# Electromagnetism (2nd year): Problems

**Problem sheet 0**: (Summer vacation 2020)

*At the start of the second year you will receive the second part of the Electromagnetism course. This vacation work contains a set of problems that will enable you to revise the material covered in the first year Electromagnetism course.*

Some of the problems below are taken from:
*Introduction to Electrodynamics*, David J. Griffiths, 4th Edition
*Electricity and Magnetism*, Edward M. Purcell and David J. Morin, 3rd Edition.
(Problems first assembled by Prof. Caroline Terquem)

## ■ *Electrostatics*

### 0.1 Field and potential from charged ring

A thin ring of radius $a$ carries a charge $q$ uniformly distributed. Consider the ring to lie in the $x$–$y$ plane with its centre at the origin.

**a)** Find the electric field $\mathbf{E}$ at a point $P$ on the $z$–axis.

**b)** Find the electric potential $V$ at $P$.

**c)** A charge $-q$ with mass $m$ is released from rest far away along the axis. Calculate its speed when it passes through the centre of the ring. (Assume that the ring is fixed in place).

### 0.2 Field from charged disc

The ring in the previous problem is replaced by a thin disc of radius $a$ carrying a charge $q$ uniformly distributed. Consider the disc to lie in the $x$–$y$ plane with its centre at the origin.

**a)** Find the electric field $\mathbf{E}$ at a point $P$ on the $z$–axis.

**b)** Check that the values of $\mathbf{E}$ at $z = 0$ and in the limit $z \gg a$ are consistent with expectations.

### 0.3 Hydrogen atom

According to quantum mechanics, the hydrogen atom in its ground state can be described by a point charge $+q$ (charge of the proton) surrounded by an electron cloud with a charge density $\rho(r) = -Ce^{-2r/a_0}$. Here $a_0$ is the Bohr radius, $0.53 \times 10^{-10}$ m, and $C$ is a constant.

**a)** Given that the total charge of the atom is zero, calculate $C$.

**b)** Calculate the electric field at a distance $r$ from the nucleus.

**c)** Calculate the electric potential, $V(r)$, at a distance $r$ from the nucleus. We give:

$$\int \left(\frac{1}{\alpha r'} + 1\right) \frac{\mathrm{e}^{-\alpha r'}}{r'} dr' = -\frac{\mathrm{e}^{-\alpha r}}{\alpha r}.$$

0.4 **Energy of a charged sphere**

We consider a solid sphere of radius $a$ and charge $Q$ uniformly distributed.

**a)** Calculate the electric field $E(r)$ and the electric potential $V(r)$ at a distance $r$ from the centre of the sphere.

Find the energy $U$ stored in the sphere three different ways:

**b)** Use the potential energy of the charge distribution due to the potential $V(r)$:

$$U = \frac{1}{2} \int_{\mathcal{V}} \rho V d\tau,$$

where $\rho$ is the charge density and the integral is over the volume $\mathcal{V}$ of the sphere.

**c)** Use the energy stored in the field produced by the charge distribution:

$$U = \int_{\text{space}} \frac{\epsilon_0 E^2}{2} d\tau,$$

where the integral is over *all space*.

**d)** Calculate the work necessary to assemble the sphere by bringing successively thin charged layers at the surface.

0.5 **Conductors**

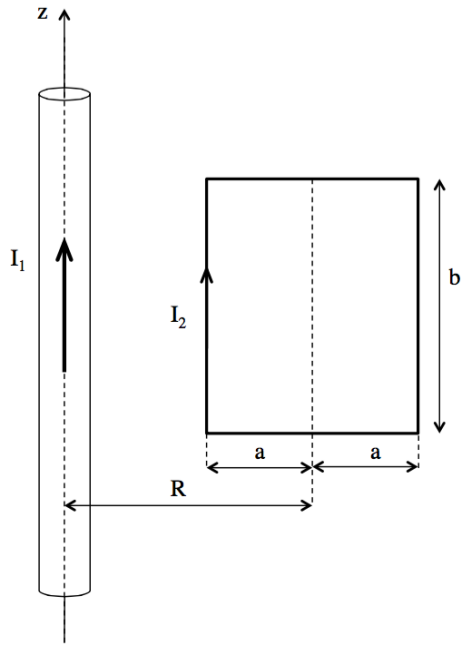
A metal sphere of radius $R_1$, carrying charge $q$, is surrounded by a thick concentric metal shell of inner and outer radii $R_2$ and $R_3$. The shell carries no net charge.

**a)** Find the surface charge densities at $R_1$, $R_2$ and $R_3$.

**b)** Find the potential at the centre, choosing $V = 0$ at infinity.

**c)** Now the outer surface is grounded. Explain how that modifies the charge distribution. How do the answers to questions (a) and (b) change?

## 0.6 **Force on a loop**



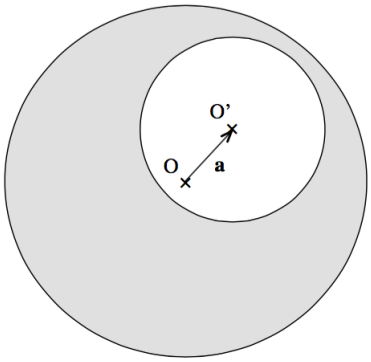

A long thin wire carries a current $I_1$ in the positive $z$–direction along the axis of a cylindrical co-ordinate system. A thin, rectangular loop of wire lies in a plane containing the axis, as represented on the figure. The loop carries a current $I_2$.

**a)** Find the magnetic field due to the long thin wire as a function of distance $r$ from the axis.

**b)** Find the vector force on each side of the loop which results from this magnetic field.

**c)** Find the resultant force on the loop.

## 0.7 **Magnetic field in off–centre hole**




A cylindrical rod carries a uniform current density $J$. A cylindrical cavity with an arbitrary radius is hollowed out from the rod at an arbitrary location. The axes of the rod and cavity are parallel. A cross section is shown on the figure. The points $O$ and $O'$ are on the axes of the rod and cavity, respectively, and we note $\mathbf{a} = \mathbf{OO'}$.

**a)** Show that the field inside a solid cylinder can be written as $\mathbf{B} = (\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{r}$, where $\hat{\mathbf{z}}$ is the unit vector along the axis and $\mathbf{r}$ is the position vector measured perpendicularly to the axis.

**b)** Show that the magnetic field inside the cylindrical cavity is uniform (in both magnitude and direction).

0.8 **Magnetic field at the centre of a sphere**
A spherical shell with radius $a$ and uniform surface charge density $\sigma$ spins with angular frequency $\omega$ around a diameter. Find the magnetic field at the centre.

0.9 **Motion of a charged particle in a magnetic field**
A long thin wire carries a current $I$ in the positive $z$–direction along the axis of a cylindrical co-ordinate system. A particle of charge $q$ and mass $m$ moves in the magnetic field produced by this wire. We will neglect the gravitational force acting on the particle as it is very small compared to the magnetic force.

**a)** Is the kinetic energy of the particle a constant of motion?

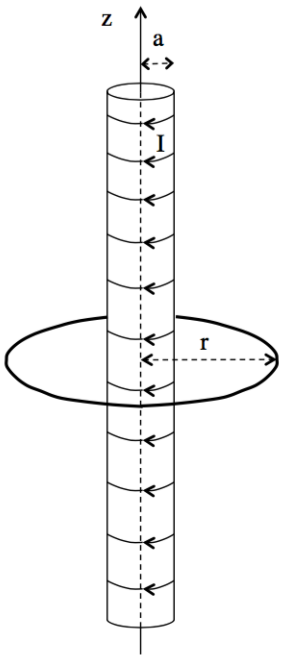
**b)** Find the force $\mathbf{F}$ on the particle, in cylindrical coordinates.

**c)** Obtain the equation of motion, $\mathbf{F} = md\mathbf{v}/dt$, in cylindrical coordinates for the particle.

**d)** Suppose the velocity in the $z$–direction is constant. Describe the motion.

■ *Electromagnetic induction*

0.10 **Growing current in a solenoid**



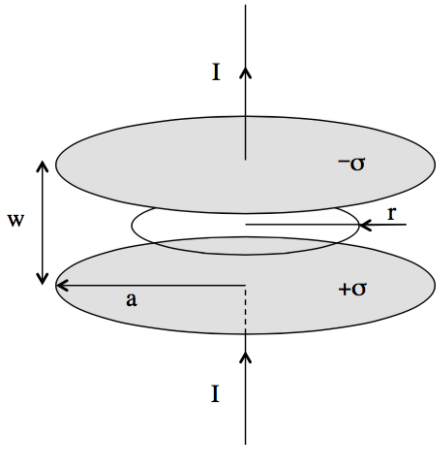

An infinite solenoid has radius $a$ and $n$ turns per unit length. The current grows linearly with time, according to $I(t) = kt$, $k > 0$. The solenoid is looped by a circular wire of radius $r$, coaxial with it. We recall that the magnetic field due to the current in the solenoid is $B = \mu_0 nI$ inside the solenoid and zero outside.

**a)** Without doing any calculation, explain which way the current induced in the loop flows.

**b)** Use the integral form of Faraday's law, which is $\oint \mathbf{E}\cdot d\mathbf{l} = -d\Phi/dt$, to find the electric field in the loop for both $r < a$ and $r > a$. Check that the orientation of $\mathbf{E}$ agrees with the answer to question (a).

**c)** Verify that your result satisfies the local form of the law, $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$.

## 0.11 **Energy flow into a capacitor**




A capacitor has circular plates with radius $a$ and is being charged by a constant current $I$. The separation of the plates is $w \ll a$. Assume that the current flows out over the plates through thin wires that connect to the centre of the plates, and in such a way that the surface charge density $\sigma$ is uniform, at any given time, and is zero at $t = 0$.

**a)** Find the electric field between the plates as a function of $t$.

**b)** Consider the circle of radius $r < a$ shown on the figure (and centered on the axis of the capacitor). Using the integral form of Maxwell's equation $\boldsymbol{\nabla}\times\mathbf{B} = \epsilon_0\mu_0\partial\mathbf{E}/\partial t$ over the surface delimited by the circle, find the magnetic field at a distance $r$ from the axis of the capacitor.

**c)** Find the energy density $u$ and the Poynting vector $\mathbf{S}$ in the gap. Check that the relation:

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{S},$$

is satisfied.

**d)** Consider a cylinder of radius $b < a$ and length $w$ inside the gap. Determine the total energy in the cylinder, as a function of time. Calculate the total power flowing into the cylinder, by integrating the Poynting vector $\mathbf{S}$ over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the cylinder.

**e)** When $b = a$, and assuming that we can still neglect edge effects in that case, check that the total power flowing into the capacitor is:

$$\frac{d}{dt}\left(\frac{1}{2}QV\right),$$

where $V$ is the voltage across the capacitor (since $QV/2$ is the energy stored in the electric field in the capacitor).

# Electromagnetism (2nd year): Solutions to Problems

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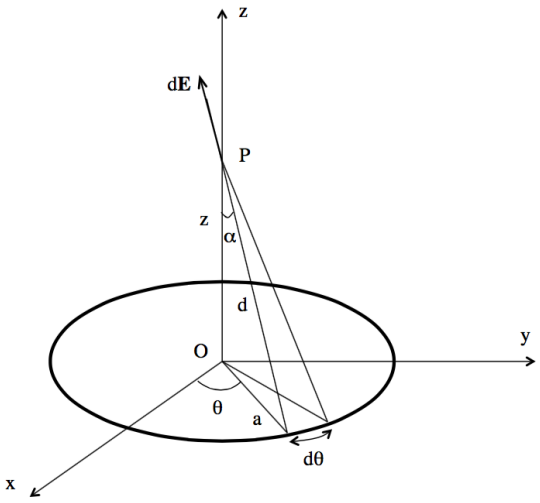
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## ■ *Electrostatics*

### 0.1 Field and potential from charged ring

*A thin ring of radius a carries a charge q uniformly distributed. Consider the ring to lie in the x–y plane with its centre at the origin.*

**a)** *Find the electric field* $\mathbf{E}$ *at a point* $P$ *on the z–axis.*




The arc along the ring that subtends the angle $d\theta$ contains a charge $dq$ that creates the field $d\mathbf{E}$ at point $P$. Given the symmetry of the charge distribution, it can be seen that the total field at $P$ is along the $z$–axis. Therefore, we are only interested in the $z$–component of $d\mathbf{E}$, which is $dE_z = dE\cos\alpha$. Coulomb's law gives:

$$dE = \frac{dq}{4\pi\epsilon_0 d^2}.$$

The charge per unit length along the ring is $\lambda = q/(2\pi a)$. With $\cos\alpha = z/d$ and $dq = \lambda a d\theta$, we obtain:

$$dE_z = \frac{\lambda a z d\theta}{4\pi\epsilon_0 d^3}.$$

Using $d = \sqrt{a^2 + z^2}$ and the above expression for $\lambda$, this yields:

$$dE_z = \frac{qzd\theta}{8\pi^2\epsilon_0\left(a^2+z^2\right)^{3/2}}$$

The field $\mathbf{E}$ due to the ring is obtained by summing over $\theta$:

$$\mathbf{E} = E_z\hat{\mathbf{z}} = \hat{\mathbf{z}}\int_0^{2\pi} dE_z = \boxed{\frac{qz}{4\pi\epsilon_0\left(a^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}},}$$

where $\hat{\mathbf{z}}$ is the unit vector along the $z$–axis.

In the limit $z \gg a$, we recover the field due to a point charge: $\mathbf{E} = q\hat{\mathbf{z}}/(4\pi\epsilon_0 z^2)$ (the ring looks like a point charge from far away).

