Fluids

University of Oxford

Third Year, Part B1

Caroline Terquem
Department of Physics

caroline.terquem@physics.ox.ac.uk

Hilary Term 2022

Contents

1 Kinematics of fluids **5**

1.1 What is a fluid? . . . 5

1.1.1 Mean free path . . . 6

1.1.2 Averaged quantities . . . 6

1.2 Eulerian and Lagrangian descriptions . . . 7

1.3 Streamlines, trajectories and streamtubes . . . 8

1.4 Material time derivative . . . 8

1.4.1 Acceleration of a fluid element: . . . 10

1.4.2 Steady flow . . . 11

1.4.3 Rate of change along a streamline . . . 11

1.5 Vorticity and strain rate . . . 12

1.5.1 Rate of strain tensor . . . 13

1.5.2 Vorticity . . . 15

1.5.3 Deformation of a fluid element in the general case . . . 16

1.6 Mass conservation . . . 17

1.6.1 Eulerian approach . . . 17

1.6.2 Lagrangian approach . . . 18

1.7 Incompressibility . . . 18

1.8 Velocity potential, circulation and stream function . . . 19

1.8.1 Velocity potential . . . 19

1.8.2 Circulation . . . 20

1.8.3 Stream function . . . 22

2 Dynamics of fluids **23**

2.1 Stress tensor . . . 23

2.1.1 Pressure and viscous forces in fluids . . . 24

2.1.2 Definition of the stress tensor . . . 25

2.1.3 Two–dimensional shear flow in a gas . . . 26

2.1.4 Stress tensor and velocity correlations . . . 28

2.1.5 Expression of the stress tensor for a Newtonian fluid . . . 29

2.2 Equation of motion for a fluid . . . 32

2.2.1 Navier–Stokes equation . . . 32

2.2.2 Reynolds number . . . 34

2.2.3 Dimensional analysis and similarity . . . 35
2.2.4 Incompressibility revisited . . . 38
2.2.5 Euler equation for an inviscid fluid . . . 39
2.3 Boundary conditions . . . 39
2.3.1 Rigid boundary . . . 39
2.3.2 Interface between two fluids . . . 41
2.3.3 Free surface . . . 44
2.4 The vorticity equation and Kelvin's theorem . . . 44
2.4.1 The vorticity equation for an incompressible viscous fluid . . . 45
2.4.2 Case of an ideal fluid and Kelvin's theorem . . . 46
2.5 Conservation of energy and Bernoulli's theorem . . . 48
2.5.1 Conservation of energy in an incompressible Newtonian fluid . . . 48
2.5.2 Conservation of energy in a steady ideal fluid: Bernoulli's theorem . . . 51
2.6 Examples of viscous flows and very viscous flows . . . 52

3 Potential flows **53**
3.1 General properties of potential flows . . . 54
3.2 Simple potential flows . . . 55
3.2.1 Uniform parallel flows . . . 55
3.2.2 Line vortex flow . . . 56
3.2.3 Sources and sinks . . . 57
3.2.4 Dipole flow . . . 57
3.2.5 Flow around a circular cylinder . . . 58
3.3 Complex velocity potential . . . 63
3.3.1 Cauchy–Riemann equations . . . 63
3.3.2 Complex potential of a flow past a cylinder . . . 64
3.3.3 Conformal mapping . . . 64
3.3.4 The Joukowski transformation . . . 67
3.3.5 Potential flow past a finite plate and the Kutta condition . . . 68
3.3.6 The Joukowski aerofoil . . . 70
3.3.7 Forces on aerofoils and the Kutta–Joukowski theorem . . . 71
3.3.8 The origin of the circulation . . . 75

4 Boundary layers **77**
4.1 The boundary layer on a flat plate . . . 78
4.1.1 Thickness of the boundary layer . . . 78
4.1.2 Equation of motion . . . 79
4.1.3 Velocity profile in the boundary layer . . . 80
4.1.4 Frictional force on a flat plate . . . 82
4.1.5 Vorticity in the boundary layer and wake . . . 83
4.1.6 Transition to turbulence . . . 83
4.2 Boundary layer separation . . . 85

4.2.1 Condition for separation . . . 85
4.2.2 Effect of the Reynolds number on the separation . . . 87
4.2.3 Enhanced drag . . . 88

5 Waves **91**
5.1 Sound waves . . . 91
5.1.1 Wave equation in a perfect gas . . . 92
5.1.2 Wave equation in a liquid . . . 93
5.1.3 The speed of sound . . . 94
5.1.4 Solutions to the wave equation . . . 95
5.1.5 Energy in sound waves . . . 95
5.2 Surface waves . . . 95
5.2.1 Equilibrium state . . . 96
5.2.2 Boundary conditions . . . 96
5.2.3 Equation and boundary conditions for the velocity potential . . . 98
5.2.4 Dispersion relation . . . 99
5.2.5 Dispersion and group velocity . . . 101
5.2.6 Surface gravity waves on deep water . . . 102
5.2.7 Surface gravity waves on water of finite depth . . . 104
5.2.8 Gravity–capillary waves . . . 105
5.3 Internal gravity waves . . . 106
5.3.1 Buoyancy frequency . . . 106
5.3.2 Dispersion relation . . . 107
5.3.3 Motion of fluid elements . . . 109

6 Instabilities and turbulence **111**
6.1 Kelvin–Helmholtz instability . . . 111
6.1.1 Boundary conditions . . . 112
6.1.2 Dispersion relation . . . 112
6.1.3 Instability condition . . . 114
6.1.4 Physics of the instability . . . 115
6.2 Rayleigh–Taylor instability . . . 116
6.3 Turbulence . . . 117
6.3.1 The Reynolds stress . . . 117
6.3.2 Mixing length theory . . . 119
6.3.3 Energy conservation . . . 119
6.3.4 Kolmogorov scaling . . . 121

A Complex variables **123**
A.1 The Cauchy–Riemann relations . . . 123
A.2 Cauchy's theorem . . . 124
A.3 Laurent series . . . 124

Fluid dynamicists were divided into hydraulic engineers who observe what cannot be explained and mathematicians who explain things that cannot be observed.

Sir Cyril Hinshelwood (in Lighthill, 1956)

These notes borrow from the following books:

E. Guyon, J.–P. Hulin, L. Petit, C. D. Mitescu, *Physical Hydrodynamics*, 2nd edition (Oxford University Press)

D. J. Acheson, *Elementary Fluid Dynamics* (Oxford University Press)

L. D. Landau, E. M. Lifshitz, *Fluid Mechanics*, 2nd edition, Volume 6 of Course of Theoretical Physics (Elsevier)

G. K. Batchelor, *An introduction to Fluid Dynamics* (Cambridge University Press)

These notes are meant to be a support for the course, but they should not replace textbooks. It is strongly advised that at least one of the books listed above is used regularly, as they provide much more details about the subject and lots of examples and problems.

Chapter 1

Kinematics of fluids

1.1 What is a fluid?

A fluid is a collection of particles that can be treated as a *continuum* and which *flows (deforms)* when acted upon by a stress.

When particles are free to move relative to each other, the description as a continuum requires their mean free path to be much smaller than the other characteristic lengths of the problem. This condition is usually met in liquids and may also be satisfied in gases and plasmas. Solids, in which particles are bound to their neighbours, can be described as a continuum on scales large compared to inter–atomic distances (e.g., theory of elasticity). In cases where the continuum approximation does not apply, kinetic theory has to be used. Hydrodynamics can actually be obtained as the limit of kinetic theory when the mean free path is much smaller than all the other characteristic lengths.

Liquids, gases and plasmas all deform under stress and therefore may be treated as fluids. In general, solids are not considered as fluids because they do not deform easily. However, some solids do flow when subject to stresses larger than their *limit of elasticity*. Examples of this are glaciers and the Earth's crust. Also, some materials behave either like solids or liquids depending on whether they are subject to a high or low frequency stress, respectively. For instance, we sink deeper into wet sand when standing up than when running. Also some polymers, which behave like solids when acted upon by a stress that varies on a short timescale, start to behave like liquids when the stress varies on a timescale long enough that the polymer can use its internal degrees of freedom to deform like a liquid. The frontier between solids and liquids can therefore be fuzzy.

1.1.1 Mean free path

The mean free path λ is the average distance travelled by a particule before it collides with another particle. It is given by:

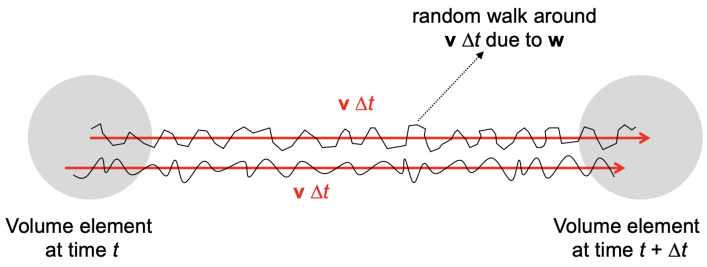
$$\lambda \sim \frac{1}{n\sigma}, \tag{1.1}$$

where n is the number density of particles and σ is the collision cross section. If all the particles are identical and with diameter d, then $\sigma = \pi d^2$.

Let us calculate the mean free path of the molecules in air at atmospheric pressure and room temperature. Air consists of 21% of O_2 and 78% of N_2 (and small amounts of other gases), which have a diameter $d \simeq 0.4$ nm. Treating the air as an ideal gas, the number density is given by $n = P/(kT)$, where P is the pressure, T is the temperature and k is the Boltzmann constant. Adopting $P = 1$ atm $= 1.01 \times 10^5$ Pa and $T = 300$ K, we obtain $n = 2.4 \times 10^{25}$ m^{-3}. The collision cross section is $\sigma = \pi d^2 = 5 \times 10^{-19}$ m^2. Therefore $\lambda \simeq 8 \times 10^{-8}$ m, which indicates that the fluid approximation applies unless we are interested in microscopic processes.

1.1.2 Averaged quantities

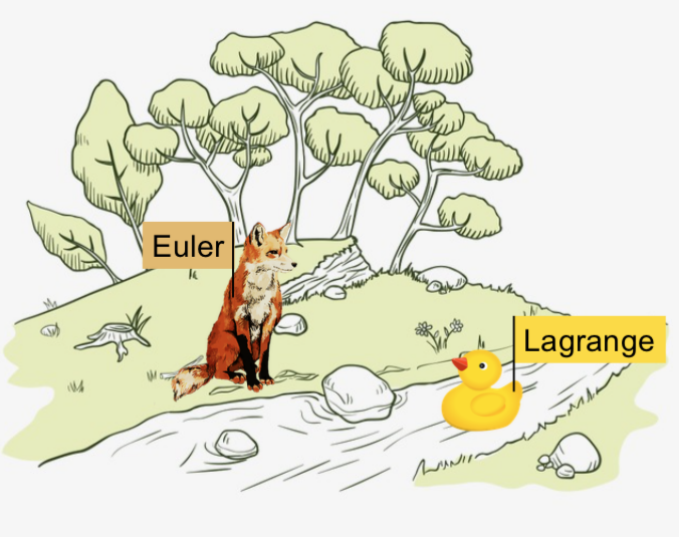
If the mean free path λ is very much smaller than the scale of interest L in the system, we can characterize a volume element with scale l such that $\lambda \ll l \ll L$ using averaged quantities. For example, the velocity of a single particle can be written as $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is the same average velocity for all the particles in the volume (since $l \ll L$), and $\mathbf{w}$ is a fluctuating part. In a volume element with $l \gg \lambda$, particles suffer a large number of collisions so that $\mathbf{w}$ changes sign very rapidly. Therefore, as illustrated in the figure below, the displacement of the particles in this volume over an interval of time Δt is given by $\mathbf{v}\Delta t$, as $\mathbf{w}$ averages to zero, and the volume element is always made of the same particles as it moves.



For most purposes, we can therefore neglect $\mathbf{w}$ and define $\mathbf{u} = \mathbf{v}$ as being the velocity of the fluid element. In the same way, the temperature, pressure, etc., can be defined as an average over the large number of particules in the volume. Thereafter, fluid elements will be sometimes loosely referred to as particles.

1.2 Eulerian and Lagrangian descriptions

A flow can be described in two different ways, depending on how the variations of the different quantities (velocity, density, temperature, etc.) are considered:



- In the Eulerian description, the variations are described as a function of time at all fixed points in the flow. The velocity $\mathbf{v}(\mathbf{r}, t)$ of a fluid element which at time t coincides with the *fixed* point located at $\mathbf{r}$ is that seen by the fox at rest on the river bank. In this description, the velocity is a vector field. Such a velocity field would be measured by *fixed* probes embedded in the fluid.

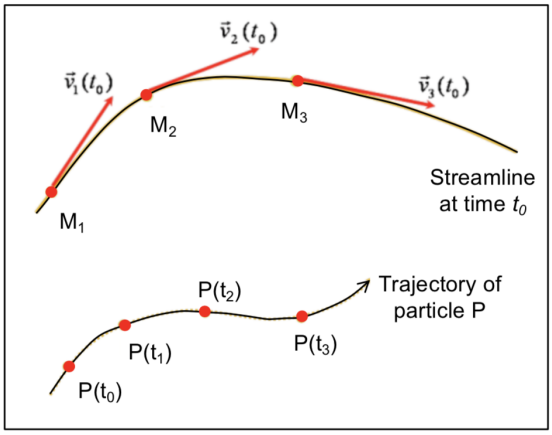
- In the Lagrangian description, one follows individual fluid elements moving with the flow and variations are described as a function of time. The velocity $\mathbf{V}(t, \mathbf{r}_0)$ of a fluid element which at some time t_0 is at position $\mathbf{r}_0$ is that of the duck in the river. (We denote the Lagrangian velocity with a capital letter to distinguish it from the Eulerian velocity). The parameter $\mathbf{r}_0$ simply 'tags' the path along which the fluid element is moving. Such a velocity can be measured by tracking (e.g., phosphorescent) tracer particles. If at a time t' the duck is a the position $\mathbf{r}'$, then its Lagrangian velocity at that time coincides with the Eulerian velocity at that point and time: $\mathbf{V}(t', \mathbf{r}_0) = \mathbf{v}(\mathbf{r}', t')$.

1.3 Streamlines, trajectories and streamtubes

A *streamline* is, at any particular time t, a curve whose tangent is everywhere parallel to the velocity vector. Let us consider a point (x, y, z) on a streamline. A small displacement $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ along the streamline is then parallel to $\mathbf{v}(x, y, z)$, which implies that:

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}. \tag{1.2}$$

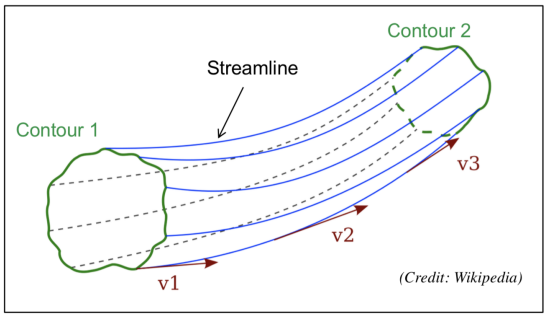
Integrating these two differential equations yields the equation of the streamlines.



Streamlines at a given time do not intersect, because a particle at a given point cannot have two different velocities at the same time.

A *trajectory* (or *pathline*) is the path followed by a particle. Trajectories can intersect.

Streamlines and trajectories only *coincide in a steady flow*. This can be seen by noting that M_1 which, at t_0, is on the streamline which is represented above, will have advanced to M_2 at a subsequent time t_1 only if the velocity does not change between t_0 and t_1.



A *streamtube* is a set of streamlines that are drawn through each point of a closed curve.

1.4 Material time derivative

We consider a Eulerian quantity Q (e.g., temperature, density, etc.), that is to say a quantity which is specified at a fixed position at a given time. For a fluid element which at time t is at a point located at $\mathbf{r}$, the value of this quantity is $Q(\mathbf{r}, t)$. If the Eulerian velocity at this point is $\mathbf{v}(\mathbf{r}, t)$, then at time $t + \delta t$ the fluid element is at $\mathbf{r} + \mathbf{v}\delta t$, where

the value of Q is $Q(\mathbf{r}+\mathbf{v}\delta t, t+\delta t)$. The time rate of change of Q for this fluid element, which we denote by $\mathrm{D}Q/\mathrm{D}t$, is therefore:

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \frac{Q(\mathbf{r}+\mathbf{v}\delta t, t+\delta t) - Q(\mathbf{r},t)}{\delta t}. \tag{1.3}$$

Performing a Taylor series expansion to first order in δt:

$$Q(\mathbf{r}+\mathbf{v}\delta t, t+\delta t) = Q(\mathbf{r},t) + \delta t \frac{\partial Q(\mathbf{r},t)}{\partial t} + \mathbf{v}\delta t \cdot \boldsymbol{\nabla} Q(\mathbf{r},t), \tag{1.4}$$

equation (1.3) becomes:

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} Q. \tag{1.5}$$

The time derivative following the motion of a fluid element is then given by the following operator:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}}, \tag{1.6}$$

which is also called *material time derivative* or *Lagrangian rate of change*. The way it has been calculated here, it has meaning only when applied to a Eulerian quantity which depends on the two independent variables $\mathbf{r}$ and t. However, this is not a unique approach[1].

From equation (1.6), we see that there are two contributions to $\mathrm{D}Q/\mathrm{D}t$: $\partial Q/\partial t$, which is the *local* rate of change due to time variations of Q at a *fixed* point, and $\mathbf{v}\cdot\boldsymbol{\nabla}Q$, which is due to the fluid element being transported to a different position along the gradient of Q (see below). The first term is the *Eulerian* rate of change, whereas the second term is the *convective* rate of change.

If Q is a *constant for every fluid element*, then $\mathrm{D}Q/\mathrm{D}t = 0$. It does not mean though that it is a constant through the fluid, as it may be a different constant for different fluid elements. It only means that a fluid element having a given value of Q at some particular time will retain this value of Q at any subsequent time.

[1]Mathematically, the operator $\mathrm{D}/\mathrm{D}t$ could also be defined as the total time derivative of a function which depends both on $\mathbf{r}(t)$ and on t explicitely. Indeed, the quantity Q could be seen as depending explicitely on time, and also on the location $\mathbf{r}$, which itself depends on time: $Q(\mathbf{r}(t),t)$. The rate of change of Q is then just its total time derivative:

$$\frac{\mathrm{d}Q(\mathbf{r}(t),t)}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial Q}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial Q}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

Using $v_x = \mathrm{d}x/\mathrm{d}t$, $v_y = \mathrm{d}y/\mathrm{d}t$ and $v_z = \mathrm{d}z/\mathrm{d}t$, we obtain:

$$\frac{\mathrm{d}Q(\mathbf{r}(t),t)}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}Q,$$

which is the same as $\mathrm{D}Q/\mathrm{D}t$. However, note that $Q(\mathbf{r}(t),t)$ is neither the Eulerian representation of Q, as this is given by $Q(\mathbf{r},t)$ where $\mathbf{r}$ is *fixed*, nor the Lagrangian representation, as this is given by $Q(\mathbf{r}_0,t)$ and is independent of $\mathbf{r}$.

1.4.1 Acceleration of a fluid element:

Above, we have calculated the material time derivative of a scalar Q, but the operator $\mathrm{D}/\mathrm{D}t$ could also be applied to a vector. For example, to calculate the acceleration of a fluid element, or *Lagrangian* acceleration, we have to calculate $\mathrm{D}\mathbf{v}/\mathrm{D}t$. Applying the operator to each of the cartesian coordinates of the velocity, it is straightforward to see that:

$$\boxed{\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}}, \tag{1.7}$$

where the operator is applied to the Eulerian velocity $\mathbf{v}(\mathbf{r},t)$. Note that $\partial\mathbf{v}/\partial t$ is not the acceleration of a fluid element at location $\mathbf{r}$ at time t, because the element is there only instantaneously.

In cartesian coordinates, the components of $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ are given by:

$$\begin{aligned} &v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z},\\ &v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z},\\ &v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}. \end{aligned}$$

To use the above formula in other coordinate systems, we have to take into account the fact that the unit vectors depend on the space coordinates. For example, in cylindrical coordinates (r,θ,z):

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \left(v_r\frac{\partial}{\partial r} + v_\theta\frac{\partial}{r\partial\theta} + v_z\frac{\partial}{\partial z}\right)\left(v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{z}}\right),$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ denote the unit vectors. Remembering that $\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}$, we see that, for example, the radial component of this expression has a term $-v_\theta^2/r$ that comes from $\partial\left(v_\theta\hat{\boldsymbol{\theta}}\right)/\partial\theta$.

The acceleration of a fluid element could also be calculated directly using the Lagrangian representation of the velocity $\mathbf{V}(\mathbf{r}_0,t)$. In this case, it is just the time derivative of the velocity along a given path, so that:

$$\boxed{\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \left(\frac{\partial\mathbf{V}}{\partial t}\right)_{\mathbf{r}_0}}. \tag{1.8}$$

Note that the velocity of a fluid element can be written as:

$$\mathbf{v} = \frac{\mathrm{D}\mathbf{r}}{\mathrm{D}t}. \tag{1.9}$$

This can be seen by writing that the operator applies to the Eulerian quantity $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, where a *hat* denotes a unit vector, so that there is no time dependence and:

$$\frac{\mathrm{D}\mathbf{r}}{\mathrm{D}t} = (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{r} = v_x\frac{\mathrm{d}x}{\mathrm{d}x}\hat{\mathbf{x}} + v_y\frac{\mathrm{d}y}{\mathrm{d}y}\hat{\mathbf{y}} + v_z\frac{\mathrm{d}z}{\mathrm{d}z}\hat{\mathbf{z}} = \mathbf{v}.$$

1.4.2 Steady flow

It is a flow in which, at any *fixed* point $\mathbf{r}$, the velocity does not depend on time, that is to say:

$$\frac{\partial\mathbf{v}}{\partial t} = 0. \tag{1.10}$$

We see from equation (1.7) that, in a steady flow, fluid elements may still be accelerated by being transported to a position where the velocity has a different value.

Consider for example a fluid in uniform rotation with angular velocity Ω, so that $v_x = -\Omega y$, $v_y = \Omega x$ and $v_z = 0$. Then:

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \left(-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\right)(-\Omega y, \Omega x, 0) = -\Omega^2(x, y, 0),$$

which is, as expected, the centripetal acceleration $-\Omega^2\mathbf{r}$.

1.4.3 Rate of change along a streamline

In a steady flow, the Lagrangian rate of change of a quantity Q is given by $\mathbf{v}\cdot\boldsymbol{\nabla}Q$. Let us denote s the coordinate (distance) along a streamline, and $\hat{\mathbf{s}}$ the unit vector associated with this coordinate. Then $\mathbf{v} = |v|\hat{\mathbf{s}}$ and

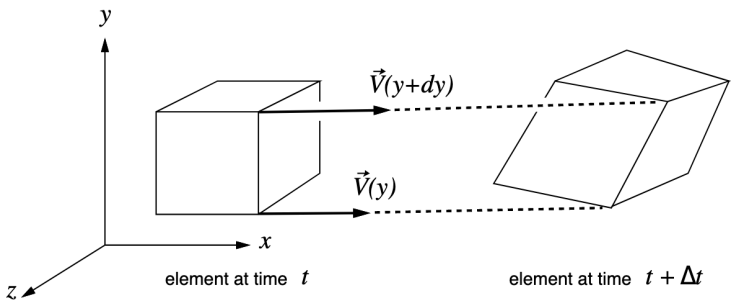
$$\mathbf{v}\cdot\boldsymbol{\nabla}Q = |v|\hat{\mathbf{s}}\cdot\boldsymbol{\nabla}Q = |v|\frac{\partial Q}{\partial s}. \tag{1.11}$$

This is the rate of change of Q with distance along the streamline times the flow speed, which gives the *rate of change of Q with time along the streamline*.

Therefore, $\mathbf{v}\cdot\boldsymbol{\nabla}Q = 0$ means that Q is constant along a streamline, that is to say for a fluid element moving along that streamline.

1.5 Vorticity and strain rate

Here we are interested in the way a small volume of fluid deforms when it moves with the flow. In the figure below, the velocity is not uniform across the volume, so that it *tilts* and *stretches* as it moves:



At time t, the velocity of a particle at location $\mathbf{r}$ is $\mathbf{v}(\mathbf{r}, t)$ and that of a particle at location $\mathbf{r} + \mathrm{d}\mathbf{r}$ is $\mathbf{v} + \mathrm{d}\mathbf{v}$. To first order in the components $\mathrm{d}x_j$ $(j = 1, 2, 3)$ of $\mathrm{d}\mathbf{r}$, we have:

$$\mathrm{d}v_i = \frac{\partial v_i}{\partial x_j}\mathrm{d}x_j. \tag{1.12}$$

Einstein notation[2] has been used in this equation and will be used throughout these notes. We use either (x_1, x_2, x_3) or (x, y, z) to denote the x–, y– and z–components.

The quantity $D_{ij} \equiv \partial v_i/\partial x_j$ is called the *deformation tensor*. If we select a corner of the cube on the figure above as a reference point, then D_{ij} tells us how the points in the cube move with respect to this reference point. Therefore, it contains information about how the cube deforms as it moves, but does not describe the overall motion of the cube with the flow. This tensor can be written as $D_{ij} = e_{ij} + \omega_{ij}$, where e_{ij} is a symmetric tensor and ω_{ij} is an anti-symmetric tensor which are given by:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \text{rate of strain tensor} \tag{1.13}$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right), \quad \text{vorticity tensor.} \tag{1.14}$$

The *strain* is a measure of the local deformation of a fluid element caused by an applied *stress*, whereas the vorticity measures the local angular velocity of the fluid element, as will be made clear below.

[2]Einstein notation implies that repeated indices within one term are summed over. Therefore,

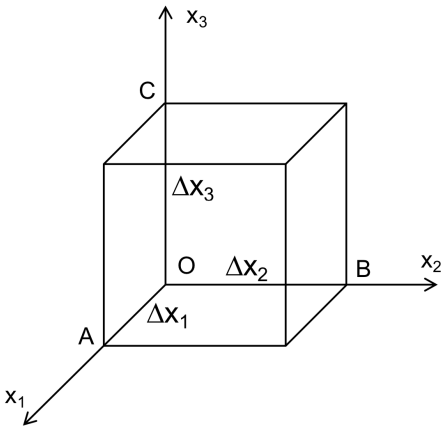
$$\frac{\partial v_i}{\partial x_j}\mathrm{d}x_j \equiv \sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\mathrm{d}x_j.$$

1.5.1 Rate of strain tensor

We are going to show that the diagonal terms of the tensor e_{ij} are associated with a change in volume whereas off-diagonal terms are associated with shear.

Diagonal terms:

We assume here that only the diagonal terms, of the form $\partial v_i/\partial x_i$, are non–zero.



Let us consider the volume element represented in the figure at time t. Its volume is $V(t) = \Delta x_1 \Delta x_2 \Delta x_3$. We are now going to calculate its volume $V(t+\delta t)$ at time $t+\delta t$.

We note $\mathbf{v}(x_1, x_2, x_3, t)$ the velocity at point (x_1, x_2, x_3) and at time t. To first order in δt, point A moves away from O at the relative velocity:

$$\mathbf{v}_{\rm rel} = \mathbf{v}(\Delta x_1, 0, 0, t) - \mathbf{v}(0, 0, 0, t) = \Delta x_1 \frac{\partial v_1}{\partial x_1} \hat{\mathbf{x}}_1,$$

where $\hat{\mathbf{x}}_1$ is the unit vector in the x_1–direction and the velocity in the derivative is evaluated at O and at time t. The distance traveled by point A relative to O between t and $t+\delta t$ is $|\mathbf{v}_{\rm rel}|\delta t$ and the distance $OA(t+\delta t)$ between O and A at $t+\delta t$ is obtained by adding Δx_1:

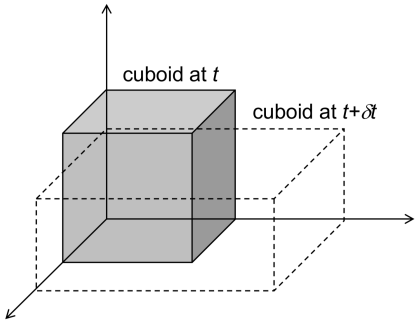
$$OA(t+\delta t) = \Delta x_1 + \Delta x_1 \frac{\partial v_1}{\partial x_1} \delta t.$$

We have assumed here that the displacement of a point which is initially at Δx_1 is $\mathbf{v}(\Delta x_1)\delta t$, that is to say we have not taken into account the variations of $\mathbf{v}$ between t and $t+\delta t$. This is only valid to first order in δt when the deformations are very small. This calculation could have been done for any point belonging to the $x = \Delta x_1$ plane, which means that this face of the cuboid is moving along the x_1–axis while staying parallel to its original plane. Similarly, along the x_2– and x_3–directions:

$$OB(t+\delta t) = \Delta x_2 + \Delta x_2 \frac{\partial v_2}{\partial x_2} \delta t,$$

$$OC(t+\delta t) = \Delta x_3 + \Delta x_3 \frac{\partial v_3}{\partial x_3} \delta t,$$

and, again, the faces of the cuboid in the $y = \Delta x_2$ and $z = \Delta x_3$ planes move while staying parallel to their original plane.



Therefore, there is no *tilting* but only *stretching*: the cuboid only dilates or contracts.

The volume at $t + \delta t$ is:

$$V(t+\delta t) = \Delta x_1 \Delta x_2 \Delta x_3 \left(1 + \frac{\partial v_1}{\partial x_1}\delta t\right)\left(1 + \frac{\partial v_2}{\partial x_2}\delta t\right)\left(1 + \frac{\partial v_3}{\partial x_3}\delta t\right),$$

which, to first order in δt, is equal to:

$$V(t+\delta t) = \Delta x_1 \Delta x_2 \Delta x_3 \left[1 + \delta t\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\right].$$

This can be written as:

$$V(t+\delta t) = V(t)\left(1 + \delta t \boldsymbol{\nabla}\cdot\mathbf{v}\right).$$

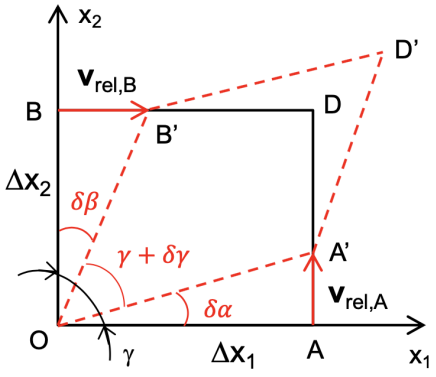
If we note δV the change in volume during δt, then the relative change in volume is:

$$\boxed{\frac{\delta V}{V} = \delta t \boldsymbol{\nabla}\cdot\mathbf{v},} \tag{1.15}$$

which expresses the fact that *the rate of volume expansion is* $\boldsymbol{\nabla}\cdot\mathbf{v}$, which is also equal to e_{ii}, the trace of the tensor e_{ij}.

Off–diagonal terms:

We now assume that only the off–diagonal terms, of the form $\partial v_i/\partial x_j$ with $j \neq i$, are non–zero. We limit the discussion to the two dimensional case to keep the analysis simpler, noting that it can easily be extended to three dimensions.



We consider the surface element $OADB$ represented in the figure at time t. Since v_1 depends only on x_2, A moves relative to O in the x_2–direction. Similarly, B moves relative to O in the x_1–direction. The dashed lines represent the surface at time $t+\delta t$ (here and thereafter we ignore the translation of the whole surface).

To first order in δt, point A moves relative to O at the velocity:

$$\mathbf{v}_{\rm rel} = \mathbf{v}(\Delta x_1, 0, 0, t) - \mathbf{v}(0, 0, 0, t) = \Delta x_1 \frac{\partial v_2}{\partial x_1} \hat{\mathbf{x}}_2,$$

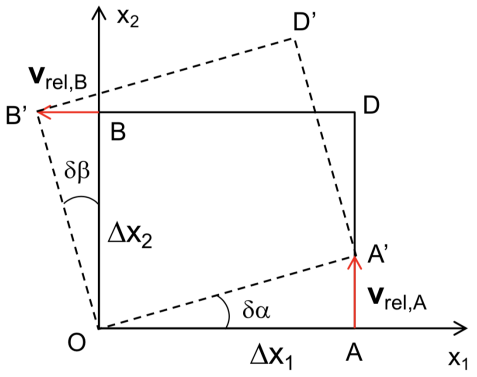
where $\hat{\mathbf{x}}_2$ is the unit vector in the x_2–direction and the velocity in the derivative is evaluated at O and at time t. Therefore, after a time δt, OA becomes OA' with $AA' = \Delta x_1 (\partial v_2/\partial x_1)\delta t$. The angle $\delta\alpha$ through which the line rotates during δt is then given by $\delta\alpha \simeq AA'/OA = (\partial v_2/\partial x_1)\delta t$ (positive angles are defined counterclockwise). Similarly, OB rotates through $\delta\beta \simeq -(\partial v_1/\partial x_2)\delta t$ (this angle is negative if, as assumed in the figure, $\partial v_1/\partial x_2 > 0$.) If $\delta\alpha = \delta\beta$, the angle γ between the lines OA and OB remains constant: there is only *rotation*. However, when $\delta\alpha \neq \delta\beta$, γ changes by $\delta\gamma = \delta\beta - \delta\alpha$ (which on the figure is negative): there is *shearing* motion. The angle $\delta\gamma$ is called the *shear strain* of the fluid element and the rate at which γ changes is called the *shear strain rate*. This can be related to the strain tensor through:

$$\frac{\delta\gamma}{\delta t} = \frac{\delta\beta - \delta\alpha}{\delta t} = -\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = -2e_{xy}. \tag{1.16}$$

Similarly, e_{xz} and e_{yz} are related to the shear strain rates in the xz and yz planes, respectively. It can be shown that the deformation of a fluid element due to the off–diagonal terms of the strain tensor *do not change its volume*.

1.5.2 Vorticity

Here again, we consider the two dimensional case for simplicity. We assume that all the components of the tensor e_{ij} are zero, so that v_1 depends only on x_2, v_2 depends only on x_1 and $\partial v_1/\partial x_2 = -\partial v_2/\partial x_1$.



We are therefore in the same situation as above but with $\delta\beta = \delta\alpha$, as illustrated in the figure. This implies that the surface is *rotating without being deformed* with the angular velocity:

$$\frac{\delta\alpha}{\delta t} = \frac{\delta\beta}{\delta t} = \frac{\delta\alpha + \delta\beta}{2\delta t} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) = \omega_{yx}.$$

The vorticity tensor is therefore related to the local angular velocity of the fluid element. If the off-diagonal components of e_{ij} are non-zero, which means that $\delta\alpha \neq \delta\beta$, then ω_{yx} represents the *average* angular velocity of the surface element around the z–axis. Similarly, ω_{zy} and ω_{xz} are the average angular velocities around the x– and y–axes, respectively.

The local angular velocity vector of a fluid element is therefore given by $\omega_{zy}\hat{\mathbf{x}} + \omega_{xz}\hat{\mathbf{y}} + \omega_{yx}\hat{\mathbf{z}}$. As will be seen throughout these notes, the quantity that appears most commonly in the

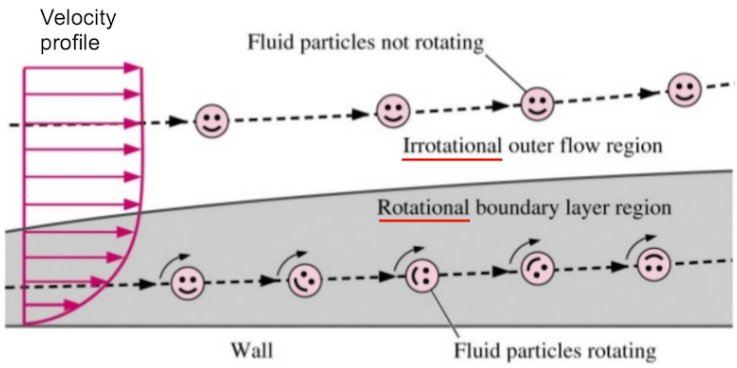
description of flows is actually twice this angular velocity vector. It is called the *vorticity vector* and is noted $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}},$$

which we recognize as:

$$\boxed{\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}}. \tag{1.17}$$

A flow is called *irrotational* if $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$ and *rotational* if $\boldsymbol{\nabla}\times\mathbf{v} \neq \mathbf{0}$. In a rotational flow, fluids elements rotate as they move, whereas in an irrotational flow they do not rotate. This is illustrated in the figure below:



1.5.3 Deformation of a fluid element in the general case

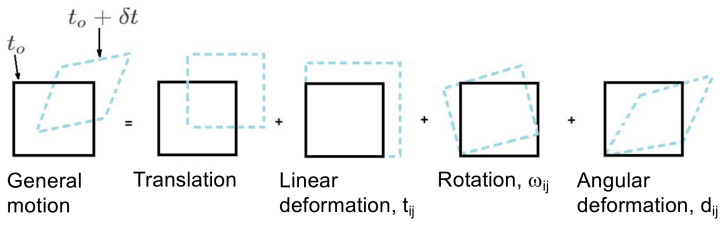
The deformation tensor $D_{ij} = \partial v_i/\partial x_j$ can be written in the form:

$$\boxed{D_{ij} = t_{ij} + d_{ij} + \omega_{ij}, \;\; \text{with} \;\; t_{ij} = \frac{1}{3}\delta_{ij}e_{kk} \;\; \text{and} \;\; d_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}.} \tag{1.18}$$

The interpretation of the different contributions for a fluid element is as follows:

- the tensor t_{ij} is diagonal and its trace is equal to the rate of volume expansion,
- the tensor d_{ij} is symmetric, its trace is zero, and it is related to the deformation of the fluid element without change of volume,
- the tensor ω_{ij} is anti-symmetric and related to the local rigid-body rotation of the fluid element.

This is illustrated in the figure below:



1.6 Mass conservation

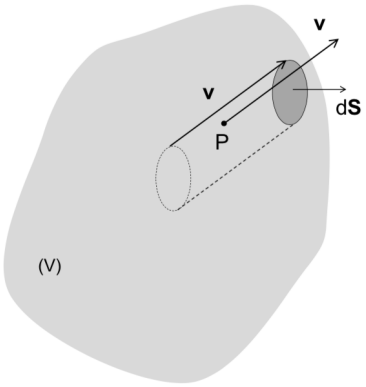
In this section, we derive the equation which expresses mass conservation.

1.6.1 Eulerian approach

We consider an arbitrary *fixed* volume V of the fluid delimited by a closed surface S, which contains the mass:

$$m = \iiint_V \rho \, \mathrm{d}V,$$

where ρ is the mass density. This mass varies due to particles entering and leaving the volume.



The particles P which cross a surface element $\mathrm{d}S$ per unit time are contained within the cylinder of cross–sectional area $\mathrm{d}S$ and length v parallel to the vector velocity $\mathbf{v}$ at this location. The volume of this cylinder is $\mathbf{v} \cdot \mathrm{d}\mathbf{S}$, where the vector $\mathrm{d}\mathbf{S}$ is perpendicular to the surface element and directed outwards. Therefore, the total mass which crosses the surface element $\mathrm{d}S$ per unit time is $\rho\mathbf{v} \cdot \mathrm{d}\mathbf{S}$. Note that this is positive if particles *leave* the volume and negative if they *enter* it.

