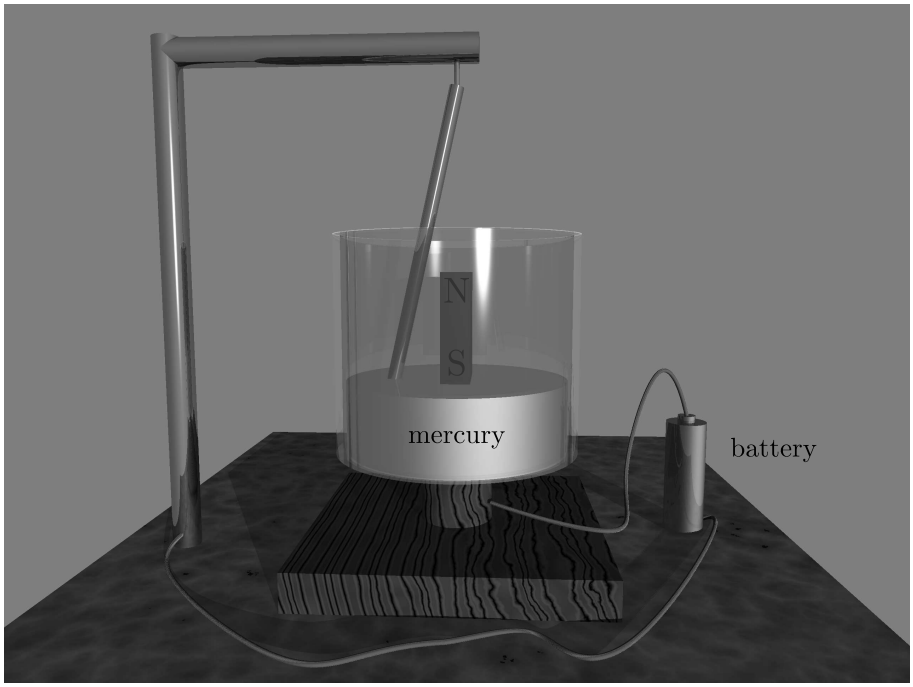
Electromagnetism

UNIVERSITY OF OXFORD

Second year physics course

Professor D. A. Uzdensky



Questions*

MICHAELMAS 2025

*I am extremely grateful to Profs. Caroline Terquem and Steven Blundell who first assembled these problem sets. Some of these problems are taken from: *Introduction to Electrodynamics*, David J. Griffiths, 4th Edition, and *Electricity and Magnetism*, Edward M. Purcell and David J. Morin, 3rd Edition.

Electromagnetism: Problems

Problem sheet 1: (Michaelmas Term 2025)

■ *Potentials*

1.1 **Laplace's equation**

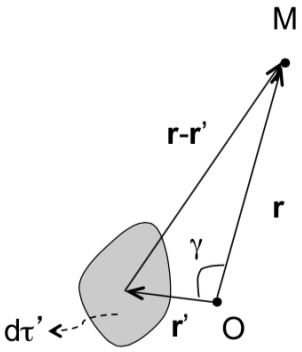
This question concerns Laplace's equation, $\nabla^2 V = 0$, which applies when there are no charges within a region. Consider a potential $V(\boldsymbol{r})$ which satisfies Laplace's equation and now imagine a sphere of radius R centred about the origin. The average value of V over the surface of the sphere is given by

$$\bar{V}_R = \frac{1}{4\pi R^2} \int_S V(\boldsymbol{r}) \mathrm{d}S,$$

where the integral is taken over the surface of the sphere. By evaluating $\partial \bar{V}_R / \partial R$, show that the potential at any point is equal to that averaged over a sphere of radius R centred at that point. (Hint: you can choose coordinates so that "any point" is the origin in those coordinates.)

1.2 **Expansion in Legendre polynomials**

This problem is useful to understand that a multipole expansion is nothing more than a development in Taylor series of $1/|\boldsymbol{r}' - \boldsymbol{r}|$.



The scalar potential V created at a point M by a localised charge distribution is:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{r}|} d\tau',$$

where the integration is over the volume $\mathcal{V}$ of the distribution and the position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ are measured from an origin O chosen arbitrarily. [Note that $\boldsymbol{r} \cdot \boldsymbol{r}' = rr' \cos\gamma$.]

Assuming $r' \ll r$, expand $1/|\boldsymbol{r}' - \boldsymbol{r}|$ in Taylor series up to 3rd order in r'/r. Use the result to give an expression for V in term of Legendre polynomials including the terms up to $l = 3$.

We give: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

1.3 **Dipole**

a) Show that the energy of a physical dipole $\boldsymbol{p}$ in an electric field $\boldsymbol{E}$ (not necessarily uniform) is given by $U = -\boldsymbol{p} \cdot \boldsymbol{E}$.

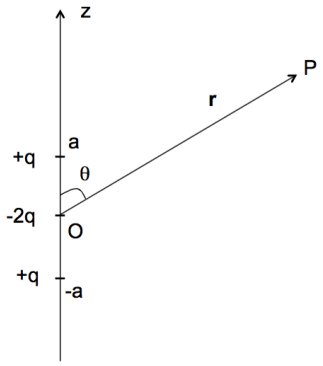
b) We consider two dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and take $\boldsymbol{r}$ to be the position vector of $\boldsymbol{p}_2$ measured from $\boldsymbol{p}_1$. Show that the interaction energy of the two dipoles is:

$$U_{\text{int}} = \frac{1}{4\pi\epsilon_0 r^3} \left[\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 - 3 \left(\boldsymbol{p}_1 \cdot \hat{\boldsymbol{r}} \right) \left(\boldsymbol{p}_2 \cdot \hat{\boldsymbol{r}} \right) \right],$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$.

c) Draw graphs showing how U_{int} depends upon the relative orientation of the dipoles in the following cases: (i) $\boldsymbol{p}_1$ is parallel to $\boldsymbol{r}$, (ii) $\boldsymbol{p}_1$ is perpendicular to $\boldsymbol{r}$.

