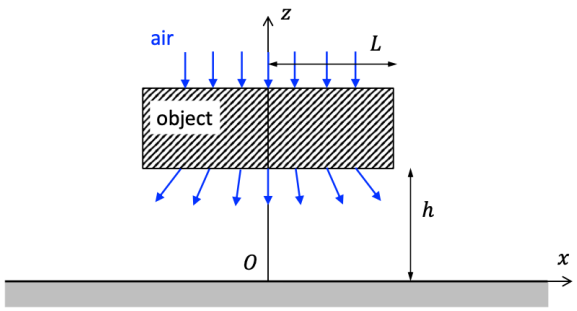
Problem set 1
Kinematics of fluids

Notations: (x, y, z), (r, θ, z) and (r, θ, φ) are the cartesian, cylindrical and spherical coordinates, respectively. We denote the unit vectors by the same letters but with a hat (e.g., $\hat{\mathbf{x}}$ is the unit vector in the x–direction in cartesian coordinates). The location of a point is $\mathbf{r}$ and the velocity is $\mathbf{v}$.

Problem 1: Eulerian and Lagrangian descriptions



A cylinder–shaped object (e.g., hockey puck) is producing an air cushion by sucking air from above and ejecting it from below. The radius of the object is L and its lower surface is at a height h above the ground. We use cylindrical coordinates and consider the flow in the region $z \leq h$ and $r \leq L$.

a) Air is ejected with the steady velocity $\mathbf{v}(r, h) = v_r(r)\hat{\mathbf{r}} - v_0\hat{\mathbf{z}}$, and we look for a velocity field in the form $\mathbf{v}(r, z) = v_r(r)\hat{\mathbf{r}} + v_z(r, z)\hat{\mathbf{z}}$. The rigid boundary at $z = 0$ requires $v_z = 0$ there.

   i) Assuming the air to be incompressible, write the mass conservation equation for a cylinder of radius r below the object, and show that $v_r(r) = rv_0/(2h)$.

   ii) Write the local mass conservation equation and show that $v_z(r, z) = -zv_0/h$.

   iii) Calculate the velocity potential ϕ.

b) Calculate the equation of the streamlines and that of the trajectories. Check that the streamlines and equipotentials are perpendicular to each other.

c) Consider a fluid element which at $t = 0$ is at $\mathbf{r}_0 = (r_0, \theta_0, z_0)$. Calculate the position of this fluid element as a function of time and check that this is consistent with the equation of the trajectories obtained in question (b).

d) Calculate the Lagrangian velocity $\mathbf{V}(t, \mathbf{r}_0)$ of the fluid element. Verify that:

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_{\mathbf{r}_0} = \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}.$$

Problem 2: Streamlines and trajectories

Consider the unsteady flow:

$$v_x = v_0, \quad v_y = kt, \quad v_z = 0,$$

where v_0 and k are positive constants.

a) Using the stream function, show that the streamlines are straight lines, and sketch them at two different times.

b) Show that any fluid particle follows a parabolic path as time proceeds.

Problem 3: Acceleration in a rotating frame

We consider a frame which rotates with angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$ with respect to a fixed inertial frame. Both frames at centered at O. We note with subscripts 'fix' and 'rot' quantities in the fixed and rotating frames, respectively. It can be shown that, for any vector $\mathbf{A}$,

$$\left(\frac{\mathrm{D}\mathbf{A}}{\mathrm{D}t}\right)_{\text{fix}} = \left(\frac{\mathrm{D}\mathbf{A}}{\mathrm{D}t}\right)_{\text{rot}} + \mathbf{\Omega}\times\mathbf{A}. \tag{1}$$

(A way to prove this relation is to note that a position vector $\mathbf{r}$ which is at rest in the rotating frame is seen in the fixed frame to precess about the origin with angular velocity $\mathbf{\Omega}$, so that $(\mathrm{D}\mathbf{r}/\mathrm{D}t)_{\text{fix}} = \mathbf{\Omega}\times\mathbf{r}$.)

a) Use equation (1) to show that the Lagrangian acceleration of a fluid element in the fixed frame is related to that in the rotating frame through the following relation:

$$\left(\frac{\mathrm{D}\mathbf{v}_{\text{fix}}}{\mathrm{D}t}\right)_{\text{fix}} = \left(\frac{\mathrm{D}\mathbf{v}_{\text{rot}}}{\mathrm{D}t}\right)_{\text{rot}} + 2\mathbf{\Omega}\times\mathbf{v}_{\text{rot}} + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{r}\right).$$

The second and third terms on the right hand side represent the opposite of the Coriolis and centrifugal forces per unit mass, respectively.

b) Write the centrifugal and Coriolis forces for a fluid element moving along a circle with constant angular velocity.

c) Sketch the effect of the Coriolis force on fluid elements moving in the atmosphere from the equator to a pole. Check that it is consistent with the following statement: *Cyclones need the Coriolis force in order to circulate. For this reason, hurricanes almost never occur in equatorial regions, and never cross the Equator itself* (from National Geographic).

Problem 4: Deformation of a fluid element

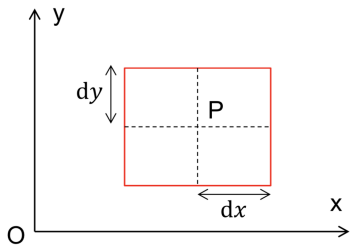
We consider the following two dimensional flow:

$$v_x = \alpha x + 2\beta y, \quad v_y = -\alpha y,$$

where α and β are two constants.

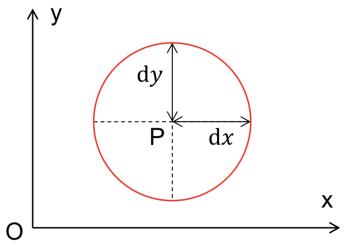
a) Decompose the deformation tensor $D_{ij} = \partial v_i / \partial x_j$ into an expansion, a deformation without change of volume, and a rigid–body rotation.

b) Case $\beta = 0$: this corresponds to a *pure strain* flow.



Consider a small surface element centered at a point P. Using $\mathrm{d}v_i = D_{ij}\mathrm{d}x_j$, indicate how the points along the contour delimiting the surface element move with respect to P and represent the surface element after a time $\mathrm{d}t$. Verify that its surface area is constant to first order in $\mathrm{d}t$.

c) Case $\alpha = 0$: this corresponds to a *simple shear* flow.