The total mass which leaves the volume V per unit time is therefore the integral of $\rho\mathbf{v} \cdot \mathrm{d}\mathbf{S}$ over the surface and this is equal to $-\mathrm{d}m/\mathrm{d}t$, so that we can write:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \, \mathrm{d}V = - \oiint_S \rho\mathbf{v} \cdot \mathrm{d}\mathbf{S}. \tag{1.19}$$

As the volume is fixed, we can move the time–derivative inside the integral on the left–hand side. By using the divergence theorem to transform the right–hand side into an integral over the volume, we then obtain:

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{v}) \right) \mathrm{d}V = 0. \tag{1.20}$$

Since this is valid for any volume V, we must have:

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{v}) = 0.} \tag{1.21}$$

This is the mass conservation equation, also called *continuity equation*.
Using equation (1.6), mass conservation can also be written as:

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0.} \tag{1.22}$$

1.6.2 Lagrangian approach

Above, we have derived the mass conservation equation by considering a *fixed* volume in the fluid. We now show that the same equation can be obtained by writing the conservation of mass for a fluid element of volume V *moving* with the flow. The calculation is slightly more complicated, but it is worth doing as it shows how conservation laws can be obtained with the two different approaches. The element distorts as it moves, but by definition its mass m stays constant: no fluid crosses the surface as the surface itself moves with the fluid. Therefore:

$$\frac{\mathrm{D}m}{\mathrm{D}t} \equiv \frac{\mathrm{d}m}{\mathrm{d}t} = 0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho(t)\mathrm{d}x(t)\mathrm{d}y(t)\mathrm{d}z(t),$$

where we make it explicit that the coordinates of the volume depend on time, as does the mass density since the volume changes as the fluid element moves. This yields:

$$\iiint_V \left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \rho\frac{\mathrm{d}(\mathrm{d}x)}{\mathrm{d}t}\mathrm{d}y\mathrm{d}z + \rho\frac{\mathrm{d}(\mathrm{d}y)}{\mathrm{d}t}\mathrm{d}x\mathrm{d}z + \rho\frac{\mathrm{d}(\mathrm{d}z)}{\mathrm{d}t}\mathrm{d}x\mathrm{d}y \right) = 0. \tag{1.23}$$

To calculate, e.g., $\mathrm{d}(\mathrm{d}x)/\mathrm{d}t$, we write $\mathrm{d}x = \left(\overrightarrow{OP}_2 - \overrightarrow{OP}_1\right) \cdot \hat{\mathbf{x}}$, where O is any fixed point, $\hat{\mathbf{x}}$ is the unit vector in the x–direction, and the coordinates of P_1 and P_2 are (x, y, z) and $(x + \mathrm{d}x, y, z)$, respectively. Therefore:

$$\frac{\mathrm{d}(\mathrm{d}x)}{\mathrm{d}t} = \left(\frac{\mathrm{d}(\overrightarrow{OP}_2)}{\mathrm{d}t} - \frac{\mathrm{d}(\overrightarrow{OP}_1)}{\mathrm{d}t} \right) \cdot \hat{\mathbf{x}} = (\mathbf{v}(x + \mathrm{d}x, y, z) - \mathbf{v}(x, y, z)) \cdot \hat{\mathbf{x}} = \frac{\partial \mathbf{v}}{\partial x}\mathrm{d}x \cdot \hat{\mathbf{x}} = \frac{\partial v_x}{\partial x}\mathrm{d}x.$$

Similarly for $\mathrm{d}(\mathrm{d}y)/\mathrm{d}t$ and $\mathrm{d}(\mathrm{d}z)/\mathrm{d}t$. Therefore, equation (1.23) becomes:

$$\iiint_V \left(\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\boldsymbol{\nabla} \cdot \mathbf{v} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0. \tag{1.24}$$

Since this satisfied for any volume V, the integrand is identically zero and we recover equation (1.22).

1.7 Incompressibility

The compressibility of a fluid is characterized by the coefficient:

$$\beta = -\frac{1}{V}\frac{\partial V}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p},$$

where the derivatives are taken at either constant temperature or entropy, depending on how compression happens. This coefficient is very small for liquids, and generally several orders of magnitude larger for gases. Water is approximately incompressible, with $\beta \sim 10^{-9}$ Pa^{-1} for a wide range of temperatures and pressures, whereas $\beta \sim 10^{-5}$ Pa^{-1} for air, and the compressiblity of air is of course what enables sound to propagate. However, we need to distinguish between an *incompressible fluid* and an *incompressible flow* as, under some circumstances, air *in motion* for example can be approximated as incompressible.

A *flow* is said to be *incompressible* if the *volume of fluid elements stays constant as they move*. In section 1.5, we established that the change δV of the volume $V = \Delta x_1 \Delta x_2 \Delta x_3$

as it moved with the flow was given by equation (1.15). Substituting $\delta V = 0$ in this equation then yields the condition for incompressibility:

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{v} = 0}. \tag{1.25}$$

As the mass of a fluid element stays constant as it moves, writing that its volume stays constant is equivalent to writing that its *density stays constant*. Therefore, incompressibility implies $\mathrm{D}\rho/\mathrm{D}t = 0$ which, from equation (1.22), yields $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, as above.

In section 2.2.4, we will give a condition for incompressibility that involves the ratio of the flow velocity to the sound speed.

Note that the density ρ is *not necessarily uniform* (the same for all fluid elements) in an incompressible flow. For example, oceans are *stratified* (higher density at the bottom) due to gradients of salinity, temperature etc., even though water can be considered as incompressible because an individual fluid element will retain its density as it moves. Stratification of an incompressible fluid, in which $\mathrm{D}\rho/\mathrm{D}t = \partial\rho/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}\rho = 0$, implies that ρ varies with time at a given location. This leads to internal waves because of buoyancy being a restoring force, as we will see later in these notes.

1.8 Velocity potential, circulation and stream function

In some cases, it is convenient to express the components of the velocity vector as the derivatives of a scalar. This can be done when the flow is either irrotational and/or incompressible.

1.8.1 Velocity potential

If the flow is *irrotational*, then $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$, which implies that there exists a scalar ϕ, called the *velocity potential*, such that:

$$\boxed{\mathbf{v} = \boldsymbol{\nabla}\phi.} \tag{1.26}$$

This is equivalent to the electrostatic potential resulting from $\boldsymbol{\nabla} \times \mathbf{E} = 0$. This equation does not uniquely define ϕ, as any function of time can be added to a solution without modifying $\mathbf{v}$. Flows in which the velocity can be written as the gradient of a scalar are also called *potential flows*. These are a very important class of flows, to which we will come back in chapter 3. If in addition the fluid is *incompressible*, then $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, which yields:

$$\boldsymbol{\nabla}^2\phi = 0, \tag{1.27}$$

that is to say ϕ satisfies Laplace's equation.

If the domain occupied by the fluid is *simply connected* (meaning any closed curve can be reduced to zero by being continuously deformed while staying in the domain, e.g., flow moving past a sphere), then, given $\mathbf{v}(\mathbf{r}, t)$, the potential $\phi(\mathbf{r}, t)$ *is a single–valued function* of position. This can be shown by writing:

$$\phi(\mathbf{r}, t) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{l}',$$

where $\mathbf{r}_0$ is an arbitrary fixed point. This integral is independent of the path from $\mathbf{r}_0$ to $\mathbf{r}$. Indeed, any two paths make up a closed curve. The circulation of $\mathbf{v}$ around that curve is equal to the flux of $\boldsymbol{\nabla}\times\mathbf{v}$ across the surface delimited by the curve (Stoke's theorem), and this is zero as the flow is irrotational. Therefore the integral is the same along the two paths, which implies that ϕ is single–valued .

The velocity potential can still be defined through equation (1.26) when the domain of the flow is not simply–connected (e.g., flow moving past an infinite cylinder), but the integral above may then depend on the path from $\mathbf{r}_0$ to $\mathbf{r}$, which means that ϕ is a multi–valued function of position. In that case, the circulation of $\mathbf{v}$ around a closed curve is not necessarily zero.

As an exemple, let us consider the so–called line vortex flow which, in cylindrical polar coordinates (r, θ, z), is given by:

$$\mathbf{v} = \frac{k}{r}\hat{\boldsymbol{\theta}},$$

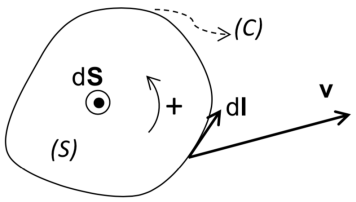
where k is a constant and $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction. It is straightforward to check that $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$ everywhere except at $r = 0$, where neither the velocity nor the vorticity are defined. If we define the flow domain to be $r \geq R$, where R is an arbitrary value, then it is not simply connected: any curve centered at the origin cannot shrunk to a point without leaving the flow domain. Integrating $\boldsymbol{\nabla}\phi = \mathbf{v}$:

$$\frac{\partial\phi}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{k}{r}, \quad \frac{\partial\phi}{\partial z} = 0,$$

we obtain $\phi = k\theta$, which is a multi–valued function of position.

1.8.2 Circulation

Circulation is a very important concept in aerodynamics, where it is used to calculate the *lift* on an object embedded in a fluid. We consider a closed curve C which delimits a surface of the fluid.



It is a mathematical convention to define the positive sense along a 2D curve as counterclockwise. Therefore, an element $\mathrm{d}\mathbf{l}$ along the curve is orientated as shown on the figure. In aerodynamics, a circulation is considered positive when it is clockwise, so in principle signs should be reversed. However, in these notes, we will use the mathematical convention.

Therefore, the circulation, which is noted Γ, is defined as:

$$\boxed{\Gamma = \oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{l}.} \tag{1.28}$$

Stokes's theorem yields:

$$\Gamma = \iint_S (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{S} \equiv \iint_S \boldsymbol{\omega}\cdot\mathrm{d}\mathbf{S}, \tag{1.29}$$

with S being the surface delimited by C. For an irrotational flow, $\boldsymbol{\omega} = \mathbf{0}$ and $\Gamma = 0$. Stokes's theorem implicitly assumes that $\boldsymbol{\omega}$ is defined everywhere over S. When the domain of the flow is not simply connected, this condition is not satisfied, and Stokes's theorem cannot be used. For example, for the line vortex flow introduced in the previous section, the circulation of the velocity along a circle C of radius $r \geq R$ centered at the origin is given by:

$$\Gamma = \oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \oint_C \frac{k}{r} r \mathrm{d}\theta = 2\pi k.$$

This is non–zero because ω is non–zero at the origin.

We now consider the circulation along a contour C which encloses an aerofoil, which is the cross-sectional shape of a wing, as represented on the figure below:



We are going to establish the following important result:

> For an irrotational flow, the circulation is the same round all simple closed contours enclosing the aerofoil. The circulation can therefore be calculated by choosing for C a circle with a radius large enough that it encloses the aerofoil.

Let us take another contour C' and show that the circulation around C is the same as that around C'. This is illustrated in the figure below:



With the orientations shown on the figure, the blue contour is $J_1 = C_1 - L_1 + C_1' - L_2$ and the red contour is $J_2 = C_2 + L_2 + C_2' + L_1$. Since they are closed and the flow is irrotational, we have:

$$\oint_{J_1} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \oint_{J_2} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0.$$

Therefore the sum of these two integrals is zero, which yields:

$$\int_{C_1 + C_1' + C_2 + C_2'} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0.$$

Since $C = C_1 + C_2$ and $C' = C'_1 + C'_2$, this means that the circulation along C is the same as that around C' if both contours are oriented in the same direction.

This is actually valid for any integral along a closed contour, not just the integral of $\mathbf{v} \cdot \mathrm{d}\mathbf{l}$.

1.8.3 Stream function

In an *incompressible* fluid, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, which implies that there exists a vector $\mathbf{A}$ such that $\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{A}$. This is equivalent to the electromagnetic vector potential resulting from $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. In cartesian coordinates, this yields:

$$v_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad v_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad v_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

These equations do not uniquely define $\mathbf{A}$: any gradient (in addition to any function of time) can be added to a solution without modifying $\mathbf{v}$.
Let us consider a two dimensional flow in the (x, y)–plane, for which there is no z–dependence. From the above equations we get:

$$\boxed{v_x = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x},} \tag{1.30}$$

where $\psi \equiv A_z$ is called the *stream function*. In this two dimensional case, the velocity vector can be characterized by this one scalar function ψ only.
The stream function can of course also be defined in polar coordinates (r, θ) through the relations:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r}. \tag{1.31}$$

In two–dimensions, the vorticity is:

$$\boldsymbol{\omega} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)\hat{\mathbf{z}} \equiv \boldsymbol{\nabla}^2 \psi \, \hat{\mathbf{z}}. \tag{1.32}$$

Therefore, if the fluid is *irrotational*, the stream function satisfies Laplace's equation:

$$\boldsymbol{\nabla}^2 \psi = 0. \tag{1.33}$$

The rate of change of ψ along a streamline is given by (see section 1.4.3):

$$\mathbf{v} \cdot \boldsymbol{\nabla} \psi = v_x \frac{\partial \psi}{\partial x} + v_y \frac{\partial \psi}{\partial y}. \tag{1.34}$$

Using equations (1.30), we see that $\mathbf{v} \cdot \boldsymbol{\nabla} \psi = 0$, which implies that ψ *is constant along a streamline*. This can be used instead of equations (1.2) to find the equations of the streamlines.

This also implies a relationship between the velocity potential and stream function for a two dimensional fluid which is both irrotational and incompressible. Indeed, since $\boldsymbol{\nabla}\phi = \mathbf{v}$, this vector is everywhere tangent to a streamline. The lines of constant ϕ, which are called *equipotential lines*, are therefore perpendicular to the streamlines ($\boldsymbol{\nabla}\phi$ cannot have a component along a line of constant ϕ). In other words, *equipotential lines (constant ϕ) and streamlines (constant ψ) are perpendicular to each other.*

Chapter 2

Dynamics of fluids

In this chapter, we focus on the transport of momentum in a moving fluid. Most of the results presented here apply to incompressible fluids only. In general, when no external forces are present, momentum can be transported by either advection and/or diffusion. Advection is a transport by the mean motion of the flow and therefore occurs in the direction of the flow. Diffusion is a transport from regions of higher momentum to regions of lower momentum and occurs perpendicularly to the direction of the flow. Diffusive transport of momentum is due to the *viscosity* of the fluid and results in *frictional* forces.

As already stated in chapter 1, the fundamental equations of fluids can be derived by considering them as either a collection of particles (kinetic theory) or as a smooth continuum. This latter approach is justified when the mean free path of the particles is very small compared to the macroscopic lengthscale of interest in the fluid. It enables to establish conservation equations more straightforwardly than kinetic theory. However, this does not lead to a precise expression for the transport coefficients, in contrast to kinetic theory. Transport of energy, mass and momentum occurs in a gas which is out of equilibrium (i.e. in which the distribution function is not a Maxwell–Boltzmann distribution) through molecular collisions. Most of the time, the departure from equilibrium is tiny, so that the distribution function is *nearly* maxwellian. Within the context of kinetic theory, in which molecular collisions are explicitly calculated, the so–called Chapman–Enskog procedure gives the transport coefficients by considering small variations of the distribution function around the Maxwell–Boltzmann distribution. Such a calculation is not possible when fluids are viewed as continua, as in this case molecular collisions are not explicitly calculated. It is possible however to get a phenomenological expression for the transport coefficients in this context, as we shall see below.

2.1 Stress tensor

When the fluid is at thermal equilibrium, there is no resultant force on any volume element within the fluid. However, when a deformation occurs (which can be measured by the rate of strain tensor e_{ij} introduced in section 1.5.1), *internal* forces are created which tend to

resist the deformation and bring the fluid back to equilibrium. Such forces, due to the deformation of the fluid, are called *internal stresses*. As we may expect, they are related to the rate of strain tensor, and also to the *viscosity* of the fluid. Internal stresses also include pressure forces, which may exist in a fluid which is at rest.

2.1.1 Pressure and viscous forces in fluids

In a gas, forces between molecules are small and pressure is due to particles colliding with each other. This can be pictured by imagining that the gas is contained within walls: molecules have random velocities due to the finite temperature, and when they hit a wall and rebound they transfer momentum to the wall. The net force communicated by the molecules is perpendicular to the wall, and its value per unit surface area is defined as the pressure. If we try to compress a gas by moving a piston, the collisions of the molecules with the piston create a pressure force that resists the compression. A similar calculation can be done by replacing the wall by an imaginary surface within the volume of the gas: the momentum communicated to the molecules on that surface yields a pressure force on the surface.
In a solid, pressure forces are due to intermolecular forces: if we try to compress a piece of wood by pushing on its surface, there is a resistance due to the force that the molecules in the wood exert on each other. The molecules are not able to move with respect to each other.
In a liquid, compression is also resisted mainly by intermolecular forces, although molecules are also able to move with respect to each other: not as much as in a gas, but more than in a solid. Intermolecular forces are strong enough to keep a given amount of liquid in a specific volume, but not strong enough to prevent the molecules from moving past each other, which enables the liquid to *flow*. In a gas, pressure forces are always present whenever there is a finite temperature. In a liquid however, there can only be pressure forces if there is gravity. In the ocean for example, pressure increases with depth: because of gravity, a layer of water at a given depth exerts a force on the layer below, and this is resisted by the pressure due to the intermolecular forces at the boundary between the two layers. Because intermolecular forces are relatively strong, liquids are almost incompressible.
Pressure forces exist in a fluid whether it moves or not. In a steady fluid, it is called *hydrostatic pressure*.

A viscous force, by contrast, is only present in moving fluids. It is the force that exists between two layers of fluid which move with respect to each other with different velocities. It is characterized by the *viscosity* of the fluid, which measures how easy it is for molecules to glide past each other. In a gas, a viscous force can be calculated in a way similar to a pressure force: molecules with different mean velocities collide with each other because of their random thermal velocity, and exchange momentum in such a way as to reduce the relative velocity between the two layers. This process is called molecular interchange. When the temperature increases, random velocities increase which leads to a higher viscosity.

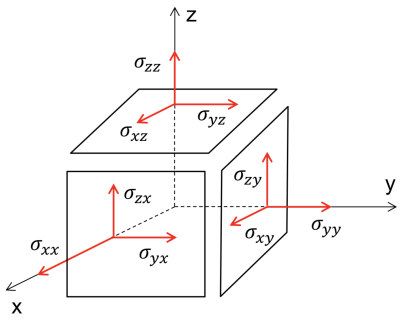
In a liquid, it is intermolecular forces that predominantly resist layers moving past each other, although there is some molecular interchange as well. Intermolecular forces become weaker at higher temperatures, and therefore the viscosity of liquids decreases when the temperature is increased.

2.1.2 Definition of the stress tensor

Transport of momentum across the surface of a volume element results in forces being exerted by the fluid located on one side of the surface onto the fluid located on the other side. When the transport of momentum is due to molecules crossing the surface and colliding with each other, these forces have a very short–range and are localized in very thin layers on both sides of the surface. Therefore, they can be viewed as being exterted onto the surface itself (like pressure forces), and we can consider the local effect of these forces by isolating a small plane surface element δS. We denote $\hat{\mathbf{n}}$ the unit vector perpendicular to this surface element.

> The local stress $\mathbf{T}$ is defined as the force per unit area exerted by the fluid located on the side of the surface element towards which $\hat{\mathbf{n}}$ points, on the fluid located on the other side.

If the range of the forces is very small compared with the linear dimensions of the surface element, then the forces are proportional to the surface area. For example, if there is no viscosity, only pressure forces are present and the stress is $-p\hat{\mathbf{n}}$. In a viscous fluid, there is an additional contribution from the viscous stress.



> We note σ_{ij} the i–component of the stress tensor on a surface element which has a normal pointing in the j–direction.

It follows that, if $i \neq j$, σ_{ij} is a tangential, or *shear stress*, whereas, if $i = j$, it is a *normal stress*.

In the particular example on the figure, the unit vectors normal to the surfaces are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, and the components of the stress on the surface which normal is $\hat{\mathbf{x}}$, for example, are $T_x = \sigma_{xx}$, $T_y = \sigma_{yx}$ and $T_z = \sigma_{zx}$. More generally, it can be shown (see appendix) that the components of the stress $\mathbf{T}$ acting on a surface which normal is along the unit vector $\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$ are given by:

$$
\begin{aligned}
T_x &= \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z, \\
T_y &= \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z, \\
T_z &= \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z.
\end{aligned}
$$

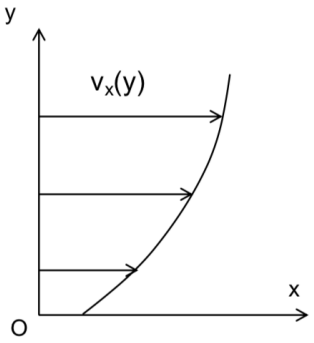
This can also be written in a compact form as:

$$\boxed{T_i = \sigma_{ij} n_j,} \tag{2.1}$$

where Einstein's notation is used.
Thereafter, we will define σ_{ij} as having contribution from viscous forces only, that is to say pressure forces will have to be added to obtain the total stress.

2.1.3 Two–dimensional shear flow in a gas

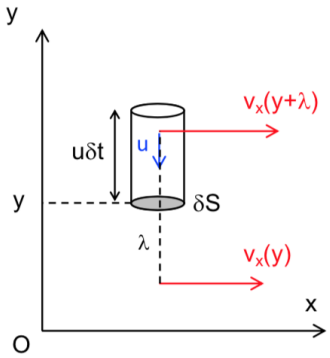
We start by revisiting a simple case which has been studied in the Statistical Physics course in second year.



Let us consider a flow with the velocity profile represented on the figure. Such a flow where adjacent layers of fluid move parallel to each other at different speeds is called *shear flow*. Shear between adjacent layers of fluids is resisted for by the *viscosity* of the fluid, which results in a frictional force between the layers.

Here we consider a gas, so that we neglect intermolecular forces and the transport of momentum is only due to particles colliding with each other. The frictional force, which we are are now going to calculate using kinetic theory, is due to the momentum transported along the y–direction by the particles in the fluid which have a random (thermal) velocity $\mathbf{u}$ relative to the mean flow.

On average, a molecule has a collision with another molecule after it travels through a distance λ, which is the mean free path of the particles. We suppose that after the collision, the momentum of the molecule is the same as that of its new environment. Let us consider the momentum which is transported during the time δt across a surface element δS perpendicular to the y–axis and with ordinate y.



The particles which cross that surface from above during δt are those contained in the cylinder of length $u\delta t$ and section δS and with a velocity $\mathbf{u}$ along $-y$. There are $nu\delta t\delta S/6$ of these particles, where n is the number density of particles and the factor 6 comes about because there are three possible directions for the particles, each with two orientations.

Each of these particles travel through λ before it suffers a collision below δS, which results in its momentum varying by:

$$m\left[v_x(y) - v_x(y+\lambda)\right] \simeq -m\lambda\frac{\mathrm{d}v_x}{\mathrm{d}y},$$

to first order in λ/L, where L is the scale of variation of the velocity. In other words, each particle carries below δS the *excess* of momentum $m\lambda \mathrm{d}v_x/\mathrm{d}y$. Here m is the mass of a particle. On the other hand, each particle traveling upward carries above δS the *deficit* of momentum $-m\lambda \mathrm{d}v_x/\mathrm{d}y$. Therefore, the net x–component of the momentum which is carried *downward* during δt by the particles crossing δS is:

$$\delta^2 p_x = 2\left(\frac{1}{6}nu\delta t\delta S\right)\left(m\lambda\frac{\mathrm{d}v_x}{\mathrm{d}t}\right) = \frac{1}{3}nmu\lambda\frac{\mathrm{d}v_x}{\mathrm{d}y}\delta S\delta t. \tag{2.2}$$

This quantity is positive if $\mathrm{d}v_x/\mathrm{d}y$ is positive. In that case, the fluid located above the surface δS accelerates the fluid located below, which means that it exerts onto this fluid a force $\delta F_x \equiv \delta^2 p_x/\delta t$ directed in the positive x–direction.
Adopting for the unit normal to the surface $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, the stress on the surface δS is the force per unit area exerted by the fluid located above the surface on the fluid located below, that is to say $\mathbf{T} = (\delta F_x/\delta S)\,\hat{\mathbf{x}} \equiv \sigma_{xy}\hat{\mathbf{x}}$. Therefore:

$$\boxed{\sigma_{xy} = \eta\frac{\mathrm{d}v_x}{\mathrm{d}y},} \tag{2.3}$$

where we have defined the *dynamic shear viscosity* η as:

$$\boxed{\eta = \frac{1}{3}nmu\lambda.} \tag{2.4}$$

Instead of η, we often use the *kinematic viscosity* ν:

$$\boxed{\nu = \frac{\eta}{\rho} = \frac{1}{3}u\lambda,} \tag{2.5}$$

where $\rho = mn$ is the fluid mass density. Note that σ_{xy} is a rate of change of momentum per unit area, which is a *flux of momentum*. The above result has been obtained for a gas, but a similar calculation could be done for a liquid by replacing the mean free path λ by a *correlation length*, which is on the order of the spatial scale over which intermolecular forces are important.

We now consider a box with horizontal faces at y and $y+\delta y$ and surface area δS. The exchange of particles across the upper face during the time δt results in the momentum $\delta^2 p_x(y+\delta y)$ being *added* to the volume, whereas the exchange across the lower surface results in the momentum $\delta^2 p_x(y)$ being *removed* from the volume. Therefore, the time rate of change of the momentum content of the box is:

$$\frac{1}{\delta t}\left[\delta^2 p_x(y+\delta y) - \delta^2 p_x(y)\right] = \frac{d}{dy}\left(\eta\frac{\mathrm{d}v_x}{\mathrm{d}y}\right)\delta S\delta y, \tag{2.6}$$

to first order in λ/L. This is also $f_{\mathrm{visc,x}}\ \delta S \delta y$, where $f_{\mathrm{visc},x}$ is the *viscous force per unit volume*. Therefore:

$$f_{\mathrm{visc},x} = \frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}y}. \tag{2.7}$$

It is important to note that viscous forces are *surface forces*, meaning that they are applied on a surface and are proportional to the area of the surface, like pressure forces. They give rise to a net force on a volume and therefore we can define a viscous force per unit volume, as done above, but when deriving boundary conditions for example they have to be explicitly written as surface forces.

2.1.4 Stress tensor and velocity correlations

Using again the simple case of the two dimensional shear flow illustrated above, we now show that the stress tensor is related to the correlation between the components of the fluctuating velocity. The components of the instantaneous velocity of a particle in the fluid are (v_x+u_x, u_y), where v_x is the mean velocity of the flow and u_x and u_y are the components of the random (thermal) velocity relative to the mean flow. We have $< u_x >=< u_y >= 0$, where the brackets denote a time average. The flux of the x–component of the momentum along the y–direction is:

$$\rho\left(v_x + u_x\right) u_y.$$

Averaged over a large number of particles, or, equivalently, over time, this gives:

$$\rho\left\langle u_x u_y \right\rangle,$$

since $< v_x u_y >= v_x < u_y >= 0$. We consider a small surface element with unit normal in the positive y–direction. The quantity above, being the upwards flux of the x–component of momentum, is the opposite of the force per unit area in the x–direction exerted by the fluid located on the side of the surface element towards which the normal points. By definition, this is $-\sigma_{xy}$. Therefore,

$$\sigma_{xy} = -\rho\left\langle u_x u_y \right\rangle. \tag{2.8}$$

This illustrates that *the momentum is transported by the fluctuations* of the velocity. For a Maxwell–Boltzmann distribution function (or any xy symmetric function), $< u_x u_y >= 0$ and there is no transport. However, in a fluid which is out of equilibrium, this correlation between the components of the fluctuating velocity may not be zero.
We denote by $\mathcal{C}$ the correlation coefficient between the velocities u_x and u_y:

$$\mathcal{C} \equiv \frac{|\langle u_x u_y \rangle|}{u^2}. \tag{2.9}$$

We see from (2.8) that $\mathcal{C}$ gives a measure of the stress tensor.

In the special case where the random fluctuations are caused by sound waves propagating with speed c_s through a gas, $u \sim c_s$ (as will be shown in section 5.1.3). In addition, using equations (2.3) and (2.8), we obtain:

$$\langle u_x u_y \rangle = -\nu \frac{dv_x}{dy}.$$

Therefore, in that case, the correlation coefficient becomes::

$$\mathcal{C} \sim \frac{\nu}{c_s^2} \left| \frac{dv_x}{dy} \right| \sim \frac{\lambda}{L} \frac{v_x}{c_s} = \frac{\lambda}{L} \mathcal{M}, \tag{2.10}$$

where we have used equation (2.5) and $dv_x/dy \sim v_x/L$, where L is a characteristic length-scale. Here $\mathcal{M} \equiv v_x/c_s$ is the Mach number. Momentum is therefore transported efficiently when the mean free path in a gas, or correlation length in a liquid, is not too small compared to L.

2.1.5 Expression of the stress tensor for a Newtonian fluid

We are now going to calculate the stress tensor in a more general case, and the calculation presented in this section applies to either a gas or a liquid. As seen above, the viscous (or friction) force in a fluid is due to an irreversible transport of momentum from regions where the velocity is higher to regions where it is lower.

Friction occurs only when different parts of the fluid have different velocities. Therefore σ_{ij} should depend on the velocity gradients. If the velocity varies on a scale large compared to the mean free path, i.e. to the scale over which molecular transport arises, one can suppose that σ_{ij} depends only on the first derivatives of the velocity with respect to the coordinates. Furthermore, we suppose that the dependence is linear, i.e. we limit ourselves to *Newtonian fluids*. The most general form of σ_{ij} is then:

$$\sigma_{ij} \propto \frac{\partial v_i}{\partial x_j} + A \frac{\partial v_j}{\partial x_i} + B \delta_{ij} \frac{\partial v_k}{\partial x_k}, \tag{2.11}$$

where A and B are constants to be determined, and δ_{ij} is the Kronecker symbol. The inclusion of the last term on the right–hand side enables the trace of the tensor σ_{ij} to be treated separately. If the flow is uniformly rotating with angular velocity Ω in the (xy)–plane for instance, we must have $\sigma_{xy} = 0$. Since $v_x = -\Omega y$ and $v_y = \Omega x$ in that case, that implies $A = 1$. Therefore the tensor σ_{ij} is symmetrical. We note that its trace is:

$$Tr\left[\sigma\right] = (2 + 3B)\, \boldsymbol{\nabla} \cdot \mathbf{v},$$

which shows that $Tr\left[\sigma\right]$ is a measure of the volume change of a fluid element. It is an experimental fact that the stresses which change the volume of a fluid element give different viscous forces than the stresses that preserve the volume. Therefore we rewrite σ_{ij} under

the form[1] :

$$\sigma_{ij} \propto \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \boldsymbol{\nabla} \cdot \mathbf{v} \right) + \left(B + \frac{2}{3} \right) \delta_{ij} \boldsymbol{\nabla} \cdot \mathbf{v},$$

where the first term in brackets on the right–hand side is trace free, i.e. does not modify the volume of a fluid element.

The *shear* and *bulk* viscosities, that we denote η and ζ respectively, are then experimentally defined as:

$$\boxed{\sigma_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \boldsymbol{\nabla} \cdot \mathbf{v} \right) + \zeta \delta_{ij} \boldsymbol{\nabla} \cdot \mathbf{v}}. \tag{2.12}$$

Since σ_{ij} has the units of a pressure, the units of η and ζ are Pa s (Pascal second). This expression for the stress tensor is not exact: it has been derived phenomenologically assuming that σ_{ij} depends only on a linear combination of the first derivatives of the velocity with respect to the coordinates. However, the kinetic theory applied to dilute gases leads to the same expression for σ_{ij}, as has been shown above in the simple case of the two dimensional shear flow. For a dilute gas, η is given by equation (2.4).
By writing that η and ζ are scalar quantities, we implicitly assume that the fluid is *isotropic*. When this is not the case, η and ζ are themselves tensors. It can be shown that, as viscosity leads to *dissipation* of energy, η is always positive. Similarly, as internal friction leads to an *increase* of entropy, ζ is also always positive.

The bulk viscosity is associated with internal degrees of freedom of the molecules in the fluid. It becomes negligible if the equipartition between these different degrees of freedom is reached over a timescale shorter than the timescale between two collisions. Furthermore, for a perfect monoatomic gas it can be shown that $\zeta = 0$.
For an *incompressible fluid*, we have simply:

$$\boxed{\sigma_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = 2\eta e_{ij}}. \tag{2.13}$$

The stress tensor is therefore *proportional to the rate of strain tensor*, that is to say to the *rate of change of the deformation* of the fluid element[2] (see section 1.5.1).

[1] I thank Prof. Steven Balbus for providing the elegant discussion presented here and leading to this expression of the stress tensor.

[2] It is interesting to contrast expressions (2.12) and (2.13) of the viscous stress tensor with that of the stress tensor obtained for solid bodies, regarded as continuous media. Within the linear theory of elasticity, that is to say in the context of small deformations, the stress tensor for isotropic bodies is given by Hooke's law:

$$\sigma_{ij} = 2\mu \left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk} \right) + K \epsilon_{kk} \delta_{ij},$$

where μ and K are the shear and bulk moduli, respectively (also called moduli of rigidity and compression). Here, ϵ_{ij} is the *strain tensor* which, for small deformations, is given by:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial \epsilon_i}{\partial x_j} + \frac{\partial \epsilon_j}{\partial x_i} \right),$$

By generalising the calculation done above (section 2.1.3) for a shear flow, we can show that the i–component of the viscous force per unit volume is given by:

$$\boxed{f_{\text{visc},i} = \frac{\partial \sigma_{ij}}{\partial x_j}.} \tag{2.14}$$

The viscous stress tensor we have calculated above, and which results from the deformation of the fluid elements, vanishes when there is no velocity gradient. In that case, the only stresses are due to pressure. We define the *total stress* Σ_{ij}, which has contributions from both viscosity and pressure, as:

$$\boxed{\Sigma_{ij} = \sigma_{ij} - p\delta_{ij},} \tag{2.15}$$

where p is the pressure. The minus sign comes from the fact that a fluid element which is at rest is under compression, and the Kronecker symbol is required because the pressure force acts perpendicularly to the surface. Pressure, therefore, only enters the component of the stress which is along the direction j in which the normal points. This expression for the total stress appears naturally in the equation of motion (see eq.[2.19] below).

Many fluids are not Newtonian, meaning there is no direct proportionality between stresses and rates of strain. This can be due to the presence in the fluid of objects which are large compared to the atomic scale, although small compared to the characteristic lengthscales of the flow. This is the case for *suspensions*, which are heterogeneous mixtures containing solid particles (e.g., muddy water, dust in air, etc.), biological fluids (e.g., blood) or molten polymeres containing macro–molecules. The study of the relation between a stress applied on a material and the resulting strains (deformation) and strain rates is called *rheology*.
There are broadly three reasons why a fluid may be non–Newtonian:

- the relation between the shear rate e_{ij} and the stress σ_{ij} is non–linear; for example, the effective viscosity η_{eff}, defined as the ratio of stress to shear rate, decreases when the shear rate increases (shampoo, wall paint, ketchup, etc.),
- the relation between the shear rate and the stress depends on time; for example, η_{eff} decreases with time, under constant stress (ketchup, cytoplasm, semen, etc.),
- the behavior is a mixture of viscous and elastic responses; the silicone *silly putty ball* is an example of such a material, as it spreads out like a liquid when left on a

where ϵ_i is the i component of the displacement vector due to the deformation. The quantity ϵ_{ij} gives the change in an element of length when the body is deformed. Therefore, in an elastic solid, the stress tensor is proportional to the strain tensor, whereas in a liquid it is proportional to the *rate* of strain tensor. In a solid body, internal stresses are due to forces of interaction between molecules which are displaced when the body is deformed. Within the theory of elasticity, the body recovers its original shape when the external applied force is removed. There is no dissipation of energy: mechanical energy is stored in the deformation and regained after the external force is removed.

table under constant stress, whereas it bounces elastically off the ground (i.e. when subject to a high stress).

Although these fluids are extremely important in a vast number of areas, we will only concentrate on Newtonian fluids thereafter.

2.2 Equation of motion for a fluid

We now write Newton's second law of motion for a fluid. This leads to the so–called *Navier–Stokes* or *Euler* equations depending on whether the fluid is viscous or not, respectively.

2.2.1 Navier–Stokes equation

We consider an arbitrary *fixed* volume V of the fluid delimited by a closed surface S, which momentum in the i–direction is:

$$\iiint_V \rho v_i \, \mathrm{d}V.$$

This momentum varies due to particles entering and leaving the volume (in other words, there is a flux of momentum advected by the fluid across the surface), and also because of forces exerted on the surface and on the volume itself. In the same way that the total mass leaving the volume V per unit time is the integral of $\rho \mathbf{v} \cdot \mathrm{d}\mathbf{S}$ over the surface (see eq. [1.19]), the i–component of the momentum advected by the fluid across the surface per unit time is the integral of $\rho v_i \mathbf{v} \cdot \mathrm{d}\mathbf{S}$ over the surface. Therefore, Newton's second law gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho v_i \, \mathrm{d}V = - \oiint_S \rho v_i \mathbf{v} \cdot \mathrm{d}\mathbf{S} + \oiint_S f_{\mathrm{surf},i} \, \mathrm{d}S + \iiint_V f_{\mathrm{vol},i} \, dV, \tag{2.16}$$

where $f_{\mathrm{surf},i}$ is the i–component of the force exerted on the surface per unit area and $f_{\mathrm{vol},i}$ is the i–component of the force exerted on the volume per unit volume. Viscous forces can be dealt with either by integrating the shear stress given by equation (2.1) over the surface, or by integrating f_{visc} given by equation (2.14) directly over the volume. Both integrals are identical, as can be shown by using the divergence theorem[3]. Here we use the force per unit volume, so that the contribution from the viscous force is:

$$\iiint_V \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V.$$

[3]For the tensor σ_{ij}, the divergence theorem can be written as:

$$\iiint_V \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V = \oiint_S \sigma_{ij} n_j \mathrm{d}S,$$

where n_j is the j–component of the unit vector normal to the surface. This surface integral is equal to $\oiint_S T_i \mathrm{d}S$ (see eq. [2.1]), where T_i is the i–component of the viscous force exterted by the fluid outside the volume element onto the surface.

Therefore the only force contributing to the surface integral is the pressure force, and we have $f_{\mathrm{surf},i}\,\mathrm{d}S = -p\,\mathrm{d}\mathbf{S}\cdot\hat{\mathbf{x}}_i$, where p is the pressure, $-p\,\mathrm{d}\mathbf{S}$ is the pressure force acting on the surface $\mathrm{d}S$, and $\hat{\mathbf{x}}_i$ is the unit vector along the i–axis.

As the volume is fixed, we can move the time–derivative inside the integral on the left–hand side of equation (2.16). By using the divergence theorem to transform the surface integrals on the right–hand side into volume integrals, we then obtain:

$$\iiint_V \frac{\partial}{\partial t}\left(\rho v_i\right)\mathrm{d}V = \\ -\iiint_V \boldsymbol{\nabla}\cdot\left(\rho v_i\mathbf{v}\right)\mathrm{d}V - \iiint_V \boldsymbol{\nabla}\cdot\left(p\hat{\mathbf{x}}_i\right)\mathrm{d}V + \iiint_V \frac{\partial\sigma_{ij}}{\partial x_j}\,dV - \iiint_V \rho g_i \mathrm{d}V, \quad (2.17)$$

where we have included the gravitational force acting on the volume in the last integral on the right–hand side, with g_i being the i–component of the acceleration due to gravity. The minus sign is due to the fact that we choose g_i to be positive. Other forces could be added as well. As this relation is satisfied for any volume V, we have:

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \boldsymbol{\nabla}\cdot\left(\rho v_i\mathbf{v}\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial\sigma_{ij}}{\partial x_j} - \rho g_i, \quad (2.18)$$

where we have used $\boldsymbol{\nabla}\cdot\left(p\hat{\mathbf{x}}_i\right) = \partial p/\partial x_i$.

We remark that this equation can also be written as:

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}\right) = -\rho g_i, \quad (2.19)$$

which makes it clear that the flux of the i component of the momentum in the j direction, $\rho v_i v_j + p\delta_{ij} - \sigma_{ij}$, has contributions from both advection (transport by the flow, $\rho v_i v_j$ term) and molecular transport (pressure and viscous forces, $p\delta_{ij} - \sigma_{ij}$ term).

The left–hand side of equation (2.18) can be written as:

$$v_i\left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\mathbf{v}\right)\right) + \rho\frac{\partial v_i}{\partial t} + \rho\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)v_i,$$

where the term in brackets is zero because of mass conservation (eq. [1.21]).
Equation (2.18) then becomes:

$$\boxed{\frac{\partial v_i}{\partial t} + \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)v_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial x_j} - g_i.} \quad (2.20)$$

We now consider the case when compressibility effects are negligible, so that σ_{ij} is given by equation (2.13). Making the sum over repeated indices explicit, we then obtain for the viscous force:

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \eta\sum_{j=1}^{3}\left[\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)\right] = \eta\nabla^2 v_i + \eta\frac{\partial}{\partial x_i}\left(\boldsymbol{\nabla}\cdot\mathbf{v}\right) = \eta\nabla^2 v_i,$$

where we have assumed that η does not depend on the space coordinates, which is valid in a homogeneous fluid.

Equation (2.20) can then be written in vectorial form as[4]:

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{v} + \mathbf{g}.} \tag{2.21}$$

where we have used $\nu = \eta/\rho$ and $\mathbf{g} = -g_i \hat{\mathbf{x}}_i$. This is the so–called *Navier–Stokes equation*, valid for an incompressible Newtonian fluid. It is a inhomogeneous non–linear partial differential equation which is first or second order depending on whether ν is zero or not, respectively.

Using equation (1.6), Navier–Stokes equation can also be written as:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{v} + \mathbf{g}. \tag{2.22}$$

This means that the Lagrangian acceleration is equal to the sum of the forces per unit mass. This equation could also have been obtained using a Lagrangian approach, that is to say by writing that the change of momentum of a fluid element moving with the fluid was equal to the sum of the forces exerted on that fluid element, as done in section 1.6.2 for mass conservation.