1.4 **Quadrupole**



A system of three charges, aligned along the z–axis, consists of a charge $-2q$ at the origin O and two $+q$ charges at $z = -a$ and $z = a$. Here we use spherical coordinates: (r, θ, φ).

a) Find the potential at $r \gg a$ using the multipole expansion with origin at O. Justify why this system of charges is called a quadrupole.
We give: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$.

b) Show that there is not translational force or couple on the quadrupole in a uniform electric field.

c) We consider a point charge Q placed at a point P a distance $r \gg a$ away from the quadrupole. Show that the torque about O on the quadrupole due to the charge is $\Gamma = 3Qqa^2 \sin 2\theta/(4\pi\epsilon_0 r^3)$ and find its direction.

d) Find the energy of the quadrupole in the potential of the charge Q. In which direction does the quadrupole rotate under the effect of the torque?

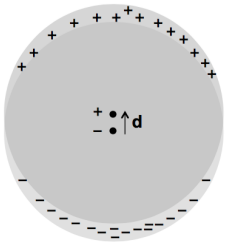
1.5 **Electric field due to a sphere with surface charge density $\propto \cos\theta$**
We consider a sphere of radius R and a spherical coordinate system (r, θ, φ) with origin at the centre of the sphere. The surface of the sphere carries a charge density $\sigma = k\cos\theta$.

a) Calculate the potential V using the fact that it satisfies Laplace's equation both inside and outside the sphere. *Hint*: write the solution of the equation in terms of Legendre polynomials and choose $V = 0$ at infinity.

b) Find the electric field $\boldsymbol{E}$ produced by the sphere.

c) Calculate the electric dipole moment $\boldsymbol{p}$ of the charge distribution. Check that the electric field outside the sphere calculated in the previous question is that due to the dipole moment $\boldsymbol{p}$. What can you conclude about the higher multipoles? Express the field inside the sphere as a function of the *polarization* of the sphere, which is the dipole moment per unit volume.

d) A surface charge density $\propto \cos\theta$ can be obtained by superimposing two solid spheres with same radius R, opposite charges Q and $-Q$ uniformly distributed over their volume, and centres slightly apart (separation $d \ll R$), as shown on the figure. Viewed from a point outside the spheres, the situation looks the same as if the charges were at the centre of the spheres.



Find the relation between Q, $\boldsymbol{d}$ and $\boldsymbol{p}$ for the electric field outside the spheres to be the same as in the previous question. Using Gauss's theorem to obtain the contribution from each sphere, calculate the field in the region where the spheres overlap and show that it is the same as in the previous question.

1.6 **Metal sphere in a uniform electric field** (*Problem 1.5 needs to be done first*)
A metal sphere of radius R with no net charge is placed in a uniform electric field $\boldsymbol{E} = E_0\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is a unit vector.

a) Explain qualitatively how the field modifies the charge distribution in the sphere and how this, in turn, affects the electric field.

b) Calculate the electric potential V outside the sphere using the fact that it satisfies Laplace's equation. *Hint*: write the solution of the equation in terms of Legendre polynomials and choose $V = 0$ at the surface of the sphere.

c) Using the boundary condition on the electric field, calculate the charge density σ at the surface of the sphere.

d) Using the results from Problem 1.5(b), give an expression for the electric field produced by σ. Find the total field inside the sphere. Comment.

1.7 **Separation of variables in cylindrical coordinates**

a) Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z.

b) Consider an infinitely long straight wire along the z–axis which carries a uniform line charge λ. Calculate the electric field using Gauss's theorem. Find the potential and check that it is a solution of Laplace's equation found in (a).

c) Consider an infinitely long metal pipe of radius R placed at right angle to a uniform electric field $\boldsymbol{E}_0$. Using the result from (a) and appropriate boundary conditions, find the potential outside the pipe. Calculate the charge density induced on the pipe.

1.8 **Magnetic vector potential**

We consider a finite segment of straight wire of length $2L$ carrying a steady current I.

a) Calculate the magnetic vector potential $\boldsymbol{A}$ at a point P a distance r from the wire along the perpendicular bisector. We give:

$$\int \frac{dz}{\sqrt{z^2+a^2}} = \ln\left(z+\sqrt{z^2+a^2}\right).$$

b) Using $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and assuming $L \gg r$, calculate the magnetic field at P. Check that your answer is consistent with what is expected from Ampère's law.

Electromagnetism: Problems

Problem sheet 2: (Michaelmas Term 2025)

■ *Electric and magnetic fields in matter*

2.1 Capacitance and dielectrics

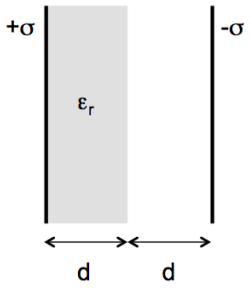
We consider a parallel plate capacitor consisting of two metal surfaces of area A separated by a distance d. We assume $d \ll A^{1/2}$, so that edge effects can be neglected and the electric field can be considered to be uniform between the plates.

a) The capacitor is connected to a battery so that a charge $+Q_0$ is brought from one plate to the other. Using Gauss's law, calculate the electric field E_0 between the plates. Find the potential difference ΔV_0 between the plates and the capacitance C_0. Calculate the potential energy U_0 stored in the capacitor.

b) We now insert a linear dielectric material between the plates while the capacitor remains connected to the battery, which supplies the potential difference ΔV_0. Experimentally, it is found that the charge Q on the plates is increased. Explain why. By representing the system as the superposition of a vacuum capacitor and a polarized dielectric slab, calculate the total electric field E in the capacitor in terms of Q and the polarization P. By substituting $P = \epsilon_0 \chi_e E$, find the relation between the displacement vector $\boldsymbol{D}$ and Q. Calculate Q in terms of Q_0 and the capacitance C in terms of C_0. Calculate the change in the stored potential energy of the capacitor in terms of U_0.

c) We now come back to question (a) and disconnect the battery before inserting a dielectric material between the capacitor plates. Experimentally, it is found that the potential difference ΔV between the plates decreases. Explain why. Calculate the capacitance C in terms of C_0 and the change in the stored potential energy of the capacitor in terms of U_0.