Indicate how the points along the circle move with respect to P and show that their motion is the superposition of a linear deformation and a rotation. Represent the deformation of a rectangle with sides parallel to the axes and again show that it is the result of a linear deformation and a rotation.

Problem 5: Vorticity

Calculate the vorticity for the following flows. In each case, draw the streamlines and consider the motion of two perpendicular line elements (or any surface element) to interpret the results qualitatively.

a) Rigid–body rotation: $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$, where $\boldsymbol{\Omega} = \Omega \hat{\mathbf{z}}$, with Ω being a constant,

b) Shear flow: $\mathbf{v} = ky\hat{\mathbf{x}}$, where k is a constant,

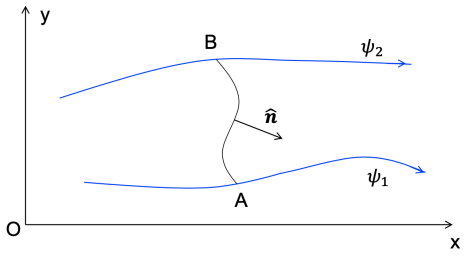
c) Line vortex: $\mathbf{v} = (k/r)\hat{\boldsymbol{\theta}}$, where k is a constant.

Please turn over...

Problem 6: Stream function and flow rate

The *flow rate*, also called *volume flux*, is the volume of fluid passing through a given surface per unit time.

a) We consider a 2D incompressible flow.



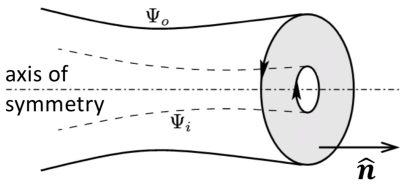
The blue lines are streamlines along which the stream function has constant values ψ_1 and ψ_2. We note s the curvilinear coordinate along the curve AB and $\hat{\mathbf{n}}$ the unit vector locally perpendicular to the curve.

The flow rate through the surface delimited by the curve AB and which has a length of unity in the direction perpendicular to the plane is: $Q = \int_A^B \mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}s$. Show that $Q = \psi_2 - \psi_1$. Hence, discuss how the velocity of the flow changes when the distance between the streamlines varies.

b) We now consider a 3D flow which is incompressible and axisymmetric. In cylindrical coordinates (with z being the axis of symmetry), we define the *Stokes stream function* $\psi(r, z, t)$ such that

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad \text{and} \quad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}.$$

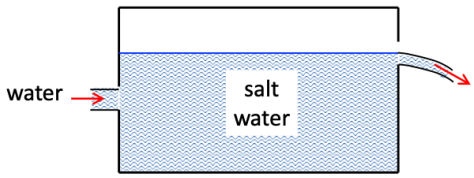
Check that (i) the incompressibility condition is satisfied, (ii) ψ is constant on streamlines and (iii) $\mathbf{v} = \boldsymbol{\nabla} \times (\psi \hat{\boldsymbol{\theta}}/r)$.



(iv) Show that the flow rate between the two streamtubes represented on the figure is $Q = 2\pi(\psi_o - \psi_i)$, where ψ_o and ψ_i are the stream functions on the surfaces of the outer and inner tubes, respectively.

Problem 7: Mass conservation

We consider a tank filled with salt water which has an initial density $\rho_i = 1.025\rho_{\mathrm{H_2O}}$.

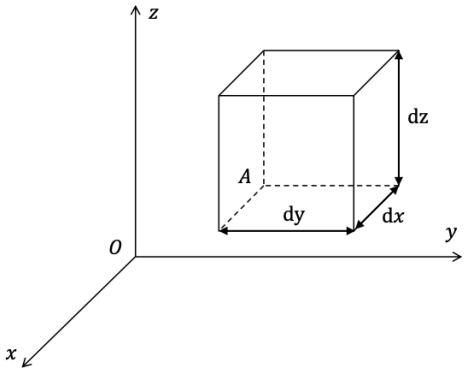


At $t = 0$, fresh water is pumped through an entrance with a volumetric flux Q, and salt water leaves the tank as represented on the figure.

Calculate the time it takes for the density of the fluid inside the tank to decrease by 1% (assume that mixing of salt water and fresh water throughout the tank happens instantaneously).

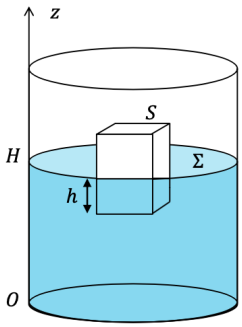
Problem 8: Hydrostatic pressure

a) Pressure force: We consider the small fluid element located at $A(x, y, z)$, as represented on the figure:



Show that the net pressure force per unit volume exerted on the fluid element is $-\boldsymbol{\nabla} P$, where $P(x, y, z)$ is the pressure in the fluid. Calculate the pressure when the fluid is at equilibrium and suject only to gravity. Assume that the surface of the fluid is at $z = H$ and in contact with the atmosphere.

b) Archimedes' principle:



An ice–cube is floating in a glass of section Σ containing a height H of a fluid. Show that the weight of the cube is equal to the weight of the volume of fluid displaced (Archimedes' principle). Calculate the change in the height of fluid when the ice melts in the following cases: (i) the fluid is fresh water, (ii) the fluid is salt water.

Caroline Terquem

Problem set 2
Dynamics of fluids

Notations: (x, y, z), (r, θ, z) and (r, θ, φ) are the cartesian, cylindrical and spherical coordinates, respectively. We denote the unit vectors by the same letters but with a hat (e.g., $\hat{\mathbf{x}}$ is the unit vector in the x–direction in cartesian coordinates). The location of a point is $\mathbf{r}$ and the velocity is $\mathbf{v}$.

Problem 1: Poiseuille flow in a cylindrical tube

A Poiseuille flow is a flow induced by a pressure gradient. Here, we consider an incompressible and Newtonian flow resulting from a pressure drop Δp in a horizontal cylindrical tube of length L and radius $R \ll L$. We use cylindrical coordinates with the z axis coinciding with the axis of the cylinder and assume that $\mathbf{v} = v_z(r)\hat{\mathbf{z}}$.

a) Write Navier–Stokes equation along the r, θ and z directions. Assuming $\partial p/\partial z$ to be a constant, show that, in steady state: $v_z(r) = \Delta p \left(R^2 - r^2\right) / (4\eta L)$.

b) Show that the flow rate Q (volume of fluid passing through a cross–section per unit time) satisfies $\Delta p = \mathcal{R} Q$ with $\mathcal{R} \propto R^{-4}$ being the 'resistance' of the fluid. This is Poiseuille's law.
Application: The flow rate of blood in a coronary artery has been reduced to half its normal value by plaque deposits. By what factor has the radius of the artery been reduced? By how much should the pressure difference increase to restore the blood flow in spite of this buildup?

c) Calculate the viscous friction force exerted by the fluid on the wall of the tube. Discuss how this force produces the profile of velocity $v_z(r)$. Calculate the viscous dissipation rate as a function of $v_{\max} = v_z(0)$. What is the effect of energy dissipation on the pressure drop along the pipe if there is no pump to maintain it?