The mass conservation equation (1.21) and Navier–Stokes equation above provide four scalar equations for five unknowns, which are the three components of the velocity, pressure and density. If the flow is incompressible, we also have equation (1.25), so that the system of equations is close. However, when the flow is not incompressible, we have to add an energy equation, or an equation of state relating p and ρ. In most of the situations studied in these notes, the density ρ will be taken as a constant, so that the mass conservation equation will automatically be satisfied. The incompressibility and Navier–Stokes equations are then sufficient for determining the properties of the flow, assuming boundary conditions (see below).

In general, the non–linear term $(\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v}$ makes it impossible to derive an exact solution to the Navier-–Stokes equation, and one has to rely on solving it numerically. Only when the non–linear term is negligible, which happens for very low speed and/or very viscous flows (see below), can an exact solution be found.

2.2.2 Reynolds number

As mentioned above, the flux of momentum is due to both advection and molecular transport, the latter manifesting itself through pressure and viscous forces. Here, we compare

[4] In cartesian coordinates, the components of the vector $\nabla^2 \mathbf{v}$ are $\nabla^2 v_x$, $\nabla^2 v_y$ and $\nabla^2 v_z$. That is to say, the definition of $\nabla^2 \mathbf{v}$ makes *explicit* reference to *cartesian* coordinates. It follows that, in cylindrical coordinates for example, as the unit vectors depend on the coordinates, the components of $\nabla^2 \mathbf{v}$ are *not* $\nabla^2 v_{r,\theta,z}$.

the advection, also called inertial, and viscous terms. Using equation (2.21), we see that:

$$\frac{\text{inertial term}}{\text{viscous term}} = \frac{|(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}|}{|\nu\nabla^2\mathbf{v}|}.$$

If U is a typical velocity of the flow and L is a characteristic lengthscale, then a spatial derivative of a component of $\mathbf{v}$ is on the order of U/L. We then obtain the approximate relation:

$$\frac{\text{inertial term}}{\text{viscous term}} \sim \frac{U^2/L}{\nu U/L^2}. \tag{2.23}$$

This ratio is called the *Reynolds number* Re and we therefore have:

$$\boxed{Re = \frac{UL}{\nu}.} \tag{2.24}$$

Note that Re can also be interpreted as the ratio of timescales. If advection is the only source of momentum transport, then $\partial\mathbf{v}/\partial t = (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$, so that the timescale for advection over a lengthscale L is $\tau_{\text{adv}} \sim L/U$. In the opposite case, when viscosity is the only source of transport, we have the diffusion equation $\partial\mathbf{v}/\partial t = \nu\nabla^2\mathbf{v}$, and the timescale for diffusion over a lengthscale L is $\tau_{\text{dif}} \sim L^2/\nu$. Therefore, $Re \sim \tau_{\text{dif}}/\tau_{\text{adv}}$.
Large Reynolds numbers correspond to flows where the advection term is dominant over the viscous term, or equivalently where the advection time is much smaller than the viscous time. Viscous effects in that case are usually negligible. However, they may still play a role in a *boundary layer* near, for example, a solid surface, where velocity gradients are very large.

It is interesting to relate the Reynolds number to the correlation coefficient between the components of the fluctuating velocity in the flow. Using equations (2.5) and (2.24), with $U \sim v_x$, we can write $Re = \mathcal{M}L/\lambda$, and therefore $\mathcal{C} = \mathcal{M}^2/Re$, where $\mathcal{C}$ is given by equation (2.10). So large Reynolds numbers correspond to a small correlation between the components of the fluctuating velocity. This is a consequence of the small value of the ratio of the mean free path to the scale of the mean flow and it means the state of the flow is not affected by the molecular transport of angular momentum. In other words, the flow and the molecular transport are completely decoupled.

2.2.3 Dimensional analysis and similarity



(Model flight in windtunnel, NASA)

Before computers were routinely used to assist design and manufacture of aircrafts, wind tunnels were a common tool to test an object in flight. Instead of the object actually flying through standing air, the air is made to move past the object which stays at rest. Although some wind tunnels were large enough to test full–size aircrafts, scale models were commonly used for tests to be carried out in smaller tunnels.

The important question is then: how to make sure that the scale model is *similar to* (i.e., has the same properties as) the prototype to be tested? First, the scale model has to be *geometrically similar* to the prototype, that is to say they can be transformed into one another by changing all linear dimensions in the same ratio.

We are now going to obtain the other conditions by first rewriting Navier–Stokes equation (2.21) under the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{v} - g\hat{\mathbf{z}}. \tag{2.25}$$

where $\hat{\mathbf{z}}$ is the unit vector in the vertical direction. Through the boundary conditions, the flow depends on the shape and velocity of a body moving through the fluid. The above equation is valid for an incompressible fluid, and we further assume that ρ is uniform. We introduce the typical velocity U of the flow, its characteristic lengthscale L (which is that of the embedded body when there is one present), a characteristic timescale T and a characteristic scale of pressure variation ΔP (the variation being measured from the hydrostatic pressure). The inertial force[5] $(\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v}$, pressure force $\boldsymbol{\nabla} p/\rho$ and viscous force $\nu \nabla^2 \mathbf{v}$ per unit mass are on the order of U^2/L, $\Delta P/(\rho L)$ and $\nu U/L^2$, respectively. We also define the following dimensionless variables:

$$\widetilde{\mathbf{r}} = \frac{\mathbf{r}}{L}, \quad \widetilde{\mathbf{v}} = \frac{\mathbf{v}}{U}, \quad \widetilde{t} = \frac{t}{T}, \quad \widetilde{p} = \frac{p}{\Delta P}. \tag{2.26}$$

Substituting these variables into equation (2.25) then yields:

$$\frac{L}{UT}\frac{\partial \widetilde{\mathbf{v}}}{\partial \widetilde{t}} + \left(\widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\nabla}}\right) \widetilde{\mathbf{v}} = -\frac{\Delta P}{\rho U^2}\widetilde{\boldsymbol{\nabla}}\widetilde{p} + \frac{\nu}{UL}\widetilde{\nabla}^2 \widetilde{\mathbf{v}} - \frac{gL}{U^2}\hat{\mathbf{z}}, \tag{2.27}$$

where $\widetilde{\boldsymbol{\nabla}} = L\boldsymbol{\nabla}$. Since L and T are the characteristic scales, the dimensionless time and space derivatives are of order unity. This equation reveals the following dimensionless numbers:

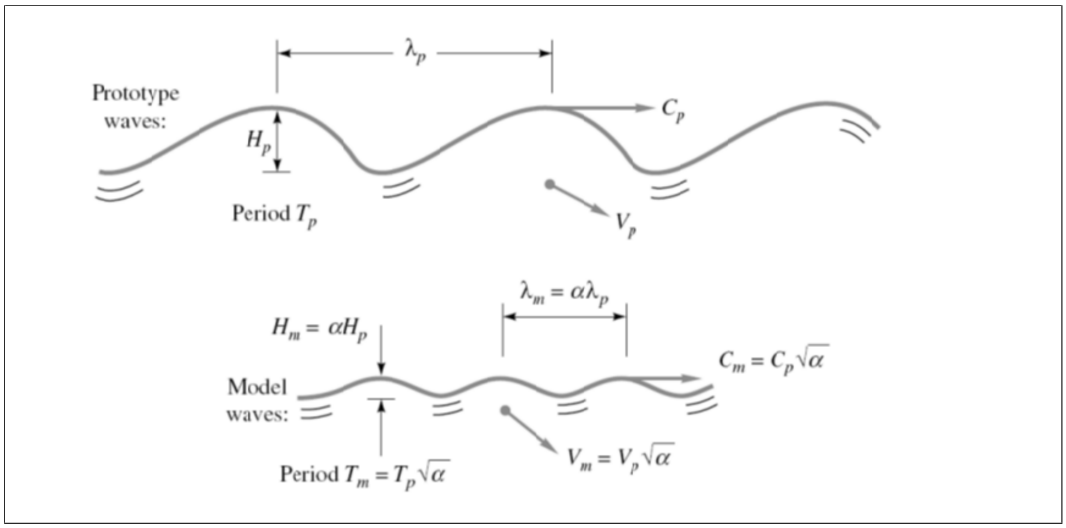
- Reynolds number $Re = UL/\nu$: ratio of the inertial force U^2/L to the viscous force $\nu U/L^2$ (already introduced above),
- Froude number (squared) $Fr^2 = U^2/(gL)$: ratio of the inertial force U^2/L to the gravitational force g,
- Euler number, or pressure coefficient, $Eu = \Delta P/(\rho U^2)$: ratio of the pressure force $\Delta P/(\rho L)$ to the inertial force U^2/L,
- Strouhal number $St = L/(UT)$: ratio of the local acceleration U/T to the inertial force U^2/L.

[5] The term $\rho\,(\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v}$ is not a physical force, but it has the dimensions of a force per unit volume. In steady state, it is the force that has to be exerted on a unit volume of the fluid to bring it into motion at the velocity v, counteracting its *inertia*, or, equivalently, to bring it to rest if it has this velocity.

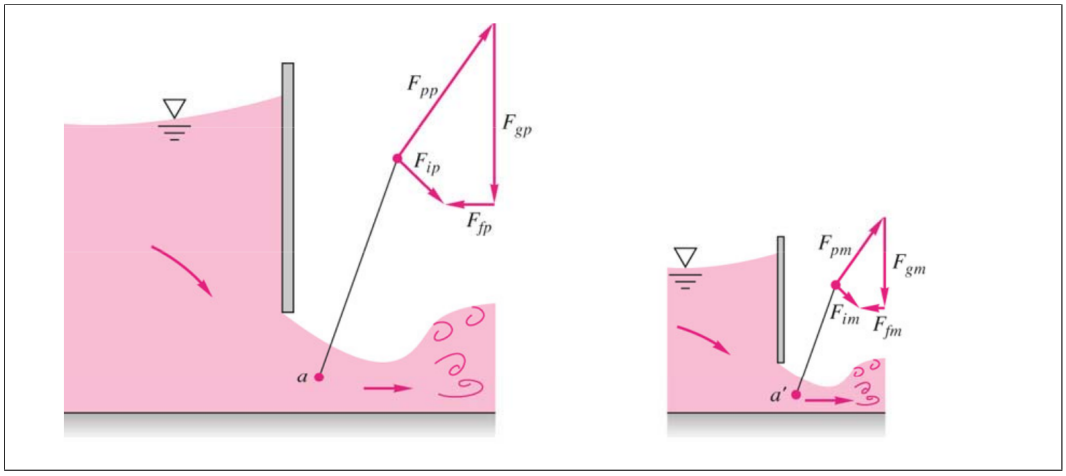
Equation (2.27) can be written in terms of these dimensionless numbers:

$$St\frac{\partial\widetilde{\mathbf{v}}}{\partial\widetilde{t}}+\left(\widetilde{\mathbf{v}}\cdot\widetilde{\boldsymbol{\nabla}}\right)\widetilde{\mathbf{v}}=-Eu\widetilde{\boldsymbol{\nabla}}\widetilde{p}+\frac{1}{Re}\widetilde{\nabla}^2\widetilde{\mathbf{v}}-\frac{1}{Fr^2}\hat{\mathbf{z}}. \tag{2.28}$$

This equation shows that, in two different flows with different viscosities moving past objects with different dimensions, the dimensionless velocities $\widetilde{\mathbf{v}}=\mathbf{v}/U$ are the *same functions* of $\widetilde{\mathbf{r}}=\mathbf{r}/L$ and $\widetilde{t}=t/T$ as long as the numbers Re, Fr, Eu and St are the same. Two such flows are called *similar*, as they can be transformed into one another by scaling the lengthscale, velocity, time and pressure.
This scaling ensures *kinematic similarity*, as illustrated on the figure below in the case of a model for waves:



Keeping the ratio of the forces constant ensures *dynamic similarity*, as shown on the figure below for a sluice gate flow in which the prototype and the model have identical Reynolds and Froude numbers:



2.2.4 Incompressibility revisited

Here we consider a steady flow (very small St) with high Reynolds number Re. Ignoring gravity and using the fact that $\left|\left(\widetilde{\mathbf{v}}\cdot\widetilde{\boldsymbol{\nabla}}\right)\widetilde{\mathbf{v}}\right|$ is on the order of unity, equation (2.28) indicates that $Eu\sim 1$, that is to say $\rho U^2\sim\Delta P$ (this is because acceleration in such a fluid is driven by pressure gradients). The pressure variation ΔP within a fluid element induces a density variation $\Delta\rho$ of this fluid element, and these are associated with sound waves propagating through the fluid with the velocity c_s such that $c_s^2\sim\Delta P/\Delta\rho$, as will be shown in a later chapter. Therefore $\rho U^2\sim\Delta P$ implies $\rho U^2\sim c_s^2\Delta\rho$ or, equivalently:

$$\frac{\Delta\rho}{\rho}\sim\mathcal{M}^2, \tag{2.29}$$

where $\mathcal{M}\equiv U/c_s$ is the Mach number.

A fluid is incompressible if the density of an element within it does not change when subject to pressure gradients. Therefore, compressibility can be neglected if $\Delta\rho/\rho\ll 1$, that is to say if $\mathcal{M}\ll 1$. In other words, *flows with velocities low compared to the sound speed in the fluid behave as if the fluid were incompressible.* This is because any surdensity created by compression of a fluid element is transported away by sound waves much more rapidly than the time it takes the fluid element to move. To a good approximation, the fluid element therefore retains a constant density as it moves[6].

[6]If a fluid element does not exchange heat with its surrounding, that is to say if the flow is *isentropic*, which is the case when viscosity can be ignored, then a pressure variation Δp results in a density variation $\Delta\rho$ such that $\Delta p/p=\gamma\Delta\rho/\rho$, with γ being the adiabatic index, or ratio of heat capacity at constant pressure to heat capacity at constant volume. Therefore, an incompressible flow corresponds to $\gamma\to\infty$.

Typically, *air is approximated to be incompressible for Mach numbers below 0.3*, for which the relative change in density is smaller than about 10%.

2.2.5 Euler equation for an inviscid fluid

Navier–Stokes equation is the equation of motion for an incompressible Newtonian (viscous) fluid. If the viscosity is zero, it reduces to *Euler equation*:

$$\boxed{\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} = -\boldsymbol{\nabla}p + \rho\mathbf{g},} \tag{2.30}$$

which is valid even if the fluid is compressible. A fluid with no viscosity is called *inviscid*. A fluid which is both inviscid and incompressible is called *ideal*, or *perfect*. It is usually an approximation, as the only ideal fluid is superfluid liquid helium: at a temperature of 2.172 K, the viscosity of He^4 vanishes (He^3 also becomes a superfluid but at a much smaller temperature).

2.3 Boundary conditions

To calculate the velocity field $\mathbf{v}(\mathbf{r},\mathbf{t})$ of a fluid in motion, Navier–Stokes (or Euler) equation has to be solved, and this requires to specify both the initial conditions and boundary conditions.

2.3.1 Rigid boundary

Here we focus on a fluid which is in contact with a solid surface.

No–penetration condition:

A fluid which moves past a solid object cannot penetrate the object, and therefore the component of the *relative* velocity perpendicular to the surface of the object is zero. In other words, the normal component $v_{n,\text{fluid}}$ of the fluid velocity at the boundary has to be equal to the normal component of the velocity of the boundary itself:

$$v_{n,\text{fluid}} = v_{n,\text{boundary}}. \tag{2.31}$$

If the boundary is at rest, then $v_{n,\text{fluid}} = 0$ there.
The above condition implies that streamlines lie on the surface of rigid boundaries and, more precisely, *in two dimensions, rigid boundaries are streamlines.*

Note that the no–penetration condition only applies if the boundary is *impermeable*. If it is *porous*, fluid can be injected into or sucked from the volume at the boundary.

No–slip condition for a viscous fluid:

If the fluid is inviscid, it generally *slips* parallel to the boundary and there is no constraint on the tangential component of the fluid velocity.

In a viscous fluid however, it is an observational fact that, in normal conditions, the *relative* velocity parallel to the surface of the object is zero. That is to say, the tangential component $v_{t,\text{fluid}}$ of the fluid velocity at the boundary is equal to the tangential component of the velocity of the boundary itself:

$$v_{t,\text{fluid}} = v_{t,\text{boundary}}. \tag{2.32}$$

This implies $\mathbf{v}_{\text{fluid}} = \mathbf{v}_{\text{boundary}}$ for a viscous fluid. This is called the *no–slip* condition, and it *holds however small the viscosity* may be. It can be understood by noting that any discontinuity of the velocity at the boundary would lead to a very large stress (as the stress depends on the gradient of the velocity) which would act to eliminate the discontinuity. In other words, molecular transport at the surface prevents any slipping of the fluid parallel to the boundary.

This condition was debated starting in the 19^{th} century, as it was realized that molecular interactions at the boundary may not lead to a stress similar to that in the interior of the fluid. Although the no–slip condition is supported by observations at macroscopic scales for simple fluids, slip over a rigid surface does occur in some cases. For example, complex fluids like polymer melts slip over solid boundaries, in a complicated process driven by the entanglement of the molecules. Boundary slip may also occur at microscopic scales (smaller than a micron), and is therefore important in microfluidics, which deals with flow of liquids through micrometer-sized channels.

In the rest of these notes, we will only deal with simple fluids at macroscopic scales and will therefore assume that the no–slip condition holds for a viscous fluid.

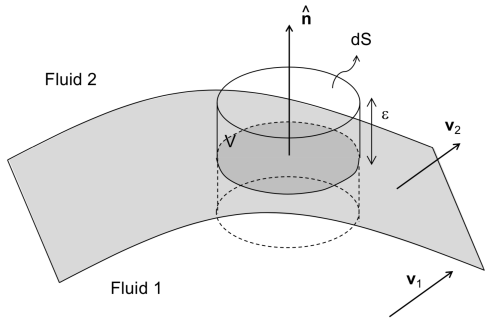
This no–slip condition has important consequences for fluids with very small viscosity (like the air) moving past an aerofoil (for example, cross-sectional shape of a wing). If the viscosity of the fluid is neglected, then the flow is found to slip over the surface. However, if the no–slip condition is enforced, the velocity varies very rapidly near the surface of the aerofoil to adjust to the boundary condition. This creates a *large stress* according to equation (2.13), *even if the viscosity is very small*, and results in a *boundary layer* in which the structure of the flow is completely different than in the inviscid case. Therefore, even though the viscosity can be neglected in the interior of the fluid, it cannot be ignored near the boundary. Boundary layers will be discussed later in these notes.

2.3.2 Interface between two fluids

We now examine the boundary conditions when the fluid is in contact with another fluid, rather than a solid surface.

Continuity of the velocity:

When there is no mass transfer between the two fluids, conservation of mass implies that the normal component of the velocity is continuous across the interface. (Mass transfer may occur, e.g., at a liquid-gas interface when the liquid evaporates.)



This can be seen by writing the mass conservation equation (1.22) for the small volume V represented on the figure and spanning the interface, where both the volume and the interface are moving with the flow :

$$\iiint_V \frac{\mathrm{D}\rho}{\mathrm{D}t}\,\mathrm{d}V + \iiint_V \rho\boldsymbol{\nabla}\cdot\mathbf{v}\,\mathrm{d}V = 0.$$

We take the limit $\epsilon \to 0$. Then the first term on the left–hand–side vanishes. The second term can be transformed into a surface integral using the divergence theorem, so that the equation becomes:

$$\oiint_S \rho\mathbf{v}\cdot\mathrm{d}\mathbf{S} = 0,$$

where S is the surface that delimits the volume V. In the limit $\epsilon \to 0$, the lateral surface does not contribute to the integral and we obtain the so–called *kinematic boundary condition*:

$$\mathbf{v}_2\cdot\hat{\mathbf{n}} - \mathbf{v}_1\cdot\hat{\mathbf{n}} = 0, \tag{2.33}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the interface and the minus sign is due to the fact that $\mathrm{d}\mathbf{S}$ in the integral has to be oriented outwards.
As in the case of a rigid boundary, the above condition implies that, *in two dimensions, the interface between two fluids is a streamline.*

If the equation of the interface is given by $z = f(x, y, t)$, then the kinematic boundary condition can be formulated in the following way:

$$v_{i,z} = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_{i,x}\frac{\partial f}{\partial x} + v_{i,y}\frac{\partial f}{\partial y}, \tag{2.34}$$

where $v_{i,x}$, $v_{i,y}$ and $v_{i,z}$ are the components of the velocity in fluid i, for $i = 1, 2$. To show that this is equivalent to the boundary condition written above, we assume that the interface depends only on one coordinate, x for example. Then $\mathbf{n} = (-\mathrm{d}f/\mathrm{d}x, 1)$ is a vector normal to the interface at x. Therefore, $\mathbf{v}_i\cdot\mathbf{n} = -v_{i,x}\,(\mathrm{d}f/\mathrm{d}x) + v_{i,z}$, and this is

equal to $\partial f/\partial t$, so that $\mathbf{v}_1 \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n}$, which is equivalent to the boundary condition (2.33).

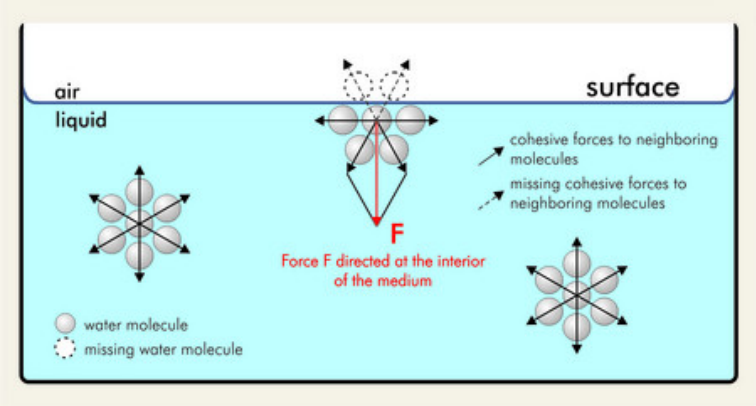
The no–slip condition described in the previous subsection in the case of an interface between a viscous fluid and a solid applies as well when the interface is between two viscous fluids. This implies that the tangential component of the velocity is also continuous across the interface. Therefore, *for viscous fluids*, the velocity is continuous across the interface, and equal to the velocity of the interface itself:

$$\mathbf{v}_{\text{fluid 1}} = \mathbf{v}_{\text{fluid 2}} = \mathbf{v}_{\text{interface}}. \tag{2.35}$$

Surface tension:

Before discussing the other boundary conditions at the interface between two fluids, we introduce the concept of surface tension, which is an important parameter for describing the surface of a fluid or the interface between two fluids.

Within the volume of a fluid, a molecule is subject to attractive forces (e.g., hydrogen, ionic or metallic bonds, van der Waals forces) from neighboring molecules in all directions, so that the total force is zero. At the surface though, forces from molecules in the fluid are not balanced and there is a net force pointing towards the interior of the fluid, as illustrated on the figure below:



(Credit: www.sita–process.com)

This force opposes any increase of the surface area. The *surface tension*, noted γ, is defined as the work per unit area that has to be done to increase the surface. For example, if the surface is a rectangle with length L, the energy that has to be supplied to stretch its width by dl is $dW = \gamma dS$, with $dS = Ldl$. This can also be written as $dW = Fdl$, where F is the force that has to be exerted to stretch the surface (and which is the opposite of the force exerted by the surface). Therefore, $F = \gamma L$, so that γ can also be defined as the force per unit length exerted by the surface. The units of the surface tension are N/m.

The surface tension refers to a fluid which surface is in contact with vacuum. If the surface is in contact with another fluid, the force exerted by this other fluid onto the

surface modifies the surface tension and we talk about *interfacial tension*[7]. In the case represented above where the interface is between a liquid and air, γ is positive. However, γ may become negative when the interface is between two fluids or between a fluid and a solid.

The effect of surface tension is to minimize the area of the interface, given of course all the constraints that the interface is subject to. This is why soap bubbles or raindrops, for which gravity is small, have a spherical shape: for a given volume, a sphere minimizes the surface.

Continuity of the stress:

We now write Navier–Stokes equation (2.22) in an integral form for the small volume element V represented on the figure used above for deriving the continuity of the velocity:

$$\iiint_V \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\,\mathrm{d}V = \oiint_S \mathbf{f}_{\mathrm{surf}}\,\mathrm{d}S + \iiint_V \mathbf{f}_{\mathrm{vol}}\,dV,$$

where $\mathbf{f}_{\mathrm{surf}}$ and $\mathbf{f}_{\mathrm{vol}}$ are the surface force per unit area and body force per unit volume (e.g., gravity) exerted on the fluid element, respectively. As pointed out in section 2.1.3, surface forces like viscous forces result in a net force on the volume of a fluid element, and can therefore be written as volume forces as was done when deriving Navier–Stokes equation. However, when the volume considered tends to zero, as here, viscous forces, like pressure forces, have to be treated as surface forces. Therefore, $\mathbf{f}_{\mathrm{surf}}$ has contribution from viscous and pressure forces. When $\epsilon \to 0$, the left–hand–side of the equation above vanishes. The first term on the right–hand–side gives $\mathbf{f}_{\mathrm{surf},1}\,\mathrm{d}S + \mathbf{f}_{\mathrm{surf},2}\,\mathrm{d}S$, where $\mathbf{f}_{\mathrm{surf},1}$ ($\mathbf{f}_{\mathrm{surf},2}$) is the force exerted by fluid 1 (fluid 2) on the area $\mathrm{d}S$ of the interface. Finally, the second term on the right–hand–side reduces to $\mathbf{f}_{\mathrm{tens}}\,\mathrm{d}S$, where $\mathbf{f}_{\mathrm{tens}}$ is the surface tension force at the interface. Therefore, we obtain at the interface:

$$\mathbf{f}_{\mathrm{surf},1} + \mathbf{f}_{\mathrm{surf},2} + \mathbf{f}_{\mathrm{tens}} = \mathbf{0}. \tag{2.36}$$

If the surface tension is negligible, this condition implies that *the total stress (including pressure) is continuous across the interface.* Projected onto the i–axis, this gives:

$$\left(\sigma_{ij}^{(2)} - p^{(2)}\delta_{ij}\right) n_j - \left(\sigma_{ij}^{(1)} - p^{(1)}\delta_{ij}\right) n_j = 0, \tag{2.37}$$

where $\sigma_{ij}^{(1)}$ and $p^{(1)}$ are the stress tensor and pressure at the interface in fluid 1, and $\sigma_{ij}^{(2)}$ and $p^{(2)}$ are the same quantities at the interface in fluid 2. Here we have used the

[7] Interfacial tension also occurs when the fluid is in contact with a solid. This leads to *capillarity*, which is the process by which a liquid in contact with a solid can rise or fall along the solid. Whether the liquid rises or falls depends on the balance between the forces of *adhesion* (interaction between the molecules in the liquid and that in the solid), *cohesion* (interaction between the molecules within the liquid) and any other force present (e.g., gravity). The same competition between adhesion and cohesion explains why in some circumstances (e.g., *hydrophobic* surface) the no–slip boundary condition does not apply.

expression (2.1) of the stress, and the minus sign in front of the second brackets comes about because the normal $\hat{\mathbf{n}}$ is oriented from fluid 1 to fluid 2.

The above condition implies that the *tangential* viscous stress is continuous across the interface, as pressure acts only perpendicularly to the interface.

If both fluids 1 and 2 are inviscid, and surface tension is negligible, there is no tangential stress and equation (2.37) implies that *pressure is continuous* across the interface.

2.3.3 Free surface

When the interface is between a liquid and a gas, it is referred to as a *free surface*. We treat it separately because it is an important particular case.

Kinematic boundary condition:

The kinematic condition (2.33) is still valid but, here again, it can be formulated in a more useful way by using the equation of the surface. For example, if this is given by $z = f(x, y, t)$, then we have at the surface:

$$v_z = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y}, \tag{2.38}$$

and this is the kinematic boundary condition. If the free surface is stationary and depends only on one coordinate, x for example, then $v_z/v_x = \mathrm{d}f/\mathrm{d}x$, which means that the slope of the streamline at the point $z = f(x)$ is equal to the slope of the free surface, which is expected as the free surface *is* a streamline. This is equivalent to the statement that the component of the velocity normal to the boundary is continuous.

Continuity of the stress:

If surface tension is negligible, the continuity of the stress at the interface, expressed by equation (2.37), is also still valid, and this implies in particular that the tangential viscous stress is continuous. Therefore, given that the viscosity of gases is usually very small, *the tangential viscous stress is zero at a free surface with no surface tension.*

In addition, if the fluid is inviscid, continuity of the perpendicular component of the stress implies that, *at the free surface, the pressure force is equal and opposite to the surface tension force.*

2.4 The vorticity equation and Kelvin's theorem

The vorticity equation expresses the conservation of angular momentum in a fluid. In this section, we write the gravitational acceleration as the gradient of a potential, $\mathbf{g} = -\boldsymbol{\nabla}\chi$. The results presented here are valid when other external forces are present, as long as they are conservative, that is to say derivable from a potential. We also assume that the fluid is incompressible and that the density ρ is constant through the fluid, although the results would also be valid if ρ were not constant but a function of pressure only.

2.4.1 The vorticity equation for an incompressible viscous fluid

With the assumptions listed above, Navier–Stokes equation (2.21) becomes:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \chi\right) + \nu\nabla^2\mathbf{v}. \tag{2.39}$$

We now use the vector identity:

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = (\boldsymbol{\nabla}\times\mathbf{v})\times\mathbf{v} + \boldsymbol{\nabla}\left(\frac{1}{2}v^2\right),$$

to transform the equation above into:

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega}\times\mathbf{v} = -\boldsymbol{\nabla}\left(\frac{1}{2}v^2 + \frac{p}{\rho} + \chi\right) + \nu\nabla^2\mathbf{v}, \tag{2.40}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$ is the vorticity. Taking the curl of this equation then yields:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{v}) = \boldsymbol{\nabla}\times\left(\nu\nabla^2\mathbf{v}\right), \tag{2.41}$$

as the curl of a gradient is zero. We now use the vector identity:

$$\boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{v}) = (\mathbf{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\mathbf{v} + \boldsymbol{\omega}\,(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathbf{v}\,(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}),$$

togeter with $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$ (the div of a curl is zero) and $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ (incompressibility), and also[8] $\boldsymbol{\nabla}\times\left(\nabla^2\mathbf{v}\right) = \boldsymbol{\nabla}^2\boldsymbol{\omega}$, to rewrite equation (2.41) as:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}, \tag{2.42}$$

which is equivalent to the so–called *vorticity equation*:

$$\boxed{\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}}. \tag{2.43}$$

This is an equation which involves $\mathbf{v}$ and $\boldsymbol{\omega}$ only, which are themselves related. It can be used instead of the Navier–Stokes equation to calculate the flow velocity, the advantage being that it does not depend on the pressure.

This equation describes the transport of vorticity in an incompressible fluid in which the external force is derivable from a potential. The term on the left–hand–side is the rate of change of vorticity in a unit volume element moving with the flow. Contributions come from $\nu\nabla^2\boldsymbol{\omega}$, which represents the rate of change due to molecular diffusion of vorticity (in the same way that $\nu\nabla^2\mathbf{v}$ represents the rate of change of momentum due to the diffusion of momentum), and from $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\mathbf{v}$, which produces vortex stretching and twisting. The process of twisting generates vorticity in a direction from a vorticity which is originally

[8] By definition of ∇^2, we have the identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nabla^2\mathbf{v}$. In an incompressible fluid, $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, so that $\nabla^2\mathbf{v} = -\boldsymbol{\nabla}\times\boldsymbol{\omega}$. Therefore, $\boldsymbol{\nabla}\times\left(\nabla^2\mathbf{v}\right) = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\omega}) = -\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}) + \nabla^2\boldsymbol{\omega}$. The div of a curl is zero, so that $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$, which yields $\boldsymbol{\nabla}\times\left(\nabla^2\mathbf{v}\right) = \nabla^2\boldsymbol{\omega}$.

in another direction. The process of stretching increases the magnitude of the vorticity if the fluid element gets elongated in the direction of its vorticity.
As can be seen from the calculation above, non–conservative forces, if present, are a source of vorticity. This is the case in particular of the Coriolis force, which appears when we write Navier–Stokes equation in a rotating frame, and which is very important in atmospheric physics. The Coriolis force, which manifests itself only over large scales (hundred or thousands of kilometers), generates hurricanes but does not affect tornadoes.

2.4.2 Case of an ideal fluid and Kelvin's theorem

When the fluid is inviscid ($\nu = 0$), the vorticity equation becomes:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\,\mathbf{v}. \tag{2.44}$$

This shows that, *if the vorticity is zero initially, then it remains zero at all times.*

This result can also be obtained by considering the circulation of the flow velocity along a contour C:

$$\Gamma = \oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{l}. \tag{2.45}$$

The rate of change of the circulation as the contour is moving with the flow is given by:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_C \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} \cdot \mathrm{d}\mathbf{l} + \oint_C \mathbf{v} \cdot \frac{\mathrm{D}\,(\mathrm{d}\mathbf{l})}{\mathrm{D}t}. \tag{2.46}$$

The first integral on the right–hand–side can be calculated using equation (2.39) with $\nu = 0$, which yields:

$$\oint_C \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} \cdot \mathrm{d}\mathbf{l} = -\oint_C \boldsymbol{\nabla}\left(\frac{p}{\rho} + \chi\right) \cdot \mathrm{d}\mathbf{l} = 0.$$

The second integral on the right–hand–side can be written as[9]:

$$\oint_C \mathbf{v} \cdot \frac{\mathrm{D}\,(\mathrm{d}\mathbf{l})}{\mathrm{D}t} = \oint_C \boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) \cdot \mathrm{d}\mathbf{l} = 0.$$

[9] We note $\mathrm{d}\mathbf{l}(t) = \overrightarrow{AB}$ where, at time t, the points A and B are at locations $\mathbf{r}$ and $\mathbf{r} + \mathrm{d}\mathbf{l}$, respectively. At time $t + \mathrm{d}t$, A has moved to A' such that $\overrightarrow{AA'} = \mathbf{v}(\mathbf{r})\mathrm{d}t$ and B has moved to B' such that

$$\overrightarrow{BB'} = \mathbf{v}(\mathbf{r} + \mathrm{d}\mathbf{l})\mathrm{d}t = \mathbf{v}(\mathbf{r})\mathrm{d}t + \frac{\partial \mathbf{v}}{\partial x_j}(\mathbf{r})\mathrm{d}l_j\mathrm{d}t.$$

We have

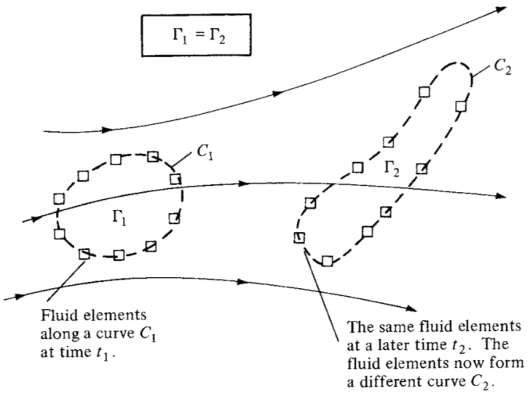
$$\overrightarrow{A'B'} = \mathrm{d}\mathbf{l}(t + \mathrm{d}t) = \mathrm{d}\mathbf{l}(t) + \frac{\mathrm{d}(\mathrm{d}\mathbf{l})}{\mathrm{d}t}(t)\mathrm{d}t,$$

with $\mathrm{d}(\mathrm{d}\mathbf{l})/\mathrm{d}t \equiv \mathrm{D}(\mathrm{d}\mathbf{l})/\mathrm{D}t$. Using $\overrightarrow{A'B'} = \overrightarrow{A'A} + \overrightarrow{AB} + \overrightarrow{BB'}$, we then get $\mathrm{D}(\mathrm{d}\mathbf{l})/\mathrm{D}t = (\partial\mathbf{v}/\partial x_j)\,\mathrm{d}l_j$, so that

$$\mathbf{v} \cdot \frac{\mathrm{D}(\mathrm{d}\mathbf{l})}{\mathrm{D}t} = \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial x_j}\mathrm{d}l_j = \boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) \cdot \mathrm{d}\mathbf{l}.$$

This yields *Kelvin's theorem*:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\left(\oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{l}\right)=0.} \tag{2.47}$$



This means that *the circulation along a closed contour C remains constant as the contour moves with the flow*, as illustrated on the figure. Kelvin's theorem is valid whether the flow domain is simply connected or not.

Using Stokes's theorem, we can also write:

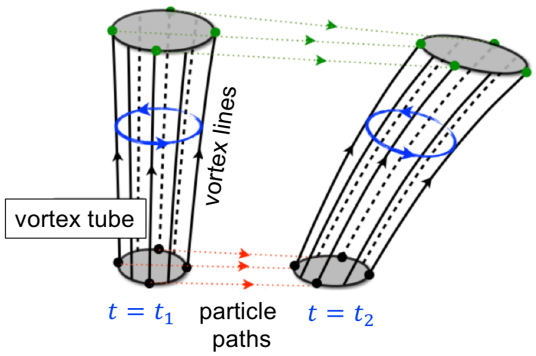
$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\left(\iint_S(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{S}\right), \tag{2.48}$$

with S being the surface delimited by C, and where we have assumed that the vorticity $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\mathbf{v}$ is defined everywhere inside C. Kelvin's theorem then implies:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\left(\iint_S \omega\cdot\mathrm{d}\mathbf{S}\right)=0.} \tag{2.49}$$

Therefore, *the flux of vorticity through a surface delimited by a contour C remains constant as the contour moves with the flow.* In particular, if the flux of the vorticity is zero initially, it remains so at all times. Since the contour C can be taken arbitrarily small, this implies that *if the vorticity is zero initially, it remains so at all times*, which is the result obtained above already.

Note that equation (2.49) is not in contradiction with equation (2.43), which shows that vorticity can be produced by stretching a fluid elements. Indeed, changes in the magnitude of the vorticity are compensated for by changes in the area so that the integral above stays constant as the element moves.



The figure illustrates Kelvin's theorem. A vortex line associated with a fluid element is a line that points in the direction of $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$, and it is therefore perpendicular to the streamline associated with that fluid element. A vortex tube is a bundle of vortex lines. A consequence of Kelvin's theorem is that the vortex lines/tubes move with the fluid.

A vortex line attached to some fluid element remains attached to that fluid element as it moves with the flow. We say that *the vorticity moves with the fluid*. This is Helmholtz's theorem.

A consequence of equation (2.49) is that *a vortex line cannot end in the fluid*: it either terminates at the boundaries or forms a closed curve. (If it ended in the fluid, there would be no flux of vorticity through the surface delimited by a closed contour, so the flux of vorticity would not be constant.)

We now examine the particular case of a two–dimensional flow in which $\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$, with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ being the unit vectors in the x and y directions, respectively, and where v_x and v_y do not depend on z. Then $\boldsymbol{\omega}$ is in the z–direction and $(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\,\mathbf{v} = \omega\,(\partial \mathbf{v}/\partial z) = 0$. Therefore

$$\boxed{\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \mathbf{0} \ \text{ for a 2D ideal flow}}, \tag{2.50}$$

assuming that it is subject to a conservative body force and it has a density ρ which is constant throughout. This means that *the vorticity of a fluid element is conserved* as the element moves with the flow. If the flow is steady, then the vorticity equation becomes $(\mathbf{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{\omega} = \mathbf{0}$, which means that the vorticity is conserved along streamlines.

2.5 Conservation of energy and Bernoulli's theorem

In the same way that the equation of motion in Mechanics leads to an equation expressing conservation of energy, the Navier–Stokes equation can be re–written to express conservation of energy in a moving fluid. In an ideal fluid, this leads to the so–called Bernoulli's theorem.

