CP4
Multiple Integrals and Vector Calculus

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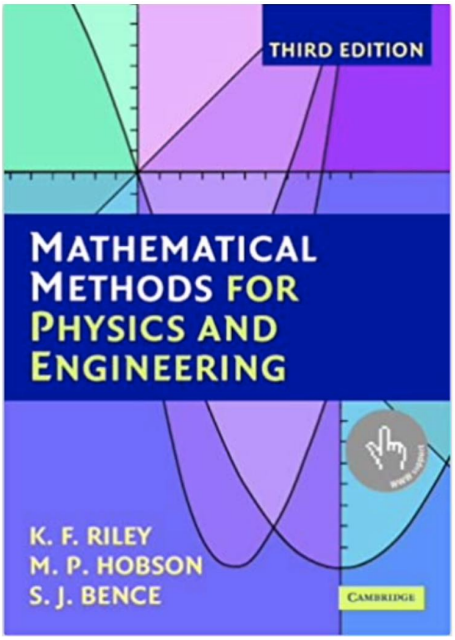
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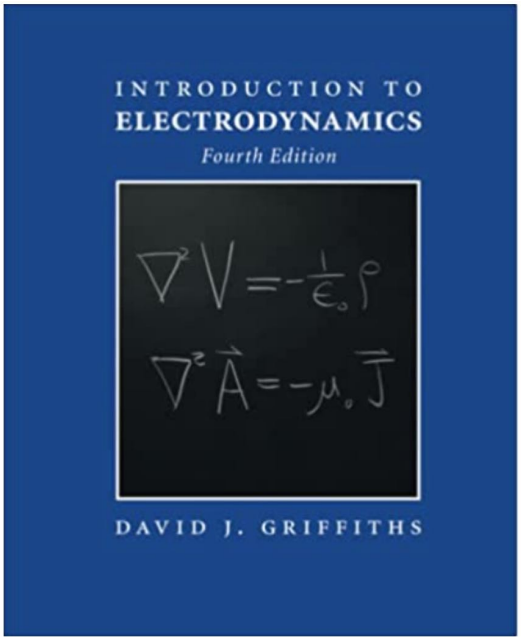
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CP4 Multiple Integrals and Vector Calculus syllabus [1]

- Double integrals and their evaluation by repeated integration in Cartesian, plane polar and other specified coordinate systems.
- Jacobians.
- Probability theory and general probability distributions.
- Line, surface and volume integrals, evaluation by change of variables (Cartesian, plane polar, spherical polar coordinates and cylindrical coordinates only unless the transformation to be used is specified).
- Integrals around closed curves and exact differentials.
- Scalar and vector fields.
- The operations of grad, div and curl and understanding and use of identities involving these.
- The statements of the theorems of Gauss and Stokes with simple applications.
- Conservative fields.

Recommended books





[1] What this looks like according to generative AI, graphically, for those interested, is depicted on the title page.

1 A refresher: vectors, planes, fields

1.1 Some basic definitions

In this course we will be interested in manipulating vector quantities of various kinds, so we should start by recalling the definition of fields, vectors and vector spaces, and setting out the preferred notation used in this course. First, a few definitions that will help clarify the language we will be using.

- A *set* is a collection of mathematical objects.
- A set G with a binary operation that combines two elements in the set to yield another element in the set is called a *group*. Mathematically this can be written as $G \times G \to G$. This operation has to have certain properties (also called the axioms), and together with the set these constitute the *algebraic structure* of the group. Specifically, for a group these are closure, associativity, $\exists$ of an identity element, and $\exists$ of inverse elements. If the operation also commutes (i.e., it does not depend on the order in which the elements are written in the operation), then the group is said to be a *commutative* or an *Abelian group*.
- A set with two binary operations of addition and multiplication is called a *field* F. These must satisfy six field axioms, namely associativity, commutativity, $\exists$ of additive and multiplicative identity, $\exists$ of additive and multiplicative inverses, and distributivity. We see the complexity of the structure is increasing; in particular, we note that a field is an Abelian group under both addition and multiplication, the two being connected via the requirement of distributivity. Note that the set of rationals $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ are all fields, but the set of integers $\mathbb{Z}$ is not a field as the multiplicative inverse of every integer is not necessarily another integer (e.g., the multiplicative inverse of 2 is $1/2$, not an integer). The integers are, instead, a *ring*. Rings generalize fields: they too have two operations, but multiplication does not have to be commutative, and multiplicative inverses need not exist.
- A *vector space* over a field F is a set V together with two binary operations: vector addition ($V \times V \to V$) and scalar multiplication ($V \times F \to V$). Note the difference with the definition of a field: the notion of multiplication between vectors is not part of the fundamental algebraic structure of a vector space. In fact, as we know there are multiple ways to multiply vectors, and the multiplicative inverse (i.e., vector division) is not defined. The elements of the field are called scalars to distinguish them from the elements of the vector space (vectors), and because the definition of multiplication acts to *scale* the vectors. We will typically be interested in vector spaces defined over $\mathbb{R}$: this is called a *real vector space*. Eight axioms must be satisfied for the operations permitted on a vector space: associativity and commutativity of addition, $\exists$ of identity elements for addition and scalar multiplication, $\exists$ of additive inverse, distributivity for scalar multiplication with respect to vector and field addition, and compatibility of scalar and field multiplication.

1.2 Vectors

Throughout this course we will be interested in n-dimensional real vector spaces and associated real vectors, defined over $\mathbb{R}^n$. We will write these vectors generally as

$$\mathbf{r} = (x_1, x_2, \cdots, x_n) \qquad\qquad \mathbf{r} \in \mathbb{R}^n,$$

where x_i are called the components of $\mathbf{r}$. Often it will be convenient to use a slightly different notation in the special cases of 2 or 3 dimensions, where instead we write

$$\mathbf{r} = (x, y) \qquad\qquad \mathbf{r} \in \mathbb{R}^2.$$
$$\mathbf{r} = (x, y, z) \qquad\qquad \mathbf{r} \in \mathbb{R}^3.$$

The modulus or norm of a vector is given by

$$|\mathbf{r}| \equiv r = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{1}$$

This allows us to define a unit vector (vector with norm of 1) as

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}. \tag{2}$$

The *standard* or *canonical* basis of $\mathbb{R}^n$ is given by the unit vectors

$$\begin{aligned} \hat{\mathbf{e}}_\mathbf{1} &= (1, 0, 0, \cdots, 0) \\ \hat{\mathbf{e}}_\mathbf{2} &= (0, 1, 0, \cdots, 0) \\ &\vdots \\ \hat{\mathbf{e}}_\mathbf{n} &= (0, 0, 0, \cdots, 1). \end{aligned}$$

In this basis we can write the a generic vector $\mathbf{r}$ as

$$\mathbf{r} = x_1\hat{\mathbf{e}}_\mathbf{1} + x_2\hat{\mathbf{e}}_\mathbf{2} + \cdots + x_n\hat{\mathbf{e}}_\mathbf{n} = \sum_{i=1}^{n} x_i\hat{\mathbf{e}}_\mathbf{i} \tag{3}$$

Note that the subscripts mean two different things: in one case they denote the component of the vector (a scalar), and in the other a specific element of the basis set (a vector). That this decomposition of a generic vector $\mathbf{r}$ can be done follows directly from the axioms of the vector space: we explicitly use both the concepts of vector sums and scalar multiplications to identify all possible elements of the vector space.

Once again, it will be convenient to use special notation for the 2 and 3 dimensional cases when discussing the sets of canonical unit vectors:

$$\begin{aligned} &(\hat{\mathbf{i}}, \hat{\mathbf{j}}) && \text{on a surface in } \mathbb{R}^2, \\ &(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) && \text{in a volume in } \mathbb{R}^3. \end{aligned}$$

With this notion a generic vector in 3D can be written as

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}. \tag{4}$$

In addition to the basic operations provided for by the axioms of a vector space, there are a few others that will prove useful. The first is the *scalar* or *dot* product. This is an operation that takes in two vectors elements of the vector space V and outputs a scalar, an element of the field F over which the vector space is defined: $V \times V \to F$. For two vectors $\mathbf{u}, \mathbf{v} \in V$, with components u_i and v_i, we write this operation as

$$\mathbf{u} \cdot \mathbf{v} \equiv \langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n. \tag{5}$$

In 2 dimensions it is easy to see that this definition can also be expressed geometrically as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\vartheta, \tag{6}$$

which allows us to define the concept to *angle between two vectors* in n dimensions as

$$\cos\vartheta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}. \tag{7}$$

We say that two vectors are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 0$. This is easy to see geometrically in the case of 2 or 3 dimensions from Eq. (6), but it holds more generally for vectors in any number of dimensions.

In 3 dimensions we will see that it will be convenient to define another product operation between vectors, but which, unlike the scalar product, outputs a vector: $V \times V \to V$. We call this the *vector* or *cross* product. For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, with components u_i and v_i, it is defined as

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}. \tag{8}$$

We note that if $\mathbf{u}, \mathbf{v}$ are linearly dependent, i.e. $\mathbf{u} = \lambda \mathbf{v}, \lambda \in \mathbb{R}$, then because of the permutation rules of the determinant the above quantity is 0. If $\mathbf{u}, \mathbf{v}$ are linearly independent, then they define a plane, and the vector product will create a new vector normal to this plane (i.e., to the entire subspace spanned by $\mathbf{u}, \mathbf{v}$). This provides insight into how the operation can be applied to vectors defined in $\mathbb{R}^2$ rather than $\mathbb{R}^3$: this is a special case where the basis has been aligned with the plane spanned by $\mathbf{u}, \mathbf{v}$, and its normal. So for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ we can pad the missing third dimension of the vectors with zeros and write

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{pmatrix} = (u_1 v_2 - v_1 u_2)\hat{\mathbf{k}}. \tag{9}$$

The vector product can be generalized to more than 3 dimensions but this is seldom useful.

With these two operations we can write the *triple product* between three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}. \tag{10}$$

This operation necessarily produces a scalar, and can be denoted as a map $V \times V \times V \to F$. Note that the parenthesis is not strictly needed, as there is only one way to get a meaningful result from the operation that is consistent with the definitions of the scalar and vector products. The triple product possesses a useful geometric property: it measures the volume of the parallelepiped formed by the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

1.3 Functions of multiple variables

With the armamentarium of vectors, fields and operations permitted on vector spaces, we are ready to consider a wide range of mappings (or functions) connecting vectors and scalars. In particular, we will be interested in the integro-differential calculus of such functions across various multidimensional domains. Because the main application of these tools lies in physics, we will focus our discussions on 2 and 3 dimensional spaces, but much of underlying theory still holds in the same form for any number of dimensions.

In general, we think of a function as a map between two sets A and B $f : A \to B$ that assigns to each element in A exactly one element in B. The set A is called the domain of f, and B is called the codomain. For a one-dimensional function defined over the reals on some domain A this would be written as $f : A \subseteq \mathbb{R} \to \mathbb{R}$. There are three main ways in which we can extend these ideas to vector calculus in multiple dimensions:

1. Functions that map multi-dimensional spaces to scalars $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$. We will call this a *multivariable function*. An example of this would be the equation of a 2D surface defined over the (x, y) plane as $z = f(x, y)$. For each point in the (x, y) plane the function returns the height of the surface above that point.

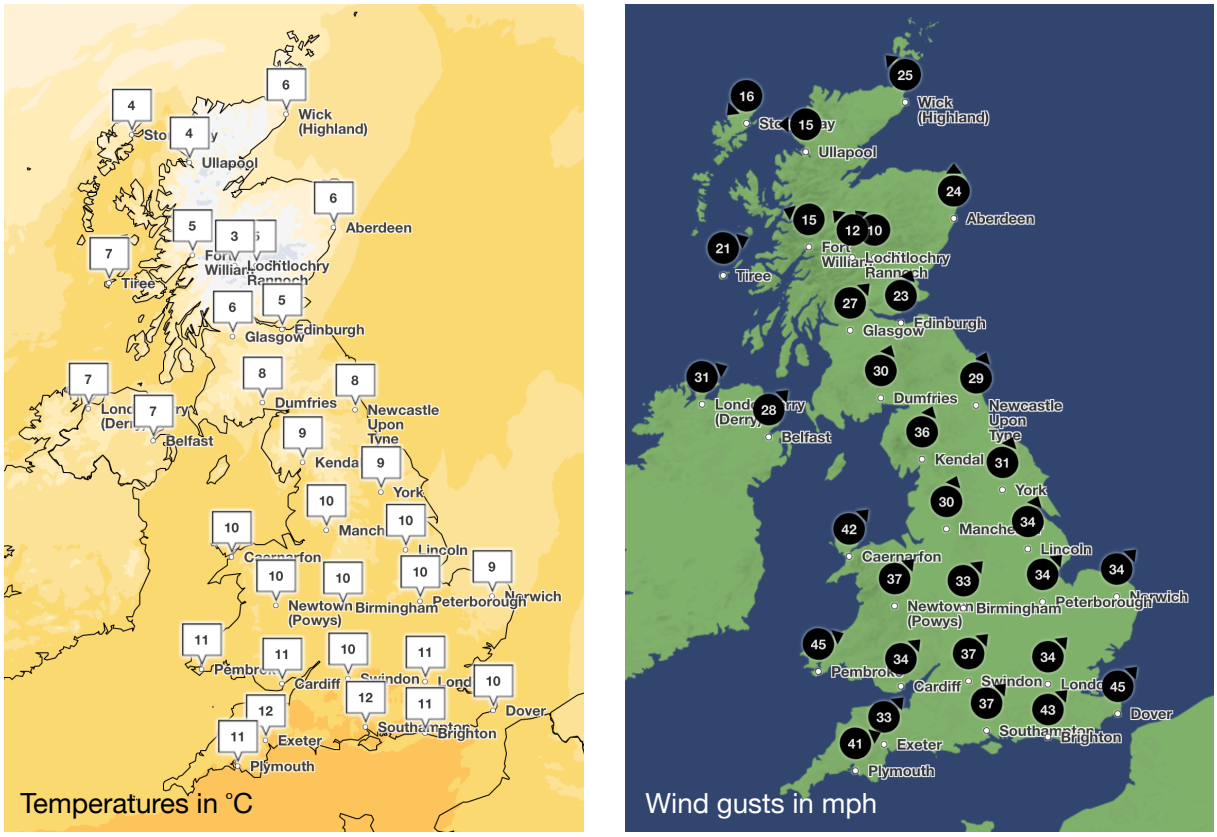


Figure 1: Temperature and wind gust maps of the UK. Temperatures represent a scalar field, while the winds must be represented by a vector quantity representing the wind speed and direction. (Source: Met Office 2022)

2. Functions that map a scalar to a vector $f : A \subseteq \mathbb{R} \to \mathbb{R}^n$. We will call this a *vector function in one variable*. An example would the equation describing the position of a particle in 3D space as a function of time. For each value of time (a scalar), the function outputs the position vector of the particle $\mathbf{r} = (x, y, z)$.

3. Functions that map multi-dimensional spaces to vectors $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. This is a *multivariable vector function*. A simple example would be an equation that relates the position of a particle in 3D space and time to the velocity of the particle, i.e. $(x, y, z, t) \to (v_x, v_y, v_z)$. Note that this is the type of function that can associate a vector (here the velocity) to every point in space-time, which leads us to an important concept in physics – that of a vector field. We will expand on this further below.

1.4 Scalar and vector fields

We will often refer to the concepts of a scalar or a vector field, both of which are of considerable importance to various physical applications. Unlike the fields mentioned above, these are not sets but rather maps. Consider, for example, the temperatures across the UK. There will be a measurable temperature in every part of the country, and so we can think of this temperature distribution as a map where we assign a value of the temperature (a scalar) to each point on the map of the UK identified via a latitude and longitude (a 2D surface). This mapping can be written as $(x, y) \to T$. We could extend this description so as to also consider how the temperature varies with altitude, in which case the mapping would associate the scalar temperature to a point in 3D space: $(x, y, z) \to T$. We could now note that the temperature changes throughout the day, and so an additional dimension is required to account for the actual

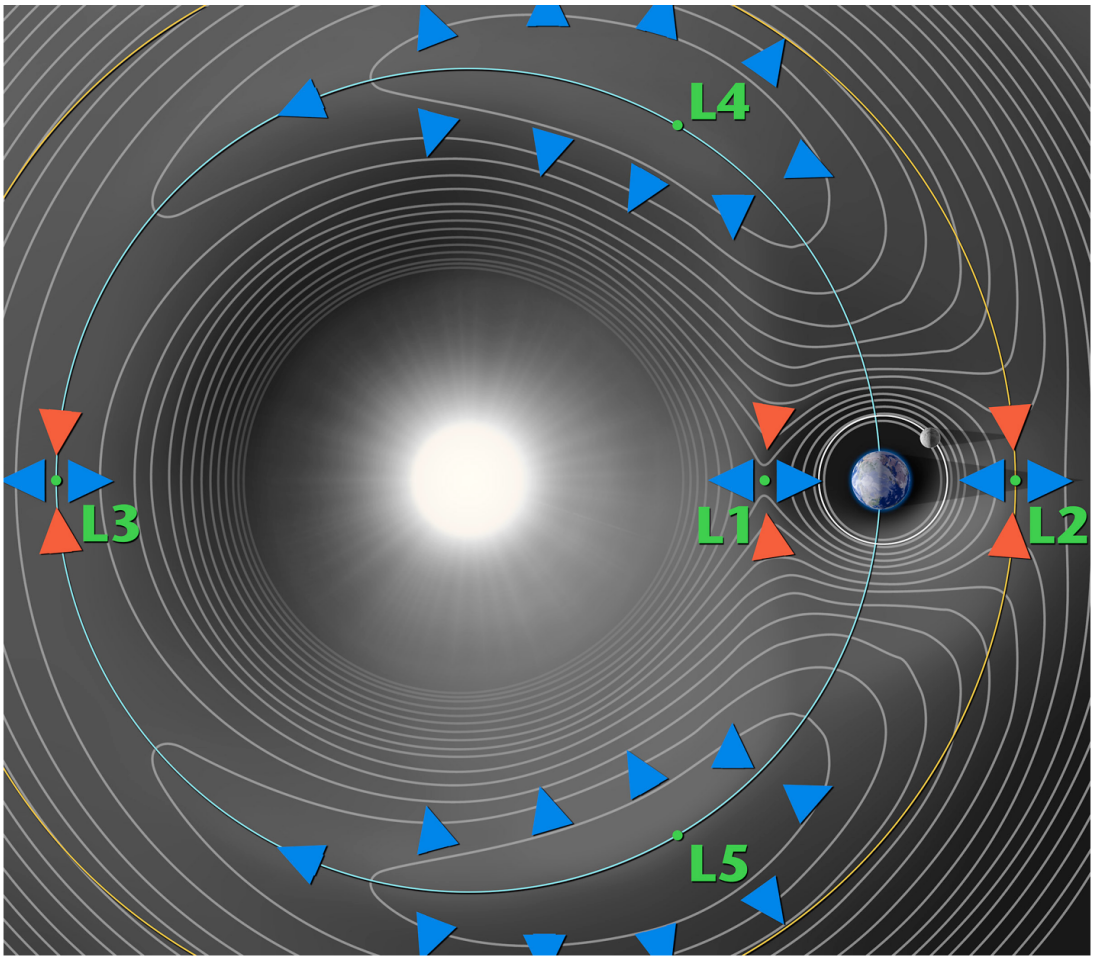


Figure 2: An object with mass travelling in the solar system will be subject to the gravitational field: a vector field where each point in space has a gravitational force associated with it. This will cause the mass to travel in some direction. There are 5 stationary points in the Earth-Sun system, called Lagrange points, where small objects can steadily orbit the Sun along with the Earth in a fixed position. One of these points, L2 (note, it's a saddle point), is the home of the recently launched James Webb telescope. (Source: NASA 2022.)

temperature distribution. A proper map would therefore look more like $(x, y, z, t) \to T$. We can generalize these ideas and define a scalar field to be a function S that assigns to each point in some n-dimensional vector space a scalar value:

$$S(\mathbf{r}) : \mathbb{R}^n \to \mathbb{R}. \tag{11}$$

If instead of the temperature we were interested in accounting for the wind speed across the country, we could do something very similar, but the scalar temperature would need to be replaced with a vector quantity accounting for the wind direction and velocity. Again, in the simplest case we may want to associate the wind direction with the latitude and longitude, and so the required mapping can be written as $(x, y) \to (v_x, v_y)$. As before, we can add the altitude or the time to our coordinate system, but now we may also include further complexity to the description of the wind, for example by adding its vertical velocity component. Then the mapping would be from a 4D to a 3D space: $(x, y, z, t) \to (v_x, v_y, v_z)$ We can generalize this by defining a vector field as a function $\mathbf{F}$ that assigns to each point in some n-dimensional vector space an m-dimensional vector:

$$\mathbf{F}(\mathbf{r}) : \mathbb{R}^n \to \mathbb{R}^m. \tag{12}$$

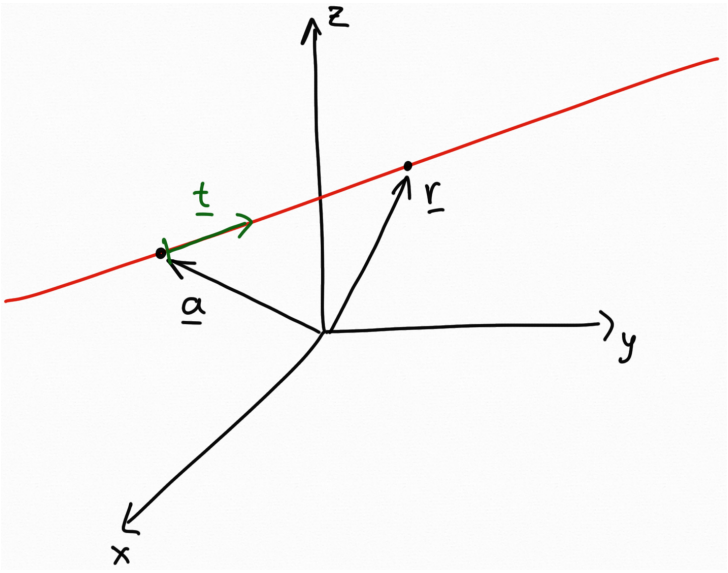


Figure 3: A line in 3D space can be fully specified by two vectors. This could either be two position vectors describing two points on the line, or by a vector along the line (the tangent vector) and a vector to any point on the line.

This is an obvious generalization of Eq. 11, since a 1D vector field reduces to a scalar field.

1.5 Lines and planes in space

Throughout this course we will be interested in shapes, primarily surfaces, in 3D space. It will therefore be convenient to briefly summarize how we can denote lines and planes using vector notation. In general, objects in some vector space (lines, surfaces, planes, etc.) are sets that will be described by equations that specify the conditions necessary for a vector to belong to the set.

A line in 3D space can be fully determined by specifying two vectors, since two (non-identical) points define a line. A set of vectors $\mathbf{r}$ denoting a line can then be specified as all vectors satisfying the following equation

$$\mathbf{r} = \mathbf{a} + \alpha\hat{\mathbf{t}}, \quad \alpha \in \mathbb{R}, \tag{13}$$

where $\mathbf{a}$ is any point on the line and $\hat{\mathbf{t}}$ is any tangent vector to the line. Unsurprisingly, the dimensionality of this curve is 1 as the equation has one degree of freedom, α. This holds regardless of the dimensionality of the space in which we operate, and the above equation extends our concept of a line to any dimension.

The set of points $\mathbf{r}$ comprising a plane can be defined in a similar manner as

$$\mathbf{r} = \mathbf{a} + \alpha\hat{\mathbf{t}}_\mathbf{1} + \beta\hat{\mathbf{t}}_\mathbf{2}, \quad \alpha, \beta \in \mathbb{R}, \tag{14}$$

where, again, $\mathbf{a}$ is any point in the plane, and $\hat{\mathbf{t}}_\mathbf{1}$ and $\hat{\mathbf{t}}_\mathbf{2}$ are any two non-colinear vectors lying in the plane. The dimensionality of this set is now 2, as we have two degrees of freedom in the equation. While not part of this course, it should be clear from these two examples how sets of higher dimensionality can be generated for vector spaces of arbitrary dimensionality.

In addition to Eq. (14), a plane is also often given in terms of three points in the plane given by the vectors **a,b,c**, noting that the difference of any two of these vectors yields a vector in the plane:

$$\mathbf{r} = \mathbf{a} + \alpha(\mathbf{b} - \mathbf{a}) + \beta(\mathbf{b} - \mathbf{c}), \quad \alpha, \beta \in \mathbb{R}. \tag{15}$$

Alternatively, a plane can also be specified using the unit vector normal to the plane $\hat{\mathbf{n}}$, and a point in the plane $\mathbf{a}$

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = d, \tag{16}$$

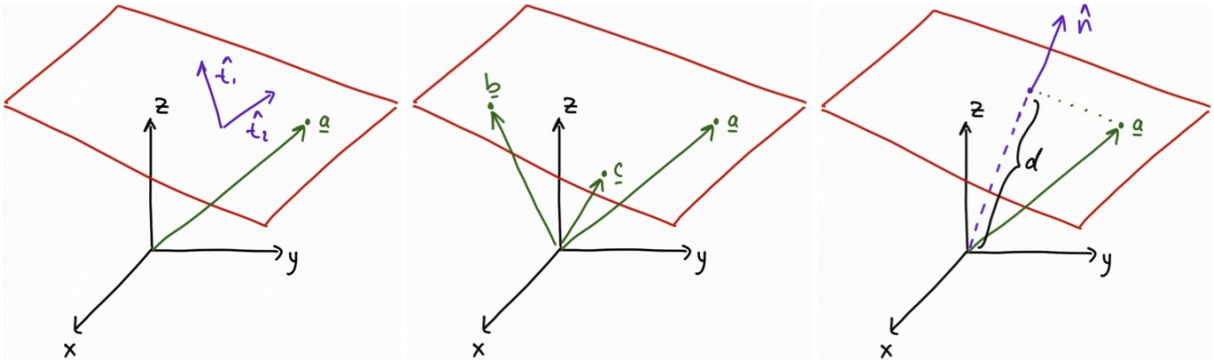


Figure 4: Three ways to define a plane in space: via two vectors in the plane and a point; via three points; and via a point and the unit normal vector to the plane.

where d is the length of the projection of any vector describing a point on the plane onto the unit normal to the plane, i.e., the distance between the plane and the origin. A summary of these ways to describe a plane is given in Fig. 4.

2 Multiple integrals

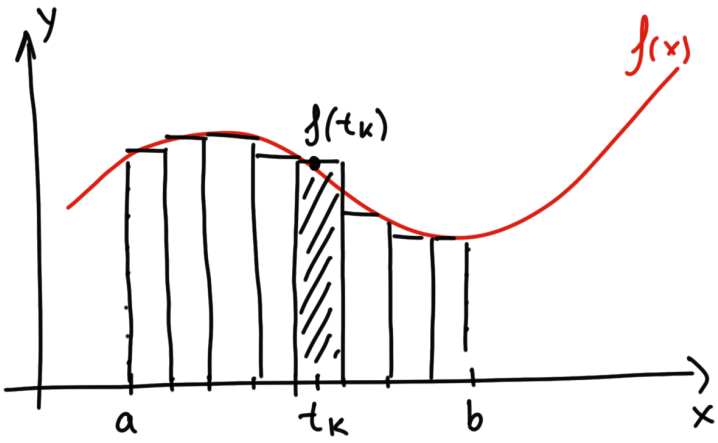


Figure 5: Riemann integration of a function in 1 dimension.

We start by recalling the well-known definition of (Riemann) integration in one dimension.

Definition 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, if we consider splitting the domain into n equal intervals of width $\frac{b-a}{n}$, and denote by t_k any point within the k-th interval, the integral of f over this domain is given by the limit of Cauchy-Riemann sums*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} s_n \equiv \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f(t_k). \tag{17}$$

If this limit exists and is finite, the function is said to be (Riemann) integrable. In practice, of course, we tend to solve integrals by guessing them; we can compute and memorize the derivatives of a wide range of commonly-encountered functions, and then use the fundamental theorem of calculus (integration is the inverse operation of differentiation) to figure out the integrals.

Conceptually, this approach can easily be extended to integrals in more dimensions, simply by considering domains that are no longer 1D intervals but higher dimensional areas, volumes, or hyper-volumes. However, on the practical side our strategy will be to reduce a multidimensional integral into a series of 1D integrals, and then solve those using standard approaches. Let us illustrate this using a simple case in two dimensions.

Let the integration domain be a rectangle $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ in the (x, y) plane, and let $f(x, y)$ be a continuous function in two variables. Intuitively, the integral of f over this domain will be some volume V contained by the rectangle between the surfaces $z = 0$ and $z = f(x, y)$, so the volume of an object in 3D space. Now let us cut this shape with a vertical plane parallel to the x-axis, as illustrated in Fig. 6. The intersection between the volume and the plane will be a flat region in the (x, z) plane, identifiable by a single value of y given by where the vertical plane intersects the y-axis. We know how to calculate the area $A = A(y)$ of this shape:

$$A(y) = \int_a^b f(x, y)\,\mathrm{d}x. \tag{18}$$

Within the integral the only variable is x since y is fixed. This is therefore a 1D integral which we can attempt to solve using known techniques. The total volume V is clearly made by varying y between the given bounds of $[c, d]$, and summing over their contributions:

$$V = \int_c^d A(y)\,\mathrm{d}y. \tag{19}$$

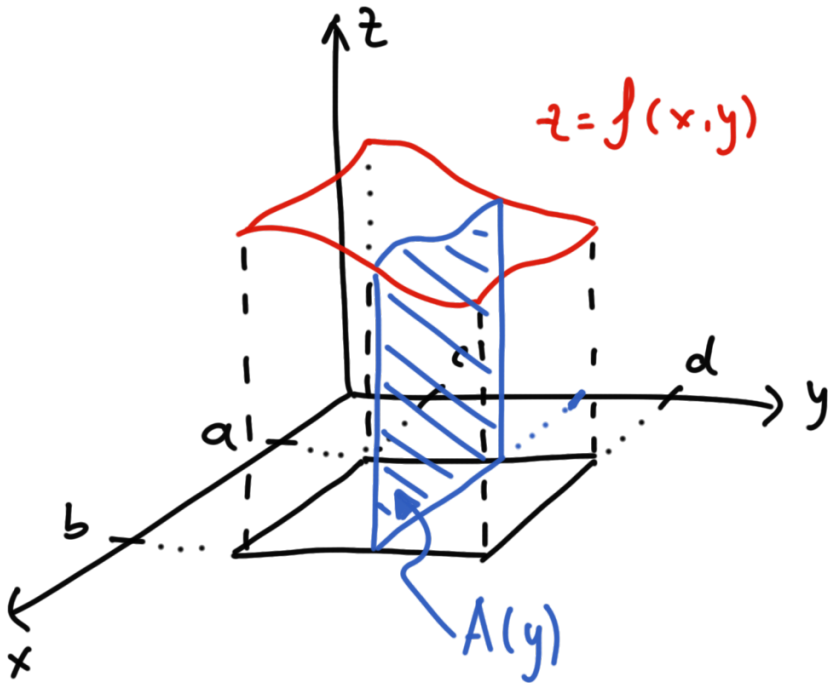


Figure 6: Integration in two dimensions.