Problem 2: Couette flow between rotating cylinders

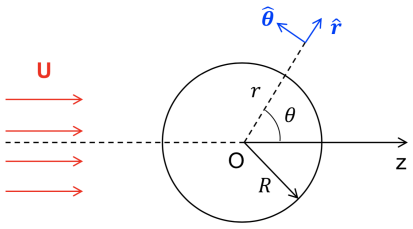
A Couette flow is a flow induced between two surfaces by the motion of one of the surfaces relative to the other. Here, we consider an incompressible and Newtonian fluid located between two coaxial cylinders of radii R_1 and $R_2 > R_1$ rotating about their axis z with respective angular velocities Ω_1 and Ω_2. No external pressure gradient is applied. At low velocities, we observe that the flow velocity and pressure are independent of θ and z.

a) Neglecting gravity, use mass conservation and Navier–Stokes equations to calculate the flow velocity. *Hint:* Solve for $\Omega = v_\theta/r$, and show that $\Omega = C_1/r^2 + C_2$, where C_1 and C_2 are constants. Discuss the different limiting cases: (i) $\Omega_1 = \Omega_2$, (ii) $\Omega_2 = 0$ and $R_2 \to \infty$, (iii) R_1 and R_2 much larger than $R_2 - R_1$ (show that we recover a simple shear flow in that case).

b) Calculate the viscous torque on the cylinders. *Application:* This apparatus is used to measure viscosities. The torque on the inner cylinder (initially at rest) indicated by a viscometer is 0.01 N m. If the diameter of the inner cylinder is 100 mm, the annular gap is 1 mm, the length is 120 mm, and the outer cylinder rotates at 40 rpm, determine the viscosity of the fluid contained in the viscometer.

Problem 3: Motion of a sphere in a very viscous fluid: Stokes law

We consider a very low Reynolds number incompressible fluid flowing past a sphere of radius R.



Far from the sphere, the fluid has a uniform velocity $\mathbf{U}$. The flow is therefore axisymmetric: $v_\varphi = 0$, $v_r(r,\theta)$ and $v_\theta(r,\theta)$. We assume that the flow is quasi–stationary (changes in the velocity occur on timescales much longer than the diffusion timescale).

a) We introduce a Stokes stream function ψ such that:

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r}.$$

Check that this satisfies the incompressibility condition. Calculate $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$ in terms of $\mathcal{O}_1(\psi)$ and $\mathcal{O}_2(\psi)$, where we define the operators:

$$\mathcal{O}_1 = \frac{\partial^2}{\partial r^2} \text{ and } \mathcal{O}_2 = \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

b) Write Navier–Stokes equation assuming gravity to be negligible, and show that it can be put under the form $\boldsymbol{\nabla} p = -\eta \boldsymbol{\nabla}\times\boldsymbol{\omega}$. By taking the curl of this equation, show that ψ satisfies:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0.$$

Hint: Express $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\omega})$ in terms of $\mathcal{O}_1(r\omega)$ and $\mathcal{O}_2(\sin\theta\,\omega)$.

c) Write the boundary conditions for ψ. We look for a solution of the form $\psi = f(r)\sin^2\theta$ with $f(r) = r^n$. Show that n can take the values $-1, 1, 2$ and 4, and calculate ψ and the velocity.

d) Calculate the pressure and show that the total stress at the surface of the sphere is $(3\eta U/(2R))\hat{\mathbf{z}} - p_\infty \hat{\mathbf{r}}$, where p_∞ is the pressure at infinity. Hence show that the force exerted by the fluid on the sphere is $6\pi\eta U R\hat{\mathbf{z}}$, which is Stokes law.

Problem 4: Coriolis force and vorticity

We have shown in Problem Set 1 that the Lagrangian acceleration of a fluid element in a fixed frame is related to that in a frame rotating with angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ through the following relation:

$$\left(\frac{\mathrm{D}\mathbf{v}_{\mathrm{fix}}}{\mathrm{D}t}\right)_{\mathrm{fix}} = \left(\frac{\mathrm{D}\mathbf{v}_{\mathrm{rot}}}{\mathrm{D}t}\right)_{\mathrm{rot}} + 2\boldsymbol{\Omega}\times\mathbf{v}_{\mathrm{rot}} + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right),$$

and also $\mathbf{v}_{\mathrm{fix}} = \mathbf{v}_{\mathrm{rot}} + \boldsymbol{\Omega}\times\mathbf{r}$.

a) Write Navier–Stokes equation in the rotating frame for an incompressible fluid. Show that the centrifugal acceleration $\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right) = -\boldsymbol{\nabla}\left(\boldsymbol{\Omega}\times\mathbf{r}\right)^2/2$, so that the centrifugal force can be subsumed into the gravitational field.

b) Assuming the mass density ρ to be constant and the flow to be inviscid, calculate the rate of change of the circulation of the flow velocity along a closed contour and show that the Coriolis force is a source of vorticity.

c) We define the *Rossby number Ro* as the ratio of the inertial force to the Coriolis force. Give an expression for *Ro* as a function of the flow characteristic velocity U and lengthscale L. For cyclones, $U \sim 10$ m s^{-1} and $L \sim 10^3$ km. For tornadoes, $U \sim 100$ m s^{-1} and $L \sim 1$ km. Is the Coriolis force important for these flows? In the movie *Escape Plan* (2013), Sylvester Stallone tries to determine the location of the prison ship within which he is trapped by watching the path of the water in a toilet as it slides into the drain. He mentions to his fellow inmates that "water goes counterclockwise above the Equator". Is this possible?