2.2 Capacitor half–filled with a dielectric



We insert a slab of linear dielectric material of dielectric constant ϵ_r and thickness d next to the positive plate in a parallel plate capacitor. The distance between the plates is $2d$, their area is A and they carry a (free) charge density $\pm\sigma$.

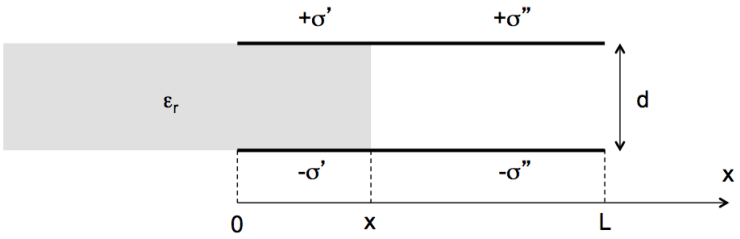
a) Find the electric displacement vector $\boldsymbol{D}$, the electric field $\boldsymbol{E}$ and the polarization $\boldsymbol{p}$ in each region.

b) Find the potential difference between the plates and the capacitance C. Show that the system can be regarded as two capacitors connected in series.

c) Find the location and amount of all bound charge. Using the distribution of free and bound charges, recalculate the electric field $\boldsymbol{E}$ in each region.

2.3 **Force on a dielectric**

We insert a portion of a slab of linear dielectric material of dielectric constant ϵ_r and thickness d on the left hand side of a parallel plate capacitor consisting of two conducting plates of length L, width w and thickness d.



a) We connect the capacitor to a battery to charge the plates, and then disconnect the battery. The total charge on each plate then remains constant equal to $\pm Q$, corresponding to surface charge densities $\pm\sigma'$ and $\pm\sigma''$ on the left and right hand sides, respectively. Using Gauss's law, write the electric fields E' and E'' on the left and right hand sides in terms of σ' and σ''. Find the relation between σ' and σ''. Using the fact that Q is constant, write σ'' in terms of x and the different constants. Find the potential difference between the plates, and then the stored potential energy U of the system, in terms of x. Write the relation between the change in energy, dU, when the dielectric is pulled out a distance dx, and the electric force F exerted by the plates on the dielectric. Calculate F in terms of x. Does this force pull the dielectric into the capacitor or push it out?

b) We now repeat the experiment but leave the capacitor connected to the battery, which supplies the potential difference ΔV. Calculate U in terms of x and the different constants. Write the relation between dU and F when the dielectric is pulled out a distance dx. Make sure to include *all* contributions to the work done in the system. Calculate F in terms of x and show that it is the same as in question (a).

c) The result obtained above can be verified experimentally. However, we have assumed in our calculation that the electric field was uniform between the plates and zero outside. If that were the case in reality, would there be a force on the dielectric? Where does this force come from, and how is it possible that we obtain the right answer given the simplified model we have used?

2.4 **Sphere with a frozen–in polarization**

A sphere of radius R made of a dielectric material carries a frozen–in polarization $\boldsymbol{P}(\boldsymbol{r}) = k\boldsymbol{r}$, where k is a constant and $\boldsymbol{r}$ is the position vector measured from the centre of the sphere. There are no free charges anywhere.

a) Calculate the bound charges.

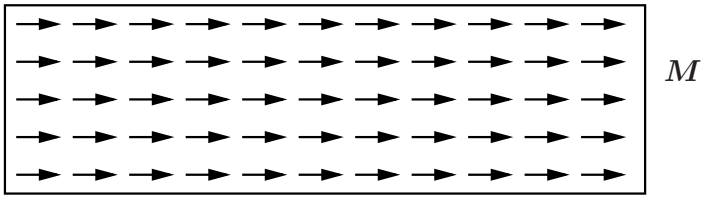
b) Using Gauss's law, find the electric field $\boldsymbol{E}$ inside and outside the sphere.

c) Using the expression of $\boldsymbol{E}$ and $\boldsymbol{P}$, find the electric displacement vector $\boldsymbol{D}$ inside and outside the sphere. Check that it satisfies Gauss's law. Is the dielectric linear?

■ *Magnetic fields in matter*

2.5 **B, H and M in a bar magnet**

Sketch $\boldsymbol{B}$, $\boldsymbol{H}$ for a rectangular bar magnet where the magnetization $\boldsymbol{M}$ is as indicated below.



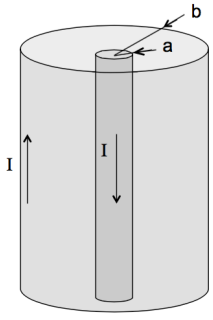
2.6 **Cylinder with a frozen–in magnetization**

An infinitely long cylinder of radius R made of a magnetic material carries a frozen–in magnetization $\boldsymbol{M} = kr\hat{\boldsymbol{z}}$, where r is the distance from the axis of the cylinder, $\hat{\boldsymbol{z}}$ is the unit vector along the axis and k is a constant. There is no free current anywhere. Find the magnetic field $\boldsymbol{B}$ inside and outside the cylinder by two different methods:

a) Locate all the bound currents and calculate the field they produce,

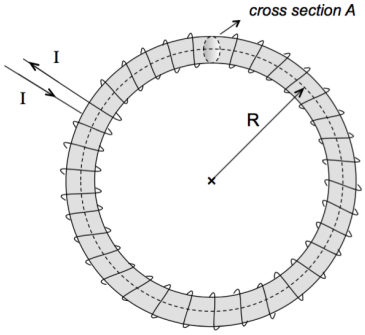
b) Use Ampère's law for $\boldsymbol{H}$ and the relationship between $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{M}$ to derive $\boldsymbol{B}$.

2.7 **Magnetic field in a coaxial cable**



A long coaxial cable of inner radius a and outer radius b is filled with an insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one, uniformly distributed over the surfaces. Find the magnetic field $\boldsymbol{B}$ in the magnetic material between the conductors. As a check, calculate the magnetization $\boldsymbol{M}$ and the bound currents, and confirm that, together with the free currents, they generate the correct field.