**b)** *Find the electric potential $V$ at $P$.*

The arc along the ring that subtends the angle $d\theta$ creates at $P$ the potential:

$$dV = \frac{dq}{4\pi\epsilon_0 d} = \frac{\lambda a d\theta}{4\pi\epsilon_0 d} = \frac{q d\theta}{8\pi^2\epsilon_0\sqrt{a^2+z^2}},$$

where we have chosen the origin of the potential at infinity. The potential $V$ due to the ring is therefore:

$$V = \int_0^{2\pi} dV = \boxed{\frac{q}{4\pi\epsilon_0\sqrt{a^2+z^2}}.}$$

Here again, we recover the potential of a point charge, $V = q/(4\pi\epsilon_0 z)$, in the limit $z \gg a$.

**c)** *A charge $-q$ with mass $m$ is released from rest far away along the axis. Calculate its speed when it passes through the centre of the ring. (Assume that the ring is fixed in place).*

The charge has no initial velocity, so its initial kinetic energy is zero. At it is initially far away from the ring, its initial potential energy, $-qV$, is also zero ($V = 0$ at infinity). The total energy of the charge is therefore zero initially. As energy is conserved, the energy of the particle when is passes through the origin has to be zero, which means $mv^2/2 - qV(0) = 0$, where $v$ is the speed of the particle. This yields:

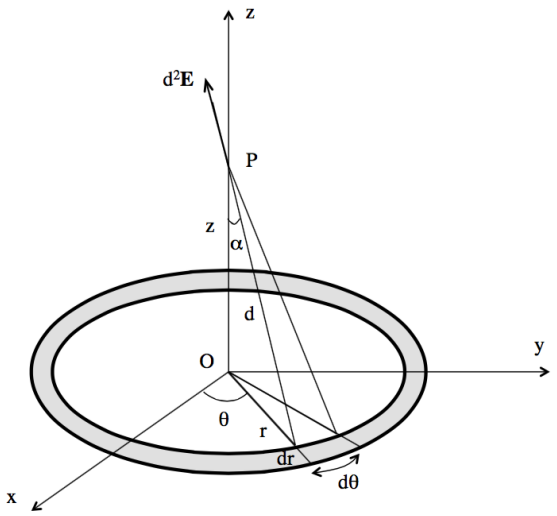
$$\frac{1}{2}mv^2 = \frac{q^2}{4\pi\epsilon_0 a}, \text{ so that } \boxed{v = \frac{q}{\sqrt{2\pi\epsilon_0 m a}}.}$$

If $a \to \infty$, $v \to 0$: the finite charge of the ring is at infinity and does not affect the charge going through the origin. If $a \to 0$, $v \to \infty$: the ring is equivalent to a charge $q$ at the origin, and the charge $-q$ is accelerated by a potential which becomes infinite close to the ring.

## 0.2 Field from charged disc

*The ring in the previous problem is replaced by a thin disc of radius $a$ carrying a charge $q$ uniformly distributed. Consider the disc to lie in the $x$–$y$ plane with its centre at the origin.*

**a)** *Find the electric field $\mathbf{E}$ at a point $P$ on the $z$–axis.*




In Problem 1.a, we have shown that the field at $P$ due to a thin ring of radius $r$ and charge $q$ is $qz\hat{\mathbf{z}}/\left[4\pi\epsilon_0\left(r^2+z^2\right)^{3/2}\right]$. Therefore, the field $d\mathbf{E}$ at $P$ due to the circular ring of radius $r$ and width $dr$ (shaded area on the figure) is:

$$d\mathbf{E} = \frac{z\,dq}{4\pi\epsilon_0\left(r^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}},$$

where $dq$ is the charge of the ring.

The charge per unit surface area on the disc is $\sigma = q/(\pi a^2)$, and $dq = 2\pi r dr\sigma$ (as the surface of the shaded area is $2\pi r dr$). This yields:

$$d\mathbf{E} = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\frac{r dr}{\left(r^2+z^2\right)^{3/2}}.$$

To obtain the total field due to the disc, we integrate over radius:

$$\mathbf{E} = \int_o^a d\mathbf{E} = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\int_o^a \frac{r dr}{\left(r^2+z^2\right)^{3/2}} = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[-\frac{1}{\sqrt{r^2+z^2}}\right]_0^a = \frac{z\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{1}{|z|}-\frac{1}{\sqrt{a^2+z^2}}\right).$$

Using the above expression for $\sigma$, we obtain:

$$\boxed{\mathbf{E} = \frac{q\hat{\mathbf{z}}}{2\pi\epsilon_0 a^2}\left(\frac{z}{|z|}-\frac{z}{\sqrt{a^2+z^2}}\right).}$$

**b)** *Check that the values of* $\mathbf{E}$ *at* $z=0$ *and in the limit* $z \gg a$ *are consistent with expectations.*

When $z \gg a$, we have:

$$\frac{1}{\sqrt{a^2+z^2}} = \frac{1}{|z|}\left(1+\frac{a^2}{z^2}\right)^{-1/2} \simeq \frac{1}{|z|}\left(1-\frac{a^2}{2z^2}\right).$$

Substituting into the above expression for $\mathbf{E}$, we obtain:

$$\mathbf{E} \simeq \frac{q\hat{\mathbf{z}}}{2\pi\epsilon_0}\frac{1}{2z|z|} = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2}\frac{z}{|z|},$$

where the term $z/|z|$ only indicates that the field reverses direction as $z$ changes sign. As expected, this is the field due to a point charge $q$ at the origin (from far away, the disc looks like a point charge).

At $z=0$, we have $\mathbf{E} = \pm\sigma\hat{\mathbf{z}}/(2\epsilon_0)$, where the $+$ and $-$ signs apply for $z>0$ and

$z < 0$, respectively. As expected, we recover the expression of the field due to an infinite charged plane.

## 0.3 Hydrogen atom

*According to quantum mechanics, the hydrogen atom in its ground state can be described by a point charge $+q$ (charge of the proton) surrounded by an electron cloud with a charge density $\rho(r) = -Ce^{-2r/a_0}$. Here $a_0$ is the Bohr radius, $0.53 \times 10^{-10}$ m, and $C$ is a constant.*

**a)** *Given that the total charge of the atom is zero, calculate $C$.*
The total negative charge is $\int_0^\infty \rho(r) 4\pi r^2 dr$ and it has to be equal to $-q$. Therefore:

$$-q = -4\pi C \int_0^\infty e^{-2r/a_0} r^2 dr.$$

The integral can be calculated using integration by parts. It is equal to $a_0^3/4$. Therefore:

$$\boxed{C = \frac{q}{\pi a_0^3}}.$$

**b)** *Calculate the electric field at a distance $r$ from the nucleus.*
Using Gauss's law:

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{q(r)}{\epsilon_0},$$

where $q(r)$ is the total charge contained in the sphere or radius $r$ and the integral is over the surface $\Sigma$ of the sphere. As the charge distribution has a spherical symmetry, the electric field is radial and depends only on $r$, so that $\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r)$. We also have:

$$q(r) = q - 4\pi C \int_0^r e^{-2r'/a_0} r'^2 dr'.$$

The integral can be calculated using integration by parts. It is equal to:

$$\frac{a_0^3}{4} - \frac{a_0}{2}\left(\frac{a_0^2}{2} + a_0 r + r^2\right) e^{-2r/a_0}.$$

Therefore:

$$\boxed{E(r) = \frac{q}{4\pi\epsilon_0 r^2}\left(1 + \frac{2r}{a_0} + \frac{2r^2}{a_0^2}\right) e^{-2r/a_0}.}$$

**c)** *Calculate the electric potential, $V(r)$, at a distance $r$ from the nucleus. We give:*

$$\int \left(\frac{1}{\alpha r'} + 1\right) \frac{e^{-\alpha r'}}{r'} dr' = -\frac{e^{-\alpha r}}{\alpha r}.$$

We have $\mathbf{E} = -\boldsymbol{\nabla} V$. As $\mathbf{E}$ is radial, this reduces to $E = -dV/dr$, or:

$$V(\infty) - V(r) = -\int_r^\infty E(r')dr'.$$

This yields:

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0 r}\left(1 + \frac{r}{a_0}\right)\mathrm{e}^{-2r/a_0},}$$

where we have chosen $V(\infty) = 0$.

## 0.4 Energy of a charged sphere

*We consider a solid sphere of radius a and charge Q uniformly distributed.*

**a)** *Calculate the electric field $E(r)$ and the electric potential $V(r)$ at a distance $r$ from the centre of the sphere.*

The electric field at a distance $r$ from the centre of the sphere can be calculated using Gauss's law:

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{q(r)}{\epsilon_0},$$

where $q(r)$ is the total charge contained in the sphere or radius $r$ and the integral is over the surface $\Sigma$ of the sphere. As the charge distribution has a spherical symmetry, the electric field is radial and depends only on $r$, so that $\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r)$. If $r > a$, $q(r) = Q$. If $r < a$, $q(r) = (4/3)\pi r^3 \rho$ with $\rho = Q/[(4/3)\pi a^3]$, which implies $q(r) = (r/a)^3 Q$. This yields:

$$\boxed{E(r) = \frac{Qr}{4\pi\epsilon_0 a^3} \text{ for } r < a} \text{ and } \boxed{E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \text{ for } r > a.}$$

We have $\mathbf{E} = -\boldsymbol{\nabla} V$. As $\mathbf{E}$ is radial, this reduces to $E = -dV/dr$. For $r \geq a$, we obtain:

$$V(\infty) - V(r) = -\int_r^\infty E(r')dr' = -\int_r^\infty \frac{Q}{4\pi\epsilon_0 r'^2}dr'.$$

Choosing $V(\infty) = 0$, this yields:

$$\boxed{V(r) = \frac{Q}{4\pi\epsilon_0 r} \text{ for } r \geq a.}$$

For $r \leq a$, we obtain:

$$V(a) - V(r) = -\int_r^a E(r')dr' = -\int_r^a \frac{Qr'}{4\pi\epsilon_0 a^3}dr' = -\frac{Q}{8\pi\epsilon_0 a^3}\left(a^2 - r^2\right).$$

The potential has to be continuous at $r = a$ (otherwise the field would diverge), so

that $V(a) = Q/(4\pi\epsilon_0 a)$. Therefore, we obtain:

$$\boxed{V(r) = \frac{Q}{8\pi\epsilon_0 a}\left(3 - \frac{r^2}{a^2}\right) \quad \text{for } r \leq a.}$$

Alternatively, $E$ can be calculated using the local form of Gauss's law: $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$. As $\mathbf{E}$ is radial,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E\right),$$

in spherical coordinates. For $r < a$, this yields:

$$\frac{d}{dr}\left(r^2 E\right) = \frac{\rho r^2}{\epsilon_0}, \text{ so that } E(r) = \frac{\rho r}{3\epsilon_0} + \frac{C}{r^2},$$

where $C$ is a constant. To avoid a singularity at $r = 0$, we have to take $C = 0$. Therefore $E(r) = Qr/(4\pi\epsilon_0 a^3)$, as above. For $r > a$, $\rho = 0$ so that

$$\frac{d}{dr}\left(r^2 E\right) = 0, \text{ so that } E(r) = \frac{C'}{r^2},$$

where $C'$ is a constant. The potential is obtained as above, and $C'$ can be calculated by requiring the potential to be continuous at $r = a$.

*Find the energy $U$ stored in the sphere three different ways:*

**b)** *Use the potential energy of the charge distribution due to the potential $V(r)$:*

$$U = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau,$$

*where $\rho$ is the charge density and the integral is over the volume $\mathcal{V}$ of the sphere.* Using the expression for $V$ found above for $r \leq a$, and with $d\tau = 4\pi r^2 dr$, we have:

$$U = \frac{Q\rho}{4\epsilon_0 a}\int_0^a \left(3r^2 - \frac{r^4}{a^2}\right) dr = \frac{Q\rho a^2}{5\epsilon_0}.$$

With $\rho = Q/[(4/3)\pi a^3]$, we obtain:

$$\boxed{U = \frac{3Q^2}{20\pi\epsilon_0 a}.}$$

**c)** *Use the energy stored in the field produced by the charge distribution:*

$$U = \int_{\text{space}} \frac{\epsilon_0 E^2}{2} d\tau,$$

*where the integral is over* all space.

Using the expressions for $E$ found above for both $r < a$ and $r > a$, and with $d\tau = 4\pi r^2 dr$, we have:

$$U = \frac{Q^2}{8\pi\epsilon_0 a^6}\int_0^a r^4 dr + \frac{Q^2}{8\pi\epsilon_0}\int_a^\infty \frac{dr}{r^2} = \frac{3Q^2}{20\pi\epsilon_0 a},$$

as above.

**d)** *Calculate the work necessary to assemble the sphere by bringing successively thin charged layers at the surface.*

Assume we have assembled the sphere up to a radius $r$. We have seen in question (a) above that the potential at the surface of a uniformly charged sphere of radius $a$ is $Q/(4\pi\epsilon_0 a)$. Therefore, the potential at the surface of the sphere of radius $r$ is $V_r = q(r)/(4\pi\epsilon_0 r) = \rho r^2/(3\epsilon_0)$, where $q(r) = (4/3)\pi r^3\rho$ is the charge of the sphere. The work necessary to add a thin layer with a uniform charge density $\rho$ between the radii $r$ and $r + dr$ is $dW = V_r dq$, where $dq$ is the charge of the layer. We have $dq = 4\pi r^2 dr\rho$, so that $dW = 4\pi\rho^2 r^4 dr/(3\epsilon_0)$. The energy of the sphere is equal to the work necessary to assemble the whole sphere, so that:

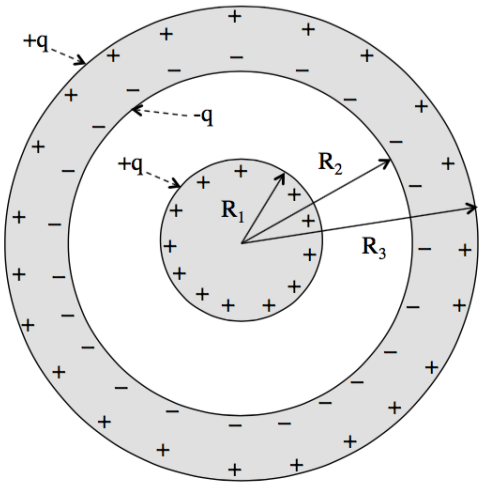
$$U = \int_0^a dW = \frac{4\pi\rho^2 a^5}{15\epsilon_0} = \frac{3Q^2}{20\pi\epsilon_0 a},$$

where we have used $\rho = Q/[(4/3)\pi a^3]$. We recover the same expression for $U$ as above.

