Electromagnetism

Course handout

> *"I have also a paper afloat, with an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns."*
> James Clerk Maxwell (1831–1879)

> *"I stood on the shoulders of Maxwell."*
> Albert Einstein (1879–1955) [on being asked if he stood on the shoulders of Newton]

In this course we develop the subject of *electromagnetism* which was introduced in the first-year course. The lecture course consists of 24 lectures, all given in Michaelmas Term. The first four weeks have 4 or 5 lectures per week so we can make a really good start, but thereafter we drop back to 3 lectures per week.

Plan of the lectures

1. **Fields and potentials** (Lectures 1–4): *Maxwell's equations, boundary conditions, Laplace and Poisson equations, Laplace equation in spherical polars and cylindrical coordinates, multipole expansion*
2. **Electric polarization** (Lectures 5–6): *Bound charges, polarization P, the electric displacement D, linear dielectrics, boundary conditions*
3. **Magnetization** (Lectures 7–9): *Field from a current loop, magnetizable materials, diamagnetism, paramagnetism, ferromagnetism, magnetization M, B and H fields, hysteresis loops*
4. **Electromagnetic waves in materials** (Lectures 10–17): *Displacement currents, Maxwell's equations in insulating dielectrics and in conductors, Poynting vector, radiation pressure, refraction, reflection, Fresnel equations, Brewster angle, plasmas, dispersion*
5. **Confined electromagnetic waves** (Lectures 17–19): *Transmission lines, waveguides*
6. **Signals and noise** (Lectures 20–22): *Fourier transforms of signals, sampling, Johnson noise, shot noise*
7. **Special relativity and electromagnetism** (Lectures 23): *Lorentz transformation of E and B fields*
8. **Examples and summary** (Lecture 24)

Books: There are lots of helpful books out there on electromagnetism. Consult your tutor for recommendations, but excellent treatments may be found in:

- Introduction to Electrodynamics, 4th Edition: David J. Griffiths, Cambridge University Press [∼£45]
  (In my view, this is easily the best book on the subject and is highly recommended. Lively writing and exceptionally clear. If you have to choose one book on electromagnetism...)

- Classical Electrodynamics, 3rd Edition: John D. Jackson [∼£60]
  (The classic textbook, detailed and thorough, but I suspect it's more useful for occasional reference.)
- Electricity and Magnetism, 3rd Edition: Edward M. Purcell and David J. Morin [∼£45]
  (Clear and careful, though quite slow. May suit some tastes. Great diagrams though.)
- The Feynman Lectures on Physics, Volume II: Richard P. Feynman, Robert B. Leighton and Matthew Sands [£52]
  (Feynman's view on a subject is always slightly unconventional but never unrewarding.)
- Electricity and Magnetism, two volumes: B. I. Bleaney and B. Bleaney (Oxford University Press) [£34, £43]
  (Thorough treatment, though a little dated. Worth looking at if it's in your college library.)
- The classical theory of fields: L. D. Landau and E. M. Lifshitz (Elsevier); Electrodynamics of continuous media: L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii (Elsevier) [>£40 each]
  (volume 2 and volume of 8 of "Landau and Lifshitz", the ten-volume course of theoretical physics developed in the grand Russian school. These volumes are translated into English but are still moderately impenetratable for beginners, but there are new insights when you already understand a particular topic. It's in cgs units though, not SI.)

Synopsis of course (as it appears in the syllabus)

In this course, we will not cover the topics in the order they are given in the syllabus.

■ Electromagnetic waves in free space. Derivation of expressions for the energy density and energy flux (Poynting vector) in an electromagnetic field. Radiation pressure. ■ Magnetic vector potential. [Non-examinable: The change of E and B fields under Lorentz transformations in simple cases.] ■ Dielectric media, polarisation density and the electric displacement D. Dielectric permittivity and susceptibility. Boundary conditions on E and D at an interface between two dielectrics. Magnetic media, magnetisation density and the magnetic field strength H. Magnetic permeability and susceptibility; properties of magnetic materials as represented by hysteresis curves. Boundary conditions on B and H at an interface between two magnetic media. Maxwell's equations in the presence of dielectric and magnetic media. ■ Electromagnetic wave equation in dielectrics: refractive index and impedance of the medium. Reflection and transmission of light at a plane interface between two dielectric media. Brewster angle. Total internal reflection . [Non- examinable: Fresnel equations] The electromagnetic wave equation in a conductor: skin depth. Electromagnetic waves in a plasma; the plasma frequency. Dispersion and absorption of electromagnetic waves, treated in terms of the response of a damped classical harmonic oscillator. ■ Treatment of electrostatic problems by solution of Poisson's equation using separation of variables in Cartesian, cylindrical or spherical coordinate systems. ■ Theory of a loss-free transmission line: characteristic impedance and wave speed. Reflection and transmission of signals at connections between transmission lines and at loads; impedance matching using a quarter-wavelength transmission line. ■ Fourier transform of signals (digital). Nyquist theorem. Interpolation and signal reconstruction. Noise (Johnson noise, shot noise, $1/f$ noise, digitization noise, interference noise).

1. Fields and potentials (Lectures 1–4):

Maxwell's equations

Maxwell's equations in free space are

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0 \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}, \tag{4}$$

and describe the interrelationships between the electric field $\boldsymbol{E}$, the magnetic induction $\boldsymbol{B}$, the charge density ρ and the current density $\boldsymbol{J}$. Equation 1 shows that electric field diverges

away from positive charges and converges into negative charges; charge density therefore acts as a source or a sink of electric field. Equation 2 shows that magnetic fields have no such divergence; thus there are no magnetic charges (monopoles) and lines of $\boldsymbol{B}$ field must just exist in loops; they can never start or stop anywhere. Equation 3 shows that you only get loops of electric field around regions in space in which there is a changing magnetic field. This leads to Faraday's law of electromagnetic induction. Equation 4, in the absence of a changing electric field, shows that loops of magnetic induction are found around electric currents.

Laplace and Poisson equations

Laplace's equation is

$$\nabla^2 V = 0. \tag{5}$$

Poisson's equation is

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{6}$$

In Cartesian components, Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{7}$$

In spherical polars, Laplace's equation becomes [it's on the exam formula sheet!]

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0. \tag{8}$$

In cylindrical coordinates, Laplace's equation becomes [also on the formula sheet]

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{9}$$

Example:

A problem set up in the lecture considers a two-dimensional region on a rectangle of sides a and b. Three sides are fixed with potential V_0, with the fourth fixed at 0. The general solution is given by

$$V(x,y) = \sum_{k\neq 0}(A_k \mathrm{e}^{kx} + B_k \mathrm{e}^{-kx})(C_k \sin ky + D_k \cos ky) + (A_0 x + B_0)(C_0 y + D_0). \tag{10}$$

The problem with this equation is that there are lots of constants, and also k can take any value. We now apply the boundary conditions and this will make things much easier to solve:

- $V(x,0) = V_0$. This fixes $D_k = 0$ and also $A_0 = 0$.
- $V(x,b) = V_0$. This can be fixed by setting $k = n\pi/b$. We also have $C_0 = 0$ and $B_0 D_0 = V_0$.
- $V(0,y) = V_0$. This implies that $A_k = -B_k$. Putting this all together results in

$$V(x,y) = V_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b}. \tag{11}$$

- $V(a,y)=0$. We can use this condition to specify that the equation to be solved is

$$V(a,y)=0=V_0+\sum_{n=1}^{\infty}\mathcal{A}_n\sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b}. \tag{12}$$