Substituting Eq. (18) into Eq. (19), and noting that performing the same operations, but choosing instead a vertical plane parallel to y, would have to yield the same result, we can write

$$V = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y \right) \mathrm{d}x. \tag{20}$$

Because the order in which we chose to integrate does not matter[2], we adopt the following notation to indicate a general (definite) integral of f over some domain D:

$$V = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

Note that if a function happens to be separable $f(x,y) = h(x) \cdot g(y)$, then the double integral above, on a rectangular domain, simplifies to the product of two 1D integrals. The following integrability theorem holds:

Theorem 1. *Let $D \subset \mathbb{R}^2$ be a regular domain and $f : D \to \mathbb{R}$ a continuous function. Then f is integrable in D.*

We used the concept of a *regular domain* here, which is worth dwelling on. A domain D is said to be regular if it is *open* (i.e., every point in D is an interior point) and if its boundary ∂D is *piecewise smooth* (i.e., represented by a finite number of functions that admit continuous derivatives). We will elaborate more on this in what follows.

We have seen an example of how a 2D integral can be used to calculate a volume in 3D. However, a double integral can also define the measure (area) of a domain. A limited domain $D \subset \mathbb{R}^2$ is said to be *measurable* if the function 1 is integrable in D. Then, the measure (or area) of D is given by

$$|D| = \iint_D 1\,\mathrm{d}x\,\mathrm{d}y. \tag{22}$$

[2]Although this holds for the integrals treated in this course, this statement is not strictly true. If you are interested to learn more, look up Fubini's theorem!

2.1 Non-rectangular domains

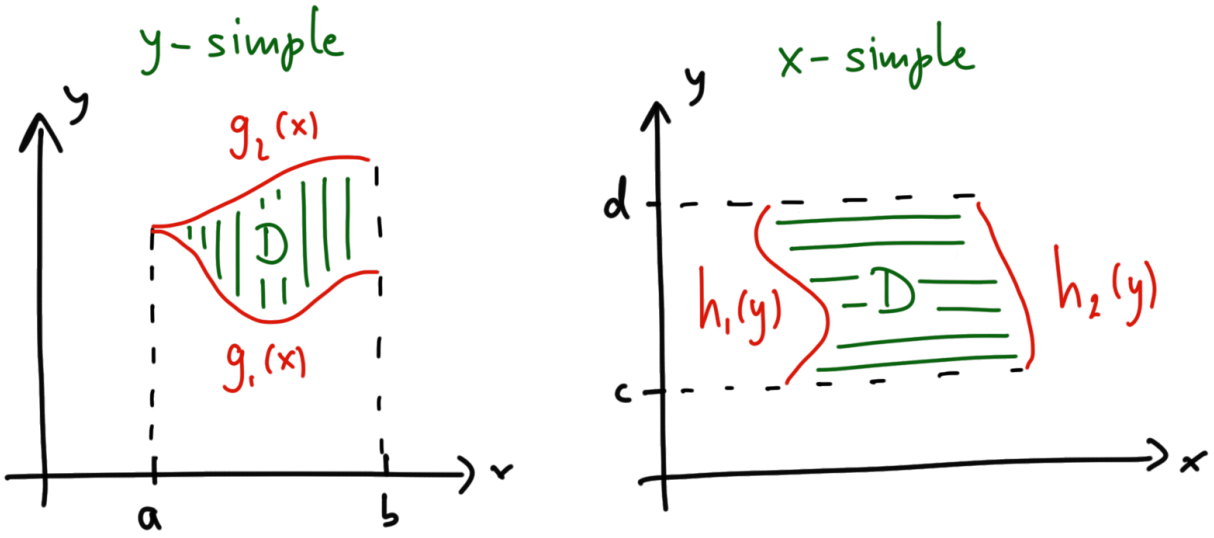


Figure 7: X-simple and y-simple domains for double integration.

The extension of double integrals with rectangular domains, as discussed above, to more complex domains, is relatively straightforward. It will be convenient to consider simple domains first. We will call a domain *x-simple* if horizontal lines always intersect the domain into a single line segment (in the domain), and y-simple if vertical lines always intersect the domain into a single line segment. The domains are illustrated in Fig. 7. The choice of horizontal and vertical here is simply a convention based on how we normally draw a 2D coordinate system. In slightly more mathematical terms,

Definition 2. *A set $D \subset \mathbb{R}^2$ is y-simple if*

$$D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], g_1(x) \leq y \leq g_2(x)\}, \tag{23}$$

where g_1 and g_2 are continuous functions.

Slicing D with a line parallel to the y axis therefore always yields a line segment, and this line segment varies continuously with a change in the position of the line. Similarly, we can also write the definition for x-simple domains as

Definition 3. *A set $D \subset \mathbb{R}^2$ is x-simple if*

$$D = \{(x, y) \in \mathbb{R}^2 : y \in [c, d], h_1(y) \leq x \leq h_2(y)\}, \tag{24}$$

where h_1 and h_2 are continuous functions.

We can now see how double integrals should be performed on such domains. Specifically, if D is y-simple, then the double integral of some function $f(x, y)$ over D can be computed by integrating over the line segments parallel to the y axis first, and is therefore given by:

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y \right) \mathrm{d}x. \tag{25}$$

Similarly, if D is x-simple, then we should integrate over the line segments along the x direction first:

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x \right) \mathrm{d}y. \tag{26}$$

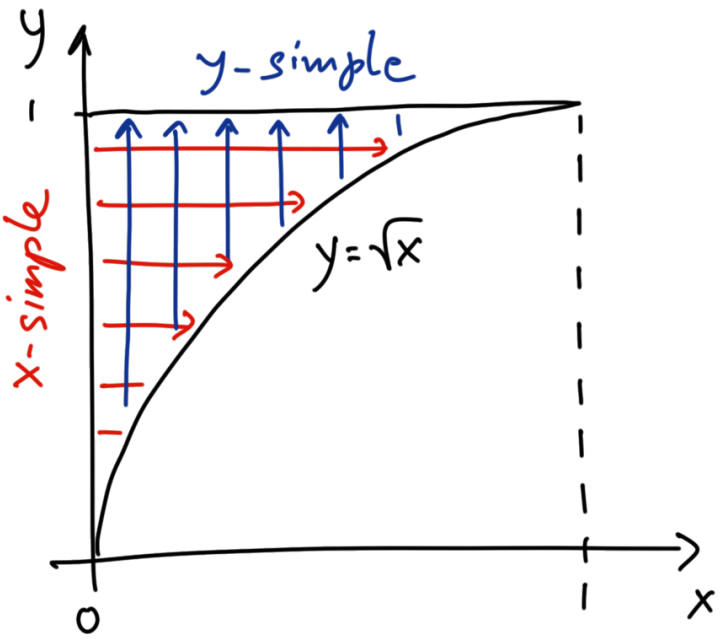


Figure 8: Domain of integral in Example 1. We see the domain is both x-simple and y-simple. The order in which we can chose to integrate is denoted by the red and blue lines.

Both approaches allow us to reduce the double integral to two single integrals, which we know how to tackle. A rectangular domain is trivially both x-simple and y-simple, but there is a further family of non-rectangular domains which are both x-simple and y-simple, such as the disc. Such domains are called *simple*. Note that if the domain is simple, then we can choose whether to partition it as x-simple or y-simple. A judicious choice here can dramatically change the difficulty of the integral.

Most domains you will encounter may not be simple, but as long as they can be partitioned into a finite number of simple domains we can still integrate across them, and the methods discussed above apply. This provides another, perhaps more intuitive, way of defining a *regular* domain: it is one that can be written as a union of a finite number of simple domains. So we see that an annulus, for example, would not be a simple domain, but would be regular (and thus allowable). We will not deal with domains that are not regular in this course.

Example 1. Evaluate the integral

$$I = \iint_D \sin\left(y^3\right) \mathrm{d}x\, \mathrm{d}y, \tag{27}$$

where D is the area between the curves $y = \sqrt{x}$ and $y = 1$, for $x \in [0, 1]$.

Solution: The domain is both x-simple and y-simple, as shown in Fig. 8, so we can choose the order of integration based on preference. Considering D to be y-simple, the integral becomes

$$I = \int_0^1 \left(\int_{y=\sqrt{x}}^{y=1} \sin\left(y^3\right) \mathrm{d}y \right) \mathrm{d}x. \tag{28}$$

Now unfortunately the integral in the parenthesis has no elementary primitive (one can write the solution using complex analysis in terms of gamma functions, a task not for the faint of heart – but do have a go if you're feeling intrepid!). A far simpler solution is to consider the

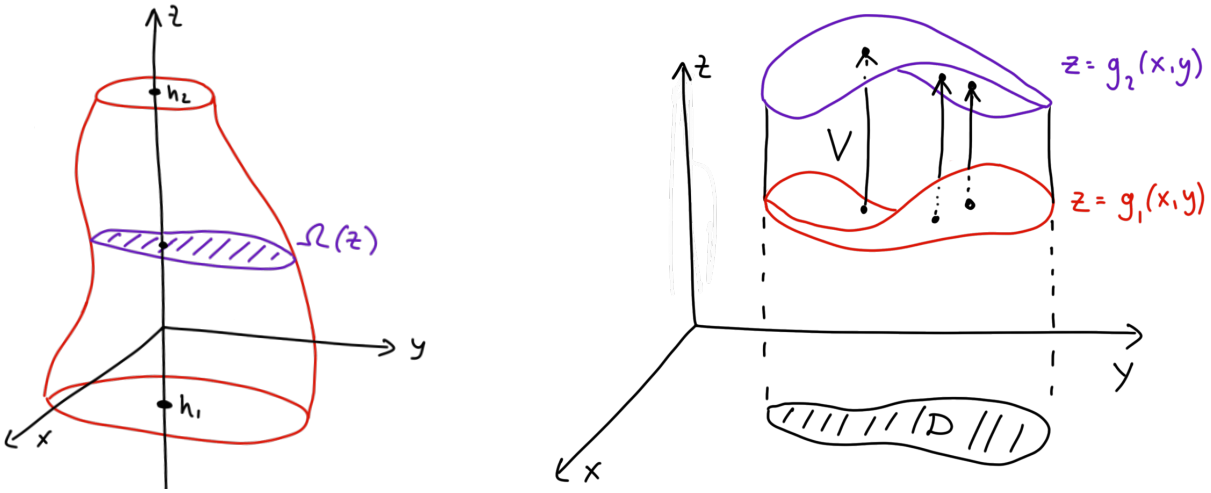


Figure 9: Separation of the domains of triple integrals based on domain type: the first domain can be split into layers of area $\Omega(z)$ for $h_1 \geq z \geq h_2$. The second integrates along the z-axis lines connecting two functions $z = g_1(x, y)$ and $z = g_2(x, y)$, for all (x, y) points in the domain D.

integral on an x-simple domain instead. Then, we can write

$$I = \int_0^1 \sin(y^3) \left(\int_{x=0}^{x=y^2} \mathrm{d}x \right) \mathrm{d}y \tag{29}$$

$$= \int_0^1 \sin(y^3)\, y^2 \, \mathrm{d}y \tag{30}$$

$$= \left[-\frac{1}{3} \cos(y^3) \right]_0^1 = \frac{1}{3}(1 - \cos(1)). \tag{31}$$

The order of integration does not impact the integrability of the function, but it does have a material impact on the ease with which one can find the solution in simple form. □

2.2 Triple integrals

We can extend the approach described above to three dimensions, allowing us to tackle triple integration. To do this, our previous definition of rectangular domains in the plane $[a, b] \times [c, d]$ must be replaced by a cuboid $[a, b] \times [c, d] \times [e, f]$ in 3D space, but otherwise the approach remains the same. We can immediately see that the main challenge with triple integrals, analogously to double integration, will be the treatment of more general, non-cuboid domains. Provided the domain of integration can be given a relatively simple analytical representation, a continuous function defined over this domain will still be integrable, and we can proceed to find the integral via appropriate iterative 1D integration. Two typical kinds of triple integration that are most commonly encountered are illustrated in Fig. 9: triple integration *by layers* and triple integration *by lines*. You may recognize the first as appropriate for the integration of domains that are x- and y-simple, while the second is used for z-simple domains. We thus can reduce the triple integral into a single and a double integral in two different ways, as given by the following two definitions:

Definition 4. *Integration by layers: Let Ω be a domain in $\mathbb{R}^3$ given by*

$$\Omega = \{(x, y, z) : h_1 \leq z \leq h_2, (x, y) \in \Omega(z)\} \tag{32}$$

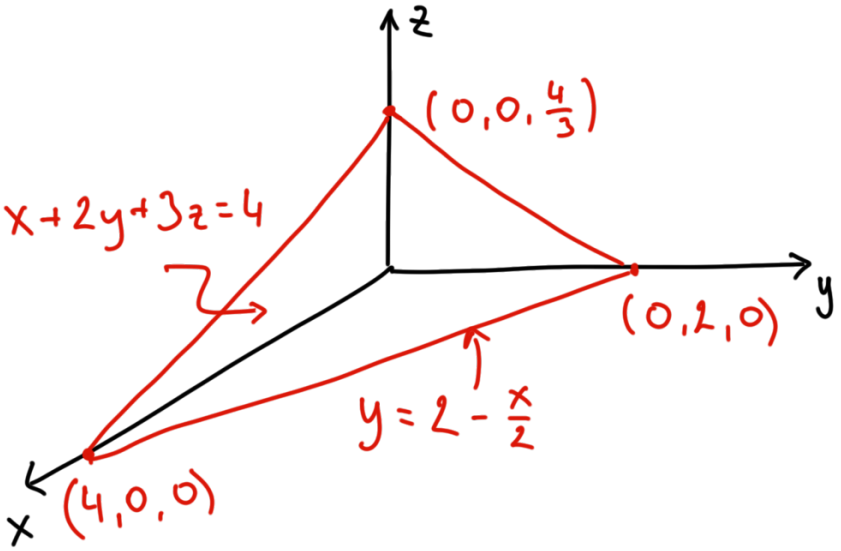


Figure 10: Tetrahedron given in Example 2.

where for every $z \in [h_1, h_2]$, $\Omega(z)$ is a plane regular domain. Then, if $f : \Omega \to \mathbb{R}$ is a continuous function, f is integrable in Ω and the integral can be calculated as

$$\iiint_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{h_1}^{h_2} \left(\iint_{\Omega(z)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}z. \tag{33}$$

Definition 5. *Integration by lines: Let Ω be a domain in $\mathbb{R}^3$ given by*

$$\Omega = \{(x,y,z) : g_1(x,y) \le z \le g_2(x,y),\ (x,y) \in D\} \tag{34}$$

where D is a regular domain in the (x,y) plane and $g_1, g_2 : D \to \mathbb{R}$ are continuous functions. Then, if $f : \Omega \to \mathbb{R}$ is a continuous function, f is integrable in Ω and the integral can be calculated as

$$\iiint_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{D} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y. \tag{35}$$

These expressions contain at most a double integral, and we can deploy the methods discussed earlier to reduce them further into single integrals that we can solve explicitly. For the purposes of this course we will not be interested in integrating in spaces having domains of dimensionality higher than three. Nevertheless, it should be fairly obvious by now how this methodology can be extended to tackle integrals in higher dimensions.

Example 2. Find the mass of the tetrahedron bounded by the three coordinate planes and the plane $x + 2y + 3z = 4$, if its density is given by $\rho(x,y,z) = \rho_0 x$.

Solution: The volume of interest is a tetrahedron that intersections the coordinate axes in points $(4,0,0)$, $(0,2,0)$, and $(0,0,4/3)$, as shown in Fig. 10. In the (x,y) plane the domain is the triangle bounded by the lines $x = 0$, $y = 0$, and $y = 2 - x/2$, found by setting $z = 0$ into the equation of the plane. For each point (x,y) in this domain, the volume can be found by

integrating along the z-axis, from $z = 0$ to $z = \frac{1}{3}(4 - x - 2y)$. We therefore have

$$\begin{aligned} m &= \iiint_V \rho(x, y, z)\, \mathrm{d}V = \\ &= \int_{x=0}^{x=4} \int_{y=0}^{y=2-x/2} \int_{z=0}^{z=(4-x-2y)/3} \rho_0 x\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \\ &= \frac{\rho_0}{3} \int_{x=0}^{x=4} \int_{y=0}^{y=2-x/2} (4 - x - 2y) x\, \mathrm{d}x\, \mathrm{d}y = \\ &= \frac{\rho_0}{3} \int_{x=0}^{x=4} \left(4x - 2x^2 + \frac{x^3}{4}\right) \mathrm{d}x = \\ &= \frac{\rho_0}{3} \left(32 - 42\frac{2}{3} + 16\right) = \frac{16}{9}\rho_0. \end{aligned}$$

□

2.3 The Jacobian matrix and determinant

It is often very useful to be able to change the integration variables to simplify the form of an integral. In fact, changing variables is one of the most common strategies used to solve 1D integrals. For example, say we are tasked with solving the integral $I = \int f(x)\, \mathrm{d}x$, which can be simplified if we change the variable $x \to t$. If x and t are related through some function g, then we have that

$$x = g(t), \quad \mathrm{d}x = \mathrm{d}g(t) = \frac{\mathrm{d}g(t)}{\mathrm{d}t}\, \mathrm{d}t, \quad f(x) = f(g(t)),$$

which allows us to rewrite the integral as

$$I = \int f(x)\, \mathrm{d}x = \int f(g(t)) \frac{\mathrm{d}g(t)}{\mathrm{d}t}\, \mathrm{d}t.$$

In this section we shall concern ourselves with how this approach can be extended to multiple dimensions. Importantly, changing variables will also change the parametrization of the domain of integration, so this can be a useful strategy to move between complex domains (which may not be simple, in the sense above) to simpler, and even rectangular, domains. We start with the simplest new case: a double integral over a 2D domain. Say we want to calculate the value of the following integral

$$\iint_D f(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{36}$$

while changing the original coordinates from (x, y) to some new set of coordinates (u, v) described by the transformation T

$$(x, y) = T(u, v). \tag{37}$$

The domain D of the integral will, necessarily, also change $T : D' \to D$ (note that given the definition of Eq. (37), T acts on (u, v)). We can can also write explicitly

$$\begin{cases} x = g(u, v) \\ y = h(u, v), \end{cases}$$

and the integrand becomes $f(g(u, v), h(u, v)) \coloneqq \tilde{f}(u, v)$.

In the 1D case, when changing variables, we were interested in finding an expression for how the infinitesimal length element changes under the transformation. In 2D, we ask how the infinitesimal area element $\mathrm{d}A$ changes with the change of variables. Consider the transformation of Eq. (37) illustrated in Fig. 11: we are interested in calculating what the infinitesimal area

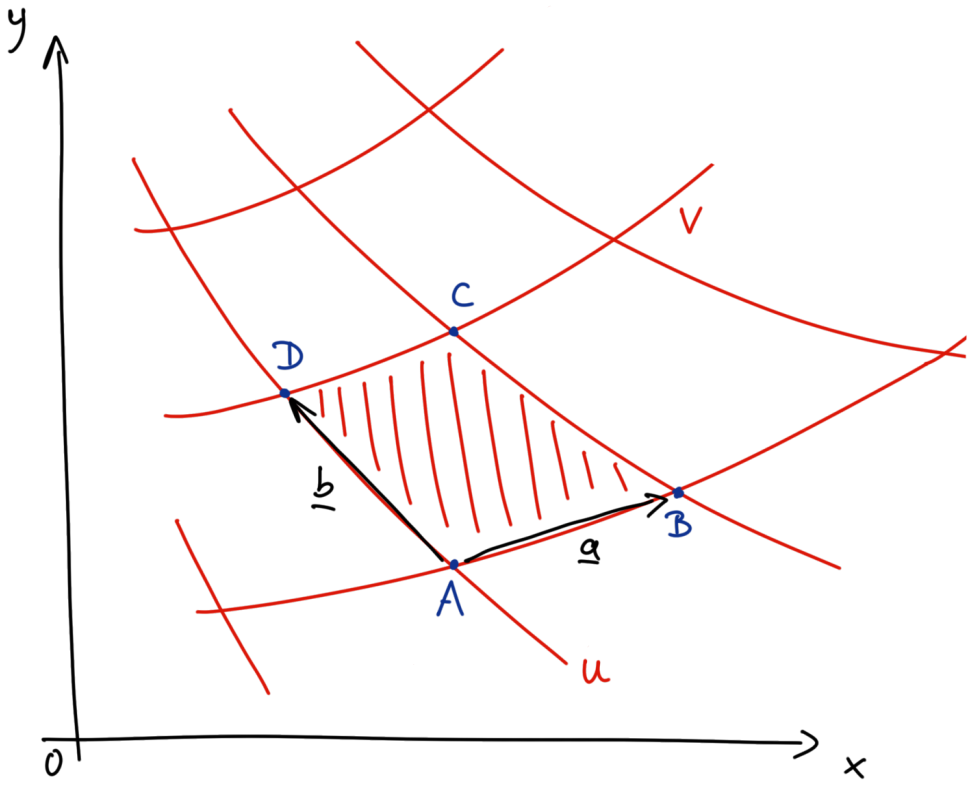


Figure 11: The infinitesimal area of a domain changes in a coordinate transformation, and must be accounted for in integration using the Jacobian. Here, (x, y) represent the standard Cartesian coordinate system, and the lines of constant u and v (the new coordinates) are drawn in red.

$\mathrm{d}x\,\mathrm{d}y$ is when written in the new coordinates $\mathrm{d}v\,\mathrm{d}u$, and vice versa. To find this, consider the area of the shaded region in Fig. 11 between the vertices A, B, C and D. For a generic transformation the sides of this shape will be curved, but in the limit of infinitesimal displacements, to the first order linear approximation, the sides can be considered straight lines and the shape tends to a parallelogram. The vertices of the parallelogram have the following (x, y) components:

A:	$g(u, v)$	$h(u, v)$
B:	$g(u + \mathrm{d}u, v)$	$h(u + \mathrm{d}u, v)$
C:	$g(u + \mathrm{d}u, v + \mathrm{d}v)$	$h(u + \mathrm{d}u, v + \mathrm{d}v)$
D:	$g(u, v + \mathrm{d}v)$	$h(u, v + \mathrm{d}v)$

From the vertices we can find the vector sides of the parallelogram

$$\mathbf{a} = (g(u + \mathrm{d}u, v), h(u + \mathrm{d}u, v)) - (g(u, v), h(u, v)) = \left(\frac{\partial g}{\partial u}\,\mathrm{d}u, \frac{\partial h}{\partial u}\,\mathrm{d}u\right)$$
$$\mathbf{b} = (g(u, v + \mathrm{d}v), h(u, v + \mathrm{d}v)) - (g(u, v), h(u, v)) = \left(\frac{\partial g}{\partial v}\,\mathrm{d}v, \frac{\partial h}{\partial v}\,\mathrm{d}v\right).$$

Simplifying the notation by writing $\frac{\partial f}{\partial \alpha} = f_\alpha$, the area of the parallelogram A is

$$\mathrm{d}A \equiv \mathrm{d}x\,\mathrm{d}y = |\mathbf{a} \times \mathbf{b}| = \det(\mathbf{a}, \mathbf{b}) = \det\begin{pmatrix} g_u\,\mathrm{d}u & h_u\,\mathrm{d}u \\ g_v\,\mathrm{d}v & h_v\,\mathrm{d}v \end{pmatrix} = (g_u h_v - g_v h_u)\,\mathrm{d}u\,\mathrm{d}v. \tag{38}$$

The relationship between the area written in the two different coordinate systems is therefore

$$\mathrm{d}x\,\mathrm{d}y = (g_u h_v - g_v h_u)\,\mathrm{d}u\,\mathrm{d}v, \tag{39}$$

or in matrix notation, which lends itself more naturally to generalization in more dimensions,

$$\mathrm{d}x\,\mathrm{d}y = \det\begin{pmatrix} g_u & h_u \\ g_v & h_v \end{pmatrix}\mathrm{d}u\,\mathrm{d}v. \tag{40}$$

This matrix containing all first-order partial derivatives is commonly encountered in vector calculus, and merits a name: the Jacobian matrix. Both the matrix and its determinant are often simply referred to as the 'Jacobian', but this is (perhaps surprisingly) rarely confusing. The Jacobian matrix is sometimes defined as the transpose of the expression above, a form originating from the requirement to define continuity for multivariable vector functions. For the use of the matrix this can be somewhat confusing at times, but as regards the determinant there is no difference since the determinant is invariant per transposition.

Definition 6. *Let $D \subset \mathbb{R}^2$ be a regular domain, let $f : D \to \mathbb{R}$ be a continuous function, and let $T : D' \to D$, $(x, y) = T(u, v)$ be a coordinate transformation with $x = g(u, v)$ and $y = h(u, v)$. Then*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(g(u,v),h(u,v))J(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{41}$$

where

$$J(u,v) = \det\begin{pmatrix} g_u & h_u \\ g_v & h_v \end{pmatrix} \tag{42}$$

is the determinant of the Jacobian matrix.

Example 3. Evaluate

$$I = \iint_D \frac{4}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y, \tag{43}$$

where D is the area enclosed by the lines $y = x - 2$, $y = x - 4$, $x = 0$, and $y = 0$.

Solution: We start by defining two new variables $u = x + y$ and $v = x - y$, or equivalently, $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. The Jacobian of this coordinate transformation is given by

$$J(u,v) = \frac{1}{4}\det\begin{pmatrix} \frac{\partial(u+v)}{\partial u} & \frac{\partial(u-v)}{\partial u} \\ \frac{\partial(u+v)}{\partial v} & \frac{\partial(u-v)}{\partial v} \end{pmatrix} = \frac{1}{4}\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -\frac{1}{2}. \tag{44}$$

We are only interested in the size of the Jacobian, not the sign (the sign will depend on the orientation choice of the vectors taken for the cross product). Taking $|J|$ we can then rewrite the integral as

$$I = \iint_{D'} \frac{4}{v^2}\cdot\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v. \tag{45}$$

Now we must analyse the domain, drawn in Fig. 12. The lines $y = x-2$ and $y = x-4$ correspond to $v = 2$ and $v = 4$, respectively. These will be the limits of the domain along v. To find the limits on u we use the remaining two lines and parametrize them in terms of (u, v). Firstly, when $x = 0$ we have that $u = y$ and $v = -y$, so $u = -v$. When $y = 0$ we find instead that $u = x = v$. Thus u will vary between $-v$ and $+v$. Therefore

$$I = \int_{v=2}^{v=4}\int_{u=-v}^{u=v}\frac{2}{v^2}\,\mathrm{d}u\,\mathrm{d}v = \int_{v=2}^{v=4}\frac{2}{v^2}\cdot 2v\,\mathrm{d}v = \ln(16). \tag{46}$$

□

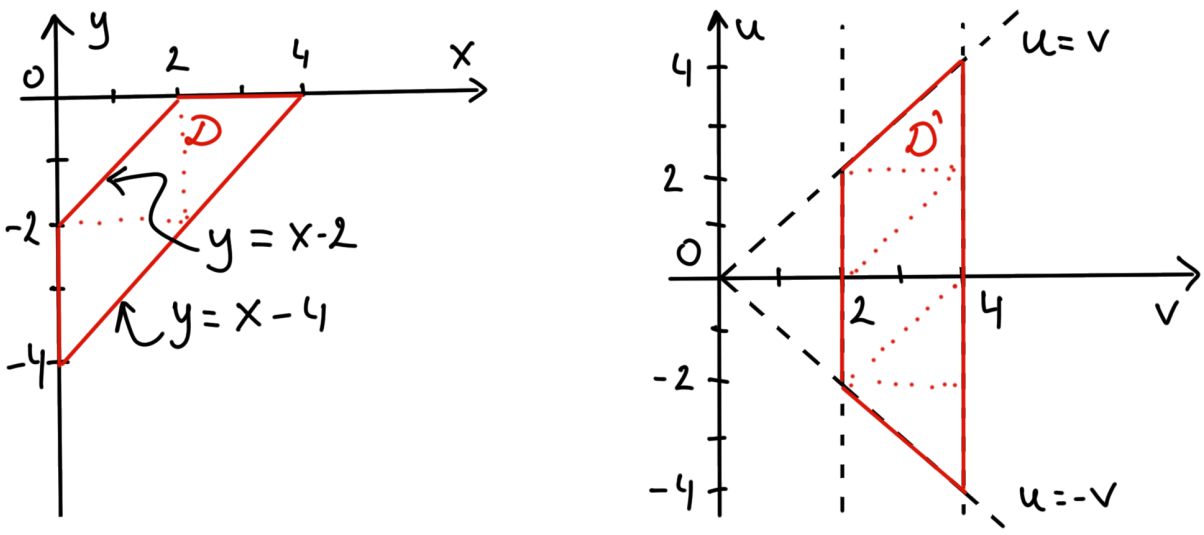


Figure 12: Integration domains for Example 3 in the (x, y) and (u, v) coordinate systems. Note the dotted triangles in both, each with an area of 2. There are 3 such triangles in the (x, y) domain but 6 in the (u, v) domain. The transformation has stretched the domain, and this is accounted for by the Jacobian of 1/2.

2.4 Curvilinear coordinate systems

Having defined the Jacobian, we are now able to move between any two coordinate systems when dealing with multiple integrals. In addition to the standard Cartesian coordinate system there are three other systems that are widely used in physics because they represent common symmetries. These are the plane polar coordinate system in 2D, and the spherical polar and cylindrical systems in 3D. We will discuss them in some detail here.

