CP4: MULTIPLE INTEGRALS AND VECTOR CALCULUS
OXFORD PHYSICS YEAR 1

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These problem sets are a slight reformatting and revision of problems developed by the previous lecturers of this course over the past several years, and should provide a solid base to help consolidate the material discussed throughout the lecture course. Students should be prepared to tackle the first problem set by the end of week 2, the second by week 3, the third by week 4, and the fourth by week 5.

All teaching materials can be found on the Oxford Physics Canvas website.

Syllabus

From the Physics Undergraduate Handbook the course will cover the following topics:

- Double integrals and their evaluation by repeated integration in Cartesian, plane polar and other specified coordinate systems.
- Jacobians.
- Probability theory and general probability distributions.
- Line, surface and volume integrals.
- Evaluation by change of variables (Cartesian, plane polar, spherical polar coordinates and cylindrical coordinates only unless the transformation to be used is specified).
- Integrals around closed curves and exact differentials.
- Scalar and vector fields.
- The operations of grad, div and curl, and understanding and use of identities involving these.
- The statements of the theorems of Gauss and Stokes with simple applications.
- Conservative fields.

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PROBLEM SET 1

1. Which of the following is best described using a vector quantity: (a) temperature; (b) magnetic field; (c) acceleration; (d) force; (e) molecular weight; (f) area; (g) angle of polarization.

2. Sketch the region of integration and evaluate:

   (a)
   $$\int_0^{\sqrt{2}} \int_{y^2}^{2} y \, \mathrm{d}x \, \mathrm{d}y.$$

   (b)
   $$\int_0^{4} \int_0^{\sqrt{x}} y\sqrt{x} \, \mathrm{d}x \, \mathrm{d}y.$$

   (c)
   $$\int_0^{1} \int_{-y}^{y^2} x \, \mathrm{d}x \, \mathrm{d}y.$$

   Evaluate the integrals again after reversing the order of integration.

3. Reverse the order of integration and evaluate
   $$\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x} \, \mathrm{d}x \, \mathrm{d}y.$$

4. Using double integration, or otherwise, evaluate
   $$\int_0^{+\infty} e^{-\alpha x^2} \, \mathrm{d}x.$$

5. Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

6. A mass distribution in the positive x region of the xy-plane and in the shape of a semicircle of radius a, centred on the origin, has mass per unit area k. Find, using polar coordinates, its

   (a) mass;
   (b) centre of mass coordinates $(\bar{x}, \bar{y})$;
   (c) moments of inertia about the x and y axes.

7. A semi-infinite sheet has mass per unit area of $\sigma = k \exp\left(-\frac{x^2+y^2}{a^2}\right)$ for $x \geq 0$, and $\sigma = 0$ for $x < 0$. Find its

   (a) mass;
   (b) centre of mass coordinates $(\bar{x}, \bar{y})$;
   (c) moments of inertia about the x and y axes.

   Comment on the comparison of these results with those of problem 6.

8. Evaluate
   $$\int_0^{a} \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) \arctan(y/x) \, \mathrm{d}x \, \mathrm{d}y.$$

PROBLEM SET 2

1. For $\mathbf{r} = (x, y, z)$, find $\nabla\phi$ in the cases:

   (a) $\phi = \ln|\mathbf{r}|$

   (b) $\phi = \frac{1}{|\mathbf{r}|}$

2. If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find $\nabla(F + G)$ and $\nabla(FG)$ at (1, 0, -2).

3. The pair of variables (x, y) are each functions of the pair of variables (u, v) and vice versa. Consider the matrices:

$$A = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

   (a) Show using the chain rule that the product AB of these two matrices equals the unit matrix $\mathbb{1}$.

   (b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, θ).

   (c) Assuming the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

4. (a) Using the change of variables $u = x + y$ and $v = x - y$, evaluate the double integral $\iint (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$, over the domain bounded by the straight lines $y = x$, $y = x + 2$, $y = -x$ and $y = -x + 2$.

   (b) Given the change of variables $u = xy$ and $v = y/x$, show that the Jacobian $J(x, y) = 2y/x$. Hence evaluate the integral

$$\iint e^{-xy}\,\mathrm{d}x\,\mathrm{d}y,$$

   over the region $x > 0$, $y > 0$, $xy < 1$, $1/2 < y/x < 2$.