## 0.5 Conductors

*A metal sphere of radius $R_1$, carrying charge $q$, is surrounded by a thick concentric metal shell of inner and outer radii $R_2$ and $R_3$. The shell carries no net charge.*

**a)** *Find the surface charge densities at $R_1$, $R_2$ and $R_3$.*




Any net charge on a conductor has to reside on the surface, otherwise there would be an electric field in the conductor. Therefore, the charge $q$ of the metal sphere is distributed at its surface. Because of the spherical symmetry, the charge is uniformly distributed. As a consequence, the surface charge density at $R_1$ is:

$$\boxed{\sigma_1 = \frac{q}{4\pi R_1^2}.}$$

For the electric field to be zero in the metal outer shell, by Gauss's law, a charge $-q$ has to be present at $R_2$. (If the charge at $R_2$ were, say, smaller than $-q$, there

would be an outward electric field in the conductor that would bring electrons toward the surface at $R_2$. They would pile up there until the field in the conductor were zero, that is to say until the total charge at $R_2$ were $-q$.) Since the charge at the surface of the inner sphere is uniformly distributed, the charge at $R_2$ is also uniformly distributed. The surface charge density at $R_2$ is therefore:

$$\boxed{\sigma_2 = \frac{-q}{4\pi R_2^2}.}$$

As the charge $-q$ has piled up at $R_2$, and the outer shell is neutral, there is a charge $+q$ left in the rest of the shell. Here again, the charge has to reside at a surface, so it has to be at $R_3$. The charge is uniformly distributed and the surface charge density at $R_3$ is therefore:

$$\boxed{\sigma_3 = \frac{q}{4\pi R_3^2}.}$$

**b)** *Find the potential at the centre, choosing $V = 0$ at infinity.*

We first calculate the electric field. At a distance $r$ from the centre of the sphere, it can be calculated using Gauss's law:

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{Q(r)}{\epsilon_0},$$

where $Q(r)$ is the total charge contained in the sphere or radius $r$ and the integral is over the surface $\Sigma$ of the sphere. As the charge distribution has a spherical symmetry, the electric field is radial and depends only on $r$, so that $\int_\Sigma \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r)$. If $r < R_1$ or $R_2 < r < R_3$, $Q(r) = 0$ so that $E = 0$ (as it should be in a conductor!). If $R_1 < r < R_2$ or $r > R_3$, $Q(r) = q$ so that $E = q/(4\pi\epsilon_0 r^2)$.

We have $\mathbf{E} = -\boldsymbol{\nabla} V$. As $\mathbf{E}$ is radial, this reduces to $E = -dV/dr$. Therefore:

$$V(\infty) - V(0) = -\int_0^\infty E dr = -\int_{R_1}^{R_2} \frac{q}{4\pi\epsilon_0 r^2} dr - \int_{R_3}^{\infty} \frac{q}{4\pi\epsilon_0 r^2} dr.$$

We take the reference point at infinity, that is to say $V(\infty) = 0$, so that we obtain:

$$\boxed{V(0) = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{R_2} + \frac{1}{R_1} + \frac{1}{R_3}\right).}$$

**c)** *Now the outer surface is grounded. Explain how that modifies the charge distribution. How do the answers to questions (a) and (b) change?*

When we connect the surface at $R_3$ to the ground, electrons flow from the ground to that surface because they are attracted by the positive charges (electrons would flow from the surface to the ground if the surface were negatively charged), until the surface becomes neutral ($\sigma_3 = 0$). So now $E = 0$, and therefore $V$ is constant, for $r > R_2$. As $V = 0$ at infinity, $V = 0$ everywhere for $r \geq R_2$.

To calculate $V(0)$, we now have to take into account the electric field only between

$R_1$ and $R_2$, so that:

$$V(\infty) - V(0) = -\int_0^\infty E dr = -\int_{R_1}^{R_2} \frac{q}{4\pi\epsilon_0 r^2} dr,$$

which yields:

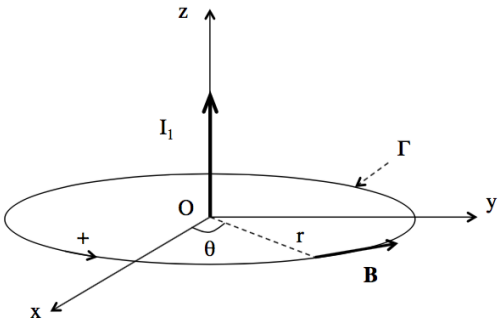
$$\boxed{V(0) = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{R_2} + \frac{1}{R_1}\right).}$$

## ■ *Magnetostatics*

### 0.6 **Force on a loop**

*A long thin wire carries a current $I_1$ in the positive $z$–direction along the axis of a cylindrical co-ordinate system. A thin, rectangular loop of wire lies in a plane containing the axis, as represented on the figure. The loop carries a current $I_2$.*

**a)** *Find the magnetic field due to the long thin wire as a function of distance $r$ from the axis.*




We consider a contour $\Gamma$ as shown on the figure. Given the direction in which $I_1$ flows, we define a "+" direction along the contour using the "right–hand rule". The Biot–Savart's law implies that the magnetic field $\mathbf{B}$ is orthoradial and the right–hand rule gives its orientation.

According to Ampère's law:

$$\int_\Gamma \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1,$$

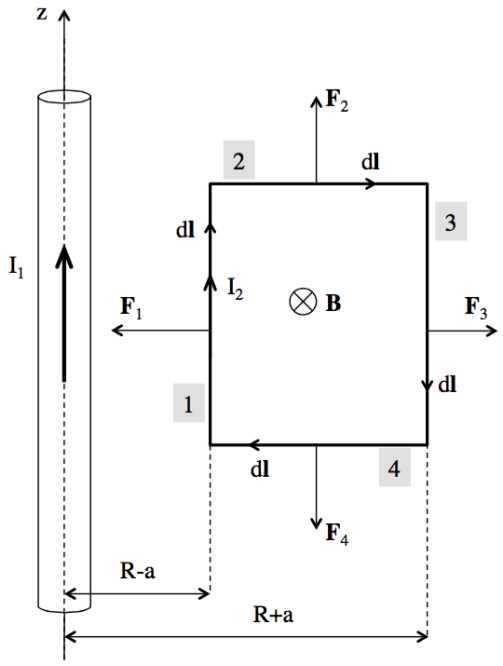
where $d\mathbf{l}$ is a small line element along the contour in the "+" direction. Then $\mathbf{B} \cdot d\mathbf{l} = B dl$. Because of the symmetry of the problem, $B$ depends only on the distance $r$ to the axis of the wire, so that it is constant along the contour. Therefore,

$$\int_\Gamma \mathbf{B} \cdot d\mathbf{l} = B(r) \int_\Gamma dl = 2\pi r B(r),$$

and Ampère's law yields:

$$\boxed{B(r) = \frac{\mu_0 I_1}{2\pi r}.}$$

**b)** *Find the vector force on each side of the loop which results from this magnetic field.*




We label the different sides $1 \ldots 4$ as shown on the figure. The force exerted on one of the sides is given by:

$$\int_{\text{side}} I_2 d\mathbf{l} \times \mathbf{B},$$

where $d\mathbf{l}$ has the same orientation as the current $I_2$ in the side we consider, and $\mathbf{B}$ is the magnetic field at the location of the side.
*Side 1*: $dl = dz$ and $B$ is evaluated at $r = R - a$, so that:

$$F_1 = \int_{z=0}^{b} \frac{\mu_0 I_1 I_2}{2\pi(R-a)} dz = \boxed{\frac{\mu_0 I_1 I_2 b}{2\pi(R-a)}.}$$

*Side 3*: Same as for side 1 but with $B$ evaluated at $r = R + a$, so that:

$$\boxed{F_3 = \frac{\mu_0 I_1 I_2 b}{2\pi(R+a)}.}$$

*Side 2*: $dl = dr$ and $B$ is evaluated between $r = R - a$ and $r = R + a$:

$$F_2 = \int_{R-a}^{R+a} \frac{\mu_0 I_1 I_2}{2\pi r} dr = \boxed{\frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{R+a}{R-a}.}$$

*Side 4*: By symmetry, $\boxed{F_4 = F_2}$, but the forces are oriented in opposite directions.

**c)** *Find the resultant force on the loop.*
As $\mathbf{F}_2 + \mathbf{F}_4 = \mathbf{0}$, the resultant force is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_3$. Given the orientation of $\mathbf{F}_1$ and $\mathbf{F}_3$, we have:

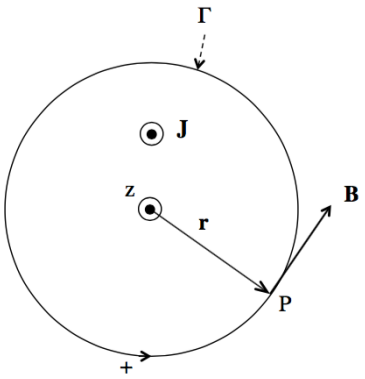
$$\boxed{F = F_1 - F_3 = \frac{\mu_0 I_1 I_2 ab}{\pi(R^2 - a^2)},}$$

and $\mathbf{F}$ is oriented toward the $z$–axis.

## 0.7 Magnetic field in off–centre hole

*A cylindrical rod carries a uniform current density $J$. A cylindrical cavity with an arbitrary radius is hollowed out from the rod at an arbitrary location. The axes of the rod and cavity are parallel. A cross section is shown on the figure. The points $O$ and $O'$ are on the axes of the rod and cavity, respectively, and we note $\mathbf{a} = \mathbf{OO'}$.*

**a)** *Show that the field inside a solid cylinder can be written as $\mathbf{B} = (\mu_0 J/2)\hat{\mathbf{z}} \times \mathbf{r}$, where $\hat{\mathbf{z}}$ is the unit vector along the axis and $\mathbf{r}$ is the position vector measured perpendicularly to the axis.*




We calculate the magnetic field in the same way as in question (a) of problem 6:

$$\int_\Gamma \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_\Sigma \mathbf{J}\cdot d\mathbf{S},$$

where $\Sigma$ is the surface enclosed by the contour $\Gamma$. If the contour is inside the cylinder, then $\int_\Sigma \mathbf{J}\cdot d\mathbf{S} = \pi r^2 J$. Therefore $B = \mu_0 \pi r^2 J/(2\pi r)$, that is to say $B = \mu_0 J r/2$.

Because of the symmetry of the problem, the field is orthoradial, pointing in the direction given by the unit vector $\hat{\mathbf{z}}\times\mathbf{r}/r$ (if we assume that $\mathbf{J}$ is in the direction of positive $z$). Using the above expression for $B$, we then obtain:

$$\boxed{\mathbf{B} = \frac{\mu_0 J}{2}\hat{\mathbf{z}}\times\mathbf{r}.}$$

**b)** *Show that the magnetic field inside the cylindrical cavity is uniform (in both magnitude and direction).*

We can view the system as the superposition of a cylinder (the rod) with current density $\mathbf{J}$ and a second cylinder (the cavity) with current density $-\mathbf{J}$, embedded in the first one. We consider a point $P$ which is inside the second cylinder (and therefore also inside the first cylinder). The first cylinder produces at point $P$ the magnetic field $\mathbf{B}_1 = (\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{OP}$. The second cylinder produces at point $P$ the magnetic field $\mathbf{B}_2 = (-\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{O'P}$. Therefore, the total field at $P$ is $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = (\mu_0 J/2)\hat{\mathbf{z}}\times(\mathbf{OP} - \mathbf{O'P})$, that is to say:

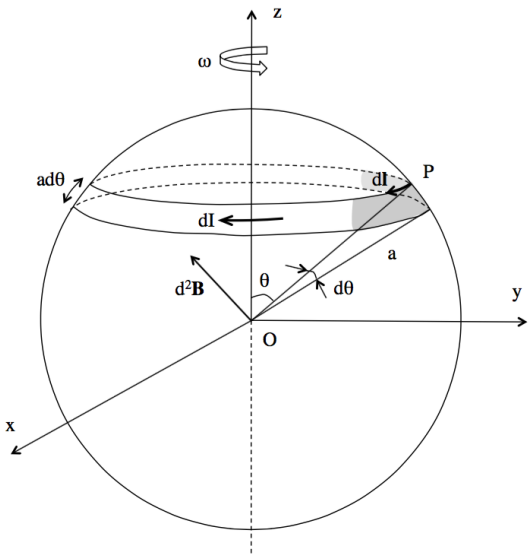
$$\boxed{\mathbf{B} = \frac{\mu_0 J}{2}\hat{\mathbf{z}}\times\mathbf{a}.}$$

It is uniform in both magnitude and direction.

## 0.8 Magnetic field at the centre of a sphere

*A spherical shell with radius $a$ and uniform surface charge density $\sigma$ spins with angular frequency $\omega$ around a diameter. Find the magnetic field at the centre.*

We choose the $z$–axis to be the axis of rotation. We consider the ring shown on the figure, and located between the angles $\theta$ and $\theta + d\theta$, measured from the $z$–axis. The radius of the ring is $R = a\sin\theta$. Since the sphere is rotating, the charges are carried around and a current $dI$ is produced at the surface of the ring.