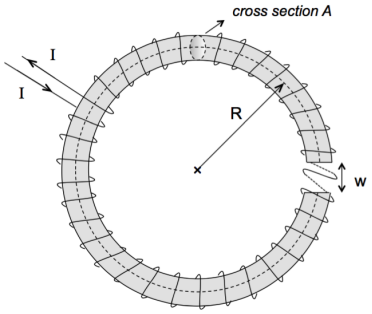
2.8 **Air gap in an inductor**



We consider a toroidal core made of an iron alloy with cross sectional area A and radius $R \gg A^{1/2}$. We make an inductor from the core by wrapping around it N turns of a wire carrying a current I, to use it as an energy storage device.

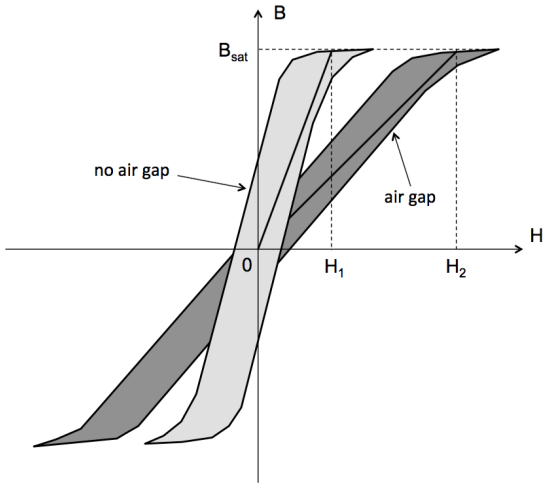
We increase the current from 0 until the material saturates, which happens when the magnetic field B inside it reaches the value B_{sat}. Before saturation is reached, the material can be regarded as being linear with relative permeability μ_r.

a) Calculate the magnetic field B in the core and the inductance L. Find the maximum current I_1 that the inductor can carry before the core saturates. Calculate the magnetic energy W_1 that is stored in the inductor when the current reaches the value I_1. If I is increased beyond I_1, how does the inductance vary?



b) To increase the energy that can be stored in the inductor, we have to find a way of increasing the saturation current while keeping L constant. This can be done by introducing an air gap of width w into the core. We take $w \ll R$, so that we can neglect magnetic fringing, that is to say the bulging out of the magnetic field lines as they enter air from the magnetic material.

Calculate the magnetic field B in the core and the number of turns of wire we now have to wrap around the core to keep L constant. Find the maximum current I_2 this second inductor can carry before the core saturates and compare it with I_1. Calculate the energy W_2 stored in the magnetic field at the point of saturation and compare it with W_1. Compare the energy stored in the magnetic material with that stored in the air gap. For numerical applications, use $R = 10$ cm, $w = 3$ mm and $\mu_r = 1500$.



c) Some of the energy stored in the inductor can be recovered by decreasing the current back to 0. The figure shows the $B - H$ curves (hysteresis loops) measured on a toroidal core without and with gap (the straight lines inside the loops are the magnetization curves).

Show on these curves how much energy is released per unit volume by the magnetic material in the core after the current is returned to 0. Comment.

d) Explain what would be the advantages of using iron, and especially *soft* iron, to make magnetic cores. Iron, however, has a major disadvantage when the inductor is used with alternating currents. Can you think what that might be?

Electromagnetism: Problems

Problem sheet 3: (Michaelmas Term 2025)

■ *Electromagnetic waves*

3.1 **Poynting vector and resistance heating**

This problem is not about waves but is useful to understand how energy is transferred in a circuit.

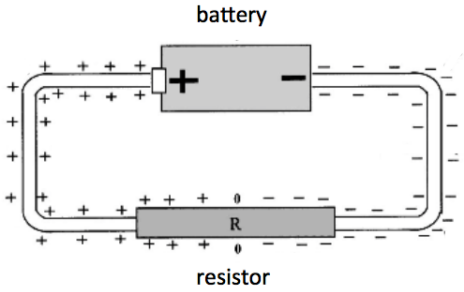
A cylindrical resistor with radius a, length L and resistance R is connected to a battery which supplies a potential difference V. The resulting steady current I which flows along the resistor is uniform through its cross-section.

a) Find the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ on the surface of the resistor.

b) Calculate the Poynting vector $\boldsymbol{S}$ on the surface of the resistor. Find the power flowing through the surface of the resistor. Compare with the rate at which work is done on the charges in the resistor by the electromagnetic field.

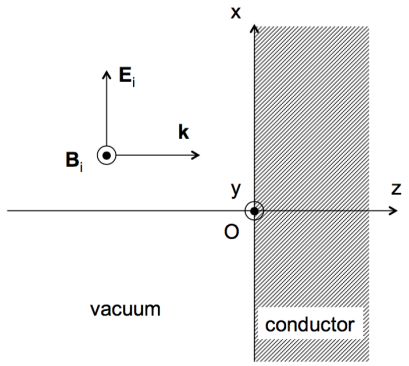
c) We have found above that there is a flow of energy going into the resistor. This energy has to come from outside the resistor, and therefore there has to be an electric field outside giving an inward radial Poynting vector. Justify that this is the case.

d) The energy entering the resistor comes from the battery: energy flows *along* the wires connecting the battery to the resistor. This requires a component of the electric field *perpendicular* to the wires. Such a field is produced by charges that have to be present on the surface of the wires for a steady current to flow. Assuming the charge distribution sketched on the figure, draw the Poynting vector in the different parts of the circuit to show how energy flows from the battery to the resistor. Comment on *where* the energy flows. Is there any energy returning to the battery?



(adapted from Galili & Goihbarg, 2005, *Energy transfer in electrical circuits: a qualitative account*, American Journal of Physics, 73, 141)

3.2 **Standing wave and radiation pressure**



Consider an electromagnetic monochromatic plane wave which travels in vacuum and hits at normal incidence the plane surface of a *perfect* conductor which fills the half–space $z > 0$. The wave is linearly polarized in the x–direction and travels in the z–direction.