2.5.1 Conservation of energy in an incompressible Newtonian fluid

The kinetic energy per unit volume is $\rho v^2/2 \equiv \rho \mathbf{v} \cdot \mathbf{v}/2$, and its Eulerian rate of change is:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) = \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{v^2}{2}\frac{\partial \rho}{\partial t} \equiv \rho v_i \frac{\partial v_i}{\partial t} + \frac{v^2}{2}\frac{\partial \rho}{\partial t}.$$

Substituting $\rho \partial v_i/\partial t$ using equation (2.20), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) = -\frac{1}{2}\rho v_j \frac{\partial v^2}{\partial x_j} - v_i \frac{\partial p}{\partial x_i} + v_i \frac{\partial \sigma_{ij}}{\partial x_j} - \rho v_i g_i + \frac{v^2}{2}\frac{\partial \rho}{\partial t}. \tag{2.51}$$

Using the fact that the fluid is incompressible, so that $\partial v_i/\partial x_i = 0$, and the mass conservation equation $\partial\rho/\partial t = -v_i \partial\rho/\partial x_i$, the equation above can be written as:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) = -\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) - \frac{\partial}{\partial x_i}\left(v_i p\right) + \frac{\partial}{\partial x_j}\left(v_i \sigma_{ij}\right) - \sigma_{ij}\frac{\partial v_i}{\partial x_j} - \rho v_i g_i. \tag{2.52}$$

Each of the first three terms on the right–hand–side is a divergence. We now integrate this equation over a fixed volume V and use the divergence theorem to transform these three terms into a surface integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \frac{\rho v^2}{2}\,\mathrm{d}V = -\oiint_S \frac{1}{2}\rho v^2 \mathbf{v}\cdot\mathrm{d}\mathbf{S} - \oiint_S p\mathbf{v}\cdot\mathrm{d}\mathbf{S} + \oiint_S v_i\sigma_{ij}n_j dS + \iiint_V \mathbf{f}_{\mathrm{vol}}\cdot\mathbf{v}\,\mathrm{d}V - \iiint_V \sigma_{ij}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}V, \tag{2.53}$$

where $\mathbf{f}_{\mathrm{vol}} \equiv \rho\mathbf{g}$ is the gravitational force (to which we may add any other body force acting on the fluid) and n_j is the j–component of the unit vector $\hat{\mathbf{n}}$ normal to the surface. Since the volume is fixed, we have moved the time derivative outside the integral on the left–hand–side. This equation means that the rate of change of kinetic energy in the volume V (term on the left–hand–side) is equal to the sum of the terms on the right–hand–side which are, in the order they are written:

- flux of kinetic energy advected by the fluid across the surface,
- work done per unit time by the pressure force $-p\mathrm{d}\mathbf{S}$ exerted on the surface,
- work done per unit time by the viscous force exerted on the surface (the i–component of the viscous force per unit surface area is $\sigma_{ij}n_j$, as given by eq. [2.1]),
- work done per unit time by the gravitational force (or any other body force) exerted on the volume,
- *irreversible* energy dissipation rate due to viscosity[10]; we denote this term D.

The work done by the pressure and gravitational forces leads to a rate of change of kinetic energy which is stored as potential energy and returned without loss to the system when the forces are removed. By contrast, the work done by viscous forces is only partially

[10]The fact that this term leads to energy dissipation can be shown by writing the equation of conservation of entropy, which takes the form:

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \sigma_{ij}\frac{\partial v_i}{\partial x_j},$$

where s is the entropy and T is the temperature (e.g., Landau & Lifshitz, Chapter V, §49).

returned when the forces are removed: an amount D of energy per unit time is irreversibly transformed into thermal energy (i.e. energy of ordered motion transferred into disordered motion).
As the stress tensor is symmetrical, and making the sum over repeated indices explicit, we can write:

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\left(\sigma_{ij}+\sigma_{ji}\right)\frac{\partial v_i}{\partial x_j} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\sigma_{ij}\frac{\partial v_i}{\partial x_j} + \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{1}{2}\sigma_{ij}\frac{\partial v_j}{\partial x_i},$$

where we have interchanged the indices in the second double sum. For a Newtonian incompressible fluid, σ_{ij} is given by equation (2.13), and therefore the rate of energy dissipation can be written as:

$$\boxed{D = \iiint_V \sigma_{ij}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}V = 2\eta\iiint_V e_{ij}^2\,\mathrm{d}V}, \tag{2.54}$$

where summation over i and j means that $e_{ij}^2 = e_{11}^2+e_{22}^2+e_{33}^2+2e_{12}^2+2e_{13}^2+2e_{23}^2$, since e_{ij} is symmetrical. As D expresses an irreversible dissipation of energy, it has to be positive, which implies $\eta > 0$. Not surprisingly, D is related to the rate of change of the deformation e_{ij} of fluid elements. In a solid, the energy D may be stored as potential energy of elastic deformation and returned to the system when the deformation is removed. However, in a fluid, this corresponds to an irreversible loss of kinetic energy.

It is interesting to compare the work W_{surf} done by the viscous forces which are exerted on the surface of a volume V with the work W_{vol} done by the resultant viscous force per unit volume given by equation (2.14). They can be written as:

$$W_{\text{surf}} = \oiint_S v_i T_i dS = \oiint_S v_i\sigma_{ij}n_j\mathrm{d}S = \iiint_V \frac{\partial}{\partial x_j}\left(v_i\sigma_{ij}\right)\,\mathrm{d}V,$$

and:

$$W_{\text{vol}} = \iiint_V v_i\frac{\partial\sigma_{ij}}{\partial x_j}\,\mathrm{d}V = W_{\text{surf}} - D.$$

Navier–Stokes equation (2.20) multiplied by v_i and integrated over a volume V yields:

$$\iiint_V v_i\frac{\mathrm{D}v_i}{\mathrm{D}t}\,\mathrm{d}V = W_{\text{p}} + W_{\text{vol}} + W_{\mathbf{g}},$$

where W_{p} and $W_{\mathbf{g}}$ are the work done by pressure and gravitational forces, respectively (note that here the work done by surface pressure forces is equal to the work done by the resultant pressure force per unit volume, as the fluid is incompressible). Therefore, the rate of change of kinetic energy is related to W_{vol}. However, this quantity does not represent the work done by viscous forces, which is given by W_{surf}. We have $W_{\text{surf}} = W_{\text{vol}} + D$, where W_{vol} is the part of the work that changes the bulk velocity of the fluid element, and is stored as potential energy, and D is the part that deforms the volume without changing its bulk velocity, and is converted into thermal energy.

2.5.2 Conservation of energy in a steady ideal fluid: Bernoulli's theorem

We now focus on the particular case of an ideal fluid, which is both incompressible and inviscid ($\nu = 0$). Although we could just use the equations above with $\sigma_{ij} = 0$, we obtain more general results by re–deriving the conservation energy equation from Euler's equation (2.30). Writing the gravitational acceleration as the gradient of a potential, $\mathbf{g} = -\boldsymbol{\nabla}\chi$, and assuming ρ to be constant throughout the fluid, this equation becomes:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\left(\boldsymbol{\nabla}\times\mathbf{v}\right)\times\mathbf{v} = -\boldsymbol{\nabla}\left(\frac{1}{2}\rho v^2 + p + \rho\chi\right). \tag{2.55}$$

(eq. [2.40]) with $\nu = 0$ and in which we have replaced $\boldsymbol{\omega}$ by $\boldsymbol{\nabla}\times\mathbf{v}$).

Bernoulli streamline theorem:

We now consider a *steady flow* and take the dot product of equation (2.55) with $\mathbf{v}$. As the vector $(\boldsymbol{\nabla}\times\mathbf{v})\times\mathbf{v}$ is perpendicular to $\mathbf{v}$, this yields:

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\left(\frac{1}{2}\rho v^2 + p + \rho\chi\right) = 0, \tag{2.56}$$

which means that the quantity

$$\boxed{H = \frac{1}{2}\rho v^2 + p + \rho\chi \text{ is constant along streamlines.}} \tag{2.57}$$

This is called Bernoulli's theorem, and is valid for a *steady*, *inviscid* and *incompressible* fluid in which ρ *is constant* throughout the fluid. It implies that, if the gravitational potential is constant along a streamline, the flow accelerates when the pressure decreases[11].

The quantity $\rho v^2/2$, which is the kinetic energy per unit volume, has the dimensions of a pressure and is called the *dynamic pressure*, and $p + \rho v^2/2$ is the *total pressure*. The fact that H is constant along streamlines can be understood by noting that, when the pressure p is increasing in the direction of the flow, fluid elements have to do work against the pressure gradient and therefore loose kinetic energy. In the opposite case, when the pressure p is decreasing, fluids elements gain kinetic energy. The quantity H is often referred to as the total enthalpy per unit volume.

[11] Bernoulli's theorem is responsible for the *cavitation* phenomenon, which is the formation of vapor bubbles in a fluid when the velocity increases sufficiently that the pressure drops down to the saturation vapor limit. This happens for example behind the blade of a rapidly rotating ship propeller. Due to the higher pressure of the surrounding medium, the bubbles subsequently collapse, emitting large amplitude shock waves which release energy locally and damage the blades of the propeller. This emission of shock waves enables to detect military submarines. Cavitation bubbles are also used in a range of medical procedures to deposit energy non–intrusively, for example to target cancer cells or break kidney stones. Cavitation bubbles are also an efficient tool for Mantis shrimps to catch their preys: the rapid motion of the claws generates bubbles which, when they subsequently collapse, produce a measurable force on the prey that can be enough to kill it.

Bernoulli's theorem for irrotational flow:

In an irrotational flow, $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$, and $\mathbf{v}$ can be written as the gradient of a velocity potential ϕ: $\mathbf{v} = \boldsymbol{\nabla}\phi$ (see section 1.8.1). Therefore, equation (2.55) implies:

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho v^2 + p + \rho\chi = C(t), \tag{2.58}$$

where C is a function of time only. The quantity on the left–hand side is therefore constant throughout the whole fluid at any given time. Since any function of time can be added to ϕ without changing the velocity, the function $C(t)$ can be subsumed into $\partial\phi/\partial t$ by adding $(1/\rho)\int_{t_0}^{t} C(t')\mathrm{d}t'$ to ϕ, where t_0 is an arbitrary constant. Therefore, Bernoulli's theorem becomes:

$$\boxed{\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho v^2 + p + \rho\chi = 0.} \tag{2.59}$$

If the flow is *steady*, then

$$\boxed{H = \frac{1}{2}\rho v^2 + p + \rho\chi \ \text{ is constant throughout the whole fluid,}} \tag{2.60}$$

but here the constant is not necessarily zero. This is an extension of Bernoulli's theorem which is valid for a *steady*, *inviscid*, *incompressible* and *irrotational* fluid in which ρ *is constant* throughout.

2.6 Examples of viscous flows and very viscous flows

Couette, Poiseuille and Stokes flows are studied in Problem Set 2.

Chapter 3

Potential flows

A potential flow is a flow which is both *inviscid* and *irrotational*. The name comes from the fact that, in an irrotational flow, the velocity can be written as the gradient of a *velocity potential*.

In the previous chapter, we have established Kelvin's theorem, which states that if an inviscid flow has no vorticity initially, it has no vorticity at all subsequent times and therefore remains a potential flow (provided the external force is derivable from a potential and the mass density is constant throughout the fluid). In particular, a flow which is at rest initially is a potential flow.

There is a difficulty near a solid boundary though, as may occur if an object is immersed in the flow. As we have already pointed out in section 2.3.1, viscosity cannot be ignored near a rigid boundary, even if it can be neglected in the interior of the fluid. The large stress that results from the flow velocity adjusting to the boundary condition creates a boundary layer in which viscous effects are important. Potential flow theory does not apply in such boundary layers, which will be studied in the next chapter. However, the flow moving past a solid object may still be calculated using the potential flow approximation in the outer regions if the Reynolds number is large there (larger than about 25), and matched to boundary layer approximations near the object.

In this chapter, we will be concerned with irrotational high Reynolds number flows, approximated as inviscid. We will also limit our study to incompressible fluids.

Important applications of potential flow theory include water waves and outer flows (away from the boundary layers) around aerofoils, which are the cross–sectional shape of an object designed to generate lift when moving through a fluid, like a wing, blade or sail. The study of the motion of air past an aerofoil is called *aerodynamics*. As pointed out in section 2.2.4, air is approximated to be incompressible for Mach numbers below 0.3. Therefore, although incompressible potential flow theory would not apply to commercial aircrafts, which cruise at speeds of Mach larger than 0.8, it is important for low–speed aerodynamics. The Reynolds number $Re = UL/\nu$ depends on the characteristic velocity U of the flow and on the characteristic lengthscale L over which parameters vary. In the case of aerodynamics, L is the size of the aerofoil. Therefore, high Reynolds numbers are achieved for high–speed flows around larger objects. In the case of waves, L is the

wavelength, so potential theory applies to large scale disturbances.

3.1 General properties of potential flows

In this chapter, we make the following assumptions:

- the flow is inviscid ($\nu = 0$),
- the flow is irrotational ($\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$),
- the flow is incompressible ($\boldsymbol{\nabla} \cdot \mathbf{v} = 0$),
- ρ is uniform throughout the fluid,
- the external force is gravity with the gravitational acceleration given by $\mathbf{g} = -\boldsymbol{\nabla}\chi$.

The condition $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$ implies that the velocity can be written as the gradient of a velocity potential ϕ: $\mathbf{v} = \boldsymbol{\nabla}\phi$. As pointed out in section 1.8.1, ϕ is a single–valued function of position only if the flow domain is simply connected. Also, ϕ is not uniquely defined by the equation above: any function of time can be added to a solution without modifying $\mathbf{v}$.

Incompressibility yields:

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{v} = \nabla^2 \phi = 0}, \tag{3.1}$$

which means that ϕ *satisfies Laplace's equation*. The boundary conditions for ϕ follow from the boundary conditions for $\mathbf{v}$. If the boundary is rigid, equation (2.31) implies that the normal derivative of ϕ on the surface (that is to say, the derivative in the direction of the normal to the boundary) is equal to the normal component of the velocity of the boundary itself:

$$\left.\frac{\partial \phi}{\partial n}\right)_{\text{surface}} = v_{n,\text{boundary}}. \tag{3.2}$$

If the boundary is an interface between two fluids, then equation (2.33) implies:

$$\left.\frac{\partial \phi}{\partial n}\right)_{\text{fluid 1, surface}} = \left.\frac{\partial \phi}{\partial n}\right)_{\text{fluid 2, surface}}. \tag{3.3}$$

The no–slip condition does not apply here as the fluid is inviscid.

It can be shown that, within a simply connected flow domain, Laplace's equation has a unique solution in the two following cases: (i) the flow domain is bounded and the boundary condition (3.2) is satisfied at each point of the boundary, (ii) the flow domain is not bounded but the boundary condition (3.2) is satisfied at any rigid surface and the velocity matches that of the fluid at infinity (e.g., flow moving past a sphere).
If the flow domain is not simply connected, other conditions have to be added for the potential to be determined uniquely.

Calculating ϕ from Laplace's equation and then $\mathbf{v}$ from $\boldsymbol{\nabla}\phi$ is much easier than calculating $\mathbf{v}$ directly from Euler's equation, because Laplace's equation is linear whereas

Euler's equation contains the non–linear term $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$. Once the velocity has been obtained, the pressure can be calculated from Euler's equation (2.30), which we recall here:

$$\rho\frac{\partial\mathbf{v}}{\partial t}+\rho\,(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}=-\boldsymbol{\nabla}p+\rho\mathbf{g}. \tag{3.4}$$

We also recall Bernoulli's theorem (2.59), as satisfied by incompressible potential flows:

$$\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho v^2+p+\rho\chi \text{ is constant throughout the whole fluid.} \tag{3.5}$$

Finally, as we have seen in section 1.8.3, if the flow is two dimensional, as it is also incompressible, we can define a stream function ψ such that:

$$v_x=\frac{\partial\psi}{\partial y},\quad\text{and}\quad v_y=-\frac{\partial\psi}{\partial x}, \tag{3.6}$$

in cartesian coordinates (x,y) or:

$$v_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad\text{and}\quad v_\theta=-\frac{\partial\psi}{\partial r}. \tag{3.7}$$

in polar coordinates (r,θ). The lines $\psi=$ constant are streamlines.
Like ϕ, ψ is not uniquely defined by the equations above: any gradient (in addition to any function of time) can be added to a solution without modifying $\mathbf{v}$.
Since the flow is irrotational, and noting $\hat{\mathbf{z}}$ the unit vector in the z–direction:

$$\boxed{\boldsymbol{\nabla}\times\mathbf{v}=-\nabla^2\psi\,\hat{\mathbf{z}}=\mathbf{0}}, \tag{3.8}$$

which means that ψ *satisfies Laplace's equation* in a two dimensional potential flow.

3.2 Simple potential flows

Because Laplace's equation is linear, different solutions for ϕ (and ψ in two dimensions) can be superposed to yield new and more complex solutions. This is why it is worth studying very simple potential flows which may not be very realistic, but can be superposed to construct more interesting flows.

3.2.1 Uniform parallel flows

This is a flow such that $\mathbf{v}=U\hat{\mathbf{x}}$ where U is a constant and $\hat{\mathbf{x}}$ is the unit vector along the x–axis. The vorticity is clearly zero for this flow. The condition $\mathbf{v}=\boldsymbol{\nabla}\phi$ then implies:

$$\phi=Ux. \tag{3.9}$$

If the flow is two dimensional, then equations (3.6) yield:

$$\psi=Uy. \tag{3.10}$$

It is easy to verify in this simple example that $\psi=$ constant, that is to say $y=$ constant, is the equation of the streamlines and that the streamlines and equipotential lines (given by $\phi=$ constant) are perpendicular to each other.

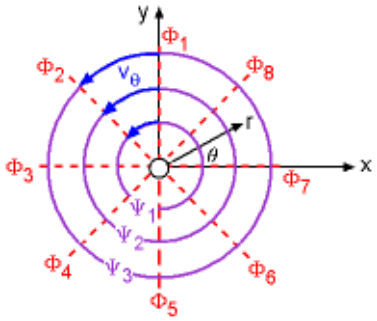
3.2.2 Line vortex flow

This a two dimensional flow such that, in polar coordinates:

$$\mathbf{v} = \frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}, \tag{3.11}$$

where Γ is a constant, called the *strength* of the vortex, and $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction. The vorticity is:

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v} = \frac{1}{r}\frac{\partial\left(rv_\theta\right)}{\partial r}\,\hat{\mathbf{z}} = 0.$$



As already calculated in section 1.8.1, the velocity potential is given by:

$$\phi = \frac{\Gamma\theta}{2\pi}, \tag{3.12}$$

which is a multi–valued function of position.

We obtain ψ by integrating equations (3.7):

$$\psi = -\frac{\Gamma}{2\pi}\ln\frac{r}{r_0}, \tag{3.13}$$

where we have added a constant to make the argument of the ln dimensionless (as ψ is defined within an arbitrary constant).
As seen in section 1.8.2, the circulation is defined as:

$$\Gamma = \oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{l},$$

and, for an irrotational flow, it is the same round all simple contours enclosing the origin. It was noted in section 1.8.2 that, for a line vortex flow, the circulation of the velocity around a contour centered at the origin is non zero, even though the vorticity is zero. This is because the flow domain is not simply connected, due to the singularity at $r = 0$. In other words, since the vorticity is not defined at $r = 0$, Stokes's theorem does not apply. One way around this difficulty is to define $\boldsymbol{\omega}$ as a Dirac delta function at the origin:

$$\boldsymbol{\omega}(r,\theta) = \Gamma\delta(0)\hat{\mathbf{z}}, \tag{3.14}$$

so that:

$$\int_{r=0}^{+\infty}\int_{\theta=0}^{2\pi}\omega r\mathrm{d}r\mathrm{d}\theta = \Gamma,$$

and Stokes's theorem applies. This is really what is meant by *line vortex*: the vorticity is non zero only along a line.

3.2.3 Sources and sinks

This is a flow which streams towards (sink) or away from (source) a point. The velocity of such a flow is given by:

$$\mathbf{v} = \frac{m}{4\pi r^2}\hat{\mathbf{r}} \text{ in 3D}, \quad \mathbf{v} = \frac{m}{2\pi r}\hat{\mathbf{r}} \text{ in 2D}, \tag{3.15}$$

where m is a constant, called the *strength* of the source or sink, and $\hat{\mathbf{r}}$ is the unit vector along the radial direction. In this case, the origin is the sink ($m < 0$) or source ($m > 0$). The vorticity is:

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v} = -\frac{1}{r}\frac{\partial v_r}{\partial\theta}\,\hat{\mathbf{z}} = 0.$$

In two dimensions, $\mathrm{d}\phi/\mathrm{d}r = m/(2\pi r)$ yields:

$$\phi = \frac{m}{2\pi}\ln\frac{r}{r_0}, \tag{3.16}$$

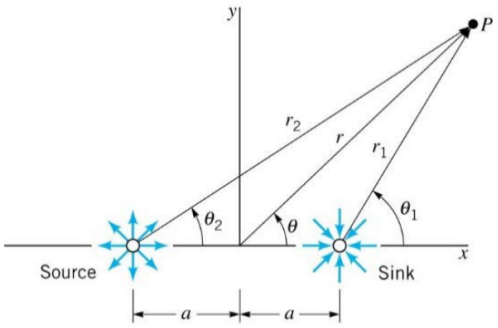
and $\mathrm{d}\psi/\mathrm{d}\theta = m/(2\pi)$ yields:

$$\psi = \frac{m\theta}{2\pi}. \tag{3.17}$$

It is similar to what what obtained for the line vortex flow, with ϕ and ψ being interchanged.

3.2.4 Dipole flow

We now consider the situation where we have both a source and a sink with the same strength m.



The dipole approximation in this context is the limit when the separation $d = 2a$ between the source and the sink goes to zero. We call *moment* of the dipole flow the vector $\mathbf{p} = -md\hat{\mathbf{x}}$ pointing from the sink to the source.

In two dimensions, the velocity potentials ϕ_1 and ϕ_2 corresponding to the sink and source, respectively, are given by:

$$\phi_1 = -\frac{m}{2\pi}\ln\frac{|\mathbf{r} - a\hat{\mathbf{x}}|}{r_0}, \quad \phi_2 = \frac{m}{2\pi}\ln\frac{|\mathbf{r} + a\hat{\mathbf{x}}|}{r_0},$$

where r_0 is an arbitrary length. Since Laplace's equation is linear, the velocity potential ϕ corresponding to the dipole flow is obtained by adding ϕ_1 and ϕ_2, so that:

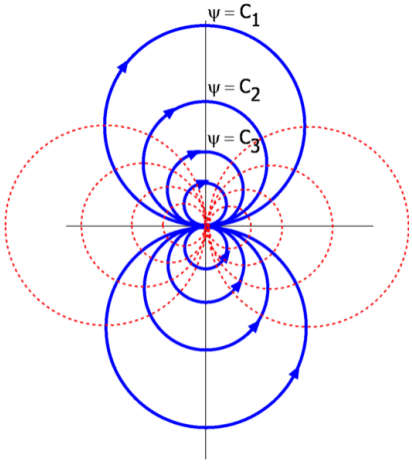
$$\phi = \frac{m}{2\pi}\left(\ln|\mathbf{r} + a\hat{\mathbf{x}}| - \ln|\mathbf{r} - a\hat{\mathbf{x}}|\right).$$

Using $a/r \ll 1$, to first order in a/r we obtain:

$$\phi = \frac{m}{2\pi}\frac{2a\cos\theta}{r} = \frac{p\cos\theta}{2\pi r} = -\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{2\pi r}, \tag{3.18}$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. The velocity is then calculated from the gradient of ϕ:

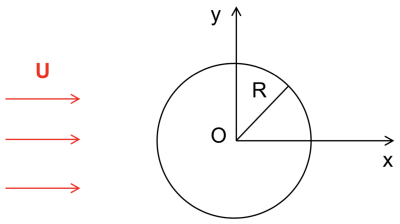
$$v_r = \frac{\partial\phi}{\partial r} = -\frac{p\cos\theta}{2\pi r^2}, \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{p\sin\theta}{2\pi r^2}. \tag{3.19}$$



The figure shows the streamlines in blue and the equipotential (lines of constant ϕ) in red. This is similar to the calculation of the electrostatic potential due to an electric dipole, as expected since in both cases the potential satisfies Laplace's equation.

3.2.5 Flow around a circular cylinder

We consider a flow moving past an infinite cylinder perpendicularly to its axis.



Far away from the cylinder, the flow velocity is $\mathbf{v} = U\hat{\mathbf{x}}$. Since there is no dependence on the coordinate along the axis of the cylinder, this is a two dimensional problem in the (x, y), or (r, θ), plane. We note R the radius of the cylinder cross–section.

Because of the presence of the cylinder, the flow domain is not simply connected (a contour encircling the cylinder cannot be shrunk to zero continuously while staying in the domain). However, it could be shown that the solution to Laplace's equation satisfying the boundary condition (3.2) at the surface of the cylinder and matching the velocity at infinity is unique if we specify the circulation around the cylinder. Below, we therefore examine two different cases corresponding to two different circulations.

Like in electrostatics, the velocity potential can be written as a multipole expansion. The general solution of Laplace's equation in two dimensions, using polar coordinates, is:

$$\phi(r,\theta) = (a_0 + b_0\ln r)(A_0 + B_0\theta) + \sum_{n=1}^{\infty}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]\left(a_n r^n + b_n r^{-n}\right). \tag{3.20}$$

We recognise the following terms: $A_0 b_0 \ln r$ corresponds to a sink or a source, $a_0 B_0 \theta$ corresponds to a line vortex, $A_1 a_1 r \cos\theta$ corresponds to a uniform flow, $A_1 b_1 \cos\theta / r$ corresponds to a dipole flow.

Cylinder with no circulation:

If there is no circulation around the cylinder, then the term corresponding to the line vortex in the above expansion of ϕ is zero. As there is also no sink nor source in the fluid, the first non zero term in the expansion is the $n = 1$ term. Since the flow velocity at infinity is proportional to $\cos\theta$, we set $B_1 = 0$ and look for solutions under the form:

$$\phi = \left(ar + \frac{b}{r} \right) \cos\theta.$$

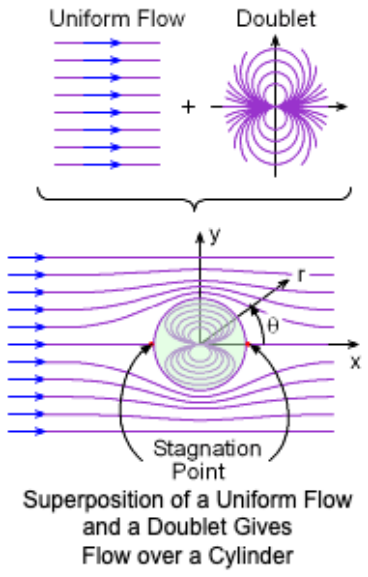
If we can find a solution under this form which satisfies the boundary conditions, then this is *the* solution as we have also specified the circulation and therefore the solution is unique. At infinity, we have a uniform flow for which $\phi = Ux = Ur\cos\theta$ as shown in section 3.2.1. Therefore $a = U$. At $r = R$, $v_r = \partial\phi/\partial r$ has to be zero to satisfy the boundary condition (3.2). Therefore $U - b/R^2 = 0$, which yields $b = UR^2$. The solution is then:

$$\phi = U \left(r + \frac{R^2}{r} \right) \cos\theta, \tag{3.21}$$

which is the sum of a uniform parallel flow and a dipole flow. The gradient of ϕ then gives the velocity:

$$v_r = U \left(1 - \frac{R^2}{r^2} \right) \cos\theta, \quad v_\theta = -U \left(1 + \frac{R^2}{r^2} \right) \sin\theta. \tag{3.22}$$

There are two *stagnation points*, where the velocity vanishes: $(R, 0)$ and (R, π).



Superposition of a Uniform Flow and a Doublet Gives Flow over a Cylinder

To obtain the equations of the streamlines (shown on the figure), we can also calculate the stream function ψ by integrating equations (3.7), which yields:

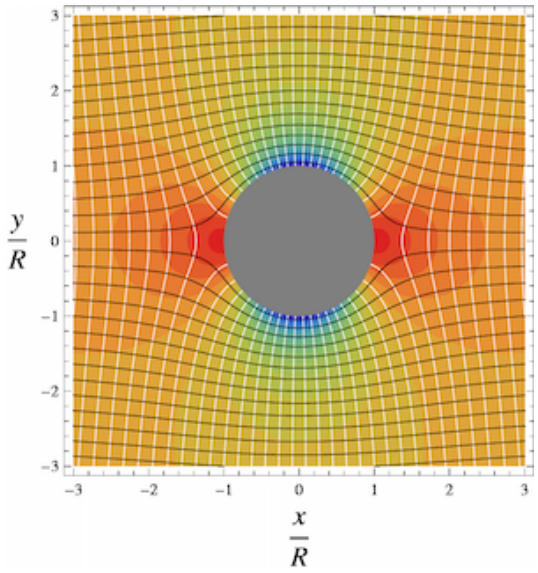
$$\psi = U \left(1 - \frac{R^2}{r^2} \right) r \sin\theta. \tag{3.23}$$

The streamline $\psi = 0$ is made of the two semi–infinite lines along the x–axis plus the circumference of the cylinder. As we have discussed in section 2.3, in a two dimensional flow, *a rigid boundary is always a streamline* as the velocity has no normal component at the boundary.

(Credit: https://www.ecourses.ou.edu)

The pressure is obtained from Bernoulli's theorem (3.5) applied to a steady flow, which states that $\rho v^2/2 + p$ is constant throughout the whole fluid, as χ is uniform. If we note p_0 the pressure at infinity, then the pressure at the surface of the cylinder is given by:

$$p(R,\theta) = p_0 + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho v^2(R,\theta) = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right). \tag{3.24}$$



In this figure, the streamlines correspond to the black curves, the equipotential lines to the white curves, and pressure is indicated by the colors (red and blue are high and low pressure, respectively).

(*Credit: Wikipedia*)

Because of the symmetry of the pressure, there is no net force acting on the cylinder. This conclusion can be shown to hold for any object, not just a cylinder. This is known as the *d'Alembert paradox*, as obviously common experience is that any object moving relative to a fluid is subject to a significant resistance, or *drag*[1]. A solution to this paradox will be presented in the next chapter.

Cylinder with circulation:

If the cylinder rotates around its axis, the relative motion of the fluid at the surface results in a finite circulation around the cylinder. Note that, as shown in section 1.8.2, for an irrotational flow, the circulation is the same round all simple contours enclosing the

[1] D'Alembert did this calculation in 1752 as a prize problem for the Berlin Academy of Science. The motion of a body relative to an inviscid and incompressible flow was important to understand the design of boats. The result however, indicating that there would be no drag, discredited completely mathematical fluid mechanics, and d'Alembert did not get the prize. This episode prompted the words of Sir Hinshelwood taken as an epigraph for these notes. Prandtl proposed a resolution of the paradox in 1904, when he found that even a very small viscosity would result in the creation of a boundary layer at the surface of the object in which viscous forces produce a drag. This is largely accepted as the right way of accounting for the drag force, as it is completely supported by experimental observations and mathematical analysis. However, there is no rigorous mathematical proof of this process and another solution to the paradox has been proposed recently, in which the drag is produced by the turbulence which results from the inviscid potential flow being unstable. No viscosity is required in that case.

cylinder. We model this situation by adding to the velocity potential and stream function found above in the case of no circulation a term corresponding to a line vortex as described in section 3.2.2. We then obtain:

$$\phi = U\left(1+\frac{R^2}{r^2}\right)r\cos\theta + \frac{\Gamma\theta}{2\pi}, \tag{3.25}$$

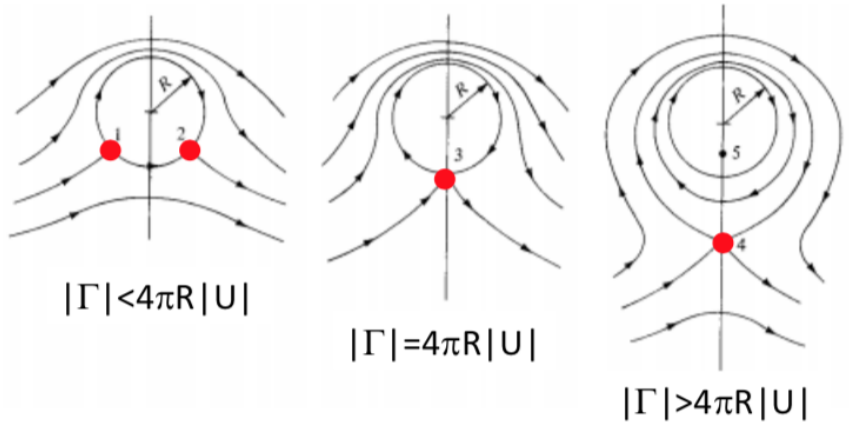
$$\psi = U\left(1-\frac{R^2}{r^2}\right)r\sin\theta - \frac{\Gamma}{2\pi}\ln\frac{r}{r_0}, \tag{3.26}$$

with r_0 being an arbitrary constant, which yield the following velocity:

$$v_r = U\left(1-\frac{R^2}{r^2}\right)\cos\theta, \quad v_\theta = -U\left(1+\frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \tag{3.27}$$

We define $\alpha \equiv \Gamma/(4\pi RU)$. The stagnation points here satisfy (i) $r = R$ and $\sin\theta = \alpha$, if $|\alpha| \leq 1$, or (ii) $\theta = \pm\pi/2$ and $r/R = |\alpha| + \sqrt{\alpha^2 - 1}$, if $|\alpha| \geq 1$ (where θ is positive if Γ and U have the same sign and negative otherwise).

The figure below shows the streamlines for different values of α. The stagnation points are indicated in red. Here $U > 0$ and $\Gamma < 0$ (clockwise), so that $\alpha < 0$.



When $|\alpha| < 1$ (that is to say, $|\Gamma| < 4\pi R|U|$), there are two distinct stagnation points. The point labelled 1 on the figure is called the *attachment* point, and it is where the oncoming flow divides into that moving above and that moving below the cylinder. The point labelled 2 is called the *separation* point.

The pressure can be calculated as above, and this yields:

$$p(R,\theta) = p_0 + \frac{1}{2}\rho U^2\left[1-\left(\frac{\Gamma}{2\pi RU} - 2\sin\theta\right)^2\right]. \tag{3.28}$$

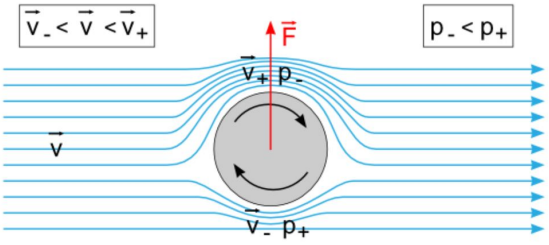
The force acting on a surface element $\mathrm{d}S = R\mathrm{d}\theta$ with a unit length along the axis of the cylinder is $\mathrm{d}\mathbf{F} = -p\mathrm{d}S\hat{\mathbf{r}}$, where p is evaluated at the location of $\mathrm{d}S$. The x– and y–components of this force are $-p\mathrm{d}S\cos\theta$ and $-p\mathrm{d}S\sin\theta$, respectively. Therefore, the x– and y–components of the total force acting on the surface of the cylinder with a unit length along the axis are:

$$F_x = \int_0^{2\pi} -p(R,\theta)R\cos\theta\mathrm{d}\theta, \quad F_y = \int_0^{2\pi} -p(R,\theta)R\sin\theta\mathrm{d}\theta. \tag{3.29}$$

The force in the x–direction, which is called the *drag*, is zero, like in the case of no circulation.
However, there is a non zero force in the y-direction, which is called the *lift*. It is due to the $\sin\theta$ term in the expression for p:

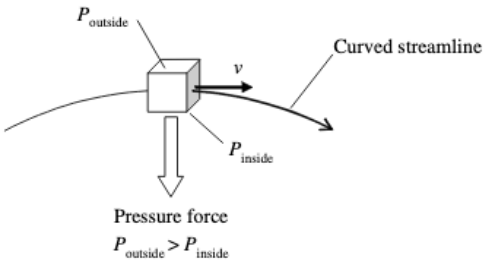
$$F_y = \int_0^{2\pi} -\frac{\rho U\Gamma}{\pi}\sin^2\theta \mathrm{d}\theta = -\rho U\Gamma. \tag{3.30}$$

In the context of a rotating object immersed in a fluid, the lift is also called the *Magnus force*. It is a well–known effect in tennis or football, as it leads to the swerving of balls which are hit with spin. On the figure below, where $U > 0$ and $\Gamma < 0$ (clockwise), $F_y > 0$ so that the cylinder is pushed upwards.
The direction of the lift can be understood in the following way:



Because the velocity of the parallel flow is in the same direction as that of the line vortex above the cylinder, and in the opposite direction below the cylinder, the flow velocity is larger above than below the cylinder.

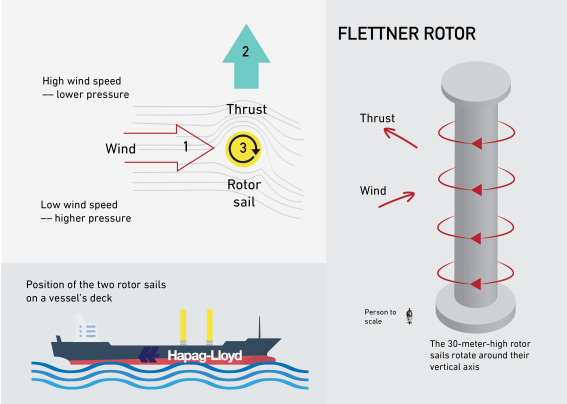
(This results in the streamlines being closer to each other above than below the cylinder, as seen in Problem Set 1). Bernoulli's theorem then implies that the pressure is larger below than above the cylinder, which results in an upwards force on the cylinder.



It is important to note that the vertical pressure gradient is due to the fact that the streamlines are curved, as illustrated on the figure.

The change of direction of the velocity $\mathbf{v}$ yields a centripetal acceleration v^2/r per unit mass, where r is the local radius of curvature, which can only be generated by the pressure force $\partial p/\partial n$, where n is the coordinate in the direction normal to the streamline. Therefore *pressure decreases towards the centre of curvature*. (This effect is the reason why the pressure at the core of a tornado is very low, and objects get 'sucked' into the sky.) In the case of the cylinder with circulation, the streamlines are curved in such a way that pressure decreases when we approach the cylinder moving vertically either from the top or from the bottom. However, because both the curvature radius is smaller and the velocity is larger above the cylinder, the drop in pressure is more significant above than below the cylinder. The lift is therefore due to an *asymmetry in the curvature of the streamlines*. Such a situation, which here is the result of the rotation of the cylinder, may

also be produced by a stationary cylinder with a sharp–edged flap at the separation point or by a wing[2].



This figure illustrates how the Magnus effect can be used on a ship to reduce fuel comsuption.

Although it is straightforward to calculate the lift in the simple case of a flow past a cylinder by directly solving for the velocity and pressure, it is much more difficult to do so for a more complicated aerofoil. However, in two dimensions, techniques based on complex variables can be used to *map* the flow past a cylinder into the same flow past a different object. This is the object of the next section.

3.3 Complex velocity potential

In this section, we consider an incompressible and irrotational two dimensional flow.

3.3.1 Cauchy–Riemann equations

The velocity may be written as a function of either the potential velocity or the stream function:

$$v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{3.31}$$

The functions $\phi(x, y)$ and $\psi(x, y)$ then satisfy the Cauchy–Riemann equations, so that the *complex potential*, defined as:

$$w(z) = \phi(x, y) + \mathrm{i}\psi(x, y), \tag{3.32}$$

with $z = x + \mathrm{i}y$, is differentiable (see appendix A). The derivative of w can then be calculated by keeping y constant:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x} = v_x - \mathrm{i}v_y. \tag{3.33}$$

[2]A very good description of the lift on aerofoils is given in *How do wings work?*, H. Babinsky, 2003, Physics Education, 38, 497. In this article, the author shows that the 'popular' explanation for the lift that fluid particles flowing across the top and the bottom surfaces must reach the trailing edge of the aerofoil at the same time, which results in higher speed and therefore lower pressure at the top, is wrong.

These results also apply in polar coordinates, using $z = re^{i\theta}$. Both ϕ and ψ, and therefore w, may be multi–valued functions. However, by specifying the values the function can take (which is known as specifying a *branch* of the function), we can make it single–valued. Since it is also differentiable, it then becomes an *analytic* function.
Cauchy–Riemann equations also imply that both ϕ and ψ satisfy Laplace's equation, as already pointed out in section 3.1.

3.3.2 Complex potential of a flow past a cylinder

In the absence of circulation, we have found above that the potential and stream function of a flow past a cylinder are: $\phi = U\left(r + R^2/r\right)\cos\theta$ and $\psi = U\left(r - R^2/r\right)\sin\theta$ (eq. [3.21] and [3.23]). Using $z = re^{i\theta}$, we obtain:

$$w(z) = U\left(ze^{-i\theta} + \frac{R^2}{z}e^{i\theta}\right)\cos\theta + iU\left(ze^{-i\theta} - \frac{R^2}{z}e^{i\theta}\right)\sin\theta = U\left(z + \frac{R^2}{z}\right). \tag{3.34}$$

For a line vortex, the potential is $\phi = \Gamma\theta/(2\pi)$ and $\psi = -(\Gamma/(2\pi))\ln(r)$ (eq. [3.12] and [3.13] where we have taken $r_0 = 1$), so that:

$$w(z) = \frac{\Gamma}{2\pi}\left(-i\ln r + \theta\right) = -\frac{i\Gamma}{2\pi}\left(\ln r + i\theta\right) \equiv -\frac{i\Gamma}{2\pi}\ln z, \tag{3.35}$$

defined up to an additive constant[3].