Problem 5: Rankine vortex

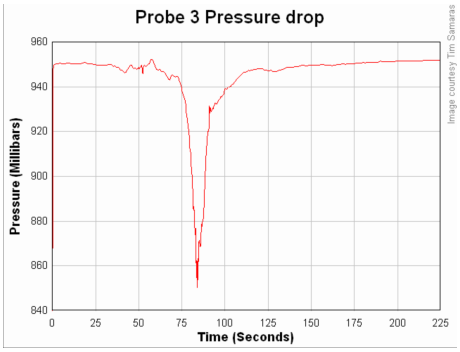
Real vortices typically have a small vortex 'core' which contains the vorticity while outside the core the flow is irrotational. This is modelled by a *Rankine vortex* which has the velocity $\mathbf{v} = v_\theta \hat{\boldsymbol{\theta}}$ in polar coordinates with:

$$v_\theta = \Omega r \quad \text{for} \quad r < R, \quad \text{and} \quad v_\theta = \frac{\Omega R^2}{r} \quad \text{for} \quad r > R,$$

where Ω and R are two constants. This is a solid–body rotation flow for $r < R$ and a line vortex flow for $r > R$. (The flow extends in the z–direction but there is no velocity along the z–axis).

a) Draw the velocity and the vorticity as a function of r.

b) *Application to tornadoes:* Neglecting gravitational effects and the viscosity of the fluid, and assuming that ρ is constant, calculate the pressure as a function of r (i) in the irrotational region (*hint*: use Bernoulli's theorem), (ii) in the solid–body rotation region.



Give an expression for $p(0) - p(\infty)$. Compare the curve $p(r)$ obtained above with the graph on this figure which shows a pressure tracing documenting the 100 mb (hPa) deficit associated with a violent tornado passing directly over an in–situ probe near Manchester, South Dakota, USA, on 24 June 2003.

c) *Application to whirlpools:* A 2 m diameter vortex appeared in Lake Texoma in Oklahoma as the Denison Dam was opened to drain the overflowing lake (2015).



We model this as a Rankine vortex. Taking gravity into account, calculate the height of the free surface $h(r)$ in the two regions of the Rankine vortex and give an expression for $h(\infty) - h(0)$. (*Hint*: use Bernoulli's theorem for $r > R$ and the continuity of the stress at the surface).

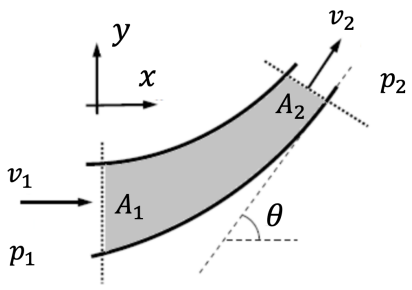
Problem set 3

Dynamics of fluids – Potential flows

Notations: (x, y, z), (r, θ, z) and (r, θ, φ) are the cartesian, cylindrical and spherical coordinates, respectively. We denote the unit vectors by the same letters but with a hat (e.g., $\hat{\mathbf{x}}$ is the unit vector in the x–direction in cartesian coordinates). The location of a point is $\mathbf{r}$ and the velocity is $\mathbf{v}$.

A – Dynamics of fluids

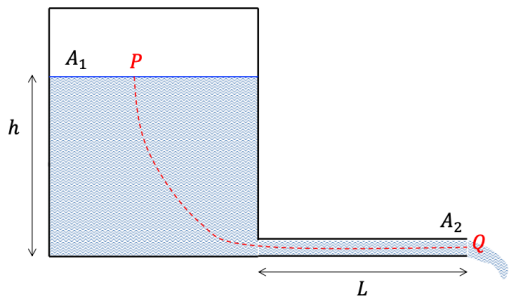
Problem 1: Force on a bent garden hose



a) Consider a pipe bend as represented on the figure. By writing the conservation of momentum for the water inside it, calculate the force exerted on the pipe by the water that flows through it (neglect gravity). Express the result in terms of v_1, p_1, A_1, A_2 and θ.

b) How is the force when the pipe is straight ($\theta = 0$)? When water rushes through, an unrestrained garden hose wriggles uncontrollably like a mad snake. This is called the *firehose* instability. Explain why this happens using the above result.

Problem 2: Drainage of a large reservoir of water



Consider a large reservoir with surface area A_1 connected to a pipe with cross section area $A_2 \ll A_1$. The initial height of water in the reservoir is h, and the length L of the pipe is at least as large as h. At $t = 0$, the pipe is opened and water starts flowing out of the reservoir.

We want to calculate the velocity $v_2(t)$ of the flow at it leaves the pipe as a function of time, using Bernoulli's theorem for an irrotational flow which states that:

$$H \equiv \rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho v^2 + p + \rho g z$$

is constant throughout the whole fluid at any given time. Here ρ is the mass density, p is the pressure, ϕ is the velocity potential, g is the acceleration of gravity and z is the vertical coordinate.

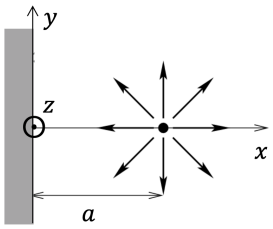
a) Justify that the flow is irrotational.

b) Consider a streamline that connects a point P at the surface of the reservoir to a point Q at the end of the pipe (red dotted line on the figure). By writing that H at P is equal to H at Q, show that $v_2(t) = \sqrt{2gh}\tanh(t/\tau)$, where τ is a constant to be determined. *Hint:* Ignore the (small) horizontal component of the velocity of the water inside the tank, and assume that the change in h can be neglected over the time interval considered.

c) Compare the value of v_2 when $t \gg \tau$ with the value that would have been obtained by using Bernoulli's theorem for a steady flow. Give the condition that should be satisfied by h, L and A_2/A_1 for the variations of h to be negligible over the time–interval during which the steady state is reached.

B – Potential flows: Method of images and conformal mapping

Problem 3: Force from a line source or vortex on a wall



Consider an infinite line parallel to the z-axis at $y = 0$ and $x = a$. An ideal flow is emitted radially from this (fixed) line with a strength m. The flow is potential and invariant along z. An infinite solid wall is located in the plane $x = 0$.

a) Show that the flow remains the same in the half–space $x > 0$ if we replace this wall by a second parallel line source at $(-a, 0, 0)$ (this is the method of images).

b) Show that the corresponding complex potential is $(m/(2\pi)) \ln\left(c^2 - a^2\right)$, where $c = x + \mathrm{i}y$.

c) Compute the flow velocity on the plane surface $x = 0$ and show that the deviation of the pressure from its value far away from the source (neglecting gravity) is:

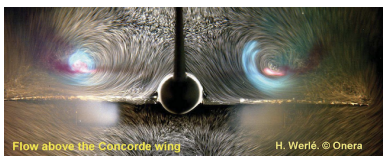
$$\delta p(0, y) = -\frac{\rho}{2}\left(\frac{m}{\pi a}\right)^2 \frac{y^2/a^2}{\left(1 + y^2/a^2\right)^2}.$$

d) Calculate the resulting force on the plane per unit length along z, using:

$$\int_{u_1}^{u_2} \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u = \left[\frac{1}{2}\left(\arctan(u) - \frac{u}{1+u^2}\right)\right]_{u_1}^{u_2}$$

In what direction is the force oriented?