The incident wave can be written in the form:

$$\begin{aligned} \tilde{\mathbf{E}}_i(z,t) &= E_0\, \mathrm{e}^{\mathrm{i}(kz-\omega t)}\, \hat{\mathbf{x}}, \\ \tilde{\mathbf{B}}_i(z,t) &= B_0\, \mathrm{e}^{\mathrm{i}(kz-\omega t)}\, \hat{\mathbf{y}}, \end{aligned}$$

where the *tilde* is used to denote complex quantities. We choose the origin of time such that E_0 (and therefore B_0) are real.

a) Find the transmitted and reflected waves in terms of the amplitude of the incident wave.

b) Write the total electric field E and magnetic field B in the half–space $z < 0$. Find the position (expressed in terms of the wavelength λ of the wave) of the *nodal* planes, where E and B are zero, and justify why this is called a *standing* wave. Draw E as a function of z at different times (expressed in terms of the period T of the wave).

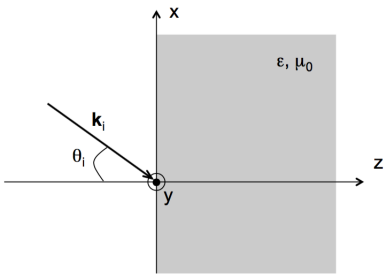
c) Calculate the Poynting vector associated with the wave. Find its value in the nodal planes and describe how the energy flows in space. Calculate the time–average of the Poynting vector over a period at position z.

d) Calculate the time–average of the energy density over a period at position z.

e) Calculate the charge and current densities at the surface of the conductor. The magnetic field due to an infinite current sheet with surface current $\boldsymbol{K}$ is $\mu_o K/2$. Find the magnetic field due to the current flowing at the surface of the conductor, and compare with the reflected magnetic field calculated in question (a). Comment. What is the source of the reflected electric field?

f) Calculate the time–average over a period of the force exerted by the electromagnetic wave on the current at the surface of the conductor. Show that it is equivalent to a *radiation pressure* $P = \epsilon_0 E^2$. [Remember that the field generated by the surface current itself does not exert a force on the surface.]

3.3 **Fresnel's formulas**



A monochromatic plane wave:

$$\tilde{\boldsymbol{E}}_i = \boldsymbol{E}_{0i}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_i\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}}_i = \frac{1}{c}\hat{\boldsymbol{k}}_i\times\tilde{\boldsymbol{E}}_i,$$

is incident on the flat surface of a linear medium with dielectric constant ϵ_r (and $\mu = \mu_0$) at an angle θ_i to the normal.

Here $\hat{\boldsymbol{k}}_i$ is the unit vector in the direction of propagation, and the *tilde* denotes complex quantities. We choose the origin of time so that the amplitude of the wave, E_{0i}, is real. This gives rise to reflected and transmitted waves:

$$\tilde{\boldsymbol{E}}_r = \tilde{\boldsymbol{E}}_{0r}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_r\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}}_r = \frac{1}{c}\hat{\boldsymbol{k}}_r\times\tilde{\boldsymbol{E}}_r, \quad \text{and} \quad \tilde{\boldsymbol{E}}_t = \tilde{\boldsymbol{E}}_{0t}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_t\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}}_t = \frac{1}{v}\hat{\boldsymbol{k}}_t\times\tilde{\boldsymbol{E}}_t,$$

where v is the velocity of the wave in the dielectric.

a) Use the dispersion relation $\omega(k)$ and the boundary conditions at the interface to find the angle of reflection θ_r (between $\boldsymbol{k}_r$ and the z–axis) and the angle of transmission (or refraction) θ_t (between $\boldsymbol{k}_t$ and the z–axis) in terms of the angle of incidence θ_i (Snell's law). If the index of refraction $n < 1$, show that there is a critical angle θ_{ic} above which there is no transmitted wave (this is called *total internal reflection*).

b) We assume that the incident wave is linearly polarized with polarization normal to the plane of incidence. We assume that the reflected and transmitted waves have the same polarization (this can be shown by assuming a more general polarization and using the boundary conditions). Use the boundary conditions to find $\tilde{E}_{0r}$ and $\tilde{E}_{0t}$ as a function of E_{0i} and θ_i (Fresnel's formulas). Plot schematically the reflection coefficient $R = \left|\tilde{E}_{0r}/E_{0i}\right|^2$ as a function of θ_i for the cases $n > 1$ and $n < 1$.

c) Suppose now that the incident wave is linearly polarized with polarization parallel to the plane of incidence. Again we assume that the reflected and transmitted waves also have a linear polarization parallel to the plane of incidence. Use the boundary conditions to find $\tilde{E}_{0r}$ and $\tilde{E}_{0t}$ as a function of E_{0i} and θ_i (Fresnel's formulas). Show that there is no reflection at a certain value of θ_i, called *Brewster's angle*. Plot schematically R as a function of θ_i for the cases $n > 1$ and $n < 1$. What happens to an unpolarized beam incident at Brewster's angle?

d) At Brewster's angle, show that $\theta_r + \theta_t = 90°$. The incident wave induces oscillations of the bound electrons at the surface of the dielectric. These electrons then act as oscillating dipoles which are the source of the reflected and transmitted waves. Given that dipoles do not radiate energy in the direction of their dipole moment, can you explain why there is no reflection at Brewster's angle?

3.4 **Waves in the ionosphere**

The ionosphere is a region of the upper atmosphere which is ionized by solar (UV) radiation. It may be simply described as a dilute gas of charged particles, composed of electrons and ionized air (N_2 and O_2) molecules. The number density n_e of free electrons is maximum at about 200 to 400 km above the surface of the Earth. At higher altitudes, n_e decreases because the density of air molecules to be ionized decreases. At lower altitudes, n_e decreases because UV radiation from the sun has been absorbed in layers above.

a) We assume that the ionosphere can be modelled as a collisionless plasma. Write the equation of motion for an electron assuming an external electric field and neglecting the local field. Show that the dielectric constant is given by $\epsilon_r = 1 - \omega_p^2/\omega^2$, with $\omega_p^2 = n_e e^2/(\epsilon_0 m)$, where $-e$ and m are the electron's charge and mass, respectively. Write the dispersion relation $\omega(k)$. Show that for $\omega < \omega_p$, waves cannot propagate in the plasma.

b) Suppose that waves are emitted at the surface of the Earth toward the ionosphere. For waves with polarization both perpendicular and parallel to the plane of incidence, show that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is a total reflection back toward the Earth. [Use the results of Problem 3].