5. Given the vector field $\mathbf{A} = (4x - y^4, -4xy^3 - 3y^2, 4)$, evaluate the line integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ from (0, 0, 0) to (1, 2, 0) along the following paths:

   (a) the straight line from (0,0,0) to (1,2,0);

   (b) on the path of straight lines joining (0, 0, 0), (0, 0, 1), (1, 0, 1), (1, 2, 1) and (1, 2, 0).

   Show that $\mathbf{A}$ is conservative and find a scalar function $V(\mathbf{r})$ such that $\mathbf{A} = \nabla V(\mathbf{r})$.

6. Given the vector field $\mathbf{A} = (3x^2 + 6y, -14yz, 20xz^2)$, evaluate the line integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ from (0, 0, 0) to (1, 1, 1) along the following paths:

   (a) the straight line from (0,0,0) to (1,1,1);

   (b) on the path of straight lines joining (0, 0, 0), (1, 0, 0), (1, 1, 0) and (1, 1, 1);

   (c) on the path parametrized by $x = t$, $y = t^2$, $z = t^3$.

   Is $\mathbf{A}$ conservative?

7. Show that the curved surface area of a hemisphere of radius R is $2\pi R^2$ by

   (a) directly integrating the element of area $R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ over the surface of the hemisphere;

   (b) projecting onto an integral taken over the x-y plane.

8. Find the area of the plane $x - 2y + 5z = 13$ cut out by the cylinder $x^2 + y^2 = 9$.

9. A uniform lamina is made of that part of the plane $x + y + z = 1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

PROBLEM SET 3

1. For spherical polar and for cylindrical coordinates, show that:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 sin\theta, \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r. \tag{1}$$

2. Calculate the solid angle of a cone of half-angle α.

3. A solid hemisphere of uniform density k occupies the volume of $x^2 + y^2 + z^2 \leq a^2, z \geq 0$. Using symmetry arguments wherever possible, find:

   (a) its total mass M;

   (b) the position $(\bar{x}, \bar{y}, \bar{z})$ of its centre of mass;

   (c) its moments and products of inertia $I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}$.

4. Air is flowing with a speed 0.4 ms^{-1} in the direction of the vector $(-1,-1,1)$. Calculate the volume of air flowing per second through the loop which consists of straight lines joining, in turn, the following points: (1,1,0), (1,0,0), (0,0,0), (0,1,1), (1,1,1) and (1,1,0).

5. If $\hat{\mathbf{n}}$ is the unit normal to the surface S, evaluate $\iint \mathbf{r} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ over

   (a) the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$;

   (b) the surface of a sphere of radius a centred at the origin.

6. Evaluate $\iint \mathbf{A} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ for the following cases:

   (a) $\mathbf{A} = (y, 2x, -z)$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

   (b) $\mathbf{A} = (x + y^2, -2x, 2yz)$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

   (c) $\mathbf{A} = (6z, 2x + y, -x)$ and S is the entire surface of the region bounded by the cylinder $x^2 + z^2 = 9$ and the planes $x = 0$, $y = 0$, $z = 0$ and $y = 8$.

7. Given the vector field $\mathbf{A} = (xy^2, x^2, yz)$

   (a) calculate the surface integral $\iint \mathbf{A} \cdot \mathrm{d}\mathbf{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = 1$.

   (b) Show that the integral $\iint \mathbf{A} \cdot \mathrm{d}\mathbf{S}$, taken outwards over the whole surface, is not zero, and equals $\iiint \nabla \cdot \mathbf{A}\, \mathrm{d}V$, calculated over the volume of the prism.

8. If $\mathbf{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, evaluate

   (a) $\nabla \cdot \mathbf{A}$;

   (b) $\mathbf{A} \cdot \nabla\phi$;

   (c) $\nabla \cdot (\phi\mathbf{A})$;

   (d) $\nabla \cdot (\nabla\phi)$.