2.4.1 2D: Plane polar coordinates

Plane polar coordinates are suitable to describe systems with axial symmetry in a plane. The transformation $(x, y) \to (r, \theta)$ is given by the following relations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

The basis vectors of the Cartesian system are unit vectors along the x- and y-axes, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. The basis vectors in the new system will be different: $\hat{\mathbf{r}}$ is the unit vector along the line through the origin at some fixed angle θ to the x-axis, while $\hat{\theta}$ is the unit vector tangent to the circumference at some fixed distance from the origin (i.e, fixed r):

$$\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta \\ \hat{\theta} &= -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta. \end{aligned} \tag{47}$$

We can calculate the Jacobian of the transformation, which will allow us to use it to simplify the calculation of some double integrals:

$$J(r,\theta) = \det\begin{pmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial r} r\sin\theta \\ \frac{\partial}{\partial \theta} r\cos\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{pmatrix} = \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} = r(\sin^2\theta + \cos^2\theta) = r. \tag{48}$$

The infinitesimal area in this coordinate system is then given by

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = J(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{49}$$

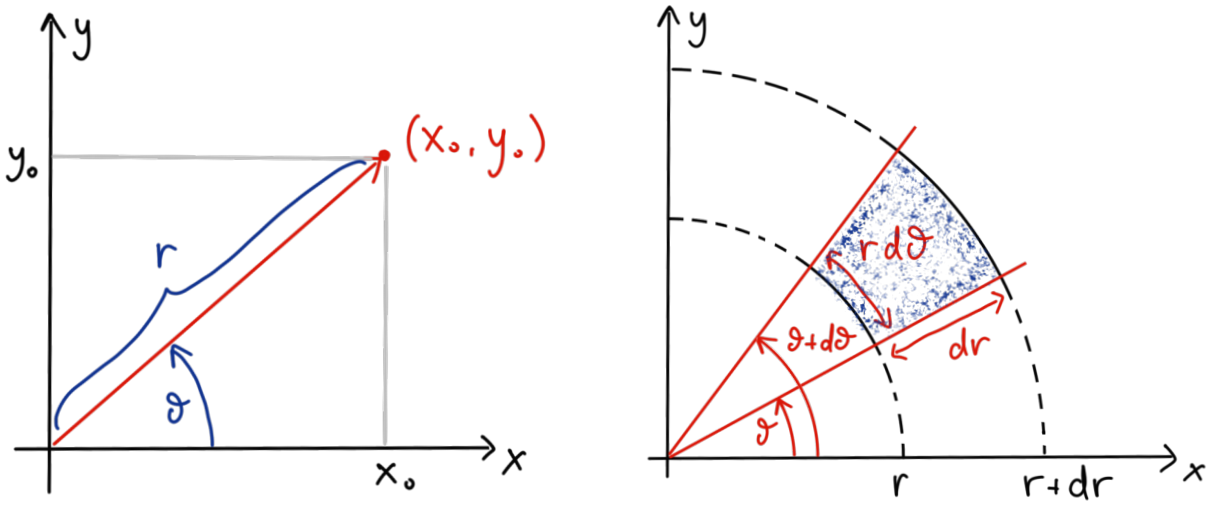


Figure 13: Plane polar coordinate system allows us to identify any point in the plane (except the origin) via two new variables: r, the Euclidean distance of the point to the origin, and the angle θ. In these system the infinitesimal surface element in has an area of $r\,\mathrm{d}r\,\mathrm{d}\theta$: it depends explicitly on the distance of the element to the origin.

The sketch in Fig. 13 shows this result more intuitively: the blue-shaded infinitesimal area, in the limit of $\mathrm{d}r, \mathrm{d}\theta \to 0$, is simply a rectangle with sides $r\,\mathrm{d}\theta$ and $\mathrm{d}r$, and therefore has an area of $r\,\mathrm{d}r\,\mathrm{d}\theta$. Interestingly, in plane polar coordinates $\mathrm{d}A$ this is not constant in all space like $\mathrm{d}x\,\mathrm{d}y$, but increases linearly with distance from the origin.

Example 4. Evaluate

$$I = \iint_D \frac{xy^2}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y, \tag{50}$$

where D is the area enclosed by the circle of radius 2 for $x \geq 0$ and $y \geq 0$.

Solution: We note that the domain is particularly simple to express in plane polar coordinates: on the (x, y) plane it represents a quarter of a disc, so we must have that $r \in [0, 2]$ and $\theta \in [0, \pi/2)$. We proceed to rewrite the integral in this coordinate system by making the substitutions $x = r\cos\theta, y = r\sin\theta$, and making sure we remember to include the Jacobian:

$$\begin{aligned}
I &= \iint_D \frac{r^3\cos\theta\sin^2\theta}{r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \\
&= \int_0^2 \int_0^{\pi/2} \frac{r^3\cos\theta\sin^2\theta}{r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \\
&= \left(\int_0^2 r^2\,\mathrm{d}r\right)\left(\int_0^{\pi/2} \sin^2\theta\cos\theta\,\mathrm{d}\theta\right) = \\
&= \left[\frac{r^3}{3}\right]_0^2 \cdot \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{8}{9}.
\end{aligned}$$

□

Example 5. Evaluate the Gaussian integral

$$G = \int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x. \tag{51}$$

Solution: The first thing to note is that the indefinite integral of e^{-x^2} does not have a simple primitive, that is to say, there is no elementary function the derivative of which is the squared exponential. However, the 2D version of the same integral is more tractable, so, for a change, we shall use double integration to help us solve an integration problem in one dimension. We start by defining the following integral

$$I(R) = \iint_D e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y, \tag{52}$$

where D is the area enclosed by the circle of radius R: $x^2 + y^2 < R$. This integral can be simplified by moving to plane polar coordinates:

$$\begin{aligned} I(R) &= \int_0^{2\pi} \left(\int_0^R e^{-r^2} r\,\mathrm{d}r \right) \mathrm{d}\theta = \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^R = \pi \left(1 - e^{-R^2} \right). \end{aligned}$$

Taking the limit of this expression for $R \to \infty$ gives us the value of the integral over the entire (x, y) plane

$$I = \lim_{R\to\infty} I(R) = \pi. \tag{53}$$

We now note that $I = G_x \times G_y = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x \right) \times \left(\int_{-\infty}^{+\infty} e^{-y^2}\,\mathrm{d}y \right)$. The two terms in x and y are identical and separable, so they must be equal. We thus conclude that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{I} = \sqrt{\pi}. \tag{54}$$

□

2.4.2 3D: Cylindrical coordinates

Cylindrical coordinates are perhaps the simplest extension of plane polar coordinates to 3 dimensions. As the name suggests, they are ideally suitable to represent physical and mathematical objects that possess some form of axial symmetry, or are at least easiest to describe within a coordinate system possessing such symmetry. The transformations from the 3D Cartesian coordinate system (x, y, z) are given by

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z. \end{aligned}$$

As shown in Fig. 14, the unit vectors in the Cartesian basis $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ are replaced by the plane polar basis vectors $\hat{\mathbf{r}}$, $\hat{\phi}$ in the (x, y) plane, while the basis vector along the z direction, $\hat{\mathbf{k}}$, remains unchanged. The relationship between basis vectors is therefore:

$$\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi \\ \hat{\phi} &= -\hat{\mathbf{i}}\sin\phi + \hat{\mathbf{j}}\cos\phi \\ \hat{\mathbf{z}} &= \hat{\mathbf{k}}. \end{aligned} \tag{55}$$

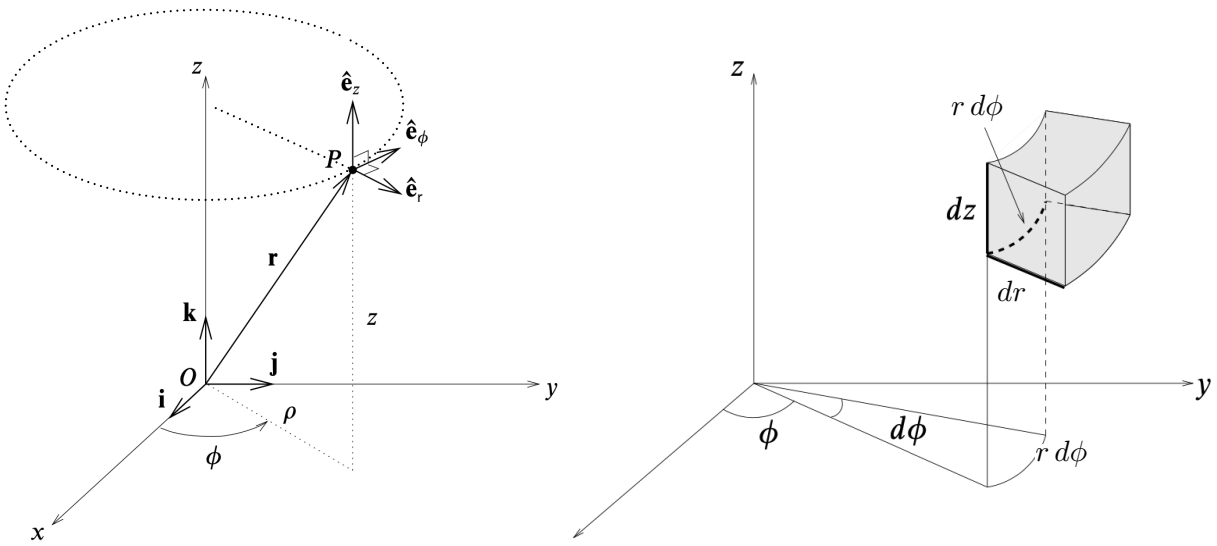


Figure 14: The cylindrical coordinate system, basis vectors, and volume element.

As illustrated in Fig. 14, the infinitesimal volume in this coordinate system tends to a cuboid with sides $\mathrm{d}r$, $\mathrm{d}z$, and $r\,\mathrm{d}\phi$. In this coordinate system we can therefore write the (square) infinitesimal line element as

$$(ds)^2 = (\mathrm{d}r)^2 + (r\,\mathrm{d}\phi)^2 + (\mathrm{d}z)^2. \tag{56}$$

Similarly, the surfaces of the cuboid have sides with areas given by

$$\mathrm{d}r\,\mathrm{d}z; \quad r\,\mathrm{d}r\,\mathrm{d}\phi; \quad r\,\mathrm{d}\phi\,\mathrm{d}z; \tag{57}$$

and the volume element is

$$\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z. \tag{58}$$

The Jacobian of this transformation is therefore $J(r, \phi, z) = r$, as can be verified easily by an explicit calculation of the determinant of the Jacobian matrix.

2.4.3 3D: Spherical polar coordinates

For systems with spherical symmetry we will find the use of the spherical polar coordinate system highly advantageous. The transformations here are given by the following relations:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta, \end{aligned}$$

where $\theta \in [0, \pi]$ is the *polar* angle, measured between the z-axis and the radial vector connecting the origin to some point, and $\phi \in [0, 2\pi)$ is the *azimuthal* angle, measured between the x-axis and the projection of the radial vector onto the xy-plane. The new basis vectors in this system are shown in Fig. 15, and are given by the relations

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi) + \hat{\mathbf{k}}\cos\theta \\ \hat{\theta} &= \cos\theta(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi) - \hat{\mathbf{k}}\sin\theta \\ \hat{\phi} &= -\hat{\mathbf{i}}\sin\phi + \hat{\mathbf{j}}\cos\phi. \end{aligned} \tag{59}$$

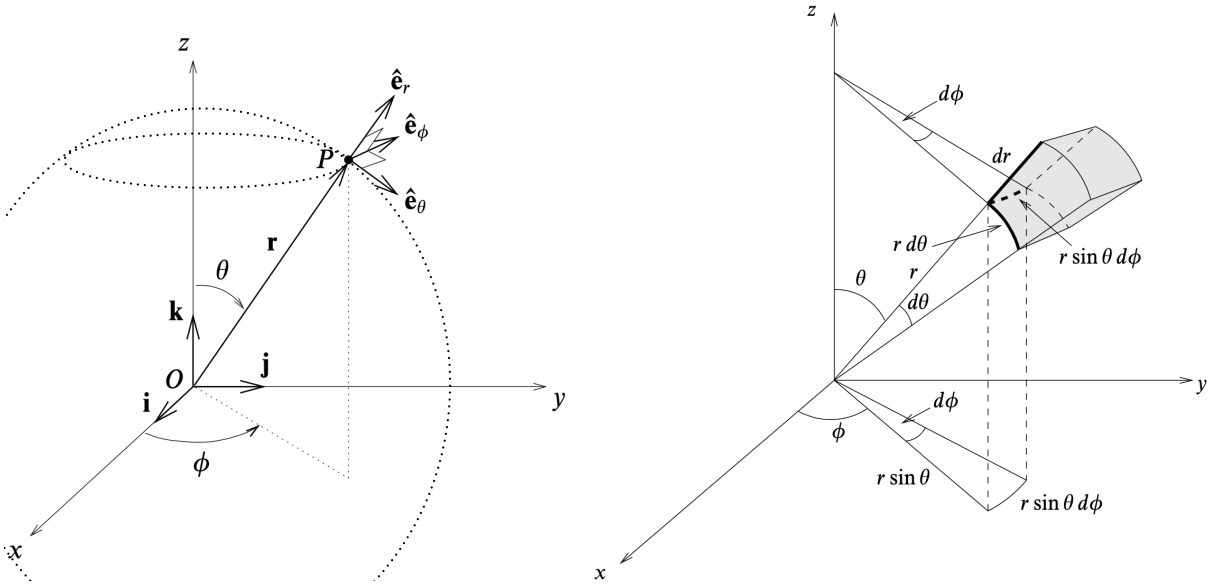


Figure 15: The spherical polar coordinate system, basis vectors, and volume element.

As illustrated in Fig. 15, the infinitesimal volume in this coordinate system tends to a cuboid with sides $\mathrm{d}r$, $r\,\mathrm{d}\theta$, and $r\sin\theta\,\mathrm{d}\phi$. The line element is therefore given by

$$(ds)^2 = (\mathrm{d}r)^2 + (r\,\mathrm{d}\theta)^2 + (r\sin\theta\,\mathrm{d}\phi)^2. \tag{60}$$

The surfaces of the cuboid have sides with areas

$$r\,\mathrm{d}r\,\mathrm{d}\theta; \quad r\sin\theta\,\mathrm{d}\phi\,\mathrm{d}r; \quad r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi; \tag{61}$$

and the volume element is

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{62}$$

The Jacobian of this transformation is therefore $J(r,\theta,\phi) = r^2\sin\theta$, as can be verified easily by an explicit calculation of the determinant of the Jacobian matrix.

Example 6. Integrate the function $f = x^2 z$ in the positive half of the sphere of radius R centred in the origin.

Solution: Our domain can be written as

$$\Omega = \{(x, y, z) : 0 \leq z \leq \sqrt{R^2 - x^2 - y^2}, x^2 + y^2 \leq R^2\},$$

which allows us to compute the integral by following the approach of Def. 5:

$$\begin{aligned}
\iiint_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iint_{x^2+y^2\leq R^2} \left(\int_0^{\sqrt{R^2-x^2-y^2}} x^2 z\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y = \\
&= \iint_{x^2+y^2\leq R^2} x^2 \left[\frac{1}{2} z^2 \right]_0^{\sqrt{R^2-x^2-y^2}} \mathrm{d}x\,\mathrm{d}y = \\
&= \iint_{x^2+y^2\leq R^2} \frac{1}{2} x^2 (R^2 - x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y = \\
&= \int_0^{2\pi} \left(\int_0^R \frac{1}{2} r^2 \cos^2\theta (R^2 - r^2) r\,\mathrm{d}r \right) \mathrm{d}\theta = \\
&= \frac{1}{2} \left(\int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta \right) \left(\int_0^R (r^3 R^2 - r^5)\,\mathrm{d}r \right) = \\
&= \frac{\pi}{2} \left[\frac{r^4}{4} R^2 - \frac{r^6}{6} \right]_0^R = \frac{\pi}{24} R^6.
\end{aligned}$$

Note how we have moved to plane polar coordinates to simply the double integral in (x, y) into a product of two single integrals in r and θ given the radial symmetry of the problem.

Alternatively, we could also consider the domain as a series of discs slicing the sphere:

$$\Omega = \{(x, y, z) : x^2 + y^2 \leq R^2 - z^2, 0 \leq z \leq R\},$$

which lends itself to splitting the triple integral according to Def. 4:

$$\begin{aligned}
\iiint_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^R \left(\iint_{x^2+y^2\leq R^2-z^2} x^2 z\,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}z = \\
&= \int_0^R \left(\int_0^{2\pi} \left(\int_0^{\sqrt{R^2-z^2}} r^2 \cos^2\theta\ z\ r\,\mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}z = \\
&= \int_0^R \left(\int_0^{\sqrt{R^2-z^2}} r^3\,\mathrm{d}r \right) \left(\int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta \right) z\,\mathrm{d}z = \\
&= \int_0^R \left[\frac{r^4}{4} \right]_0^{\sqrt{R^2-z^2}} \pi z\,\mathrm{d}z = \\
&= \frac{\pi}{4} \int_0^R z(R^2 - z^2)^2\,\mathrm{d}z = \frac{\pi}{4} \int_0^R (zR^4 + z^5 + 2R^2 z^3)\,\mathrm{d}z = \\
&= \frac{\pi}{4} \left[\frac{R^4 z^2}{2} + \frac{z^6}{6} - 2R^2 \frac{z^4}{4} \right]_0^R = \frac{\pi}{24} R^6.
\end{aligned}$$

Now let us use spherical polar coordinates instead.

$$\begin{aligned}\iiint_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_{\Omega} r^3\sin^2\theta cos^2\phi\cos\theta\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \\ &= \int_0^R\int_0^{\pi/2}\int_0^{2\pi} r^5\sin^3\theta cos^2\phi\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \\ &= \int_0^R r^5\,\mathrm{d}r\cdot\int_0^{\pi/2}\sin^3\theta\cos\theta\,\mathrm{d}\theta\cdot\int_0^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \\ &= \frac{R^6}{6}\cdot\frac{1}{4}\cdot\pi = \frac{\pi}{24}R^6.\end{aligned}$$

We see that the use of polar coordinates makes our domain a cuboid, and so the triple integral becomes a simple product of three independent one-dimensional integrals.

□

3 Statistical distributions

Consider we wish to perform an experiment, the outcome of which is well-defined (and ideally measurable), but *a priori* unknown. There is value in being able to make quantitative predictions on what the outcome of the experiment may be, without having to perform it. The mathematical tools that can assist us in making such predictions in real-world experiments fall under the general headings of probability and statistics. Of course this is a huge topic; in this course we will explore a very small part of it, that of continuous random distributions.

First, we should define some basic terminology. The experiment mentioned earlier will be called a *trial*, and the result of the experiment the *outcome*. The set of all possible outcomes of the trial, i.e. our space of possibilities, will be called the *sample space*. The sample space can be discrete, with a finite or infinite number of possible outcomes, or continuous, with a necessarily infinite number of possible outcomes. The trial of flipping a coin has a discrete number of outcomes if we are interested in which side ends facing upwards (heads, tails, or, possibly, it landing on its side), but a continuous set of outcomes if we were interested in predicting, for example, where on the floor it will land. If instead of considering an individual outcome we are interested in set of outcomes, itself a subset of the sample space, we will call such a subset an *event*. These too can be continuous or discrete. In this course we will be interested in continuous sample spaces, which will allow us to apply our vector calculus methods to the study of statistics.

How likely is a certain event, compared to all others? One way to think about this is to imagine repeating the trial many times over, and to count the frequency of measuring a certain event compared to all other outcomes. It turns out that this relative frequency tends to be approximately the same every time a large set of trials is performed, which spurs us to call this ratio the *probability* P of the event taking place. If we call the event A, then we can write

$$P(A) = \frac{n_A}{n_{\text{tot}}}, \tag{63}$$

where n_A is the number of outcomes of event A and n_{tot} is the number of total outcomes.

3.1 Continuous random variables

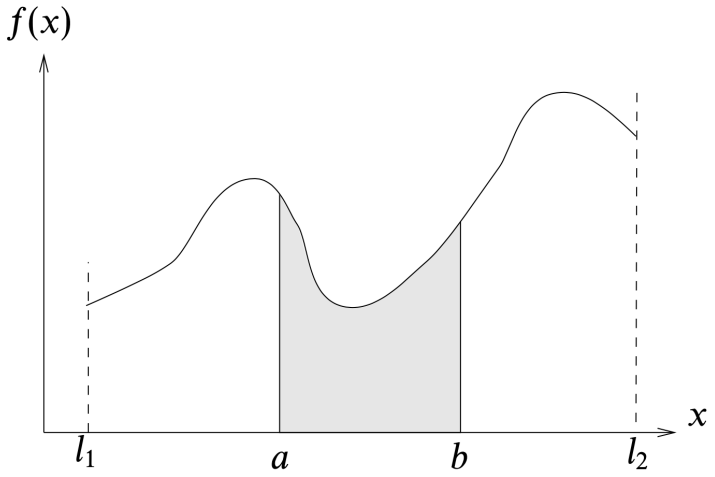


Figure 16: Probability density function (PDF) for a variable X that can take values between l_1 and l_2]. The probability of finding X between a and b is given by the integral of the PDF over that range, as given in Eq. (65).

A random variable in one dimension X is said to have a continuous distribution if X can assume any real value over some domain of the reals: $X \in [a, b] \subset \mathbb{R}$. The number of possible

outcomes is therefore equal to the number of values present on some finite interval of the reals. As this is infinite, we cannot assign a finite probability to individual outcomes: Eq. (63) shows that $P \to 0$ if $n_{\text{tot}} \to \infty$ for any finite n_A. Instead of considering individual outcomes, when dealing with continuous variables we must therefore consider outcomes over a finite continuous subset of the sample space.

We can use this insight to define a *probability density function* (PDF) f of a continuous random variable X as

$$P(x < X \leq x + \mathrm{d}x) = f(x)\,\mathrm{d}x. \tag{64}$$

Here $f(x)\,\mathrm{d}x$ is the probability of finding X in the interval between x and $\mathrm{d}x$. More generally:

$$P(a < X \leq b) = \int_a^b f(x)\,\mathrm{d}x. \tag{65}$$

We know that some outcome of a trial will take place with certainty, and so if we take the range to be the entire sample space S, the probability of finding X there must be 1 (i.e., 100%), by definition. This leads to the requirement for f to be integrable over S, and normalized to 1:

$$\int_S f(x)\,\mathrm{d}x = 1.$$

Furthermore, a probability must be a real number between 0 and 1, and can never be negative. For $f(x)$ to be a probability density we therefore further require f to be a non-negative function of $x, \forall x \in S$.

For practical reasons it will be useful to define the *cumulative probability distribution function* (CDF), denoted by $F(x)$, which is given by the following integral:

$$F(x) = \int_{l_1}^{x} f(u)\,\mathrm{d}u. \tag{66}$$

Clearly, this is the probability that X is less than, or equal to, x, i.e., $F(x) = P(X \leq x)$. We can turn back to Eq. (65), and rewrite it in terms of the CDF as

$$P(a < X \leq b) = \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{67}$$

We can see why the CDF is useful – if known, it provides us with a way to quickly determine the probability of the random variable lying in some range, without needing to evaluate an integral. From the fundamental theorem of calculus $f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$.

Example 7. Consider the PDF provided in Fig. 17, describing the seasonal maximum air temperature in SE England according to two climate modelling scenarios. Estimate the likelihood that the maximum temperatures exceed 2°C and 6°C, respectively, in the two scenarios. What is the probability the maximum temperature increase will be in the range 2°C-6°C?

Solution: The likelihood of the temperature exceeding 6°C is easy to glean directly from the PDF. It approaches 0 under the RCP2.6 scenario (the entire distribution lies below the 6° mark), but is around 50% in the scenario RCP8.5. The probability that the temperature will exceed 2°C is a little harder to extract from the PDF figure without having the numerical values, but can instead be found from the CDF:

$$P(X > 2)_{\text{RCP2.6}} \approx 1 - 35\% = 65\%,$$
$$P(X > 2)_{\text{RCP8.5}} \approx 1 - 5\% = 95\%.$$

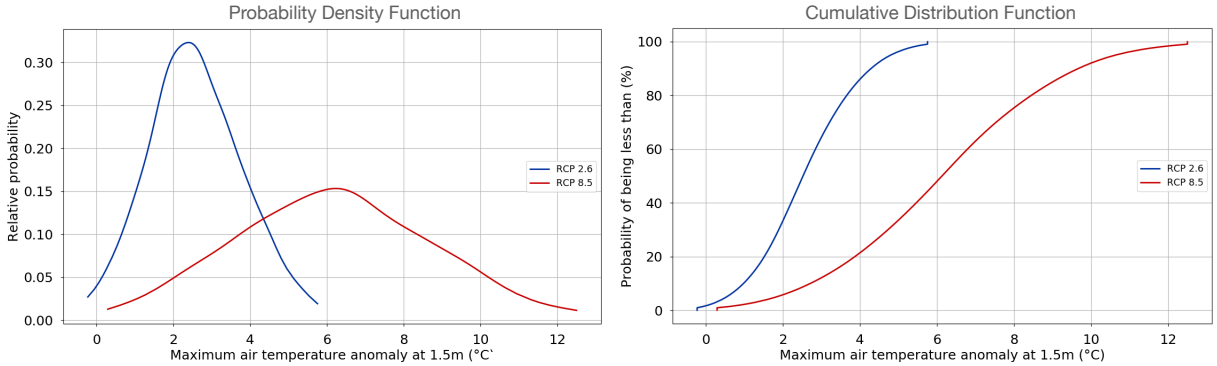


Figure 17: Seasonal average Maximum air temperature anomaly at 1.5m (in °C) for June-July-August in years 2070 up to and including 2098, in SE England using baseline 1961-1990. RCP stands for Representative Concentration Pathway. RCP8.5 is a scenario where we make little effort to curb emissions, while under RCP2.6 a concerted effort is made. Source: Met Office.

To find the probability of the temperatures ending up in the range 2°C-6°C we can simply subtract the CDFs at the two values as shown in Eq. (67): we find:

$$P(2 < X \leq 6)_{\text{RCP2.6}} \approx 100\% - 35\% = 65\%,$$
$$P(2 < X \leq 6)_{\text{RCP8.5}} \approx 50\% - 5\% = 45\%.$$

□

3.1.1 Expectation value

Definition 7. *The expectation value of any function $g(X)$ of the random variable X for a continuous distribution f is defined as*

$$E[g(X)] = \int_S g(x) f(x)\,\mathrm{d}x, \tag{68}$$

where S is the sample space, and assuming the integral exists. We note that it is a linear functional that depends on the functions f and g, rather than being a function of the random variable X.

The most common types of functions for which we will want to calculate the expectation value will be powers of the variable itself. Because of this, these expectation values are given a special name to distinguish them – they are called *moments* of the distribution, with

$$E[X^k] = \int_S x^k f(x)\,\mathrm{d}x \tag{69}$$

denoting the k-th moment of the distribution. Of these, the first moment – the expectation of the variable itself – is most widely used, and is more commonly called the mean:

$$E[X] = \int_S x f(x)\,\mathrm{d}x. \tag{70}$$

This quantity is often denoted as $\langle x \rangle$, $\bar{x}$, or by the greek letter μ. In addition to this, distributions are also commonly characterized by the *mode* and the *median*. The mode of a distribution is the value of the random variable X at which the probability density function $f(x)$ has its greatest value. Note that it is possible to have multiple modes in a distribution. The median of a distribution is the value of the random variable X at which the cumulative probability function F(x) takes the value 1/2: it divides the PDF into two, equally probably halves.

Another important expectation value is the *variance*:

Definition 8. *The variance of a distribution, σ^2, is defined as*

$$E[(X - \langle x \rangle)^2] = \sigma^2 = \int_S (x - \langle x \rangle)^2 f(x)\,\mathrm{d}x, \tag{71}$$

where S is the sample space, and assuming the integral exists.

The variance of a distribution is always positive. Its positive square root is known as the standard deviation of the distribution and is often denoted by σ. Roughly speaking, σ measures the spread of the values that X can assume around the mean. Often the variance is given in a slightly different form:

$$\begin{aligned}\sigma^2 &= E[(X - \langle x \rangle)^2] = \\ &= E[X^2] - 2\langle x \rangle E[X] + \langle x \rangle^2 = \\ &= E[X^2] - \langle x \rangle^2 \equiv E[X^2] - E[X]^2.\end{aligned}$$

It follows that the variance is the difference between the second moment and the square of the first moment of a distribution.

3.2 Distribution functions

There are some commonly encountered continuous distribution functions you should be familiar with. We list some of their PDFs here, and show them in Fig. 18, alongside their CDFs.

- **Uniform distribution**

  $$f(x) = \text{constant}$$

  This is the simplest PDF, and we note that the domain must be limited to some finite subset of $\mathbb{R}$ or the integral will not converge. Despite its simplicity, it is still widely used, for example:

  - For random number generation.
  - In Monte Carlo simulations, as a model for inputs where random samples from a uniform distribution are used to estimate results.
  - In sampling, as a prior distribution in Bayesian analysis when little prior knowledge about a parameter is available.
  - For hypothesis testing to compare the uniformity of a sample against a known uniform distribution.
  - In computer graphics, to distribute random points on a surface, or to determine the brightness of pixels in an image.

- **Exponential distribution**

  $$f(x) = \lambda e^{-\lambda x}$$

  The exponential distribution models the time between events in a Poisson process. Some common uses include:

  - In reliability engineering, to model the lifetime of a device or system, and to calculate its reliability and failure rate.
  - In survival analysis, to model the time to an event, such as the time to death, failure or diagnosis of a disease.
  - In finance, to model the time between financial events, such as the time between stock trades or the time between defaults in a portfolio.
  - In statistical inference, as a conjugate prior distribution in Bayesian analysis.

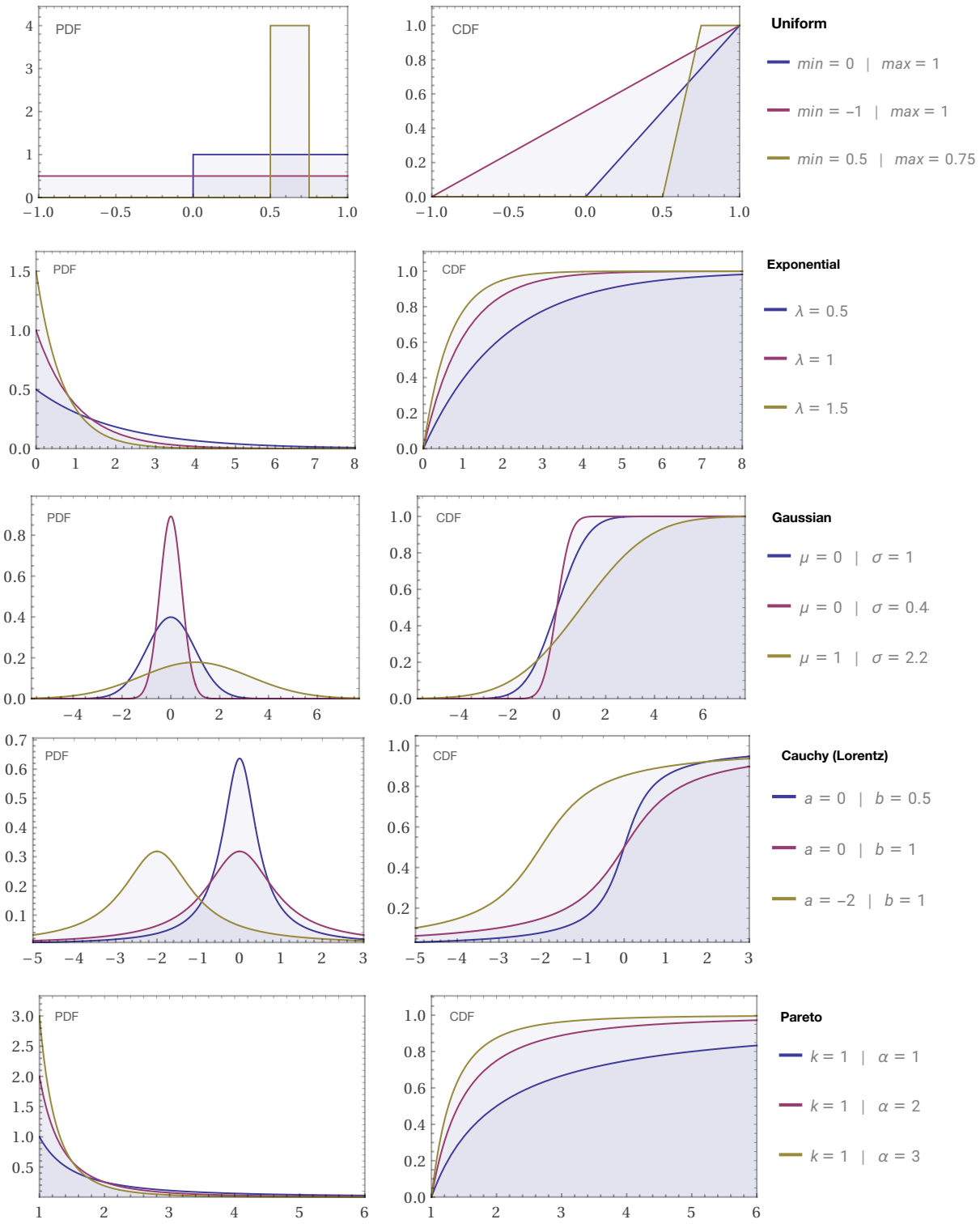


Figure 18: Commonly encountered probability density functions (PDFs) and cumulative distribution functions (CDFs). The equations are given in the main text.