c) A radio amateur emits a wave with a wavelength of 21 m (corresponding to high frequency or shortwave broadcast) in the early evening. This wave is received by another radio amateur located 1000 km away, but not by other amateurs located closer. Assuming that the wave is being reflected by the so–called F layer of the ionosphere at a height of 300 km (the altitude at which reflection occurs can be determined from the travel time of the signal), calculate the electron density n_e. Compare with the known maximum and minimum F layer densities of 2×10^6 cm^{-3} in the daytime and (2–4) $\times 10^5$ cm^{-3} at night.

d) The cutoff frequency ω_p corresponding to $n_e \sim 10^5$–10^6 cm^{-3} is of a few 10^7 rad s^{-1}. FM radio and television signals have frequencies much larger than this value. What happens to these signals when they hit the ionosphere? Why are AM signals, with frequencies between 3 and 30 MHz, used for long distance (including intercontinental) communication?

e) From the BBC website: *AM reception can vary a great deal from day to night because of differences in the atmosphere. You may get good, clear reception during the day, but after sunset the signal may fade or become distorted. Signals travel further at night, so you may get interference from other transmitters, and you may even hear stations from outside the UK.* Why do radio signals travel farther at night than in the day?

3.5 **Waves in a good conductor**

We consider an ohmic conductor with conductivity σ, permittivity ϵ_0 and permeability μ_0. We assume that the conductor is excellent, that is to say $\sigma \gg \omega\epsilon_0$.

a) Show that the conductor can be described as a dielectric with complex dielectric constant $\tilde{\epsilon}_r = 1 + \mathrm{i}\sigma/(\omega\epsilon_0)$. An electromagnetic monochromatic plane wave propagates through the conductor. Write the dispersion relation and find the wavenumber in terms of the penetration depth $\delta = \sqrt{2/(\omega\mu_0\sigma)}$. Compare the group and phase velocities. Is dispersion normal or anomalous? Show that the impedance is $Z = (1 - \mathrm{i})/(\delta\sigma)$.

b) Calculate the time–average over a period of the Poynting vector, $\langle \boldsymbol{S} \rangle$, at a location z along the direction of propagation of the wave. Calculate the time–average over a period of the rate at which work is done on the charges contained in the volume delimited by the surfaces at $z = z_1$ and $z = z_2$. Verify that it is equal to the time–average over a period of the flux of the Poynting vector into the volume.

c) Calculate the time–average over a period of the energy density, $\langle u \rangle$. Is the energy carried mainly by the electric field or by the magnetic field? Calculate the velocity at which energy is transported (it is given by $|\langle \boldsymbol{S} \rangle| / \langle u \rangle$). Compare with the group velocity. Comment.

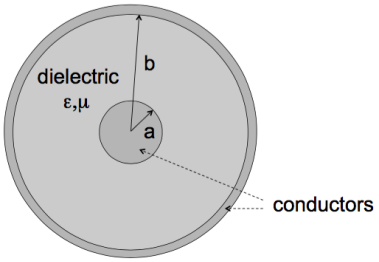
Electromagnetism: Problems

Problem sheet 4: (Michaelmas Term 2025)

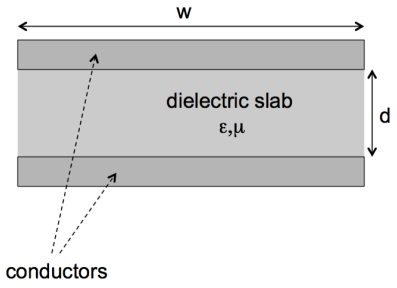
■ *Transmission lines*

4.1 **Practical types of transmission lines**
Calculate the capacitance per unit length, the inductance per unit length and the speed at which signals propagate for the following transmission lines. In each case, the conductors are separated by a material of electric permittivity ϵ and magnetic permeability μ. The figures show a cross section of the lines, which extend in the direction perpendicular to the figure.

a) Coaxial cable:



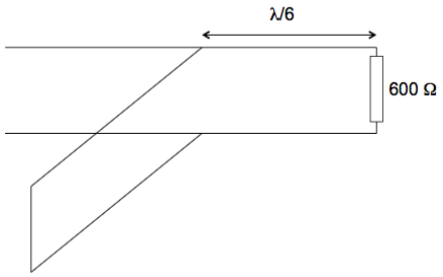
b) Strip line:



4.2 **Short and open circuited transmission lines**
Calculate the input impedance Z_1 of a length l of a loss-free transmission line of characteristic impedance Z_0 terminated by an open circuit. Calculate the input impedance Z_2 of the same transmission line terminated by a short circuit. Show that the lines can be used to create an equivalent inductor or capacitor. Show that $Z_1 Z_2 = Z_0^2$. Discuss the applications of such transmission lines (also called *stubs*).

4.3 **Power transmitted into a load**
A wave travels along a loss–free transmission line of characteristic impedance Z_1 which is terminated by a load of impedance Z_2. Show that the fraction of the incident power time–averaged over a wave period transmitted into the load is $t = 4Z_1 \,\mathrm{Re}(Z_2)/\left|Z_1 + Z_2\right|^2$.

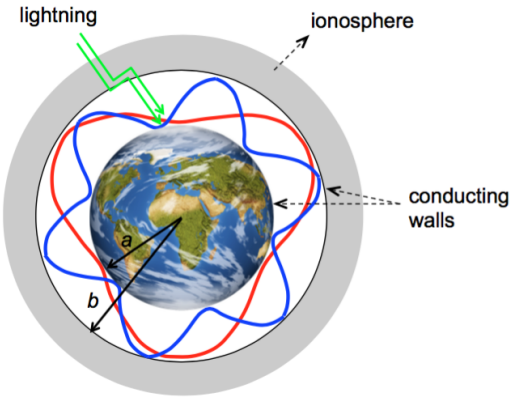
4.4 **Transmission lines in parallel**
A transmission line with a characteristic impedance $Z_0 = 200\ \Omega$ is terminated by a resistor $R = 600\ \Omega$. Calculate the input admittance Y (inverse of the impedance) of the line at one–sixth of a wavelength from the end.