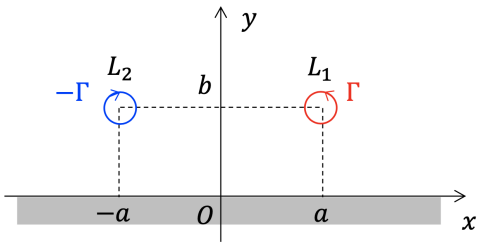
e) Would the direction of the force be changed if the line source were replaced by a sink or a vortex?



Some airplanes have sharp edge wings designed to generate vortices above them at high angles of attack. Comment on the advantages.

Please turn over...

Problem 4: Line vortices in motion



We consider the two line vortices L_1 and L_2 of strength Γ and $-\Gamma$ represented on the figure at time $t = 0$. They generate a flow in an inviscid and incompressible fluid in the region $y \geq 0$ bounded by a rigid boundary at $y = 0$.

a) Show that the flow remains the same in the half–space $y \geq 0$ if we replace the wall by two other line vortices in the half–space $y < 0$ (method of images).

b) Due to Kelvin's theorem, each line vortex moves at the local flow velocity due to everything other than itself. Is the symmetry of the problem preserved as the line vortices move with respect to each other? Calculate the complex potential at the location $z_1 = a + \mathrm{i}b$ of the line vortex L_1 and show that:

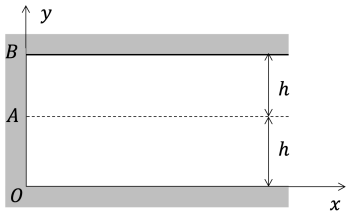
$$v_x(z_1) - \mathrm{i}v_y(z_1) = \frac{\mathrm{i}\Gamma}{2\pi}\left[\frac{1}{z_1 + z_1^*} + \frac{1}{z_1 - z_1^*} - \frac{1}{2z_1}\right].$$

c) Hence show that the path taken by the line vortex L_1 is $1/a^2 + 1/b^2 = \text{constant}$.

d) When an aircraft takes off, the two vortices that trail from its wing–tips are observed to move downwards under each other's influence and then to move further apart as they approach the ground. Why is this?
Note that this result could also be obtained by calculating the force exerted by a line vortex on another one: vortices which circulate in opposite directions repel, whereas they attract each other and eventually merge if they circulate in the same direction.

Problem 5: Conformal mapping: Vortex in a semi–infinite channel

Preliminary: Show that a two dimensional line vortex of strength Γ located at a point z_1 is transformed by a conformal mapping $Z = f(z)$ into a line vortex of the same strength at the point $Z_1 = f(z_1)$. *Hint:* Transform the complex potential into the Z–plane and expand for $Z \to Z_1$. Alternatively, show that the integral of ϕ along a contour encircling z_1 is unchanged when transformed into the Z–plane.



We consider a semi–infinite channel located between the horizontal planes $y = 0$ and $y = 2h$ in the $x > 0$ half–plane.

a) Show that the transformation $f(z) = \cosh \frac{\pi z}{2h}$, with $z = x + \mathrm{i}y$, maps this semi–infinite channel into the $Y > 0$ half–plane. Determine how the boundaries in the z–plane are transformed into the Z–plane. Show that f is not conformal at O and B and comment on the mapping of the local angles.

b) A (fixed) line vortex of strength Γ is located at $x = a$ and $y = h$ in the z–plane. Calculate the complex potential due to this vortex in the Z–plane. *Hint*: use the method of images. Transform back into the z–plane and show that:

$$v_x - \mathrm{i}v_y = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\Gamma}{2h} \frac{\sinh \frac{\pi z}{2h} \sinh \frac{\pi a}{2h}}{\cosh^2 \frac{\pi z}{2h} + \sinh^2 \frac{\pi a}{2h}}.$$

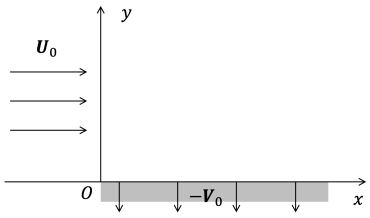
Check that the no–penetration condition is satisfied at the boundaries.

Problem set 4

Boundary layers – Thin films

A – Boundary layers

Problem 1: The boundary layer on a flat plate with suction



We consider a steady, two–dimensional, incompressible flow with high Reynolds number in the (x, y)–plane past a thin porous plate located at $y = 0$. Suction is applied to the plate, which is modelled by the boundary condition $v_y = -V_0$ at $y = 0$. We look for a solution $v_x(y)$ independent of x and which can be approximated by $v_x = U_0$ far enough from the plate.

a) Neglecting gravity, write the mass conservation and Navier–Stokes equations, and show that v_x satisfies: $\nu(\partial^2 v_x/\partial y^2) + V_0(\partial v_x/\partial y) = 0$. Solve this equation for the boundary conditions stated above.

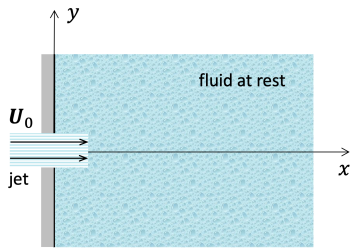
b) We define the thickness δ of the boundary layer as the distance from the plate where $v_x = 0.9U_0$. Show that $\delta \sim \nu/V_0$, which is independent of x. Write the different timescales associated with the advection and diffusion of vorticity, and discuss how they control the thickness of the boundary layer. Find the distance x_0 from the leading edge of the plate beyond which the boundary layer has the constant thickness calculated above. What is the advantage of a boundary layer with constant thickness?

c) Calculate the frictional force exerted on a plate which has a length L in the x–direction and compare with the result obtained when there is no suction (section 4.1.4 of the Lecture Notes).



d) In a test carried in the 1990s, a porous titanium glove was installed on the left wing of the F-16XL, which flew at speed of Mach 2 at an altitude of 15 km, where the viscosity of air is $\nu \simeq 7 \times 10^{-5}$ m^2 s^{-1}. For which values of V_0/U_0 would the boundary layer stay laminar? (NASA Photo EC96-43548-7)

Problem 2: Boundary layer: Two–dimensional laminar jet



A jet emerges with a velocity U_0 from a slit in a wall into a fluid which is at rest. The slit, and therefore the jet, is very thin along the y–direction and extended in the z–direction. The flow is two dimensional in the $x > 0$ half space, with $v_x(x, y)$ and $v_y(x, y)$, and is steady, laminar, incompressible and with high Reynolds number.