Therefore the complex potential for a flow past a cylinder with circulation Γ is:

$$w(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z. \tag{3.36}$$

3.3.3 Conformal mapping

This is a technique in which a flow problem (or more generally a Laplace equation problem) in simple geometries is transformed into more complicated geometries by using an analytic mapping function that preserves the angles.

We consider a pair of complex variables $z = x + iy$ and $Z = X + iY$, each defined in their own complex plane, such that:

$$Z = f(z) \tag{3.37}$$

is an *analytic* function of z. This is a transformation that maps a point (x, y) in the z–plane into a point (X, Y) in the Z–plane. The inverse $z = F(Z)$ is also an analytic function of Z if $f'(z) \neq 0$, as $F'(Z) \equiv \left(f^{-1}\right)'(Z) = 1/f'\left(f^{-1}(Z)\right) \equiv 1/f'(z)$.

[3]The logarithm of a complex number $z = re^{i\theta}$ is defined as follows:

$$\ln(z) = \ln r + i\left(\theta + 2\pi k\right),$$

where k is an integer. Although this is a multi–valued function, we can specify the branch as $-\pi \leq \mathrm{Im}\left[\ln(z)\right] \leq \pi$ so that the function (called *principal value*) becomes single–valued.

It can be shown that:

> At the points in the z–plane where the first derivative of f is non–zero, the transformation preserves the angles and is called *conformal mapping*.

This means that two short segments which are perpendicular in the z–plane are mapped into two short segments which are also perpendicular in the Z–plane. More generally, if the first non–zero derivative of f at a point z is the n^{th} derivative, then a local angle at z is multiplied by n when mapped into the Z–plane. (See, e.g., section 4.6 in Acheson for a proof).

If $w(z) = \phi(x, y) + \mathrm{i}\psi(x, y)$ is the complex potential of a two dimensional incompressible potential flow in the z–plane, and $Z = f(z)$ is a conformal transformation, then:

$$W(Z) = w\left(F(Z)\right) = \Phi(X, Y) + \mathrm{i}\Psi(X, Y)$$

is an analytic function of Z. It follows that Φ and Ψ satisfy the Cauchy–Riemann relations, which implies that they satisfy Laplace's equation[4], and that $V_X(X, Y) = \partial\Phi/\partial X = \partial\Psi/\partial Y$ and $V_Y(X, Y) = \partial\Phi/\partial Y = -\partial\Psi/\partial X$ represent the velocity components of an incompressible potential flow in the Z–plane, with again $\mathrm{d}W/\mathrm{d}Z = V_X - \mathrm{i}V_Y$. In addition, since $W(Z) = w\left(F(Z)\right)$, the value of ϕ at (x, y) is the same at that of Φ at (X, Y) when those two points are related by $z = F(Z)$, with similar result for ψ and Ψ. This implies that streamlines in the (x, y)–plane are mapped into streamlines in the (X, Y)–plane through this transformation. Furthermore, as rigid boundaries are themselves streamlines, they too map into rigid boundaries in the (X, Y)–plane.

> In other words, if we know the streamlines and equipotentials of a flow in a given geometry, we can obtain the streamlines and equipotentials corresponding to any conformal mapping of this geometry.

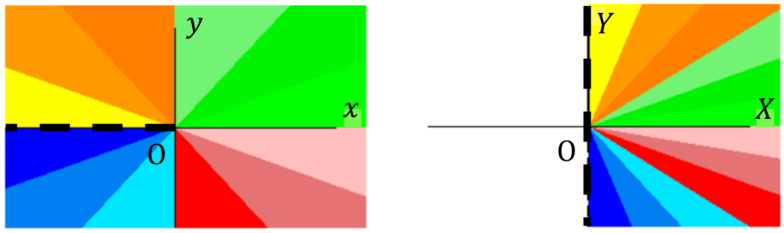
Note that, since $W(Z) = w\left(F(Z)\right)$, the streamline $\psi(x, y) = C_1$, for example, transforms into the streamline $\Psi(X, Y) = C_1$. We would still obtain a flow in the Z–plane with $W(Z) = 2w\left(F(Z)\right)$, for example, but that would lead to the velocity satisfying different boundary conditions. The complex potential in the Z–plane is defined to be $W(Z) = w\left(F(Z)\right)$ because, in the case of the flow past an aerofoil, as studied below, the velocity satisfies the same boundary condition at infinity in both the z and Z–planes.

[4]The transformation needs to be conformal for Laplace's equation in the z–plane to be transformed into Laplace's equation in the Z–plane because, e.g.,

$$\boldsymbol{\nabla}^2\Phi = \frac{1}{\left|f'(z)\right|^2}\boldsymbol{\nabla}^2\phi,$$

which requires $f'(z) \neq 0$.

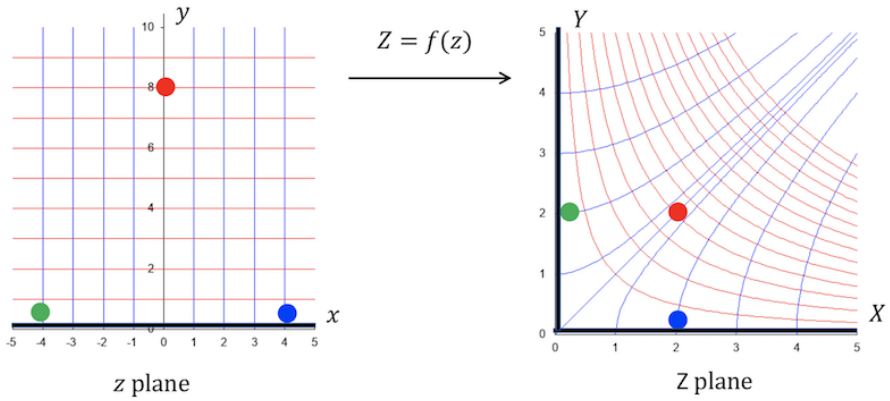
We now illustrate conformal mapping with the simple case of a uniform parallel flow. We consider the transformation $Z = f(z) = \sqrt{z}$, with the inverse being $z = F(Z) = Z^2$. Writing $z = |z|e^{i\theta}$, with $-\pi \leq \theta \leq \pi$, $f(z)$ has two *branch* functions, $f_+(z) = \sqrt{|z|}e^{i\theta/2}$ and $f_-(z) = -\sqrt{|z|}e^{i\theta/2}$. Each of these functions is single–valued. However, they are not continuous at points along the axis $\theta = \pm\pi$ ($x \leq 0$ axis), which is called the *branch cut*. The end–point of the branch cut, which is $z = 0$ and where f_+ and f_- are both 0, is called a *branch point* and is a singularity. If we exclude the *branch cut*, then each of these functions is analytic. In what follows we consider f_+, which is called the *principal value*. Then the z–plane is mapped into the the right half of the Z–plane, in the way illustrated in the figure below, where the thick dashed black line indicates the branch cut :



For a uniform parallel flow, the potential and stream function are given by $\phi(x, y) = Ux$ and $\psi(x, y) = Uy$ (see eq.[3.9] and [3.10]). Therefore, the complex potential is $w(z) = U(x + iy) = Uz$. For simplicity we take $U = 1$. Using the transformation $f_+(z)$, the complex potential in the Z–plane can be written as:

$$W(Z) = w\left(Z^2\right) = Z^2 = (X + iY)^2 = X^2 - Y^2 + 2iXY.$$

Therefore $\Phi(X, Y) = X^2 - Y^2$ and $\Psi(X, Y) = 2XY$. In the z-plane, the streamlines $\psi = y =$ constant are horizontal lines, and the equipotentials $\phi = x =$ constant are vertical lines. In the Z-plane, the streamlines become $\Psi = 2XY =$ constant, which are hyperbolas with the coordinate axes as asymptotes, and the equipotentials are $\Phi = X^2 - Y^2 =$ constant, which are hyperbolas with either the X– or Y–axis as axis of symmetry. The figure below shows the streamlines in red and the equipotentials in blue for this flow in the z– and Z–planes. The colored dots illustrate how different points are mapped. We can think of the thick black line just above the x–axis in the z–plane as a rigid boundary, which maps into the thick black rigid boundary shown in the Z–plane. Therefore, *this mapping has transformed a uniform parallel flow into a flow in a right–angle corner*, as could be confirmed by calculating directly the potential and stream function in this latter case.



We notice on the figure above that the streamlines and equipotentials, which are perpendicular in the z–plane, are still perpendicular in the Z–plane, as they should from the discussion in section 1.8. The transformation is indeed conformal as f' does not vanish, and conformal mapping always preserves the angles.

We now discuss a transformation which is very important in the context of aerodynamics.

3.3.4 The Joukowski transformation

The Joukowski transformation enables a two dimensional flow around a cylinder to be mapped onto a two dimensional flow around the wing of an airplane. It is given by:

$$Z = f(z) = z + \frac{c^2}{z}, \tag{3.38}$$

where c is a real number. The inverse is then:

$$z = F(Z) = \frac{1}{2}\left(Z + \sqrt{Z^2 - 4c^2}\right), \tag{3.39}$$

where we have not inserted $\pm$ in front of the square root as it is understood to have two branch functions. To make this function continuous we exclude the branch cut which is the segment along the X–axis between $-2c$ and $2c$. We also make it single–valued by selecting the principal value of the square root. Therefore $z \sim Z$ when $|Z|$ is large. Excluding $Z = \pm 2c$ is equivalent to excluding $z = \pm c$, which are the only points where $f'(z) = 0$. Therefore, the transformation is conformal in the domain which excludes the branch cut.

Using $z = r\mathrm{e}^{\mathrm{i}\theta}$, the transformation can be written as:

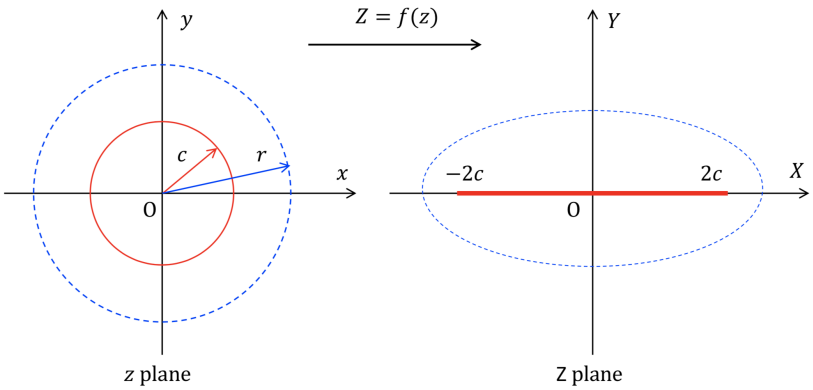
$$Z = r\mathrm{e}^{\mathrm{i}\theta} + \frac{c^2}{r}\mathrm{e}^{-\mathrm{i}\theta} = \left(r + \frac{c^2}{r}\right)\cos\theta + \mathrm{i}\left(r - \frac{c^2}{r}\right)\sin\theta. \tag{3.40}$$

The real and imaginary parts are equal to X and Y, respectively. Writing $\cos^2\theta + \sin^2\theta = 1$, we then obtain:

$$\frac{X^2}{\left(r + c^2/r\right)^2} + \frac{Y^2}{\left(r - c^2/r\right)^2} = 1.$$

If r is a constant, that is to say z moves along a circle of radius r centered at the origin in the z–plane, then this equation is that of an ellipse. If $r = c$, $Z = 2c\cos\theta$, which is a

segment along the X–axis, as illustrated on the figure below:



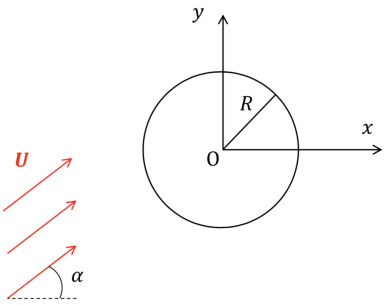
Here again, we can think of the circle of radius c in the z–plane as a rigid boundary which maps into a plate segment which is also a rigid boundary in the Z–plane. Let us consider a uniform parallel flow past this plate. The complex potential for this flow, as seen above (section 3.3.3), is $W(Z) = UZ$, with U being the velocity. We can map this flow back into the flow around the circle in the z–plane:

$$w(z) = W\left(f(z)\right) = U\left(z + \frac{c^2}{z}\right). \tag{3.41}$$

This is an elegant way to recover the result we have already obtained above (eq. [3.34]).

3.3.5 Potential flow past a finite plate and the Kutta condition

The complex potential corresponding to a flow past a circle of radius R when there is a finite circulation is given by equation (3.36). This corresponds to the case when the oncoming uniform flow is parallel to the x–axis.



If, instead, this flow was coming from below the x–axis at an *angle of attack* α, the complex potential would be obtained from equation (3.36) by changing θ to $\theta - \alpha$ or, equivalently, by changing $z = re^{i\theta}$ to $ze^{-i\alpha}$. This yields:

$$w(z) = U\left(ze^{-i\alpha} + \frac{R^2}{z}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\ln z, \tag{3.42}$$

where we have ignored $e^{-i\alpha}$ in the ln as it amounts to just adding a constant to the potential. If we now map this flow into the Z–plane using Joukowski transformation (3.38) with $c = R$, we get the complex potential $W(Z)$ of a flow incident at an angle α onto a plate segment, since the angles are preserved. The expression for $W(Z)$ is rather complicated, so instead of calculating it we derive the velocity components V_X and V_Y in the Z–plane

using $V_X - \mathrm{i}V_Y = \mathrm{d}W/\mathrm{d}Z$. Since $W(Z) = w\,(F(Z))$, with $z = F(Z)$, we obtain:

$$V_X - \mathrm{i}V_Y = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}Z} = \frac{\mathrm{d}w}{\mathrm{d}z}\left(\frac{\mathrm{d}Z}{\mathrm{d}z}\right)^{-1}. \tag{3.43}$$

Using equations (3.38) with $c = R$ and (3.42), this yields:

$$V_X - \mathrm{i}V_Y = \left(U\mathrm{e}^{-\mathrm{i}\alpha} - U\mathrm{e}^{\mathrm{i}\alpha}\frac{R^2}{z^2} - \frac{\mathrm{i}\Gamma}{2\pi z}\right)\frac{z^2}{z^2 - R^2}. \tag{3.44}$$

These velocities represent the flow in the Z–plane and therefore need to be expressed as a function of Z. However, keeping in mind the relation (3.38) between z and Z, we can get useful information from the expression above.

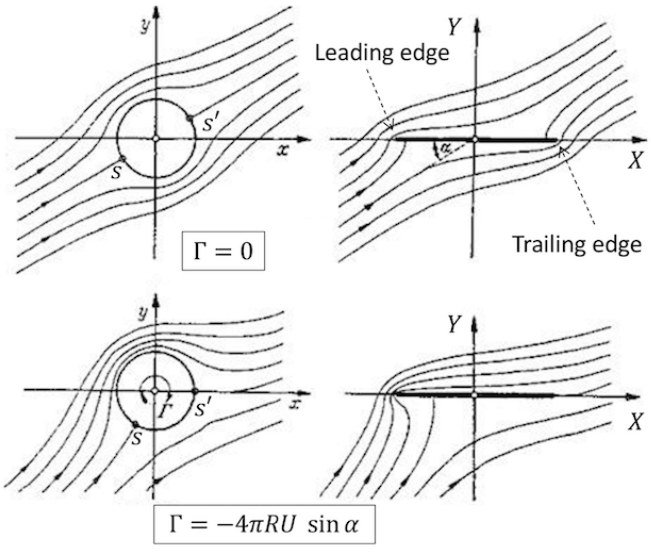
We see that, if the term inside the brackets in the above equation is non zero, the flow speed is infinite when $z = \pm R$, which corresponds to $Z = \pm 2R$, that is to say the ends of the plate. We call *trailing edge* the downstream edge of the plate, located at $X = 2R$, and *leading edge* the upstream edge, located at $X = -2R$. The velocity at the trailing edge can be made finite if the term in brackets vanishes for $z = R$, which implies:

$$U\mathrm{e}^{-\mathrm{i}\alpha} - U\mathrm{e}^{\mathrm{i}\alpha} - \frac{\mathrm{i}\Gamma}{2\pi R} = 0, \tag{3.45}$$

that is to say:

$$\Gamma = -4\pi R U \sin\alpha. \tag{3.46}$$

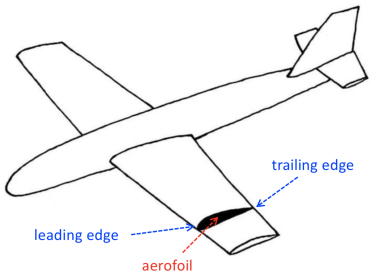
This is called the *Kutta condition*, and its meaning will be discussed in the next section in the context of more realistic aerofoils. Writing $z = R + \epsilon$, with $\epsilon \ll R$, equation (3.44) then yields $V_X \to U\cos\alpha$ and $V_Y \to 0$ when $\epsilon \to 0$. The fluid is therefore parallel to the plate at the trailing edge. The figure below shows the streamlines around the circle in the z–plane and around the finite plate in the Z–plane for both the case where there is no circulation and the case where there is circulation and the condition above it satisfied:



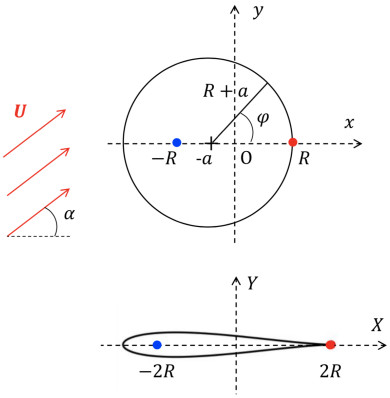
The Kutta condition implies $|\Gamma| < 4\pi R|U|$, so that in the z–plane there are two distinct stagnation points s and s' on the surface of the circle (see section 3.2.5), and since $\mathrm{d}w/\mathrm{d}z = 0$ when $z = R$ (eq. [3.45]), s' is on the x–axis when the Kutta condition is met.

3.3.6 The Joukowski aerofoil

In the case of the plate considered above, the velocity at the trailing edge is finite when the Kutta condition is met, but the velocity at the leading edge is still infinite. This is why flat plates with sharp edges are not used to make aerofoils.



Instead, aerofoils have a rounded leading edge and a sharp trailing edge, which yields a finite velocity everywhere, as we are now going to show.



We can generate such an aerofoil by applying Joukowski mapping on a circle in the z–plane which center is not at the origin, as illustrated on the figure. Since $z = -a + (R+a)\mathrm{e}^{\mathrm{i}\varphi}$ along the circle, Joukowski transformation (3.38) with $c = R$ yields the shape of the co–called *Joukowski aerofoil* in the Z–plane as:

$$Z = -a + (R+a)\mathrm{e}^{\mathrm{i}\varphi} + \frac{R^2}{-a + (R+a)\mathrm{e}^{\mathrm{i}\varphi}}, \qquad (3.47)$$

which is represented on the figure.

The complex potential in the z–plane for an angle of attack α is the same as that given by equation (3.42) but with z shifted to $z + a$ and the radius of the circle changed from R to $R + a$:

$$w(z) = U\left[(z+a)\mathrm{e}^{-\mathrm{i}\alpha} + \frac{(R+a)^2}{z+a}\mathrm{e}^{\mathrm{i}\alpha}\right] - \frac{\mathrm{i}\Gamma}{2\pi}\ln(z+a)\,. \qquad (3.48)$$

We then redo the same calculation as above to obtain the velocities:

$$V_X - \mathrm{i}V_Y = \frac{\mathrm{d}w}{\mathrm{d}z}\left(\frac{\mathrm{d}Z}{\mathrm{d}z}\right)^{-1}, \qquad (3.49)$$

where $w(z)$ is now given by equation (3.48) and Z is still given by equation (3.38) with $c = R$. This yields:

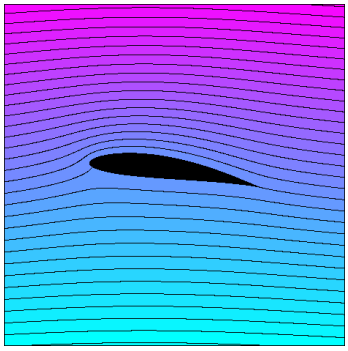
$$V_X - \mathrm{i}V_Y = \left(U\mathrm{e}^{-\mathrm{i}\alpha} - U\mathrm{e}^{\mathrm{i}\alpha}\frac{(R+a)^2}{(z+a)^2} - \frac{\mathrm{i}\Gamma}{2\pi(z+a)}\right)\frac{z^2}{z^2 - R^2}. \qquad (3.50)$$

The denominator still vanishes at $z = \pm R$, which still corresponds to $Z = \pm 2R$. From equations (3.38) and (3.39), we see that points on the x–axis in the z–plane map into points on the X–axis in the Z–plane. Therefore, the aerofoil intersect the X–axis at points given by equation (3.47) with $\mathrm{e}^{\mathrm{i}\varphi} = \pm 1$, that is to say at $Z = 2R$ and $Z = -2R\left(1 + 2a^2/(2aR + R^2)\right)$. This means that the point $Z = -2R$ is *inside* the aerofoil,

and there is only one point in the flow where the denominator of equation (3.50) vanishes, which is the trailing edge of the aerofoil. As in the case of the plate, this singularity is removed if the term in brackets in equation (3.50) vanishes as well for $z = R$, which requires:

$$\boxed{\Gamma = -4\pi(R + a)U \sin \alpha.} \tag{3.51}$$

This is the Kutta condition for the Joukowski aerofoil. As above, writing $z = R + \epsilon$, with $\epsilon \ll R$, equation (3.50) then yields $V_X \to U \cos\alpha/(1 + a/R)$ and $V_Y \to 0$ when $\epsilon \to 0$. The fluid is therefore parallel to the plate at the trailing edge. *When the Kutta condition is met, there are no singularities and the flow is smooth everywhere.*



The figure shows the streamlines relative to a Joukowski aerofoil which is inclined with respect to the oncoming flow. (*Credit: Wikimedia*)

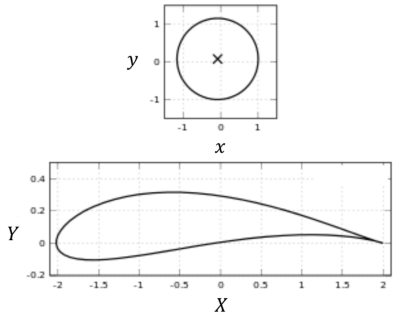
We can now rephrase the Kutta condition by saying that, for an aerofoil moving through a fluid with an angle of attack α, *the circulation will adjust itself so that the flow leaving the trailing edge of the aerofoil is smooth.*

In other words, there are, mathematically, a number of solutions which are attainable for the velocity of the flow around the aerofoil, but those giving an infinite velocity are unphysical. Nature picks up the only physical solution by adjusting the circulation around the aerofoil. We will describe below how this is achieved.

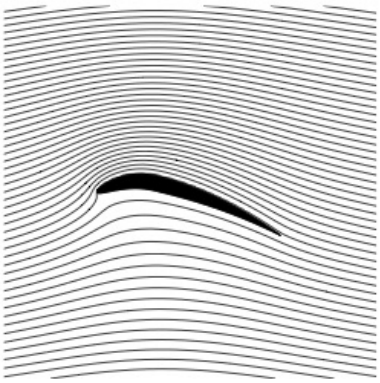
We have found above that V_X is non zero at the trailing edge of the plate. However, this is due to the particular symmetrical geometry of the aerofoil. For more realistic aerofoils, the velocity does vanish at the trailing edge, which therefore coincides with a stagnation point.

3.3.7 Forces on aerofoils and the Kutta–Joukowski theorem

A symmetrical aerofoil like that described above would not be the most efficient way of getting a lift. As pointed out in section 3.2.5, the lift is due to the curvature of the streamlines, with pressure decreasing towards the centre of curvature. If the symmetrical aerofoil is not inclined with respect to the incoming flow, the strealimes are identical above and below the aerofoil and there is no lift. When the aerofoil is inclined however, a lift is produced because the curvature is larger above than below the aerofoil, as seen on the figure above. But a larger lift would be obtained if the curvature below the aerofoil were giving a pressure that increases as we approach the aerofoil. This is why *cambered* aerofoils are used.

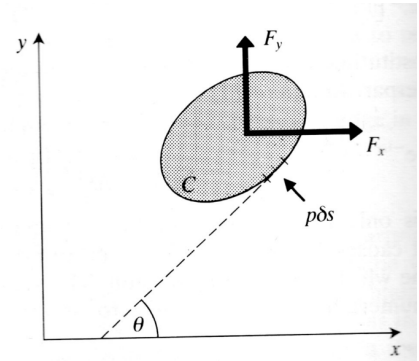


Cambered aerofoils can be generated using Joukowski transformation by moving the center of the circle not just along the x–axis, as done above, but also along the y-axis. In this figure the center has been moved in the upper left quadrant.



As illustrated on the figure, this aerofoil gives a much larger lift than a symmetrical aerofoil. The lift is also larger if the aerofoil is thinner, which is why birds have thin wings. Airplanes however tend to have thicker wings because there are other considerations coming into play, such as design and fuel storage.

We are now going to calculate the force exerted by the flow on an aerofoil with *any shape*. Here we assume that the Kutta condition is met, so that there are no singularities in the flow. Although this condition has been derived above for a symmetrical aerofoil, it is valid for any kind of aerofoil as could be shown by making a Joukowski transformation of a circle with a centre not on the x–axis.



We take the x–axis to be along the direction of the flow velocity at infinity and note C the (closed) contour of the object. We consider a small element δs along the boundary, as represented on the figure (*Credit: Acheson*), where the dashed line is tangent to the curve.

The components δF_x and δF_y of the force exerted on δs by the flow are given by $(\delta F_x, \delta F_y) = (-p\sin\theta, p\cos\theta)\delta s$, so that: $\delta F_x - \mathrm{i}\delta F_y = -p\,(\sin\theta + \mathrm{i}\cos\theta)\,\delta s = -p\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\delta s$. Since C is a rigid boundary, the flow velocity at δs is tangent to the curve, which implies $v_x = v\cos\theta$ and $v_y = v\sin\theta$. This yields $v_x - \mathrm{i}v_y = v\mathrm{e}^{-\mathrm{i}\theta}$, and since $v_x - \mathrm{i}v_y = \mathrm{d}w/\mathrm{d}z$, we obtain:

$$v^2 = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{e}^{2\mathrm{i}\theta},$$

along C. Assuming the flow to be steady, Bernoulli's theorem (3.5) implies that $\rho v^2/2 + p$

is constant throughout the fluid, as the gravitational potential is uniform throughout the domain of the flow. We note k the constant, so that the expression above for $\delta F_x - \mathrm{i}\delta F_y$ becomes:

$$\delta F_x - \mathrm{i}\delta F_y = \left(\frac{1}{2}\rho v^2 - k\right) \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\delta s = \frac{\mathrm{i}}{2}\rho \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{e}^{\mathrm{i}\theta}\delta s - \mathrm{i}k\mathrm{e}^{-\mathrm{i}\theta}\delta s.$$

By integrating over the closed contour C, we then obtain:

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}}{2}\rho \oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{e}^{\mathrm{i}\theta}\delta s - \mathrm{i}k \oint_C \mathrm{e}^{-\mathrm{i}\theta}\delta s,$$

where F_x and F_y are the components of the total force exerted by the flow on the object (the force here is implicitly meant to be per unit length in the direction perpendicular to the plane of the flow). We have $\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\delta s = \mathrm{i}\delta s\cos\theta + \delta s \sin\theta = \mathrm{i}\delta x + \delta y$, so that the second integral on the right–hand side of the above expression is zero. We can re–write the first integral by noting that $\mathrm{e}^{\mathrm{i}\theta}\delta s = \delta s\cos\theta + \mathrm{i}\delta s\sin\theta = \delta x + \mathrm{i}\delta y \equiv \delta z$, which yields:

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}}{2}\rho \oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \delta z. \tag{3.52}$$

This is known as *Blasius's theorem*. As the function $\mathrm{d}w/\mathrm{d}z$ is analytic everywhere in the domain of the flow, Cauchy's theorem and a deformation of contours (see appendix A) imply that the value of the integral in equation (3.52) is actually the same if we replace C by any closed contour surrounding the aerofoil.

We now choose the origin O of our coordinate system inside the aerofoil and define η as the radius of the smallest circle centered on O which encloses the aerofoil. Because $\mathrm{d}w/\mathrm{d}z$ is analytic in the flow domain, it can be expressed as a Laurent series which converges on the domain $|z| > \eta$:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \sum_{n=1}^{+\infty} \frac{a_n}{z^n} + \sum_{n=0}^{+\infty} b_n z^n,$$

where a_n and b_n are constant complex numbers (see appendix A). Since $\mathrm{d}w/\mathrm{d}z = v_x - \mathrm{i}v_y$ and the velocity stays finite in the flow domain, $b_n = 0$ for all $n \geq 1$. At infinity, $v_x = U$ and $v_y = 0$, so that $b_0 = U$. The $1/z$ term corresponds to the vortex flow, so $a_1 = -\mathrm{i}\Gamma/(2\pi)$ (see eq. [3.35]). Therefore the series becomes:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = U - \frac{\mathrm{i}\Gamma}{2\pi z} + \frac{a_2}{z^2} + \ldots$$

which is the superposition of a uniform flow, vortex flow, dipole flow, etc. Blasius's theorem (3.52) with the contour C changed to a contour C' that encloses the circle of radius η can then be written as:

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}}{2}\rho \oint_{C'} \left(U - \frac{\mathrm{i}\Gamma}{2\pi z} + \frac{a_2}{z^2} + \ldots\right)^2 \delta z.$$

According to the residue theorem (see appendix A), only the $1/z$ term contributes to the integral, which is equal to $2\pi\mathrm{i}$ times the coefficient of this term. Therefore:

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}}{2}\rho \times 2\pi\mathrm{i} \times \left(\frac{-\mathrm{i}U\Gamma}{\pi}\right) = \mathrm{i}\rho U\Gamma,$$

which implies the so–called *Kutta–Joukowski lift theorem*:

$$\boxed{F_x = 0, \quad F_y = -\rho U \Gamma.} \tag{3.53}$$

This theorem generalizes the result which was obtained for the particular case of a flow past a cylinder (eq. [3.30]). Note that the lift is the component of the force perpendicular to the direction of the flow at infinity (which is inclined with respect to the aerofoil for non–zero angles of attack), whereas the drag is the component of the force in that direction. Here again, the absence of a drag is due to the fact that viscosity has been neglected. Since Γ satisfies the Kutta condition (3.51), the lift can also be written as:

$$\boxed{F_y = 4\pi\rho(R+a)U^2 \sin\alpha.} \tag{3.54}$$

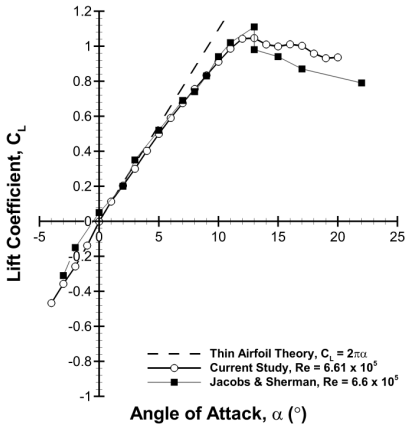
We define the dimensionless lift coefficient as:

$$C_L = \frac{F_y}{\frac{1}{2}\rho U^2 d}, \tag{3.55}$$

where d is the dimension of the aerofoil (in three dimensions, we would replace d by the surface of the aerofoil, so we can think of C_L above as being a lift coefficient per unit length in the direction perpendicular to the flow). Using the above expression for F_y, we obtain:

$$C_L = 8\pi \frac{R+a}{d} \sin\alpha. \tag{3.56}$$

This coefficient is used to compare the performance of different aerofoils with different dimensions and flying at different speeds. It can be obtained from measurements done in wind tunnels.

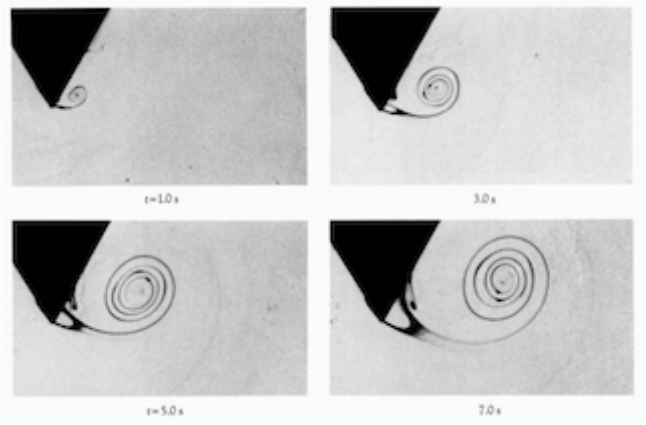


The figure shows C_L *versus* α. The dashed line is the theoretical curve and the other curves correspond to measurements. It can be seen that C_L decreases for values of α larger than about 12°: the plane stalls! This cannot be captured by the theory developed here because we have neglected viscosity. Viscous forces result in a boundary layer which detaches from the wing for larger angles. This will be studied in section 4.2.3 in the following chapter. (*Credit: Daniel Heathcote, PhD thesis, 2017.*)

3.3.8 The origin of the circulation

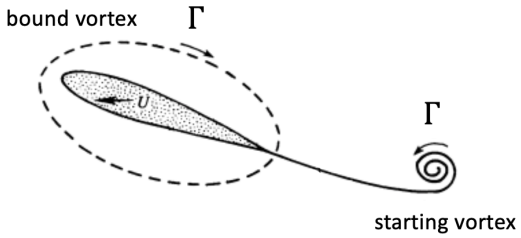
As we have mentioned above, the flow adjusts itself and create a circulation so that the velocity at the trailing edge of the aerofoil stays finite. We now describe how this happens.

When an aerofoil, e.g. the wing of an airplane, is accelerated from rest in a fluid, the flow around the sharp edge forms a boundary layer in which viscous forces are important. They produce a separated flow which moves from below the wing up around the trailing edge to the top, because pressure has become higher at the bottom of the wing than at the top. This induces a so–called *starting vortex*.



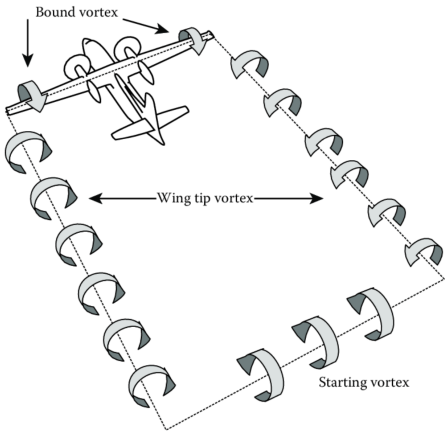
From Van Dyke, *Album of Fluid Motion*. The figure shows a starting vortex produced when a fluid is moved impulsively past a wedge. Here the fluid is viscous, but a starting vortex is produced in a boundary layer even if the viscosity of the fluid further away from the object is negligible.

As the wing accelerates, this vortex is left behind. Initially, the circulation around any closed contour in the flow is zero, as there is no vorticity if viscosity is negligible. After the aerofoil moves, the circulation around a contour enclosing the starting vortex becomes non zero, as the flow in some part along the contour has been affected by viscous forces in the boundary layer. However, due to Kelvin's theorem, the circulation around a contour enclosing both the aerofoil and the starting vortex has to remain zero, as the flow along this contour has not been affected by viscous forces. This implies that the flow produces a circulation in opposite direction to the starting vortex around the aerofoil. This can be seen as a fictitious vortex and is *bound* to the wing.



These two vortices are shown in the figure. Here the aerofoil was moved from right to left.

The air also moves from below the wing up around the tips to the top, and this creates *trailing*, or *tip* vortices that extend for miles behind the wings. The trailing vortices on both sides rotate in opposite directions due to Kelvin's theorem. As illustrated on the figure below, these vortices are such that the resultant vortex line is closed, as expected from Helmholtz's theorem.





It is the bound vortex that creates the circulation around the wing required to produce a finite velocity at the trailing edge and which results in a lift. Note that although a vortex is the cause of the circulation, it is localized at the surface of the wing and therefore the flow is still irrotational away from this boundary layer, which justifies the approximation of potential and steady flow used in this chapter.

Chapter 4

Boundary layers

As pointed out in section 2.3.1 of chapter 2, the no–slip boundary condition at a rigid boundary holds *however small the viscosity* may be: molecular transport at the surface prevents any slipping of the fluid parallel to the boundary.

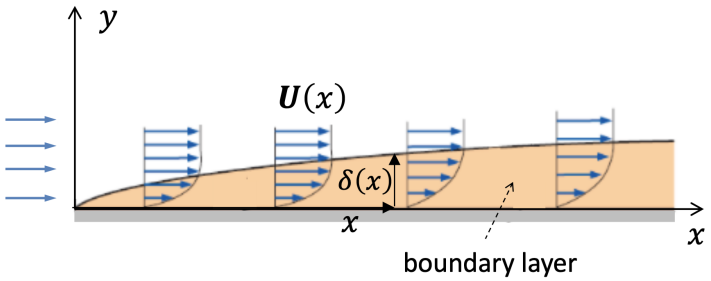
This has important consequences for fluids with very small viscosity (like the air) moving past an aerofoil, as the velocity varies very rapidly near the surface of the aerofoil to adjust to the boundary condition. This creates a significant stress according to equation (2.13), even if the viscosity is very small, and results in a so–called *boundary layer* in which viscous forces cannot be neglected. As boundary layers are thin and localized near rigid boundaries, a high Reynolds number flow can be approximated as ideal away from the boundary, where techniques applicable to potential flows can be used if the flow is irrotational. The solutions obtained for the ideal fluid can then be matched to those in the boundary layers. Boundary layers were first studied experimentally and mathematically by Prandtl (1905).

A boundary layer will develop whenever there is a boundary at which the flow has to adjust to a condition which is different than what an inviscid solution would give. Although in this chapter we will focus on flows near a rigid boundary, boundary layers may then also occur, e.g., at a free surface, where the tangential stress is zero (as shown in section 2.3.3).

The next section describes the boundary layer that develops in a uniform flow passing past a thin plate. Although this is an idealization, it is a good approximation to the case of a flow past an aerofoil.

4.1 The boundary layer on a flat plate

We consider a steady two–dimensional incompressible flow with high Reynolds number in the (x, y)–plane past a thin plate located at $y = 0$:



As the flow passes the leading edge of the plate (located at $x = 0$), it slows down. The flow closest to the plate slows down first, and in turn slows down the flow higher up, so that a layer with a velocity gradient is formed. If $U(x)$ does not increase significantly with x, the thickness of the boundary layer increases with distance from the leading edge because of the cumulative effect of the frictional force from the plate: the momentum lost by the flow increases with x. However, if U increases enough with x, the acceleration of the flow in the boundary layer by the outer flow also increases with x and the boundary layer becomes thinner. Another way to describe the boundary layer is through the vorticity of the flow.

> The vorticity is non–zero in the boundary layer because of the shear: the rigid boundary acts as a source of vorticity which spreads out vertically by viscous diffusion while being advected horizontally with the flow.

Outside the boundary layer, the flow is potential (viscosity being negligible) with a velocity $\mathbf{U}(x)$ along the x–axis. Across the boundary layer, the velocity varies from 0 at the surface of the plate to $\mathbf{U}(x)$.

4.1.1 Thickness of the boundary layer

The vorticity generated by the rigid boundary between 0 and x moves downstream of x in a time $t \sim x/U(x)$, as advection along the x direction in the boundary layer occurs at a velocity which is of order of magnitude $U(x)$. During this time t when it is between 0 and x, the vorticity diffuses viscously over a vertical distance[1] $\delta \sim \sqrt{\nu t}$. By definition, this is the thickness of the boundary layer, and it can be written as:

[1] In the vertical direction, momentum is transported by viscosity only (as the average flow is going in the horizontal direction). We neglect pressure to obtain an order of magnitude estimate of the diffusion lengthscale. Navier–Stokes equation then gives gives $\partial v_y/\partial t = \nu \nabla^2 v_y$, from which it can be seen that diffusion over a scale δ takes the time $t \sim \delta^2/\nu$.

$$\delta(x) \sim \sqrt{\frac{\nu x}{U(x)}}. \tag{4.1}$$

We note L the scale over which the velocity along x changes significantly (that would typically be the length of the plate if it had a finite size). The Reynolds number for the flow is defined as $Re = UL/\nu$ (see eq. [2.24]), and the expression above can be written under the form:

$$\boxed{\frac{\delta(x)}{x} \sim \sqrt{\frac{L}{x}}\frac{1}{\sqrt{Re}} \ll 1,} \tag{4.2}$$

since L and x are typically of the same order of magnitude. This is the core of the boundary layer approximation: *its scale is much smaller than the horizontal scale.*

The estimate above is only valid if the boundary layer is *laminar*, that it to say there is no turbulence. When the flow is turbulent, there is convective in addition to molecular transport of vorticity in the vertical direction, and the thickness of the boundary layer becomes much larger. This will be discussed further below.

