Form)} = -kc^2 \text{ (FRW Form)}$$

FRW metric, R dimensions of length, $k = 0, \pm 1$:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1 - kr^2} + R^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Curvature form, with $R_0 = 1$ and R dimensionless, a^2 positive or negative:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1 - r^2/a^2} + R^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

1a.) *A big bang, but in empty space you say. Really?* Show that the dynamical field equation for the scale factor $R(t)$ for an *empty space* $\rho = 0$ leads to an FRW metric of the form

$$-d\tau^2 = -dt^2 + \frac{t^2 dr^2}{1 + r^2} + r^2 t^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Use $c = 1$ for this problem!

1b.) Wait...Surely empty space must be Minkowski spacetime. Though this metric does not look static, there *must* be a coordinate transformation that turns this metric into a static Minkowski form. In other words, we ought to be able to find two functions, s and T,

$$s = s(r,t), \quad T = T(r,t) \qquad \text{or equivalently} \qquad r = r(s,T), \quad t = t(s,T)$$

that transform the metric of part (1a) into an old friend:

$$-d\tau^2 = -dT^2 + ds^2 + s^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

By inspection, we must have

$$s(r,t) = rt.$$

Why "by inspection?" Explain convincingly why it is as simple as this, in just one to two sentences.

1c.) Using $s = rt$, and by then demanding that the coefficient of dT^2 be -1 after the coordinate change, show that $T = \sqrt{s^2 + t^2}$ (up to an additive function of s which you may safely discard), and thereby derive the second coordinate transformation:

$$T = t\sqrt{1 + r^2}.$$

Give the explicit functional forms for $r(s,T)$ and $t(s,T)$.

1d.) Complete the full coordinate transformation for $d\tau^2$ and verify in detail that the Minkowski metric emerges. You may find it to your advantage to express $\partial t/\partial s$ and $\partial r/\partial s$ in terms of r and t, and $\partial r/\partial T$ in terms of $\partial t/\partial T$, before you begin. This is a valuable lesson: it is easy to be fooled by coordinates.

2.) *A radiation/matter universe.* Repent now, or face a calculation for an Eternal, Infinite Universe of Fire and Brimstone! Well...radiation and matter, actually. Much the same. Anyway, it's too late to repent, the calculation begins. Solve the dynamical cosmological equation (Energy form) for $R(t)$ for the case of an arbitrary mixture of radiation and non-relativistic matter in a spatially flat universe ($E = 0$). Assume a current energy density of $\rho_{\gamma 0}c^2$, and a matter density ρ_{m0}. In terms of the "inferno ratio" $I = \rho_{\gamma 0}/\rho_{m0}$, you should find

$$(R+I)^{3/2} - 3I(R+I)^{1/2} + 2I^{3/2} = \frac{3\Omega_{m0}^{1/2} H_0 t}{2}$$

(Note: This cubic equation is simple enough that the analytic solution is useful. Here it is [no need to prove]:

$$R = 4I\cos^2\left[\frac{1}{3}\cos^{-1} Q\right] - I, \quad Q = \frac{3H_0 t\Omega_{m0}^{1/2}}{4I^{3/2}} - 1$$

This holds as long as $-1 \le Q < 1$. When $Q \ge 1$, replace cos and $\cos^{-1}$ with cosh and $\cosh^{-1}$.)

3.) *A bullet in an E-dS universe.* Shoot a bullet into an Einstein-de Sitter universe at start of time. Nothing is actually pushing or pulling the bullet, but each comoving observer will see the bullet fly by at a different velocity as it passes. The question is, how far does the bullet get? More precisely, what is the largest comoving coordinate distance r the bullet attains if it starts at $r = 0$, $R = 0$? The metric is standard E-dS:

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2 dr^2 + R^2 r^2 d\Omega^2$$

$R(t)$ is the usual scale factor. We will use $d\varpi = R dr$ for the proper physical distance. Other standard notation and results for reference: t_0 is the current age of the universe, $R = (t/t_0)^{2/3}$ for E-dS, $H_0 \equiv \dot{R}_0$.