We can multiply both sides by $\sin m\pi y/b$ and integrate between 0 and b. This gives

$$0=V_0\int_0^b\sin\frac{m\pi y}{b}\,\mathrm{d}y+\sum_{n=1}^{\infty}\mathcal{A}_n\sinh\frac{n\pi a}{b}\int_0^b\sin\frac{m\pi y}{b}\sin\frac{n\pi y}{b}\,\mathrm{d}y. \tag{13}$$

The integrals are

$$\int_0^b\sin\frac{m\pi y}{b}\,\mathrm{d}y=\frac{b}{m\pi}\left[-\cos\frac{m\pi y}{b}\right]_0^b=\begin{cases}0 & m\,\text{even}\\ \frac{2b}{m\pi} & m\,\text{odd}\end{cases} \tag{14}$$

and

$$\int_0^b\sin\frac{m\pi y}{b}\sin\frac{n\pi y}{b}\,\mathrm{d}y=\frac{b}{2}\delta_{mn}. \tag{15}$$

Thus

$$\mathcal{A}_m=\begin{cases}0 & m\,\text{even}\\ -\frac{4V_0}{m\pi}\frac{1}{\sinh\frac{m\pi a}{b}} & m\,\text{odd}.\end{cases} \tag{16}$$

The final result can then be written

$$V(x,y)=V_0\left(1-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sinh\frac{(2k+1)\pi x}{b}\sin\frac{(2k+1)\pi y}{b}}{(2k+1)\sinh\frac{(2k+1)\pi a}{b}}\right). \tag{17}$$

Solution of Laplace's equation in spherical polars

Use separation of variables. Write $V(r,\theta,\psi)=R(r)\Theta(\theta)\Phi(\phi)$, substitute it into eqn 8 and divide by V and multiply by $r^2\sin^2\theta$:

$$\sin^2\theta\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right]+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=0. \tag{18}$$

The final term on the LHS doesn't depend on r or θ, so the first two can't either, and they don't depend on ϕ, so the final one also can't. Thus the final term must be a constant, independent of r, θ and ϕ. Let's call it $-m^2$, so that

$$\frac{\partial^2\Phi}{\partial\phi^2}=-m^2\Phi\,\text{and hence}\,\Phi=\mathrm{e}^{\pm\mathrm{i}m\phi},\,\text{with}\,m\,\text{an integer so that}\,\Phi\,\text{is single-valued.} \tag{19}$$

We can tackle the r and θ parts the same way, and here we will define the constant value they take as $\ell(\ell+1)$, so that we now have to solve

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=\ell(\ell+1)R. \tag{20}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0, \tag{21}$$

To solve eqn 20, try $R = r^s$ which gives $s = \ell$ or $s = -(\ell+1)$, so that $R(r) = Ar^\ell + Br^{-(\ell+1)}$, where A and B are constants. Equation 21 is harder to solve, but in fact it is the general Legendre equation (this determined our choise of $\ell(\ell+1)$ as the constant) and so the solution can be written down as $\Theta(\theta) = P_\ell^m(\cos\theta)$, and is an associated Legendre polynomial (see below). Let us now assume azimuthal symmetry, so that $m = 0$, and the associated Legendre polynomial reduces to a Legendre polynomial (see below). Thus the general solution is (summing over all possible values of ℓ)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{22}$$

Legendre and associated Legendre polynomials (for reference only)

Legendre polynomials $P_\ell(x)$ are an orthogonal set of polynomials that can be derived from the formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell.$$

They are a solution to Legendre's differential equation which is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_\ell(x)}{\mathrm{d}x}\right] + \ell(\ell+1)P_\ell(x) = 0. \tag{23}$$

The first few polynomials are:

$P_0(x) =$	1
$P_1(x) =$	x
$P_2(x) =$	$\frac{1}{2}(3x^2-1)$
$P_3(x) =$	$\frac{1}{2}(5x^3-3x)$
$P_4(x) =$	$\frac{1}{8}(35x^4-30x^2+3)$

The **associated Legendre polynomials** $P_\ell^m(x)$ can be obtained via

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m P_\ell(x)}{dx^m} = \frac{1}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell.$$

They are a solution to the general Legendre differential equation which is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_\ell^m(x)}{\mathrm{d}x}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m(x) = 0. \tag{24}$$

The first few polynomials are:

$P_0^0(x) = 1$				
$P_1^{-1}(x) = \frac{1}{2}(1-x^2)^{1/2}$	$P_1^0(x) = x$	$P_1^1(x) = -(1-x^2)^{1/2}$		
$P_2^{-2}(x) = \frac{1}{8}(1-x^2)$	$P_2^{-1}(x) = \frac{x}{2}(1-x^2)^{1/2}$	$P_2^0(x) = \frac{1}{2}(3x^2-1)$	$P_2^1(x) = -3x(1-x^2)^{1/2}$	$P_2^2(x) = 3(1-x^2)$

When $m = 0$, they reduce to the ordinary Legendre polynomials: $P_\ell^0(x) = P_\ell(x)$. As functions of $\cos\theta$ they become:

$P_0^0(\cos\theta) = 1$				
$P_1^{-1}(\cos\theta) = \frac{1}{2}\sin\theta$	$P_1^0(\cos\theta) = \cos\theta$	$P_1^1(\cos\theta) = -\sin\theta$		
$P_2^{-2}(\cos\theta) = \frac{1}{8}\sin^2\theta$	$P_2^{-1}(\cos\theta) = \frac{1}{2}\cos\theta\sin\theta$	$P_2^0(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$	$P_2^1(\cos\theta) = -3\cos\theta\sin\theta$	$P_2^2(\cos\theta) = 3\sin^2\theta$

2. Electric polarization (Lectures 5–6):

- The potential $V(\boldsymbol{r})$ from a point dipole is $V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} = -\boldsymbol{p}\cdot\nabla\left(\frac{1}{4\pi\epsilon_0 r}\right)$.
- The polarization $\boldsymbol{P}$ is the dipole moment per unit volume, so that $\boldsymbol{P} = n\boldsymbol{p}$ where n is the number density of atoms and $\boldsymbol{p}$ is the dipole moment of each atom.
- The potential from a box of dipoles is as if there is a volume charge density $\rho_\mathrm{b} = -\boldsymbol{\nabla}\cdot\boldsymbol{P}$ and surface charge density $\sigma_\mathrm{b} = \boldsymbol{P}\cdot\hat{\boldsymbol{n}}$. These are bound charges.
- The electric displacement $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}.$$

  This implies that $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_\mathrm{f}$, where ρ_f is the free charge density.
- For a linear dielectric, $P = \epsilon_0\chi_\mathrm{e}E$, where χ_e is the electric susceptibility. $D = \epsilon_0\epsilon_\mathrm{r}E$ and so $\epsilon_\mathrm{r} = 1 + \chi_\mathrm{e}$ is the relative permittivity.
- The boundary conditions are: $E_\parallel$ and $D_\perp$ are continuous.

