CP4 Mathematical Methods 2 - Mock Paper

Preliminary Examination Style

Time allowed: 2.5 hours. **Calculators:** not required.

Instructions: Answer all of Section A and three questions from Section B. Start each Section B answer on a new page. Marks are shown in square brackets. This mock is designed to cover the recurring topics in the uploaded 2015–2019 and 2021–2025 CP4 papers.

Section A - answer all questions

A1. Evaluate the following limits:

$$L_1 = \lim_{x\to 0} \frac{\ln(1+x) - x + x^2/2}{x^3}, \qquad L_2 = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{1/x^2}.$$

[5]

A2. Find the value of the constant a for which

$$(y\cos x + a\sin y)\,\mathrm{d}x + (\sin x + x\cos y + y^2)\,\mathrm{d}y$$

is an exact differential. For that value of a, find a potential function $f(x,y)$ such that $\mathrm{d}f$ is the expression above. [5]

A3. A joint PDF is defined by

$$f_{X,Y}(x,y) = Cx, \qquad 0 \le y \le x \le 1,$$

and is zero otherwise.

(a) Find C.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(c) Calculate $\mathbb{E}[Y]$.

[6]

A4. Consider the vector field

$$\mathbf{F} = (2xy + z^2,\ x^2,\ 2xz + 3z^2).$$

Show that it is conservative and evaluate the work done in moving a particle from $A = (0,1,1)$ to $B = (2,0,3)$. [5]

A5. The surface S is the part of the paraboloid $z = b(x^2+y^2)$ with $0 \le z \le bR^2$, where $b > 0$. Evaluate the surface area of S. [6]

A6. Use the change of variables

$$u = x + y, \qquad v = \frac{y}{x+y}$$

to evaluate

$$I = \iint_R (x+y)\exp\left(\frac{y}{x+y}\right)\mathrm{d}x\,\mathrm{d}y,$$

where R is the triangular region bounded by $x = 0$, $y = 0$ and $x + y = 1$. [6]

A7. Waves obey the dispersion relation

$$\omega^2 = \omega_0^2 + c^2 k^2,$$

where ω_0 and c are positive constants.

(a) Derive v_p and v_g.

(b) Find the product $v_p v_g$ and the limiting forms for $k \ll \omega_0/c$ and $k \gg \omega_0/c$.

(c) Briefly explain which wave packets spread more strongly: long-wavelength or short-wavelength packets.

[7]

Section B - answer three questions

B8. Toroidal coordinates are defined by

$$x = (R_0 + r\cos\theta)\cos\phi, \quad y = (R_0 + r\cos\theta)\sin\phi, \quad z = r\sin\theta,$$

where $R_0 > a > 0$.

(a) Calculate the scale factors and the Jacobian determinant. [6]

(b) Compute the volume of the half torus $0 \le r \le a$, $z \ge 0$, and the area of its curved outer surface $r = a$, $z \ge 0$. [6]

(c) State the divergence theorem. Use it to evaluate the flux of $\mathbf{F} = (0, 0, z)$ through the closed boundary of the half torus. [8]

B9. A uniform string of density ρ and tension T supports small transverse waves $y(x, t)$.

(a) Derive the wave equation and identify the wave speed. [4]

(b) Two semi-infinite strings of densities ρ_1 and ρ_2 are joined at $x = 0$ under the same tension. A harmonic wave is incident from $x < 0$. Derive the reflected and transmitted amplitude ratios and the reflected and transmitted power ratios. [8]

(c) A point mass M is attached at the junction. Derive the modified boundary condition and show that the reflection amplitude is

$$\frac{A_r}{A_i} = \frac{T(k_1 - k_2) + iM\omega^2}{T(k_1 + k_2) - iM\omega^2}.$$

Discuss the limits $\omega \to 0$ and $\omega \to \infty$. [8]

B10. Two masses $m_1 = M$ and $m_2 = 2M$ are connected to fixed walls and to each other by springs with $k_1 = \kappa$ and $k_2 = k_3 = 2\kappa$. Their displacements from equilibrium are x_1 and x_2.

(a) Write the equations of motion. [4]

(b) Find the normal mode frequencies and the corresponding relative displacement vectors. [8]

(c) The masses are released from rest with $x_1(0) = \alpha$ and $x_2(0) = 0$. Find $x_1(t)$ and $x_2(t)$. [4]

(d) The masses instead start from equilibrium with $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = v$. Find $x_1(t)$ and $x_2(t)$. [4]

B11. (a) Write the second-order Taylor expansion of $f(x, y)$ about (a, b) and derive the Hessian criteria for classifying stationary points. [5]

(b) Find and classify all stationary points of

$$f(x, y) = xye^{x-y}.$$

[7]

(c) State the two-dimensional divergence theorem

$$\iint_A \nabla \cdot \mathbf{F}\,\mathrm{d}A = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}l = \oint_C M\,\mathrm{d}y - N\,\mathrm{d}x,$$

where $\mathbf{F} = (M, N)$. Verify it for $\mathbf{F} = (x^2, y^2)$ on the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, oriented counter-clockwise. [8]

Outline Mark Scheme

This is a compact checking scheme, not a full model solution.

Section A

A1. $\ln(1+x) = x - x^2/2 + x^3/3 + O(x^4)$, so $L_1 = 1/3$. Also $\ln(\sin x/x) = -x^2/6 + O(x^4)$, so $L_2 = e^{-1/6}$.

