CP4 Mathematical Methods 2 复习知识点与高频考点

基于已上传 past papers: 2015–2019, 2021–2025

使用说明。这份材料按“题型触发词 - 必会公式 - 典型失分点”组织。已覆盖你上传的所有年份；2020 paper 不在上传文件中，所以没有把 2020 计入频率统计。近三年 2023–2025 已连续出现 no-calculator 条件，因此复习时默认用解析推导、级数展开、变量代换和结构化解题，而不是数值代入。

1 结论版：高频排序与复习优先级

知识块	频率	必须会做的题型
向量分析与积分定理	10/10	Gauss/divergence theorem, Stokes/Green, curl/divergence, conservative field, line/surface/volume integrals, vector identities, topology/singularity.
波动、弦、色散	10/10	wave equation derivation, d'Alembert solution, standing waves, reflection/transmission, power/energy, phase/group velocity, wave packet spreading.
耦合振子与 normal modes	10/10	写出运动方程，矩阵特征值，频率/eigenvector，初值分解，normal coordinates，能量分配，beats/weak coupling.
多变量微积分与 stationary points	9–10/10	Taylor expansion, Hessian test, degenerate stationary point, contour sketch, gradient direction, implicit/chain-rule derivatives.
变量代换、Jacobian、曲线/曲面/体积	10/10	polar/cylindrical/spherical, ellipsoid/ellipse/toroidal, surface area element, curve length, region sketch and integral limits.
极限、级数、渐近展开	9/10	Maclaurin/Taylor, logarithm trick for powers, rationalisation, dominant balance, no-calculator exact limits.
概率密度与随机变量变换	6/10，近年 5/5	normalisation, marginal PDFs, CDF/PDF relation, expectation, coordinate transformation in probability.
Exact differentials, path independence	5–7/10	$M\,\mathrm{d}x + N\,\mathrm{d}y$ exactness, potential function, conservative field, work depending/not depending on path.
PDE/material derivative/continuity	近年高频	Taylor expansion to diffusion equation, material derivative D/Dt, divergence theorem as conservation law.

复习顺序建议：先刷向量分析、波、normal modes 三大块；再补多变量 Taylor/Hessian、Jacobian/积分；最后做概率、exact differential、单变量极限的短题速度训练。CP4 的 Section B 几乎总是从“向量分析 / normal modes / waves / stationary points 或坐标积分”中抽三到四个大题。

2 逐年考点地图

Year	Section A 高频点	Section B 高频点
2015	Gauss theorem; limits; series of $\tan x$ and $e^{\tan x}$; central-field divergence; phase/group velocity; double integral; polar integral; wave equation change of variables.	Two-variable stationary points and Hessian; two-mass normal modes and forced absorber; string wave equation, power, reflection/transmission.
2016	Arc length; limits; implicit chain rule; flux through paraboloid; Taylor linearisation; conservative field and line integral; stationary points.	Stokes theorem and vector integral identity; triatomic normal modes; surface area and surface integrals; fixed-string modes and energy.
2017	Exact differential; ellipsoid volume via Jacobian; Taylor/Maclaurin; Stokes as area formula; derivative under integral; fixed-free standing waves; variable transformation chain rule.	Napkin-ring volume and divergence theorem; coupled pendula normal modes and beats; string pulse reflection/energy; stationary points and contours.
2018	Stokes line integral; curve length; asymptotic limit; hyperbolic substitution area; implicit derivatives/infinite slopes; $\int \sec\theta \, \mathrm{d}\theta$; semi-infinite string pulse.	Stationary points plus substitution integral; divergence-free vector identity; constrained spring-pendulum normal modes; dispersive PDE and phase/group limits.
2019	Taylor series; solid angle; one-variable stationary point; PDF/CDF transform; chain rule derivative; divergence theorem over box; string power/energy.	Curl/vorticity and Stokes topology; wave equation with variable tension and waveguide dispersion; cylindrical/spherical coordinates and surface integrals; three-mass normal modes.
2021	One-variable stationary points; PDF mean/variance and marginals; helix path length; angle between surfaces; double integral; line integral; wave equation and fixed-end reflection.	Coupled oscillator frequencies and weak-coupling sketch; Stokes/Green with cylindrical topology; divergence theorem over sphere; dispersion and wave packets.
2022	Limits; derivatives; exact differentials; Taylor linearisation; plane surface flux; joint PDF expectation; elliptical-coordinate volume; phase/group speed relation.	Vector identities, Stokes and path independence; stationary points plus 2D divergence theorem; double-pendulum normal modes; fixed-string modes, energy and recurrence.
2023	Change-of-variables integral; work/path dependence; surface integral and divergence theorem; Maclaurin/limit; partial derivative transformations; water-wave dispersion; time-weighted probability distribution.	Two-mass springs with initial conditions; Stokes identity; string power and density junction; Taylor/Hessian and contour classification.