Relative permittivity ϵ_r and polarizability α

substance	ϵ_r
vacuum	1
He	1.000065
Ne	1.00013
H_2	1.0000254
Ar	1.000517
N_2 (gas)	1.000548
N_2 (liquid)	1.45
polyethylene	2.25
diamond	5.7–5.9
NaCl	5.9
Si	11.7
H_2O (liquid)	80.1
$SrTiO_3$	∼300
$CaCu_3Ti_4O_{12}$	12000

substance	$\alpha/(4\pi\epsilon_0)$ $(10^{-30}\,\mathrm{m}^3)$
H	0.667
Be	5.59
B	3.04
C	1.67
N	1.10
O	0.79
F	0.55
Li	24.3
Na	24.1
K	43.4
Rb	47.4
Cs	59.4
He	0.205
Ne	0.40
Ar	1.64
Kr	2.49
Xe	4.05

3. Magnetization (Lectures 7–9):

Here is the proof of a useful vector identity. First recall Stokes' theorem which states that

$$\int \boldsymbol{\nabla}\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}. \tag{25}$$

Let us suppose that A takes the form $\boldsymbol{A}(\boldsymbol{r}) = f(\boldsymbol{r})\boldsymbol{c}$ where $\boldsymbol{c} \neq \boldsymbol{c}(\boldsymbol{r})$ is a constant vector. Hence we have

$$\boldsymbol{\nabla}\times\boldsymbol{A} = f\boldsymbol{\nabla}\times\boldsymbol{c} - \boldsymbol{c}\times\boldsymbol{\nabla}f = -\boldsymbol{c}\times\boldsymbol{\nabla}f \tag{26}$$

since $\boldsymbol{\nabla} \times \boldsymbol{c} = 0$ (because $\boldsymbol{c}$ is a constant vector, its derivative vanishes). Stokes' theorem then gives us

$$\oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \boldsymbol{c} \cdot \oint f \, \mathrm{d}\boldsymbol{l} = -\int \boldsymbol{c} \times \boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{S} = -\boldsymbol{c} \cdot \int \boldsymbol{\nabla} f \times \mathrm{d}\boldsymbol{S}, \tag{27}$$

but this is true for any $\boldsymbol{c}$ and so

$$\boxed{\oint f \, \mathrm{d}\boldsymbol{l} = -\int \boldsymbol{\nabla} f \times \mathrm{d}\boldsymbol{S}.} \tag{28}$$

Now let us choose $f = \hat{\boldsymbol{r}} \cdot \boldsymbol{r}'$ and take the gradient and integral around a closed loop in the primed coordinates:

$$\oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') \, \mathrm{d}\boldsymbol{l}' = -\int \boldsymbol{\nabla}'(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') \times \mathrm{d}\boldsymbol{S}' = -\hat{\boldsymbol{r}} \times \int \mathrm{d}\boldsymbol{S}' \tag{29}$$

which works because $\nabla'(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') = \hat{\boldsymbol{r}}$. In summary:

$$\boxed{\oint \hat{\boldsymbol{r}} \cdot \boldsymbol{r}' \, \mathrm{d}\boldsymbol{l}' = \left(\int \mathrm{d}\boldsymbol{S}' \right) \times \hat{\boldsymbol{r}}.} \tag{30}$$

We will use this to show that the magnetic vector potential from a magnetic dipole is given by

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r^2} \boldsymbol{m} \times \hat{\boldsymbol{r}}, \tag{31}$$

where $\boldsymbol{m}$ is the magnetic dipole moment.

For example, if we put $\boldsymbol{m}$ parallel to z and use spherical polars, we have that

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r^2} m \sin\theta \, \hat{\boldsymbol{\phi}}, \tag{32}$$

and so

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(r \sin\theta \, A_\phi)\hat{\boldsymbol{r}} - \frac{1}{r \sin\theta} \frac{\partial}{\partial r}(r \sin\theta \, A_\phi)\hat{\boldsymbol{\theta}}, \tag{33}$$

because $A_r = A_\theta = 0$ and hence

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi} \left(\frac{2\cos\theta}{r^3} \hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3} \hat{\boldsymbol{\theta}} \right), \tag{34}$$

which is the same form as an electric dipole.

- The magnetization $\boldsymbol{M}$ is the magnetic dipole moment per unit volume, so that $\boldsymbol{M} = n\boldsymbol{m}$ where n is the number density of atoms and $\boldsymbol{m}$ is the magnetic dipole moment of each atom.

- The potential from a box of magnetic dipoles is as if there is a volume (bound) current density $\boldsymbol{J}_\mathrm{b} = -\boldsymbol{\nabla} \times \boldsymbol{M}$ and surface (bound) current density $\boldsymbol{K}_\mathrm{b} = \boldsymbol{M} \times \hat{\boldsymbol{n}}$.

- The magnetic field strength $\boldsymbol{H}$ is given in terms of the magnetic flux density $\boldsymbol{B}$ by

  $$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}).$$

  This implies that $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f}$, where $\boldsymbol{J}_\mathrm{f}$ is the free current density.

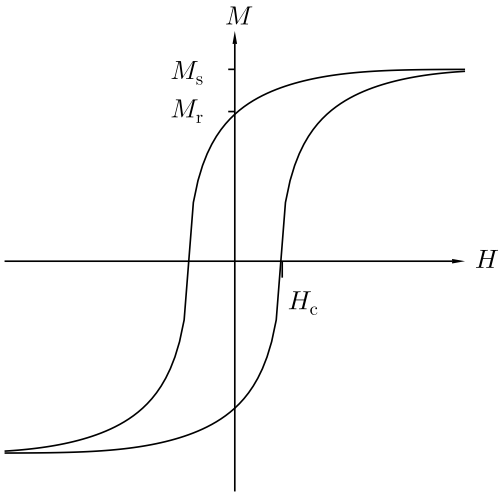
- For a linear material, $M = \chi_{\rm m} H$, where $\chi_{\rm m}$ is the magnetic susceptibility. $B = \mu_0(H + M) = \mu_0\mu_{\rm r} H$ and so $\mu_{\rm r} = 1 + \chi_{\rm m}$ is the relative permeability.
- The boundary conditions are: $H_\parallel$ and $B_\perp$ are continuous.

Magnetic scalar potential $\phi_{\rm m}$

If $\boldsymbol{J}_{\rm f} = 0$ everywhere, $\boldsymbol{\nabla} \times \boldsymbol{H} = 0$ and we can write $\boldsymbol{H} = -\boldsymbol{\nabla}\phi_{\rm m}$ and (for a linear material) $\boldsymbol{B} = -\mu_o\mu_{\rm r}\boldsymbol{\nabla}\phi_{\rm m}$. Away from interfaces, $\boldsymbol{\nabla}\cdot\boldsymbol{H} = 0$ and so $\boldsymbol{\nabla}^2\phi_{\rm m} = 0$ and we can use Laplace's equation formalism.

Relative permeability $\mu_{\rm r}$

substance	$\mu_{\rm r}$
air	1.00000037
platinum	1.000265
iron	∼5000
permalloy ($Ni_{0.80}Fe_{0.20}$)	up to 10^5



Coercive field $H_{\rm c}$

Soft magnetic materials: $H_{\rm c} \sim 0.1\,{\rm A\,m^{-1}}$. It is easy to move domain walls and these materials are useful for transformers, motors etc.

Hard magnetic materials: $H_{\rm c} \sim 10\,{\rm MA\,m^{-1}}$. It is hard to move domain walls and these materials are useful for permanent magnets.

The figure above shows a hysteresis loop with the coercive field $H_{\rm c}$, remanent magnetization $M_{\rm r}$, and saturation magnetization $M_{\rm s}$. This is obviously non-linear and so if you model it as $M = (\mu_{\rm r} - 1)H$ then $\mu_{\rm r} = \mu_{\rm r}(H)$.