A short–circuited stub of line (with the same characteristic impedance) is added in parallel at this point. What should be its length in order to cancel the reflected signal travelling back towards the source? [*Hint:* Match the admittances.]

■ *Resonant cavities*

4.5 **Earth and ionosphere as a resonant cavity: Schumann resonances**
This problem is taken from section 8.9 of Jackson, Classical Electrodynamics
This is a harder problem and is for enthusiasts only



On a global scale, the Earth and the lower ionosphere can be idealized as two perfectly conducting, concentric spheres with radii a and b. Therefore, they form a resonant electromagnetic cavity which can support resonant standing waves, as sketched on the figure. The resonant frequencies are called *Schumann resonances*, as they were first predicted by the physicist Winfried Otto Schumann in 1952 (although Tesla may have observed them before 1900).

The characteristic wavelength of the lowest frequency mode is on the order of the largest dimension of the cavity, that is to say on the order of magnitude of the Earth's circumference. The modes therefore have extremely low frequencies. Such modes can be excited by lightning and they are detected at various research stations around the world. The aim of this problem is to calculate the resonant frequencies of the Earth–ionosphere cavity. We will focus on TM modes, as they have the lowest frequencies (in spherical coordinates, TM and TE modes have no *radial* component of the magnetic and electric field, respectively). We note (r, θ, φ) the spherical coordinates and assume that the problem has azimuthal symmetry (no φ–dependence).

a) Using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ and Faraday's law $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, show that $B_\theta = 0$ and $E_\varphi = 0$. Write the boundary conditions at $r = a$ and $r = b$.

b) Assuming a time–dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, show that $\boldsymbol{B}$ satisfies the following equation:

$$\frac{\omega^2}{c^2}\boldsymbol{B} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{0},$$

and that the φ–component of this equation can be written in the form:

$$\frac{\omega^2}{c^2}\left(rB_\varphi\right)+\frac{\partial^2}{\partial r^2}\left(rB_\varphi\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\left(rB_\varphi\right)}{\partial\theta}\right)-\frac{rB_\varphi}{\sin^2\theta}\right]=0.$$

c) We look for solutions to this equation of the form:

$$B_\varphi=\frac{F(r)}{r}G(\theta).$$

Show that $G(\theta)$ is the associated Legendre polynomial $P_l^m(\theta)$ with $m=\pm1$. Since the two solutions P_l^{-1} and P_l^1 only differ by a constant of proportionality, we fix $m=1$. Write the components of the fields in terms of $F(r)$, $P_l^1(\cos\theta)$ and $P_l(\cos\theta)$, and the boundary conditions satisfied by F at $r=a$ and $r=b$. You may use the equation:

$$\frac{d}{d\theta}\left[\sin\theta P_l^1(\cos\theta)\right]=-l(l+1)\sin\theta P_l(\cos\theta),$$

where $P_l(\cos\theta)$ is a Legendre polynomial.

d) Show that the differential equation satisfied by $F(r)$ is:

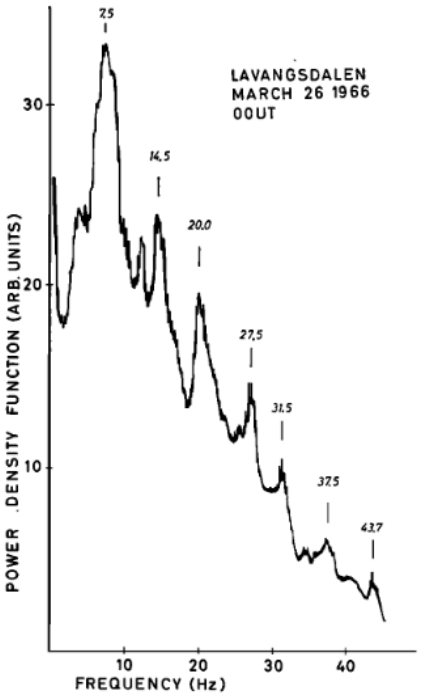
$$\frac{d^2F}{dr^2}+\left[\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2}\right]F=0.$$

We assume that the height of the ionosphere is small compared to the radius of the Earth ($h=b-a\ll a$) so that $1/r^2$ can be approximated by $1/a^2$ in the above equation. Calculate the solution of the above equation that satisfies the boundary conditions at $r=a$ and $r=b$. Show that the (Schumann) resonant frequencies are:

$$\omega_{ln}=\frac{c}{a}\sqrt{l(l+1)+n^2\pi^2\frac{a^2}{h^2}},$$

where $n\geq0$ is an integer.

e) The Earth–ionosphere cavity is excited by lightning bolts which generate radial electric fields and contain a large spectrum of frequencies. The components with a frequency close to a Schumann frequency have the largest amplitude as they correspond to normal modes of the cavity.



This spectrum shows the first seven resonance modes (corresponding to $n = 0$). The resonant frequencies are 7.5, 14.5, 20.0, 27.5, 31.5, 37.5 and 43.7 Hz. The measurements shown on this figure were carried out at Lavangsdalen, in Norway (geographic latitude 70°N) in March 1966 (from T. R. Larsen & A. Egeland, *Fine Structure of the Earth–Ionosphere Cavity Resonances*, 1968, Journal of Geophysical Research, 73, 15).

Compare the measured frequencies with those calculated above. Comment.

Electromagnetism: Problems

Problem sheet 5: (Michaelmas Term 2025)

■ *Signals and noise*

5.1 An audio signal contains frequencies from 20 Hz to 20 kHz. What is the minimum sampling frequency that would need to be able to capture this audio signal without any loss of information?

5.2 A time-varying voltage signal is sampled at 100 Hz. The original voltage signal contains a mixture of frequencies: 25 Hz, 80 Hz, and 314 Hz. What frequencies will be in the sampled signal?