3a.) The quantity $d\varpi/dt$ measures the bullet's velocity relative to expanding, comoving observers who are all moving away. Show that if the bullet has a measured velocity V_1 at some instant when it passes one such observer, then when the bullet overtakes another observer, a tiny distance $d\varpi$ farther away, the velocity V_2 this observer measures is

$$V_2 = V_1 - \frac{\dot{R}\, d\varpi}{R}\left(1 - \frac{V_1^2}{c^2}\right)$$

to first order in $d\varpi$. (You will need the special relativity velocity addition formula and Hubble's law. Full special relativity works locally because $d\varpi$ is a tiny distance, and in this tiny *comoving* frame special relativity holds. The relativity bit only matters when V_1 is

comparable to c, but we allow for that!) From this equation, show that the rate at which the measured $V = d\varpi/dt$ is changing with cosmic time is given by the differential equation

$$\frac{\dot{V}}{V(1-V^2/c^2)} = -\frac{\dot{R}}{R}$$

where $\dot{V} = (V_2 - V_1)/dt = dV/dt$. Solve this equation and show that with $V = V_0$ at $t = t_0$, the solution is

$$\frac{V}{\sqrt{1-V^2/c^2}} = \frac{U_0}{R}$$

where U_0 is the spatial component of the bullet 4-velocity corresponding to V_0 at time t_0. (N.B.: In this problem, subscript 0 will always denote "current time," not the 4-vector time-like component.)

3b.) The result of (3a.) shows that the product $\mathcal{P}R$ is constant, where $\mathcal{P}$ is the spatial component of the bullet 4-momentum. Show that, in this form, this is equivalent to an adiabatic expansion, either of photons (extreme relativistic particles), or classical particles (classical nonrelativistic gas). [Cosmic adiabatic expansion for photons correponds to the temperature T obeying $TR \sim \textit{constant}$, while for a classical gas, adiabatic behaviour is $T\rho^{-2/3} \sim \textit{constant}$, where ρ is the mass (or in this case number) density.] In other words, a gas of bullets would "cool" like an ordinary gas!

3c.) Solve the equation $d\varpi/dt = V(R)$ for the comoving coordinate r in an E-dS universe to obtain for our problem:

$$r(R) = \frac{c}{H_0}\int_0^R \frac{dx}{[x + c^2x^3/U_0^2]^{1/2}}$$

and show therefore that as $R \to \infty$, the comoving coordinate $r \to r_{max}$, where

$$r_{max} = \frac{3.708\sqrt{U_0 c}}{H_0}$$

The numerical factor is

$$3.708 = \int_0^\infty \frac{dy}{(y + y^3)^{1/2}}$$

Even after an infinite amount of time, and even though this universe is decelerating, a fired bullet only reaches a finite value of comoving coordinate r for any finite U_0. But the bullet can reach arbitrarily large r, if V_0 approaches the speed of light.

4.) *Schwarzschild and FRW geometries.* How long does it take a classical matter dominated closed universe to collapse, starting at its maximum extent? Express your answer two ways: in terms of the current value of the density ρ_0 and Ω_{m0}, and then in terms of the density at maximum extent ρ_m. Now, suppose we take all the mass in a small sphere of radius r_0 with density ρ_m (the sphere is small so that we don't have to worry about non-Euclidian curvature: the mass is just $4\pi r_0^3\rho_m/3$), and turn the matter into a Schwarzschild black hole. Calculate the *proper time* for a test particle to fall into the hole from radial coordinate r_0 in a Schwarzschild geometry. You should find exactly the same answer for the universe as a whole. (Sections 6.5 and 10.5 in the notes will be useful.) Can you account for this amazing agreement in a simple way?

5a.) *There and back again: a photon's tale.* For a closed, matter-dominated universe with current mass density ρ_0, show that

$$H_0^2(\Omega_{M0} - 1) = c^2/a^2$$

where

$$\Omega_{M0} = \frac{8\pi G\rho_0}{3H_0^2}$$

5b.) Consider the path of a photon (null geodesic) through this universe. With η defined in §10.5 in the notes:

$$R = \frac{1 - \cos\eta}{2(1 - \Omega_{M0}^{-1})}$$

show that

$$\eta = \sin^{-1}(r/a)$$

where r follows the proper coordinate of the photon. In other words, r goes from zero to a and back again to zero (and R goes from zero to a maximum), as η advances by π. How many times could a photon travel around such a universe?