A2. Exactness requires $M_y = N_x$: $\cos x + a\cos y = \cos x + \cos y$, hence $a = 1$. A potential is

$$f = y\sin x + x\sin y + \frac{y^3}{3} + C.$$

A3. Normalisation gives $C\int_0^1 \int_0^x x\,\mathrm{d}y\,\mathrm{d}x = C/3 = 1$, so $C = 3$. Then

$$f_X(x) = 3x^2 \ (0 \le x \le 1), \quad f_Y(y) = \int_y^1 3x\,\mathrm{d}x = \frac{3}{2}(1-y^2) \ (0 \le y \le 1), \quad \mathbb{E}[Y] = \frac{3}{8}.$$

A4. $\mathbf{F} = \nabla\phi$ with $\phi = x^2y + xz^2 + z^3$. Work $= \phi(2,0,3) - \phi(0,1,1) = 45 - 1 = 44$.

A5. In polar coordinates, $z = br^2$, $0 \le r \le R$, and

$$A = 2\pi\int_0^R r\sqrt{1+4b^2r^2}\,\mathrm{d}r = \frac{\pi}{6b^2}\left[(1+4b^2R^2)^{3/2} - 1\right].$$

A6. $x = u(1-v), y = uv$, $0 \le u \le 1$, $0 \le v \le 1$, and $|J| = u$. Therefore

$$I = \int_0^1\int_0^1 u^2e^v\,\mathrm{d}v\,\mathrm{d}u = \frac{e-1}{3}.$$

A7.

$$v_p = \frac{\sqrt{\omega_0^2 + c^2k^2}}{k}, \quad v_g = \frac{c^2k}{\sqrt{\omega_0^2 + c^2k^2}}, \quad v_pv_g = c^2.$$

For small k, $v_p \sim \omega_0/k$, $v_g \sim c^2k/\omega_0$; for large k, both tend to c. Long-wavelength packets are more dispersive.

Section B

B8. Scale factors are $h_r = 1$, $h_\theta = r$, $h_\phi = R_0 + r\cos\theta$, so

$$\mathrm{d}V = r(R_0 + r\cos\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

For $z \ge 0$, take $0 \le \theta \le \pi$:

$$V = \int_0^a\int_0^\pi\int_0^{2\pi} r(R_0 + r\cos\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r = \pi^2R_0a^2.$$

Outer curved area:

$$A = \int_0^\pi\int_0^{2\pi} a(R_0 + a\cos\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta = 2\pi^2R_0a.$$

Since $\nabla\cdot(0,0,z) = 1$, the flux through the closed boundary is $\pi^2R_0a^2$.

B9. Wave equation: $y_{tt} = c^2 y_{xx}$, $c = \sqrt{T/\rho}$. At a massless junction,

$$y_1(0,t) = y_2(0,t), \qquad y_{1x}(0,t) = y_{2x}(0,t).$$

With incident from medium 1,

$$r = \frac{k_1 - k_2}{k_1 + k_2}, \qquad t = \frac{2k_1}{k_1 + k_2}.$$

Power ratios:

$$R = r^2, \qquad T_{\text{pow}} = \frac{k_2}{k_1} t^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

Point mass condition:

$$T[y_x(0^+,t) - y_x(0^-,t)] = M y_{tt}(0,t).$$

Solving gives the stated r. As $\omega \to 0$, r tends to the massless-junction value; as $\omega \to \infty$, $r \to -1$ and reflection becomes total with phase reversal.

B10. Equations:

$$M\ddot{x}_1 = -3\kappa x_1 + 2\kappa x_2, \qquad 2M\ddot{x}_2 = 2\kappa x_1 - 4\kappa x_2.$$

With $\lambda = M\omega^2/\kappa$,

$$\det \begin{pmatrix} 3-\lambda & -2 \\ -2 & 4-2\lambda \end{pmatrix} = 0 \Rightarrow \lambda = 1, 4.$$

Thus $\omega_a = \sqrt{\kappa/M}$ with vector $(1,1)$, and $\omega_b = 2\sqrt{\kappa/M}$ with vector $(1,-1/2)$. Let $\omega = \sqrt{\kappa/M}$. For $x(0) = (\alpha, 0)$,

$$x_1 = \frac{\alpha}{3}\cos\omega t + \frac{2\alpha}{3}\cos(2\omega t), \quad x_2 = \frac{\alpha}{3}\cos\omega t - \frac{\alpha}{3}\cos(2\omega t).$$

For $\dot{x}(0) = (0, v)$,

$$x_1 = \frac{2v}{3\omega}\sin\omega t - \frac{v}{3\omega}\sin(2\omega t), \quad x_2 = \frac{2v}{3\omega}\sin\omega t + \frac{v}{6\omega}\sin(2\omega t).$$

B11. Taylor expansion and Hessian criteria as in the formula sheet. For $f = xye^{x-y}$,

$$f_x = ye^{x-y}(1+x), \qquad f_y = xe^{x-y}(1-y).$$

Stationary points: $(0,0)$ and $(-1,1)$. At $(0,0)$ the leading term is xy, so it is a saddle. At $(-1,1)$,

$$f_{xx} = e^{-2}, \quad f_{yy} = e^{-2}, \quad f_{xy} = 0,$$

so it is a local minimum. For the triangle and $\mathbf{F} = (x^2, y^2)$,

$$\iint_A (2x + 2y)\,\mathrm{d}A = \frac{2}{3}.$$

On the boundary, only the sloping edge contributes nonzero net value:

$$\oint_C x^2\,\mathrm{d}y - y^2\,\mathrm{d}x = \int_0^1 \left[x^2 + (1-x)^2\right]\,\mathrm{d}x = \frac{2}{3}.$$

The two sides agree.