4. Electromagnetic waves in materials (Lectures 10–17):

The Maxwell's equation $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\rm f}$ can be "upgraded" in the case of time-dependent fields to read

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\rm f} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{35}$$

This equation can also be written as

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}, \tag{36}$$

where

$$\boldsymbol{J} = \boldsymbol{J}_{\rm f} + \boldsymbol{J}_{\rm b} + \boldsymbol{J}_{\rm p} = \boldsymbol{J}_{\rm f} + \boldsymbol{\nabla} \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}. \tag{37}$$

In an insulating dielectric, Maxwell's equations give electromagnetic waves with speed c/n where $c = 1/\sqrt{\epsilon_0\mu_0}$ and refractive index $n = \sqrt{\epsilon_{\rm r}\mu_{\rm r}}$. The impedance Z is given by

$$Z = \sqrt{\frac{\mu_{\rm r}}{\epsilon_{\rm r}}} Z_0 \quad \text{where the impedance of free space is} \quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega. \tag{38}$$

Conductors (Lecture 11)
In a conductor we have that $\boldsymbol{J} = \sigma \boldsymbol{E}$ where σ is the conductivity. We then have

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \mu_\mathrm{r} \sigma \boldsymbol{E} + \mu_0 \mu_\mathrm{r} \epsilon_0 \epsilon_\mathrm{r} \frac{\partial \boldsymbol{E}}{\partial t}, \tag{39}$$

which is a sum of the conduction current and the displacement current. It is a good conductor if the first term dominates ($\omega \lesssim \sigma/(\epsilon_0 \epsilon_\mathrm{r})$ and a poor conductor if the second term dominates. The wave equation becomes

$$\nabla^2 \boldsymbol{E} = \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{40}$$

(where $\epsilon = \epsilon_0 \epsilon_\mathrm{r}$ and $\mu = \mu_0 \mu_\mathrm{r}$) and that yields wave solutions such as

$$\boldsymbol{E}(z,t) = E_0 \hat{\boldsymbol{x}} \mathrm{e}^{\mathrm{i}(\tilde{k} z - \omega t)}, \tag{41}$$

with $\tilde{k} = k + \mathrm{i}\kappa$. Hence

$$\tilde{k}^2 = \mu\epsilon\omega^2 + \mathrm{i}\mu\sigma\omega. \tag{42}$$

Note however that

$$\tilde{k}^2 = (k + \mathrm{i}\kappa)^2 = k^2 - \kappa^2 + \mathrm{i}(2k\kappa), \tag{43}$$

and so equating real and imaginary parts gives

$$k^2 - \kappa^2 = \mu\epsilon\omega^2, \tag{44}$$

$$2k\kappa = \mu\sigma\omega \tag{45}$$

and hence eqn 45 gives $k = \mu\sigma\omega/(2\kappa)$, which when substituted back into eqn 44 gives a quadratic in κ^2, i.e.

$$\kappa^4 + \mu\epsilon\omega^2\kappa^2 - \frac{1}{4}(\mu\sigma\omega)^2 = 0. \tag{46}$$

The solution of this yields

$$\kappa = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2} \tag{47}$$

$$k = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2} \tag{48}$$

and these values can be inserted into

$$\boldsymbol{E}(z,t) = E_0 \hat{\boldsymbol{x}} \mathrm{e}^{-\kappa z} \mathrm{e}^{\mathrm{i}(kz - \omega t)}. \tag{49}$$

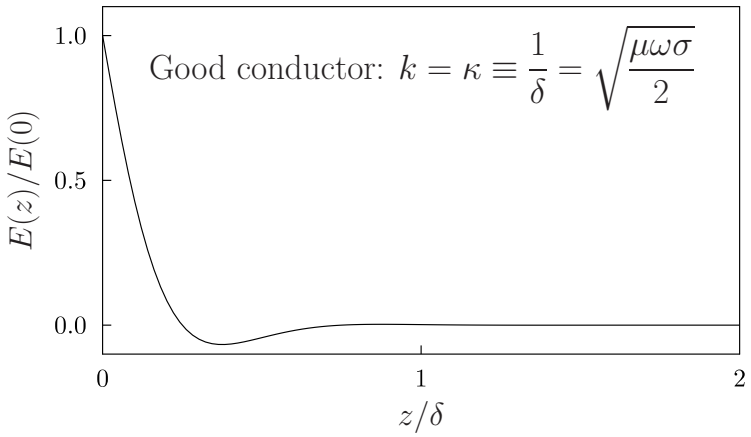
Note the wave contains an oscillatory part and a decaying part, with the decaying part behaving like $\mathrm{e}^{-z/\delta}$ where $\delta = 1/\kappa$ is the skin depth.

For a good conductor $\sigma \gg \epsilon\omega$, we have that

$$k = \kappa = \sqrt{\frac{\mu\omega\sigma}{2}} \tag{50}$$

and so the skin depth is

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}. \tag{51}$$



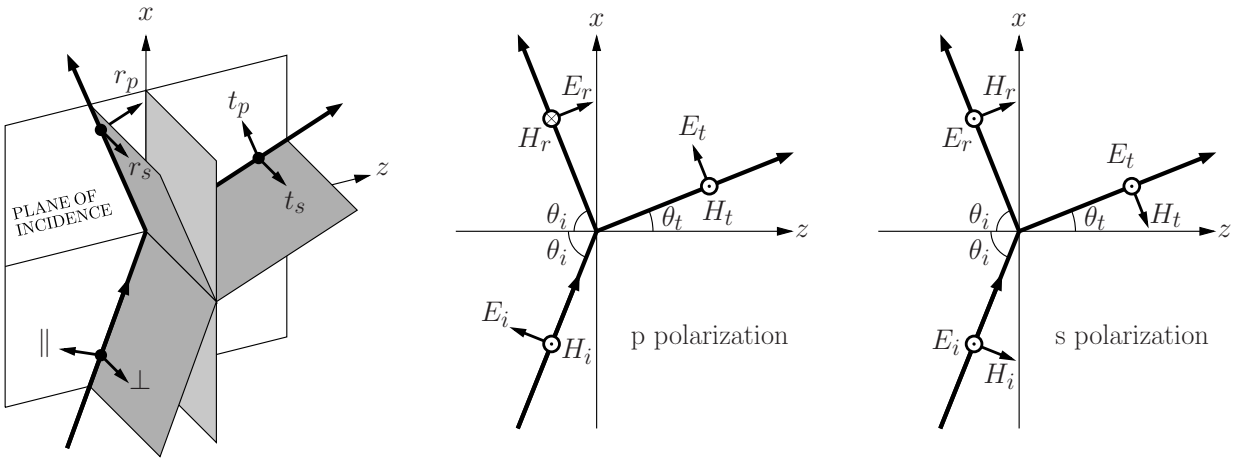
Poynting vector and radiation pressure (Lecture 12)

- The Poynting vector is given by $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ and measures the rate of flow of energy per unit area.
- Because electromagnetic waves are made up of photons with energy $E = pc$, transport of energy is accompanied by transport of momentum. The radiation pressure is given by $\langle S \rangle / c$ for radiation incident on a perfect absorber, or $2\langle S \rangle / c$ for radiation incident on a perfect reflector.