- **Gausssian distribution**

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

The Gaussian (or normal) distribution is the most widely used probability distribution, and has many applications across various fields, including:

  - In physics, to quantify uncertainties and to describe random processes.
  - In statistics, to model the distribution of random variables, and as a basis for hypothesis testing.
  - In machine learning, in algorithms such as Gaussian Naive Bayes and Gaussian Mixture Models.
  - In signal processing, to model random noise and to perform signal smoothing.
  - In image processing, in edge detection, image smoothing, and deblurring.
  - In economics, to model stock prices and other financial data.
  - In biology, to model the distribution of various physical and biological phenomena, such as the sizes of organisms in a population.

- **Cauchy or Lorentzian distribution**

$$f(x) = \frac{1}{\pi b}\left(\frac{(x-a)^2}{b^2} + 1\right)^{-1}$$

The Lorentzian distribution is similar in shape to a Gaussian distribution, but it has a "heavier tail", which means it decays slower to zero (as $1/x^2$, rather than square-exponentially) and thus gives more weight to outliers than the Gaussian. Its uses include:

  - In spectral analysis, to model the line shapes of spectral peaks, such as those observed in emission spectroscopy, nuclear magnetic resonance (NMR), and Raman spectroscopy.
  - In robust statistics, as a model for outliers, as it is less sensitive to outliers than the normal distribution.
  - In signal processing, to model resonances in signals, such as those observed in vibration analysis.
  - In fluid mechanics, to model the turbulence in fluid flow.

- **Pareto distribution**

$$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}}$$

The Pareto distribution is another heavy-tailed distribution, popular in the social sciences. Its uses include:

  - In economics, to model the distribution of wealth.
  - In urban economics, to model the distribution of city sizes.
  - In computer networking, to model the distribution of internet traffic.
  - In software engineering, to model the distribution of software faults.
  - In the modelling of certain types of stochastic processes, such as Lévy flights (a type of random walk).

3.3 Joint probability distributions

The discussions above can be extended to multiple random variables, which may or may not be independent. Consider the case of two random variables X and Y, described by the join probability density function $f_{XY}(x, y)$. The two variables are said to be *independent* if we can factorize their joint PDF: $f_{XY}(x, y) = g_X(x)h_Y(y)$. Otherwise, the variables are interdependent, or coupled. Similarly to Eq. (64), we can write the probability for the variables to fall within some infinitesimal range as

$$\mathrm{d}P(X, Y) = f_{XY}(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{72}$$

which, again, must be normalized (or at least normalizable) over the space of all possible outcomes S:

$$\iint_S f_{XY}(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1. \tag{73}$$

The probability of an event is then simply a double integral over the 2D domain D of desired outcomes of the random variables X and Y:

$$P((X, Y) \in D) = \iint_D f_{XY}(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{74}$$

Example 8. Let $f_{XY}(x, y) = x+cy^2$ be a joint probability density function of two random variables X and Y, defined over the support $x \in [0, 1]$ and $y \in [0, 1]$. Find the appropriate normalization constant c, and determine the probability that both X and Y are less than, or equal to, $1/2$.

Solution: For the normalization we write

$$\begin{aligned} 1 = \int_0^1 \int_0^1 (x + cy^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \left(xy + \frac{cy^3}{3}\right)_0^1 \mathrm{d}x = \\ &= \int_0^1 \left(x + \frac{c}{3}\right)\mathrm{d}x = \\ &= \left(\frac{x^2}{2} + \frac{cx}{3}\right)_0^1 = \frac{1}{2} + \frac{c}{3}. \end{aligned}$$

Therefore, $c = 3/2$. To find the required probability we then need to integrate the normalized PDF over the domain of required outcomes:

$$P(0 < X \leq 1/2, 0 < Y \leq 1/2) = \int_0^{1/2} \int_0^{1/2} \left(x + \frac{3}{2}y^2\right)\mathrm{d}x\,\mathrm{d}y = \frac{3}{32}.$$

□

3.4 Marginal and conditional distributions

Given a joint PDF $f(x, y)$, we may be interested only in the probability function for X irrespective of the value of Y (or vice versa). This is called the *marginal distribution* of X (or Y), and is obtained by integrating the joint PDF over all allowed values of Y (or X):

$$f_X(x) = \int f(x, y)\,\mathrm{d}y; \qquad f_Y(y) = \int f(x, y)\,\mathrm{d}x. \tag{75}$$

Alternatively, we might be interested in the probability function of X given that Y takes some specific value $Y = y_0$, $P(X|Y = y_0)$. Such a probability is given by the *conditional distribution* $\mathcal{C}(x)$:

$$\mathcal{C}(x) = \frac{f(x, y_0)}{\int f(x, y_0)\,\mathrm{d}x} = \frac{f(x, y_0)}{f_Y(y_0)}.$$

An important point to highlight here is the need to renormalize this distribution. This is because the overall joint PDF is normalized assuming all values of Y are accessible. As we can see, the normalization needed is provided by the marginal distribution for $Y = y_0$.

Example 9. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$ for the joint PDF given in Example 8 $f_{XY}(x, y) = x + \frac{3}{2}y^2$. Find the probability that $Y > 1/2$ if $X = 1/3$.

Solution: We use Eq. (75) to find:

$$f_X(x) = \int_0^1 \left(x + \frac{3}{2}y^2\right) \mathrm{d}y = x + \frac{1}{2};$$
$$f_Y(y) = \int_0^1 \left(x + \frac{3}{2}y^2\right) \mathrm{d}x = \frac{1}{2} + \frac{3}{2}y^2.$$

For the second question we first need to find the conditional distribution

$$\mathcal{C}(y) = \frac{f(\frac{1}{3}, y)}{f_X(\frac{1}{3})} = \frac{\frac{1}{3} + \frac{3}{2}y^2}{\frac{5}{6}} = \frac{2}{5} + \frac{9}{5}y^2.$$

We note that the nominator is a single variable PDF, but one that is no longer normalized, hence the need for the denominator. The resulting conditional distribution is a proper PDF. To find the required probability we now simply need to integrate this conditional distribution over the required range for Y:

$$P(1/2 < Y \leq 1) = \int_{1/2}^1 \mathcal{C}(y)\, \mathrm{d}y = \int_{1/2}^1 \left(\frac{2}{5} + \frac{9}{5}y^2\right) \mathrm{d}y = \frac{29}{40}.$$

□

3.5 The Dirac Delta

So far we have considered real 1D functions $f : \mathbb{R} \to \mathbb{R}$, which are essentially objects that assign the numbers $f(x)$ to some subset of real numbers x. It turns out this is too restrictive for certain aspects of differential calculus, and a generalization in terms of a new class of object is useful. The idea is to reinterpret f as some object that assigns the number $\int f(x)\phi(x)\, \mathrm{d}x$ to every suitably chosen ϕ belonging to some set of continuous functions. An f defined as such a map is called a *distribution* or a *generalized function*.

We can intuitively see why this definition of f would lend itself to be called a distribution: in order to get a number from f one must compute the integral over the entire function, not just use a single point on the real axis. The distribution f can thus be thought of as a linear functional on a space of functions ϕ. Clearly, all continuous functions must also be distributions, but a distribution need not be a function since a point-wise definition of f is not required. Distributions thus extend the space of objects we can play with within the realm of differential calculus. The formal maths surrounding distributions is not trivial, nor a part of the syllabus for this course, so we will not discuss it in any detail. However, one distribution that is not also a function plays a vital role in physics (and in the differential calculus used by physicists): the Dirac delta δ. With some care one can often manipulate it as if it were a function, but it is important to keep in mind the differences to avoid confusion. We list some of the useful characteristic of the Dirac delta function below. For a proper discussion of the more general analysis of distributions the interested student may wish to consult chapter 6 of Rudin's Functional Analysis for an excellent, if quite advanced, exposition of the topic.

The Dirac delta function has several definitions, but the most useful one for our purposes is, in one dimension:

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\, \mathrm{d}x = f(0), \tag{76}$$

for all continuous functions f on $\mathbb{R}$. We will not address the details of the meaning of the symbol $\int$ in Eq. (76), but note that it cannot be the Riemann integral given by Eq. (17) (and sketched in Fig. 5)).

While it is tempting to consider $\delta(x)$ as a function, i.e., a map between reals, this would be incorrect and can lead to contradictions. From the definition we see that for the particular case of $f(x) = 1$ we must have

$$\int_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x = 1. \tag{77}$$

This fits with our concept of continuous probability distributions: the integrand must be evaluated over some finite range rather than in a point. In fact, by definition this integral of the delta is always 1 for any interval containing the point 0, and 0 otherwise.

The δ can be shifted around the real axis generalizing Eq. (76) to a generic point x_0, where for convenience we use the standard notation for functions:

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0). \tag{78}$$

We can also define the derivative of a distribution, here of the Dirac delta, as

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)\,\mathrm{d}x = -f'(0), \tag{79}$$

which can be shown to have all the required properties. The expression above can be understood in terms of integration by parts, i.e.

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)\,\mathrm{d}x = \delta(x)f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty} f'(x)\delta(x)\,\mathrm{d}x = -f'(0). \tag{80}$$

This is, of course, a terrible abuse of notation, and the term with the delta outside of the integral should give you indigestion. Nevertheless, it is a useful aide-mémoire, even if not a rigorous proof. It is important to note that these integrals make sense whether δ is differentiable or not (it is not, in the normal sense that we have used so far). All that is needed is that the function f is differentiable. By setting $f(x) = 1$ we then find that

$$\int_{-\infty}^{+\infty} \delta'(x)\,\mathrm{d}x = 0. \tag{81}$$

One of the many uses in physics of the Dirac delta is that it allows us to move between discrete and continuous variables and distribution functions with ease. Let X be a discrete random variable which can take values $x_1, x_2, ..., x_n$ with related probabilities $p_1, p_2, ..., p_n$. Then X can be described as a continuous variable having a PDF given by:

$$f(x) = \sum_{i=1}^{n} p_i \delta(x - x_i). \tag{82}$$

Using the PDF to evaluate the probability of X lying in some interval $[a, b]$ yields

$$P(a < X \leq b) = \int_a^b f(x)\,\mathrm{d}x = \sum_{i=1}^{n} p_i \int_a^b \delta(x - x_i)\,\mathrm{d}x = \sum_{i|x_i \in [a,b]} p_i, \tag{83}$$

as expected. Note that this can considerably simplify dealing with discrete quantities, which are almost always harder to treat than continuous functions.

4 Curves and vector functions

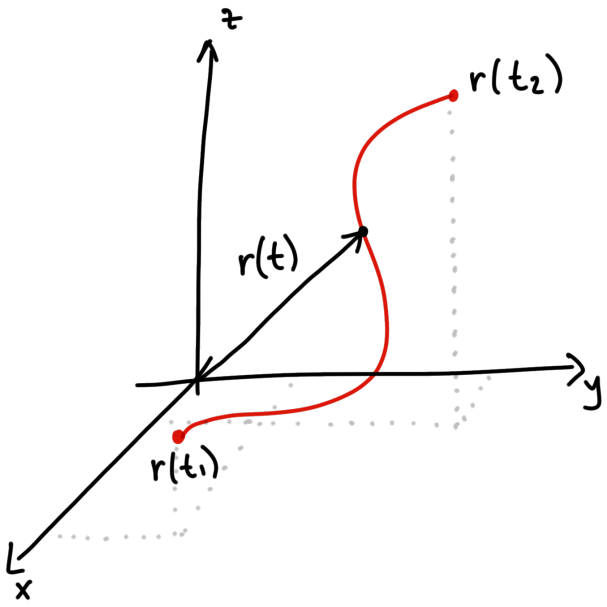


Figure 19: Parametrization of a curve in 3D space by a vector $\mathbf{r}$ that is a function of a continuous variable $t \in [t_1, t_2]$.

Vector functions of one variable are useful to represent curves in multidimensional space. To be specific, we will focus here on the case of 2 or 3 dimensions, but the generalization to higher dimensions is straightforward. Consider a point moving in 3D space along a certain trajectory. Its positional coordinates (x, y, z) will be a function of time t, and thus specified by a function of the form $\mathbf{r} : [t_1, t_2] \to \mathbb{R}^3$, with $\mathbf{r} = (x, y, z)$ and

$$\begin{cases} x = x(t) \\ y = y(t) \qquad t \in [t_1, t_2] \\ z = z(t). \end{cases}$$

We can also write the function explicitly in vector form as $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$.

Definition 9. *Let I be an interval $[a, b]$ in $\mathbb{R}$, and let $\mathbf{r} : I \to \mathbb{R}^3$ be a continuous map. Then $\mathbf{r}$ is called a curve in $\mathbb{R}^3$. The image, $\mathbf{r}(I)$, is called the trace of $\mathbf{r}$.*
If $\mathbf{r}(a) = \mathbf{r}(b)$, the curve is said to be closed. If, with the exception of its end points, the curve never passes the same point twice (i.e., never crosses itself), the curve is said to be simple.

The curve $\mathbf{r}$ written explicitly as a function of some parameter (here t) is called the parametrized form of the curve. The same curve can also be given in non-parametrized version, purely as a function of the coordinate variables (x, y, z).

Example 10. Let the curve $\mathbf{c}$ be a circumference in 2D, centered in the origin, of radius r. Write the curve in parametric and non-parametric form.

Solution: In parametric form the curve is given by $\mathbf{c} = (x, y)$ with

$$\begin{cases} x = r\cos(t) \\ y = r\sin(t) \qquad t \in [0, 2\pi]. \end{cases}$$

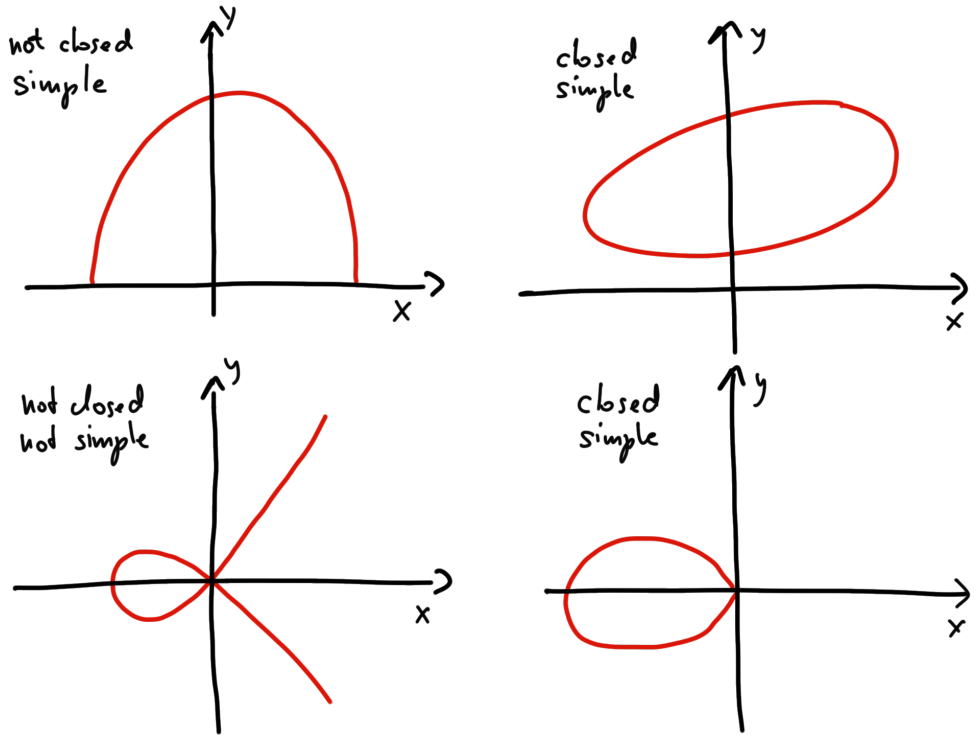


Figure 20: Examples of curves that are closed, open, simple, and not simple.

To find the non-parametric form, we must eliminate t from the equations above. The simplest way to do this is to square x and y, and add them together, to find explicitly the relationship between x and y: $x^2 + y^2 = r^2$. □

4.1 Derivative of a vector function

We now wish to generalize the concept of derivative for the vector function $\mathbf{r}$ defined above. We can immediately see that the standard definition of differentiation via the limit of increments remains valid:

Definition 10. *Let* $\mathbf{r} : I \to \mathbb{R}^n$*, and* $t_0 \in I$*. Then the derivative of* $\mathbf{r}$ *in* t_0 *is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t)\bigg|_{t=t_0} \equiv \mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}, \tag{84}$$

provided the limit exists and is finite.

The limit of a vector quantity is straightforward to compute: it is simply the limit of each component of the vector. This follows directly from the definition of a vector space. Hence we see that the derivative of a vector function is the vector containing the derivatives of its components:

$$\mathbf{r}'(t) = (r_1'(t), r_2'(t), \cdots, r_n'(t)). \tag{85}$$

A curve $\mathbf{r}$ is said to be *regular* if $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in I$, and *singular* otherwise. From Eq. (84) we also see how we can define the *differential* of a vector:

$$\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t. \tag{86}$$

So far we have only considered vectors written in the Cartesian basis, where the basis vectors do not change from one point to another. This is not the case for polar coordinates, and in this

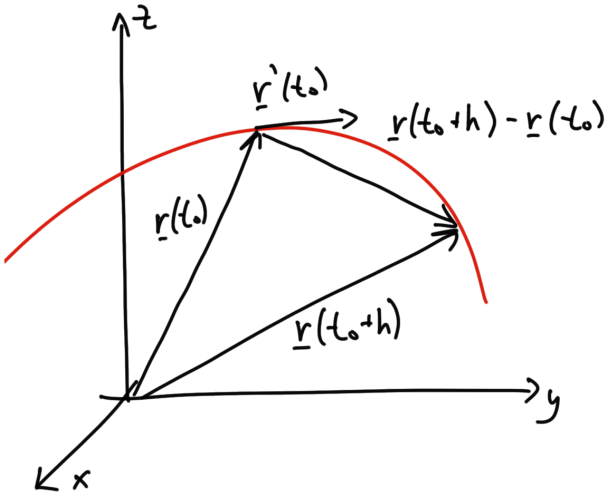


Figure 21: Graphical representation of the derivative of a vector $\mathbf{r}(t)$, taken in the limit of infinitesimal increment h of the parameter t.

case we cannot assume that the basis vectors remain constant. To see this, recall the definition of the basis vectors in plane polar coordinates from Eq. (47):

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$
$$\hat{\theta} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}.$$

By writing them in the Cartesian basis we can find their derivatives explicitly:

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{i}} + \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{j}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}$$
$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = -\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{i}} - \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{j}} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{r}}.$$

So some care is needed in general curvilinear coordinate systems, but the standard results and the chain rule all continue to work as expected.

Example 11. Let $(\mathrm{d}s)^2 = \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}y)^2$ indicate an infinitesimal displacement written in Cartesian coordinates. Find the expression for $(\mathrm{d}s)^2$ in spherical polar coordinates.

Solution: We make use of the definition of a vector differential from Eq. (86), for the transformation $(x, y, z) \to (r, \theta, \phi)$:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

Because $\mathbf{r} = \mathbf{r}(r, \theta, \phi)$ depends on three variables, we have:

$$(\mathrm{d}s)^2 = |\,\mathrm{d}\mathbf{r}|^2 = \left|\frac{\partial\mathbf{r}}{\partial r}\right|^2(\mathrm{d}r)^2 + \left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^2(\mathrm{d}\theta)^2 + \left|\frac{\partial\mathbf{r}}{\partial\phi}\right|^2(\mathrm{d}\phi)^2 =$$
$$= \left|\begin{matrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{matrix}\right|^2(\mathrm{d}r)^2 + \left|\begin{matrix}r\cos\theta\cos\phi\\ r\cos\theta\sin\phi\\ -r\sin\theta\end{matrix}\right|^2(\mathrm{d}\theta)^2 + \left|\begin{matrix}-r\sin\theta\sin\phi\\ r\sin\theta\cos\phi\\ 0\end{matrix}\right|^2(\mathrm{d}\phi)^2 =$$
$$= 1(\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\phi)^2.$$

We recover the result for spherical polar coordinates of Eq. (60), found using a geometric approach. □

4.2 Integral of a vector function

Like the derivative, the definition of the integral of a vector function poses no particular difficulties, and is a simple vector extension of known results in 1D.

Definition 11. *Let* $\mathbf{r} : [a, b] \to \mathbb{R}^n$. *The curve* $\mathbf{r}$ *is integrable on* $[a, b]$ *if every component of* $\mathbf{r}$, r_i, *is integrable on* $[a, b]$. *Then we can write*

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left(\int_a^b r_1(t)\,\mathrm{d}t, \int_a^b r_2(t)\,\mathrm{d}t, \cdots, \int_a^b r_n(t)\,\mathrm{d}t\right). \tag{87}$$

With this definition we can recover the fundamental theorem of calculus for vector functions:

Theorem 2. *Let* $\mathbf{r} : [a, b] \to \mathbb{R}^n$ *be a differentiable map with continuous first derivatives* $\mathbf{r}'$. *Then,*

$$\int_a^b \mathbf{r}'(t)\,\mathrm{d}t = \mathbf{r}(b) - \mathbf{r}(a). \tag{88}$$

Clearly, for regular closed curves this integral is always zero.

4.3 Arc length

Let $\mathbf{r} : [a, b] \to \mathbb{R}^m$ be a regular curve. How can we find the length of an arc between two points a and b on the curve (a quantity sometimes also referred to as the metric distance)? To fix ideas let us start with the simplest non-trivial case of a curve in 2D ($m = 2$), as illustrated in Fig. 22. Imagine we split the curve into a finite number of segments, and then connect the points delimitating the segments with straight lines. The length of each such segment, Δs_i, can be found easily via Pythagoras' theorem as $\Delta s_i^2 = \Delta x_i^2 + \Delta y_i^2$ (i.e., making use of the Euclidean metric for the distance between points). Summing all the lengths across all the segments provides an approximation to the length of the curve between the two limiting points $s \approx \sum_i \Delta s_i$. If we then take the limit for $\Delta x \to \mathrm{d}x$, the arc length s can be written as an integral in terms of the infinitesimal line element $\mathrm{d}s$

$$s = \int_a^b \mathrm{d}s. \tag{89}$$

This equation holds for curves embedded in any number of dimensions, the only thing that changes is the number of terms contained in $\mathrm{d}s$. The integral seems straightforward to evaluate, and it is, provided the curve is parametrized in terms of the arc length $\mathrm{d}s$. Unfortunately, this is almost never the case. Fortunately, it is not difficult to see how the above integral can be written in a more general form, which better lends itself to be applied to curves of any parametrization.

To see this, let $\mathbf{r}(t)$ be some curve in 3D space, given in parametric form as $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$, and consider an infinitesimal displacement $\mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}} + \mathrm{d}z\,\hat{\mathbf{k}}$ along the curve. The square of the infinitesimal distance moved is given by

$$(\mathrm{d}s)^2 = \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2. \tag{90}$$

If instead of taking the differential $\mathrm{d}\mathbf{r}$ we consider the derivative $\mathrm{d}\mathbf{r}/\,\mathrm{d}t$, this relationship can we rewritten as

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2. \tag{91}$$

In particular, we can write

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left|\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\right| = |\mathbf{r}'(t)| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}. \tag{92}$$

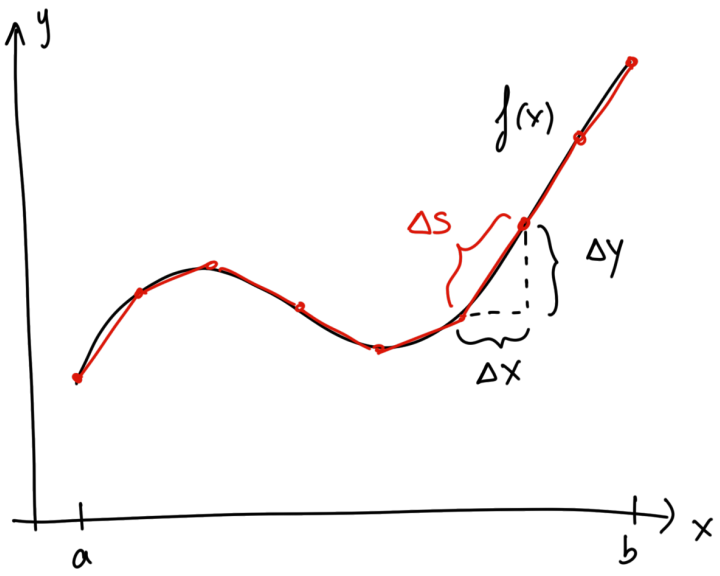


Figure 22: Calculation of the length of a curve via segments.

The arc length given by the integral in Eq. (89) can then be rewritten as

$$s = \int_a^b \left|\mathbf{r}'(t)\right| \mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t. \tag{93}$$

If the curve is provided in non-parametric form, then we need to find how $\mathrm{d}s$ varies with $\mathrm{d}x$, $\mathrm{d}y$ or $\mathrm{d}z$:

$$\begin{aligned}
\mathrm{d}s =& \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2} = \\
=& \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \\
=& \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)^2}\, \mathrm{d}y \\
=& \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}\, \mathrm{d}z.
\end{aligned}$$

Which one of the last three expressions we choose to substitute into Eq. (89) will largely depend on the functional expressions of the curve, and on the interdependences between them, as some derivatives may be easier to integrate than others.

These expressions yield a particularly simple, yet useful, case for curves $\mathbf{r}$ that can be written as a 1D function in the (x, y) plane by an equation of the form $y = f(x)$. The arc length can then be written as directly as a function of f, or of its inverse f^{-1}, if it exists, as:

$$s = \int_{a_x}^{b_x} \sqrt{1 + f'(x)^2}\, \mathrm{d}x = \int_{a_y}^{b_y} \sqrt{1 + f^{-1\prime}(y)^2}\, \mathrm{d}y. \tag{94}$$

Example 12. Find the arc length of a cylindrical helix given by the parametric equations

$$\begin{aligned} x &= R\cos t, \\ y &= R\sin t, \\ z &= \alpha t, \end{aligned}$$

where R and α are real constants, and $t \in [0, 2\pi]$.

Solution: We start by calculating $|\mathbf{r}'(t)|$:

$$\begin{aligned} \mathbf{r}'(t) &= (-R\sin t, R\cos t, \alpha), \\ |\mathbf{r}'(t)| &= \sqrt{R^2\sin^2 t + R^2\cos^2 t + \alpha^2} = \sqrt{R^2+\alpha^2}. \end{aligned}$$

The length of arc is then given by the integral

$$l = \int_0^{2\pi} |\mathbf{r}'(t)|\,\mathrm{d}t = \int_0^{2\pi} \sqrt{R^2+\alpha^2}\,\mathrm{d}t = 2\pi\sqrt{R^2+\alpha^2}.$$

□

4.4 Frenet-Serret coordinate system

We have introduced the concept of a curve in space in vector form with some parametrization as $\mathbf{r}(t)$, and of the tangent vector to the curve $\mathbf{r}'(t)$. We want to extend this to describe the way the curve changes direction more generally, and define a new orthogonal coordinate system that is intrinsic to the curve and not fixed by an external reference frame.

First of all, it is easy to show that if $|\mathbf{r}(t)|$ is constant, that is to say the length of the vector does not change along the curve, then $\mathbf{r}(t) \perp \mathbf{r}'(t) \quad \forall t$:

$$\begin{aligned} &|\mathbf{r}(t)|^2 = \text{const.} \\ &\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)|^2 = 0 = 2\,\mathbf{r}'(t)\cdot\mathbf{r}(t) \quad \Rightarrow \quad \mathbf{r}' \perp \mathbf{r}. \end{aligned}$$

From Eq. (92) we have that $\mathrm{d}s = |\mathbf{r}'(t)|\,\mathrm{d}t$. The size of the tangent $|\mathbf{r}'(t)|$ assumes the role of a speed: it tells us how quickly distance mapped out by s changes with t. Clearly, if we chose our parametrization t to be the arc length s, then $\mathrm{d}s = |\mathbf{r}'(s)|\,\mathrm{d}s$, and the tangent in this parametrization must have a modulus of 1 for all s: $\mathbf{r}'(s)$ is a unit tangent vector to the curve, which we shall call $\hat{\mathbf{t}}$.

The rate at which the unit tangent $\hat{\mathbf{t}}$ changes with respect to s is given by $\mathrm{d}\hat{\mathbf{t}}/\,\mathrm{d}s$, and its magnitude is defined as the *curvature* κ of the curve at a given point

$$\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2}\right|. \tag{95}$$

Note that for a straight line, $\kappa = 0$. The inverse of the curvature, the radius of curvature $\rho = 1/\kappa$, is often the more useful quantity to quantify the "curvedness" of a curve, as shown in Fig. 23. Because $\hat{\mathbf{t}}$ is a unit vector, its derivative (of size κ) is orthogonal to it. The unit vector in this direction is called the *principal normal* $\hat{\mathbf{n}}$:

$$\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s} = \kappa\hat{\mathbf{n}}. \tag{96}$$

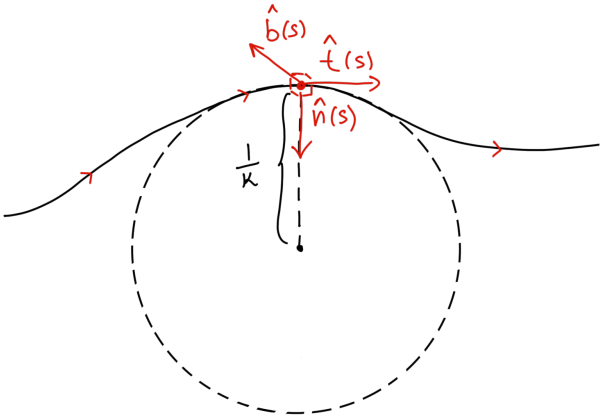


Figure 23: Unit vectors defining a orthonormal coordinate system on a curve in space. The derivatives of these three vectors with respect to the arc length along the curve s are given by the Frenet-Serret formulae.