2024	No-calculator limits; stationary points; paraboloid surface area; cyclotron-wave dispersion and packet spreading; line integral; Gaussian-type joint PDF; Jacobian integral.	Coupled pendula; fixed-string modes and energy; Stokes over ellipse/hemisphere; material derivative and divergence theorem for a density field.
2025	Limits; two-wave superposition and group velocity; inverse partial derivatives; joint PDF and marginals; conical surface integral; work by polynomial plus inverse-square force; Taylor expansion to diffusion equation.	Toroidal coordinates and divergence theorem; string junction with point mass; two-mass springs, normal coordinates and energy; vector identity, vorticity and circulation.

3 核心知识点清单

3.1 1. 极限、级数与渐近

触发词： evaluate the limit, Maclaurin/Taylor series, no calculators, $x \to 0$, $x \to \infty$, power of a function.

- 常用展开：

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),$$
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6),$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5),$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4),$$
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + O(x^3).$$

- 幂极限：若 $L = \lim f(x)^{g(x)}$，先取

$$\ln L = \lim g(x) \ln f(x).$$

  例如 $\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$。

- 无计算器题常考“保留到第一个非零项”。不要只写 $O(x^n)$；必须给出抵消后的首项。

- $x \to \infty$ 根式差：先有理化，或提出最高幂。典型结构：

$$\sqrt{x^3+ax^2} - \sqrt{x^3+bx^2} = \frac{(a-b)x^2}{\sqrt{x^3+ax^2}+\sqrt{x^3+bx^2}}.$$

3.2 2. 多变量 Taylor、stationary points、Hessian

触发词： stationary points, classify, contour plot, Taylor expansion to second order, degenerate.

$$f(a+h,b+k) = f(a,b) + hf_x + kf_y + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right) + O(3).$$

Stationary point 满足 $f_x = f_y = 0$。令

$$D = f_{xx}f_{yy} - f_{xy}^2.$$

若 $D>0, f_{xx}>0$ 为 local minimum；$D>0, f_{xx}<0$ 为 local maximum；$D<0$ 为 saddle；$D=0$ 二阶判别失效，必须看更高阶或沿不同路径测试。

Contour sketch 方法：先找零线，再找 stationary points；用 ∇f 指向函数增大方向；saddle 附近 contour 形如双曲线，minimum/maximum 附近闭合。

常见陷阱：

- 只解出一部分 stationary points。例如 xye^{x-y} 类题会由乘积因子给出多个分支。
- 把“函数最低值所在集合”误认为“local minima”。周期函数或径向函数会出现这种区别。
- Hessian test 只适用于 interior stationary points；边界或约束区域还要检查边界。

3.3 3. Chain rule, implicit differentiation, exact differentials

Total differential 与 Jacobian 语言是 CP4 的核心。若 $u=u(s,t), v=v(s,t)$,

$$\begin{pmatrix}\mathrm{d}u\\ \mathrm{d}v\end{pmatrix}=\begin{pmatrix}u_s & u_t\\ v_s & v_t\end{pmatrix}\begin{pmatrix}\mathrm{d}s\\ \mathrm{d}t\end{pmatrix},\qquad J=\frac{\partial u}{\partial s}\frac{\partial v}{\partial t}-\frac{\partial u}{\partial t}\frac{\partial v}{\partial s}.$$