Electromagnetic waves – reflection and refraction (Lectures 13–15)

The boundary conditions ($E_\parallel$ and $H_\parallel$ being continuous) at the interface between two dielectric materials are used to give the well-known laws of reflection ($\theta_\mathrm{i} = \theta_\mathrm{r}$) and refraction, the latter of which is Snell's law:

$$\frac{\sin\theta_\mathrm{t}}{\sin\theta_\mathrm{i}} = \frac{k_1}{k_2} = \frac{n_1}{n_2}. \tag{52}$$



The reflection and transmission coefficients will be derived for the two cases of

- $\boldsymbol{E}$ in the plane of incidence (p-polarization, coming from "parallel")

- $\boldsymbol{E}$ perpendicular to the plane of incidence (s-polarization, coming from "*senkrecht*", the German for perpendicular)

The results are the **Fresnel equations**:

$\boldsymbol{E} \parallel$ plane of incidence	$\boldsymbol{E} \perp$ plane of incidence
$r_p = \dfrac{E_\mathrm{r}}{E_\mathrm{i}} = \dfrac{Z_2 \cos\theta_\mathrm{t} - Z_1 \cos\theta_\mathrm{i}}{Z_2 \cos\theta_\mathrm{t} + Z_1 \cos\theta_\mathrm{i}}$	$r_s = \dfrac{E_\mathrm{r}}{E_\mathrm{i}} = \dfrac{Z_2 \cos\theta_\mathrm{i} - Z_1 \cos\theta_\mathrm{t}}{Z_2 \cos\theta_\mathrm{i} + Z_1 \cos\theta_\mathrm{t}}$
$t_p = \dfrac{E_\mathrm{t}}{E_\mathrm{i}} = \dfrac{2Z_2 \cos\theta_\mathrm{i}}{Z_2 \cos\theta_\mathrm{t} + Z_1 \cos\theta_\mathrm{i}}$	$t_s = \dfrac{E_\mathrm{t}}{E_\mathrm{i}} = \dfrac{2Z_2 \cos\theta_\mathrm{i}}{Z_2 \cos\theta_\mathrm{i} + Z_1 \cos\theta_\mathrm{t}}$

If we set $\mu_\mathrm{r} = 1$, then $Z = \sqrt{\mu_\mathrm{r}\mu_0/\epsilon_\mathrm{r}\epsilon_0} = Z_0/n$ where $n = \sqrt{\epsilon_\mathrm{r}}$ is the refractive index and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. We can replace Z_i with $1/n_i$ in expressions involving ratios of impedances, and hence obtain:

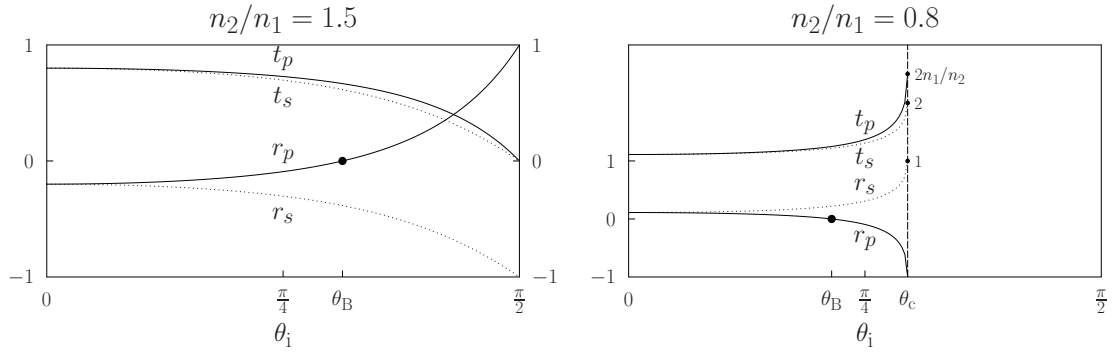
$r_p = \dfrac{n_1 \cos\theta_\mathrm{t} - n_2 \cos\theta_\mathrm{i}}{n_1 \cos\theta_\mathrm{t} + n_2 \cos\theta_\mathrm{i}}$	$r_s = \dfrac{n_1 \cos\theta_\mathrm{i} - n_2 \cos\theta_\mathrm{t}}{n_1 \cos\theta_\mathrm{i} + n_2 \cos\theta_\mathrm{t}}$
$t_p = \dfrac{2n_1 \cos\theta_\mathrm{i}}{n_1 \cos\theta_\mathrm{t} + n_2 \cos\theta_\mathrm{i}}$	$t_s = \dfrac{2n_1 \cos\theta_\mathrm{i}}{n_1 \cos\theta_\mathrm{i} + n_2 \cos\theta_\mathrm{t}}$

Using Snell's law ($n_1 \sin\theta_\mathrm{i} = n_2 \sin\theta_\mathrm{t}$) can lead to further simplifications. A particularly compact form can be obtained using

$$\alpha = \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}} = \frac{1}{\cos\theta_\mathrm{i}}\sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_i\right)^2} \quad \text{and} \quad \beta = \frac{n_2}{n_1}$$

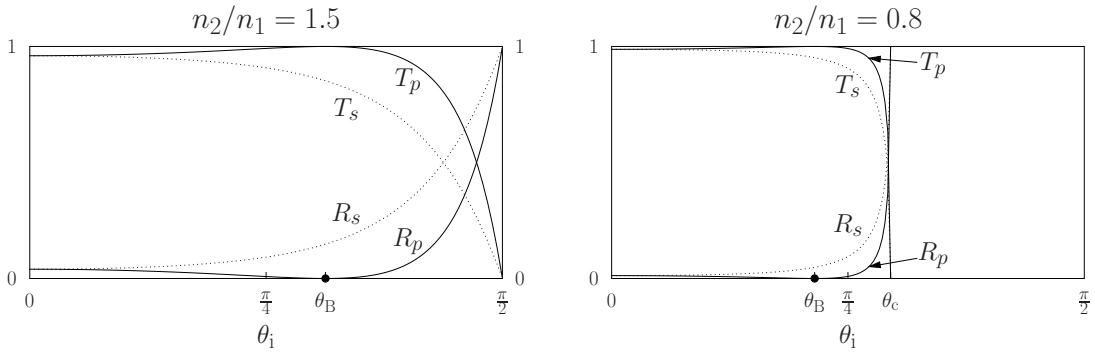
yielding

$r_p = \dfrac{\alpha - \beta}{\alpha + \beta}$	$r_s = \dfrac{1 - \alpha\beta}{1 + \alpha\beta}$
$t_p = \dfrac{2}{\alpha + \beta}$	$t_s = \dfrac{2}{1 + \alpha\beta}$



The intensity reflection and transmission coefficients are given as:

$R_p = \left(\dfrac{\alpha - \beta}{\alpha + \beta}\right)^2$	$R_s = \left(\dfrac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2$
$T_p = \alpha\beta\left(\dfrac{2}{\alpha + \beta}\right)^2$	$T_s = \alpha\beta\left(\dfrac{2}{1 + \alpha\beta}\right)^2$



Note the existence of the *Brewster angle* θ_B at which $R_p = r_p = 0$. Note also the critical angle for total internal reflection θ_c (relevant only for $n_2 < n_1$).