Finally, we can define a third unit vector, the *binormal* to the curve, as $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Since $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$ we can write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left[\hat{\mathbf{b}} \cdot \hat{\mathbf{t}}\right] &= \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s} \\ &= \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\hat{\mathbf{n}} = 0. \end{aligned}$$

Since $\hat{\mathbf{b}} \perp \hat{\mathbf{n}}$ the last term is zero, the first term must be zero as well. The vector $\mathrm{d}\hat{\mathbf{b}}/\,\mathrm{d}s$ is thus $\perp$ to both $\hat{\mathbf{t}}$ and to $\hat{\mathbf{b}}$ (because $|\hat{\mathbf{b}}| = 1$), and must therefore point in the direction of $\hat{\mathbf{n}}$. To state this explicitly, we write

$$\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s} = -\tau\hat{\mathbf{n}}, \tag{97}$$

where τ is called the *torsion* of the curve, and the negative sign is a matter of convention.

To find $\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s}$ we note that this vector must be $\perp$ to $\hat{\mathbf{n}}$. It can thus be written as some linear combination of the vectors $\hat{\mathbf{t}}$ and $\hat{\mathbf{b}}$:

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = \alpha\hat{\mathbf{t}} + \beta\hat{\mathbf{b}}. \tag{98}$$

Taking the scalar product of this expression with $\hat{\mathbf{t}}$ we can see that $\alpha = \frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} \cdot \hat{\mathbf{t}}$. We then note that since $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$,

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} \cdot \hat{\mathbf{t}} = -\hat{\mathbf{n}} \cdot \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s}.$$

Using Eq. (96) we can conclude that $\alpha = -\kappa$. Similarly, taking the scalar product of Eq. (98) with $\hat{\mathbf{b}}$ we find $\beta = \frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} \cdot \hat{\mathbf{b}}$, and since $\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = 0$, we again have that

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} \cdot \hat{\mathbf{b}} = -\hat{\mathbf{n}} \cdot \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s},$$

from which, using Eq. (97), we conclude that $\beta = \tau$. Therefore, Eq. (98) can be rewritten as

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\kappa\hat{\mathbf{t}} + \tau\hat{\mathbf{b}}. \tag{99}$$

The trio $(\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}})$ form a right-handed orthonormal coordinate system at any given point on the curve, but note that this system changes as one moves along the curve. They are sketched

together in Fig. 23. The equations relating the vectors of the Frenet frame and their derivatives with respect to the arc length along the curve s are given by the famous Frenet-Serret formulæ:

$$\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s} = \kappa\hat{\mathbf{n}}, \qquad \frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}}, \qquad \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s} = -\tau\hat{\mathbf{n}}. \tag{100}$$

They describe by how much the intrinsic coordinate system changes orientation as one moves along the space curve parametrized by s.

We can see that the shape of a curve is completely determined by κ and τ. In particular, the necessary and sufficient condition for a curve to be planar is for it to have no torsion, i.e. $\tau = 0$, at arbitrary s. In contrast, a general helix is defined as a curve in which $\hat{\mathbf{t}}$ makes a constant angle with a fixed direction, that is, when either κ or τ are independent of s. If both κ and τ are constant, the helix is called a circular helix.

We can use these results to distinguish three different kinds of coordinate systems:

- Fixed coordinate system, e.g. Cartesian $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$.
- Varying coordinate system, e.g. spherical polar $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi})$.
- Intrinsic coordinate system, e.g. Frenet-Serret $(\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}})$.

4.5 Line integrals

The results shown above allow us to consider a new type of integral: a line or path integral. The idea is that we want to find a general way to integrate some quantity (typically a vector or a scalar function) along some specific pathway in space, and which does not necessarily align with any of the coordinate axes. Such integrals are common in physics. Suppose we wish to calculate the total mass of a inhomogeneous string of length L, for which we know the linear density. If we can parametrize the string using the arc length $\mathbf{r} = \mathbf{r}(s), s \in [0, L]$, and if the linear density is given as $\rho = \rho(s)$, then the total mass m will be given by the integral

$$m = \int_0^L \rho(s)\,\mathrm{d}s. \tag{101}$$

In practice, parametrizing the curve as a function of the arc length can be quite challenging, but fortunately we can write the same integral for a generic parametrization $\mathbf{r} = \mathbf{r}(t), t \in [a, b]$ using a change of variables transformation

$$\begin{aligned} s &= s(t) \\ \mathrm{d}s &= |\mathbf{r}'(t)|\,\mathrm{d}t \end{aligned}$$

$$m = \int_a^b \rho(s(t))|\mathbf{r}'(t)|\,\mathrm{d}t = \int_a^b \tilde{\rho}(t)|\mathbf{r}'(t)|\,\mathrm{d}t. \tag{102}$$

Definition 12. *Let $\mathbf{r} : [a, b] \to \mathbb{R}^m$ be a regular curve with trace γ and let f be a real-valued function defined on a subset of $\mathbb{R}^m$ which contains γ ($f : A \subset \mathbb{R}^m \to \mathbb{R}, \gamma \subset A$). The line (or curvilinear) integral of f along γ is the integral*

$$\int_\gamma f\,\mathrm{d}s \equiv \int_a^b f[\mathbf{r}(t)]\;|\mathbf{r}'(t)|\,\mathrm{d}t. \tag{103}$$

This is the natural generalization of the standard 1-dimensional integral to a generic curve in space. The challenge is typically not so much in computing the integral, but rather in finding a suitable parametrization for the curve γ.

Example 13. Evaluate the line integral

$$\int_\gamma xy^4 \, \mathrm{d}s,$$

where γ is the right-half of the circumference given by the equation $x^2 + y^2 = 16$.

Solution: We start by parametrizing the variables in the most convenient way:

$$\begin{aligned} x &= 4\cos\theta, \\ y &= 4\sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \end{aligned}$$

and therefore $\mathrm{d}x = -4\sin\theta\,\mathrm{d}\theta$ and $\mathrm{d}y = 4\cos\theta\,\mathrm{d}\theta$. We can use Eq. (92) to find $\mathrm{d}s$:

$$\begin{aligned} \mathrm{d}s &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\ &= \sqrt{16\sin^2\theta + 16\cos^2\theta}\,\mathrm{d}\theta = 4\,\mathrm{d}\theta. \end{aligned}$$

The integral becomes

$$\int_\gamma xy^4 \, \mathrm{d}s = \int_{-\pi/2}^{+\pi/2} (4\cos\theta)(4\sin\theta)^4 4\,\mathrm{d}\theta = 4^6 \left.\frac{\sin^5\theta}{5}\right|_{-\pi/2}^{+\pi/2} = 4^6\frac{2}{5}.$$

□

4.5.1 Line integrals of scalar and vector fields

It is often convenient to consider line integrals where we integrate with respect to the infinitesimal vector displacement $\mathrm{d}\mathbf{r}$, rather than with respect to the arc length parameter $\mathrm{d}s = |\,\mathrm{d}\mathbf{r}|$. This can be done using Eq. (86), since the two quantities are related: $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\,\mathrm{d}s$. In this case we will be presented with integrals of the form:

$$\int_\gamma f(\mathbf{r})\,\mathrm{d}\mathbf{r} = \int_\gamma f(x,y,z)(\hat{\mathbf{i}}\;\mathrm{d}x + \hat{\mathbf{j}}\;\mathrm{d}y + \hat{\mathbf{k}}\;\mathrm{d}z), \tag{104}$$

where $f(x,y,z)$ is a scalar function or field. For basis vectors that do not change with the coordinates we can take them out of the integral and write the result as a vector with integrals $\int_\gamma f(x,y,z)\,\mathrm{d}x_i$ as its components.

If instead of integrating a scalar function along some curve γ we integrate a vector function, then we will find integrals of the following form:

$$\int_\gamma \mathbf{A}(\mathbf{r})\cdot\mathrm{d}\mathbf{r} = \int_\gamma A_x(x,y,z)\,\mathrm{d}x + \int_\gamma A_y(x,y,z)\,\mathrm{d}y + \int_\gamma A_z(x,y,z)\,\mathrm{d}z. \tag{105}$$

where $\mathbf{A}(\mathbf{r}) = (A_x, A_y, A_z)$ is the vector field, and the dot indicates the scalar product, so the outcome of the integral must now be a scalar. Integrals of this form are common in physics, and are used, for example, to find the total work done, W, by a force $\mathbf{F}$ moving along some path γ:

$$W = \int_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r}. \tag{106}$$

Example 14. Evaluate the line integral

$$I = \int_\gamma \mathbf{A}(\mathbf{r})\cdot\mathrm{d}\mathbf{r},$$

where γ is the line segment connecting points $(0,2)$ and $(1,4)$, and $\mathbf{A} = (yx^2, \sin\pi y)$.

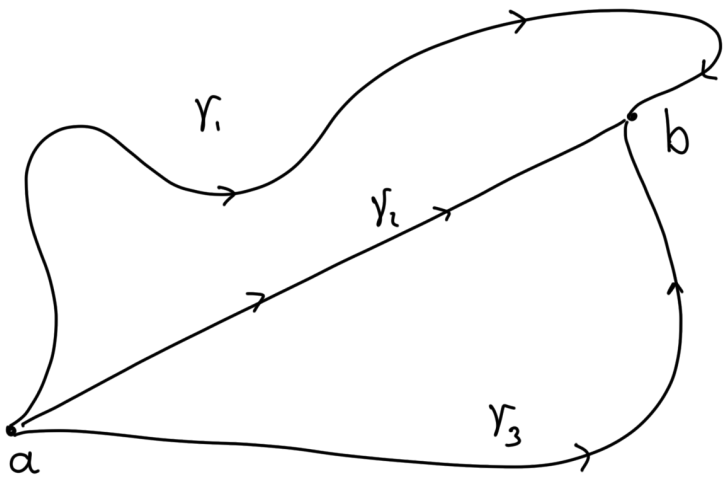


Figure 24: Three different paths taken between points **a** and **b**. If the vector field is conservative, then the line integral of the vector field will not depend on the path taken, but only on the points **a** and **b**.

Solution: We start by parametrizing the curve γ. Taking $(0, 2)$ as the point on the line, and noting it has a slope of 2, we can write the line equation as

$$\mathbf{r}(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t \in [0, 1].$$

From this we find the most convenient parametrization setting $x = t$, and $y = 2t + 2$, which implies $\mathrm{d}x = \mathrm{d}t$, and $\mathrm{d}y = 2\,\mathrm{d}t$. We can now rewrite the integral in a more tractable form:

$$\begin{aligned} I &= \int_\gamma (yx^2\,\mathrm{d}x + \sin \pi y\,\mathrm{d}y) = \\ &= \int_0^1 (2t+2)t^2\,\mathrm{d}t + \int_0^1 \sin[\pi(2t+2)]\,2\,\mathrm{d}t = \\ &= \left(\frac{t^4}{2} + \frac{2}{3}t^3\right)_0^1 - \left(\frac{2\cos[\pi(2t+2)]}{2\pi}\right)_0^1 = \frac{7}{6}. \end{aligned}$$

□

4.6 Conservative fields

The line integral of a vector field (or vector function) $\mathbf{A}(\mathbf{r})$ given by

$$\int_\gamma \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

in general depends on $\mathbf{A}$ in all points $\mathbf{r}$ traversed by the path γ: it is a path-dependent quantity. However, if $\mathbf{A}$ can be expressed as a gradient of some scalar function $\mathbf{A}(\mathbf{r}) = \nabla\phi(\mathbf{r})$, the integral becomes dependent only on the end points of the curve, and independent on the specific path taken between these two end points. A vector field $\mathbf{A}$ for which this holds is called a *conservative field*, and is sometimes referred to as a *vector potential*.

To see this explicitly, we can substitute the expression for the gradient of a scalar field into

the expression for the line integral:

$$\int_\gamma \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_\gamma \nabla\phi(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \tag{107}$$

$$= \int_\gamma \left(\frac{\partial\phi}{\partial x}\,\mathrm{d}x + \frac{\partial\phi}{\partial y}\,\mathrm{d}y + \frac{\partial\phi}{\partial z}\,\mathrm{d}z \right) = \tag{108}$$

$$= \int_\gamma \mathrm{d}\phi(\mathbf{r}) = \phi(\mathbf{r})\Big|_{\mathbf{a}}^{\mathbf{b}} = \phi(\mathbf{b}) - \phi(\mathbf{a}), \tag{109}$$

where $\mathbf{a}$ and $\mathbf{b}$ indicate the start and end points of the curve γ, as illustrated in Fig. 24. The integral is path-independent, because we are integrating an exact (also called perfect) differential. A consequence of this is that every closed line integral of such a vector field must necessarily be zero, since closed integrals have the same starting and ending point:

$$\oint_\gamma \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0, \quad \text{if } \mathbf{A}(\mathbf{r}) = \nabla\phi(\mathbf{r}). \tag{110}$$

Importantly, the converse statement also holds. If $\mathbf{A}$ is such that its integral vanishes around every closed curve in some region, then there exists a scalar function ϕ such that $\mathbf{A} = \nabla\phi$. In physics, the line integral of a vector field around a closed curve is often also called the *circulation* of the field.

There is a small but important caveat to the statement above: it holds provided the region R in which we select our curves γ is *simply connected*: we need to be able to continuously transform every path between the two points into any other such path in R without leaving R. An alternative way to say this is to require that every simple closed curve in R can be continuously shrunk to a point. In practice this means the domain must not have holes: i.e., regions for which $\mathbf{A}(\mathbf{r})$ and its derivatives are ill-defined. If the region R contains a hole then there exist simple curves that cannot be shrunk to a point without leaving R. Such a domain has two boundaries (the outer boundary, and the boundary around the hole), and is thus called doubly-connected.

Definition 13. *A plane region R is said to be simply connected if every simple closed curve within R can be continuously shrunk to a point without leaving the region. A region with n holes is said to be $(n+1)$-fold connected, or multiply connected.*

In practice it is much simpler to identify exact differential forms than it is to prove general path independence of some vector field. To see this, consider the integrand in Eq. (108), which must be an exact differential, in terms of the components of $\mathbf{A}$:

$$\frac{\partial\phi}{\partial x}\,\mathrm{d}x + \frac{\partial\phi}{\partial y}\,\mathrm{d}y + \frac{\partial\phi}{\partial z}\,\mathrm{d}z = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_z\,\mathrm{d}z. \tag{111}$$

Now note that the second mixed partial derivatives of ϕ must be independent of the order in which the derivatives are taken

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}, \qquad \frac{\partial^2\phi}{\partial x\partial z} = \frac{\partial^2\phi}{\partial z\partial x}, \qquad \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y}. \tag{112}$$

Written in terms of $\mathbf{A} = (A_x, A_y, A_z)$, these expressions provide a useful test for whether a generic vector field is conservative – the following relations must hold for its components:

$$\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x}, \qquad \frac{\partial A_x}{\partial z} = \frac{\partial A_z}{\partial x}, \qquad \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y}. \tag{113}$$

Example 15. Determine if $\mathbf{A}$ represents a conservative vector field for the following cases:

$$\text{(a) } \mathbf{A} = \mathbf{r}\, f(r) \qquad\qquad \text{(b) } \mathbf{A} = \mathbf{c}\, f(r),$$

where $\mathbf{r}$ is the position vector, r its modulus, and $\mathbf{c}$ a constant vector.

Solution: We need to check that the relations in Eq. (113) hold for the vector fields provided. Starting with case (a), we have

$$\frac{\partial A_x}{\partial y} = \frac{\partial (x f(r))}{\partial y} = x f'(r) \frac{\partial r}{\partial y} = f'(r) \frac{xy}{r}$$
$$\frac{\partial A_y}{\partial x} = \frac{\partial (y f(r))}{\partial x} = y f'(r) \frac{\partial r}{\partial x} = f'(r) \frac{yx}{r}.$$

These are the same; proceeding similarly we can find the other two equalities hold, hence $\mathrm{d}\phi = f(r)(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)$ is an exact differential. The associated scalar potential ϕ can only be given in integral form, since we have no information on the functional form of f. It is simply $\phi = \int r f(r) dr$.

For case (b) the calculation is very similar. Writing $\mathbf{c} = (c_x, c_y, c_z)$ we have

$$\frac{\partial A_x}{\partial y} = \frac{\partial (c_x f(r))}{\partial y} = c_x f'(r) \frac{\partial r}{\partial y} = c_x f'(r) \frac{y}{r}$$
$$\frac{\partial A_y}{\partial x} = \frac{\partial (c_y f(r))}{\partial x} = c_y f'(r) \frac{\partial r}{\partial x} = c_y f'(r) \frac{x}{r}.$$

These clearly differ, and so we conclude that $f(r)(c_x\,\mathrm{d}x + c_y\,\mathrm{d}y + c_z\,\mathrm{d}z)$ cannot be an exact differential. □

Example 16. The gravitational force acting on a particle of mass m near the Earth's surface can be represented by the vector $\mathbf{F} = (0, 0, -mg)$, where the z-axis points vertically upwards, and g is some constant. Find a general expression for the work done by the force as the particle is displaced from height h_1 to h_2, and show that the work done is path independent.

Solution: We start by checking if the force is conservative, i.e., if the vector field can be represented as the gradient of some scalar field. The conditions of Eq. (113) are trivially verified, since $\mathbf{F}$ is a constant vector and all the partial derivatives of its components are 0. The differential is thus exact, and given by

$$\mathrm{d}\phi(x, y, z) = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -mg\,\mathrm{d}z.$$

We recognize the scalar field ϕ as the potential energy. Integrating both sides between the initial and final points, with corresponding heights h_1 to h_2, we find the work done

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}\phi(\mathbf{r}) = \int_{h_1}^{h_2} -mg\,\mathrm{d}z =$$
$$= \phi(h_2) - \phi(h_1) = -mg(h_2 - h_1).$$

This is obviously path-independent, since it is given by the difference in values of $\phi(z)$ calculated in two different points. □

4.7 Green's theorem

Green's theorem allows us to make a connection between two seemingly different mathematical objects: a line integral around some closed curve in a plane, and the double integral over the region bounded by that closed curve, as shown in Fig. 25. This is the first of several such relationships that we will study in this course.

Theorem 3. *Let $P(x, y)$ and $Q(x, y)$ be two continuous functions with continuous partial derivatives, well-defined inside and on the boundary γ of a simply connected region R in the xy-plane. Then the following relationship holds*

$$\oint_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y. \tag{114}$$

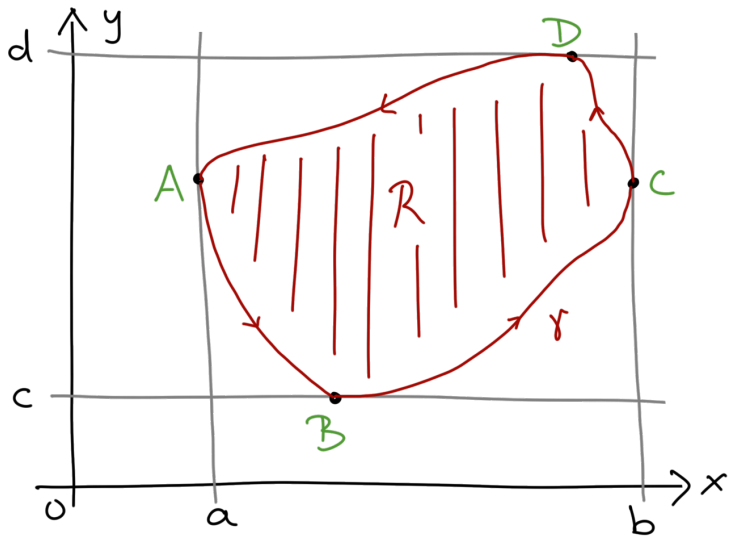


Figure 25: A domain R of the xy-plane bounded by a curve γ, used in Green's theorem. The domain is simple, and bounded by the box $x \in [a, b]$ and $y \in [c, d]$.

The proof of Green's theorem is relatively straightforward if we restrict ourselves to simple domains. Note that this is a convenience, not a requirement. A non-simple domain can be sliced into multiple smaller domains which can be made to be simple. The process is only more laborious. Consider the expression on the right hand side of Eq. (114), and the domain R and bounding curve γ shown in Fig. 25. For convenience we will write the curve connecting the points ABC as $y_1(x)$, and that connecting ADC $y_2(x)$. The two allow us to consider the double integral over R as a y-simple domain, and together parametrize the entire boundary curve γ. Similarly, taking the domain as x-simple we can parametrize the curve γ as two functions, now of y, and call the curve connecting BAD $x_1(y)$, and that connecting BCD $x_2(y)$. Our strategy will be to break down the double integral using known methods. Starting with the term containing P we have:

$$\begin{aligned}\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}\,\mathrm{d}y = \int_a^b \mathrm{d}x \Big[P(x,y)\Big]_{y=y_1(x)}^{y=y_2(x)} \\ &= \int_a^b [P(x,y_2(x)) - P(x,y_1(x))]\,\mathrm{d}x = -\oint_\gamma P\,\mathrm{d}x.\end{aligned}$$

Similarly, for the Q term:

$$\begin{aligned}\iint_R \frac{\partial Q}{\partial x}\,\mathrm{d}x\,\mathrm{d}y &= \int_c^d \mathrm{d}y \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\,\mathrm{d}x = \int_c^d \mathrm{d}y \Big[Q(x,y)\Big]_{x=x_1(y)}^{x=x_2(y)} \\ &= \int_c^d [Q(x_2(y),y) - Q(x_1(y),y)]\,\mathrm{d}y = \oint_\gamma Q\,\mathrm{d}y.\end{aligned}$$

Adding the two integrals yields Green's theorem.

Example 17. Use Green's theorem to calculate the area of an ellipse centred at the origin with semi-axes a and b.

Solution: Let R be the region in the xy-plane bounded by the closed curve describing the ellipse in question. The area A of the ellipse will be given by the double integral $A = \iint_R \mathrm{d}x\,\mathrm{d}y$. We must therefore choose the integrand of the right hand side of Eq. 114 such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

which restricts our choice of P and Q. There are many ways to satisfy this requirement; for example, if we set $P = 0$, then $\frac{\partial Q}{\partial x} = 1$, and $Q = x$. Inserting these into Green's theorem:

$$A = \oint_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \oint_\gamma x\,\mathrm{d}y.$$

The ellipse in question can be written in parametric form as

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta, \qquad \theta \in [0, 2\pi]. \end{aligned}$$

Substituting this parametrization into the expression for the area gives

$$A = \int_0^{2\pi} a\cos\theta\; b\cos\theta\,\mathrm{d}\theta = \pi ab.$$

□

Example 18. Evaluate the integral $\oint_\gamma \mathbf{A}\cdot\mathrm{d}\mathbf{r}$, where $\mathbf{A} = (2xy, x^2)$, and γ is the triangle connecting points (0,0), (1,0) and (0,2).

Solution: We start by solving this problem by performing the integral along the line. We have

$$\begin{aligned} \oint_\gamma \mathbf{A}\cdot\mathrm{d}\mathbf{r} &= \oint_\gamma (2xy\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= \int_{(0,0)}^{(1,0)} (2xy\,\mathrm{d}x + x^2\,\mathrm{d}y) + \int_{(1,0)}^{(0,2)} (2xy\,\mathrm{d}x + x^2\,\mathrm{d}y) + \int_{(0,2)}^{(0,0)} (2xy\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= 0 + \int_1^0 2x(-2x+2)\,\mathrm{d}x + x^2(-2)\,\mathrm{d}x + 0 = \Big[-2x^3 + 2x^2\Big]_1^0 = 0, \end{aligned}$$

where for the middle segment connecting points (1,0) and (0,2) we used the parametrization $y = -2x + 2$, $\mathrm{d}y = -2\,\mathrm{d}x$.

Another way to proceed is to use Green's theorem. We identify $P = 2xy$, and $Q = x^2$, both well-behaved differentiable functions on the xy-plane. Their derivatives are

$$\frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x.$$

As these are the same, from Green's theorem we can immediately see that the integral must be 0, irrespective of the path taken. So not only is the integral along the triangle 0, all closed integrals will integrate to 0. We learn more by doing less.

The observed equality of mixed derivatives above is of course nothing but the known result for an exact differential, and so it should be possible to write the vector field provided as the gradient of some scalar function ϕ, such that $\mathbf{A} = \nabla\phi$. To find ϕ we note that

$$\begin{aligned} P = \frac{\partial\phi}{\partial x} \quad &\Rightarrow \quad \int \mathrm{d}\phi = \int 2xy\,\mathrm{d}x \quad \Rightarrow \quad \phi = x^2y + C(y), \\ Q = \frac{\partial\phi}{\partial y} \quad &\Rightarrow \quad \int \mathrm{d}\phi = \int x^2\,\mathrm{d}y \quad \Rightarrow \quad \phi = x^2y + C(x). \end{aligned}$$

Setting $C(x) = C(y) = 0$ provides the scalar function sought. Its existence means that the closed line integral must be zero, as found previously. □

5 Surfaces

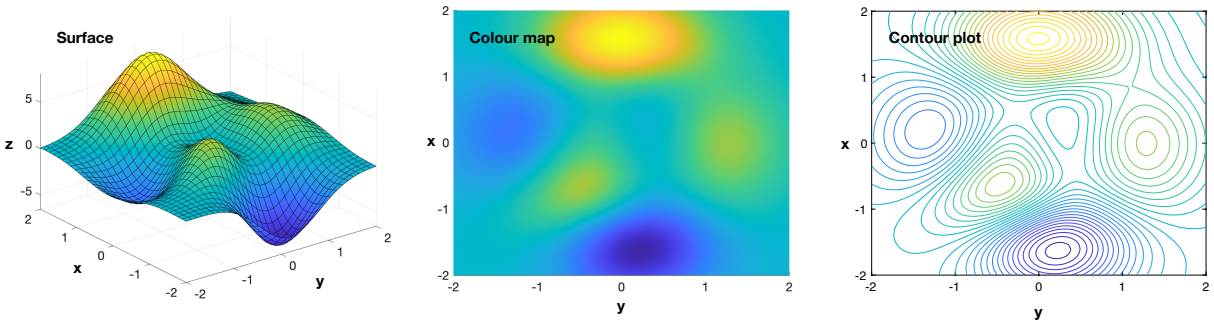


Figure 26: Three common ways to present and plot a 2D surface.

In this course we are interested in studying functions of multiple variables, alongside their derivates and integrals, and surfaces are among the most common and useful examples of such functions. Recall that a general function of multiple variables is given by the map

$$f : D \subset \mathbb{R}^n \to \mathbb{R}.$$

In 2D such a map, if continuous, represents a surface and is commonly written as $f(x, y) = z$. The surface itself can be thought of as an object with two degrees of freedom embedded in 3D space, and can then be described by a 3D vector with components

$$\mathbf{r} = (x, y, z = f(x, y)).$$

This expression suggests a convenient general parametric way of describing surfaces along the same lines used to define curves:

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)). \tag{115}$$

Here $\mathbf{r}$ is a vector function of the parameters (u, v) which vary within a certain domain D in the parametric uv-plane.

Example 19. Write the surface of the sphere $x^2 + y^2 + z^2 = R^2$ in parametric form.

Solution: The convenient parameter choice here is to use spherical polar coordinates, where $(u, v) = (\theta, \phi)$ are the polar and azimuthal angles. We know that

$$\begin{aligned} x &= R \sin\theta \cos\phi \\ y &= R \sin\theta \sin\phi \\ z &= R \cos\theta, \end{aligned}$$

and so the parametric form of the surface of a sphere is simply

$$\mathbf{r}(\theta, \phi) = R(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta). \tag{116}$$

□

5.1 The Gradient

Let us now extend the definition of a derivative to functions in more than one dimension. This will allow us to apply it more broadly, and in particular to study surfaces and volumes in 3D space. Recall the definition for the derivative of a function

$$\frac{\mathrm{d}f}{\mathrm{d}x} \equiv f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{117}$$

How should we extend this to multiple dimensions? The first difficulty is in the definition of the increment h: in multiple dimensions the increment will necessarily become a vector, as it will have an orientation depending on what directions we want to move in to evaluate the change in the function. However, simply replacing h by a vector $\mathbf{h}$ will not work as we cannot divide by a vector in the definition of a limit in Eq. (117). The solution will be to consider, instead, directional derivatives. In particular, for a function in n dimensions $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, ..., x_i, ..., x_n)$, we fix $n-1$ dimensions and vary only one, x_i, at a time. Along this single dimension the function is one-dimensional, and the known results for taking derivatives apply. However, to keep in mind that the function is free to vary in multiple dimensions, we will call the derivative of the function f along the single dimension x_i the partial derivative of f with respect to x_i, denoted by

$$\frac{\partial f(\mathbf{x})}{\partial x_i} \equiv \partial_i f(\mathbf{x}) \equiv f_{x_i}.$$

Clearly, for a function in n dimensions we will have n partial derivatives. These form a vector space, and it will be convenient to write them explicitly in vector form. We will call this the *gradient* of the function f, calculated in some point $\mathbf{x}$:

$$\mathrm{grad} f(\mathbf{x}) := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) = \nabla f(\mathbf{x}). \tag{118}$$

The last equality introduces a new vector operator sufficiently important to merit its own new symbol, ∇, called nabla:

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right). \tag{119}$$

This is an n-dimensional vector operator that takes a (multidimensional) *scalar* function as an input, and returns a *vector* containing all the partial derivatives of the scalar function. Of particular importance to this course is the gradient in two and three dimensions:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \qquad\qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{120}$$

5.1.1 Directional derivatives

For a continuous and differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ the partial derivative $\partial f/\partial x_i$ provides a measure of how quickly the function varies in along the direction of x_i. We can use this to find the rate of change of the function along any direction, making use of the gradient. Suppose we have some surface as shown in Fig. 27, and that we are interested in evaluating how the function changes along some direction parametrized by the line

$$\mathbf{x}(t) = \mathbf{x_0} + t\hat{\mathbf{v}}.$$

The change in the value of the function will be

$$\Delta f = f(\mathbf{x_0} + t\hat{\mathbf{v}}) - f(\mathbf{x_0}),$$

which along the line is now a 1D problem. In the limit of $t \to 0$ this is simply the directional derivative of f along $\hat{\mathbf{v}}$:

$$\frac{\partial f}{\partial x_v} \equiv \lim_{t\to 0} \frac{f(\mathbf{x_0} + t\hat{\mathbf{v}}) - f(\mathbf{x_0})}{t}. \tag{121}$$

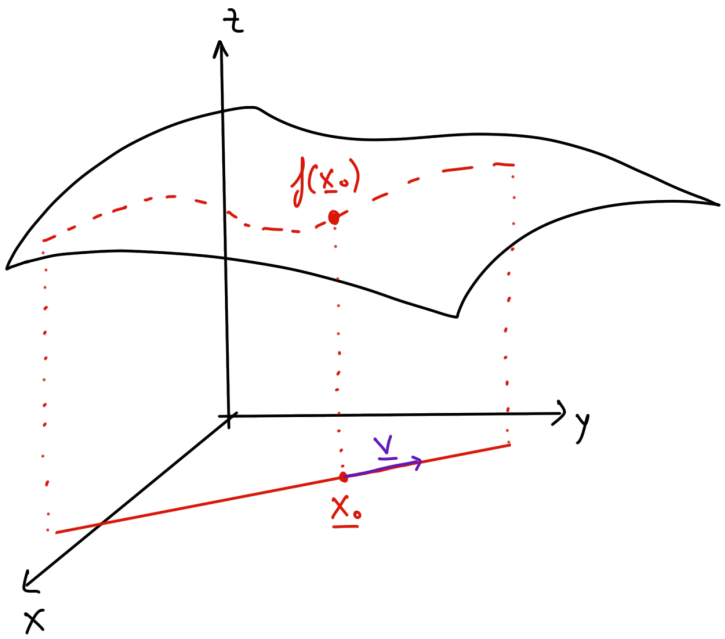


Figure 27: Variation of a function along some specific direction $\hat{\mathbf{v}}$.