反解为

$$\mathrm{d}s=\frac{v_t\,\mathrm{d}u-u_t\,\mathrm{d}v}{J},\qquad \mathrm{d}t=\frac{-v_s\,\mathrm{d}u+u_s\,\mathrm{d}v}{J}.$$

所以

$$\boxed{\left(\frac{\partial s}{\partial v}\right)_u=-\frac{u_t}{u_sv_t-u_tv_s}}.$$

二维 exact differential:

$$\mathrm{d}f=M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y\quad\Longleftrightarrow\quad M_y=N_x$$

在 simply connected 区域内成立。求势函数时先积分 M 或 N，再用另一个分量定“积分常数函数”。

隐函数求导：若 $F_1(x,y,z)=0, F_2(x,y,z)=0$，要求 $\mathrm{d}y/\mathrm{d}x, \mathrm{d}z/\mathrm{d}x$，写

$$F_{1x}+F_{1y}y'+F_{1z}z'=0,\qquad F_{2x}+F_{2y}y'+F_{2z}z'=0$$

解 2×2 线性方程组。导数发散通常对应 Jacobian determinant 为零，即投影曲线有 vertical tangent 或参数化退化。

3.4 4. Jacobian、区域变换、几何积分

变量代换:

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\iint_{R'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v.$$

三维同理。典型坐标:

polar:	$x=r\cos\theta,\ y=r\sin\theta,$	$\mathrm{d}A=r\,\mathrm{d}r\,\mathrm{d}\theta,$
cylindrical:	$x=r\cos\theta,\ y=r\sin\theta,\ z=z,$	$\mathrm{d}V=r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z,$
spherical:	$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta,$	$\mathrm{d}V=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$

2025 toroidal coordinates:

$$x=(R_0+r\cos\theta)\cos\phi,\quad y=(R_0+r\cos\theta)\sin\phi,\quad z=r\sin\theta,$$

a volume element is

$$\mathrm{d}V = r(R_0 + r\cos\theta)\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

with torus surface area and volume

$$A = 4\pi^2 R_0 a, \qquad V = 2\pi^2 R_0 a^2.$$

Surface $z = f(x, y)$:

$$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Parametric surface $\mathbf{r}(u, v)$:

$$\mathrm{d}\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v, \qquad \mathrm{d}S = |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}u\,\mathrm{d}v.$$

Curve length:

$$L = \int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t.$$

Angle between surfaces $F = 0, G = 0$:

$$\cos\alpha = \frac{|\nabla F \cdot \nabla G|}{|\nabla F||\nabla G|}.$$

Solid angle of cone half-angle θ:

$$\Omega = 2\pi(1 - \cos\theta) \simeq \pi\theta^2 \quad (\theta \ll 1).$$

3.5 5. Vector calculus and integral theorems

Divergence theorem:

$$\iiint_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S.$$

Use it when surface is closed or can be closed cheaply. Watch outward normal.

Stokes theorem:

$$\oint_{\partial S} \mathbf{F} \cdot\,\mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\,\mathrm{d}S.$$

Orientation rule: fingers follow boundary direction, thumb gives $\hat{\mathbf{n}}$.

Green theorem in the plane:

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}A.$$

Area formula:

$$A = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Conservative fields:

$$\mathbf{F} = \nabla\phi \quad \Rightarrow \quad \int_A^B \mathbf{F} \cdot\,\mathrm{d}\mathbf{r} = \phi(B) - \phi(A).$$

In simply connected regions, $\nabla \times \mathbf{F} = 0$ implies conservative. Singularities matter: $\mathbf{F} = (-y, x)/(x^2 + y^2)$ has zero curl away from origin but nonzero circulation around origin.