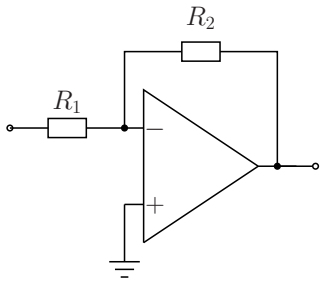
5.3 (a) Calculate the Johnson noise in a 1 kΩ resistor, expressing your answer in $\mathrm{nV/(Hz)^{1/2}}$. If this is placed in a system with a bandwidth of 100 MHz, what is the mean and rms voltage measured across the resistor?
(b) Calculate the shot noise through a diode, carrying a current of 1 mA (the signal), connected into a measuring device with bandwidth of 10 kHz. What is the signal to noise ratio?

5.4 The formula for Johnson noise across a resistor R is $\sqrt{4k_{\mathrm{B}}TR\,\Delta f}$. A perfect measuring device would presumably measure with infinite bandwidth, across all frequencies, and the noise voltage would seemingly go infinite, thereby blowing up your perfect measuring device. Where has this argument gone wrong?

5.5 Show that the noise voltage from an inverting amplifier made from the op-amp circuit shown is

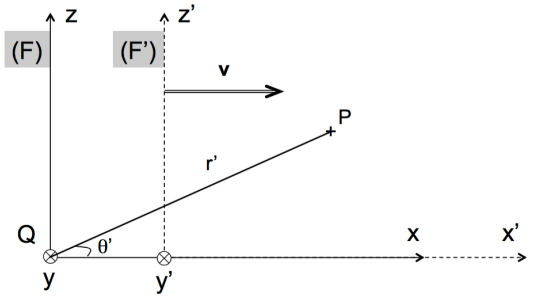
$$\langle V^2 \rangle = \int \mathrm{d}f\, 4k_{\mathrm{B}}T \frac{R_2}{R_1}(R_1 + R_2),$$

neglecting the noise contribution from the op-amp itself.



5.6 **Electric field of a point charge moving with constant velocity**

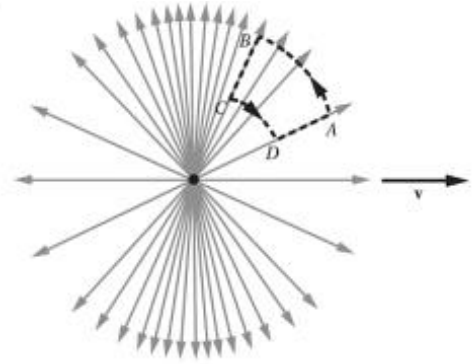
A point charge Q is at rest at the origin in an inertial frame (F). At a point P with Cartesian coordinates $(x, 0, z)$ the field measured in (F) is $\boldsymbol{E}$. We consider another inertial frame (F') which moves in the positive x–direction with speed $\boldsymbol{v}$ with respect to (F).



a) Find the electric field $\boldsymbol{E}'$ produced at P by this charge as measured in (F') and in terms of the polar coordinates (r', θ') in (F') centered on the charge. Show that the field is parallel to the unit vector $\hat{\boldsymbol{r}}'$. Find $\boldsymbol{E}'$ in the limit $v \ll c$.

b) We are going to show that Gauss's law is satisfied in (F'). Choose as a Gaussian surface the surface of a sphere at rest in (F') and centered on the charge at some particular time t'. Use spherical coordinates such that the polar angle θ' is measured from the x'–axis, so that there is azimuthal symmetry. Calculate the flux of $\boldsymbol{E}'$ through the surface of this sphere and show that it is equal to Q/ϵ_0. You may use the equation:

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}}.$$

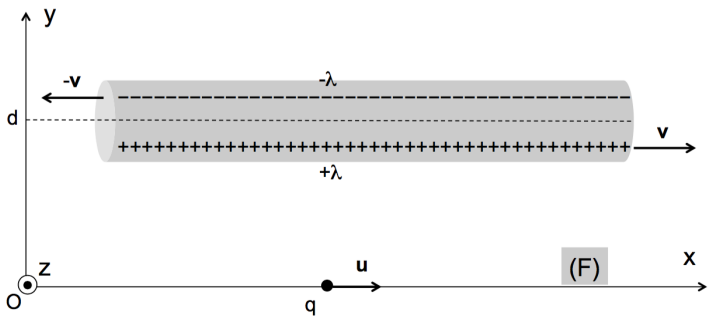


c) The figure (from E. Purcell & D. Morin, *Electricity and Magnetism*) shows a representation of the electric field in (F') (the dot in the centre represents the charge, and the density of field lines indicates the intensity of the field). Justify that the circulation of the electric field along the closed path $ABCDA$ is non zero. Comment.

d) Calculate the magnetic field $\boldsymbol{B}'$ of the charge Q as measured in (F'). [*Hint:* Use the fact that the magnetic field is zero in (F).] Find $\boldsymbol{B}'$ in the limit $v \ll c$. What is the magnetic field of a charge moving at constant velocity $\boldsymbol{v}$?

5.7 **Interaction between a moving charge and other moving charges**

In an inertial frame (F), we consider a line of positive charges all moving to the right with constant speed v. There is also a line of negative charges all moving to the left with the same speed v. The total charges of the positive and negative lines are equal and opposite.



We view the lines as continuous distributions with linear charge densities $+\lambda$ and $-\lambda$, as measured in (F). This is an ideal representation of a wire containing both ions and electrons. At a distance d from the axis of the wire, there is a charge q which moves to the right with speed $u < v$.

a) Calculate the net charge and the net current in the wire. Find the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ at the position of the charge, and the force $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B})$ on q, as measured in (F).

b) We now consider the inertial frame (F') which moves to the right at speed u. The charge q is at rest in (F'). Find the velocities v'_+ and v'_- of the positive and negative lines in (F') and show that $\gamma_\pm \equiv 1/\sqrt{1 - v'^2_\pm/c^2} = \gamma\,(1 \mp uv/c^2)\,/\sqrt{1 - u^2/c^2}$, where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$. Calculate the charge densities $\lambda'_\pm$ of the lines as measured in (F') and the net charge $\lambda' = \lambda'_+ + \lambda'_-$ in terms of λ, u, v and c. Find the electric field $\boldsymbol{E}'$ at the position of the charge, and the force $\boldsymbol{f}' = q\boldsymbol{E}'$ on q, as measured in (F'). Compare $\boldsymbol{f}$ and $\boldsymbol{f}'$.