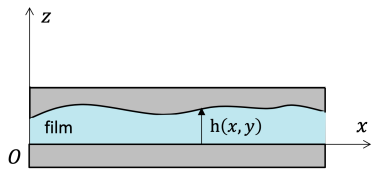
a) Describe how the fluid which is initially at rest is brought into motion by the jet and compare the situation with that of a uniform flow past a plate. Sketch the velocity profile along the x–direction. Show that the semi–thickness of the jet can be defined as $\delta(x) = \sqrt{\nu x / v_x(x, 0)}$ and justify that the boundary layer approximations apply across the jet.

b) By integrating the boundary layer (Prandtl's) equation across the jet, show that the momentum flux $M \equiv \int \rho v_x^2 \mathrm{d}y$ is constant (independent of x). Show that this implies $\delta(x) \simeq (2\rho/M)^{1/3}(\nu x)^{2/3}$ and $v_x(x, 0) \simeq (M/(2\rho))^{2/3}(\nu x)^{-1/3}$, and that the mass flux at the slit is very small.

B – Thin films

Thin films are amenable to similar approximations to boundary layers, although they are usually important at low Reynolds numbers.

Problem 3: Thin film approximation



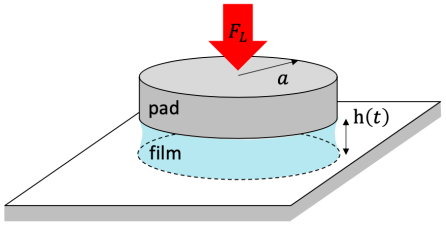
We consider a thin film of a viscous, incompressible and steady flow between two rigid boundaries at $z = 0$ and $z = h(x, y)$. We note L the horizontal scale over which variations of h and $\mathbf{v}$ occur and U a typical horizontal flow speed. We assume that $h \ll L$.

a) Show that $\nabla^2 \mathbf{v} \simeq \partial^2 \mathbf{v}/\partial z^2$ and that, when $Re \ll (L/h)^2$, the term $(\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v}$ can be neglected in Navier–Stokes equation. This is the *thin film approximation.*

b) Using Navier–Stokes equation, show that $\partial p/\partial z$ is negligible compared to the horizontal gradients. (This implies that $\partial^2 v_z/\partial z^2$ cannot be calculated within the thin film approximation.)

c) Show that the tangential stress is small compared with the normal stress (which includes pressure).

Problem 4: Thin film: Bearing pad and viscous adhesion

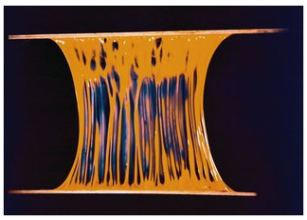


We consider a circular bearing pad which is isolated from a plane surface by a thin incompressible viscous film with dynamic viscosity η. A load force F_L is applied, which is supported only by the pressure of the moving fluid. It causes h to decrease *slowly*. Since $h \ll a$, the thin film approximations derived in Problem 3 apply.

a) Explain how slowly should h decrease for the quasi–steady state approximation to hold. Calculate $v_r(r, z)$ by solving Navier–Stokes equation in cylindrical coordinates. Then, by integrating the incompressibility equation over the thickness of the film, calculate the speed $\mathrm{d}h/\mathrm{d}t$ at which the pad moves. Integrate again to find p as a function of r and $\mathrm{d}h/\mathrm{d}t$. Finally, by writing the quasi–equilibrium condition for the pad, show that:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{2}{3\pi}\frac{h^3 F_L}{\eta a^4}.$$

b) Assuming a constant load, integrate this equation to find the relation between h and the time t for an initial thickness h_0. If the film is made of heavy oil with kinematic viscosity $\nu = 10^{-3}$ m^2 s^{-1} and mass density $\rho = 10^3$ kg m^{-3}, calculate $\mathrm{d}h/\mathrm{d}t$ initially and the time it takes for h to decrease by half of its initial value for $h_0 = 0.003$ cm, $a = 0.15$ m and a load of 10 kg. Discuss the respective roles of pressure and viscosity. Is the quasi–steady state approximation justified?

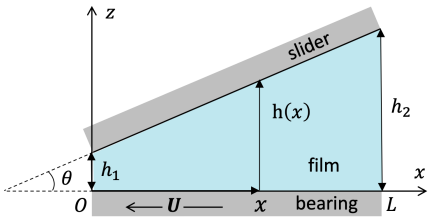


c) We now assume that, instead of being acted upon by F_L, the pad is pulled upwards by a constant force F. Calculate the time it takes for the pad to be pulled away. This is the basis for PSA (pressure-sensitive adhesives), like Scotch tape. (*Credit: Henkel*)

Please turn over...

Problem 5: Lubrication theory: Slider bearing

We have seen in Problem 3 that, in thin films, tangential stresses are very small compared to normal stresses. Therefore, a thin layer of fluid can support large normal load while offering little resistance to tangential motion. This is the basis for *lubrication*: sliding between two surfaces is made much easier if a thin film of fluid is inserted in between them. A common example is the sliding of a sheet of paper across a smooth table. In the lubrication theory, the space between the two surfaces is small so that the *thin film approximation is valid.*



Here we consider a *slider bearing*, in which the plane at $z = 0$ (the bearing) moves with constant velocity U past a stationary plane (the slider). The planes are almost parallel to each other: $h(x) \ll x$ and $\mathrm{d}h/\mathrm{d}x \simeq \theta$. The space between them is filled with a two dimensional, steady, viscous and incompressible fluid.

a) Calculate $v_x(x, z)$ by solving Navier–Stokes equation.

b) Mass conservation implies that the flow rate $Q = \int_0^{h(x)} v_x \mathrm{d}z$ is constant (independent of x). Write Q as a function of $\mathrm{d}p/\mathrm{d}x$. Using $\mathrm{d}p/\mathrm{d}x = \theta \mathrm{d}p/\mathrm{d}h$ and $p(h_1) = p_a$ (atmospheric pressure), integrate to get p as a function of $h(x)$, h_1, Q and other appropriate constants. Writing $p(h_2) = p_a$, calculate Q as a function of h_1, h_2 and U, and show that:

$$p(x) = p_a + \frac{6\eta U}{\theta} \frac{(h_2 - h)(h - h_1)}{h^2(h_1 + h_2)}.$$

c) Calculate the net force on the slider.

d) The process described above is what makes ice slippery: friction on ice produces a layer of meltwater, which has recently been shown to be up to 100 times more viscous than pristine water (see http://www.cnrs.fr/en/why-ice-so-slippery for a fascinating summary of ongoing research).