Vector identities that keep recurring:

$$\nabla \cdot (\phi\mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{A},$$
$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A},$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

3.6 6. Probability

Continuous PDF:

$$\int f_X(x)\,\mathrm{d}x = 1, \quad F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(s)\,\mathrm{d}s, \quad f_X = F_X'.$$

Joint PDF:

$$f_X(x) = \int f_{X,Y}(x,y)\,\mathrm{d}y, \qquad f_Y(y) = \int f_{X,Y}(x,y)\,\mathrm{d}x.$$

Expectation:

$$\mathbb{E}[g(X,Y)] = \iint g(x,y) f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Transformation $Y = g(X)$:

$$f_Y(y) = \sum_{x_i : g(x_i) = y} f_X(x_i) \left| \frac{\mathrm{d}x_i}{\mathrm{d}y} \right|.$$

For 2D transformations use the absolute Jacobian. Always sketch the support first; most CP4 probability marks are actually integration-limit marks.

3.7 7. Waves on strings and dispersion

Small transverse displacement on string:

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}, \qquad c = \sqrt{T/\rho}.$$

Small-amplitude approximation enters through $\sin\theta \simeq \tan\theta \simeq \frac{\partial y}{\partial x}$ and constant tension along the string.

D'Alembert solution:

$$y(x,t) = f(x - ct) + g(x + ct).$$

Fixed end: $y = 0$ gives inverted reflection. Free end: $y_x = 0$ gives non-inverted reflection.

Fixed-fixed string:

$$k_n = \frac{n\pi}{L}, \qquad \omega_n = ck_n, \qquad y(x,t) = \sum_{n=1}^{\infty} (A_n \cos\omega_n t + B_n \sin\omega_n t) \sin\frac{n\pi x}{L}.$$

Fixed-free string:

$$k_n = \frac{(n + 1/2)\pi}{L}, \quad n = 0, 1, 2, \ldots$$

Energy and power of sinusoidal wave $y = A\cos(kx - \omega t)$:

$$\langle P \rangle = \frac{1}{2} T\omega k A^2 = \frac{T}{2c} A^2\omega^2 = \frac{1}{2}\rho c A^2 \omega^2.$$

Interface of two strings under the same tension, incident from medium 1:

$$\frac{A_r}{A_i} = \frac{k_1 - k_2}{k_1 + k_2} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}, \qquad \frac{A_t}{A_i} = \frac{2k_1}{k_1 + k_2} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}.$$

Power ratios:

$$R = \left| \frac{A_r}{A_i} \right|^2, \qquad T_{\text{pow}} = \frac{k_2}{k_1} \left| \frac{A_t}{A_i} \right|^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

Point mass M at a string junction modifies the slope condition:

$$T\left[y_x(0^+, t) - y_x(0^-, t)\right] = M\, y_{tt}(0, t).$$

Phase and group velocities:

$$v_p = \frac{\omega}{k}, \qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

Wave packets travel at v_g; crests travel at v_p. Dispersion means v_g or v_p depends on k, so packet shape changes.

Important examples:

$$\omega^2 = \Omega^2 + c_s^2 k^2: \quad v_p = \frac{\sqrt{\Omega^2 + c_s^2 k^2}}{k}, \quad v_g = \frac{c_s^2 k}{\sqrt{\Omega^2 + c_s^2 k^2}}, \quad v_p v_g = c_s^2.$$

Water waves:

$$\omega^2 = gk \tanh(kh).$$

Shallow water $kh \ll 1$: $v_p = v_g = \sqrt{gh}$. Deep water $kh \gg 1$: $v_p = \sqrt{g/k}$ and $v_g = \frac{1}{2} v_p$.

3.8 8. Normal modes and coupled oscillators

For linear small oscillations:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0, \quad \mathbf{x} = \mathbf{a}e^{i\omega t} \quad \Longrightarrow \quad (K - \omega^2 M)\mathbf{a} = 0.$$

Frequencies satisfy

$$\det(K - \omega^2 M) = 0.$$

Eigenvectors give relative amplitudes and phases. General solution:

$$\mathbf{x}(t) = \sum_j (A_j \cos \omega_j t + B_j \sin \omega_j t)\, \mathbf{a}_j.$$

Use initial positions to solve A_j and initial velocities to solve $B_j \omega_j$. For unequal masses, use mass-weighted orthogonality:

$$\mathbf{a}_i^T M \mathbf{a}_j = 0 \quad (i \neq j).$$

Normal coordinates q_j diagonalise kinetic and potential energy:

$$T = \frac{1}{2} \sum_j \dot{q}_j^2, \qquad V = \frac{1}{2} \sum_j \omega_j^2 q_j^2$$

when coordinates are mass-normalised.