The shaded area along the ring around the point $P$ (shown on the figure) produces the field $d^2\mathbf{B}$ at the centre. From the Biot–Savart's law:

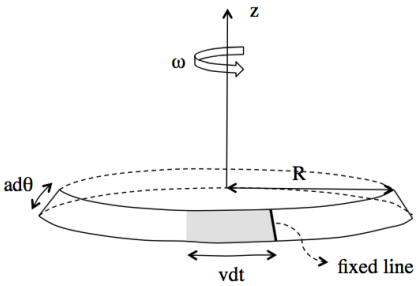
$$d^2\mathbf{B} = \frac{\mu_0 dI}{4\pi}\frac{d\mathbf{l}\times\mathbf{PO}}{PO^3}.$$

As $d\mathbf{l}$ and $\mathbf{PO}$ are perpendicular (they point in the directions of two unit vectors of the spherical coordinate system), we obtain $d^2B = (\mu_0 dI/4\pi)dl/a^2$. It can be seen that, because of the symmetry of the problem, the field produced by the entire ring is vertical. Therefore we are only interested in the vertical component of $d^2\mathbf{B}$, which is $d^2B_z = d^2B\sin\theta$. The magnetic field due to the ring is therefore:

$$d\mathbf{B} = \int_{\text{ring}} d^2B_z\,\hat{\mathbf{z}} = \int_{\text{ring}} \frac{\mu_0 dI}{4\pi a^2} dl\sin\theta\,\hat{\mathbf{z}} = \frac{\mu_0 dI}{4\pi a^2}\sin\theta\,\hat{\mathbf{z}}\int_{\text{ring}} dl = \frac{\mu_0 dI}{2a^2}R\sin\theta\,\hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the unit vector in the vertical direction. With $R = a\sin\theta$, we obtain:

$$d\mathbf{B} = \frac{\mu_0 dI}{2a}\sin^2\theta\,\hat{\mathbf{z}}.$$




To calculate the current $dI$, we consider a fixed (non rotating) line on the ring. The amount of charges $dq$ that passes through that line during a time interval $dt$ is the amount of charges contained in the surface area of length $vdt$ and width $ad\theta$ (shaded area on the figure), where $v$ is the velocity of the charges. Therefore, $dq = \sigma a v d\theta dt$. With $v = R\omega = a\omega\sin\theta$, we obtain: $dI = dq/dt = \sigma\omega a^2\sin\theta d\theta$.

Therefore:

$$d\mathbf{B} = \frac{\mu_0 \sigma \omega a}{2} \sin^3\theta d\theta \, \hat{\mathbf{z}},$$

and the magnetic field produced by the sphere at the centre is:

$$\mathbf{B} = \int_{\theta=0}^{\pi} d\mathbf{B} = \frac{\mu_0 \sigma \omega a}{2} \, \hat{\mathbf{z}} \int_0^{\pi} \sin^3\theta d\theta.$$

We can calculate the integral by using $\sin^3\theta = \sin\theta(1 - \cos^2\theta)$:

$$\int_0^{\pi} \sin^3\theta d\theta = \int_0^{\pi} \left(\sin\theta - \sin\theta\cos^2\theta\right) d\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{4}{3},$$

so that:

$$\boxed{\mathbf{B} = \frac{2}{3}\mu_0 \sigma \omega a \, \hat{\mathbf{z}}.}$$

## 0.9 Motion of a charged particle in a magnetic field

*A long thin wire carries a current $I$ in the positive $z$–direction along the axis of a cylindrical co-ordinate system. A particle of charge $q$ and mass $m$ moves in the magnetic field produced by this wire. We will neglect the gravitational force acting on the particle as it is very small compared to the magnetic force.*

**a)** *Is the kinetic energy of the particle a constant of motion?*

Yes, because the magnetic force does not work. It can be shown in the following way. From Newton's law: $m d\mathbf{v}/dt = \mathbf{F}$, where $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$ is the magnetic force. Therefore:

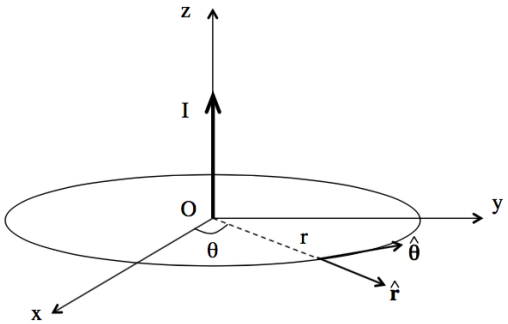
$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = q(\mathbf{v}\times\mathbf{B}) \cdot \mathbf{v}.$$

Since $\mathbf{v}\times\mathbf{B}$ is perpendicular to $\mathbf{v}$, the right–hand side of this equation is zero. We then obtain:

$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{1}{2} m \frac{dv^2}{dt} = 0,$$

which means that the kinetic energy $mv^2/2$ is constant.

**b)** *Find the force $\mathbf{F}$ on the particle, in cylindrical coordinates.*




We use the cylindrical coordinates $(r, \theta, z)$ and note $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ the unit vectors. The magnetic field has been calculated in question (a) of problem 6, and it is equal to:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}.$$

The position of the particle is $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}$, and therefore its velocity is $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\boldsymbol{\theta}}} + \dot{z}\hat{\mathbf{z}}$, where the dot denotes a time derivative. Using the above expression for $\mathbf{B}$, we obtain:

$$\boxed{\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) = \frac{\mu_0 I q}{2\pi r}\left(-\dot{z}\hat{\mathbf{r}} + \dot{r}\hat{\mathbf{z}}\right).}$$

**c)** *Obtain the equation of motion, $\mathbf{F} = m d\mathbf{v}/dt$, in cylindrical coordinates for the particle.*

The acceleration in cylindrical coordinates is $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}$. Therefore the equation of motion $m\mathbf{a} = \mathbf{F}$ yields the three equations:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu_0 I q}{2\pi m r}\dot{z}, \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0, \\ \ddot{z} &= \frac{\mu_0 I q}{2\pi m r}\dot{r}. \end{aligned}$$

**d)** *Suppose the velocity in the $z$–direction is constant. Describe the motion.*

We assume that $\dot{z} = v_0$ is a constant. Then the third equation above yields $\dot{r} = 0$, that is to say $r$ is a constant which we note $r_0$. The first equation then becomes:

$$\dot{\theta}^2 = \frac{\mu_0 I q}{2\pi m r_0^2} v_0.$$

This implies that $v_0 > 0$ if $q > 0$ and $v_0 < 0$ if $q < 0$. The particle is then rotating around the $z$–axis with an angular velocity

$$\omega \equiv \dot{\theta} = \sqrt{\frac{\mu_0 I q}{2\pi m r_0^2} v_0}.$$

The particle is moving along a helix which axis is the $z$–axis. If $q > 0$ the particle is moving toward increasing $z$, whereas it is moving toward decreasing $z$ if $q < 0$.

The second equation in question (c) means that the vertical component of the angular momentum $\mathbf{J}$ of the particle is constant. Indeed, $\mathbf{J} = m\mathbf{r}\times\mathbf{v} = m(-r\dot{z}\hat{\boldsymbol{\theta}} + r^2\dot{\theta}\hat{\mathbf{z}})$. Therefore $\dot{J}_z = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$ and this is zero from question (c). Note that we also have $\dot{J}_\theta = 0$, as $\dot{r} = \ddot{z} = 0$. The fact that $\dot{\mathbf{J}} = \mathbf{0}$ is a consequence of the force $\mathbf{F}$ being radial (see question b with $\dot{r} = 0$). Indeed, $m\dot{\mathbf{v}} = \mathbf{F}$ implies $m\mathbf{r}\times\dot{\mathbf{v}} = \mathbf{r}\times\mathbf{F}$. As $\mathbf{F}$ and $\mathbf{r}$ are parallel, the right–hand side of this equation is zero. Then $\dot{\mathbf{J}} = m\, d(\mathbf{r}\times\mathbf{v})/dt = m\mathbf{r}\times\dot{\mathbf{v}} = \mathbf{0}$.

## ■ *Electromagnetic induction*

### 0.10 Growing current in a solenoid

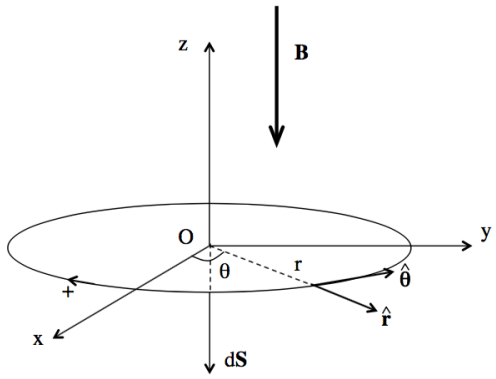
*An infinite solenoid has radius $a$ and $n$ turns per unit length. The current grows linearly with time, according to $I(t) = kt$, $k > 0$. The solenoid is looped by a circular wire of*

*radius $r$, coaxial with it. We recall that the magnetic field due to the current in the solenoid is $B = \mu_0 nI$ inside the solenoid and zero outside.*

**a)** *Without doing any calculation, explain which way the current induced in the loop flows.*

The magnetic field in the solenoid is $B = \mu_0 nI$. Therefore, an increase of $I$ leads to an increase of the magnetic flux through the surface delimited by the loop. By Lenz's law, the current induced in the loop will produce a field that will oppose this increase of the flux. Therefore, within the loop, it will point in the opposite direction to the field produced by the solenoid. This implies that the current induced in the loop flows in opposite way to the current in the solenoid.

**b)** *Use the integral form of Faraday's law, which is $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt$, to find the electric field in the loop for both $r < a$ and $r > a$. Check that the orientation of $\mathbf{E}$ agrees with the answer to question (a).*




We use the cylindrical coordinates $(r, \theta, z)$ and note $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ the unit vectors, with the $z$–axis pointing upward. The magnetic field due to the current in the solenoid is $\mathbf{B} = -\mu_0 nI\hat{\mathbf{z}}$ inside the solenoid, and zero outside. We *choose* a positive orientation along the loop as indicated on the figure. Then the orientation of $d\mathbf{S}$, which is given by the right–hand rule, is such that the flux of $\mathbf{B}$ through the surface $\Sigma$ delimited by the loop is positive.

Faraday's law:

$$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{S},$$

where $d\mathbf{l}$ is oriented in the positive direction chosen along the loop (we could have chosen the positive direction in the opposite way, but then *both* $d\mathbf{S}$ and $d\mathbf{l}$ would have had to be reversed). The electric field induced in the loop is orthoradial: $\mathbf{E} = E_\theta \hat{\boldsymbol{\theta}}$, where $E_\theta$ can be either positive or negative. In addition, because of the symmetry of the problem, $E_\theta$ depends only on $r$. With $d\mathbf{l} = -dl\hat{\boldsymbol{\theta}}$, we obtain $\mathbf{E} \cdot d\mathbf{l} = -E_\theta \, dl$, and $\oint \mathbf{E} \cdot d\mathbf{l} = -2\pi r E_\theta$. If $r > a$, the flux of $\mathbf{B}$ through the loop is $\pi a^2 \mu_0 nI$, whereas it is $\pi r^2 \mu_0 nI$ if $r < a$. Therefore, Faraday's law yields: $2\pi r E_\theta = \pi a^2 \mu_0 nk$ is $r > a$ and $2\pi r E_\theta = \pi r^2 \mu_0 nk$ is $r < a$. The electric field is then given by:

$$\boxed{\mathbf{E} = \frac{\mu_0 nka^2}{2r}\hat{\boldsymbol{\theta}} \text{ for } r > a, \text{ and } \mathbf{E} = \frac{\mu_0 nkr}{2}\hat{\boldsymbol{\theta}} \text{ for } r < a.}$$

Because the electric field is in the direction of $\hat{\boldsymbol{\theta}}$, the electric force on the electrons in the loop points in the $-\hat{\boldsymbol{\theta}}$ direction, and the current in the loop points in the $+\hat{\boldsymbol{\theta}}$ direction. As the current in the solenoid is opposite, we recover the result of

question (a).

**c)** *Verify that your result satisfies the local form of the law,* $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$.

In cylindrical coordinates, and given that $\mathbf{E}$ is orthoradial and depends only on $r$:

$$\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{r}\frac{d(rE_\theta)}{dr}\hat{\mathbf{z}}.$$

By using the above expressions for $\mathbf{E}$, it is straightforward to verify that $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$.

■ *Maxwell's equations*

0.11 **Energy flow into a capacitor**

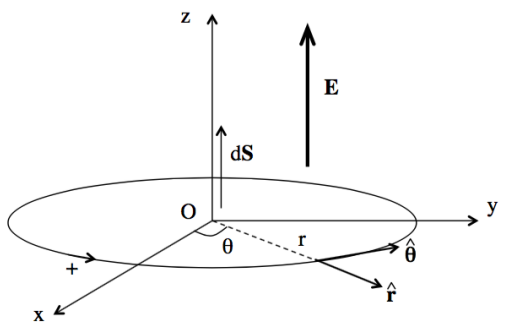
*A capacitor has circular plates with radius $a$ and is being charged by a constant current $I$. The separation of the plates is $w \ll a$. Assume that the current flows out over the plates through thin wires that connect to the centre of the plates, and in such a way that the surface charge density $\sigma$ is uniform, at any given time, and is zero at $t = 0$.*

**a)** *Find the electric field between the plates as a function of $t$.*

Since $w \ll a$, we can ignore edge effects, and the electric field produced by the lower plate is the same as that produced by an infinite plane with surface density $\sigma$, that is to say $\hat{\mathbf{z}}\sigma/(2\epsilon_0)$, where $\hat{\mathbf{z}}$ is the unit vector in the vertical direction, pointing upward. The electric field produced by the upper plate is the same as that produced by the lower plate. Therefore, the total electric field is $\mathbf{E} = \hat{\mathbf{z}}\sigma/\epsilon_0$. The total charge on the lower plate is $Q = \pi a^2\sigma$, and we have $I = dQ/dt$. As $I$ is constant, this yields $Q = It + Q_0$, where $Q_0$ is the charge on the plate at $t = 0$. As $Q_0 = 0$, we obtain $Q = It$. Therefore:

$$\boxed{\mathbf{E} = \frac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$$

**b)** *Consider the circle of radius $r < a$ shown on the figure (and centered on the axis of the capacitor). Using the integral form of Maxwell's equation $\boldsymbol{\nabla}\times\mathbf{B} = \epsilon_0\mu_0\partial\mathbf{E}/\partial t$ over the surface delimited by the circle, find the magnetic field at a distance $r$ from the axis of the capacitor.*




The integral form of Maxwell's equation is:

$$\iint_\Sigma (\boldsymbol{\nabla}\times\mathbf{B})\cdot d\mathbf{S} = \iint_\Sigma \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\cdot d\mathbf{S},$$

where the integral is over the surface $\Sigma$ delimited by the circle.