This process is also responsible for *hydroplaning*, which is called *aquaplaning* when the film is made of water.

We consider the case when a car drives over a film of engine oil that has leaked on the road, for which $h_2 \sim 2$ mm and $L \sim 20$ cm (tyre imprint). Given that $\eta \simeq 10^{-1}$ Pa s and $\nu \simeq 10^{-4}$ m^2 s^{-1} for engine oil at 20°, at what speed does a car with mass $M \simeq 10^3$ kg experience hydroplaning? (Use $h_2 \gg h_1$).

Third–Year Fluids **Hilary Term 2022**

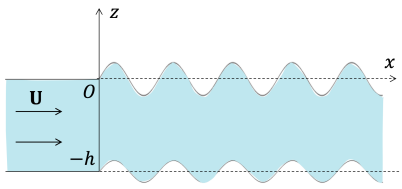
Caroline Terquem

Problem set 5

Waves, instabilities and turbulence

Problem 1: Steady flow over a wavy wall

Water flow over a wavy bottom is important in many civil engineering applications, as sediment transport in such a geometry affects coastal structures, harbors, rivers etc.



We consider a stream of water of depth h and velocity $U\hat{\mathbf{x}}$ in the $x < 0$ half–plane which flows over a corrugated bed $z = -h + \epsilon \sin(kx)$ located in the $x > 0$ half–plane, where $\epsilon \ll h$. This produces a perturbation of the free surface which is displaced from $z = 0$ to $z = \eta(x)$.

The density of the flow is constant equal to ρ_0, the pressure in the $x > 0$ half–plane is $p = p_0 + p_1$, where P_0 is the pressure of the undisturbed flow, and we note $\phi(x, z)$ the velocity potential of the perturbation.

a) Linearize Euler's equation and show that, neglecting surface tension, the continuity of the stress at the free surface yields:

$$U\frac{\partial \phi}{\partial x} + g\eta = 0 \ \text{ at } \ z = 0. \tag{1}$$

Show that the linearized kinematics boundary conditions are:

$$U\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\partial \phi}{\partial z} \ \text{ at } \ z = 0, \tag{2}$$

$$\frac{\partial \phi}{\partial z} = Uk\epsilon \cos(kx) \ \text{ at } \ z = -h. \tag{3}$$

b) We look for a solution to Laplace's equation of the form $\phi(x, z) = f(z)\cos(kx)$. Show that:

$$\eta(x) = \frac{\epsilon \sin(kx)}{\cosh(kh)\,(1 - c^2/U^2)}, \tag{4}$$

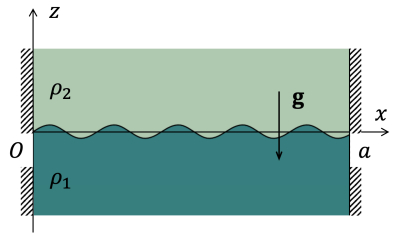
where:

$$c = \sqrt{\frac{g}{k}\tanh(kh)},$$

is the phase velocity of surface gravity waves with wavenumber k in water of depth h. Discuss the relative position of the crests and troughs of the surface and bottom. Is the analysis still valid when $U = c$?

Problem 2: Rayleigh–Taylor instability

It is an instability that occurs when a light fluid pushes on a heavy fluid. Examples include water above oil in the Earth gravity, or supernova explosions during which gas expands and is accelerated into a denser surrounding shell.



We consider a fluid of density ρ_2 above a fluid of density ρ_1 contained between two walls at $x = 0$ and $x = a$. Both fluids are inviscid and incompressible. At equilibrium, the interface between the two fluids is flat and in the $z = 0$ plane.

The surface tension at the interface is γ. We assume that the height of the vertical walls is large compared to a, so that capillary effects along these walls are negligible. As a result of a perturbation, the interface is moved from $z = 0$ to $z = \eta(x, t)$. We note $\phi_i(x, z, t)$ the velocity potential of the perturbation in fluid i.

a) Linearize Euler's equation and show that the continuity of the stress at the interface yields:

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t}(x, 0, t) + g\eta(x, t) \right) - \rho_2 \left(\frac{\partial \phi_2}{\partial t}(x, 0, t) + g\eta(x, t) \right) = \gamma \frac{\partial^2 \eta}{\partial x^2}(x, t). \quad (5)$$

Write the linearized kinematics boundary conditions at the interface and the boundary conditions at the walls at $x = 0$ and $x = a$.

b) We look for solutions to Laplace's equation of the form $\phi_i(x, z, t) = f_i(x) g_i(z) \mathrm{e}^{\mathrm{i}\omega t}$, where $i = 1, 2$. Calculate f_i and g_i assuming that the depth of fluids on both sides of the interface is very large compared to a. Show that the solutions are normal modes which frequencies ω satisfy:

$$(\rho_1 + \rho_2)\omega^2 = \frac{n\pi}{a} \left[(\rho_1 - \rho_2)g + \gamma \left(\frac{n\pi}{a} \right)^2 \right], \quad (6)$$

where n is an integer.

c) Discuss the stability of the system. A tube containing water is turned upside down. Calculate what the diameter of the tube a should be for water to remain in the tube ($\gamma = 0.07$ N m^{-1} for air–water interface).

Problem 3: Stability of a self–gravitating rotating cylinder

This is a very simplified model of star formation, which results from the gravitational instability of rotating molecular clouds.