Small-displacement pendula: horizontal displacement $x \simeq \ell\theta$ and restoring acceleration $-(g/\ell)x$. Coupling springs add terms proportional to displacement differences, e.g. $-\Omega^2(x_1 - x_2)$.

Beats: if two close frequencies ω_+ and ω_- are excited,

$$\cos \omega_+ t - \cos \omega_- t = -2 \sin \frac{(\omega_+ + \omega_-)t}{2} \sin \frac{(\omega_+ - \omega_-)t}{2}.$$

Fast oscillation is modulated by slow envelope.

3.9 9. PDEs, material derivative, conservation

Taylor expansion of a nearest-neighbour model such as

$$\rho(x,t) = \frac{1}{2}\rho(x+\lambda, t-\tau) + \frac{1}{2}\rho(x-\lambda, t-\tau)$$

gives, to lowest non-trivial order,

$$\rho_t = \frac{\lambda^2}{2\tau}\rho_{xx}.$$

At a critical point, $\rho_t = 0$ and $\rho_x = 0$ imply $\rho_{xx} = 0$ for this PDE, so the usual second-derivative classification is inconclusive.

Material derivative:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho, \qquad \mathbf{u} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Continuity form:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

when density is conserved along an incompressible flow with appropriate assumptions. Integrating over a volume and using divergence theorem turns it into a flux-through-boundary calculation.

4 高分答题模板

题型	标准解题骨架
Flux over closed surface	Check whether surface is closed. Compute $\nabla\cdot F$. Use Gauss. If part of surface missing, close it and subtract added pieces.
Stokes/Green	State theorem with orientation. Choose easiest surface. Compute curl or scalar Green integrand. Check singularities/topology.
Line integral/work	Test conservative by curl or exactness. If conservative, find potential. If not, parameterise path.
Surface area/integral	Parameterise or use $z = f(x,y)$. Write $\mathrm{d}S$ or vector $\mathrm{d}\mathbf{S}$ before substituting limits.
Normal modes	Write $M\ddot{x} + Kx = 0$. Insert $x = ae^{i\omega t}$. Set determinant zero. Solve eigenvectors. Decompose initial conditions.
String interface	Write incident/reflected/transmitted waves. Apply displacement continuity and force/slope condition. Compute amplitudes, then power ratios.
Dispersion	Start with $v_p = \omega/k$, $v_g = \mathrm{d}\omega/\mathrm{d}k$. Evaluate limiting forms and interpret packet spreading.
Probability	Draw support. Normalise. Marginalise by correct variable limits. For transformed variable use CDF or Jacobian.
Stationary point	Solve $\nabla f = 0$. Compute Hessian. If degenerate, use higher order or path tests. Sketch zeros and gradient directions.

5 最后一周检查表

1. 能否不看 notes 写出 Gauss, Stokes, Green 的公式、条件和 orientation?
2. 能否 3 分钟内判断一个 vector field 是否 conservative，并找 potential?
3. 能否从任意 2×2 或 3×3 small-oscillation system 写出 characteristic equation?
4. 能否把初始条件投影到 normal modes?
5. 能否推导弦 wave equation，并解释 small-amplitude approximation 在哪里用到?
6. 能否熟练做 string junction 的 A_r/A_i, A_t/A_i, power ratios?
7. 能否对给定 dispersion relation 画出 $v_p(k)$ 和 $v_g(k)$ 的极限?
8. 能否正确设置 2D/3D Jacobian 和积分边界?
9. 能否处理 PDF support，尤其是 marginal 的积分上下限?
10. 能否在 no-calculator 条件下完成常见极限的展开?