Plasmas (Lecture 16)

A plasma is a neutral gas of ionised particles. Plasmas are found in lots of places. Examples include

- the electron gas in a metal
- the ionosphere
- stars
- fusion reactors
- interstellar gas / intergalactic medium
- supernova remnants
- radio galaxies and quasars
- lightning, aurorae
- fire
- plasma displays

We will show that the permittivity of a cold plasma is given by

$$\epsilon_\mathrm{r}(\omega) = 1 - \left(\frac{\omega_\mathrm{p}}{\omega}\right)^2, \tag{53}$$

where the plasma frequency ω_p is

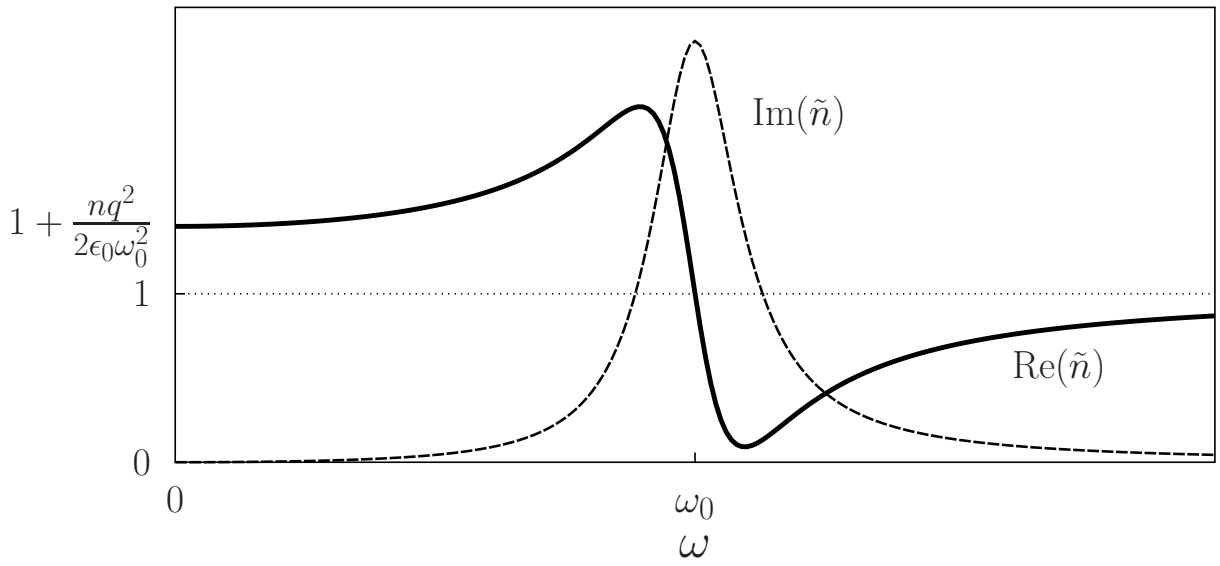
$$\omega_\mathrm{p}^2 = \frac{ne^2}{\epsilon_0 m} \tag{54}$$

and is the frequency of longitudinal vibrations in the plasma. Transverse waves are also possible, with dispersion relation

$$\omega^2 = c^2k^2 + \omega_\mathrm{p}^2. \tag{55}$$

Dispersion (Lecture 17)

We will model the response of electrons in a material to an applied electromagnetic wave using a damped simple harmonic oscillator.



The drop in $\mathrm{Re}(\tilde{n})$ is called anomalous dispersion (because generally the refractive index increases with frequency).

5. Confined electromagnetic waves (Lectures 17–19):

Transmission lines (Lectures 17 [second half], 18)

A transmission line has capacitance C and inductance L, *both per unit length*, leading to a wave equation

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}, \tag{56}$$

and hence waves with speed $1/\sqrt{LC}$. The characteristic impedance Z of the transmission line is given by

$$Z = \pm\sqrt{\frac{L}{C}}. \tag{57}$$

One example of a transmission line is a coaxial cable which has

$$C = \frac{2\pi\epsilon_0\epsilon_\mathrm{r}}{\ln\frac{b}{a}} \qquad L = \frac{\mu_0\mu_\mathrm{r}}{2\pi}\ln\frac{b}{a}. \tag{58}$$

The reflection and transmission coefficients at the boundary between two transmission lines with impedances Z_1 and Z_2 are

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \qquad t = \frac{2Z_2}{Z_2 + Z_1}. \tag{59}$$

- We have normally been using the 'physics' convention for waves in this course, writing the complex wave as being proportional to $\mathrm{e}^{\mathrm{i}(kz-\omega t)}$. (The choice here is in the sign of the term in the exponential.) Physicists like this since, for example, it gives the correct answers in the Schrödinger equation when you write the momentum operator as $\hat{p} = -\mathrm{i}\hbar\,\partial/\partial x$. For transmssion lines, we will instead use the 'circuit theory' convention: $\mathrm{e}^{\mathrm{i}(\omega t-kz)}$. This has the virtue of making positive imaginary impedances correspond to inductances, with negative imaginary impedances come out as capacitances.

For a transmission line with characeristic impedance Z_c terminated by a load Z_T, one can write

$$r = \frac{Z_\mathrm{T} - Z_\mathrm{c}}{Z_\mathrm{T} + Z_\mathrm{c}}, \qquad t = \frac{2Z_\mathrm{T}}{Z_\mathrm{T} + Z_\mathrm{c}}. \tag{60}$$

A short section of transmission line (characteristic impedance Z_c, length ℓ, terminated by Z_T) has input impedance

$$Z_\mathrm{in} = Z_\mathrm{c}\left(\frac{Z_\mathrm{T}\cos k\ell + \mathrm{i}Z_\mathrm{c}\sin k\ell}{Z_\mathrm{c}\cos k\ell + \mathrm{i}Z_\mathrm{T}\sin k\ell}\right) \tag{61}$$

- If $Z_\mathrm{T} = 0$ then $Z_\mathrm{in} = \mathrm{i}Z_\mathrm{c}\tan k\ell$.
- If $Z_\mathrm{T} = \infty$ then $Z_\mathrm{in} = -\mathrm{i}Z_\mathrm{c}\cot k\ell$.

Waveguides (Lecture 19)

Consider a rectangular waveguide with a cross section in the xy plane and defined by $0 \le x \le a$ and $0 \le y \le b$, so that waves can travel along the z-direction. The walls are conducting and since E and B are both time varying ($\sim \mathrm{e}^{-\mathrm{i}\omega t}$), then $E = 0$ and $B = 0$ inside the conductor. Since $B_\perp$ and $E_\parallel$ are continuous at an interface, this means that both $B_\perp$ and $E_\parallel$ must be zero at the surface of the waveguide. This provides the boundary conditions we need.

One set of solutions have the electric field transverse to the waveguide (these are called **TE modes**) and so $E_z = 0$. Since $E_\parallel = 0$ at the walls, this means that we have $E_x = 0$ at $y = 0, b$ and $E_y = 0$ at $x = 0, a$. Maxwell's equations *inside* the waveguide are

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} = \mathrm{i}\omega\boldsymbol{B}, \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = -\mathrm{i}\frac{\omega}{c^2}\boldsymbol{E}, \tag{62}$$

and so

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \boldsymbol{\nabla}^2\boldsymbol{E} = \mathrm{i}\omega\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{\omega^2}{c^2}\boldsymbol{E}, \tag{63}$$

and hence

$$\boldsymbol{\nabla}^2\boldsymbol{E} + \frac{\omega^2}{c^2}\boldsymbol{E} = 0. \tag{64}$$

Since, for a TE mode, $\boldsymbol{E}$ is purely transverse, we can write this as

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2}E_x = 0 \tag{65}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + \frac{\omega^2}{c^2}E_y = 0, \tag{66}$$

and $E_z = 0$. The equations for E_x and E_y can be solved by separation of variables, and the z part factors out. Thus we obtain

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{\mathcal{E}}(x, y)\mathrm{e}^{\mathrm{i}(k_\mathrm{g} z - \omega t)}, \tag{67}$$

where $\boldsymbol{\mathcal{E}}(x, y) = E_x(x, y)\hat{\boldsymbol{x}} + E_y(x, y)\hat{\boldsymbol{y}}$ is a vector field that lies in the xy plane and only depends on x and y. Therefore