We choose a positive orientation on the circle, as indicated on the figure, and the orientation of $d\mathbf{S}$ is then given by the right–hand rule. We use Stokes's theorem:

$$\iint_{\Sigma} (\boldsymbol{\nabla}\times\mathbf{B})\cdot d\mathbf{S} = \int_{\text{circle}} \mathbf{B}\cdot d\mathbf{l},$$

where $d\mathbf{l} = dl\hat{\boldsymbol{\theta}}$ is oriented in the positive direction along the circle. The magnetic field is orthoradial, that it to say $\mathbf{B} = B_\theta\hat{\boldsymbol{\theta}}$ and, because of the symmetry of the problem, depends only on $r$. Therefore, in the integral above, $\mathbf{B}\cdot d\mathbf{l} = B_\theta(r)dl$. Maxwell's equation then yields:

$$2\pi r B_\theta(r) = \epsilon_0\mu_0\pi r^2\frac{\partial E_z}{\partial t},$$

where we have written $\mathbf{E} = E_z\hat{\mathbf{z}}$. With the expression of $E_z$ found in question (a), we obtain:

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 I r}{2\pi a^2}\,\hat{\boldsymbol{\theta}}.}$$

**c)** *Find the energy density $u$ and the Poynting vector $\mathbf{S}$ in the gap. Check that the relation:*

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{S},$$

*is satisfied.*

The energy density is given by:

$$u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}.$$

With the expressions of $E$ and $B$ found above, we obtain:

$$\boxed{u = \frac{I^2}{2\pi^2 a^4}\left(\frac{\mu_0 r^2}{4} + \frac{t^2}{\epsilon_0}\right).}$$

The Poynting vector is given by:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}E_z\hat{\mathbf{z}}\times B_\theta\hat{\boldsymbol{\theta}} = -\frac{1}{\mu_0}E_z B_\theta\hat{\mathbf{r}} = \boxed{-\frac{I^2 r t}{2\pi^2\epsilon_0 a^4}\hat{\mathbf{r}}.}$$

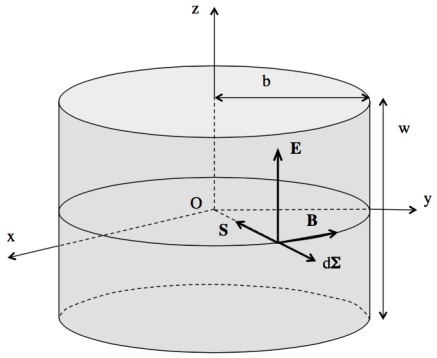
The Poynting vector points radially inward, which means that energy *enters* the capacitor. This makes sense as $E$, and therefore $u$, is increasing with time.

The above expression for $u$ yields $\partial u/\partial t = I^2 t/(\pi^2\epsilon_0 a^4)$. Given that $\mathbf{S}$ is radial and depends only on the coordinate $r$, its divergence in cylindrical coordinates is:

$$\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{1}{r}\frac{d(rS)}{dr},$$

and it is straightforward to check that $\partial u/\partial t = -\boldsymbol{\nabla}\cdot\mathbf{S}$.

**d)** *Consider a cylinder of radius $b < a$ and length $w$ inside the gap. Determine the total energy in the cylinder, as a function of time. Calculate the total power flowing into the cylinder, by integrating the Poynting vector* $\mathbf{S}$ *over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the cylinder.*




The total energy $U$ in the cylinder is:

$$U = \iiint_{\text{cylinder}} urd\theta drdz = 2\pi w \int_0^b urdr,$$

$$\boxed{U = \frac{I^2 b^2 w}{2\pi a^4}\left(\frac{\mu_0 b^2}{8} + \frac{t^2}{\epsilon_0}\right).}$$

The Poynting vector is an energy flux density, which means an energy per unit time per unit surface area. Therefore, the total power (energy per unit time) flowing into the cylinder is:

$$P = \left| \iint_\Sigma \mathbf{S} \cdot d\mathbf{\Sigma} \right|,$$

where the integral is over the surface $\Sigma$ of the cylinder and $d\mathbf{\Sigma}$ is perpendicular to a surface element of the cylinder and pointing outward. As $\mathbf{S}$ is radial and pointing inward, $\mathbf{S} \cdot d\mathbf{\Sigma} = -S\, d\Sigma$ on the lateral surface of the cylinder and zero on the horizontal surfaces. Since $S$ is uniform over the lateral surface of the cylinder, $P = S(b) \times 2\pi bw$. With the above expression for $S$, we obtain:

$$\boxed{P = \frac{I^2 b^2 wt}{\pi \epsilon_0 a^4}.}$$

As expected from energy conservation, we check that $\partial U/\partial t = P$.

**e)** *When $b = a$, and assuming that we can still neglect edge effects in that case, check that the total power flowing into the capacitor is:*

$$\frac{d}{dt}\left(\frac{1}{2}QV\right),$$

*where $V$ is the voltage across the capacitor (since $QV/2$ is the energy stored in the electric field in the capacitor).*

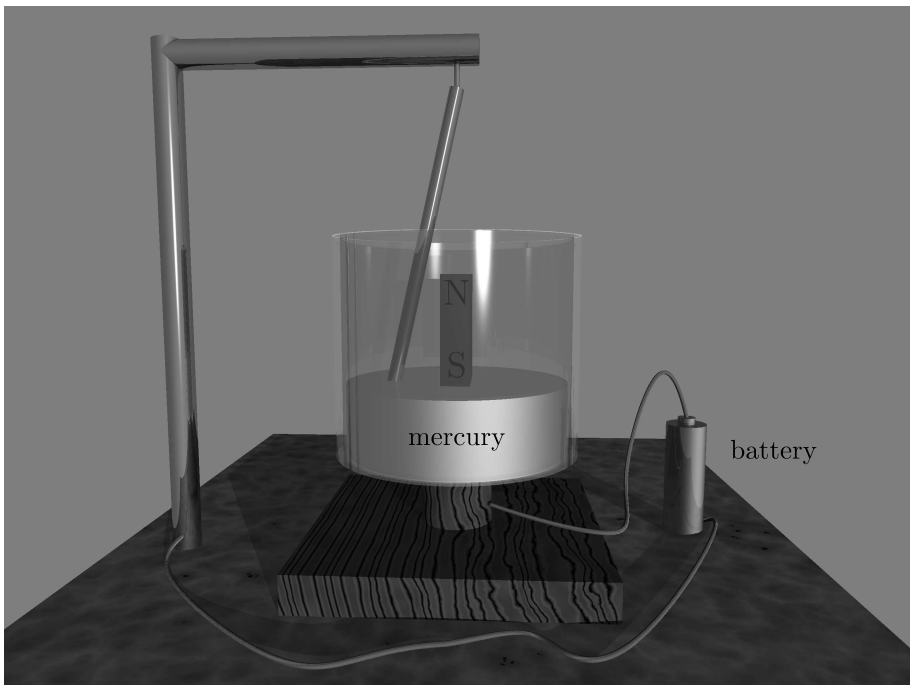
When $b = a$, the above expression for $P$ becomes $P = I^2 wt/(\pi\epsilon_0 a^2)$. With $Q = It$, $V = Ew$ and $E$ given in question (a), we obtain $QV/2 = wI^2 t^2/(2\pi\epsilon_0 a^2)$. Therefore $d(QV/2)/dt = P$, as expected.

# Electromagnetism

UNIVERSITY OF OXFORD

**Second year physics course**

Professor S. J. Blundell




# Questions*

MICHAELMAS 2021

*I am grateful to Prof. Caroline Terquem who first assembled these sheets. Some of these problems are taken from: *Introduction to Electrodynamics*, David J. Griffiths, 4th Edition, and *Electricity and Magnetism*, Edward M. Purcell and David J. Morin, 3rd Edition.

# Electromagnetism: Problems

**Problem sheet 1**: (Michaelmas Term 2021)

■ *Potentials*

1.1 **Laplace's equation**

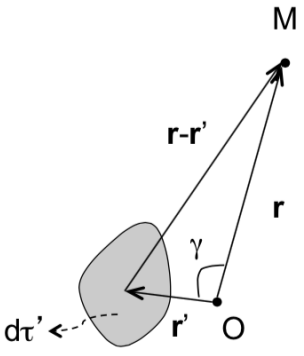
This question concerns Laplace's equation, $\nabla^2 V = 0$, which applies when there are no charges within a region. Consider a potential $V(\boldsymbol{r})$ which satisfies Laplace's equation and now imagine a sphere of radius $R$ centred about the origin. The average value of $V$ over the surface of the sphere is given by

$$\bar{V}_R = \frac{1}{4\pi R^2} \int_S V(\boldsymbol{r}) \mathrm{d}S,$$

where the integral is taken over the surface of the sphere. By evaluating $\partial \bar{V}_R / \partial R$, show that the potential at any point is equal to that averaged over a sphere of radius $R$ centred at that point. (Hint: you can choose coordinates so that "any point" is the origin in those coordinates.)

1.2 **Expansion in Legendre polynomials**

*This problem is useful to understand that a multipole expansion is nothing more than a development in Taylor series of* $1/|\boldsymbol{r}' - \boldsymbol{r}|$.




The scalar potential $V$ created at a point $M$ by a localised charge distribution is:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{r}|} d\tau',$$

where the integration is over the volume $\mathcal{V}$ of the distribution and the position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ are measured from an origin $O$ chosen arbitrarily. [Note that $\boldsymbol{r} \cdot \boldsymbol{r}' = rr' \cos\gamma$.]

Assuming $r' \ll r$, expand $1/|\boldsymbol{r}' - \boldsymbol{r}|$ in Taylor series up to 3rd order in $r'/r$. Use the result to give an expression for $V$ in term of Legendre polynomials including the terms up to $l = 3$.

We give: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

## 1.3 **Dipole**

**a)** Show that the energy of a physical dipole $\boldsymbol{p}$ in an electric field $\boldsymbol{E}$ (not necessarily uniform) is given by $U = -\boldsymbol{p} \cdot \boldsymbol{E}$.

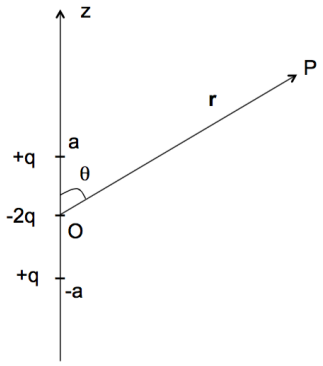
**b)** We consider two dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and take $\boldsymbol{r}$ to be the position vector of $\boldsymbol{p}_2$ measured from $\boldsymbol{p}_1$. Show that the interaction energy of the two dipoles is:

$$U_{\text{int}} = \frac{1}{4\pi\epsilon_0 r^3} \left[ \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 - 3 \left( \boldsymbol{p}_1 \cdot \hat{\boldsymbol{r}} \right) \left( \boldsymbol{p}_2 \cdot \hat{\boldsymbol{r}} \right) \right],$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$.

**c)** Draw graphs showing how $U_{\text{int}}$ depends upon the relative orientation of the dipoles in the following cases: (i) $\boldsymbol{p}_1$ is parallel to $\boldsymbol{r}$, (ii) $\boldsymbol{p}_1$ is perpendicular to $\boldsymbol{r}$.

## 1.4 **Quadrupole**




A system of three charges, aligned along the $z$–axis, consists of a charge $-2q$ at the origin $O$ and two $+q$ charges at $z = -a$ and $z = a$. Here we use spherical coordinates: $(r, \theta, \varphi)$.

**a)** Find the potential at $r \gg a$ using the multipole expansion with origin at $O$. Justify why this system of charges is called a quadrupole.
We give: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$.

**b)** Show that there is not translational force or couple on the quadrupole in a uniform electric field.

**c)** We consider a point charge $Q$ placed at a point $P$ a distance $r \gg a$ away from the quadrupole. Show that the torque about $P$ on the quadrupole due to the charge is $\Gamma = 3Qqa^2 \sin 2\theta/(4\pi\epsilon_0 r^3)$ and find its direction.

**d)** Find the energy of the quadrupole in the potential of the charge $Q$. In which direction does the quadrupole rotate under the effect of the torque?

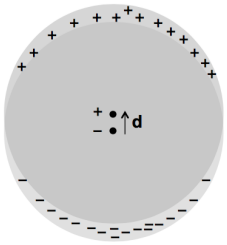
1.5 **Electric field due to a sphere with surface charge density $\propto \cos\theta$**
We consider a sphere of radius $R$ and a spherical coordinate system $(r, \theta, \varphi)$ with origin at the centre of the sphere. The surface of the sphere carries a charge density $\sigma = k\cos\theta$.

**a)** Calculate the potential $V$ using the fact that it satisfies Laplace's equation both inside and outside the sphere. *Hint*: write the solution of the equation in terms of Legendre polynomials and choose $V = 0$ at infinity.