4.1.2 Equation of motion

Ignoring gravity, Navier–Stokes equation (2.21) in the x– and y–directions in the boundary layer gives:

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right), \tag{4.3}$$

$$v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right). \tag{4.4}$$

The incompressibility condition also has to be satisfied:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{4.5}$$

Since L and δ (where the x–dependence is implied) are the characteristic scales over which the velocity changes in the x and y–directions, respectively, we have $|\partial v_x/\partial x| \sim v_x/L$ and $|\partial v_y/\partial y| \sim v_y/\delta$. Therefore, equation (4.5) yields:

$$\boxed{v_y \sim \frac{\delta}{L}v_x \ll v_x}. \tag{4.6}$$

Also,

$$\left|\frac{\partial^2 v_x}{\partial x^2}\right| \sim \frac{v_x}{L^2} \ll \frac{v_x}{\delta^2} \sim \left|\frac{\partial^2 v_x}{\partial y^2}\right|,$$

with similar result for v_y. This yields:

$$\nabla^2 v_x \simeq \frac{\partial^2 v_x}{\partial y^2}, \quad \text{and} \quad \nabla^2 v_y \simeq \frac{\partial^2 v_y}{\partial y^2}. \tag{4.7}$$

In equation (4.3), all the terms involving the velocity are on the order of v_x^2/L (since $\nu \sim U\delta^2/L$ from eq. [4.1], and $v_x \sim U$); therefore, $\partial p/\partial x = \mathcal{O}\left(\rho v_x^2/L\right)$. Similarly, in equation (4.4), all the terms involving the velocity are on the order of $v_x^2\delta/L^2$; therefore, $\partial p/\partial y = \mathcal{O}\left(\rho v_x^2\delta/L^2\right)$. It follows that $|\partial p/\partial y| \ll |\partial p/\partial x|$, so that

> p can be regarded as a function of x only, which means that the pressure is approximately uniform across the boundary layer.

The pressure can therefore be calculated from the velocity outside the boundary layer, where $\nu = 0$ and Navier–Stokes (or Euler) equation yields:

$$U(x)\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{4.8}$$

Using the approximation (4.7), Navier–Stokes equation (4.3) can be written under the form:

$$\boxed{v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 v_x}{\partial y^2}.} \tag{4.9}$$

This is the boundary layer equation, also called *Prandtl's equation*. It has to be solved with the following boundary conditions:

$$v_x = v_y = 0 \;\text{ at }\; y = 0, \quad \text{and} \quad v_x(x,y) = U(x) \;\text{ for }\; y/\delta \to +\infty. \tag{4.10}$$

The tangential shear stress in the boundary layer is:

$$\sigma_{xy} = \rho\nu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right).$$

Since $|\partial v_x/\partial y| \sim v_x/\delta$ and $|\partial v_y/\partial x| \sim v_x\delta/L^2 = (v_x/\delta)(\delta^2/L^2) \ll v_x/\delta$, we obtain:

$$\boxed{\sigma_{xy} \simeq \rho\nu\frac{\partial v_x}{\partial y}.} \tag{4.11}$$

Although the above results have been derived for a flow past a flat plate, they still apply if the boundary is curved. The coordinates x and y then represent the coordinates along and perpendicular to the boundary, respectively. In that case, there is a pressure gradient $\partial p/\partial y$ comparable to $\partial p/\partial x$, to balance the centripetal acceleration induced by the curvature of the boundary. However, changes of p along y in the boundary layer are still δ/L smaller than changes of p along x, so that p can be regarded as uniform across the (very thin) boundary layer.

4.1.3 Velocity profile in the boundary layer

Because the flow outside of the boundary layer is approximated as being inviscid, it is unaffected by the presence of the plate. Therefore, if the velocity upstream of the plate is uniform, it stays uniform everywhere outside of the boundary layer, which means $U(x) =$

U_0, where U_0 is a constant. In that case, equation (4.8) yields $\mathrm{d}p/\mathrm{d}x = 0$, so that Prandtl's equation (4.9) becomes:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}. \tag{4.12}$$

We take an arbitrary length scale x_0 in the horizontal direction, associated with the vertical length scale $\delta(x_0) = \sqrt{\nu x_0/U_0}$, and define the dimensionless variables:

$$x' = \frac{x}{x_0}, \quad y' = \frac{y}{\delta(x_0)} \quad v'_x = \frac{v_x}{U_0}, \quad \text{and} \quad v'_y = \frac{v_y}{U_0}\frac{x_0}{\delta(x_0)}. \tag{4.13}$$

The scaling is chosen so that x' and y' on the one hand, and v'_x and v'_y on the other hand, are of the same order of magnitude (for v'_y and v'_x, this follows from eq. [4.6] which shows that $v_y \sim v_x(\delta/L)$). The equation above then becomes:

$$v'_x \frac{\partial v'_x}{\partial x'} + v'_y \frac{\partial v'_x}{\partial y'} = \frac{\partial^2 v'_x}{\partial y'^2}. \tag{4.14}$$

This has to be solved together with the mass conservation equation (4.5) which, in dimensionless form, is:

$$\frac{\partial v'_x}{\partial x'} + \frac{\partial v'_y}{\partial y'} = 0. \tag{4.15}$$

Using the dimensionless variables, the boundary conditions (4.10) can be expressed as:

$$v'_x = v'_y = 0 \ \text{ at } \ y' = 0, \quad \text{and} \quad v'_x(x', y') = 1 \ \text{ for } \ y' \to +\infty. \tag{4.16}$$

As Re does not enter the equations nor the boundary conditions, the solutions do not depend on Re. *The Reynolds number affects only the thickness of the boundary layer, not the velocity profile.*

The velocity cannot depend on the length scale x_0 either, as it is arbitrary. Therefore, since $v_x(x, y) = U_0 v'_x(x', y')$, v'_x cannot depend on x' and y' independently, but has to depend on a combination of these two variables which cancels out x_0. As $x' \propto 1/x_0$ and $y' \propto 1/\delta(x_0) \propto 1/\sqrt{x_0}$, the simplest combination that does not depend on x_0 is:

$$\eta = \frac{y'}{\sqrt{x'}} = \sqrt{\frac{U_0}{\nu x}}\, y, \tag{4.17}$$

which can also be written as $\eta = y/\delta(x)$. We therefore look for solutions of the form $v'_x = g(\eta)$, which is equivalent to:

$$v_x = U_0\, g\,(\eta) = U_0\, g\left(\sqrt{\frac{U_0}{\nu x}}\, y\right). \tag{4.18}$$

Similarly, $v_y(x, y) = U_0(\delta(x_0)/x_0)\; v'_y(x', y') = \sqrt{U_0\nu/x_0}\; v'_y(x', y')$ is independent of x_0 if $v'_y \propto \sqrt{x_0}$, which is achieved by taking $v'_y(x', y') = h(\eta)/\sqrt{x'}$. This then corresponds to:

$$v_y = \sqrt{\frac{U_0\nu}{x}}\, h(\eta) = \sqrt{\frac{U_0\nu}{x}}\, h\left(\sqrt{\frac{U_0}{\nu x}}\, y\right). \tag{4.19}$$

Such solutions, which have the same y–dependence, and therefore the same shape, at all values of x except for a different scale factor $\sqrt{U_0/(\nu x)} = 1/\delta(x)$, are called *self–similar*.

Since the flow is incompressible, we can introduce a stream function ψ such that $v_x = \partial\psi/\partial y$ and $v_y = -\partial\psi/\partial x$. With the above expressions (4.18) and (4.19) for v_x and v_y, ψ has to be of the form:

$$\psi = \sqrt{U_0 \nu x}\, f(\eta).$$

We then have:

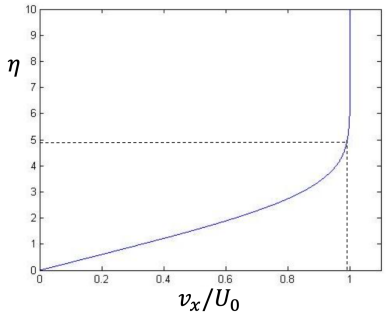
$$\begin{aligned} v_x &= \frac{\partial\psi}{\partial y} = \sqrt{U_0\nu x}\, f'(\eta)\frac{\partial\eta}{\partial y} = U_0\, f'(\eta), \\ v_y &= -\frac{\partial\psi}{\partial x} = -\frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\, f(\eta) - \sqrt{U_0\nu x}\, f'(\eta)\frac{\partial\eta}{\partial x} = \frac{1}{2}\sqrt{\frac{U_0\nu}{x}}\left(-f(\eta) + \eta\, f'(\eta)\right), \end{aligned}$$

where $f'(\eta) \equiv \mathrm{d}f/\mathrm{d}\eta$. Substituting these expressions in Prandtl's equation (4.12) then yields the so–called Blasius's equation:

$$f''' + \frac{1}{2} f f'' = 0, \tag{4.20}$$

while the boundary conditions (4.10) become:

$$f = f' = 0 \;\text{ at }\; \eta = 0, \quad \text{and} \quad f' = 1 \;\text{ for }\; \eta \to +\infty. \tag{4.21}$$



This boundary value problem has to be solved numerically. The figure shows the solution obtained for v_x/U_0 as a function of $\eta = y/\delta(x)$. The fact that $v_x/U_0 = 0.99$ at $\eta = 4.9$ confirms quantitatively that $\delta(x)$ gives a measure of the thickness of the boundary layer (this had been obtained by a qualitative argument only in section 4.1.1).

4.1.4 Frictional force on a flat plate

We orientate the normal to the plate $\hat{\mathbf{n}} = \hat{\mathbf{y}}$ in the positive y–direction, to be consistent with the analysis above. Therefore, the viscous stress on the plate is given by equation (2.1) as $\mathbf{T} = \sigma_{xy}\hat{\mathbf{x}} + \sigma_{yy}\hat{\mathbf{y}} + \sigma_{zy}\hat{\mathbf{z}}$, where the components of the stress tensor are evaluated at $y = 0$. As $v_z = 0$ and v_y does not depend on z, equation (2.13) shows that $\sigma_{zy} = 0$, so that the tangential stress is $\sigma_{xy}\hat{\mathbf{x}}$ with:

$$\sigma_{xy} = \rho\nu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)_{y=0} = \rho U_0\sqrt{\frac{U_0\nu}{x}}\, f''(0), \tag{4.22}$$

where $f''(0) = 0.33$.

If the plate is immersed in the flow, there is a boundary layer on each side, so that the total frictional force $\mathbf{F} = F_x\hat{\mathbf{x}}$ on a plate of length L per unit length in the z–direction is

twice the force exerted on one side:

$$\boxed{F_x = 2\int_0^L T_x \mathrm{d}x = 1.33\rho U_0\sqrt{U_0\nu L} = 1.33\,\frac{\rho U_0^2 L}{\sqrt{Re}}}, \tag{4.23}$$

where we have used the fact that the scale over which v_x varies is on the order of the length of the plate, so that $Re = U_0 L/\nu$. This force varies as $\sqrt{L}$ rather than L because the stress tensor decreases with x, due to the thickening of the boundary layer wich reduces the velocity gradient. Here the plate is at rest and the fluid is moving. If instead the fluid were initially at rest and the plate moving towards the left, the situation would be the same and the force would oppose the motion of the plate: it is therefore a *drag*.

The presence of a boundary layer therefore accounts for the existence of a drag on an object moving relative to a fluid, and solves the d'Alembert paradox presented in section 3.2.5.

4.1.5 Vorticity in the boundary layer and wake

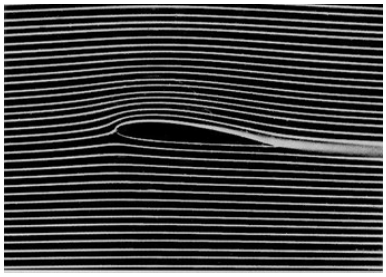
The vorticity is given by:

$$\boldsymbol{\omega} = \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right)\hat{\mathbf{z}}. \tag{4.24}$$

We have shown when deriving equation (4.11) that the first term on the right side of the equation above dominates over the second term, which yields:

$$\omega \simeq \frac{\partial v_x}{\partial y} = U_0\sqrt{\frac{U_0}{\nu x}}f''(\eta). \tag{4.25}$$

This vorticity, which is generated where the fluid is in contact with the rigid boundary, is both diffused vertically by viscosity and advected downstream by the flow.



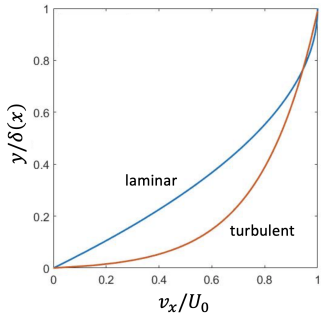
Therefore, the flow *behind* the plate also has vorticity. It is called the *wake*. It can be seen together with the boundary layer on the figure which shows streamlines around an aerofoil immersed in a fluid.

4.1.6 Transition to turbulence

The velocity profile and drag obtained above are in excellent agreement with experiments as long as the flow is laminar. However, it is an experimental fact that the flow becomes turbulent when the Reynolds number exceeds a critical value. This can be understood by noting that, for lower Reynolds numbers, viscosity is efficient enough that any small disturbances present in the fluid are damped out. At higher Reynolds numbers however, viscosity becomes more ineffective and small perturbations can grow. The flow then becomes unstable and there is a transition to turbulence.

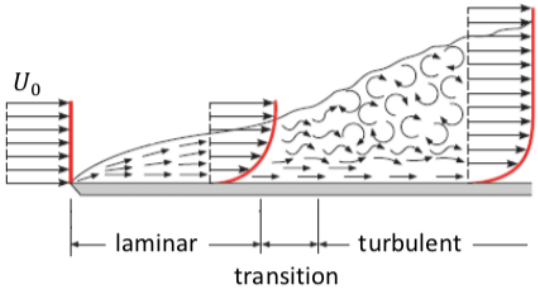
The flow becomes unstable when the *local* Reynolds number, defined as $Re_x = U\delta(x)/\nu$, becomes larger than about 600. This typically corresponds to the global Reynolds number $Re = UL/\nu$ being larger than a few 10^5. Because $Re_x \propto \delta(x) \propto \sqrt{x}$, the transition occurs at some distance from the leading edge of the plate.

As already pointed out above, when the flow becomes turbulent, the boundary layer is much thicker, because vertical transport of vorticity by turbulent motion is much more effective than molecular diffusion. This can be understood in the following way. We have seen that the transport of momentum perpendicularly to the direction of the main flow is due to the correlation between the fluctuations of the velocity in the fluid (see eq. [2.8]). In the case of molecular transport, which yields the viscous stress, these fluctuations are caused by the kinetic energy of the molecules associated with the finite temperature. When the flow is turbulent though, these fluctuations and their correlation are usually much larger. The associated stress, which is called the *Reynolds stress*, is then usually orders of magnitude larger than the viscous stress.

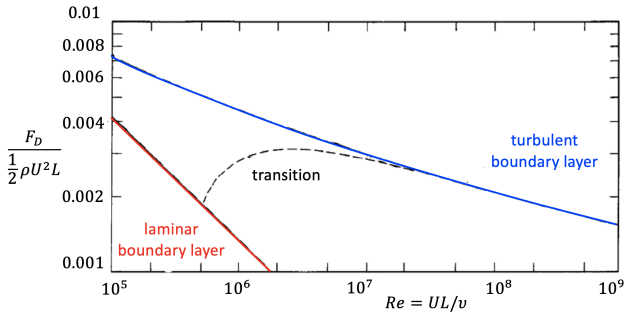


Therefore, vertical mixing is much more efficient in the turbulent case, and the flow is accelerated much more efficiently through the boundary layer, as shown by the velocity profile on the figure. (*Credit: https://aerospaceengineeringblog.com/boundary-layers/*)

The figure below gives a schematic view of the boundary layer when the flow becomes turbulent:



When the flow is turbulent, there is still a laminar sub–layer near the plate. This is because the flow is stationary at the boundary so that turbulent eddies cannot exist very close to it. Therefore, the tangential stress on the plate is still the viscous stress $\sigma_{xy}\hat{\mathbf{x}}$, with $\sigma_{xy} = \rho\nu(\partial v_x/\partial y)$ evaluated at $y = 0$ (eq. [4.11]). Since vertical mixing is much more efficient in a turbulent boundary layer, the increase of velocity near the boundary is much sharper in the turbulent than in the laminar case. *Therefore, the drag is much larger in the turbulent case*, and it decreases less sharply when the Reynolds number increases.



The figure shows the drag F_D normalized to $\rho U^2 L/2$ as a function of the Reynolds number in a turbulent boundary layer.

As the lift increases with the velocity (see eq. [3.54]), commercial aircrafts fly at high velocities to produce a lift sufficient to balance their weight. That leads to high Reynolds numbers at which the boundary layers over the wings are turbulent, which yields an enhanced drag. Keeping a boundary layer laminar though is a very complicated problem, because all laminar boundary layers can be made to 'trip' (that it to say, to transition) to turbulence by any inhomogeneities (like insects) on the wing. Tests in wind tunnels are therefore not very helpful as they tend to be too idealised. One way of preventing the boundary layer from becoming turbulent is to remove a small amount of air by suction through porous materials, multiple narrow surface slots or small perforations. That way, the thickness of the boundary layer, and hence the Reynolds number, does not increase along the wing, and the flow stays laminar. This has been used for some supersonic cruise aircrafts (see Problem Set 4).

4.2 Boundary layer separation

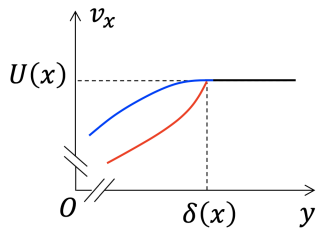
In the analysis above, we have assumed that the flow velocity was uniform outside of the boundary layer, so that the pressure gradient $\partial p/\partial x$ was zero. Let us now assume that there is a finite pressure gradient.

4.2.1 Condition for separation

As shown in section 4.1.2, p only depends on x, so that the pressure gradient inside the boundary layer is the same as that in the outer flow.

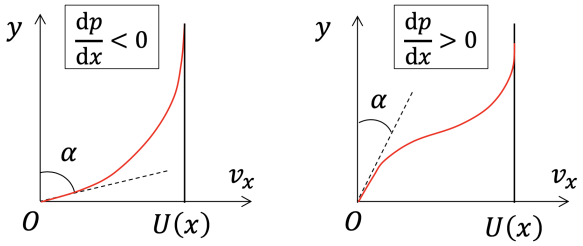
Because $v_x = v_y = 0$ at $y = 0$, Prandtl's equation (4.9) near the rigid boundary gives:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \rho\nu\frac{\partial^2 v_x}{\partial y^2}. \tag{4.26}$$



At the surface $y = \delta$ of the boundary layer, v_x matches the velocity $U(x)$ of the outer flow. For the stress to be continuous at $y = \delta$ (and therefore volume forces to stay finite), v_x has to approach $U(x)$ following the blue curve and not the red curve.

This implies that $\partial^2 v_x/\partial y^2 < 0$ at the surface of the boundary layer. Therefore, depending on whether $\mathrm{d}p/\mathrm{d}x$ is positive or negative, we can have either of the two cases represented on the figure below (where $\alpha = \partial v_x/\partial y$ at $y = 0$):



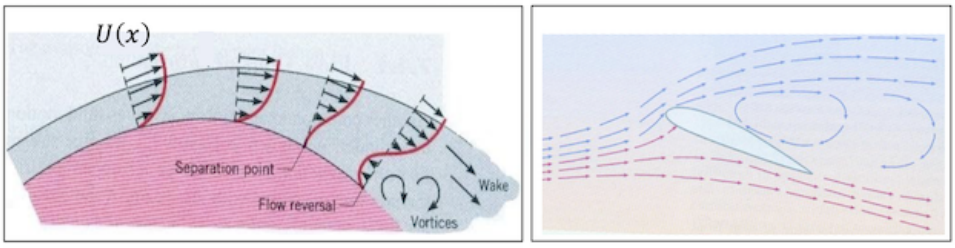
The fluid near the rigid boundary is subject to the following forces: (i) it is decelerated by the frictional force from the boundary, (ii) it is accelerated by the viscous stress from the upper layers, that is to say by the transfer of momentum from the outer flow which moves at velocity $U(x)$, (iii) it is accelerated or decelerated by the pressure gradient, depending on whether it is negative or positive.

When $\mathrm{d}p/\mathrm{d}x < 0$, $U(x)$ increases with x, and the transfer of momentum from the upper layers down to the flow in the boundary layer increases as the flow moves along the boundary. In addition, in that case, the pressure gradient also accelerates the flow directly in the boundary layer. Therefore, the boundary layer becomes thinner as the flow moves along x.

By contrast, when $\mathrm{d}p/\mathrm{d}x > 0$, $U(x)$ decreases with x. The transfer of momentum from the upper layers then decreases with x, and the flow in the boundary layer is also directly decelerated by the pressure gradient. Therefore, at some point along x, the positive acceleration from the outer layer is cancelled out by the adverse pressure gradient and viscous force from the boundary. The acceleration then becomes negative, so that the flow velocity near the boundary decreases. At some point x_s (stagnation point), the velocity becomes zero and the flow stalls. Further downstream, the velocity is negative so that the flow reverses direction. At x_s, the angle α on the right–hand side panel of the figure above is zero. The fact that v_x changes sign for $x > x_s$ is due to the fact that the inflexion point which has to be present on the graph above for $\mathrm{d}p/\mathrm{d}x > 0$ can only exist if v_x becomes negative near the boundary.

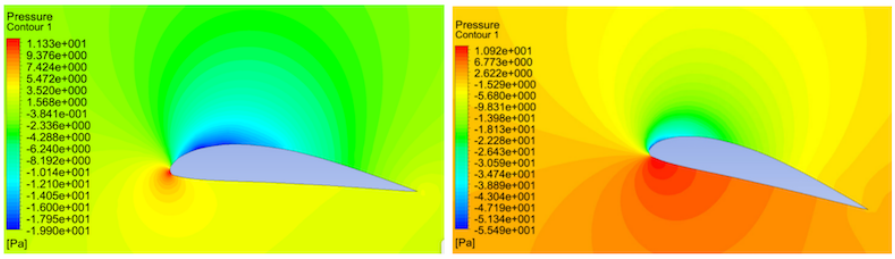
When $\mathrm{d}p/\mathrm{d}x > 0$, *separation* of the boundary layer from the rigid boundary at some distance x_s from the leading edge of the plate is therefore unavoidable.

This is illustrated on the figure below, which shows both the evolution of the velocity profile in the boundary layer along the rigid boundary for $\mathrm{d}p/\mathrm{d}x > 0$, and the streamlines around an aerofoil from which the boundary layer has separated. As indicated on the figure, the combination of flows in two directions induces a *wake of turbulent vortices*.



When separation occurs, Prandtl's equation (4.9) does not apply anymore, as v_x is no longer very large compared to v_y. In fact, separation occurs at the point where this equation becomes singular (see Landau & Lifshitz, Chapter IV, §40, for more details).

As we have seen in the previous chapter, an object immersed in a flow does induce a pressure gradient in the flow, so that the conditions for boundary layer separation may be created by the object itself. For example, as shown in section 3.2.5, the pressure around a cylinder immersed in a flow first decreases along the rigid boundary as the flow 'climbs' from the leading stagnation point up to the top of the cylinder, and then increases as the flow goes from the top down to the trailing stagnation point. Similar pressure gradients are created on the upper surface of an aerofoil, as can be seen from the numerical simulations displayed below, and which show the pressure contours for an angle for attak of 5° on the left panel and 15° on the right panel (*Credit: Kandil & Elnady, Int. J. Aerospace Sci., 2017, 5, 1*):



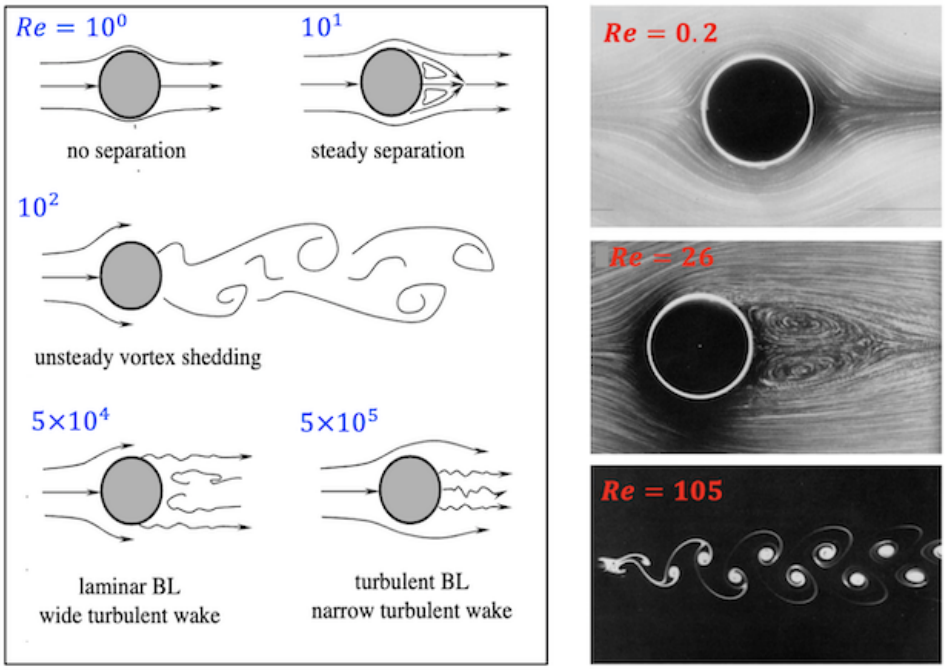
The pressure is maximum at the edges, where there are stagnation points, and has a minimum on the upper side. Therefore, *we expect the boundary layer to separate at some point along the upper surface of the aerofoil*, and separation occurs closer to the leading edge for higher angles of attack.

4.2.2 Effect of the Reynolds number on the separation

When the Reynolds number is low, molecular transport of momentum from the outer layers down into the boundary layer is more efficient. Therefore, *separation occurs further downstream or can be prevented altogether when the Reynolds number is small enough.* Similarly, in a turbulent flow, as pointed out above, the flow near the rigid boundary is accelerated by transport of momentum from the outer layers more efficiently than in a laminar boundary layer. Therefore, *turbulence also results in separation occurring further downstream.* Separation is said to be *delayed.* This yields to a narrower wake behind the object.

The different situations which arise depending on the Reynolds number are indicated on the figure below, which also shows the flow around a cylinder for Reynolds numbers of

0.2, 26 and 105 (from *An Album of Fluid Motion*, M. Van Dyke):



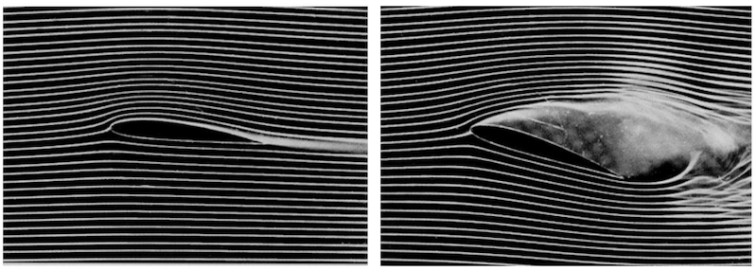
What is seen at $Re = 105$ is called a *von Kármán vortex street*. It appears when vortices are emitted alternatively by the flow on top and on the bottom of the object. The emission is periodic. This leads to a lift on the cylinder which oscillates with time. In a realistic set up, when the frequency of the oscillation matches some natural frequency of the object, structural damage to the object may occur[2].

4.2.3 Enhanced drag

As pointed out above, separation of the boundary layer results in the creation of a turbulent wake on the downstream side of the object. Because of turbulent mixing, the pressure in this wake is uniform and equal to that at the point where separation occurs. Therefore, *pressure in the flow downstream is lower than when there is no separation*. This results in a larger *pressure drag* on the object. This comes in addition to the viscous drag calculated above and due to the laminar sublayer at the surface of the plate which is always present because of the no–slip condition. As seen on one of the figures in section 4.2.1, the pressure distribution around aerofoils is such that boundary layer separation happens closer to the leading edge at higher angles of attack, which causes airplanes to stall (see

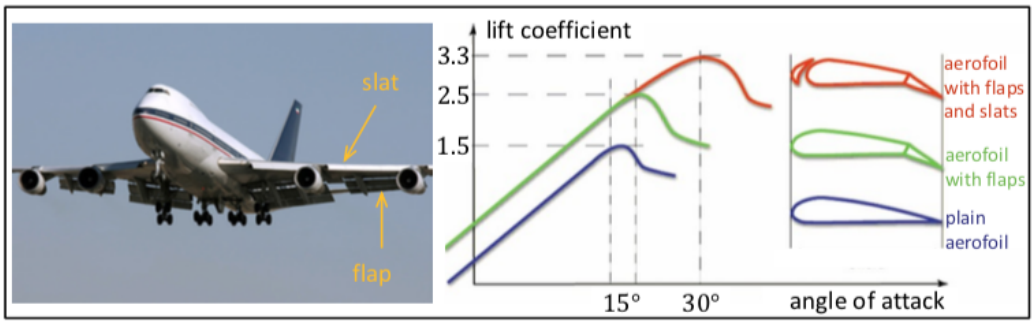
[2]This led to the collapse of the Tacoma Narrows span Bridge in the USA in 1940, four months after it had been opened. In that case, vortices were shed by the cylindrical cables in a periodic manner when strong cross–wind were present. The ensuing resonance vibrations in the bridge resulted in its collapse. It is this event that led von Kármán to discover this periodic shedding of vortices. One of the insurance policies could not be collected by the state of Washington, where the bridge was, because the insurance agent had pocketed the premium believing that the bridge would never collapse...

the discussion in section 3.3.7). This is illustrated on the figure below, which shows the streamlines around an aerofoil for two different angles of attack:



Delaying boundary layer separation is therefore a major goal in the design of airplanes or fast vehicles. This can be done by streamlining the aerofoil (or any fast vehicles), that is to say by aligning its back surface with the unperturbed streamlines of the outer flow. That is why many sea mammals or fishes have tapered caudal fin shaped like a crescent moon.

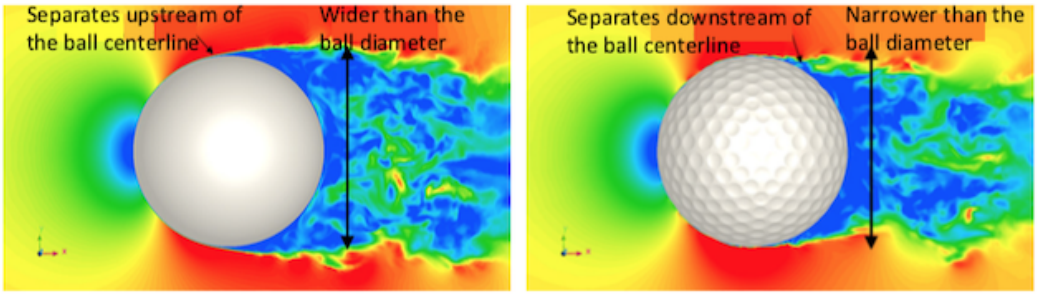
However, even if boundary layer separation over a wing can be prevented at small angles of attack α, it does eventually occur when this angle gets larger than some critical value α_c, as suggested by the figure above. When an aircraft cruises, it usually keeps a low angle of attack. For take off or landing though, equation (3.54) indicates that α has to be increased for a lift to be produced at low velocity, which is required to keep the lenghts of runaways reasonably short. In order to get a lift as large as possible, *wing flaps and slats* are used. These are small curved aerofoils that are fitted near either the leading or trailing edge of the wing. When the flap with slat is at the leading edge, the flow coming from below the wing and which passes through the gap between the flap and the wing replenishes the boundary layer with high speed fluid, which opposes the adverse pressure gradient and delays the separation of the boundary layer, increasing α_c. When the flap is near the trailing edge, the air passing through the gap adds momentum to the flow which circulates around the wing and which is slowed down by the adverse pressure gradient. This does not change α_c significantly, but it increases the circulation and therefore the lift, so that higher altitudes can be reached before the plane stalls.



Another strategy is to delay boundary separation by making the flow turbulent. We have seen that drag from a non–separated boundary layer is smaller when it is laminar. However, the total drag produced by a laminar boundary layer which ends up separating is

usually larger than the total drag produced by a turbulent boundary layer which separates later. Therefore, devices tripping the boundary layer to turbulence can be installed near the leading edge of the wings, so that separation is delayed and the drag can be controlled.

It is also to generate a turbulent boundary layer and reduce the pressure drag that dimples are added on golf balls, for example. As can be seen on the figure below, which shows the result of a numerical simulation, the separation occurs earlier and the wake is larger when the ball is smooth (*Credit: https://www.cradle-cfd.com/media/column/a170*):



Adding dimples reduce the drag by a factor of about two.

Chapter 5

Waves

A wave is produced when a system at equilibrium is perturbed and subsequently oscillates around its equilibrium state, as a result of a balance between the inertial force, which is $\rho \mathrm{D}\mathbf{v}/\mathrm{D}t$ for a unit volume of a fluid, and a restoring force. In this chapter, we will consider sound waves, gravity waves, and gravity–capillary waves, for which the restoring force is the compressibility of the fluid, gravity, and a combination of surface tension and gravity, respectively. We will restrict ourselves to the *linear* theory of waves, which is valid for small amplitude disturbances.

Gravity waves occur when a fluid at equilibrium is displaced vertically and subsequently oscillates under the action of its intertial force and the external gravitational force. These waves only exist if the fluid is stratified, that is to say if the density varies vertically.

When the variation is *continuous*, with the density decreasing with increasing altitude, as in the atmosphere or in the oceans because of the gradients of temperature and/or salinity, *internal gravity waves* are produced.

When the variation is discontinuous and confined to a surface, as at the interface between the ocean and the atmosphere, *surface gravity waves* are produced. For these waves, gravity is the only restoring force. However, if the wavelength of the disturbance is short, surface tension becomes important and acts as an additional restoring force. The waves are then called *gravity–capillary waves*. Both gravity and gravity–capillary waves are produced by wind on the surface of oceans. In the opposite regime of a long wavelength disturbance, as caused by the gravitational attraction of the Moon on the oceans, the Coriolis force acts as an additional restoring force and we have *tidal waves*.

5.1 Sound waves

As the restoring force for sound waves is the compressibility of the fluid itself, we consider the simplest case of an inviscid fluid with no external force. It is described by Euler equation (2.30) where only the pressure force is present:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \, \mathbf{v} \right) = -\boldsymbol{\nabla} p, \tag{5.1}$$

and the mass conservation equation (1.21):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0. \tag{5.2}$$

We consider an equilibrium state in which the pressure and density are constant throughout the fluid: $\mathbf{v} = \mathbf{0}$, $p = p_0$ and $\rho = \rho_0$. This equilibrium is then perturbed, meaning that the fluid is displaced and some fluid elements are compressed while others are expanded. As we have four scalar equations for five variables, which are the three components of the velocity, pressure and density, an energy equation is needed to close the system of equations. We now consider in turn the case of a perfect gas like the air and the case of a liquid like water.

5.1.1 Wave equation in a perfect gas

It was originally proposed by Newton that heat would flow very rapidly from the compressed to the rarefied regions so that the temperature would stay essentially constant and the perturbation would be isothermal. However, as argued later by Laplace, the opposite actually occurs: as long as the scale over which the fluid is perturbed (that is to say, the wavelength of the perturbation) is large compared to the mean free path of the molecules in the fluid, heat flows from compressed to rarefied regions on a timescale long compared to the period of the oscillations, so that the perturbation is essentially adiabatic (there is negligible exchange of energy). Fluid elements therefore retain their entropy as they move, which means that $p\rho^{-\gamma}$ stays constant for each fluid element, where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. (In the case where the wavelength is smaller than the mean free path, viscous forces cannot be neglected and disturbances are damped).

If the perturbation is small, we can write $\mathbf{v} = \mathbf{v}_1$, $p = p_0 + p_1$ and $\rho = \rho_0 + \rho_1$, with $p_1 \ll p_0$ and $\rho_1 \ll \rho_0$, and where the perturbed quantities depend on location $\mathbf{r}$ and time t. Inserting into equation (5.1) yields:

$$(\rho_0 + \rho_1) \left(\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \boldsymbol{\nabla}) \mathbf{v}_1 \right) = -\boldsymbol{\nabla} (p_0 + p_1) .$$

For small perturbations, we linearize the equation[1] by retaining only the terms which are first order in the perturbed quantities $\mathbf{v}_1$, p_1 and ρ_1. Therefore, we neglect $\rho_1 (\partial \mathbf{v}_1 / \partial t)$ and $(\mathbf{v}_1 \cdot \boldsymbol{\nabla}) \mathbf{v}_1$, which are both quadratic in the perturbation. Since in addition p_0 is uniform, the equation above gives:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\boldsymbol{\nabla} p_1. \tag{5.3}$$

[1] In principle, we should write the equation in a dimensionless form and do an expansion using a small parameter. However, the result would be the same as that obtained by the linearization process presented here.

Similarly, linearization of the mass conservation equation (5.2) yields:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0. \tag{5.4}$$

Each fluid element retains its value of $p\rho^{-\gamma}$ as it moves with the flow. Since this is equal to $p_0\rho_0^{-\gamma}$ initially, before the fluid is perturbed, it means that $p\rho^{-\gamma}$ stays constant throughout the fluid and equal to $p_0\rho_0^{-\gamma}$, yielding:

$$(p_0 + p_1)\,(\rho_0 + \rho_1)^{-\gamma} = p_0\rho_0^{-\gamma}.$$

This can also be written as:

$$\left(1 + \frac{p_1}{p_0}\right)\left(1 + \frac{\rho_1}{\rho_0}\right)^{-\gamma} = 1,$$

or, to first order in the perturbed quantities:

$$\frac{p_1}{p_0} - \gamma\frac{\rho_1}{\rho_0} = 0. \tag{5.5}$$

We now take the divergence of equation (5.3) and substitute $\boldsymbol{\nabla} \cdot \mathbf{v}_1$ using equation (5.4), to obtain:

$$\frac{\partial^2 \rho_1}{\partial t^2} = \nabla^2 p_1.$$

Replacing ρ_1 by its expression as a function of p_1 as given by equation (5.5) then yields the wave equation:

$$\boxed{\frac{\partial^2 p_1}{\partial t^2} - \frac{\gamma p_0}{\rho_0}\nabla^2 p_1 = 0.} \tag{5.6}$$

5.1.2 Wave equation in a liquid

Variations in pressure and variations in density are related to each other through the *bulk modulus*, which measures the liquid's resistance to compression and is defined as:

$$K = \rho\frac{\mathrm{d}p}{\mathrm{d}\rho}. \tag{5.7}$$

K has the dimension of a pressure. Integrating this equation between a time when the system is at equilibrium, with $p = p_0$ and $\rho = \rho_0$, and a subsequent time, when $p = p_0 + p_1$ and $\rho = \rho_0 + \rho_1$, gives:

$$p_1 = K\frac{\rho_1}{\rho_0}, \tag{5.8}$$

where we have used $\rho_1/\rho_0 \ll 1$. We can therefore carry out the same calculation as for the perfect gas, but replacing equation (5.5) by equation (5.8). This yields the wave equation:

$$\boxed{\frac{\partial^2 p_1}{\partial t^2} - \frac{K}{\rho_0}\nabla^2 p_1 = 0.} \tag{5.9}$$

5.1.3 The speed of sound

Equations (5.6) and (5.9) show that the perturbed pressure p_1 propagates as a wave with the phase velocity:

$$\boxed{c_s = \sqrt{\frac{K'}{\rho_0}}}, \tag{5.10}$$

which is the speed of sound in the fluid. Here K' is the bulk modulus K if the fluid is a liquid, and $K' = \gamma p_0$ if the fluid is a gas (this is also called the bulk modulus for a gas). The speed of sound is a constant, that is to say it does not depend on the perturbation itself, so that sound waves are *non–dispersive* (they keep their shape as they propagate).