Clearly, the vector $\hat{\mathbf{v}}$ can be written as a linear combination of the basis vectors in the domain. In the example shown in Fig. 27, this would correspond to some linear combination of the unit vectors in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions. It follows that the directional derivatives are linear combinations of the partial derivatives in the standard basis, and we can thus relate the directional derivatives to our gradient operator via the following theorem:

Theorem 4. *Gradient Theorem.*
Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with f differentiable in $\mathbf{x_0} \in D$. For every unit vector $\hat{\mathbf{v}}$ there exists a directional derivative $\partial_v f(\mathbf{x_0})$, and the following identity holds:

$$\partial_v f(\mathbf{x_0}) = \nabla f(\mathbf{x_0}) \cdot \hat{\mathbf{v}} = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x_0}} v_i. \tag{122}$$

The directional derivative is the scalar product of the gradient of the function and the direction vector. This relatively simple result leads to two importance consequences. Firstly, the expression $\nabla f \cdot \hat{\mathbf{v}}$ is a scalar product of two vectors that will be maximized when $\nabla f \parallel \hat{\mathbf{v}}$. The gradient of a function thus always points in the direction where the change in the function is maximal. In contrast, contour lines are defined as lines along which there is no change in the value of the function, i.e, where $\partial_v f = 0$. This occurs when $\nabla f \perp \hat{\mathbf{v}}$, and the gradient ∇f is thus always perpendicular to the contour lines. The gradient of the function shown in Fig. 26 is given in Fig. 28 (red arrows), alongside the contour lines of the function to aid the eye. We see that the gradient is indeed perpendicular to the contour lines, and that the gradient always points to the direction of maximum ascent. As intuitively expected, the gradient goes to zero in the extremum points.

Example 20. Find the unit normal vector to the surface $z = x^2 + y^2$ in the point $(1, 1, 2)$.

Solution: Recall that the gradient of a scalar field is always perpendicular to the isosurfaces of that field. As we have seen, for a 2D field these will be (contour) lines, but in higher dimensions

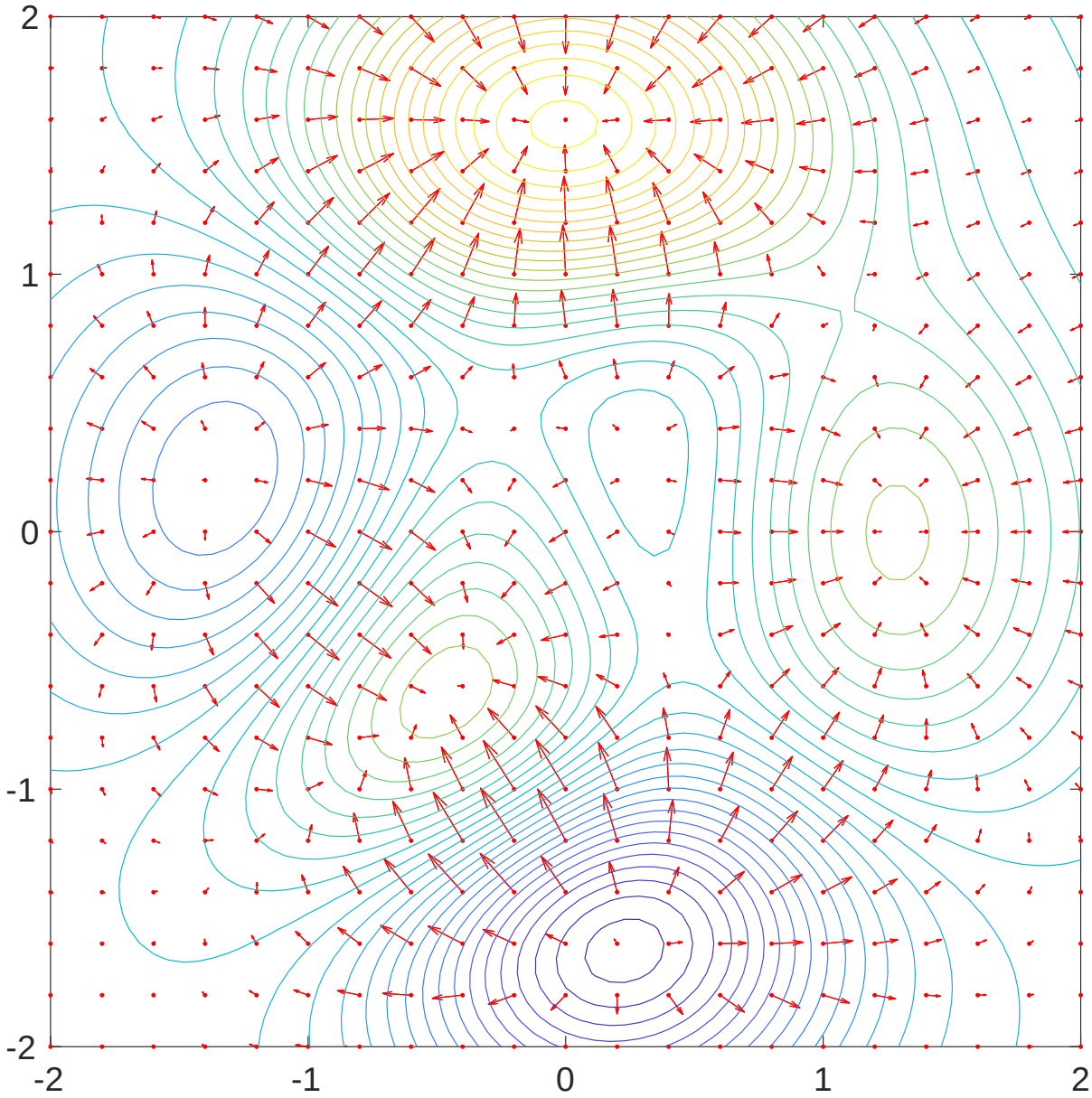


Figure 28: Gradient of the function shown in Fig. 26. The gradient is perpendicular to the contour lines, and always points in the direction of maximum ascent.

they will be (hyper-)surfaces representing points where the field has some constant value. We therefore rewrite the surface above as an isosurface of a 3D scalar field

$$U(x, y, z) = x^2 + y^2 - z = 0,$$

and compute its 3D gradient:

$$\begin{aligned}\nabla U(x, y, z) &= \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \\ &= (2x, 2y, -1).\end{aligned}$$

This is a vector in 3D space normal to the surface in question. In the point $(1, 1, 2)$ on the surface we have $\mathbf{n} = (2, 2, -1)$, and $\hat{\mathbf{n}} = \frac{1}{3}(2, 2, -1)$. The gradient thus allows us to quickly and conveniently compute the normal vectors to surfaces and hypersurfaces by considering them as isosurfaces of objects in higher dimensional space. □

Example 21. The 2D gravitational potential of a point mass M placed in the origin of the coordinate system is given by

$$V(\mathbf{r}) = -\frac{GM}{r},$$

where $\mathbf{r} = (x, y)$ is the position vector, $r \equiv |\mathbf{r}| = \sqrt{x^2 + y^2}$ its modulus, and G and M are constants. This represents a 2D scalar field. Find the gradient of V.

Solution:

$$\begin{aligned}\nabla V(\mathbf{r}) &= -\nabla\left(\frac{GM}{r}\right) = \\ &= -GM\left(\frac{\partial}{\partial x}(x^2 + y^2)^{-1/2}, \frac{\partial}{\partial y}(x^2 + y^2)^{-1/2}\right) = \\ &= -GM\left(\frac{-x}{(x^2 + y^2)^{3/2}}, \frac{-y}{(x^2 + y^2)^{3/2}}\right) = \\ &= GM\frac{\mathbf{r}}{r^3} = GM\frac{\hat{\mathbf{r}}}{r^2}.\end{aligned}$$

Both the scalar potential V and its gradient (a radial vector field) are shown in Fig. 29. As expected, the gradient is just the negative of the gravitational force. □

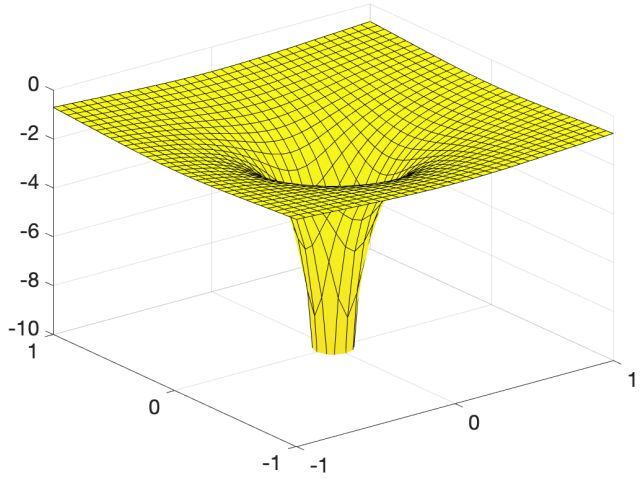
5.1.2 The gradient in curvilinear coordinate systems

Consider a multi-dimensional scalar function $f(\mathbf{x})$ (in physics we often refer to this as a scalar field), and assume f is differentiable so that its gradient, the vector ∇f, exists. We are interested in finding how to compute ∇f, or more precisely, its components, in a generic, non-Cartesian coordinate system.

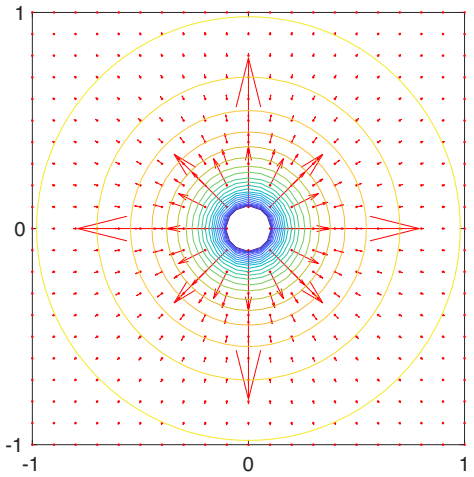
In Cartesian coordinates the components of ∇f are given by Eq. (118), and can be represented concisely in index notation as

$$(\nabla f)_i = \partial_i f = \frac{\partial f}{\partial x_i}. \tag{123}$$

The three forms are, of course, equivalent. In order to move to a new coordinate system $\mathbf{x} \to \mathbf{u}$, we need to find how to express the derivates of f with respect to the new variables u_i. Given



$$V(\mathbf{r}) = -\frac{GM}{r}$$



$$\nabla V(\mathbf{r}) = GM\frac{\mathbf{r}}{r^3}.$$

Figure 29: Relationship between the scalar gravitation potential field, and the vector field given by its gradient.

that the new coordinates are a function of the old, $\mathbf{u} = \mathbf{u}(\mathbf{x})$, we can use the chain rule to compute this:

$$\frac{\partial f}{\partial u_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_i} + ... + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial u_i} = \sum_j \frac{\partial x_j}{\partial u_i}\frac{\partial f}{\partial x_j}. \tag{124}$$

We recognize this expression as a matrix equation of the form $\nabla_u f = J\ \nabla_x f$, where J is the matrix containing the first partial derivatives of the two coordinate systems. For a 3D system, where $\mathbf{x} = (x, y, z)$ and $\mathbf{u} = (u, v, w)$, we can write the 3×3 matrix explicitly

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}, \tag{125}$$

which we recognize as the transpose of the Jacobian matrix. This should come as no surprise, given that we are essentially performing basis transformations on a differential operator.

Example 22. Calculate the gradient of the function $f(x, y) = 3(x^2 + y^2)$ in both the Cartesian and the plane polar coordinate basis.

Solution: In Cartesian coordinates the gradient is simply given by

$$\nabla f = \left(\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right) = (6x, 6y). \tag{126}$$

For plane polar coordinates we have $\mathbf{x} = (x, y)$ and $\mathbf{u} = (r, \theta)$, which are related by the expressions

$$x = r\cos\theta$$
$$y = r\sin\theta$$

To find the gradient in the new coordinates we make use of Eq. (124), which here becomes:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

We can invert this matrix to find an explicit expression for the gradient in Cartesian coordinates

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{-1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix},$$

and can write

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \\ \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta} \end{pmatrix}. \tag{127}$$

In polar coordinates the function can be written as $f = 3r^2$, so that its partial derivatives with respect to (r, θ) are $\partial f/\partial r = 6r$ and $\partial f/\partial\theta = 0$. Substituting these in the expression above yields:

$$\nabla f = (6r\cos\theta, 6r\sin\theta),$$

which is equal to our first result, as given in Eq. (126). These expressions are all written in the standard Cartesian basis, even if we have changed the variables. That is to say, calculating ∇f using Eq. (127) is, explicitly in terms of the basis vectors:

$$\begin{aligned} \nabla f &= \left[\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}\right]\hat{\mathbf{i}} + \left[\sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta}\right]\hat{\mathbf{j}} \\ &= \frac{\partial f}{\partial r}[\hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta] + \frac{\partial f}{\partial \theta}\frac{1}{r}[-\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta]. \end{aligned}$$

Recalling Eq. (47) for the basis vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$, we can rewrite this expression in the basis (r, θ):

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} = 6r\ \mathbf{r} = (6r, 0), \tag{128}$$

which is the gradient written in the basis of the plane polar coordinate system. □

5.1.3 Gradient expressions

We summarize here the results for the gradient written in curvilinear systems. For plane polar coordinates, as we have seen in the example above, we have:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} \equiv \left(\partial_r, \frac{1}{r}\partial_\theta\right) f. \tag{129}$$

Cylindrical coordinates in 3D are a simple extension of the plane polar system, and the form of the gradient can be guessed – we just need to add the z component, but this axis is the same as in the now familiar Cartesian system. Therefore:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \equiv \left(\partial_r, \frac{1}{r}\partial_\theta, \partial_z\right) f. \tag{130}$$

Spherical coordinates require a little more work. Following the same approach as above, and using the relationships introduced in Eq. (59), we find:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \equiv \left(\partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\phi\right) f. \tag{131}$$

where θ is the polar angle and ϕ is the azimuthal angle.

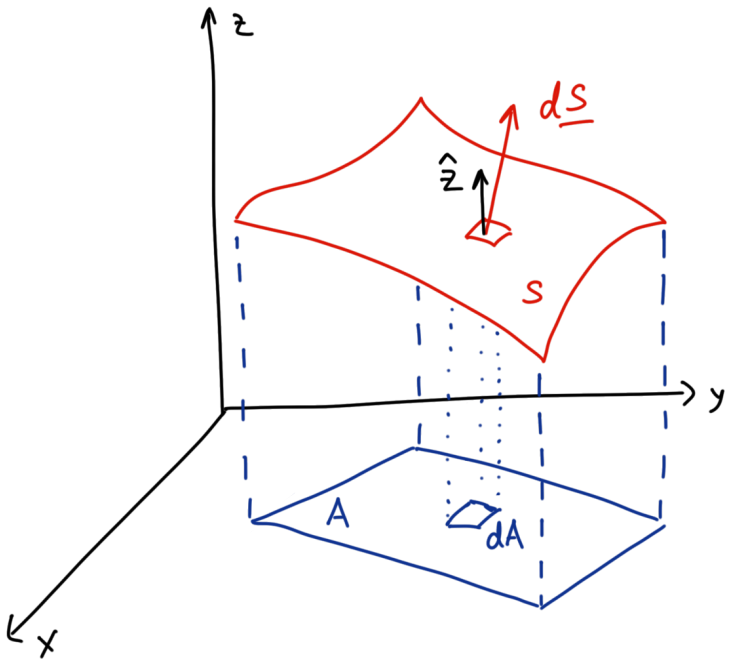


Figure 30: Bounded surface S and its projection A on the xy-plane.

5.2 Surface integrals

We now turn to the development of techniques necessary to perform surface integrals. These are similar to the double integrals you have already encountered, but have the added complication that the surface – which is the domain of integration – is not a flat xy-plane, but some generic curved surface. Such integrals are common in physics, and we have two main ways to tackle them. For surfaces which can be parametrized conveniently in some new set of coordinates (e.g., spherical polar or cylindrical systems) one can often perform the integral directly following a direct coordinate transformation. However, finding a simple and suitable transformation can prove challenging for more complex surfaces, and we often resort to the alternative: we project the surface onto some convenient Cartesian coordinate plane and perform the integration there. We proceed to show how these two approaches can be deployed in practice. Surface integrals will typically come in four general forms. The two that produce scalar outcomes are

$$\iint_S \phi(\mathbf{r})\,\mathrm{d}S, \qquad \iint_S \mathbf{A}(\mathbf{r})\cdot\hat{\mathbf{n}}\,\mathrm{d}S, \tag{132}$$

while the integrals producing a vector output can be written as:

$$\iint_S \phi(\mathbf{r})\hat{\mathbf{n}}\,\mathrm{d}S, \qquad \iint_S \mathbf{A}(\mathbf{r})\times\hat{\mathbf{n}}\,\mathrm{d}S. \tag{133}$$

Here $\hat{\mathbf{n}}$ denotes the normal vector to the surface, ϕ denotes a scalar function, and $\mathbf{A}$ a vector function (or field).

It will be convenient to consider surfaces in terms of their vector areas, defined as $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$. This is particularly convenient in the context of surface integrals of scalar and vector fields, which are commonly used to describe physical quantities and processes. We proceed by finding convenient expressions for the terms $\hat{\mathbf{n}}$, $\mathrm{d}S$, and $\mathrm{d}\mathbf{S}$.

5.2.1 Vector area

The vector area $\mathbf{S}$ is easiest to understand for a finite planar surface of scalar area S (i.e., the normal area) and unit normal $\hat{\mathbf{n}}$:

$$\mathbf{S} = S\hat{\mathbf{n}}. \tag{134}$$

It is simply a vector parallel to the plane normal, with modulus corresponding to the (scalar) surface area. We can extend this notation to a more general surface composed of a set $\{S_i\}$ of flat facet areas with corresponding normal vectors $\hat{\mathbf{n}}_i$ as:

$$\mathbf{S} = \sum_i S_i \hat{\mathbf{n}}_i. \tag{135}$$

Taking the limit of this expression for infinitely small facets allows us to define the concept of a vector area of a bounded differentiable surface

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S \quad \Rightarrow \quad \mathbf{S} = \int_S \mathrm{d}\mathbf{S} \equiv \iint_S \hat{\mathbf{n}}\,\mathrm{d}S. \tag{136}$$

It is worth noting from this expression that the vector and scalar areas are of the same size only for a plane surface. For a curved surface, the modulus of the vector area will always be less than the scalar area.

Consider a general surface S as shown in Fig. 30 that has some projection A onto the xy-plane. An infinitesimal vector area in some point $\mathrm{d}\mathbf{S}$ has a projection $\mathrm{d}A$, which can be found by taking the scalar product of the vector area and the unit normal to the xy-plane, $\hat{\mathbf{k}}$:

$$\mathrm{d}A = \hat{\mathbf{k}} \cdot \mathrm{d}\mathbf{S} = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = |\cos\alpha|\,\mathrm{d}S, \tag{137}$$

where α is the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$. Clearly the entire surface S can be tiled in this way and the total vector area is then given by integrating over the entire surface $\mathbf{S} = \int_S \mathrm{d}\mathbf{S}$.

Example 23. Find the vector areas of a hemisphere and of a sphere of radius R.

Solution: Given the symmetry of the problem, we place the sphere in the centre of the coordinate system and choose spherical polar coordinates to represent the surface of the sphere as (see Example 19):

$$\mathbf{r}(\theta, \phi) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

We know this vector is normal to the surface of the sphere, so the unit normal is given simply by the same vector, but normalized:

$$\hat{\mathbf{n}}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Taking the hemisphere the surface for positive z, we can calculate the total vector surface as

$$\mathbf{S} = \iint \hat{\mathbf{n}}\,\mathrm{d}S = \int_0^{\pi/2}\int_0^{2\pi} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \begin{pmatrix} 0 \\ 0 \\ \pi R^2 \end{pmatrix}.$$

As expected, the vector only has a component along the z-axis, and the size of the vector area is the projected area – the circle – in the xy-plane. To find the vector area of the entire sphere we simply need to change the limits of the polar integral from $\int_0^{\pi/2} \to \int_{-\pi/2}^{+\pi/2}$, which leads to $\mathbf{S} = 0$. This is a general result: the vector area of any closed surface is the 0 vector. □

This result leads to an interesting (and useful) consequence, and that is that the vector area of an open surface must depend only on its boundary curve. To see this, consider two surfaces, S_1 and S_2, which share a boundary curve. Taking the difference of the surfaces $S_1 - S_2$ must yield a closed surface, for which we know the vector area is 0. The two surfaces must therefore have the same vector area.

We can use this observation to obtain an expression of the vector area for an open surface directly from the line integral around its boundary curve γ. Consider the open surface illustrated

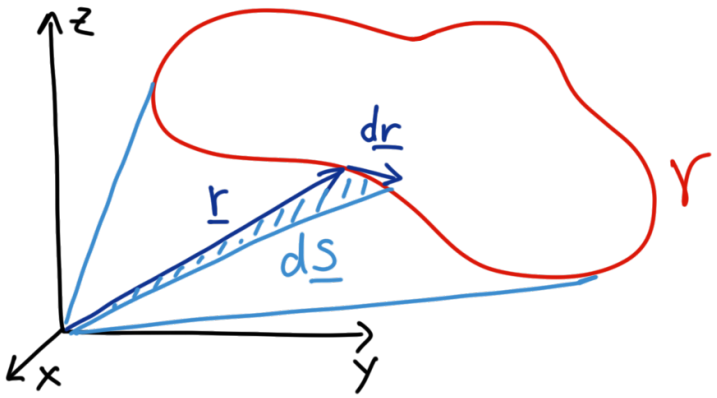


Figure 31: Vector area defined as a cone connecting the origin to some closed curve γ. The vector area element $\mathrm{d}\mathbf{S}$ can be found by taking the cross product of the vector $\mathbf{r}$ describing the curve, and its differential $\mathrm{d}\mathbf{r}$.

in Fig. 31, created by the cone connecting the origin to some closed curve γ. The triangular vector area element $\mathrm{d}\mathbf{S}$ is given by

$$\mathrm{d}\mathbf{S} = \frac{1}{2}\mathbf{r} \times \mathrm{d}\mathbf{r},$$

and adding up all of these triangles along the curve γ gives us the total vector area of the surface:

$$\mathbf{S} = \frac{1}{2}\oint_\gamma \mathbf{r} \times \mathrm{d}\mathbf{r}. \tag{138}$$

The vector area of any surface bound by a closed curve γ can therefore be found by performing a line integral of this kind.

5.2.2 Normal vector to a surface

A convenient way to describe the normal to a general surface is in terms of its gradient. Consider the surface shown in Fig. 30, with some projection on the xy-plane A. For a 2D surface $f(x, y)$ we have already seen how this can be done by considering an isosurface in a higher dimension in Example 20, i.e. $g(x, y, z) = f(x, y) - z = 0$. The unit normal at any point on the surface is then given by the normalized gradient

$$\hat{\mathbf{n}} = \frac{\nabla g}{|\nabla g|},$$

evaluated in the point of interest. Explicitly,

$$\nabla g = \nabla f(x, y) - \nabla z = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right), \tag{139}$$

$$|\nabla g| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}. \tag{140}$$

We see from Eq. (136) that in order to evaluate a surface integral we need to find expressions for both $\hat{\mathbf{n}}$ and $\mathrm{d}S$. To find a more convenient expression for the surface element we can use Eq. (137) to write

$$\mathrm{d}S = \frac{\mathrm{d}A}{|\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}|} = \frac{|\nabla g|\,\mathrm{d}A}{|\nabla g \cdot \hat{\mathbf{k}}|} = |\nabla g|\,\mathrm{d}A. \tag{141}$$

We can use these expressions to relate any surface integral over S to a double integral over the projected region in the xy-plane A:

$$\int_S \mathrm{d}\mathbf{S} = \iint_S \hat{\mathbf{n}}\,\mathrm{d}S = \iint_A \frac{\nabla g}{|\nabla g|}|\nabla g|\,\mathrm{d}A = \iint_A \nabla g\,\mathrm{d}A. \tag{142}$$

The convenient cancelation of the normalization of the gradient of g make this expression relatively simple to manipulate. Note the loose similarities to the chain rule, considering that $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$.

An alternative approach is to calculate the vector normal to the surface directly, taking the cross product of two linearly independent vectors lying in the tangent plane to some point on the surface. This approach is convenient for surfaces given in parametric form $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. As we have seen when discussing vector derivatives and the gradient, the vectors $\frac{\partial \mathbf{r}(u,v)}{\partial u}$ and $\frac{\partial \mathbf{r}(u,v)}{\partial v}$, evaluated in some point, are guaranteed to lie in the tangent plane to the surface in that point. The unit normal vector to the surface (and to the tangent plane) can then be obtained from the cross product:

$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|}. \tag{143}$$

Following the same steps taken to find the Jacobian for a coordinate transformation we can also find the size of the area element for a parametric surface as

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v. \tag{144}$$

Again, the normalization term will cancel when inserting these into a surface integral

$$\int_S \mathrm{d}\mathbf{S} = \iint_S \hat{\mathbf{n}}\,\mathrm{d}S = \iint_A \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v. \tag{145}$$

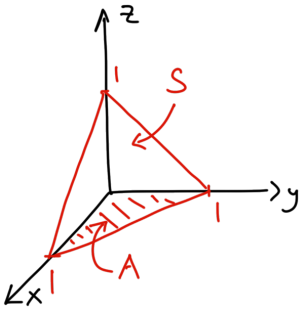
5.2.3 Surface integrals of scalar fields

We can use the results from the preceding section to evaluate integrals of some scalar field over a surface. There are two main approaches to follow here, depending on whether the surface is given in parametric form, or if it is provided non-parametrically. The easiest way to get to grips with this is via a couple of examples.

Example 24. Evaluate

$$\iint_S 6xy\,\mathrm{d}S$$

over the plane $x + y + z = 1$ in the first octant (i.e., for positive (x, y, z)).



Solution: We need to find a convenient way to parametrize the surface element $\mathrm{d}S$. Let's use the gradient, via Eq. (141). We write:

$$g = x + y + z - 1 = 0 \quad \Rightarrow \quad \nabla g = (1, 1, 1) \quad \Rightarrow \quad |\nabla g| = \sqrt{3}.$$

Therefore,

$$\mathrm{d}S = \sqrt{3}\,\mathrm{d}x\,\mathrm{d}y,$$

where we have projected the integral over the surface S onto the xy-plane. The domain of integration is now the triangle bounded by $x = 0$, $y = 0$, and $y = 1 - x$ (all in the xy-plane), and we can write the integral as

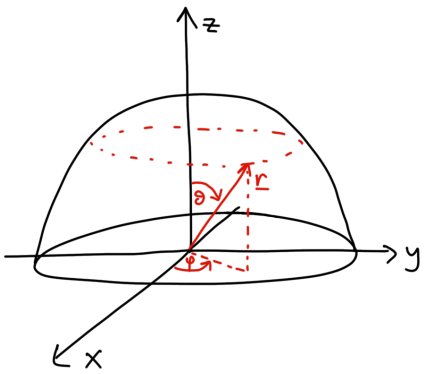
$$\begin{aligned}\iint_S 6xy\,\mathrm{d}S &= \iint_A 6xy|\nabla g|\,\mathrm{d}A \\ &= \sqrt{3}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\; 6xy \\ &= 6\sqrt{3}\int_0^1 \mathrm{d}x\; x\frac{(1-x)^2}{2} \\ &= \frac{\sqrt{3}}{4}.\end{aligned}$$

□

Example 25. Evaluate

$$\iint_S z\,\mathrm{d}S$$

over the surface $x^2 + y^2 + z^2 = 4$, for $z \geq 0$.



Solution: The surface is the positive hemisphere, which can be conveniently parametrized using spherical polar coordinates. Let's take this approach in evaluating the integral. The vector to the surface is given by (see Eq. (116))

$$\mathbf{r}(\theta, \phi) = 2(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \theta \in [0, \pi/2], \phi \in [0, 2\pi]. \tag{146}$$

To find $\mathrm{d}S$, we use Eq. (144), with $u = \theta$ and $v = \phi$. Let's first find the two vectors in the tangent plane to the surface:

$$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} 2\cos\theta\cos\phi \\ 2\cos\theta\sin\phi \\ -2\sin\theta \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial \phi} = \begin{pmatrix} -2\sin\theta\sin\phi \\ 2\sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

The modulus of the cross product is

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = 4 \sin\theta, \tag{147}$$

which should come as little surprise: we did not need to do this calculation again, as we know the Jacobian for spherical polar coordinates is $r^2 \sin\theta$, i.e., the result found here. However, for a more general parametrization for which the Jacobian is not known, the cross product will need to be evaluated directly. We now have all the ingredients needed to evaluate the integral:

$$\begin{aligned}\iint_S z \, \mathrm{d}S &= \iint_S z(\theta) \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \mathrm{d}\theta \; 2\cos\theta \, 4\sin\theta = 8\pi.\end{aligned}$$

□

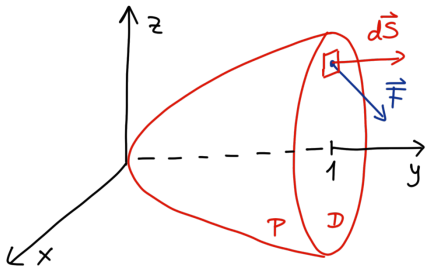
5.2.4 Surface integrals of vector fields

Surface integrals of vector fields are widely in physics as we often want to evaluate vector quantities over some surface in three-dimensional space. They provide a way to calculate the *total flow* of a vector field across a surface; if the surface is closed, then we will talk about the flow into, or out of, some region of space. We have a choice in the direction of the normal vector to a surface, which is important as it determines the direction of the vector flow. The normal vector is conventionally chosen to point *outwards* of a closed surface. For an open surface, the convention is to follow the right-hand rule tracing the curve that forms the surface boundary.