**b)** Find the electric field $\boldsymbol{E}$ produced by the sphere.

**c)** Calculate the electric dipole moment $\boldsymbol{p}$ of the charge distribution. Check that the electric field outside the sphere calculated in the previous question is that due to the dipole moment $\boldsymbol{p}$. What can you conclude about the higher multipoles? Express the field inside the sphere as a function of the *polarization* of the sphere, which is the dipole moment per unit volume.

**d)** A surface charge density $\propto \cos\theta$ can be obtained by superimposing two solid spheres with same radius $R$, opposite charges $Q$ and $-Q$ uniformly distributed over their volume, and centres slightly apart (separation $d \ll R$), as shown on the figure. Viewed from a point outside the spheres, the situation looks the same as if the charges were at the centre of the spheres.




Find the relation between $Q$, $\boldsymbol{d}$ and $\boldsymbol{p}$ for the electric field outside the spheres to be the same as in the previous question. Using Gauss's theorem to obtain the contribution from each sphere, calculate the field in the region where the spheres overlap and show that it is the same as in the previous question.

1.6 **Metal sphere in a uniform electric field** (*Problem 1.5 needs to be done first*)
A metal sphere of radius $R$ with no net charge is placed in a uniform electric field $\boldsymbol{E} = E_0\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is a unit vector.

**a)** Explain qualitatively how the field modifies the charge distribution in the sphere and how this, in turn, affects the electric field.

**b)** Calculate the electric potential $V$ outside the sphere using the fact that it satisfies Laplace's equation. *Hint*: write the solution of the equation in terms of Legendre polynomials and choose $V = 0$ at the surface of the sphere.

**c)** Using the boundary condition on the electric field, calculate the charge density $\sigma$ at the surface of the sphere.

**d)** Using the results from Problem 1.5(b), give an expression for the electric field produced by $\sigma$. Find the total field inside the sphere. Comment.

1.7 **Separation of variables in cylindrical coordinates**

**a)** Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on $z$.

**b)** Consider an infinitely long straight wire along the $z$–axis which carries a uniform line charge $\lambda$. Calculate the electric field using Gauss's theorem. Find the potential and check that it is a solution of Laplace's equation found in (a).

**c)** Consider an infinitely long metal pipe of radius $R$ placed at right angle to a uniform electric field $\boldsymbol{E}_0$. Using the result from (a) and appropriate boundary conditions, find the potential outside the pipe. Calculate the charge density induced on the pipe.

1.8 **Magnetic vector potential**

We consider a finite segment of straight wire of length $2L$ carrying a steady current $I$.

**a)** Calculate the magnetic vector potential $\boldsymbol{A}$ at a point $P$ a distance $r$ from the wire along the perpendicular bisector. We give:

$$\int \frac{dz}{\sqrt{z^2+a^2}} = \ln\left(z+\sqrt{z^2+a^2}\right).$$

**b)** Using $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and assuming $L \gg r$, calculate the magnetic field at $P$. Check that your answer is consistent with what is expected from Ampère's law.

# Electromagnetism: Problems

**Problem sheet 2**: (Michaelmas Term 2021)

■ *Electric and magnetic fields in matter*

## 2.1 Capacitance and dielectrics

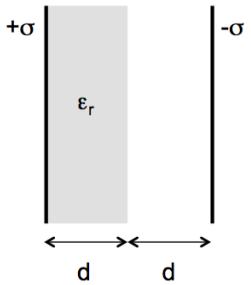
We consider a parallel plate capacitor consisting of two metal surfaces of area $A$ separated by a distance $d$. We assume $d \ll A^{1/2}$, so that edge effects can be neglected and the electric field can be considered to be uniform between the plates.

**a)** The capacitor is connected to a battery so that a charge $+Q_0$ is brought from one plate to the other. Using Gauss's law, calculate the electric field $E_0$ between the plates. Find the potential difference $\Delta V_0$ between the plates and the capacitance $C_0$. Calculate the potential energy $U_0$ stored in the capacitor.

**b)** We now insert a linear dielectric material between the plates while the capacitor remains connected to the battery, which supplies the potential difference $\Delta V_0$. Experimentally, it is found that the charge $Q$ on the plates is increased. Explain why. By representing the system as the superposition of a vacuum capacitor and a polarized dielectric slab, calculate the total electric field $E$ in the capacitor in terms of $Q$ and the polarization $P$. By substituting $P = \epsilon_0 \chi_e E$, find the relation between the displacement vector $\boldsymbol{D}$ and $Q$. Calculate $Q$ in terms of $Q_0$ and the capacitance $C$ in terms of $C_0$. Calculate the change in the stored potential energy of the capacitor in terms of $U_0$.

**c)** We now come back to question (a) and disconnect the battery before inserting a dielectric material between the capacitor plates. Experimentally, it is found that the potential difference $\Delta V$ between the plates decreases. Explain why. Calculate the capacitance $C$ in terms of $C_0$ and the change in the stored potential energy of the capacitor in terms of $U_0$.

## 2.2 Capacitor half–filled with a dielectric




We insert a slab of linear dielectric material of dielectric constant $\epsilon_r$ and thickness $d$ next to the positive plate in a parallel plate capacitor. The distance between the plates is $2d$, their area is $A$ and they carry a (free) charge density $\pm\sigma$ .

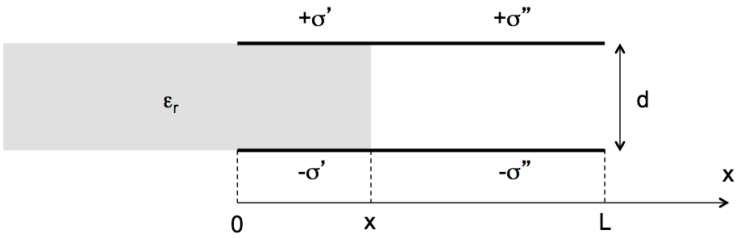
**a)** Find the electric displacement vector $\boldsymbol{D}$, the electric field $\boldsymbol{E}$ and the polarization $\boldsymbol{p}$ in each region.

**b)** Find the potential difference between the plates and the capacitance $C$. Show that the system can be regarded as two capacitors connected in series.

**c)** Find the location and amount of all bound charge. Using the distribution of free and bound charges, recalculate the electric field $\boldsymbol{E}$ in each region.

2.3 **Force on a dielectric**

We insert a portion of a slab of linear dielectric material of dielectric constant $\epsilon_r$ and thickness $d$ on the left hand side of a parallel plate capacitor consisting of two conducting plates of length $L$, width $w$ and thickness $d$.




**a)** We connect the capacitor to a battery to charge the plates, and then disconnect the battery. The total charge on each plate then remains constant equal to $\pm Q$, corresponding to surface charge densities $\pm\sigma'$ and $\pm\sigma''$ on the left and right hand sides, respectively. Using Gauss's law, write the electric fields $E'$ and $E''$ on the left and right hand sides in terms of $\sigma'$ and $\sigma''$. Find the relation between $\sigma'$ and $\sigma''$. Using the fact that $Q$ is constant, write $\sigma''$ in terms of $x$ and the different constants. Find the potential difference between the plates, and then the stored potential energy $U$ of the system, in terms of $x$. Write the relation between the change in energy, $dU$, when the dielectric is pulled out a distance $dx$, and the electric force $F$ exerted by the plates on the dielectric. Calculate $F$ in terms of $x$. Does this force pull the dielectric into the capacitor or push it out?

**b)** We now repeat the experiment but leave the capacitor connected to the battery, which supplies the potential difference $\Delta V$. Calculate $U$ in terms of $x$ and the different constants. Write the relation between $dU$ and $F$ when the dielectric is pulled out a distance $dx$. Make sure to include *all* contributions to the work done in the system. Calculate $F$ in terms of $x$ and show that it is the same as in question (a).

**c)** The result obtained above can be verified experimentally. However, we have assumed in our calculation that the electric field was uniform between the plates and zero outside. If that were the case in reality, would there be a force on the dielectric? Where does this force come from, and how is it possible that we obtain the right answer given the simplified model we have used?

2.4 **Sphere with a frozen–in polarization**

A sphere of radius $R$ made of a dielectric material carries a frozen–in polarization $\boldsymbol{P}(\boldsymbol{r}) = k\boldsymbol{r}$, where $k$ is a constant and $\boldsymbol{r}$ is the position vector measured from the centre of the sphere. There are no free charges anywhere.

**a)** Calculate the bound charges.

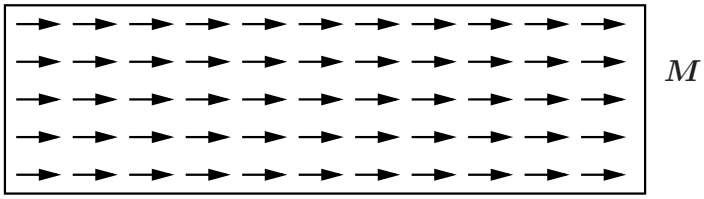
**b)** Using Gauss's law, find the electric field $\boldsymbol{E}$ inside and outside the sphere.

**c)** Using the expression of $\boldsymbol{E}$ and $\boldsymbol{P}$, find the electric displacement vector $\boldsymbol{D}$ inside and outside the sphere. Check that it satisfies Gauss's law. Is the dielectric linear?

## ■ *Magnetic fields in matter*

2.5 **B, H and M in a bar magnet**

Sketch $\boldsymbol{B}$, $\boldsymbol{H}$ for a rectangular bar magnet where the magnetization $\boldsymbol{M}$ is as indicated below.



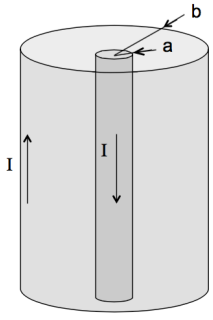

2.6 **Cylinder with a frozen–in magnetization**

An infinitely long cylinder of radius $R$ made of a magnetic material carries a frozen–in magnetization $\boldsymbol{M} = kr\hat{\boldsymbol{z}}$, where $r$ is the distance from the axis of the cylinder, $\hat{\boldsymbol{z}}$ is the unit vector along the axis and $k$ is a constant. There is no free current anywhere. Find the magnetic field $\boldsymbol{B}$ inside and outside the cylinder by two different methods:

**a)** Locate all the bound currents and calculate the field they produce,

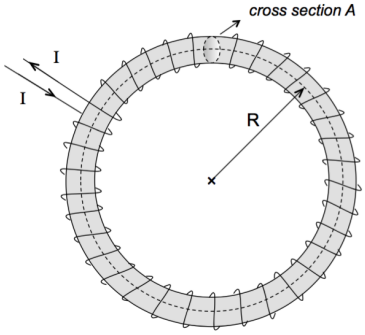
**b)** Use Ampère's law for $\boldsymbol{H}$ and the relationship between $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{M}$ to derive $\boldsymbol{B}$.

2.7 **Magnetic field in a coaxial cable**




A long coaxial cable of inner radius $a$ and outer radius $b$ is filled with an insulating material of magnetic susceptibility $\chi_m$. A current $I$ flows down the inner conductor and returns along the outer one, uniformly distributed over the surfaces. Find the magnetic field $\boldsymbol{B}$ in the magnetic material between the conductors. As a check, calculate the magnetization $\boldsymbol{M}$ and the bound currents, and confirm that, together with the free currents, they generate the correct field.

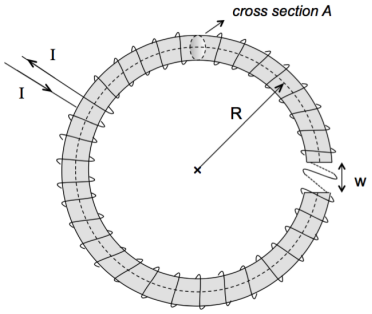
## 2.8 **Air gap in an inductor**




We consider a toroidal core made of an iron alloy with cross sectional area $A$ and radius $R \gg A^{1/2}$. We make an inductor from the core by wrapping around it $N$ turns of a wire carrying a current $I$, to use it as an energy storage device.

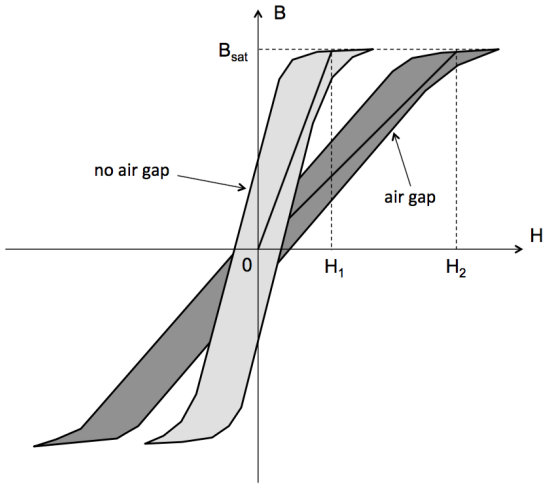
We increase the current from 0 until the material saturates, which happens when the magnetic field $B$ inside it reaches the value $B_{\text{sat}}$. Before saturation is reached, the material can be regarded as being linear with relative permeability $\mu_r$.

**a)** Calculate the magnetic field $B$ in the core and the inductance $L$. Find the maximum current $I_1$ that the inductor can carry before the core saturates. Calculate the magnetic energy $W_1$ that is stored in the inductor when the current reaches the value $I_1$. If $I$ is increased beyond $I_1$, how does the inductance vary?




**b)** To increase the energy that can be stored in the inductor, we have to find a way of increasing the saturation current while keeping $L$ constant. This can be done by introducing an air gap of width $w$ into the core. We take $w \ll R$, so that we can neglect magnetic fringing, that is to say the bulging out of the magnetic field lines as they enter air from the magnetic material.