In a perfect gas:

Air at sea level has a pressure $P_0 = 1$ atm $\simeq 10^5$ Pa and a density $\rho_0 = 1.2$ kg m^{-3}. Using $\gamma = 7/5$ (diatomic gas) then yields $c_s = 342$ m s^{-1}, in excellent agreement with experimental values.

For a perfect gas, $P_0 = Nk_BT/V$, where V is the volume occupied by the gas, N is the number of molecules in this volume, T is the temperature and k_B is the Boltzmann constant. Since $\rho_0 V$ is equal to the mass Nm of gas in the volume, with m being the mass of a molecule, equation (5.10) becomes:

$$c_s = \sqrt{\frac{\gamma k_B T}{m}}.$$

Therefore, *the sound speed in a perfect gas only depends on temperature.* Due to the finite temperature, the molecules have a random velocity with a mean square value $\langle u^2 \rangle$ given by:

$$k_B T = \frac{1}{3} m \langle u^2 \rangle ,$$

so that we obtain:

$$\boxed{c_s = \sqrt{\frac{\gamma}{3}} \sqrt{\langle u^2 \rangle}.} \tag{5.11}$$

For $\gamma = 5/3$ (monoatomic gas) or $\gamma = 7/5$ (diatomic gas), this gives $c_s \simeq 0.7\sqrt{\langle u^2 \rangle}$. This result is not surprising since it is the motion of the molecules itself which propagates disturbances involving compression of fluid elements.

This also justifies *a posteriori* that we have identified the fluctuating velocity of the molecules in a gas with the sound speed in section 2.1.4.

In water:

The bulk modulus of water is $K = 2.2 \times 10^9$ Pa at a temperature of 20° C. The high value of K is a consequence of the near incompressibility of water. It is nearly constant for temperatures up to about 300 K and decreases at higher temperatures. Since for water $\rho_0 = 10^3$ kg m^{-3}, we obtain $c_s = 1483$ m s^{-1}, which again is in excellent agreement with experimental values.

5.1.4 Solutions to the wave equation

For a one dimensional wave $p(x,t)$, corresponding to a compression occurring along the x–axis, the wave equation becomes:

$$\frac{\partial^2 p_1}{\partial t^2} - c_s^2 \frac{\partial^2 p_1}{\partial x^2} = 0. \tag{5.12}$$

The solutions are plane waves propagating with the velocity c_s:

$$p_1(x,t) = f(x - c_s t) + g(x + c_s t), \tag{5.13}$$

where f and g describe waves propagating towards x positive and x negative, respectively.

If the compression is spherically symmetric, then $p(r,t)$ and the wave equation becomes:

$$\frac{\partial^2 p_1}{\partial t^2} - c_s^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial p_1}{\partial r} \right) = 0. \tag{5.14}$$

Using the new variable $h(r,t) \equiv r p_1(r,t)$ yields:

$$\frac{\partial^2 h}{\partial t^2} - c_s^2 \frac{\partial^2 h}{\partial r^2} = 0, \tag{5.15}$$

which solutions are the same as above, after replacing x with r. Therefore:

$$p_1(r,t) = \frac{1}{r} \left[f(r - c_s t) + g(r + c_s t) \right]. \tag{5.16}$$

If the wave is produced by a sound emitted at some location, then the wave propagates towards increasing values of r and $g = 0$.

As can be seen from equation (5.3), the fluid elements oscillate in the direction of the pressure gradient, which is also the direction of propagation of the wave. Therefore, sound waves are *longitudinal*.

5.1.5 Energy in sound waves

Although waves do not transport matter, they transport energy. Establishing an energy equation for linear waves, which are calculated to first order in the perturbation, is subtle, as the terms describing energy are second order in the perturbation. This will not be presented here, but details can be found in *Waves in Fluids*, by James Lighthill (CUP).

In linear sound waves, the energy is equally divided between kinetic energy and the potential energy associated with the restoring force, due to compressibility. This potential energy is supplied by the work done by the excess pressure $p - p_0$ on a fluid which density is increased from ρ_0 to ρ. The total energy is transported by the wave at the group velocity, which is equal to the phase velocity c_s as sound waves are non–dispersive.

5.2 Surface waves

As mentioned in the introduction to this chapter, surface waves are produced at the (free) surface of a liquid when the liquid–gas interface is perturbed. If gravity is the only

restoring force, we obtain gravity waves. At short wavelengths however, the surface tension is important and the waves become gravity–capillary waves. We start this section by a general analysis of surface waves, and then consider in turn these different types of waves.

The velocity and pressure satisfy Euler equation (2.30) with both pressure and gravitational forces included:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}\right)=-\boldsymbol{\nabla}p-\rho g\hat{\mathbf{z}}, \tag{5.17}$$

where $g>0$. Surface tension forces only act at the free surface, and therefore only enter the boundary conditions.

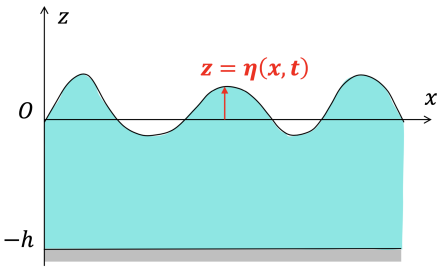
5.2.1 Equilibrium state

We consider an interface between water and air which is flat when at equilibrium. Water can be approximated to be incompressible, and here we take its density $\rho=\rho_0$ to be constant. Viscosity is also negligible, and we assume that the flow is irrotational. This is the case, for example, if the fluid is at rest initially (at equilibrium) as, according to Kelvin's theorem, a flow with constant density and subject to conservative forces retains its vorticity at all times.

In the equilibrium state, $\mathbf{v}_0=\mathbf{0}$, so that Euler equation (5.17) in the z–direction gives $\partial p_0/\partial z=-\rho_0 g$, where p_0 is the pressure at equilibrium. Since the surface is flat, there are no surface tension forces acting along its normal. Therefore, continuity of the stress at the surface (see section 2.3.3) implies $p_0(z=0)=p_{\text{atm}}$, where p_{atm} is the gas pressure at the interface. The solution of Euler equation is then:

$$p_0(z)=-\rho_0 g z+p_{\text{atm}}. \tag{5.18}$$

5.2.2 Boundary conditions



We consider two dimensional waves. The axis $z=0$ corresponds to the flat surface at equilibrium and the equation of the perturbed free surface is $z=\eta(x,t)$. The depth of water is h.

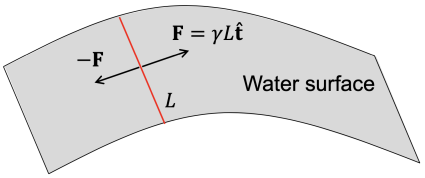
The boundary conditions are as follows:

- the velocity at $z=-h$ satisfies the no–penetration condition $v_z=0$,
- the velocity at the surface satisfies the kinematic boundary condition (2.38):

$$v_z\,(x,\eta,t)=\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial\eta}{\partial t}+v_x\frac{\partial\eta}{\partial x}, \tag{5.19}$$

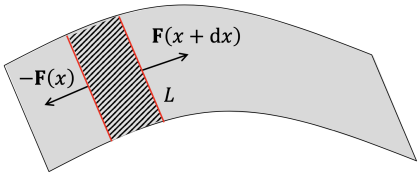
- as seen in section 2.3.3, when both the fluid and the gas are inviscid, continuity of the stress implies that, *at the free surface, the net pressure force is equal and opposite to the component of the surface tension force normal to the surface.*

We now calculate explicitly this latter boundary condition.

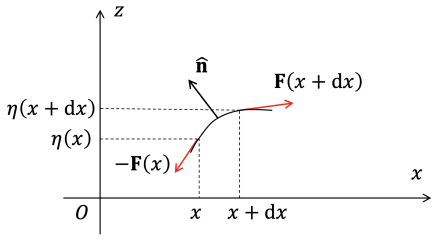


We consider a line of length L at the surface of the water. The molecules on one side of this line exert a pulling force (tension) on the molecules on the other side.

This force $\mathbf{F}$ is tangent to the surface and perpendicular to the line. Its value per unit length is the surface tension γ.



We now consider the forces exerted on the dashed surface element which is between x and $x + \mathrm{d}x$ and for which the length L is along the y–axis.



Tension forces give a resultant force $\mathrm{d}\mathbf{F} = \mathbf{F}(x + \mathrm{d}x) - \mathbf{F}(x)$ which component normal to the surface element has to be balanced by pressure forces, as seen in section 2.3.3.

In the figure above, the amplitude of the wave has been exaggerated for clarity but, in the linear regime, this amplitude is small, so that the normal $\hat{\mathbf{n}}$ to the surface element is equal to $\hat{\mathbf{z}}$ to first order. Therefore, the component of the tension force normal to the surface element can be approximated by $\mathrm{d}F_z = \mathbf{F}(x + \mathrm{d}x) \cdot \hat{\mathbf{z}} - \mathbf{F}(x) \cdot \hat{\mathbf{z}}$. Since $\mathbf{F}(x)$ is tangent to the curve $\eta(x)$, the angle between $\mathbf{F}(x)$ and the x–axis is θ such that $\tan\theta = \partial\eta/\partial x$. This angle is small, so that $\tan\theta \simeq \sin\theta \simeq \theta$, and $\mathbf{F}(x) \cdot \hat{\mathbf{z}} = F(x)\,(\partial\eta/\partial x)$. Doing a similar calculation for the force at $x + \mathrm{d}x$, and using $F(x) = F(x + \mathrm{d}x) = \gamma L$, we obtain:

$$\mathrm{d}F_z = \gamma L \left[\frac{\partial \eta}{\partial x} \right)_{x+\mathrm{d}x} - \frac{\partial \eta}{\partial x} \Big)_{x} \Bigg] = \gamma L \mathrm{d}x \frac{\partial^2 \eta}{\partial x^2}. \tag{5.20}$$

The pressure force acting on the surface element is $(p(x, \eta, t) - p_{\mathrm{atm}})\, L\mathrm{d}x\hat{\mathbf{z}}$. Since the sum of the forces in the vertical direction is zero (see eq. [2.36]), we obtain the following boundary condition:

$$\boxed{p(x, \eta, t) - p_{\mathrm{atm}} = -\gamma \frac{\partial^2 \eta}{\partial x^2}}. \tag{5.21}$$

If surface tension is negligible, that is to say $\gamma = 0$, the equation above indicates that the pressure in the fluid is equal to the pressure in the air at the free surface.

5.2.3 Equation and boundary conditions for the velocity potential

Since the flow is both irrotational and incompressible, we can define a velocity potential ϕ which satisfies Laplace's equation:

$$\boxed{\nabla^2\phi = 0.} \tag{5.22}$$

Instead of solving Euler and the incompressibility equations for $\mathbf{v}$ and p, we can solve Laplace's equation for ϕ. However, this requires to derive boundary conditions for ϕ from the boundary conditions written above for $\mathbf{v}$ and p. This is the object of this section.

As for sound waves, we assume that the velocity and pressure in the perturbed state can be written as $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1(x,z,t) = \mathbf{v}_1(x,z,t)$ and $p = p_0(z) + p_1(x,z,t)$. Then, Euler equation (5.17) yields:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\boldsymbol{\nabla}\left(p_0 + p_1\right) - \rho_0 g\hat{\mathbf{z}}, \tag{5.23}$$

where we have neglected the quadratic terms in the perturbation on the left–hand side. Since the pressure at equilibrium satisfies $-\boldsymbol{\nabla}p_0 - \rho_0 g\hat{\mathbf{z}} = \mathbf{0}$, the zeroth order terms cancel out and the equation above becomes:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\boldsymbol{\nabla}p_1. \tag{5.24}$$

Using $\mathbf{v}_1 = \boldsymbol{\nabla}\phi$, this equation can be written as:

$$\boldsymbol{\nabla}\left(\rho_0 \frac{\partial \phi}{\partial t} + p_1\right) = \mathbf{0}, \tag{5.25}$$

which gives:

$$p_1(x,z,t) = -\rho_0 \frac{\partial \phi}{\partial t}(x,z,t). \tag{5.26}$$

Any function of time could be added to p_1, but it does not need to be written explicitly as any function of time could be added to ϕ anyway without changing the velocity. At the surface $z = \eta(x,t)$, the pressure $p(x,\eta,t) \equiv p_0(\eta) + p_1(x,\eta,t)$ is given by the boundary condition (5.21). As $p_0(\eta)$ is given by equation (5.18) with $z = \eta$, this yields:

$$p_1(x,\eta,t) = \rho_0 g\eta(x,t) - \gamma \frac{\partial^2 \eta}{\partial x^2}, \tag{5.27}$$

so that equation (5.26) at $z = \eta$ can be written as:

$$\frac{\partial \phi}{\partial t}(x,\eta,t) = -g\eta(x,t) + \frac{\gamma}{\rho_0}\frac{\partial^2 \eta}{\partial x^2}. \tag{5.28}$$

Expanding the left–hand side in Taylor series gives:

$$\frac{\partial \phi}{\partial t}(x,\eta,t) = \frac{\partial \phi}{\partial t}(x,0,t) + \eta \frac{\partial^2 \phi}{\partial z \partial t}(x,0,t) + \cdots \tag{5.29}$$

Since $\boldsymbol{\nabla}\phi = \mathbf{v}_1$, ϕ is first order in the perturbation, and the second term on the right hand–side of the equation above is quadratic in the perturbation. To first order, we then neglect it and substitute into equation (5.28) to obtain:

$$\boxed{\frac{\partial \phi}{\partial t}(x,0,t) = -g\eta(x,t) + \frac{\gamma}{\rho_0}\frac{\partial^2 \eta}{\partial x^2}}. \tag{5.30}$$

We now use the boundary condition (5.19) which, to first order in the perturbation, becomes:

$$v_{1,z}\left(x,\eta,t\right)=\frac{\partial\eta}{\partial t}. \tag{5.31}$$

(Both the displacement η and the velocity $v_{1,x}$ are small quantities, so $v_{1,x}(\partial\eta/\partial x)$ is quadratic in the perturbation). Expanding the left–hand side in Taylor series gives:

$$v_{1,z}\left(x,\eta,t\right)=v_{1,z}\left(x,0,t\right)+\eta\frac{\partial v_{1,z}}{\partial z}(x,0,t)+\cdots \tag{5.32}$$

Identifying with equation (5.31) and keeping only first order terms in the perturbation then yields:

$$v_{1,z}\left(x,0,t\right)=\frac{\partial\eta}{\partial t}. \tag{5.33}$$

Using $v_{1,z}=\partial\phi/\partial z$, this equation can be written as:

$$\boxed{\frac{\partial\phi}{\partial z}\left(x,0,t\right)=\frac{\partial\eta}{\partial t}}. \tag{5.34}$$

Finally, the no–penetration condition at the bottom of the water becomes:

$$\boxed{\frac{\partial\phi}{\partial z}(x,-h,t)=0}. \tag{5.35}$$

Surface gravity waves are the solutions of Laplace's equation (5.22) for $-h\leq z\leq 0$ subject to the boundary conditions (5.30) and (5.34) at $z=0$ and (5.35) at $z=-h$.

5.2.4 Dispersion relation

We look for a surface displacement under the form of a sinusoidal wave travelling in the positive x–direction, that it to say:

$$\eta(x,t)=A\cos\left(kx-\omega t\right), \tag{5.36}$$

where A is the amplitude of the displacement, $k>0$ is the wavenumber and ω is the frequency. Note that here we cannot think of the equilibrium state being perturbed at some time $t=0$ for example, as η cannot be zero at all values of x at a given time. The form of η above tells us about the perturbation after it has been set up, but not about how it was set up.

With η of the form above, equation (5.30) becomes:

$$\frac{\partial\phi}{\partial t}(x,0,t)=-\left(g+\frac{\gamma k^2}{\rho_0}\right)A\cos\left(kx-\omega t\right), \tag{5.37}$$

and equation (5.34) becomes:

$$\frac{\partial\phi}{\partial z}(x,0,t)=A\omega\sin\left(kx-\omega t\right). \tag{5.38}$$

This indicates that ϕ is of the form:

$$\phi(x,z,t)=f(z)\sin\left(kx-\omega t\right). \tag{5.39}$$

Substituting into Laplace's equation (5.22), we obtain:

$$f''(z) - k^2 f(z) = 0.$$

The solutions are $f(z) = C_1 \mathrm{e}^{kz} + C_2 \mathrm{e}^{-kz}$, where C_1 and C_2 are two constants. The no–penetration boundary condition (5.35) then yields: $C_1 \mathrm{e}^{-kh} - C_2 \mathrm{e}^{kh} = 0$, so that $f(z) = C \cosh\left[k(z+h)\right]$, where C is a constant. This gives:

$$\phi(x, z, t) = C \cosh\left[k(z+h)\right] \sin\left(kx - \omega t\right). \tag{5.40}$$

Equation (5.37) then yields:

$$C = \frac{A}{\omega \cosh(kh)} \left(g + \frac{\gamma k^2}{\rho_0}\right). \tag{5.41}$$

Substituting this expression for ϕ into equation (5.38) then gives the *dispersion relation:*

$$\boxed{\omega^2 = gk\left(1 + \lambda_c^2 k^2\right) \tanh(kh),} \tag{5.42}$$

where we have introduced the capillary length:

$$\boxed{\lambda_c = \sqrt{\frac{\gamma}{g\rho_0}}.} \tag{5.43}$$

It is the length below which surface tension forces dominate over gravity[2]. For water, $\lambda_c \simeq 3 \times 10^{-3}$ m at $T \simeq 20°\mathrm{C}$, and it varies only weakly with temperature.

At some time t and some position x along the x–axis, the phase of the wave is $kx - \omega t$. At a time $t + \mathrm{d}t$, this phase has advanced to $x + \mathrm{d}x$ such that $k(x+\mathrm{d}x) - \omega(t+\mathrm{d}t) = kx - \omega t$. Therefore, $k\mathrm{d}x = \omega \mathrm{d}t$, and the *phase velocity* (speed at which the phase of the wave travels) is given by $v_\varphi = \mathrm{d}x/\mathrm{d}t = \omega/k$. Using the dispersion relation above, this gives:

$$\boxed{v_\varphi = \sqrt{\frac{g}{k}\left(1 + \lambda_c^2 k^2\right) \tanh(kh).}} \tag{5.44}$$

The waves are *dispersive*: waves with different wavenumbers travel at different speeds.

Let us assume that $\lambda_c \ll h$, which is reasonable since for water λ_c is a few mm. We then have the following different regimes:

- capillary waves: $\lambda \ll \lambda_c \ll h$,
- gravity–capillary waves on deep water: $\lambda \simeq \lambda_c \ll h$,
- gravity waves on deep water: $\lambda_c \ll \lambda \ll h$,
- gravity waves on shallow water: $\lambda_c \ll h \ll \lambda$.

Since $\mathbf{v}_1 = \boldsymbol{\nabla}\phi$ and ϕ given by equation (5.40) depends both on x and z, fluid elements oscillate in both the x and z directions so that *surface waves are neither longitudinal nor transverse.*

[2]For example, water droplets with a radius smaller than the capillary length have a shape determined solely by surface tension effects, and are therefore spherical.

5.2.5 Dispersion and group velocity

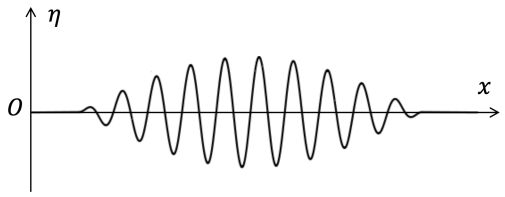
To understand the effect of dispersion, we now consider a disturbance which is the superposition of sinusoidal waves travelling in the positive x–direction[3]:

$$\eta(x,t) = \int_0^{+\infty} |a(k)| \cos\left(kx - \omega t\right) \mathrm{d}k. \tag{5.45}$$

This is the real part of:

$$\tilde{\eta}(x,t) = \int_0^{+\infty} a(k) \mathrm{e}^{\mathrm{i}(kx-\omega t)} \mathrm{d}k. \tag{5.46}$$

Since $\omega(k)$, each Fourier component travels with a different speed velocity.



We assume that the coefficients $a(k)$ are non zero only in a narrow interval of wavenumbers centered on k_0. Such a disturbance is called a *wave packet* and is represented on the figure.

Performing a Taylor expansion of ω in the vicinity of k_0 to first order yields:

$$\omega(k) = \omega_0 + (k - k_0) v_g(k_0), \tag{5.47}$$

where we have defined:

$$\boxed{v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.} \tag{5.48}$$

Therefore, equation (5.46) becomes:

$$\tilde{\eta}(x,t) = \mathcal{A}(x,t) \mathrm{e}^{\mathrm{i}(k_0 x - \omega(k_0) t)}, \tag{5.49}$$

with:

$$\mathcal{A}(x,t) = \int_0^{+\infty} a(k) \mathrm{e}^{\mathrm{i}(k-k_0)(x - v_g(k_0)t)} \mathrm{d}k, \tag{5.50}$$

where we have kept the integral from 0 to $+\infty$, even though the Taylor expansion above is valid only in the vicinity of k_0, because the values of k outside of a narrow band centered on k_0 correspond to negligible $a(k)$ and therefore do not contribute to the integral. This expression shows that $\tilde{\eta}$ can be approximated by a sinusoidal wave with wavenumber k_0 and frequency $\omega(k_0)$, travelling at the speed $v_\varphi = \omega(k_0)/k_0$, which amplitude $\mathcal{A}$ is a function of x and t through the combination $x - v_g(k_0)t$.

[3] According to the Fourier integral theorem, any function $f(x)$ which is absolutely integrable can be written as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k) \mathrm{e}^{\mathrm{i}kx} dk,$$

with

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{-\mathrm{i}kx} dx.$$

The function $\hat{f}$ is the Fourier transform of f, and f is the inverse Fourier transform to $\hat{f}$.

Therefore, the amplitude, which is called the *envelope* of the wave packet, is also a wave and it travels at the velocity $v_g(k_0)$, which is called the *group velocity* of the wave packet.

If the wave is non–dispersive, then $v_\varphi = \omega/k$ is a constant, which implies that $\omega \propto k$ and therefore $v_g = v_\varphi$. In that case, the envelope travels at the same speed as the phase of the individual components.

The effect of dispersion can be seen by continuing the Taylor expansion of $\omega(k)$ in equation (5.47) to second order in $k - k_0$:

$$\omega(k) = \omega_0 + (k - k_0)v_g(k_0) + \frac{1}{2}(k - k_0)^2\omega''(k_0). \tag{5.51}$$

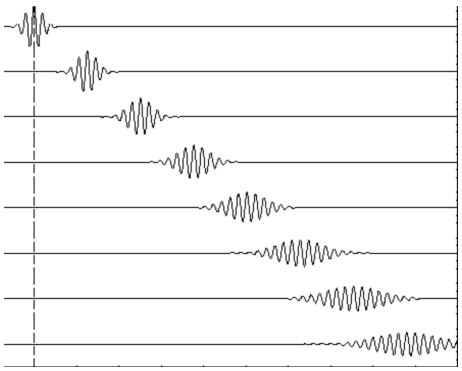
With this extra term included, the amplitude $\mathcal{A}$ becomes:

$$\mathcal{A}(x,t) = \int_0^{+\infty} a(k)\mathrm{e}^{\mathrm{i}(k-k_0)\left\{x-\left[v_g(k_0)+\frac{1}{2}(k-k_0)\omega''(k_0)\right]t\right\}}\mathrm{d}k. \tag{5.52}$$

This shows that the velocity at which the envelope travels is:

$$v_g(k_0) + \frac{1}{2}(k - k_0)\omega''(k_0).$$

Remembering that k is in a narrow interval centered on k_0, this means that the velocity is $v_g(k_0)$ for the main component which is at $k = k_0$, but that the velocity is slightly decreased or increased for values of k on either side of k_0. Therefore, the different Fourier components of the wave packet travel with slightly different velocities, which results in the envelope spreading with time: this is the effect of *dispersion*. If $\omega \propto k$, $\omega'' = 0$ and there is no dispersion.



The figure below shows a wave packet travelling in the x–direction. Dispersion results in a flattening and broadening of the envelope.

(*Credit:*
http://www.jick.net/ jess/hr/skept/GWP/)

5.2.6 Surface gravity waves on deep water

Here we focus on waves such that $\lambda_c \ll \lambda \ll h$, that is to say for which the surface tension does not play a role and which wavelength is small compared to the depth of water.

Phase and group velocities:

When $kh \gg 1$, $\tanh(kh) \simeq 1$. Therefore, the dispersion relation (5.42) can be approximated by:

$$\boxed{\omega^2 = gk.} \tag{5.53}$$

This yields the phase velocity:

$$\boxed{v_\varphi = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}.} \tag{5.54}$$

This shows that longer wavelengths travel faster than shorter wavelengths. The difference between the speeds of different wavelengths is significant, as $\sqrt{g/(2\pi)} = 1.25\ \mathrm{m}^{1/2}\ \mathrm{s}^{-1}$. Surface gravity waves in the oceans have wavelength typically in the range 1–100 m, for which the phase velocity varies from 1.25 to 12.5 m s^{-1}. The corresponding periods are $T = \lambda/v_\varphi$ in the range 0.8 to 8 s.

Differentiating the dispersion relation above yields $2\omega \mathrm{d}\omega = g\mathrm{d}k$, and therefore the group velocity (5.48) is $v_g = \mathrm{d}\omega/\mathrm{d}k = g/(2\omega)$, which can also be written as:

$$\boxed{v_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{v_\varphi}{2}.} \tag{5.55}$$

As with sound waves, the energy in linear gravity waves is equally divided between kinetic energy and the potential energy associated with the restoring force, which here is the gravitational potential energy, and is transported at the group velocity.

Motion of fluid elements:

The velocity potential ϕ is given by equations (5.40), in which C is given by equation (5.41) with the surface tension term being negligible. Since $kh \gg 1$, we have $\cosh(kh) \simeq \sinh(kh) \simeq \mathrm{e}^{kh}/2$, so that:

$$\frac{\cosh\left[k(z+h)\right]}{\cosh(kh)} = \frac{\cosh(kz)\cosh(kh) + \sinh(kz)\sinh(kh)}{\cosh(kh)} \simeq \mathrm{e}^{kz}.$$

Therefore:

$$\phi(x,z,t) = \frac{Ag}{\omega}\mathrm{e}^{kz}\sin\left(kx - \omega t\right). \tag{5.56}$$

The components of the velocity $\mathbf{v}_1 = \boldsymbol{\nabla}\phi$ are then given by:

$$v_{1,x}(x,z,t) = A\omega\mathrm{e}^{kz}\cos\left(kx-\omega t\right), \quad v_{1,z}(x,z,t) = A\omega\mathrm{e}^{kz}\sin\left(kx-\omega t\right), \tag{5.57}$$

where we have used $gk = \omega^2$. We consider a fluid element which at equilibrium is at (x_0, z_0). It is displaced by the perturbation so that its location becomes $(x = x_0 + x_1, z = z_0 + z_1)$, where $|x_1|$ and $|z_1|$ are small. We perform a Taylor expansion for $v_{1,x}$:

$$v_{1,x}(x,z,t) = v_{1,x}(x_0,z_0,t) + x_1\frac{\partial v_{1,x}}{\partial x}(x_0,z_0,t) + z_1\frac{\partial v_{1,x}}{\partial z}(x_0,z_0,t) + \cdots$$

Therefore, to first order in the perturbation, $v_{1,x}(x,z,t) = v_{1,x}(x_0,z_0,t)$, and similarly for $v_{1,z}$. The velocities then only depend on time, and we can replace x by x_0 and z by z_0 in equations (5.57) to obtain:

$$v_{1,x}(t) = A\omega e^{kz_0}\cos(kx_0 - \omega t), \quad v_{1,z}(t) = A\omega e^{kz_0}\sin(kx_0 - \omega t). \tag{5.58}$$

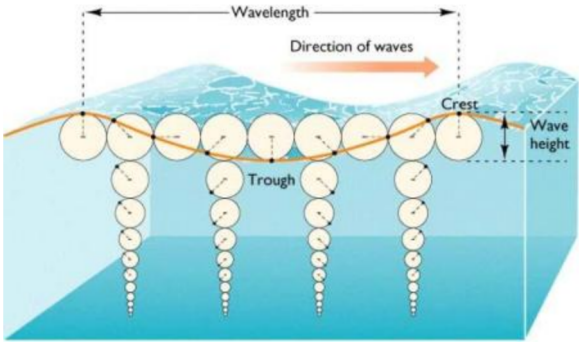
As $v_{1,x} = \mathrm{d}x_1/\mathrm{d}t$ and $v_{1,z} = \mathrm{d}z_1/\mathrm{d}t$, we then obtain:

$$x_1(t) = -Ae^{kz_0}\sin(kx_0 - \omega t) + D_1, \quad z_1(t) = Ae^{kz_0}\cos(kx_0 - \omega t) + D_2, \tag{5.59}$$

where D_1 and D_2 are two constants. As mentioned above, given the form of η we have assumed, the system is not at equilibrium at $t = 0$, and therefore we cannot use this condition to calculate D_1 and D_2. However, the equations above yield:

$$(x_1(t) - D_1)^2 + (z_1(t) - D_2)^2 = A^2 e^{2kz_0}, \tag{5.60}$$

which indicates that the perturbed fluid elements move along a circle centered on (D_1, D_2). Physically, this center has to be the equilibrium position of the fluid element, since x_1 and z_1 are departure from equilibrium. Therefore, $D_1 = x_0$ and $D_2 = z_0$, and *a fluid element which is displaced from its equilibrium position* (x_0, z_0) *subsequently moves along a circle centered on this point and with radius* Ae^{kz_0}.



The figure illustrates the path of fluid elements displaced from their equilibrium position. The radius of the circles decreases exponentially with depth. (*Credit: https://www.nortekgroup.com/*)

5.2.7 Surface gravity waves on water of finite depth

We now consider the case where we still have $\lambda_c \ll \lambda$, that is to say surface tension does not play a role, but when the wavelength is not small compared to the depth of water.

The dispersion relation (5.42) then becomes:

$$\boxed{\omega^2 = gk\tanh(kh),} \tag{5.61}$$

and the phase velocity is:

$$\boxed{v_\varphi = \sqrt{\frac{g}{k}\tanh(kh)},} \tag{5.62}$$

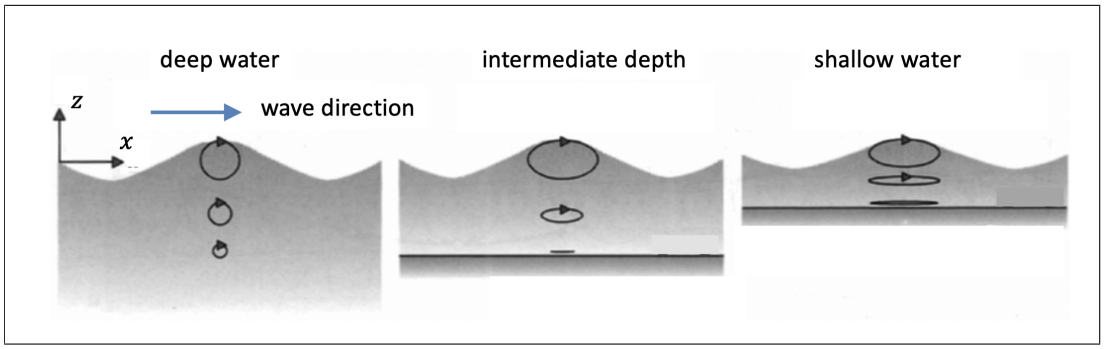
so that here again the waves are dispersive.

The case corresponding to $h \ll \lambda$, that is to say $kh \ll 1$, is called the *shallow water* regime. Using $\tanh(kh) \simeq kh$ then yields:

$$\boxed{v_\varphi = \sqrt{gh},} \tag{5.63}$$

which indicates that *surface gravity waves on shallow water are non–dispersive.* The fact that v_φ decreases when the depth of water decreases explains the *breaking of waves* on a steep beach. This happens because, as the wave approaches the beach, its velocity decreases, and therefore so does its wavelength. The wave is then 'squeezed' into a narrower volume, and its amplitude has to increase for mass to be conserved (this can also be understood from the point of view of energy conservation, as potential energy has to increase to compensate for the decrease in kinetic energy). This process is called *shoaling.* When the amplitude of the wave is large enough, a crest overtakes the trough which is directly in front of it and goes slower (as it reaches shallower depths first), which results in the wave spilling or plunging forward: the wave breaks.

In the same way that we have shown that fluid elements displaced from their equilibrium position move along a circle in the case of deep–water, it could be shown that the fluid elements move along ellipses in the case of finite depth. This is illustrated on the figure below (*from Ahmed et al., JMST, 2010, 24, 943*), which shows that the ellipses become more flattened as the depth of water decreases.

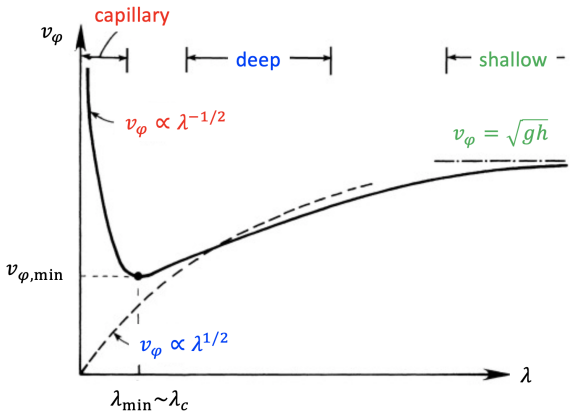


5.2.8 Gravity–capillary waves

Surface tension becomes important when λ is on the order of or smaller than the capillary length λ_c. Such waves are called *ripples.* As λ_c is a few mm in water, the *deep water* approximation is usually relevant, so that $kh \gg 1$. Therefore, the dispersion relation (5.42) becomes:

$$\boxed{\omega^2 = gk\left(1 + \lambda_c^2 k^2\right).} \tag{5.64}$$

When $\lambda_c k \gg 1$, gravity is negligible and we have *capillary waves*, for which the dispersion relation is $\omega^2 = g\lambda_c^2 k^3 = \gamma k^3/\rho_0$, where we have used equation (5.43) for λ_c. Their phase and group velocities are $v_\varphi = \sqrt{\gamma k/\rho}$ and $v_g = 3v_\varphi/2$, respectively, so that the envelope of wave packets travel faster than the phase of the individual Fourier components.



This figure shows the phase velocity v_φ as a function of wavelength λ for the different type of surface waves mentioned above. The phase velocity has a minimum for $\lambda \sim \lambda_c$.

5.3 Internal gravity waves

We now consider the waves for which the restoring force is gravity acting in a fluid in which the density varies *continuously* with height. In general, the density *decreases* with height, and this is what we assume here.

The flow satisfies Euler equation with both pressure and gravity included:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} \right) = -\boldsymbol{\nabla} p - \rho g \hat{\mathbf{z}}. \tag{5.65}$$

We assume that the fluid is incompressible, so that we have:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{5.66}$$

Using this condition, the mass conservation equation (1.22) yields:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \rho = 0. \tag{5.67}$$

It expresses the fact that, in an incompressible flow, fluid elements retain their density as they move. This is a good approximation for describing internal gravity waves in the oceans, but not in the atmosphere where, instead of retaining their mass density, fluid elements retain their entropy as they move, as in the case of sound waves (section 5.1.1).

5.3.1 Buoyancy frequency

We consider a fluid element located at z and which moves upwards to $z + \delta z$ while conserving its density $\rho_0(z)$, where ρ_0 is the density at equilibrium. The density of the fluid surrounding it at $z + \delta z$ is smaller, being $\rho_0(z) + (\mathrm{d}\rho_0/\mathrm{d}z)\delta z$. The fluid element, therefore, has an excess density over that of the surrounding fluid which is $-(\mathrm{d}\rho_0/\mathrm{d}z)\delta z$. This results in an excess weight $g(\mathrm{d}\rho_0/\mathrm{d}z)\delta z$ (directed downwards) per unit volume of fluid for the element, so that the equation of motion for the fluid element can be written as:

$$\rho_0 \frac{\mathrm{d}^2(\delta z)}{\mathrm{d}t^2} = g \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \delta z. \tag{5.68}$$

Therefore, the fluid element oscillates with the *buoyancy frequency* N defined as:

$$\boxed{N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}.} \tag{5.69}$$

Here we have assumed that the displacement of the fluid element was purely vertical. As we will see below, any additional horizontal component would reduce the frequency of the oscillations, so that N is the maximum frequency for oscillations under gravity. The analysis above is still valid if $\mathrm{d}\rho_0/\mathrm{d}z > 0$, but in that case $N^2 < 0$ and, as will be discussed in chapter 6, the displacement is then *unstable*: the fluid element keeps moving upwards after it is displaced[4].

5.3.2 Dispersion relation

We now solve the full equations above satisfied by the flow for a general linear perturbation. We consider an equilibrium in which the velocity is zero. The density ρ_0 and pressure p_0 vary only with z, so that Euler equation gives:

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} + \rho_0(z)g = 0. \tag{5.70}$$

As for surface waves, we consider two dimensional small perturbations in the (x, z) plane so that $\mathbf{v}(x, z, t) = \mathbf{v}_1(x, z, t)$, $p(x, z, t) = p_0(z) + p_1(x, z, t)$ and $\rho(x, z, t) = \rho_0(z) + \rho_1(x, z, t)$, with $p_1 \ll p_0$ and $\rho_1 \ll \rho_0$. Substituting in equations (5.65), (5.66) and (5.67) and keeping only terms of first order in the perturbed quantities yield:

$$\begin{aligned} \rho_0\frac{\partial v_{1,x}}{\partial t} &= -\frac{\partial p_1}{\partial x} \quad , \qquad \rho_0\frac{\partial v_{1,z}}{\partial t} = -\frac{\partial p_1}{\partial z} - \rho_1 g, \\ \frac{\partial v_{1,x}}{\partial x} + \frac{\partial v_{1,z}}{\partial z} &= 0 \quad , \qquad \frac{\partial \rho_1}{\partial t} + v_{1,z}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = 0. \end{aligned} \tag{5.71}$$

Note that the zeroth order terms in equation (5.65) drop as they themselves satisfy Euler equation.

[4]Here we have considered an incompressible fluid in which fluid elements retain their mass density as they move. As mentioned already, this is a good approximation in the oceans, but not in the atmosphere or in a star, where the fluid is a gas. A similar analysis can still be done by considering that fluid elements retain their entropy as they move, for the same reason that we have assumed adiabatic perturbations in the case of sound waves in section 5.1.1. On the other hand, pressure balance between the fluid element and its new surrounding as it is displaced is achieved on a timescale shorter than the timescale on which the displacement occurs (as it is given by the sound crossing time through the fluid element). Writing the condition for stability in that context leads a similar result as that described in this section, but with the buoyancy frequency (5.69) replaced by the so–called *Brunt–Väisälä frequency* N and defined as:

$$N^2 = g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}r}\right),$$

where r is the local radius and g the local gravitational acceleration. When $N^2 < 0$, the equilibrium is unstable and *convective motions* persist. The Sun, for example, has a large convective envelope in its outer parts in which the flow rises from below as described here.

We look for plane wave solutions of the form:

$$\tilde{v}_{1,x}(x,z,t) = \tilde{V}_x \, \mathrm{e}^{\mathrm{i}(k_x x + k_z z - \omega t)}, \tag{5.72}$$

$$\tilde{v}_{1,z}(x,z,t) = \tilde{V}_z \, \mathrm{e}^{\mathrm{i}(k_x x + k_z z - \omega t)}, \tag{5.73}$$

$$\tilde{p}_1(x,z,t) = \tilde{P} \, \mathrm{e}^{\mathrm{i}(k_x x + k_z z - \omega t)}, \tag{5.74}$$

$$\tilde{\rho}_1(x,z,t) = \tilde{R} \, \mathrm{e}^{\mathrm{i}(k_x x + k_z z - \omega t)}, \tag{5.75}$$

where a *tilde* indicates a complex quantity. The real parts have to be taken to obtain the physical quantities. Substituting into the linearized equations (5.71) yields:

$$\omega \rho_o \tilde{V}_x - k_x \tilde{P} = 0, \tag{5.76}$$

$$\mathrm{i}\omega\rho_0 \tilde{V}_z - \mathrm{i}k_z \tilde{P} - g\tilde{R} = 0, \tag{5.77}$$

$$k_x \tilde{V}_x + k_z \tilde{V}_z = 0, \tag{5.78}$$

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}z} \tilde{V}_z - \mathrm{i}\omega \tilde{R} = 0. \tag{5.79}$$

This system has a non–trivial solution only if the determinant of the coefficient matrix is zero, which yields the *dispersion relation*:

$$\boxed{\omega = \frac{N k_x}{k} = N \sin\theta,} \tag{5.80}$$

where $k = \sqrt{k_x^2 + k_z^2}$ and θ is the angle between the wavenumber $\mathbf{k} = k_x \hat{\mathbf{x}} + k_z \hat{\mathbf{z}}$ and the z–axis. If there is no stratification, $N = 0$ and $\omega = 0$, which makes it clear that the restoring force is due to stratification. If $k_z = 0$, $\omega = N$, as expected since in that case equation (5.78) implies that $v_x = 0$ and the fluid elements oscillate as described in section 5.3.1. If $k_x = 0$, $\omega = 0$, again as expected since in that case equation (5.78) implies that $v_z = 0$, which means that the motion is in the x–direction in which there is no restoring force. For that reason as well, the frequency of the oscillations is smaller when the motion is not purely vertical, so that $\omega \leq N$.