Example 26. Evaluate

$$\iint_S \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{S}$$

over the surface bounded by the paraboloid $y = x^2 + z^2$ for $y \in [0, 1]$, and the disc $x^2 + z^2 \leq 1$ at $y = 1$, for $\mathbf{F} = (0, y, -z)$.



Solution: We will use the gradient approach to solve this problem, and will want to parametrize the surface to write it in the form given in Eq. (142), so that

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_A \mathbf{F} \cdot \nabla g \, \mathrm{d}A.$$

We will consider the paraboloid and disc surfaces separately. Because of the orientation of the paraboloid it will be most convenient to choose A in the xz-plane. We can then write $y = f(x, z) = x^2 + z^2$, so that $g(x, y, z) = y - f(x, z) = y - x^2 - z^2$. The gradient of g is given by

$$\nabla g = (-2x, 1, -2z),$$

which represents a vector normal to the surface of the paraboloid. This vector points *inward* to the surface, and so we should change its sign to adhere to the normal convention of outward-pointing normals. The integral over the surface of the paraboloid then becomes

$$\begin{aligned}\iint_P \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iint_A \begin{pmatrix}0\\ y\\ -z\end{pmatrix}\cdot\begin{pmatrix}2x\\ -1\\ 2z\end{pmatrix}\mathrm{d}x\,\mathrm{d}z\\ &= \iint_A (-y-2z^2)\,\mathrm{d}x\,\mathrm{d}z\\ &= \iint_A (-x^2-3z^2)\,\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

The domain A is the projection of the paraboloid on the xz-plane, and is simply the disc of radius 1 in that plane. Plane polar coordinates are therefore ideally suited to compute this double integral. Setting $x = r\cos\theta$ and $z = r\sin\theta$ for $\theta\in[0,2\pi]$ and $r\in[0,1]$, we have:

$$\begin{aligned}\iint_A (-x^2-3z^2)\,\mathrm{d}x\,\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^1 \mathrm{d}r\; r(-r^2\cos^2\theta - 3r^2\sin^2\theta)\\ &= -\int_0^{2\pi}\mathrm{d}\theta(\cos^2\theta+3\sin^2\theta)\int_0^1 r^3\,\mathrm{d}r\\ &= -\frac{1}{4}\int_0^{2\pi}\mathrm{d}\theta\left(\frac{1}{2}(\cos 2\theta-1)+\frac{3}{2}(1-\cos 2\theta)\right) = -\pi.\end{aligned}$$

We now need to integrate the vector field over the disc. This part is much simpler. The normal is obviously along the y-axis, and so the integral becomes:

$$\begin{aligned}\iint_D \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iint_D \begin{pmatrix}0\\ y\\ -z\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\mathrm{d}x\,\mathrm{d}z\\ &= \iint_D y\,\mathrm{d}x\,\mathrm{d}z = \pi,\end{aligned}$$

where the results follows directly since $y=1$ on the disc, and so the integral is simply the area of a disc of radius 1. Adding the two contributions we find that the total integral over the closed surface is 0. □

Example 27. Evaluate

$$\iint_S \mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{S}$$

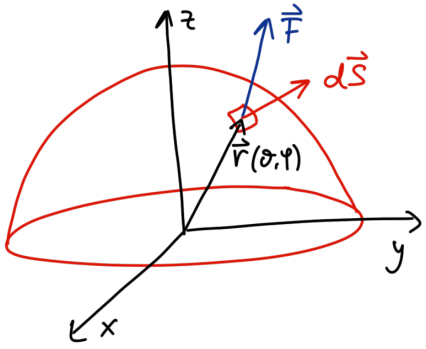
over the upper half of the sphere $x^2+y^2+z^2=9$ and for $\mathbf{F}=(x,y,z^4)$.

Solution: We will solve this using the parametric approach, given that the surface is a sphere. We follow the same initial approach as in Example 25. The surface is given parametrically by the vector equation

$$\mathbf{r}(\theta,\phi) = 3(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\quad \theta\in[0,\pi/2],\phi\in[0,2\pi]. \tag{148}$$

We can then find the normal vector from the cross product of two vectors in the tangent plane, found by taking the derivatives of $\mathbf{r}$ with respect to the two parameters (θ,ϕ):

$$\frac{\partial\mathbf{r}}{\partial\theta} = \begin{pmatrix}3\cos\theta\cos\phi\\ 3\cos\theta\sin\phi\\ -3\sin\theta\end{pmatrix},\quad \frac{\partial\mathbf{r}}{\partial\phi} = \begin{pmatrix}-3\sin\theta\sin\phi\\ 3\sin\theta\cos\phi\\ 0\end{pmatrix},$$



and the cross product is

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = 9 \begin{pmatrix} \sin^2\theta \cos\phi \\ \sin^2\theta \sin\phi \\ \sin\theta \cos\theta \end{pmatrix}.$$

The vector function $\mathbf{F}$ written in spherical polar coordinates is:

$$\mathbf{F} = \begin{pmatrix} 3\sin\theta\cos\phi \\ 3\sin\theta\sin\phi \\ 3^4\cos^4\theta \end{pmatrix}.$$

We can now evaluate the integral using the expression in Eq. (145):

$$\begin{aligned} \iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \iint_A \mathbf{F}\cdot\left(\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right)\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathrm{d}\theta\,\begin{pmatrix} 3\sin\theta\cos\phi \\ 3\sin\theta\sin\phi \\ 3^4\cos^4\theta \end{pmatrix}\cdot 9\begin{pmatrix} \sin^2\theta\cos\phi \\ \sin^2\theta\sin\phi \\ \sin\theta\cos\theta \end{pmatrix} \\ &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathrm{d}\theta\,\left(27\sin^3\theta + 729\sin\theta\cos^5\theta\right) = 279\pi. \end{aligned}$$

□

6 Vector fields and operators

In addition to the gradient that was introduced in the previous chapter, which we have seen is a differential operator on scalar fields, there are two more differential operators important for the study of vector fields: the *divergence* and the *curl*. As we shall see, both can be expressed in terms of the differential operator ∇.

6.1 The Divergence

The divergence of a vector field $F(\mathbf{r})$, with $\mathbf{r} = (x, y, z)$, is defined as

$$\operatorname{div} \mathbf{F}(\mathbf{r}) := \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \nabla \cdot \mathbf{F}(\mathbf{r}), \tag{149}$$

where F_x, F_y, F_z are the components of $\mathbf{F}$. A vector field $\mathbf{F}$ for which $\nabla \cdot \mathbf{F} = 0$ is called *solenoidal* or *incompressible*.

The divergence produces a scalar output (which is a function, or a field) that measures the rate of change of a vector field at a given point, or, perhaps easier to imagine, in some infinitesimal volume around a point. As such, the divergence of a vector field can be thought of as a measure of how much the vectors in the field are spreading out or converging. Because of this, it is often used to describe the flow of a fluid, heat or electricity, or other similar physical quantities.

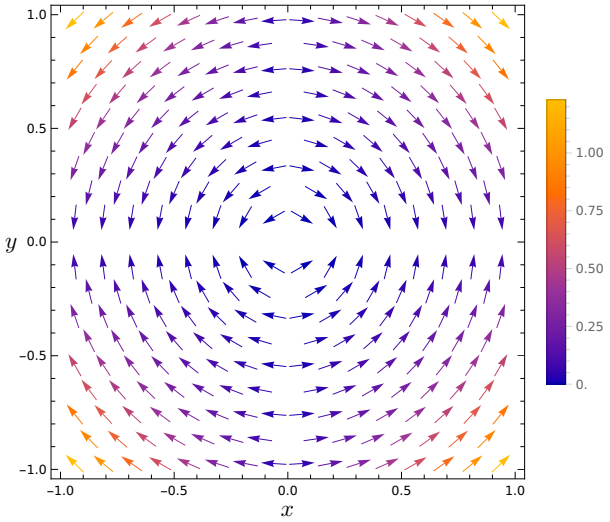


Figure 32: Plot of the 2D vector field $\mathbf{F} = (xy^2, -yx^2)$.

To see this in a simple 2D example, consider the vector field $\mathbf{F} = (xy^2, -yx^2)$, shown in Fig. 32. The divergence of this field is simple to calculate: $\nabla \cdot \mathbf{F} = y^2 - x^2$, which is the hyperboloid surface shown in Fig. 33. We can see that the divergence is increasingly large and positive along the y-axis around $x = 0$, indicating a flow of the vector field *away* from this region, and increasingly negative along the x-axis, indicating a flow *toward* this region. This is, of course, fully consistent with the direction of the arrows in Fig. 32.

Example 28. Evaluate the divergence of the vector field

$$\mathbf{F}(x, y, z) = \left(3xy^2, \frac{2x}{(y-1)}, -\frac{xy}{z}\right).$$

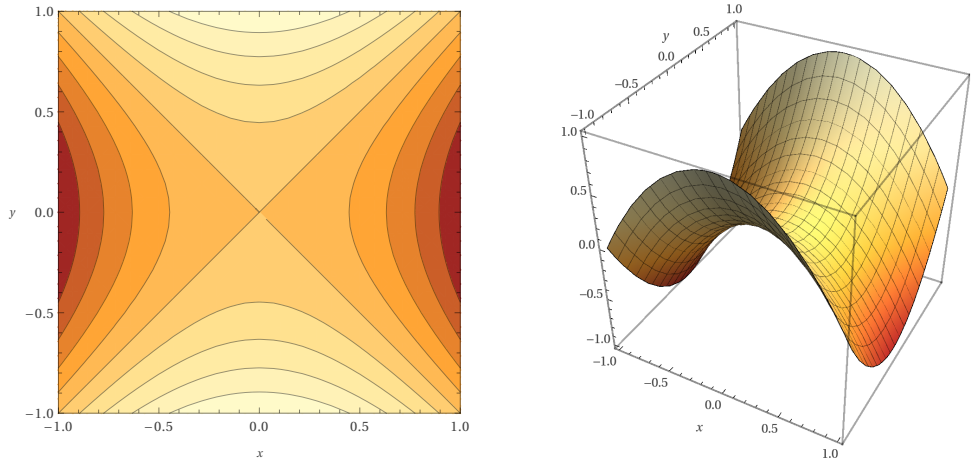


Figure 33: Divergence of the vector field $\mathbf{F} = (xy^2, -yx^2)$ shown in Fig. 32.

Solution: We note that the field is well-behaved except for $y \neq 1$ and $z \neq 0$. Everywhere else,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}\left(\frac{2x}{(y-1)}\right) - \frac{\partial}{\partial z}\left(\frac{xy}{z}\right) \\ &= 3y - \frac{2x}{(y-1)^2} + \frac{xy}{z^2}.\end{aligned}$$

□

If a vector field $\mathbf{F}$ can be written as the gradient of some scalar potential ϕ, i.e., it is conservative and $\mathbf{F} = \nabla\phi$, then we can write the divergence directly in terms of the scalar potential as

$$\nabla \cdot \mathbf{F} = \nabla \cdot \nabla\phi = \nabla^2\phi. \tag{150}$$

This differential operator is again sufficiently common to merit its own symbol and name – it is called the *Laplacian*:

$$\Delta := \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{151}$$

The usage of the delta to denote the Laplacian, while common in maths, is not common in physics.

Example 29. Evaluate the Laplacian of the scalar field

$$\phi = xy^2z^3.$$

Solution:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2xz^3 + 6xy^2z.$$

□

6.2 The Divergence Theorem

The divergence theorem, also known as Gauss's theorem or the Gauss-Ostrogradsky theorem, is an important mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.

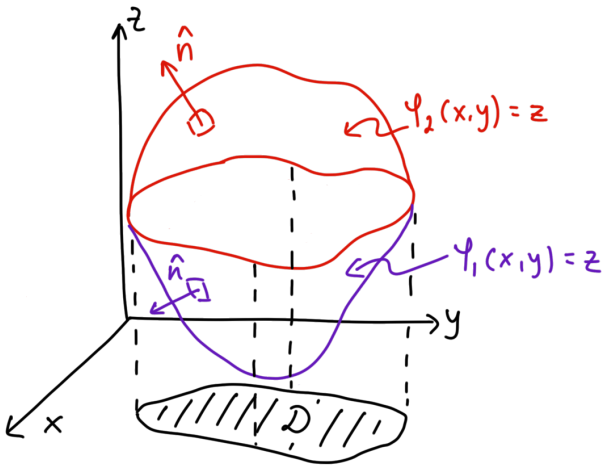


Figure 34: A volume V enclosed by a surface S used in the divergence theorem. The surface can be split into two surfaces which can be parametrized as $z = \varphi(x, y)$.

Theorem 5. *Let $V \subset \mathbb{R}^3$ be a simple region in space bounded by a regular closed surface S, let $\hat{\mathbf{n}}$ be a vector normal to S pointing outwards, and let $\mathbf{F}$ be a differentiable vector field with continuous first derivatives. Then,*

$$\iiint_V \nabla \cdot \mathbf{F} \; \mathrm{d}V = \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S. \tag{152}$$

That is to say, the flux of a vector field leaving a closed surface is equal to the divergence of the field integrated over the volume bounded by that surface.

To prove the divergence theorem, we proceed as follows. Let us first assume the volume V is z-simple, which is to say it can be written as

$$V = \{(x, y, z) : \varphi_1(x, y) < z < \varphi_2(x, y), (x, y) \in D \subset \mathbb{R}^2\},$$

with φ_1 and φ_2 differentiable functions with continuous first derivatives (also called C^1 functions on D). This volume is illustrated in Fig. 34. Using this, and taking the vector field components explicitly $\mathbf{F} = (F_1, F_2, F_3)$, we can rewrite the last term of the triple integral on the left hand side of Eq. (152) as

$$\begin{aligned}
\iiint_V \frac{\partial F_3}{\partial z} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \iint_D \mathrm{d}x \, \mathrm{d}y \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial F_3}{\partial z} \, \mathrm{d}z \\
&= \iint_D [F_3(x, y, \varphi_2(x, y)) - F_3(x, y, \varphi_1(x, y))] \, \mathrm{d}x \, \mathrm{d}y \\
&= \iint_{\varphi_2} F_3 \, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \iint_{\varphi_1} F_3 \, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \oiint_S F_3 \, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.
\end{aligned}$$

The last line follows from Eq. (137) as we move from an integral over the xy-domain back to an integral over the open surfaces φ_1 and φ_1. The closed surface S is then reconstructed from the two open surfaces φ_1 and φ_1.

Following the same reasoning, we see that if, instead, V is y-simple, then

$$\iiint_V \frac{\partial F_2}{\partial y} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \oiint_S F_2 \, \hat{\mathbf{j}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S,$$

or if V is x-simple, then

$$\iiint_V \frac{\partial F_1}{\partial x} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \oiint_S F_1 \, \hat{\mathbf{i}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

A bounded volume can always be broken down into sections that are simple along some direction, and since integration is linear, summing the three components gives the general result, which is the divergence theorem. □

A valuable way to see the physical meaning of the divergence operator is to take the divergence theorem in the limit of infinitesimal small volumes. For this purpose, let us consider the volume V to be that of a sphere of radius r centred in some point $\mathbf{r}_0$, and let us divide the divergence theorem expressions by the volume $|V(r)|$ of this sphere:

$$\frac{1}{|V(r)|}\iiint_V \nabla\cdot\mathbf{F}\ \mathrm{d}V = \frac{1}{|V(r)|}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S.$$

We now take the limit of these expressions for $r\to 0$. The left-hand side of the equation simply tends to the divergence of $\mathbf{F}$, evaluated in the point $\mathbf{r}_0$:

$$\nabla\cdot\mathbf{F}(\mathbf{r}_0) = \lim_{r\to 0}\frac{1}{|V(r)|}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S. \tag{153}$$

The right-hand side of the equation shows that the divergence of a vector field in some point gives the flux density of the vector field leaving that point, per unit volume. It is therefore intimately related to the concept of sources and sinks of a vector field. The expression given in Eq.(153) is also known as the integral form of the divergence operator. It has the advantage of defining the divergence operator in terms of the geometry of the system, and is not dependent on the coordinate system used (unlike Eq. (149)). It is, however, typically more challenging to use for solving problems in practice.

Example 30. Evaluate the divergence of the gravitational field (also called the gravitational acceleration) generated by a point mass placed in the origin

$$\mathbf{F}(\mathbf{r}) = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\begin{pmatrix}x\\y\\z\end{pmatrix},$$

where $\mathbf{r}=(x,y,z)$ and $r=\sqrt{x^2+y^2+z^2}$ is the standard Euclidean distance.

Solution: We note that the field is not defined in the origin, but is well behaved elsewhere. For $r\neq 0$ we have

$$\begin{aligned}\nabla\cdot\mathbf{F} &= -GM\left\{\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right\}\\ &= -GM\left\{\frac{r^3-3x^2r}{r^6}+\frac{r^3-3y^2r}{r^6}+\frac{r^3-3z^2r}{r^6}\right\}\\ &= -\frac{GM}{r^6}\left\{3r^3-3(x^2+y^2+z^2)r\right\}=0.\end{aligned}$$

We conclude that the gravitational field is solenoidal everywhere except in the origin.

This result may surprise you, as you may have expected the field to be a diverging one, given that it has a source in the origin. To dig into this result a little deeper, consider the same problem, but let us now use the divergence theorem given in Eq. (152), where we evaluate the surface integral considering a sphere of radius r centred in the origin:

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= -GM\oiint_S \frac{\hat{\mathbf{r}}}{r^2}\,\hat{\mathbf{r}}\,\mathrm{d}S\\ &= -GM\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\frac{|\hat{\mathbf{r}}|^2}{r^2}\,r^2\sin\theta\,\mathrm{d}\theta\\ &= -GM\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = -4\pi GM.\end{aligned}$$

This expression does not depend on r, which means that if we take a volume between two spherical shells some Δr apart, then the amount of vector flux traversing each surface must be the same. We can see why from the second line of the equation above: the scaling of the field with distance as $1/r^2$ exactly cancels out the r^2 increase of the surface with distance (which comes from the Jacobian). We thus understand intuitively why the field must have zero divergence outside of the origin, and can extend this result to any vector field that scales as $1/r^2$, e.g. the electric field.

Interestingly, the result above also implies that

$$\iiint_V \nabla \cdot \mathbf{F} \; \mathrm{d}V = -4\pi GM, \tag{154}$$

which cannot hold if $\nabla \cdot \mathbf{F} = 0$ everywhere. The problem is of course in the origin, where the field is not defined. Equation (153) provides further insight on this conundrum; in the limit of $r \to 0$ the divergence can be written as

$$\nabla \cdot \mathbf{F}(\mathbf{r} \to 0) = \lim_{r \to 0} \frac{-4\pi GM}{|V(r)|}. \tag{155}$$

This limit diverges for any $M \neq 0$. It would thus seem that the divergence of our inverse-square-law field should be 0 everywhere except in the origin, where it diverges. We have already seen an object that can describe such physics: the Dirac delta. We can then rewrite the divergence of the gravitational field as

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi GM\delta(\mathbf{r}), \tag{156}$$

where $\delta(\mathbf{r}) := \delta(x)\delta(y)\delta(z)$ is the 3D generalization of the 1D Dirac delta:

$$\iiint_V \delta(\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}z \; \delta(x)\delta(y)\delta(z) = 1. \tag{157}$$

With this expression the divergence in any region of space excluding the origin $r = 0$ is indeed 0, but the divergence returns the result of Eq. (154) whenever the origin is included in the integral. By introducing the Dirac delta the divergence theorem can be made valid even for non-differentiable point sources. □

We note that the gravitational field can be expressed as the gradient of a scalar potential, $\mathbf{F} = -\nabla\phi$. Finding ϕ is straightforward. We write

$$\begin{pmatrix} \frac{\partial}{\partial x}\phi \\ \frac{\partial}{\partial y}\phi \\ \frac{\partial}{\partial z}\phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which we can integrate directly to find

$$\phi(r) = \frac{GM}{r} + \text{const.}$$

By imposing that the scalar potential should tend to 0 for $r \to \infty$ we can set the constant to 0.

It is often convenient to deal with mass densities $\rho = \rho(x, y, z)$ rather than with point masses, where the total mass M contained in some volume V of space is given by the integral

$$M = \iiint_V \rho \, \mathrm{d}V.$$

In terms of the mass density, the divergence of the gravitational field can be written as

$$\nabla \cdot \mathbf{F} = -4\pi G\rho. \tag{158}$$

Writing this expression in terms of the scalar potential leads to the well-known Poisson equation, which relates a distribution of matter of density ρ with the gravitational potential ϕ:

$$\nabla^2\phi = 4\pi G\rho. \tag{159}$$

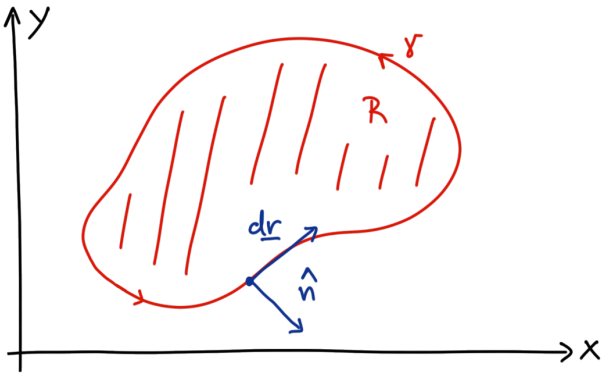


Figure 35: A region of the xy-plane bounded by the closed curve γ. For a point on the curve described by the vector $\mathbf{r}$ we can identify a tangent vector $\mathrm{d}\mathbf{r}$ and a normal vector $\hat{\mathbf{n}}$.

6.2.1 Divergence theorem in 2D

The ideas described above were developed in 3D space, but the divergence theorem can also be useful in two dimensions. Consider a two-dimensional planar region R in the xy-plane bounded by some closed curve γ, as shown in Fig. 35. As we have seen when discussing space curves (in particular, recall Fig. 23 and related discussion), the tangent and normal vectors to any point on the bounding curve γ are given by $\mathrm{d}\mathbf{r} = (\mathrm{d}x, \mathrm{d}y)$ and $\hat{\mathbf{n}} = (\mathrm{d}y, -\mathrm{d}x)$, where we chose $\hat{\mathbf{n}}$ to point outwards of the region R (note: to find $\hat{\mathbf{n}}$ simply look for a vector in the plane for which $\mathrm{d}\mathbf{r} \cdot \hat{\mathbf{n}} = 0$; clearly the above satisfies this requirement). If $\mathbf{F} = (F_x, F_y)$ is a continuous and differentiable vector field on R, then we can rewrite Eq. (152) as:

$$\iint_R \nabla \cdot \mathbf{F} \; \mathrm{d}x \, \mathrm{d}y = \oint_\gamma \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}s, \tag{160}$$

which, written explicitly in terms of the components of $\mathbf{F}$, is

$$\iint_R \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \oint_\gamma (F_x \, \mathrm{d}y - F_y \, \mathrm{d}x). \tag{161}$$

By setting $P = -F_y$ and $Q = F_x$ we rediscover Green's theorem in the plane, as defined in Eq. (114), which we can now consider simply as a special case of the more general divergence theorem.

6.3 The Curl

We now want to find an expression to help us evaluate the *circulation* of a vector field. We already encountered the concept of circulation – the line integral of a vector field around a closed curve – in the context of conservative fields, and will come back to this application shortly. Let's denote the circulation of some continuously differentiable vector field $\mathbf{F}$ as the curl of $\mathbf{F}$, or $\operatorname{curl} \mathbf{F}$. This operation will necessarily produce another vector field, since the curl will have both a size (i.e., the amount of circulation) and a direction (i.e., axis around which the circulation occurs defines a direction). Furthermore, we are interested in the microscopic circulation, that is to say the value of the curl in some infinitesimal neighbourhood of some point in space. To define this, we start by picking some point P, and some direction given by a vector $\hat{\mathbf{n}}$ in the point P. Then the projection of the curl of $\mathbf{F}$ near P along $\hat{\mathbf{n}}$ will be given by

$$(\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}} = \lim_{A \to 0} \frac{1}{A} \oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r}, \tag{162}$$

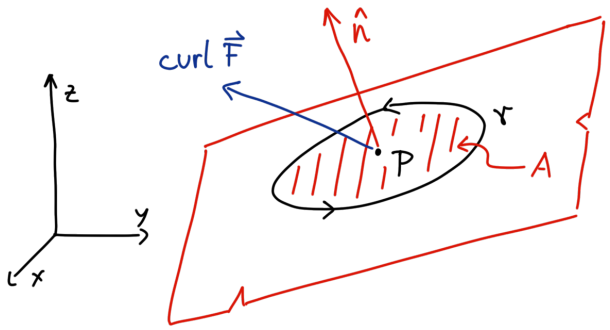


Figure 36: The curl of a vector field $\mathbf{F}$ in some point P along the direction $\hat{\mathbf{n}}$ is given by the limit of a closed line integral representing the circulation of the field.

where the closed line integral is the circulation of $\mathbf{F}$ along the curve γ that lies in the plane with normal vector $\hat{\mathbf{n}}$ circling point P, and A is the area of the plane bounded by γ, as shown in Fig. 36. In the limit of $A \to 0$ we obtain the component of the curl along $\hat{\mathbf{n}}$ in the point P.

To find a more convenient expression for the curl, let's pick a particular case where $\hat{\mathbf{n}}$ lies along the z axis, i.e., $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. The curve γ will then lie in the xy-plane, and let us consider γ to be a rectangle with one edge in point $P(a, b)$ and with sides Δx and Δy, as illustrated in Fig. 37. Since $\mathbf{F} = (F_x, F_y, F_z)$, we can rewrite $\mathbf{F} \cdot \mathrm{d}\mathbf{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$.

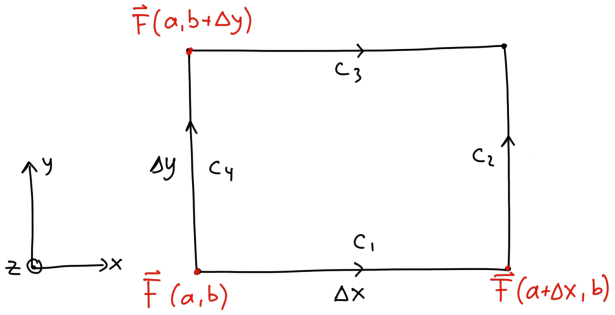


Figure 37: Rectangular curve with area $\Delta x \Delta y$, split into straight segments C_1, C_2, C_3 and C_4. Given the orientation, note that the closed curve $\gamma = C_1 + C_2 - C_3 - C_4$. We omit writing the z component of $\mathbf{F}$ explicitly as it is constant in the plane.

Given that $\mathrm{d}z = 0$ along any path in the xy-plane, Eq. (162) becomes:

$$
\begin{aligned}
(\operatorname{curl}\mathbf{F}) \cdot \hat{\mathbf{k}} &= \lim_{A\to 0} \frac{1}{A} \oint_\gamma (F_x\,\mathrm{d}x + F_y\,\mathrm{d}y) \\
&= \lim_{A\to 0} \frac{1}{A} \left(\int_{C_1} F_x\,\mathrm{d}x + \int_{C_2} F_y\,\mathrm{d}y - \int_{C_3} F_x\,\mathrm{d}x - \int_{C_4} F_y\,\mathrm{d}y \right) \\
&= \lim_{A\to 0} \frac{1}{A} [F_x(a,b)\Delta x + F_y(a+\Delta x, b)\Delta y - F_x(a, b+\Delta y)\Delta x - F_y(a,b)\Delta y].
\end{aligned}
$$

Note that in the last two lines we changed the direction of two of the the integrals to use the three vertices shown in the Fig. 37, and have also omitted writing the z component of $\mathbf{F}$ explicitly

to simplify notation. Noting that $A = \Delta x \Delta y$, and rearranging, we have

$$\begin{aligned}(\operatorname{curl}\mathbf{F})\cdot\hat{\mathbf{k}} &= \lim_{A\to 0}\left\{\frac{[F_y(a+\Delta x,b)-F_y(a,b)]\Delta y}{\Delta x\Delta y} - \frac{[F_x(a,b+\Delta y)-F_x(a,b)]\Delta x}{\Delta x\Delta y}\right\}\\ &= \lim_{\Delta x\to 0}\frac{[F_y(a+\Delta x,b)-F_y(a,b)]}{\Delta x} - \lim_{\Delta y\to 0}\frac{[F_x(a,b+\Delta y)-F_x(a,b)]}{\Delta y}\\ &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.\end{aligned}$$

The choice of axis is, of course, arbitrary, so repeating the same exercise with $\hat{\mathbf{n}}$ along the x and y axes gives

$$\begin{aligned}(\operatorname{curl}\mathbf{F})\cdot\hat{\mathbf{i}} &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\\ (\operatorname{curl}\mathbf{F})\cdot\hat{\mathbf{j}} &= \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}.\end{aligned}$$

These projections along the coordinate axes provide the curl vector in the Cartesian basis. We recognize them as the components of a vector product, as defined in Eq. (8). Using the differential operator ∇ we can thus write the curl in a more convenient form for calculations

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}} \equiv \nabla\times\mathbf{F}. \tag{163}$$

A vector field $\mathbf{F}$ for which $\nabla\times\mathbf{F} = 0$ is called *irrotational*.

Example 31. Find the curl of the vector field

$$\mathbf{F} = (xy, x^2z, xyz).$$

Solution:

$$\nabla\times\mathbf{F} = \det\begin{pmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & x^2z & xyz\end{pmatrix} = \begin{pmatrix}xz - x^2\\ -yz\\ 2xz - x\end{pmatrix}.$$

□

From the definition of the curl we can see that it measures the microscopic rather than some macroscopic circulation of a vector field. It represents the tendency of a fictional microscopic paddle wheel, placed in some point in the field, to rotate around the axis that points in the direction of the curl. Note that the micro and macroscopic circulations of a field are typically different. To see this, consider a solid body rotating around the z axis with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$. In the xy-plane, some point at $\mathbf{r} = (x, y)$ will then have a velocity

$$\mathbf{v} = \vec{\omega}\times\mathbf{r} = \det\begin{pmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0 & \omega\\ x & y & 0\end{pmatrix} = \begin{pmatrix}-y\omega\\ +x\omega\\ 0\end{pmatrix}.$$

This velocity is a vector field that represents the macroscopic rotation of the solid body. Let's calculate the curl of this field to find the microscopic circulation, for comparison:

$$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \det\begin{pmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -y\omega & x\omega & 0\end{pmatrix} = 2\omega\hat{\mathbf{k}}.$$

We see that the curl is twice the angular velocity vector of the solid body around its axis of rotation. The velocity field and its curl are shown in Fig. 38.