Calculate the magnetic field $B$ in the core and the number of turns of wire we now have to wrap around the core to keep $L$ constant. Find the maximum current $I_2$ this second inductor can carry before the core saturates and compare it with $I_1$. Calculate the energy $W_2$ stored in the magnetic field at the point of saturation and compare it with $W_1$. Compare the energy stored in the magnetic material with that stored in the air gap. For numerical applications, use $R = 10$ cm, $w = 3$ mm and $\mu_r = 1500$.




**c)** Some of the energy stored in the inductor can be recovered by decreasing the current back to 0. The figure shows the $B - H$ curves (hysteresis loops) measured on a toroidal core without and with gap (the straight lines inside the loops are the magnetization curves).

Show on these curves how much energy is released per unit volume by the magnetic material in the core after the current is returned to 0. Comment.

**d)** Explain what would be the advantages of using iron, and especially *soft* iron, to make magnetic cores. Iron, however, has a major disadvantage when the inductor is used with alternating currents. Can you think what that might be?

# Electromagnetism: Problems

**Problem sheet 3**: (Michaelmas Term 2021)

■ *Electromagnetic waves*

3.1 **Poynting vector and resistance heating**

*This problem is not about waves but is useful to understand how energy is transferred in a circuit.*

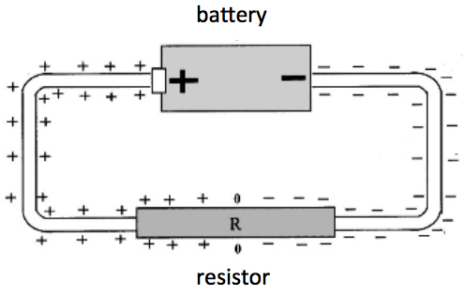
A cylindrical resistor with radius $a$, length $L$ and resistance $R$ is connected to a battery which supplies a potential difference $V$. The resulting steady current $I$ which flows along the resistor is uniform through its cross-section.

**a)** Find the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ on the surface of the resistor.

**b)** Calculate the Poynting vector $\boldsymbol{S}$ on the surface of the resistor. Find the power flowing through the surface of the resistor. Compare with the rate at which work is done on the charges in the resistor by the electromagnetic field.

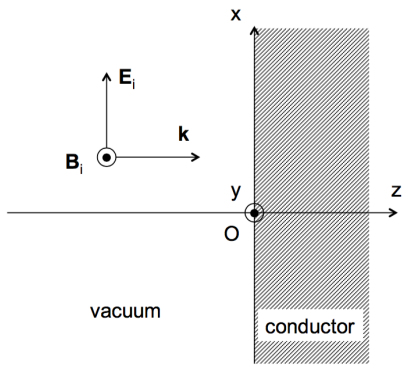
**c)** We have found above that there is a flow of energy going into the resistor. This energy has to come from outside the resistor, and therefore there has to be an electric field outside giving an inward radial Poynting vector. Justify that this is the case.

**d)** The energy entering the resistor comes from the battery: energy flows *along* the wires connecting the battery to the resistor. This requires a component of the electric field *perpendicular* to the wires. Such a field is produced by charges that have to be present on the surface of the wires for a steady current to flow. Assuming the charge distribution sketched on the figure, draw the Poynting vector in the different parts of the circuit to show how energy flows from the battery to the resistor. Comment on *where* the energy flows. Is there any energy returning to the battery?




(adapted from Galili & Goihbarg, 2005, *Energy transfer in electrical circuits: a qualitative account*, American Journal of Physics, 73, 141)

## 3.2 **Standing wave and radiation pressure**




Consider an electromagnetic monochromatic plane wave which travels in vacuum and hits at normal incidence the plane surface of a *perfect* conductor which fills the half–space $z > 0$. The wave is linearly polarized in the $x$–direction and travels in the $z$–direction.

The incident wave can be written in the form:

$$\begin{aligned} \tilde{\mathbf{E}}_i(z,t) &= E_0\, \mathrm{e}^{\mathrm{i}(kz-\omega t)}\, \hat{\mathbf{x}}, \\ \tilde{\mathbf{B}}_i(z,t) &= B_0\, \mathrm{e}^{\mathrm{i}(kz-\omega t)}\, \hat{\mathbf{y}}, \end{aligned}$$

where the *tilde* is used to denote complex quantities. We choose the origin of time such that $E_0$ (and therefore $B_0$) are real.

**a)** Find the transmitted and reflected waves in terms of the amplitude of the incident wave.

**b)** Write the total electric field $E$ and magnetic field $B$ in the half–space $z < 0$. Find the position (expressed in terms of the wavelength $\lambda$ of the wave) of the *nodal* planes, where $E$ and $B$ are zero, and justify why this is called a *standing* wave. Draw $E$ as a function of $z$ at different times (expressed in terms of the period $T$ of the wave).

**c)** Calculate the Poynting vector associated with the wave. Find its value in the nodal planes and describe how the energy flows in space. Calculate the time–average of the Poynting vector over a period at position $z$.

**d)** Calculate the time–average of the energy density over a period at position $z$.

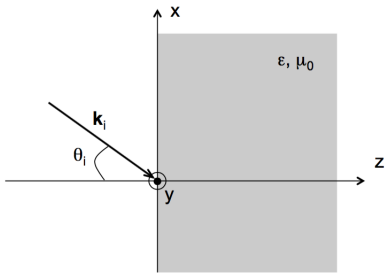
**e)** Calculate the charge and current densities at the surface of the conductor. The magnetic field due to an infinite current sheet with surface current $\boldsymbol{K}$ is $\mu_o K/2$. Find the magnetic field due to the current flowing at the surface of the conductor, and compare with the reflected magnetic field calculated in question (a). Comment. What is the source of the reflected electric field?

**f)** Calculate the time–average over a period of the force exerted by the electromagnetic wave on the current at the surface of the conductor. Show that it is equivalent to a *radiation pressure* $P = \epsilon_0 E^2$. [Remember that the field generated by the surface current itself does not exert a force on the surface.]

**g)** The result from the previous question indicates that the radiation pressure is equal to the energy density of the electromagnetic wave. We are now going to show why this is the case. We suppose that the conductor contracts by an average distance $dz$ under the action of the radiation pressure. Find the resulting increase of the

electromagnetic energy. Calculate the work done by the electromagnetic field on the conductor to make it contract. What can you conclude? [Here we only consider the time–average of the different quantities over a period.]

### 3.3 **Fresnel's formulas**




A monochromatic plane wave:

$$\tilde{\boldsymbol{E}}_i = \boldsymbol{E}_{0i}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_i\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}}_i = \frac{1}{c}\hat{\boldsymbol{k}}_i\times\tilde{\boldsymbol{E}}_i,$$

is incident on the flat surface of a linear medium with dielectric constant $\epsilon_r$ (and $\mu = \mu_0$) at an angle $\theta_i$ to the normal.

Here $\hat{\boldsymbol{k}}_i$ is the unit vector in the direction of propagation, and the *tilde* denotes complex quantities. We choose the origin of time so that the amplitude of the wave, $E_{0i}$, is real. This gives rise to reflected and transmitted waves:

$$\tilde{\boldsymbol{E}}_r = \tilde{\boldsymbol{E}}_{0r}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_r\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}}_r = \frac{1}{c}\hat{\boldsymbol{k}}_r\times\tilde{\boldsymbol{E}}_r, \quad \text{and} \quad \tilde{\boldsymbol{E}}_t = \tilde{\boldsymbol{E}}_{0t}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_t\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}}_t = \frac{1}{v}\hat{\boldsymbol{k}}_t\times\tilde{\boldsymbol{E}}_t,$$

where $v$ is the velocity of the wave in the dielectric.

**a)** Use the dispersion relation $\omega(k)$ and the boundary conditions at the interface to find the angle of reflection $\theta_r$ (between $\boldsymbol{k}_r$ and the $z$–axis) and the angle of transmission (or refraction) $\theta_t$ (between $\boldsymbol{k}_t$ and the $z$–axis) in terms of the angle of incidence $\theta_i$ (Snell's law). If the index of refraction $n < 1$, show that there is a critical angle $\theta_{ic}$ above which there is no transmitted wave (this is called *total internal reflection*).

**b)** We assume that the incident wave is linearly polarized with polarization normal to the plane of incidence. We assume that the reflected and transmitted waves have the same polarization (this can be shown by assuming a more general polarization and using the boundary conditions). Use the boundary conditions to find $\tilde{E}_{0r}$ and $\tilde{E}_{0t}$ as a function of $E_{0i}$ and $\theta_i$ (Fresnel's formulas). Plot schematically the reflection coefficient $R = \left|\tilde{E}_{0r}/E_{0i}\right|^2$ as a function of $\theta_i$ for the cases $n > 1$ and $n < 1$.

**c)** Suppose now that the incident wave is linearly polarized with polarization parallel to the plane of incidence. Again we assume that the reflected and transmitted waves also have a linear polarization parallel to the plane of incidence. Use the boundary conditions to find $\tilde{E}_{0r}$ and $\tilde{E}_{0t}$ as a function of $E_{0i}$ and $\theta_i$ (Fresnel's formulas). Show that there is no reflection at a certain value of $\theta_i$, called *Brewster's angle*. Plot schematically $R$ as a function of $\theta_i$ for the cases $n > 1$ and $n < 1$. What happens to an unpolarized beam incident at Brewster's angle?

**d)** At Brewster's angle, show that $\theta_r + \theta_t = 90°$. The incident wave induces oscillations of the bound electrons at the surface of the dielectric. These electrons then act as oscillating dipoles which are the source of the reflected and transmitted waves. Given that dipoles do not radiate energy in the direction of their dipole moment, can you explain why there is no reflection at Brewster's angle?

3.4 **Waves in the ionosphere**

The ionosphere is a region of the upper atmosphere which is ionized by solar (UV) radiation. It may be simply described as a dilute gas of charged particles, composed of electrons and ionized air ($N_2$ and $O_2$) molecules. The number density $n_e$ of free electrons is maximum at about 200 to 400 km above the surface of the Earth. At higher altitudes, $n_e$ decreases because the density of air molecules to be ionized decreases. At lower altitudes, $n_e$ decreases because UV radiation from the sun has been absorbed in layers above.

**a)** We assume that the ionosphere can be modelled as a collisionless plasma. Write the equation of motion for an electron assuming an external electric field and neglecting the local field. Show that the dielectric constant is given by $\epsilon_r = 1 - \omega_p^2/\omega^2$, with $\omega_p^2 = n_e e^2/(\epsilon_0 m)$, where $-e$ and $m$ are the electron's charge and mass, respectively. Write the dispersion relation $\omega(k)$. Show that for $\omega < \omega_p$, waves cannot propagate in the plasma.

**b)** Suppose that waves are emitted at the surface of the Earth toward the ionosphere. For waves with polarization both perpendicular and parallel to the plane of incidence, show that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is a total reflection back toward the Earth. [Use the results of Problem 3].

**c)** A radio amateur emits a wave with a wavelength of 21 m (corresponding to high frequency or shortwave broadcast) in the early evening. This wave is received by another radio amateur located 1000 km away, but not by other amateurs located closer. Assuming that the wave is being reflected by the so–called F layer of the ionosphere at a height of 300 km (the altitude at which reflection occurs can be determined from the travel time of the signal), calculate the electron density $n_e$. Compare with the known maximum and minimum F layer densities of $2 \times 10^6$ cm$^{-3}$ in the daytime and (2–4) $\times 10^5$ cm$^{-3}$ at night.

**d)** The cutoff frequency $\omega_p$ corresponding to $n_e \sim 10^5$–$10^6$ cm$^{-3}$ is of a few $10^7$ rad s$^{-1}$. FM radio and television signals have frequencies much larger than this value. What happens to these signals when they hit the ionosphere? Why are AM signals, with frequencies between 3 and 30 MHz, used for long distance (including intercontinental) communication?

**e)** From the BBC website: *AM reception can vary a great deal from day to night because of differences in the atmosphere. You may get good, clear reception during the day, but after sunset the signal may fade or become distorted. Signals travel further at night, so you may get interference from other transmitters, and you may even hear stations from outside the UK.* Why do radio signals travel farther at night than in the day?

3.5 **Waves in a good conductor**

We consider an ohmic conductor with conductivity $\sigma$, permittivity $\epsilon_0$ and permeability $\mu_0$. We assume that the conductor is excellent, that is to say $\sigma \gg \omega\epsilon_0$.

**a)** Show that the conductor can be described as a dielectric with complex dielectric constant $\tilde{\epsilon}_r = 1 + \mathrm{i}\sigma/(\omega\epsilon_0)$. An electromagnetic monochromatic plane wave propagates through the conductor. Write the dispersion relation and find the wavenumber in terms of the penetration depth $\delta = \sqrt{2/(\omega\mu_0\sigma)}$. Compare the group and phase velocities. Is dispersion normal or anomalous? Show that the impedance is $Z = (1 - \mathrm{i})/(\delta\sigma)$.

**b)** Calculate the time–average over a period of the Poynting vector, $\langle \boldsymbol{S} \rangle$, at a location $z$ along the direction of propagation of the wave. Calculate the time–average over a period of the rate at which work is done on the charges contained in the volume delimited by the surfaces at $z = z_1$ and $z = z_2$. Verify that it is equal to the time–average over a period of the flux of the Poynting vector into the volume.

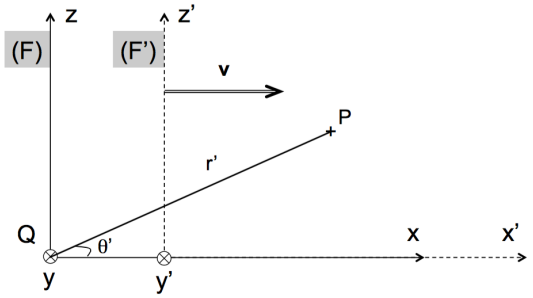
**c)** Calculate the time–average over a period of the energy density, $\langle u \rangle$. Is the energy carried mainly by the electric field or by the magnetic field? Calculate the velocity at which energy is transported (it is given by $|\langle \boldsymbol{S} \rangle| / \langle u \rangle$). Compare with the group velocity. Comment.