Here we have considered a *free oscillation*, that is to say an oscillation that results from some initial disturbance and occurs with the system own natural frequency. If instead the oscillation were forced, then both k and ω would be given by the forcing process. For example, when the waves are produced by water flowing over a ridge in the ocean or air moving over hills in the atmosphere, k is given by the geometry of the ridge/hills and ω by the speed of the flow. The result above then shows that *only waves with frequencies smaller than N can propagate.*

The phase velocity is $v_\varphi = \omega/k$ and is in the direction of $\mathbf{k}$, so that we can write:

$$\boxed{v_\varphi = \frac{N k_x}{k^2} \frac{\mathbf{k}}{k}.} \tag{5.81}$$

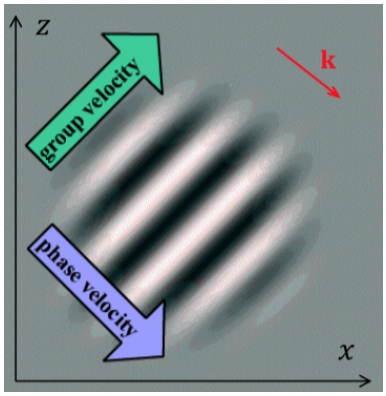
In the same way that we have shown that the group velocity for a one dimensional wave is $v_g = \mathrm{d}\omega/\mathrm{d}k$ (section 5.2.5), we could show that, in two dimensions:

$$\mathbf{v}_g = \frac{\partial \omega}{\partial k_x} \hat{\mathbf{x}} + \frac{\partial \omega}{\partial k_z} \hat{\mathbf{z}}. \tag{5.82}$$

Therefore, here, we obtain:

$$\boxed{\mathbf{v}_g = \frac{Nk_z}{k^3}\left(k_z\hat{\mathbf{x}} - k_x\hat{\mathbf{z}}\right).} \tag{5.83}$$

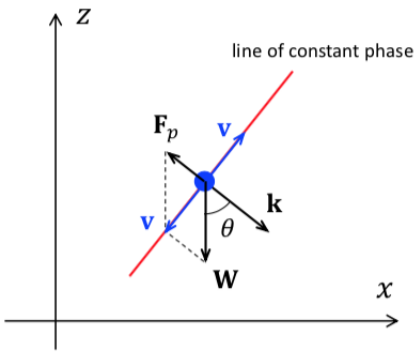
This shows that the group velocity is *perpendicular* to the direction in which the wave propagates!



In other words, the envelope of wave packets and the energy propagate in a direction which is perpendicular to the direction of propagation of the crests and troughs, as illustrated on the figure.
(*Credit: http://www.po.gso.uri.edu*)

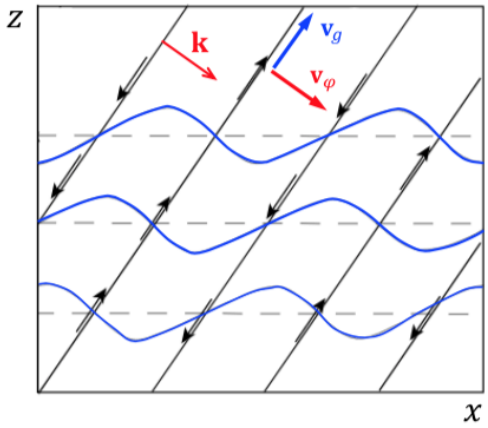
5.3.3 Motion of fluid elements

Equation (5.78) can also be written as $\mathbf{k}\cdot\tilde{\mathbf{v}}_1 = 0$, which means that the waves are *transverse*: fluid elements oscillate along lines perpendicular to the direction of propagation $\mathbf{k}$, that is to say along lines of constant phase.



The figure illustrates the motion of a fluid element (represented by the blue dot). The pressure force $\mathbf{F}_p = -\boldsymbol{\nabla} p = -\mathbf{k}p$ is perpendicular to the lines of constant phase and directed upwards (since pressure decreases upwards). The weight $\mathbf{W}$ is downwards, and the resultant net force is along the line of constant phase, as this is the line along which the fluid element moves.

We can recover the frequency of the oscillations by doing the same analysis as in section 5.3.1. A displacement δl along a line of constant phase corresponds to a vertical displacement $\delta z = \delta l \sin\theta$. Therefore, the weight W acting on the excess mass that the fluid element has relative to its surrounding is $W = -N^2\delta z = -N^2\delta l \sin\theta$ per unit mass of the fluid element. The component of the weight perpendicular to the line of constant phase balances the pressure force, so the net force acting on the fluid element is $W\sin\theta = -N^2\delta l \sin^2\theta$. Therefore, writing an equation of motion similar to equation (5.68) (replacing δz by δl), we obtain that the frequency of the oscillations is $N\sin\theta$, in agreement with equation (5.80).



The figure illustrates the motion of fluid elements and propagating internal gravity waves.
(*Credit: http://www-eaps.mit.edu*)

Chapter 6

Instabilities and turbulence

In chapter 5, we have discussed the propagation of small perturbations in a fluid. As already mentioned, using the Fourier integral theorem, it is always possible to write a perturbation as a linear combination of Fourier components $e^{i(kx-\omega t)}$, the physical quantity being the real part of this complex function. We obtain a relation between ω and k (dispersion relation) by writing that the perturbed quantities satisfy Euler's equation, the incompressibility condition and the boundary conditions. For surface waves, we have found that ω was always real, which means that the perturbation is a wave propagating in the positive x direction. However, in the case of internal gravity waves, when N^2 given by equation (5.69) is negative, which is the case when the mass density increases with height, the dispersion relation (5.80) yields the imaginary frequency $\omega = \pm i\sqrt{-N^2}k_x/k$. The term $e^{|\omega|t}$ is then present in the expression of the perturbation, which means that the perturbation grows with time: it is *unstable*. This is not surprising as a heavy fluid element displaced downwards, where the fluid is lighter, will continue to sink instead of oscillating around its equilibrium position.

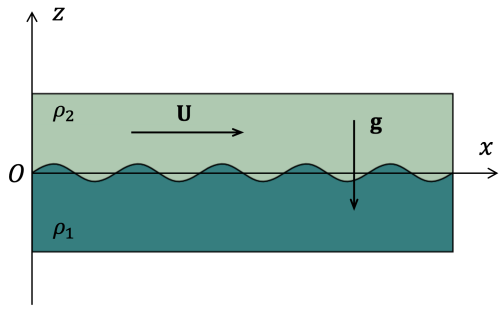
Linear stability theory deals with the growth of very small perturbations which can be treated in the same way as we have treated waves in the previous chapter, by keeping only first order terms in the perturbation in the equations. Of course, after the perturbations grow beyond some amplitude, linear theory is not valid anymore. Therefore, this theory only allows to study the onset of instabilities, and cannot be used to describe the development of the instability when non–linear terms become important. As we are going to see in this chapter, there are a number of instabilities which develop starting with very small perturbations. However, in some cases, instabilities arise only when perturbations reach a certain amplitude. These are called *finite amplitude instabilities* and will not be studied here.

6.1 Kelvin–Helmholtz instability

We consider a fluid of density ρ_2 which is above another fluid or density $\rho_1 > \rho_2$. Both fluids are inviscid and incompressible, and we take their densities to be constant. When the interface between them is perturbed, we obtain stable surface gravity waves, as studied

in chapter 5. However, the perturbation may become unstable if the two fluids move with respect to each other. In other words, a gravitationally stable stratification may be destabilized by shear. This is called a Kelvin–Helmholtz instability and is the focus of this section.

We assume that the fluid on top moves with velocity $U\hat{\mathbf{x}}$, where U is a constant, whereas the fluid at the bottom is at rest. At equilibrium, the interface between the two fluids is flat and in the $z = 0$ plane.



The interface between the two fluids is perturbed such that its equation becomes $z = \eta(x, t)$, with η being the real part of:

$$\tilde{\eta}(x, t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}. \tag{6.1}$$

6.1.1 Boundary conditions

The analysis presented here follows closely that done for surface waves in section 5.2. We take into account gravitational, pressure and surface tension forces. Both fluids satisfy Euler equation (2.30):

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}\right) = -\boldsymbol{\nabla}p - \rho g\hat{\mathbf{z}}, \tag{6.2}$$

where $g > 0$.
The boundary conditions at the interface are as follows:

- the kinematic condition (2.34) yields:

$$v_{i,z} = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial \eta}{\partial t} + v_{i,x}\frac{\partial \eta}{\partial x}, \tag{6.3}$$

  where $v_{i,x}$ and $v_{i,z}$ are the components of the velocity in fluid i, for $i = 1, 2$;

- pressure forces are equal and opposite to surface tension forces (see section 2.3.3). As shown in chapter 5, this yields (see eq. [5.21]):

$$p_1(x, \eta, t) - p_2(x, \eta, t) = -\gamma\frac{\partial^2 \eta}{\partial x^2}, \tag{6.4}$$

  where p_1 and p_2 are the pressures in fluids 1 and 2, respectively, and γ is the surface tension.

6.1.2 Dispersion relation

We note $\mathbf{v}_i = \mathbf{v}_{i,0} + \mathbf{v}_i'$ the velocity in fluid i, where $\mathbf{v}_{i,0}$ is the velocity at equilibrium and $\mathbf{v}_i'$ is the perturbed velocity. We have $\mathbf{v}_{2,0} = U\hat{\mathbf{x}}$ and $\mathbf{v}_{1,0} = \mathbf{0}$. Since the fluids are

incompressible, we can then define the velocity potential ϕ_i such that $\mathbf{v}_i = \boldsymbol{\nabla}\phi_i$. Given that the fluids at equilibrium are also incompressible, we can write $\phi_i = \phi_{i,0} + \phi'_i$, with $\mathbf{v}_{i,0} = \boldsymbol{\nabla}\phi_{i,0}$ and $\mathbf{v}'_i = \boldsymbol{\nabla}\phi'_i$. Similarly, we have $p_i = p_{i,0} + p'_i$, where $p_{i,0}$ is the pressure at equilibrium and p'_i is the perturbed pressure in fluid i.

In equation (6.3), $v_{i,z}$ is the perturbed velocity since the equilibrium velocity has no vertical component. Therefore, this equation gives:

$$\frac{\partial \phi'_1}{\partial z}(x,\eta,t) = \frac{\partial \eta}{\partial t}, \tag{6.5}$$

$$\frac{\partial \phi'_2}{\partial z}(x,\eta,t) = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x}, \tag{6.6}$$

to first order in the perturbation. To this order, we also have that $\partial\phi'_i/\partial z$ at (x,η,t) is equal to its value at $(x,0,t)$ (see section 5.2.3). Therefore, we obtain:

$$\boxed{\frac{\partial \phi'_1}{\partial z}(x,0,t) = \frac{\partial \eta}{\partial t},} \tag{6.7}$$

$$\boxed{\frac{\partial \phi'_2}{\partial z}(x,0,t) = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x}.} \tag{6.8}$$

Since the flows are irrotational and with constant densities, Bernoulli's theorem can be expressed as (eq. [2.59]):

$$\rho_i \frac{\partial \phi_i}{\partial t} + \frac{1}{2}\rho_i v_i^2 + p_i + \rho_i g z = 0, \tag{6.9}$$

which can be written as:

$$\rho_i \frac{\partial}{\partial t}\left(\phi_{i,0} + \phi'_i\right) + \frac{1}{2}\rho_i\left[\left(v_{i,0} + v'_{i,x}\right)^2 + v'^2_{i,z}\right] + p_i + \rho_i g z = 0. \tag{6.10}$$

For fluid 1, this gives at $z = \eta$:

$$\rho_1 \frac{\partial \phi'_1}{\partial t}(x,\eta,t) + p_1(x,\eta,t) + \rho_1 g\eta(x,t) = 0, \tag{6.11}$$

where we have neglected $v'^2_{1,z}$ and $v'^2_{1,x}$, as they are second order in the perturbation. To first order, $\partial\phi'_1/\partial t$ at (x,η,t) is equal to its value at $(x,0,t)$ (see section 5.2.3). Therefore, equation (6.11) yields:

$$p_1(x,\eta,t) = -\rho_1\left(\frac{\partial \phi'_1}{\partial t}(x,0,t) + g\eta(x,t)\right). \tag{6.12}$$

For fluid 2, equation (6.10) at $z = \eta$ becomes:

$$\rho_2 \frac{\partial \phi'_2}{\partial t}(x,\eta,t) + \frac{1}{2}\rho_2 U^2 + \rho_2 U\frac{\partial \phi'_2}{\partial x}(x,\eta,t) + p_2(x,\eta,t) + \rho_2 g\eta(x,t) = 0, \tag{6.13}$$

where we have used $v'_{2,x} = \partial\phi'_2/\partial x$ and neglected second order terms. Since any function of time can be added to ϕ'_2 without changing the velocities, the term $\rho_2 U^2/2$ can be subsumed into $\partial\phi'_2/\partial t$ by adding $U^2 t/2$ to ϕ'_2. Replacing $\partial\phi'_2/\partial t$ and $\partial\phi'_2/\partial x$ at $(x,\eta,t)$by their values at $(x,0,t)$ then yields:

$$p_2(x,\eta,t) = -\rho_2\left(\frac{\partial \phi'_2}{\partial t}(x,0,t) + U\frac{\partial \phi'_2}{\partial x}(x,0,t) + g\eta(x,t)\right). \tag{6.14}$$

Substituting equations (6.12) and (6.14) into the boundary condition (6.4) then gives:

$$\boxed{\rho_1 \left(\frac{\partial \phi'_1}{\partial t}(x,0,t) + g\eta \right) - \rho_2 \left(\frac{\partial \phi'_2}{\partial t}(x,0,t) + U \frac{\partial \phi'_2}{\partial x}(x,0,t) + g\eta \right) = \gamma \frac{\partial^2 \eta}{\partial x^2}.} \tag{6.15}$$

As in the case of surface waves (section 5.2.4), we look for complex solutions under the form:

$$\tilde{\phi'}_1 = f_1(z) \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \quad \tilde{\phi'}_2 = f_2(z) \mathrm{e}^{\mathrm{i}(kx - \omega t)}. \tag{6.16}$$

Since both ϕ'_1 and ϕ'_2 satisfy Laplace's equation, we have:

$$f''_i(z) - k^2 f_i(z) = 0.$$

The solutions are a linear combination of e^{kz} and e^{-kz}. We assume that the height of fluid on both sides of the interface is large compared to the wavelength of the perturbation, so that ϕ'_1 and ϕ'_2 go to zero when z go to $-\infty$ and $+\infty$, respectively. This implies $f_1 = K_1 \mathrm{e}^{kz}$ and $f_2 = K_2 \mathrm{e}^{-kz}$, where K_1 and K_2 are two constants.

Inserting in equations (6.7), (6.8) and (6.15) then yields:

$$K_1 k = -\mathrm{i}\omega A,$$

$$-K_2 k = \mathrm{i}\omega A(-\omega + Uk),$$

$$\rho_1 \left(-\mathrm{i}\omega K_1 + gA\right) - \rho_2 \left(-\mathrm{i}\omega K_2 + \mathrm{i}UK_2 k + gA\right) = -\gamma k^2 A.$$

This system has a non–trivial solution for K_1, K_2 and A only if the following *dispersion relation* is satisfied:

$$\boxed{(\rho_1 + \rho_2)\,\omega = \rho_2 U k \pm \left\{(\rho_1 + \rho_2)\left[\gamma k^3 + (\rho_1 - \rho_2)\, gk\right] - \rho_1 \rho_2 U^2 k^2\right\}^{1/2}.} \tag{6.17}$$

6.1.3 Instability condition

If the term in the square root in equation (6.17) is negative, then ω is complex and can be written as $\omega = \omega_R \pm \mathrm{i}\omega_I$. In that case, the term $\mathrm{e}^{|\omega_I| t}$ is present in the expression of $\tilde{\phi'}_1$ and $\tilde{\phi'}_2$ given by equation (6.16): the perturbation grows exponentially with time, which means *it is unstable*[1].

Therefore, the *instability condition* is:

$$(\rho_1 + \rho_2)\left[\gamma k^3 + (\rho_1 - \rho_2)\, gk\right] - \rho_1 \rho_2 U^2 k^2 < 0, \tag{6.18}$$

[1] Any general disturbance can be written as:

$$\int_{-\infty}^{+\infty} a(k) \mathrm{e}^{\mathrm{i}(kx - \omega(k) t)} \mathrm{d}k.$$

Therefore, if there exists a value of k for which $\omega(k)$ is complex, the perturbation will grow exponentially with time. Instability can only be avoided if $a(k) = 0$ for the unstable wavenumbers, which requires the perturbation to select particular values of k.

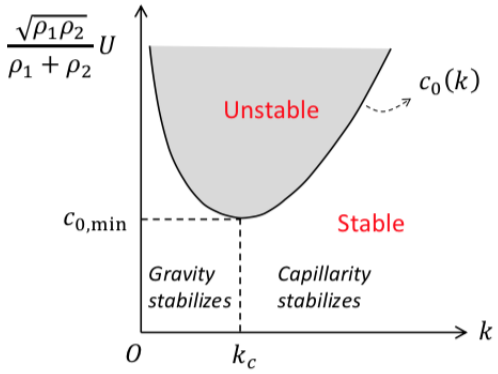
which can also be written as:

$$\boxed{\frac{\rho_1 \rho_2 U^2}{(\rho_1 + \rho_2)^2} > c_0^2,} \tag{6.19}$$

where we have defined:

$$c_0^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \frac{g}{k} \left(1 + \frac{\gamma k^2}{g\,(\rho_1 - \rho_2)} \right). \tag{6.20}$$

c_0 is the phase velocity of the surface wave which propagates as a result of the perturbation when $U = 0$ (when $\rho_2 \ll \rho_1$, we recover ω/k with ω given by eq. [5.64]).



The figure shows the region of instability in the (k, U) plane.

The minimum velocity $c_{0,\min}$ is given by:

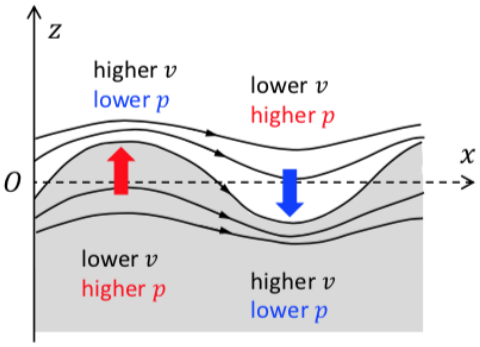
$$c_{0,\min} = \frac{2}{\rho_1 + \rho_2} \sqrt{\gamma g\,(\rho_1 - \rho_2)}, \tag{6.21}$$

and it corresponds to the wavenumber $k_c = 1/\lambda_c$, where λ_c is the capillary length:

$$\lambda_c = \sqrt{\frac{\gamma}{g\,(\rho_1 - \rho_2)}}. \tag{6.22}$$

This is equivalent to equation (5.43) when $\rho_2 \ll \rho_1$.

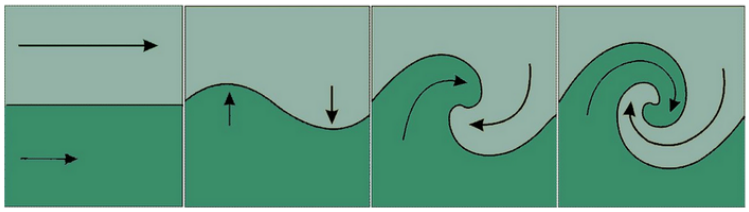
6.1.4 Physics of the instability



When the interface is perturbed, the streamlines are brought closer to each other above the crests and below the troughs, so that the flow velocity increases there (due to mass conservation), which implies that the pressure decreases (due to Bernoulli's theorem).

Similarly, the velocity decreases and the pressure increases under the crests and above the troughs. Therefore, pressure forces amplify the perturbation, which is the basis of the instability. Gravitational forces tend to stabilize the flow, as they oppose the upwards motion of the heavier fluid which is at the bottom. Similarly, surface tension stabilizes the flow by opposing the deformation of the interface. Gravitation and capillarity are more effective at large and small wavelengths, respectively, with comparable contribution at $\lambda \sim \lambda_c$.

When the perturbation starts growing, the upper part of the interface moves with the velocity of the fluid at the top whereas the lower part moves with the velocity of the fluid at the bottom, which results in the deformation of the fronts and produces a rolling up of the interface, as indicated on the figure below:



The evolution of the flow can also be understood in terms of the vorticity. At equilibrium, the circulation $\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{l}$ around a rectangular contour C across the interface with two sides parallel to the interface is equal to $\pm UL$, where L is the length of the sides along the x–axis. Using Stokes theorem, this is equal to the flux of vorticity through the surface delimited by the contour. Since the flows on both sides are irrotational at equilibrium, the vorticity is localized on the interface, where the shear is present (as U varies discontinuously at the interface). When the interface is perturbed, this *vorticity sheet* is deformed, and the subsequent evolution is constrained by Kelvin's theorem.



This instability may also occur when there is a continuous gradient of density and velocity, instead of a sudden jump at an interface, as in layers of clouds.

6.2 Rayleigh–Taylor instability

See Problem Set 5.

6.3 Turbulence

In some cases, the growth of perturbations due to instabilities in the non–linear regime leads to a transition to turbulence. The instabilities described in the above sections are due to a particular gradient of density or velocity. However, it is an experimental fact that every shear flow becomes turbulent when its Reynolds number exceeds a critical value $Re_c \sim 10^3$. The structure of a turbulent flow is very complex and non predictable. However, statistical methods can be used to derive some properties of turbulence. In this section, we assume that the density ρ is constant (in space and time), which implies that the fluid is incompressible.

6.3.1 The Reynolds stress

In a turbulent flow, velocities have random fluctuations around a mean value. This is expressed by the co–called *Reynolds decomposition*:

$$\mathbf{v} = \mathbf{V} + \mathbf{v}', \tag{6.23}$$

where $\mathbf{V} = \langle \mathbf{v} \rangle$ and $\langle \mathbf{v}' \rangle = \mathbf{0}$, with the brackets denoting an average. The different quantities depend both on location $\mathbf{r}$ and on time t. In general, if the flow is time–dependent, $\langle \mathbf{v} \rangle$ corresponds to the average of the velocity over a large ensemble of flows with the same properties. However, if the flow varies on a timescale T long compared to the characteristic timescale τ of the fluctuations, or if the flow is time–independent (infinite T), the average may be calculated over a time large compared to τ and small compared to T. A similar decomposition can be made for the pressure and the viscous stress tensor (2.13):

$$p = P + p', \quad \sigma_{ij} = S_{ij} + \sigma'_{ij}, \tag{6.24}$$

where $P = \langle p \rangle$, $S_{ij} = \langle \sigma_{ij} \rangle$ and $\langle p' \rangle = \left\langle \sigma'_{ij} \right\rangle = 0$. Given the way average quantities are calculated, it is straightforward to show that the spatial derivative of an average quantity is equal to the average of the spatial derivative of that quantity:

$$\frac{\partial \langle \rangle}{\partial x_i} = \left\langle \frac{\partial}{\partial x_i} \right\rangle .$$

This is also valid for the time–derivative:

$$\frac{\partial \langle \rangle}{\partial t} = \left\langle \frac{\partial}{\partial t} \right\rangle ,$$

but, when the flow is time–dependent, this requires $\tau \ll T$.
Interchanging the average and space derivatives, we can write the average of the viscous stress as:

$$S_{ij} = \rho\nu \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) . \tag{6.25}$$

The flow satisfies Navier–Stokes equation (2.20):

$$\rho \frac{\partial v_i}{\partial t} + \rho \left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) v_i = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \tag{6.26}$$

where $\mathbf{f}$ includes all the forces per unit volume which act on the fluid. Using the Reynolds decomposition above, this equation can be re–written as:

$$\rho\frac{\partial}{\partial t}\left(V_i+v_i'\right)+\rho\left[\left(\mathbf{V}+\mathbf{v}'\right)\cdot\boldsymbol{\nabla}\right]\left(V_i+v_i'\right)=-\frac{\partial}{\partial x_i}\left(P+p'\right)+\frac{\partial}{\partial x_j}\left(S_{ij}+\sigma_{ij}'\right)+f_i. \quad (6.27)$$

We now average this equation over a time long compared to τ but small compared to T. Using the fact that the time and space derivatives can be interchanged with the averages, and that the average of the fluctuations is zero, this yields:

$$\rho\frac{\partial V_i}{\partial t}+\rho\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)V_i+\rho\left\langle\left(\mathbf{v}'\cdot\boldsymbol{\nabla}\right)v_i'\right\rangle=-\frac{\partial P}{\partial x_i}+\frac{\partial S_{ij}}{\partial x_j}+f_i. \quad (6.28)$$

We have assumed that the flow is incompressible, that is to say:

$$\boldsymbol{\nabla}\cdot\left(\mathbf{V}+\mathbf{v}'\right)=0. \quad (6.29)$$

Taking a time–average of this equation yields:

$$\boldsymbol{\nabla}\cdot\mathbf{V}=0, \quad (6.30)$$

which means that *the average flow is incompressible*. Subtracting from equation (6.29) then gives:

$$\boldsymbol{\nabla}\cdot\mathbf{v}'=0, \quad (6.31)$$

which means that *the fluctuations are also incompressible*. Therefore, the term $\rho\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)V_i$ in equation (6.28) can be written as:

$$\rho V_j\frac{\partial V_i}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\rho V_jV_i\right),$$

where we have used the incompressibility of the average velocity. Similarly, the term $\rho\left\langle\left(\mathbf{v}'\cdot\boldsymbol{\nabla}\right)v_i'\right\rangle$ in equation (6.28) can be written as:

$$\rho\left\langle v_j'\frac{\partial v_i'}{\partial x_j}\right\rangle=\left\langle\frac{\partial}{\partial x_j}\left(\rho v_j'v_i'\right)\right\rangle=\frac{\partial}{\partial x_j}\left\langle\rho v_j'v_i'\right\rangle,$$

where we have used the incompressibility of the fluctuations and interchanged the space derivatives and average. Therefore, equation (6.28) becomes:

$$\rho\frac{\partial V_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho V_jV_i+\rho\left\langle v_j'v_i'\right\rangle\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial S_{ij}}{\partial x_j}+f_i, \quad (6.32)$$

which can also be written in the form:

$$\boxed{\rho\frac{\partial V_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho V_jV_i+\rho\left\langle v_j'v_i'\right\rangle+P\delta_{ij}-S_{ij}\right)=f_i.} \quad (6.33)$$

This equation expresses the fact that the rate of change of momentum in a fixed volume ($\partial/\partial t$ term) is due to the divergence of the *momentum flux* ($\partial/\partial x_j$ term) and to the forces acting on the volume. This is similar to equation (2.19), but with the addition of the *Reynolds stress,* also called *turbulent stress*:

$$\boxed{\tau_{ij}=-\rho\left\langle v_j'v_i'\right\rangle.} \quad (6.34)$$

Therefore, in a turbulent flow, momentum is transported by the mean velocities, by viscosity and pressure, as in a laminar flow, but also by the correlations between the components of the fluctuations.

6.3.2 Mixing length theory

As we have no expression for the components of τ_{ij} (six of which are independent, as the tensor is symmetrical), the problem has more unknowns than equations. This is the well–known closure problem of turbulence. Since τ_{ij} appears in the same way as $S_{ij} = \langle \sigma_{ij} \rangle$ in equation (6.33), it is tempting to express τ_{ij} by analogy with $\langle \sigma_{ij} \rangle$, which is given by equation (6.25). This is the basis for the *mixing length theory*, in which τ_{ij} is written as:

$$\tau_{ij} = \rho \nu_t \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right). \tag{6.35}$$

where ν_T is the so–called *turbulent, or eddy viscosity*. By analogy with expression (2.5) for the molecular viscosity, it is supposed that:

$$\nu_T \sim v_T \Lambda, \tag{6.36}$$

where v_t is a characteristic velocity of the turbulent eddies and Λ is the so–called *mixing length*, which is the "mean free path" of the eddies, i.e. the distance they travel through before they mix with their environment. Both v_t and Λ depend on the flow under consideration. Mixing length theory is widely used, for example to model the convective zone of stars. However, although in this context it describes well the transport of energy by the turbulent flow, it fails to give an accurate description of the transport of momentum. This is a problem when modelling the interaction between the convective flow and tidal oscillations excited by a stellar companion in binary systems, for example. One of the limitation of the model is that the scale of the largest turbulent eddies is comparable to the scale on which the average quantities vary. This is in constrast to kinetic theory, where the calculation of the viscosity relies on the separation between the scale on which random motions occur and the scale over which quantities are averaged.

6.3.3 Energy conservation

The kinetic energy per unit mass is:

$$\frac{1}{2} v_i^2 = \frac{1}{2} \left(V_i + v_i' \right)^2 = \frac{1}{2} \left(V_i^2 + v_i'^2 + 2 V_i v_i' \right).$$

Therefore, the average kinetic energy per unit mass is:

$$\frac{1}{2} \left(V_i^2 + \langle v_i'^2 \rangle \right),$$

which is the sum of the kinetic energy of the mean flow and that of the fluctuations.

We obtain an energy conservation equation for the mean flow by multiplying equation (6.28) by V_i:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} V_i^2 \right) + V_i V_j \frac{\partial V_i}{\partial x_j} + V_i \left\langle v_j' \frac{\partial v_i'}{\partial x_j} \right\rangle = -\frac{V_i}{\rho} \frac{\partial P}{\partial x_i} + \nu V_i \frac{\partial}{\partial x_j} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right), \tag{6.37}$$

where we have neglected gravity, and where summation over i and j is implicit. Using the incompressibility of the average velocity (eq. [6.30]), we have:

$$\frac{V_i}{\rho}\frac{\partial P}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{V_i P}{\rho}\right),$$

and:

$$V_i \frac{\partial}{\partial x_j}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) = V_i \frac{\partial^2 V_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(V_i \frac{\partial V_i}{\partial x_j}\right) - \left(\frac{\partial V_i}{\partial x_j}\right)^2.$$

Using the incompressibility of the fluctuations (eq. [6.31]) yields:

$$V_i \left\langle v_j' \frac{\partial v_i'}{\partial x_j} \right\rangle = V_i \left\langle \frac{\partial}{\partial x_j}\left(v_j' v_i'\right) \right\rangle.$$

Interchanging the average and space derivatives, this can be written as:

$$V_i \frac{\partial}{\partial x_j}\left\langle v_j' v_i' \right\rangle = \frac{\partial}{\partial x_j}\left(V_i \left\langle v_j' v_i' \right\rangle\right) - \left\langle v_j' v_i' \right\rangle \frac{\partial V_i}{\partial x_j}.$$

Therefore, equation (6.37) becomes:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{V_i^2}{2}\right) + V_j \frac{\partial}{\partial x_j}\left(\frac{V_i^2}{2}\right) = \\ &\qquad \frac{\partial}{\partial x_j}\left(-\frac{V_j P}{\rho} + \nu V_i \frac{\partial V_i}{\partial x_j} - V_i \left\langle v_j' v_i' \right\rangle\right) - \nu \left(\frac{\partial V_i}{\partial x_j}\right)^2 + \left\langle v_j' v_i' \right\rangle \frac{\partial V_i}{\partial x_j}.\end{aligned}} \tag{6.38}$$

This equation expresses the conservation of kinetic energy for the mean flow. It indicates that the Lagrangian derivative of $V_i^2/2$ (left hand–side) is equal to the divergence of a flux, which represents the work done by pressure forces, viscous and Reynolds stresses, plus a term expressing dissipation of energy in the mean flow due to viscosity, which is $\nu\left(\partial V_i/\partial x_j\right)^2$, plus a term which represents the *transport of mean momentum by the Reynolds stress*, which is $\left\langle v_j' v_i' \right\rangle \left(\partial V_i/\partial x_j\right)$.

A similar conservation equation for the fluctuations can be obtained by multiplying equation (6.26) by v_i':

$$v_i' \frac{\partial v_i}{\partial t} + v_i' v_j \frac{\partial v_i}{\partial x_j} = -\frac{v_i'}{\rho}\frac{\partial p}{\partial x_i} + \nu v_i' \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{6.39}$$

Using the Reynolds decomposition and averaging over time yields:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{v_i'^2}{2}\right) + \left\langle v_i' V_j \frac{\partial v_i'}{\partial x_j} \right\rangle + \left\langle v_i' v_j' \frac{\partial v_i'}{\partial x_j} \right\rangle + \left\langle v_i' v_j' \frac{\partial V_i}{\partial x_j} \right\rangle = \\ - \left\langle \frac{v_i'}{\rho}\frac{\partial p'}{\partial x_i} \right\rangle + \nu \left\langle v_i' \frac{\partial}{\partial x_j}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right) \right\rangle.\end{aligned} \tag{6.40}$$

As above, we use the incompressibility of the average and fluctuating velocities to re–write this equation as:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\left\langle v_i'^2 \right\rangle}{2}\right) + V_j \frac{\partial}{\partial x_j}\left(\frac{\left\langle v_i'^2 \right\rangle}{2}\right) = \\ &\frac{\partial}{\partial x_j}\left(-\frac{\left\langle v_j' p' \right\rangle}{\rho} + \nu \left\langle v_i' \frac{\partial v_i'}{\partial x_j} \right\rangle - \frac{\left\langle v_j' v_i'^2 \right\rangle}{2}\right) - \nu \left\langle \left(\frac{\partial v_i'}{\partial x_j}\right)^2 \right\rangle - \left\langle v_j' v_i' \right\rangle \frac{\partial V_i}{\partial x_j}.\end{aligned}} \tag{6.41}$$

Here again, this equation expresses the conservation of kinetic energy for the fluctuations. It indicates that the Lagrangian derivative of $\langle v_i'^2 \rangle / 2$ (left hand–side) is equal to the divergence of a flux, which represents the average of the work done by the fluctuating pressure forces, viscous and Reynolds stresses, plus a term expressing dissipation of energy in the fluctuations due to viscosity, which is $\nu \left\langle (\partial v_i'/\partial x_j)^2 \right\rangle$, plus the same term as in equation (6.38) which represents the *transport of mean momentum by the Reynolds stress*, which is $-\left\langle v_j' v_i' \right\rangle (\partial V_i/\partial x_j)$. This term has a positive sign in equation (6.38), and a negative sign here. It means that *energy is transferred from the mean flow to the fluctuations.*

6.3.4 Kolmogorov scaling

When the fluctuations are stationary and vary slowly in space, equation (6.41) gives:

$$\langle v_j' v_i' \rangle \frac{\partial V_i}{\partial x_j} = -\nu \left\langle \left(\frac{\partial v_i'}{\partial x_j} \right)^2 \right\rangle, \tag{6.42}$$

which shows that the energy received by the fluctuations from the mean flow (left hand side) is dissipated by viscosity (right hand side).

We now present the Kolmogorov model, which describes how the energy which is fed into the fluctuations from the mean flow is ultimately dissipated by viscosity. The turbulence can be modelled as a superposition of vortices, also called *turbulent eddies*, of different sizes. The Kolmogorov model, which applies to homogeneous and isotropic turbulence, assumes that the energy from the mean flow is fed into the largest eddies, and that it subsequently cascades down to the smallest eddies where it is dissipated by viscosity.

The largest eddies have a characteristic size and velocity that we note l and v_l, respectively. Let ϵ be the energy per unit mass which is transferred by the mean flow to the largest eddies per unit time. The units of ϵ are J kg^{-1} s^{-1}, which is equal to m^2 s^{-3}. Dimensional analysis then yields:

$$\epsilon \sim v_l^3 / l. \tag{6.43}$$

This energy is then passed on continuously to smaller eddies. Therefore, the same argument applies to an eddy of size λ and velocity v_λ receiving the energy ϵ per unit mass and per unit time, so that:

$$\epsilon \sim v_\lambda^3 / \lambda. \tag{6.44}$$

When the energy reaches the smalles eddies, which have a size l_0 and a velocity v_0, it is dissipated by viscosity, Therefore, ϵ is equal to the term on the right hand side of equation (6.42) evaluated for the smallest eddies, which gives:

$$\epsilon \sim \nu v_0^2 / l_0^2. \tag{6.45}$$

Writing equation (6.44) for $v_\lambda = v_0$ and $\lambda = l_0$ and comparing with equation (6.45) then yields the *Komolgorov length and velocity*::

$$l_0 \sim \nu^{3/4} \epsilon^{-1/4}, \quad v_0 \sim (\nu \epsilon)^{1/4}. \tag{6.46}$$

The kinetic energy per unit mass of an eddy of size λ and velocity v_λ is $v_\lambda^2/2$, which according to equation (6.44) is proportional to $(\epsilon\lambda)^{2/3}$. Therefore, *larger eddies have more energy.*

The range of scales $\lambda \sim l$ is called the *energy range*, as this is where most of the energy is. The scales $\lambda \sim l_0$ are called *dissipation range*, and the intermediate scales $l_0 \ll \lambda \ll l$ are called the *inertial range.*

A very important law in the theory of turbulence is the so–called *Kolmogorov's similarity law*, which is obtained by introducing the wavenumber k such that $\lambda \sim 1/k$. We then note $E(k)\mathrm{d}k$ the kinetic energy per unit mass in the eddies with wavenumber between k and $k + \mathrm{d}k$. The quantity $E(k)$ is the spectral energy density and is associated with the Fourier decomposition of the turbulent velocities. In the inertial range, this energy cannot depend on l, l_0 or ν. Therefore, it can only depend on ϵ and on k. Since it has the dimensions $\mathrm{m}^3\ \mathrm{s}^{-2}$, dimensional analysis yields:

$$\boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}.} \tag{6.47}$$

Note that, for $k = 1/\lambda$:

$$\int_k^{+\infty} E(k')\mathrm{d}k' \propto \epsilon^{2/3} k^{-2/3} \sim (\epsilon\lambda)^{2/3} \sim v_\lambda^2.$$

This means that the total energy in all the eddies with sizes smaller than λ is roughly equal to the energy of the eddies with size λ, which is consistent with the result that the energy is in the largest eddies.

Appendix A

Complex variables

A.1 The Cauchy–Riemann relations

A function is *analytic* in a domain if it is single–valued and differentiable at all points in this domain. For the function $w(z)$ to be differentiable, the limit:

$$L = \lim_{\Delta z \to 0} \frac{w(z + \Delta z) - w(z)}{\Delta z},$$

must exist and be unique. With $z = x + \mathrm{i}y$ and $w(z) = \phi(x, y) + \mathrm{i}\psi(x, y)$, this limit can also be written as:

$$L = \lim_{\Delta x, \Delta y \to 0} \frac{\phi(x + \Delta x, y + \Delta y) + \mathrm{i}\psi(x + \Delta x, y + \Delta y) - \phi(x, y) - \mathrm{i}\psi(x, y)}{\Delta x + \mathrm{i}\Delta y,}.$$

This limit should not depend on the direction along which we approach the point (x, y). Therefore, the result corresponding to Δz being purely real ($\Delta y = 0$), which is:

$$L = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y) + \mathrm{i}\psi(x + \Delta x, y) - \phi(x, y) - \mathrm{i}\psi(x, y)}{\Delta x} = \frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x},$$

should be identical to the result corresponding to Δz being purely imaginary ($\Delta x = 0$), which is:

$$L = \lim_{\Delta y \to 0} \frac{\phi(x, y + \Delta y) + \mathrm{i}\psi(x, y + \Delta y) - \phi(x, y) - \mathrm{i}\psi(x, y)}{\mathrm{i}\Delta y} = -\mathrm{i}\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}.$$

By equating the real and imaginary parts of these two expressions, we see that ϕ and ψ have to satisfy the *Cauchy–Riemann* relations:

$$\boxed{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}.} \tag{A.1}$$

It is straightfoward to see that these relations imply $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$, and similarly for ψ, so that *both the real and imaginary parts of an analytic function satisfy Laplace's equation.* A function that satisfies Laplace's equation is called a *harmonic function.*

The derivative of w can then be calculated by keeping y constant:

$$\boxed{\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x}.} \tag{A.2}$$

A.2 Cauchy's theorem

A.3 Laurent series