We consider an inviscid fluid with density ρ, pressure P and velocity $\mathbf{v}$. It is acted on by its own gravity which gives a gravitational potential Ψ. At equilibrium, $\rho = \rho_0$ and $P = P_0$ are uniform, and the fluid rotates with angular velocity Ω around the z–axis. A small perturbation is applied to this fluid so that $\rho = \rho_0 + \rho_1$, $P = P_0 + P_1$, $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ and $\Psi = \Psi_0 + \Psi_1$, where the quantities with subscript '1' are small. The fluid satisfies:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0, \quad \text{(mass conservation)} \tag{7}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\frac{1}{\rho} \boldsymbol{\nabla} P - \boldsymbol{\nabla} \Psi, \quad \text{(Euler}'\text{s equation)} \tag{8}$$

$$\nabla^2 \Psi = 4\pi G \rho, \quad \text{(Poisson}'\text{s equation)} \tag{9}$$

$$P \rho^{-\gamma} = \text{constant}, \quad \text{(equation of state)} \tag{10}$$

where $\gamma = 1$ if the perturbation is isothermal, and γ is the ratio of the heat capacity at constant pressure to heat capacity at constant volume if the perturbation is adiabatic.

a) Using cylindrical coordinates, show that $\Psi_0 = r^2 \Omega^2 / 2$ and $\Omega^2 = 2\pi G \rho_0$.

b) We study a perturbation which propagates along the z–axis, with no dependence on r nor θ, so that the perturbed quantities are proportional to $\mathrm{e}^{\mathrm{i}(k_z z - \omega t)}$. Linearize equations (7)–(10) and derive the dispersion relations:

$$\omega = \pm 2\Omega, \tag{11}$$

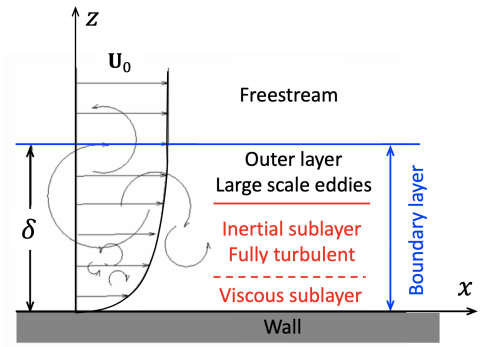
$$\omega^2 = k_z^2 c_s^2 - 4\pi G \rho_0, \tag{12}$$

where $c_s = \sqrt{\gamma P_0 / \rho_0}$ is the (isothermal or adiabatic) sound speed. Show that equation (11) corresponds to fluid elements moving along *epicycles* (circles centered on the equilibrium position, which itself moves along a circle). Derive the so–called *Jeans's instability criterion* from equation (12).

c) We consider a sphere of fluid of radius $\sim 1/k_z$ at equilibrium. If this sphere were slightly compressed, pressure waves would tend to transport the surdensity away whereas self–gravity would tend to induce a collapse of the sphere. Show that Jeans's criterion can be recovered by comparing the timescales involved.

Problem 4: Turbulent boundary layer

We consider an incompressible turbulent flow with an average velocity along x between two parallel plates located at $z = 0$ and $z = h$.



In the turbulent boundary layer near, e.g., the plate at $z = 0$, we can distinguish three layers: (i) a viscous sublayer, where the boundary conditions make the flow laminar, (ii) an inertial intermediate layer, where turbulent transport dominates over viscous transport and where still $z \ll h$, (iii) an outer layer with large scale turbulent eddies.

We use the Reynolds decomposition $v_x = V_x + v'_x$, $v_z = V_z + v'_z$, $p = P + p'$ and consider a fully developed steady flow in which $V_z = 0$ and all the statistical properties of the turbulent fluctuations (e.g., V_x, $\langle v'^2_x \rangle$, $\langle v'_x v'_y \rangle$, etc., where the brackets denote a time average) are independent of x. There is however a pressure gradient which is externally applied.

a) Neglecting gravity, show that Navier–Stokes equation yields:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\nu\frac{\partial V_x}{\partial z} - \langle v'_x v'_z \rangle\right) - \frac{1}{\rho}\frac{\partial P}{\partial x} = 0, \tag{13}$$

$$\frac{\mathrm{d}\langle v'^2_z \rangle}{\mathrm{d}z} + \frac{1}{\rho}\frac{\partial P}{\partial z} = 0, \tag{14}$$

where ν is the kinematic viscosity.

b) Show that these equations can be integrated to give:

$$\rho\nu\frac{\mathrm{d}V_x}{\mathrm{d}z} - \rho\langle v'_x v'_z \rangle = \frac{\partial P}{\partial x}z + \tau_0, \tag{15}$$

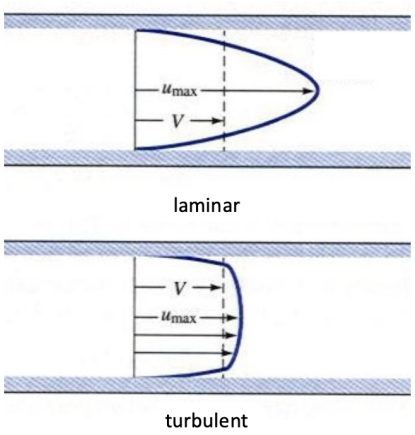
where τ_0 is a constant.

c) In the viscous sublayer, the velocity fluctuations are negligible (as the flow is stationary at the boundary) and z is very small compared to all the other scales. Show that this implies $V_x \propto z$.

d) In the inertial sublayer, transport of momentum is dominated by turbulence, and z is still very small compared to all the other scales. Using mixing length theory, assuming that the mixing length $\Lambda = \kappa z$, where κ is the so–called *von Karman* constant, and $|v'_z| \sim |v'_x| \sim u$, where u is a characteristic value, show that V_x varies logarithmically with z. Experimentally, it is found that $1/\kappa \simeq 2.5$.

e) We now consider the whole flow between two identical plates at $z = 0$ and $z = h$. By writing that the viscous and turbulent stresses are comparable at the boundary between the viscous and inertial layers, express the viscous stress at $z = 0$ and $z = h$ in terms of u. By integrating equation (13) between $z = 0$ and $z = h$, show that $u \sim \sqrt{h\Delta P/(\rho L)}$, where L is the length of the plates and ΔP is the pressure drop along the pipe.

f) The figure below shows the velocity profiles for both laminar and turbulent flows.



In the turbulent case, the average velocity in the central region of the flow is roughly constant (*plug flow*) and comparable to that in the inertial sublayer. For a laminar flow in a circular pipe of radius R, it was shown in Problem Set 2 that the velocity is given by $\left(R^2 - r^2\right)\Delta P/(4\rho\nu L)$. Give an estimate for the ratio of the maximum velocities corresponding to each case in terms of the local Reynolds number $Re = uh/\nu$.