# Electromagnetism: Problems

**Problem sheet 4**: (Michaelmas Term 2021)

■ *Electromagnetism and special relativity* (*non-examinable)

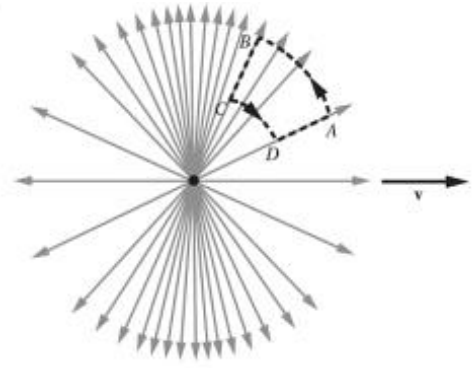
4.1 **Electric field of a point charge moving with constant velocity**
A point charge $Q$ is at rest at the origin in an inertial frame $(F)$. At a point $P$ with Cartesian coordinates $(x, 0, z)$ the field measured in $(F)$ is $\boldsymbol{E}$. We consider another inertial frame $(F')$ which moves in the positive $x$–direction with speed $\boldsymbol{v}$ with respect to $(F)$.




**a)** Find the electric field $\boldsymbol{E}'$ produced at $P$ by this charge as measured in $(F')$ and in terms of the polar coordinates $(r', \theta')$ in $(F')$ centered on the charge. Show that the field is parallel to the unit vector $\hat{\boldsymbol{r}}'$. Find $\boldsymbol{E}'$ in the limit $v \ll c$.

**b)** We are going to show that Gauss's law is satisfied in $(F')$. Choose as a Gaussian surface the surface of a sphere at rest in $(F')$ and centered on the charge at some particular time $t'$. Use spherical coordinates such that the polar angle $\theta'$ is measured from the $x'$–axis, so that there is azimuthal symmetry. Calculate the flux of $\boldsymbol{E}'$ through the surface of this sphere and show that it is equal to $Q/\epsilon_0$. You may use the equation:

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}}.$$

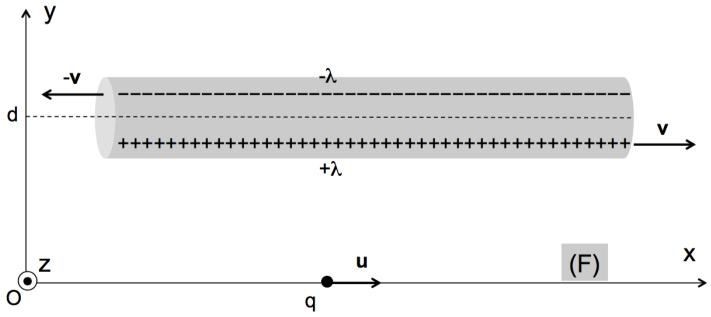



**c)** The figure (from E. Purcell & D. Morin, *Electricity and Magnetism*) shows a representation of the electric field in $(F')$ (the dot in the centre represents the charge, and the density of field lines indicates the intensity of the field). Justify that the circulation of the electric field along the closed path $ABCDA$ is non zero. Comment.

**d)** Calculate the magnetic field $\boldsymbol{B}'$ of the charge $Q$ as measured in $(F')$. [*Hint:* Use the fact that the magnetic field is zero in $(F)$.] Find $\boldsymbol{B}'$ in the limit $v \ll c$. What is the magnetic field of a charge moving at constant velocity $\boldsymbol{v}$?

4.2 **Interaction between a moving charge and other moving charges**

In an inertial frame $(F)$, we consider a line of positive charges all moving to the right with constant speed $v$. There is also a line of negative charges all moving to the left with the same speed $v$. The total charges of the positive and negative lines are equal and opposite.




We view the lines as continuous distributions with linear charge densities $+\lambda$ and $-\lambda$, as measured in $(F)$. This is an ideal representation of a wire containing both ions and electrons. At a distance $d$ from the axis of the wire, there is a charge $q$ which moves to the right with speed $u < v$.

**a)** Calculate the net charge and the net current in the wire. Find the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ at the position of the charge, and the force $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ on $q$, as measured in $(F)$.

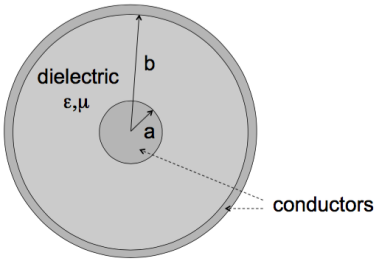
**b)** We now consider the inertial frame $(F')$ which moves to the right at speed $u$. The charge $q$ is at rest in $(F')$. Find the velocities $v'_+$ and $v'_-$ of the positive and negative lines in $(F')$ and show that $\gamma_\pm \equiv 1/\sqrt{1 - v'^2_\pm/c^2} = \gamma\,(1 \mp uv/c^2)\,/\sqrt{1 - u^2/c^2}$, where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$. Calculate the charge densities $\lambda'_\pm$ of the lines as measured in $(F')$ and the net charge $\lambda' = \lambda'_+ + \lambda'_-$ in terms of $\lambda$, $u$, $v$ and $c$. Find the electric field $\boldsymbol{E}'$ at the position of the charge, and the force $\boldsymbol{f}' = q\boldsymbol{E}'$ on $q$, as measured in $(F')$. Compare $\boldsymbol{f}$ and $\boldsymbol{f}'$.

■ *Transmission lines*

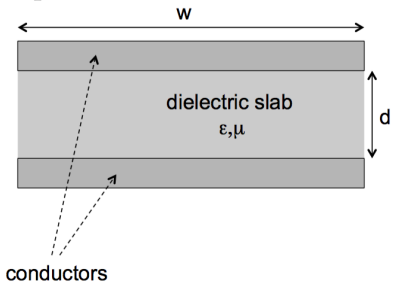
4.3 **Practical types of transmission lines**

Calculate the capacitance per unit length, the inductance per unit length and the speed at which signals propagate for the following transmission lines. In each case, the conductors are separated by a material of electric permittivity $\epsilon$ and magnetic permeability $\mu$. The figures show a cross section of the lines, which extend in the direction perpendicular to the figure.

**a)** Coaxial cable:




**b)** Strip line:



4.4 **Short and open circuited transmission lines**

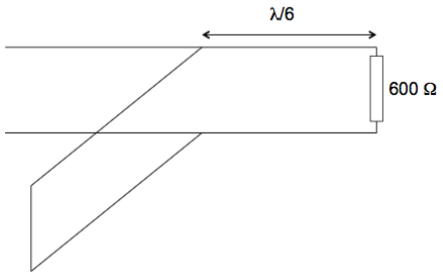
Calculate the input impedance $Z_1$ of a length $l$ of a loss-free transmission line of characteristic impedance $Z_0$ terminated by an open circuit. Calculate the input impedance $Z_2$ of the same transmission line terminated by a short circuit. Show that the lines can be used to create an equivalent inductor or capacitor. Show that $Z_1 Z_2 = Z_0^2$. Discuss the applications of such transmission lines (also called *stubs*).

4.5 **Power transmitted into a load**

A wave travels along a loss–free transmission line of characteristic impedance $Z_1$ which is terminated by a load of impedance $Z_2$. Show that the fraction of the incident power time–averaged over a wave period transmitted into the load is $t = 4Z_1 \,\mathrm{Re}(Z_2)/\left|Z_1 + Z_2\right|^2$.

4.6 **Transmission lines in parallel**

A transmission line with a characteristic impedance $Z_0 = 200\ \Omega$ is terminated by a resistor $R = 600\ \Omega$. Calculate the input admittance $Y$ (inverse of the impedance) of the line at one–sixth of a wavelength from the end.

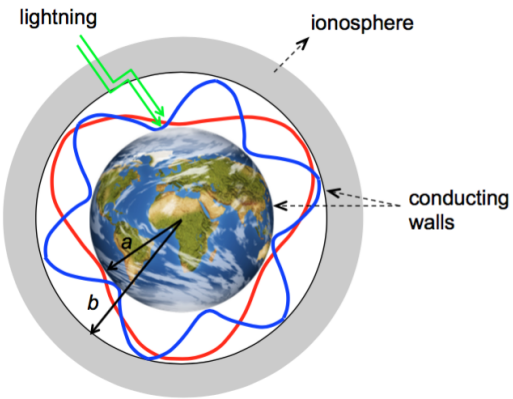



A short–circuited stub of line (with the same characteristic impedance) is added in parallel at this point. What should be its length in order to cancel the reflected signal travelling back towards the source? [*Hint:* Match the admittances.]

■ *Resonant cavities*

4.7 **Earth and ionosphere as a resonant cavity: Schumann resonances**

*This problem is taken from section 8.9 of Jackson, Classical Electrodynamics*
*This is a harder problem and is for enthusiasts only*




On a global scale, the Earth and the lower ionosphere can be idealized as two perfectly conducting, concentric spheres with radii $a$ and $b$. Therefore, they form a resonant electromagnetic cavity which can support resonant standing waves, as sketched on the figure. The resonant frequencies are called *Schumann resonances*, as they were first predicted by the physicist Winfried Otto Schumann in 1952 (although Tesla may have observed them before 1900).

The characteristic wavelength of the lowest frequency mode is on the order of the largest dimension of the cavity, that is to say on the order of magnitude of the Earth's circumference. The modes therefore have extremely low frequencies. Such modes can be excited by lightning and they are detected at various research stations around the world. The aim of this problem is to calculate the resonant frequencies of the Earth–ionosphere cavity. We will focus on TM modes, as they have the lowest frequencies (in spherical coordinates, TM and TE modes have no *radial* component of the magnetic and electric field, respectively). We note $(r, \theta, \varphi)$ the spherical coordinates and assume that the problem has azimuthal symmetry (no $\varphi$–dependence).

**a)** Using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ and Faraday's law $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, show that $B_\theta = 0$ and $E_\varphi = 0$. Write the boundary conditions at $r = a$ and $r = b$.

**b)** Assuming a time–dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, show that $\boldsymbol{B}$ satisfies the following equation:

$$\frac{\omega^2}{c^2}\boldsymbol{B} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{0},$$

and that the $\varphi$–component of this equation can be written in the form:

$$\frac{\omega^2}{c^2}(rB_\varphi) + \frac{\partial^2}{\partial r^2}(rB_\varphi) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(rB_\varphi)}{\partial\theta}\right) - \frac{rB_\varphi}{\sin^2\theta}\right] = 0.$$

**c)** We look for solutions to this equation of the form:

$$B_\varphi = \frac{F(r)}{r}G(\theta).$$

Show that $G(\theta)$ is the associated Legendre polynomial $P_l^m(\theta)$ with $m = \pm 1$. Since the two solutions $P_l^{-1}$ and $P_l^1$ only differ by a constant of proportionality, we fix $m = 1$. Write the components of the fields in terms of $F(r)$, $P_l^1(\cos\theta)$ and $P_l(\cos\theta)$, and the boundary conditions satisfied by $F$ at $r = a$ and $r = b$. You may use the equation:

$$\frac{d}{d\theta}\left[\sin\theta P_l^1(\cos\theta)\right] = -l(l+1)\sin\theta P_l(\cos\theta),$$

where $P_l(\cos\theta)$ is a Legendre polynomial.

**d)** Show that the differential equation satisfied by $F(r)$ is:

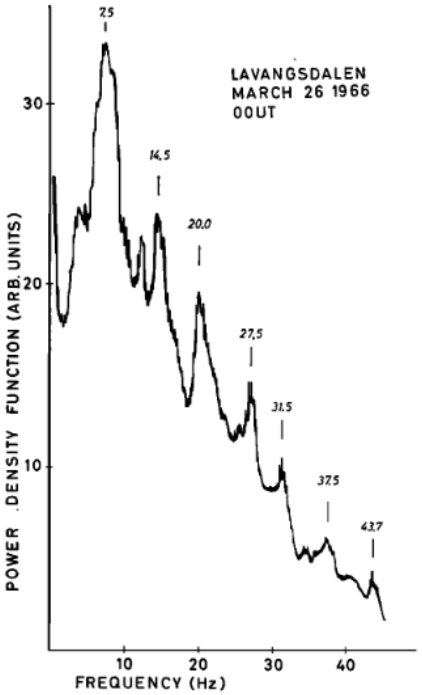
$$\frac{d^2F}{dr^2} + \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right]F = 0.$$

We assume that the height of the ionosphere is small compared to the radius of the Earth ($h = b - a \ll a$) so that $1/r^2$ can be approximated by $1/a^2$ in the above equation. Calculate the solution of the above equation that satisfies the boundary conditions at $r = a$ and $r = b$. Show that the (Schumann) resonant frequencies are:

$$\omega_{ln} = \frac{c}{a}\sqrt{l(l+1) + n^2\pi^2\frac{a^2}{h^2}},$$

where $n$ is a positive integer.

**e)** The Earth–ionosphere cavity is excited by lightning bolts which generate radial electric fields and contain a large spectrum of frequencies. The components with a frequency close to a Schumann frequency have the largest amplitude as they correspond to normal modes of the cavity.




This spectrum shows the first seven resonance modes (corresponding to $n = 0$). The resonant frequencies are 7.5, 14.5, 20.0, 27.5, 31.5, 37.5 and 43.7 Hz. The measurements shown on this figure were carried out at Lavangsdalen, in Norway (geographic latitude 70°N) in March 1966 (from T. R. Larsen & A. Egeland, *Fine Structure of the Earth–Ionosphere Cavity Resonances*, 1968, Journal of Geophysical Research, 73, 15).

Compare the measured frequencies with those calculated above. Comment.