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_\mathrm{c}^2\right]\boldsymbol{\mathcal{E}}(x, y) = 0, \tag{68}$$

where $k_c^2 = \frac{\omega^2}{c^2} - k_g^2$. Putting in the boundary condition for this equation gives solutions

$$E_x \propto \sin \frac{n\pi y}{b} \tag{69}$$

$$E_y \propto \sin \frac{m\pi x}{a}, \tag{70}$$

where m and n are integers. Thus we can write

$$E_x = f(x) \sin \frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)} \tag{71}$$

$$E_y = g(y) \sin \frac{m\pi x}{a} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)}, \tag{72}$$

where $f(x)$ and $g(y)$ are functions to be determined. The equation $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ gives us

$$\frac{\partial f}{\partial x} \sin \frac{n\pi y}{b} + \frac{\partial g}{\partial y} \sin \frac{m\pi x}{a} = 0, \tag{73}$$

and we can make the left hand side vanish by choosing

$$\frac{\partial f}{\partial x} = -\sin \frac{m\pi x}{a} \quad \text{and} \quad \frac{\partial g}{\partial y} = \sin \frac{n\pi y}{b}, \tag{74}$$

and integrating gives

$$f = \frac{a}{m\pi} \cos \frac{m\pi x}{a} \quad \text{and} \quad g = -\frac{b}{n\pi} \cos \frac{n\pi y}{b}, \tag{75}$$

Substituting these in and multiplying f and g by $\frac{m\pi}{a}\frac{n\pi}{b}A$ (where A is a constant) gives

$$E_x = A\frac{n\pi}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)}, \tag{76}$$

$$E_y = -A\frac{m\pi}{a} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)}, \tag{77}$$

$$E_z = 0, \tag{78}$$

where A is a constant and $m \geq 0$ and $n \geq 0$ are integers. These solutions lead to the identification

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{79}$$

The magnetic field components can be obtained using $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}} = \mathrm{i}\omega \boldsymbol{B}$:

$$B_x = A\frac{k_g}{\omega}\frac{m\pi}{a} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)}, \tag{80}$$

$$B_y = A\frac{k_g}{\omega}\frac{n\pi}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)}, \tag{81}$$

$$B_z = -\frac{A}{\mathrm{i}\omega}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}(k_g z - \omega t)}. \tag{82}$$

The dispersion relation of the waveguide is

$$\omega^2 = c^2(k_g^2 + k_c^2). \tag{83}$$

The phase velocity is $v_p = \frac{\omega}{k} = c\left(1 + \frac{k_c^2}{k_g^2}\right)^{1/2}$ and the group velocity is $v_g = \frac{\partial \omega}{\partial k} = c\left(1 + \frac{k_c^2}{k_g^2}\right)^{-1/2}$. The TE modes are labelled by the indices m and n and so are often written TE_{mn}. TE_{00} does not exist, but the lowest energy mode is TE_{10} (assuming $(a > b)$. It is also possible to find TM modes (for which the magnetic field is transverse) but these are TM_{mn} with both $m > 0$ and $n > 0$ because $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ always (for TE modes, you can have free electric charges on the waveguide walls, but you can't have magnetic monopoles anywhere).

5. Signals and noise (Lectures 20–22):

Fourier transforms of signals

A time-dependent function $f(t)$, such as a voltage signal, is related to its Fourier transform $\tilde{f}(\omega)$ by

$$\tilde{f}(\omega) = A \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t. \tag{84}$$

The inverse Fourier transform is then

$$f(t) = B \int_{-\infty}^{\infty} \tilde{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{85}$$

The constants A and B satisfy $AB = 1/(2\pi)$, so we often choose

$$A = B = \frac{1}{\sqrt{2\pi}}. \tag{86}$$

This works well in quantum mechanics, but here we will choose $A = 1$ and $B = 1/(2\pi)$, so that

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \tag{87}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{88}$$

We say that a signal is *band limited* to $\omega_{\max}$ if the highest frequencies in the signal are $\omega_{\max}$. A delta function $\delta(t)$ has Fourier transform $\tilde{\delta}(\omega) = 1$ and is not band limited.

If a signal $f(t)$ is band limited to $\omega_{\max} = 2\pi f_{\max}$, then it can be completely determined by a sampled version with a sampling period T that satisfied

$$T \leq \frac{1}{2f_{\max}}. \tag{89}$$

(Equivalently, the sampling frequency $f_{\mathrm{s}} = 1/T$ must satisfy $f_{\mathrm{s}} \geq 2f_{\max}$.) Another way of stating this result is that the highest frequency in a signal that can be accurately represented by sampling is the *Nyquist frequency* given by

$$f_{\mathrm{Nyquist}} = \frac{f_{\mathrm{Nyquist}}}{2} = \frac{1}{2T}, \tag{90}$$

which can also be written

$$\omega_{\mathrm{Nyquist}} = 2\pi f_{\mathrm{s}} = \frac{\pi}{T}. \tag{91}$$

The autocorrelation of a function $f(t)$ is defined by

$$\gamma(t) = \int f(\tau) f^*(\tau - t)\, \mathrm{d}\tau. \tag{92}$$

Its Fourier transform is then

$$\begin{aligned} \tilde{\gamma}(\omega) &= \int \int f(\tau) f^*(\tau - t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\tau\, \mathrm{d}t \\ &= \int f(\tau) \mathrm{e}^{-\mathrm{i}\omega\tau}\, \mathrm{d}\tau \int f^*(\tau - t) \mathrm{e}^{\mathrm{i}\omega(\tau - t)}\, \mathrm{d}t \\ &= \tilde{f}(\omega)\tilde{f}^*(\omega) = |\tilde{f}(\omega)|^2. \end{aligned} \tag{93}$$

This is the power spectrum of $f(t)$. Thus, the Fourier transform of the autocorrelation function is the power spectrum, a result which is known as the Wiener-Khinchin theorem.

Noise

There are various sources of noise that can affect electrical measurements.

- The *Johnson noise* in a resistor R at temperature T is given by

$$\langle V^2 \rangle = 4k_{\rm B}TR\Delta f, \tag{94}$$

  where Δf is the frequency interval in which the noise is being measured and where $k_{\rm B}$ is Boltzmann's constant (see the Thermal Physics course). This implies a spread of voltage values about an average $\langle V \rangle = 0$, so that the rms voltage is $V_{\rm rms} = \sqrt{\langle V^2 \rangle} = \sqrt{4k_{\rm B}TR\Delta f}$. This is a thermal effect and comes from the result that the mean thermal energy available for an individual degree of freedom at temperature T is given by

$$k_{\rm B}T. \tag{95}$$

  If one includes quantum mechanics, then the factor $k_{\rm B}T$ can be replaced by

$$\frac{\hbar\omega}{{\rm e}^{\hbar\omega/k_{\rm B}T} - 1}. \tag{96}$$

- *Shot noise* leads to a frequency independent contribution to the current density given by

$$\langle I^2 \rangle = 2e\langle I \rangle \, \Delta f, \tag{97}$$

  where this corresponds to a spread in values about a non-zero current flow (average value $\langle I \rangle \neq 0$).

- Other internal sources of noise (that is, internal to the circuit) include $1/f$ *noise* and *digitization noise* which will be discussed in the lectures.

- There are also *external* sources of noise, which can introduce unwanted voltages into the circuit of interest by capacitive or inductive coupling. The ground of the circuit can ride up and down if there is a shared current path on a common ground (these are known as *ground loops*). The DC supply of your circuit might have a ripple on it due to an imperfect transformation of AC power to the DC supply. In this case, there could be unwanted noise at 50 Hz (and perhaps also at some integer multiple of it), where 50 Hz is the frequency of the mains electricity (in the UK and many other countries). Often, external sources give rise to noise at specific frequencies. Such sources of *interference noise* can often be eliminated by careful filtering if the interference is at a specific frequency.