9. The magnetic field $\mathbf{B}$ at a distance r from a straight wire carrying a current I has a magnitude $\mu_0 I/(2\pi r)$. The lines of force are circles centred on the wire lying in planes perpendicular to it. Evaluate $\nabla \cdot \mathbf{B}$.

PROBLEM SET 4

1. O is the origin and A, B, C are points with position vectors $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (1, 1, 1)$ and $\mathbf{c} = (0, 2, 0)$, respectively. Find the vector area $\mathbf{S}$ of the loop OABCO

   (a) by drawing the loop in projection onto the yz, zx and xy planes and calculating the components of $\mathbf{S}$;

   (b) by filling the loop with plane polygons, ascribing a vector area to each, and taking the resultant.

   Calculate the projected area of the loop

   (a) when seen from the direction which makes it appear as large as possible;

   (b) when seen from the direction of the vector $(0, -1, 1)$.

   What are the corresponding answers for the loop OACBO?

2. Sketch the vector fields $\mathbf{A} = (x, y, 0)$ and $\mathbf{B} = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

3. Verify Stokes' Theorem for the hemispherical surface $|\mathbf{r}| = 1, z \geq 0$ for the vector field $\mathbf{A} = (y, -x, z)$.
   [Note: $\mathbf{r} \equiv (x, y, z)$]

4. For $\mathbf{A} = (y, -x, 0)$, find $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ for any closed loop on the surface of the cylinder $(x - 3)^2 + y^2 = 2$.

5. A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\mathbf{v}(\mathbf{r})$ as if it were a rigid body.

   (a) Calculate $\nabla \cdot \mathbf{v}$ and interpret the result.

   (b) Calculate $\nabla \times \mathbf{v}$. Can the flow be represented in terms of a scalar velocity potential ϕ such that $\mathbf{v} = \nabla\phi$? If so, find ϕ.

6. Without using index notation (i.e. by writing out in components), prove:

   (a) $\nabla \times (\nabla\phi) = 0$;

   (b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

7. Evaluate the line integrals

   (a) $\int_\gamma \phi \, \mathrm{d}\mathbf{r}$;

   (b) $\int_\gamma \mathbf{F} \times \mathrm{d}\mathbf{r}$;

   for $\phi = 2xyz^2$ and $\mathbf{F} = (xy, -z, x^2)$ if γ is the curve given by $x = t^2$, $y = 2t$, $z = t^3$, for $t \in [0, 1]$.

8. Using index notation [Note: $\mathbf{r} = (x, y, z)$, and $\mathbf{a}, \mathbf{b}$ are constant vectors]:

   (a) What scalar or vector quantities do the following expressions correspond to:

   $$a_i b_j c_i; \quad a_i b_j c_j d_i; \quad \delta_{ij} a_i a_j; \quad \delta_{ij}\delta_{ij}; \quad \epsilon_{ijk} a_i b_k; \quad \epsilon_{ijk}\delta_{ij}$$

   (b) Find the grad of the scalar functions:

   i. $|\mathbf{r}|^n$

   ii. $\mathbf{a} \cdot \mathbf{r}$

   (c) Find the div and curl of the vector functions:

   i. $\mathbf{r}$

   ii. $|\mathbf{r}|^n \mathbf{r}$

   iii. $(\mathbf{a} \cdot \mathbf{r})\mathbf{b}$

   iv. $\mathbf{a} \times \mathbf{r}$

   (d) Prove (i.e. repeat question 7 above by hopefully by a faster method):

   i. $\nabla \times (\nabla\phi) = 0$;

   ii. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

9. Let X be a continuous random variable with probability density function (PDF) $f(x) = ce^{-x}$ for $x \geq 0$ and zero otherwise.

(a) Determine the value of the normalization constant c.

(b) Find the cumulative distribution function $F(x)$.

(c) Calculate the mean and median.

(d) Find $P(1 < X < 3)$.

10. Let X and Y be two jointly continuous random variables with a joint PDF $f(x, y) = cx^2y$ for $0 \leq y \leq x \leq 1$ and zero otherwise.

(a) Sketch the region in the (x, y) plane for which the PDF is non-zero.

(b) Determine the value of the normalization constant c.

(c) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(d) Find $P(Y \leq X/2)$.