Different noise sources add in quadrature to give the total noise in the circuit. A useful metric is the *signal-to-noise ratio* (SNR) which is obtained by

$$\text{SNR} = \frac{V_{\rm RMS}^{\rm signal}}{V_{RMS}^{\rm noise}}. \tag{98}$$

6. Relativity (Lecture 23)

Some revision: Many quantities in special relativity are defined in terms of **four vectors**. The components of the four vectors transform according to the **Lorentz transformation**. The dot product between two four vectors is **invariant**, meaning that the dot product takes the same value in all inertial frames.

The spacetime four vector x^μ can be written

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \boldsymbol{r}), \tag{99}$$

where $\boldsymbol{r} = (x^1, x^2, x^3) = (x, y, z)$ is the spatial position. The index μ takes the values $0, 1, 2, 3$ for the four spacetime coordinates. The dot product between x^μ and itself is written

$$x^\mu x_\mu = c^2 t^2 - r^2, \tag{100}$$

where the $x^\mu x_\mu$ notation (rather than writing $x^\mu x^\mu$) is used for a similar reason to the $\langle\psi|\psi\rangle$ notation in quantum mechanics (one can write $x_\mu = (ct, -\boldsymbol{r})$). Note also that (i) the summation convention is applied; (ii) there is an alternative convention (mainly used in general relativity) in which the dot product takes the opposite sign.

Other four vectors include:

- The velocity four vector u^μ is given by

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \gamma(c, \boldsymbol{v}), \tag{101}$$

  where $\tau = t/\gamma$ is the proper time. The dot product between u^μ and itself is given by

$$u^\mu u_\mu = \gamma^2(c^2 - v^2) = c^2. \tag{102}$$

- The momentum four vector p^μ is written as

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right), \tag{103}$$

  and the invariant is given by $p^\mu p_\mu = E^2/c^2 - p^2 = m^2 c^2$ where m is the rest mass. The momentum four vector $p^\mu = m u^\mu$ and so $p^\mu = \gamma m(c, \boldsymbol{v})$ giving

$$E = \gamma m c^2 \quad \text{and} \quad \boldsymbol{p} = \gamma m \boldsymbol{v}. \tag{104}$$

- The differential operator ∂_μ (the four-dimensional generalisation of ∇) is written

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \tag{105}$$

  and note that this operator is written with a downstairs, not an upstairs, index.

- The current density four vector J^μ is given by

$$J^\mu = (c\rho, \boldsymbol{J}), \tag{106}$$

where ρ is the charge density and $\boldsymbol{J} = \rho\boldsymbol{v}$ is the usual current density three vector. If we write $\rho = \gamma\rho_0$ where ρ_0 is the rest frame charge density, then

$$J^\mu = \rho(c, \boldsymbol{v}) = \gamma\rho_0(c, \boldsymbol{v}) = \rho_0 u^\mu, \tag{107}$$

and hence $J^\mu J_\mu = \rho_0^2 c^2$. This obeys a continuity equation

$$\partial_\mu J^\mu = 0 \qquad \text{or, in components,} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0. \tag{108}$$

- The force four vector F^μ is given by

$$F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}p^\mu}{\mathrm{d}t} = \gamma\left(\frac{\dot{E}}{c}, \boldsymbol{F}\right), \tag{109}$$

where $\boldsymbol{F}$ is the usual force three vector. The force four vector can also be written as $F^\mu = m\mathrm{d}u^\mu/\mathrm{d}\tau$. Thus the dot product $F^\mu u_\mu$ is

$$F^\mu u_\mu = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(u^\mu u_\mu) = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}c^2 = 0, \tag{110}$$

but equivalently

$$F^\mu u_\mu = \gamma^2(\dot{E} - \boldsymbol{F}\cdot\boldsymbol{v}), \tag{111}$$

and so $\dot{E} = \boldsymbol{F}\cdot\boldsymbol{v}$, and hence

$$F^\mu = \gamma(\frac{\boldsymbol{F}\cdot\boldsymbol{v}}{c}, \boldsymbol{F}). \tag{112}$$

- The potential four vector A^μ is given by

$$A^\mu = \left(\frac{V}{c}, \boldsymbol{A}\right). \tag{113}$$

The components of A^μ are related to the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ by

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \qquad \text{and} \qquad \boldsymbol{E} = -\boldsymbol{\nabla}V - \frac{\partial\boldsymbol{A}}{\partial t}, \tag{114}$$

but the same fields are obtained if $\boldsymbol{A}$ are V are subject to a gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi \qquad \text{and} \qquad V \to V - \frac{\partial\chi}{\partial t}, \tag{115}$$

which is equivalent to $A_\mu \to A_\mu - \partial_\mu\chi$, where χ is a scalar function. This analysis leads to the conclusion that the electric and magnetic fields cannot be combined into a four vector (they actually have to be expressed in terms of a four tensor).

Some example questions (Lecture 24)

1. Consider the hypothetical model of a Universe where the magnetic field $\boldsymbol{B}$ is generated by a magnetic charge via a modified Maxwell's equation

$$\nabla \cdot \boldsymbol{B} = \mu_0 \rho_{\rm m} \tag{116}$$

where μ_0 is the vacuum permeability and $\rho_{\rm m}$ the magnetic charge density.

Assuming that the magnetic charge is conserved, derive a relation between $\rho_{\rm m}$ and the magnetic current density $\boldsymbol{J}_{\rm m}$. *[3 marks]*

Deduce how Faraday's law needs to be modified in order to be consistent with the existence of a magnetic charge. *[5 marks]*

2. A lump of matter of volume V consists of an ensemble of N particles of charge q moving with velocity $\boldsymbol{v}$ (assuming non-relativistic motion). Both an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$ are applied to this lump of matter. Starting from the Lorentz force, show that energy conservation implies that the electromagnetic field obeys an energy equation of the form

$$-\frac{\partial u}{\partial t} = \nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \boldsymbol{J}, \tag{117}$$

where u is the energy density of the electromagnetic field, $\boldsymbol{S}$ is the Poynting vector and $\boldsymbol{J}$ is the current density. Using Maxwell's equations, find an expression for u and $\boldsymbol{S}$. *[7 marks]*

Using the fact that momentum density carried by the electromagnetic wave is $\langle \boldsymbol{g} \rangle = \langle \boldsymbol{S} \rangle / c^2$, where c is the speed of light and $\langle \ldots \rangle$ refers to time averaged quantities, show that the radiation pressure exerted by an electromagnetic wave incident onto a surface at an angle θ with respect to its normal is $P_{\rm rad} = (1 + R) u \cos^2 \theta$, where R is the reflection coefficient and u the energy density of the radiation field. Hence show that, in the case of an isotropic radiation field and for a perfectly reflecting surface, the average pressure is $P_{\rm rad} = u/3$. *[9 marks]*

Consider a stationary, rapidly rotating, spherical dust grain of radius r_c and density $\rho = 103$ kg/m^3 within the Solar System. The grain acts as a perfect absorber of the Sun's light. Given the Sun's luminosity of $L_\odot = 4 \times 10^{26}$ W and neglecting the pressure exerted by the solar wind ions, find the radius r_c of those grains that are neither expelled out of the Solar System nor attracted towards the Sun. *[4 marks]*

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