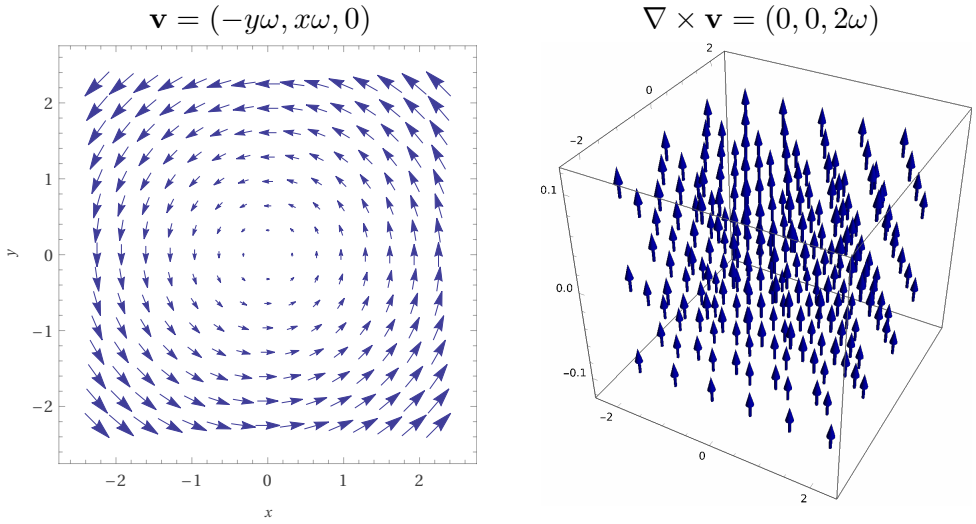


Figure 38: Velocity field of a solid body in rotation, and its curl.

Recall that we have already encountered the terms forming the components of $\nabla \times \mathbf{F}$ when discussing exact differentials in Section 4.6, in Eq. (113). There we saw that given some vector field $\mathbf{F}$, if all the components of $\nabla \times \mathbf{F} = 0$, then the differential form $\mathbf{F} \cdot \mathrm{d}\mathbf{r} = F_x \, \mathrm{d}x + F_y \, \mathrm{d}y + F_z \, \mathrm{d}z$ is an exact differential. We showed how line integrals of such a differential must be path-independent, and that closed line integrals are always zero. A vector field for which this holds is called a conservative field. Using the curl formalism defined here we now have a convenient way to check whether a field is conservative or not. This should come as little surprise given the definition of the curl in Eq. (162): if $\mathbf{F}$ is conservative then the closed line integral will be zero for any choice of $\hat{\mathbf{n}}$.

It is important to note that these observations require the region bounded by the curve along which we perform the line integral to be simply connected. From our derivation of the curl it should be obvious why: in taking the limit for the area bounded by the curve, we explicitly require that the vector field be well-defined in the point (and its neighbourhood) in which the curl is being evaluated. We discussed this already in Section 4.6, and reiterate it here. If all closed line integrals on some domain are always zero then the field is certainly irrotational, but if we find an irrotational field on a multiply-connected domain we are not guaranteed that all closed line integrals will be zero. Here is an example of such a case.

Example 32. Find the curl of the vector field

$$\mathbf{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Is $\mathbf{F}$ conservative?

Solution: We proceed in the normal way by writing

$$\nabla \times \mathbf{F} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} = \left(\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2} \right) \hat{\mathbf{k}} = 0.$$

The field $\mathbf{F}$ is shown in Fig. 39. There is a clearly visible macroscopic circulation, but the curl shows us that the microscopic circulation is zero. So if we imagine a paddle wheel moving around in this field it would follow the field lines, but would not itself rotate. If we compare this with

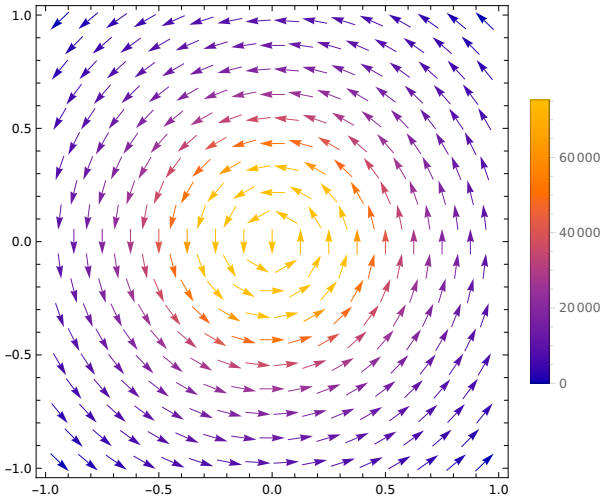


Figure 39: Vector field given in Example 32. While there is clear macroscopic circulation, there is no microscopic circulation: the curl of this field is 0.

the solid body case, we can see that the only difference between the two is how quickly the field drops off as we move away from the origin: the inverse square rate here is such that the fields on the inner and outer side of our imaginary paddle wheel cancel out exactly. It is clearly possible to have very different behaviour in micro and macroscopic circulation of a vector field.

Importantly, this field is defined everywhere except for the origin (where it diverges), so the xy-plane is not simply connected. Let's calculate the circulation of $\mathbf{F}$ along a circumference centred in the origin of radius 1. Moving to plane polar coordinates, we have $x^2 + y^2 = r^2 = 1$ so $\mathbf{F} = (-\sin\theta, \cos\theta)$, and

$$\oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta = 2\pi \neq 0.$$

We see that while $\nabla \times \mathbf{F} = 0$, not all closed line integrals are zero. They are zero only if the path doesn't enclose the origin. Otherwise, the integral will be 2π. So $\mathbf{F}$ is (locally) conservative over any domain in the xy-plane that excludes the origin, but is globally non-conservative. □

6.4 Stokes' theorem

Stokes' theorem is a mathematical statement that allows us to relate the *circulation* of a vector field along some curve γ, to its *vorticity* (a measure of the amount of rotation of the field in a given region, given by the curl operator) within the surface bounded by γ.

Theorem 6. *Let S be a regular and oriented surface with unit normal vector $\hat{\mathbf{n}}$, bounded by a closed regular curve γ, and let $\mathbf{F} = (F_x, F_y, F_z)$ be a continuously differentiable vector field defined over S and γ. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{164}$$

That is to say, the flux of the curl of a vector field through some open surface S equals the circulation of the field along the surface's boundary. We note that the surface may be closed; if so, the integrals must equal zero.

To see why Stokes' theorem holds, consider the surface S bounded by a closed curve γ as illustrated in Fig. 40. The circulation of some well-behaved vector field $\mathbf{F}$ along γ is given by

$$\oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

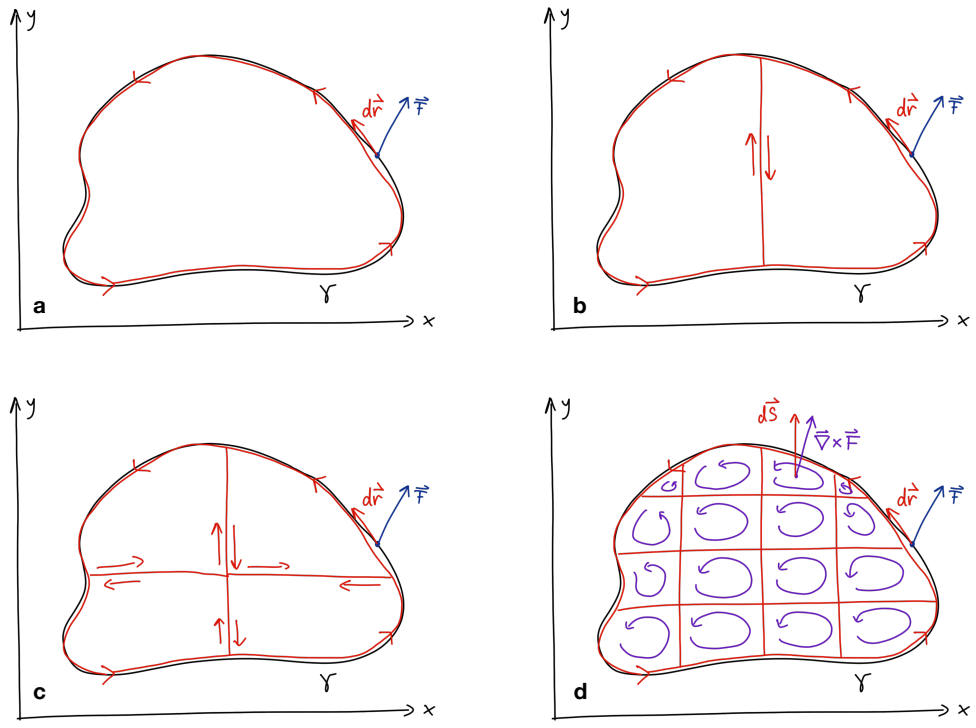


Figure 40: Slicing of some surface parametrized by the variables (x, y) for Stokes' theorem. The line integrals on all internal grid lines cancel because they are traversed in opposite directions, leaving only the line integral along the boundary curve γ. Each small circulation element in the final panel (d) resembles the setup shown in Fig. 36: the circulation corresponds to the scalar product of the curl of the field and the vector area.

We now divide the area enclosed by γ in two parts, bounded by two new curves γ_1 and γ_2. Clearly,

$$\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_{\gamma_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \oint_{\gamma_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r},$$

since the internal grid line is traversed twice, in opposite directions, and thus cancels exactly – see Fig. 40(b). We can continue this process to build an increasingly dense tiling, so that

$$\begin{aligned}\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \sum_{i=1}^{n} \oint_{\gamma_i} \mathbf{F}\cdot\mathrm{d}\mathbf{r} \\ &\approx \sum_{i=1}^{n} (\nabla\times\mathbf{F})\cdot\mathbf{S}_i,\end{aligned}$$

where in the last line we have used the definition of the curl from Eq. (162), and where $\mathbf{S}_i$ denotes the vector area of the region bounded by γ_i. Since the vector area has not been taken in the infinitesimal limit, the curl expression is only approximate. The relationship becomes exact in the limit of $\mathbf{S}_i \to 0$, i.e., infinite regions of infinitesimal size, and the sum over the regions is replaced by a double integral over the surface S

$$\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathrm{d}\mathbf{S}, \tag{165}$$

which we recognize as Stokes' theorem. Note that in two dimensions with $\mathbf{F} = (F_x, F_y)$ we once again recover Green's theorem:

$$\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathrm{d}S. \tag{166}$$

Just as the divergence theorem can be used to relate volume and surface integrals for certain types of integrands, Stokes' theorem connects surface integrals with line integrals.

Example 33. Evaluate the line integral

$$\oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{F} = (y+z, z+x, x+y)$, and the curve γ is given by the intersection of the sphere $x^2+y^2+z^2 = R^2$ with the plane $x+y+z=0$.

Solution: The vector field is well-defined in all $\mathbb{R}^3$ and γ is a closed regular curve so we can use Stokes' theorem to transform the line integral into a surface integral:

$$\oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S,$$

where S is the portion of the plane bounded by the intersection with the sphere. The normal vector is thus simply that of the plane: $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(1,1,1)$. Calculating the curl we have

$$\nabla \times \mathbf{F} = \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and so we conclude that $\mathbf{F}$ is conservative in $\mathbb{R}^3$, $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is an exact differential, and that therefore $\oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$. □

Example 34. Show that Stokes' theorem holds for the vector field $\mathbf{F} = (x, 0, y)$ over the surface S given parametrically as

$$\begin{aligned} x &= R \sin\theta \cos\phi \\ y &= R \sin\theta \sin\phi \\ z &= R \cos\theta \end{aligned}$$

for $\theta \in [0, \pi/2]$ and $\phi \in [-\pi/2, +\pi/2]$ (a quarter sphere).

Solution: We proceed to verify

$$\oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S,$$

by calculating both integrals separately, and showing the results agree.
For the surface integral we have

$$\nabla \times \mathbf{F} = (1, 0, 0).$$

The normal vector to the sphere is (see Eq. (116)) $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and we know from the Jacobian that $\mathrm{d}S = R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$. Therefore

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \int_{-\pi/2}^{+\pi/2} \mathrm{d}\phi \int_0^{\pi/2} \sin\theta\cos\phi \, R^2 \sin\theta \, \mathrm{d}\theta \\ &= R^2 \int_{-\pi/2}^{+\pi/2} \cos\phi \, \mathrm{d}\phi \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta \\ &= R^2 \cdot 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} R^2. \end{aligned}$$

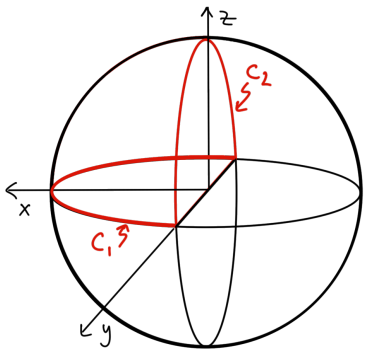


Figure 41: The quarter sphere surface used in Example 34, bounded by the two semi-circumferences C_1 and C_2.

To find the circulation, we must first parametrize the bounding curve γ. As can be seen from Fig. 41, γ will be given by the sum of two half-circumferences on the sphere. The first can be parametrized as:

$$\begin{aligned} x &= R\cos t \\ y &= R\sin t \\ z &= 0 \end{aligned}$$

for $t \in [-\pi/2, +\pi/2]$. The contribution to the total circulation from this segment is

$$\begin{aligned} C_1 &= \int_{-\pi/2}^{+\pi/2} (F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z) \\ &= \int_{-\pi/2}^{+\pi/2} F_x\,\mathrm{d}x \\ &= \int_{-\pi/2}^{+\pi/2} R\cos t(-R\sin t)\,\mathrm{d}t = 0. \end{aligned}$$

The second half-circumference can be parametrized as

$$\begin{aligned} x &= 0 \\ y &= R\sin t \\ z &= R\cos t \end{aligned}$$

for $t \in [-\pi/2, +\pi/2]$. The contribution to the total circulation from this segment is (note the change in direction to make this a closed curve)

$$\begin{aligned} C_2 &= \int_{+\pi/2}^{-\pi/2} (F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z) \\ &= \int_{+\pi/2}^{-\pi/2} F_z\,\mathrm{d}z \\ &= \int_{+\pi/2}^{-\pi/2} R\sin t(-R\sin t)\,\mathrm{d}t = \frac{\pi}{2}R^2. \end{aligned}$$

So the total circulation is

$$\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = C_1 + C_2 = \frac{\pi}{2}R^2.$$

□

6.5 Some differential identities

Differential operators are often most conveniently handled in index notation. Given a differentiable scalar field ϕ and a differentiable vector field $\mathbf{F}$, we can write:

$$\nabla\phi = \frac{\partial}{\partial x_i}\phi = \partial_i\phi$$

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x_i}F_i = \partial_i F_i$$

$$\nabla\times\mathbf{F} = \varepsilon_{ijk}\frac{\partial}{\partial x_j}F_k = \varepsilon_{ijk}\partial_j F_k$$

where we remember the convention that repeated indices are summed over. The number of free indices (not repeated) indicate the rank of the remaining tensor: in the first case it is 1, so we have a vector, in the second 0 (i is repeated) so a scalar, and in the third again 1, so a vector (j and k are repeated). Here ε_{ijk} is the Levi-Civita symbol, a rank 3 tensor.

Given the definitions of the gradient, divergence and curl, there are certain relationships between them that will prove useful. Note that only some compositions of these operators are meaningful:

1. div grad $\phi \equiv \nabla\cdot(\nabla\phi) = \nabla^2\phi$. This is the Laplacian.
2. curl grad $\phi \equiv \nabla\times(\nabla\phi) = 0$. Equivalent to computing the curl of a conservative field.
3. div curl $\mathbf{F} \equiv \nabla\cdot(\nabla\times\mathbf{F}) = 0$. The curl is solenoidal by construction.
4. grad div $\mathbf{F} \equiv \nabla(\nabla\cdot\mathbf{F})$
5. curl curl $\mathbf{F} \equiv \nabla\times(\nabla\times\mathbf{F})$

Example 35. Prove the identity

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}.$$

Solution: We proceed using index notation:

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{F}) &= \varepsilon_{ijk}\,\partial_j\,(\nabla\times\mathbf{F})_k \\
&= \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,\partial_l\,F_m \\
&= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j\,\partial_l\,F_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\,\partial_l\,F_m \\
&= \delta_{il}\delta_{jm}\,\partial_j\,\partial_l\,F_m - \delta_{im}\delta_{jl}\,\partial_j\,\partial_l\,F_m \\
&= \partial_i\,\partial_m\,F_m - \partial_j\,\partial_j\,F_i \\
&= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}.
\end{aligned}$$

7 Revision

7.1 Summary of some key points

The arc length s of a space curve parametrized by a vector $\mathbf{r}(x(t), y(t), z(t))$ is given by

$$s = \int_a^b \left|\mathbf{r}'(t)\right| \mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$

We have three main kinds of line integrals:

1. Line integral of a scalar function f along curve γ, produces a scalar:

$$\int_\gamma f(\mathbf{r})\, \mathrm{d}s = \int_a^b f[\mathbf{r}(t)]\, |\mathbf{r}'(t)|\, \mathrm{d}t.$$

2. Line integral of a scalar function f along curve γ with respect to an infinitesimal vector displacement, produces a vector:

$$\int_\gamma f(\mathbf{r})\, \mathrm{d}\mathbf{r} = \int_\gamma f(x, y, z)(\hat{\mathbf{i}}\ \mathrm{d}x + \hat{\mathbf{j}}\ \mathrm{d}y + \hat{\mathbf{k}}\ \mathrm{d}z).$$

3. Line integral of a vector function $\mathbf{F}$ along curve γ with respect to an infinitesimal vector displacement, produces a scalar:

$$\int_\gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_\gamma F_x(x, y, z)\, \mathrm{d}x + \int_\gamma F_y(x, y, z)\, \mathrm{d}y + \int_\gamma F_z(x, y, z)\, \mathrm{d}z.$$

   A common example is the work W done by a force $\mathbf{F}$ moving along a curvilinear path γ.

The three main differential operators are:

$$\begin{aligned}
\mathrm{grad} f(\mathbf{r}) &= \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right). \\
\mathrm{div}\, \mathbf{F}(\mathbf{r}) &= \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \\
\mathrm{curl}\, \mathbf{F}(\mathbf{r}) &= \nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}.
\end{aligned}$$

We can use these to relate integrals along lines, surfaces and volumes of scalar and vector functions.

1. Divergence Theorem:

$$\iiint_V \nabla \cdot \mathbf{F}\ \mathrm{d}V = \oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S.$$

2. Stokes' Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

3. In 2D, the divergence and Stokes' theorems reduce to Green's Theorem:

$$\oint_\gamma (P\, \mathrm{d}x + Q\, \mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y.$$

For surface integrals, where the surface is parametrized by the vector $\mathbf{r}(u, v)$, we have

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v; \qquad \mathrm{d}\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v. \tag{167}$$

For surface integrals, where the surface is given by the equation $z = f(x, y)$, we can write $g(x, y, z) = f(x, y) - z = 0$, and use the gradient of g to find the normal vector to the surface:

$$\mathrm{d}S = |\nabla g|\,\mathrm{d}A = \sqrt{f_x^2 + f_y^2 + 1}\,\mathrm{d}A; \qquad \mathrm{d}\mathbf{S} = \nabla g\,\mathrm{d}A = (f_x, f_y, -1)\,\mathrm{d}A, \tag{168}$$

where the subscripts indicate first partial derivatives.

7.2 Worked examples

Example 36. Viviani's window is the part of the surface of the sphere of radius R centred in the origin, contained within the right cylinder with base given by a circumference of radius $R/2$, centred in $(x, y) = (R/2, 0)$, for $z \geq 0$ (see Fig. 42). Find the area S of Viviani's window.

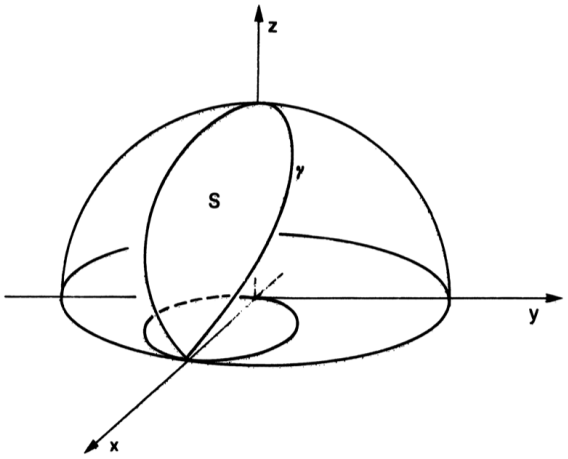


Figure 42: Viviani's window. The surface S is bounded by the space curve γ, also known as Viviani's curve.

Solution: We are tasked with solving the following integral for the area S:

$$S = \iint_S \mathrm{d}S.$$

Finding the normal vector to the surface of the sphere, and thus $\mathrm{d}S$ according to the standard expressions, is simple enough – we have already solved this problem in Example 25. However, the challenge is then to find the limits of the two variables (θ, ϕ) parametrizing the surface of the sphere over which we should integrate. This is not trivial in this case, so we shall proceed with a different parametrization of the surface that is more convenient, using cylindrical coordinates for a cylinder not centred in the origin. Specifically, we write

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z(r, \theta). \end{aligned}$$

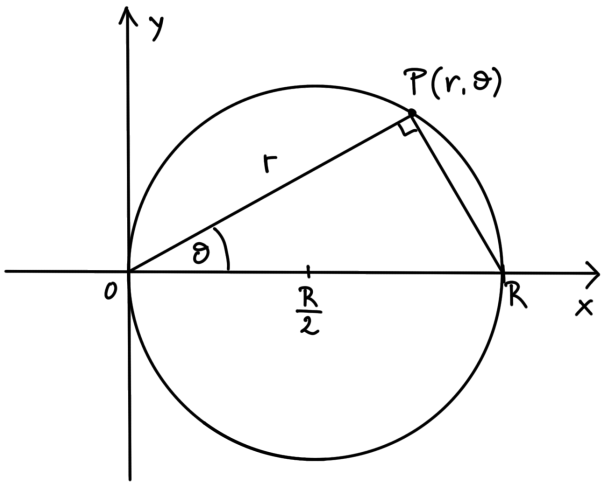


Figure 43: Parametrization of an offset circle in plane-polar coordinates. A point P on the circle is given by the parameters (r, θ), with $r = R\cos\theta$.

To find the range of (r, θ), consider the base of the cylinder in the xy-plane, as shown in Fig. 43. Clearly, in order to describe the circle we see that $-\pi/2 < \theta < \pi/2$, while the length of the vector tracing the circle for some given value of θ must be $r = R\cos\theta$. The entire base of the cylinder can then be described by setting $0 \le r \le R\cos\theta$.

All points on the surface S must obey the equation of the sphere, and thus

$$R^2 = x^2 + y^2 + z^2 = (r\cos\theta)^2 + (r\sin\theta)^2 + z^2 = r^2 + z^2.$$

It follows that $z = \sqrt{R^2 - r^2}$. So the parametrization we need is:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= \sqrt{R^2 - r^2}, \qquad\qquad \text{for } 0 \le r \le R\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}. \end{aligned}$$

To find a convenient form for $\mathrm{d}S$ we recall Eq. (144), where our surface is traced out by the vector $\mathbf{r} = (x, y, z)$, parametrized in terms of the two variables (r, θ). We therefore have

$$\iint_S \mathrm{d}S = \iint_S \left| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| \mathrm{d}r\,\mathrm{d}\theta.$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix} \\ &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{R^2 - r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} = \begin{pmatrix} \frac{r^2\cos\theta}{\sqrt{R^2 - r^2}} \\ \frac{-r^2\sin\theta}{\sqrt{R^2 - r^2}} \\ r \end{pmatrix}. \end{aligned}$$

The modulus of this vector is

$$\left| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{\frac{(r^2\cos\theta)^2}{R^2 - r^2} + \frac{(r^2\sin\theta)^2}{R^2 - r^2} + r^2} = \frac{rR}{\sqrt{R^2 - r^2}}.$$

We now have all the ingredients needed to perform the integral:

$$\begin{aligned} S = \iint_S \mathrm{d}S &= \int_{-\pi/2}^{+\pi/2} \mathrm{d}\theta \int_0^{R\cos\theta} \frac{rR}{\sqrt{R^2 - r^2}}\,\mathrm{d}r \\ &= \int_{-\pi/2}^{+\pi/2} R\left(-\sqrt{R^2 - r^2}\right)_0^{R\cos\theta} \mathrm{d}\theta \\ &= \int_{-\pi/2}^{+\pi/2} R^2(1 - |\sin\theta|)\,\mathrm{d}\theta \\ &= R^2\left(\pi - 2\int_0^{+\pi/2} \sin\theta\,\mathrm{d}\theta\right) \\ &= R^2(\pi - 2). \end{aligned}$$

□

Example 37. Alice and Bob agree to meet for lunch at some time between noon and 1 pm. They are both willing to wait up to 10 min for the other to arrive, but will leave otherwise. Find the probability that they meet for lunch, if

a) their arrival times are independent and uniformly distributed;

b) their arrival times are independent, but while Alice's probability of arrival is uniform, Bob is quadratically more likely to arrive later in the hour.

Does Bob's tardiness help or hinder the likelihood of the two enjoying a meal together?

Solution: Let X indicate the arrival time for Alice, and Y the arrival time for Bob. We're told the two events are independent, so the joint PDF for the arrival of Alice and Bob is obtained by multiplying their respective single-variable PDFs.

Part a) We're told Alice's and Bob's arrival time is uniformly distributed, so the PDF is a constant function, the integral of which, over the domain of interest (a 1 hour window), should be 1:

$$\int_{1\mathrm{h}} f_X(x)\,\mathrm{d}x = \int_{1\mathrm{h}} f_Y(y)\,\mathrm{d}y = \int_{1\mathrm{h}} C\,\mathrm{d}x = 1.$$

The constant is therefore simply the inverse of the time window. For convenience we can choose to work in the units of minutes, in which case $C = 1/60$. The joint PDF is then

$$f_{XY}(x, y) = \left(\frac{1}{60}\right)^2.$$

The domain over which the joint PDF is defined is shown in Fig. 44: a square of total area 3600 square minutes. We're interested in finding the probably that they meet. Since this requires $|Y - X| \leq 10$, this probability is given by

$$P(|Y - X| \leq 10) = \iint_{|Y-X|\leq 10} f_{XY}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

In this case the integral simply needs to evaluate the red area shown in Fig. 44. We do not need to do the integral to find the answer: the total area of the box is $60^2 = 3600$, while the area of the two blue triangles is $50^2 = 2500$, and so

$$P(|Y - X| \leq 10) = \frac{3600 - 2500}{3600} = \frac{11}{36} \approx 31\%.$$

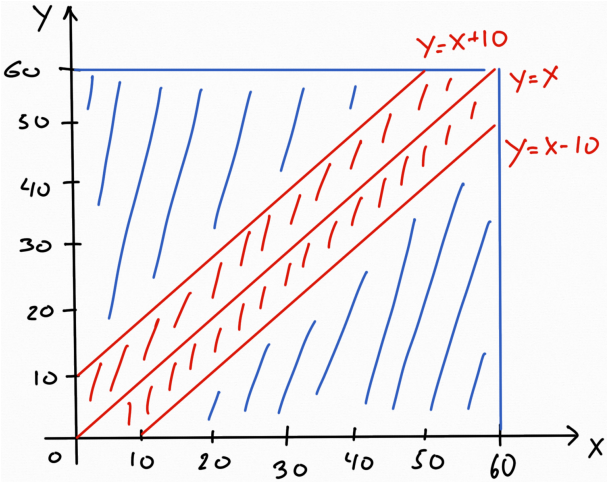


Figure 44: Probability domain for Alice and Bob meeting.

Part b) Here the PDFs for Alice and Bob are not the same. We still have

$$f_X(x) = \frac{1}{60}$$

for Alice, but for Bob we're told that $f_Y(y) \propto y^2$. To find the normalization constant we must integrate this function over the domain of interest:

$$\int_0^{60} Cy^2 \, \mathrm{d}y = \frac{Cy^3}{3}\bigg|_0^{60} = \frac{60^3}{3}C := 1; \quad \Rightarrow C = \frac{3}{60^3}.$$

The joint PDF is therefore:

$$f_{XY}(x, y) = \frac{3}{60^4}y^2,$$

and the probability for the two to meet in a 10 minute window is

$$P(|Y - X| \leq 10) = \iint_{|Y-X|\leq 10} f_{XY}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint_{|Y-X|\leq 10} \frac{3}{60^4}y^2 \, \mathrm{d}x \, \mathrm{d}y,$$

where the domain is the same as before, i.e., the red region in Fig. 44. Due to the shape of the domain it will be easier to integrate the blue regions instead (that is, the cases where the two do not meet), so we write

$$P(|Y - X| \leq 10) = 1 - P((Y - X) > 10) - P((Y - X) < -10),$$

where the last two terms correspond to the upper and lower triangles, respectively. Note that

these two are no longer the same. We have

$$\begin{aligned}
P((Y-X)>10) &= \int_0^{50} \mathrm{d}x \int_{x+10}^{60} \frac{3}{60^4} y^2 \,\mathrm{d}y \\
&= \int_0^{50} \mathrm{d}x \, \frac{3}{60^4} \left(\frac{60^3}{3} - \frac{(x+10)^3}{3} \right) \\
&= \frac{x}{60}\Big|_0^{50} - \frac{1}{60^4} \frac{(x+10)^4}{4}\Big|_0^{50} \\
&= \frac{7}{12} + \frac{1}{4}\left(\frac{1}{6}\right)^4 . \\
P((Y-X)<-10) &= \int_{10}^{60} \mathrm{d}x \int_0^{x-10} \frac{3}{60^4} y^2 \,\mathrm{d}y \\
&= \int_{10}^{60} \mathrm{d}x \, \frac{3}{60^4} \frac{(x-10)^3}{3} \\
&= \frac{1}{60} \frac{(x-10)^4}{4}\Big|_{10}^{60} \\
&= \frac{1}{4}\left(\frac{5}{6}\right)^4 .
\end{aligned}$$

Therefore, we find

$$P(|Y-X| \leq 10) = 1 - \frac{7}{12} - \frac{1}{4}\left(\frac{1}{6}\right)^4 - \frac{1}{4}\left(\frac{5}{6}\right)^4 = \frac{767}{2592} \approx 30\%.$$

This is slightly lower than the result from part a), but only by around 1%. Bob's preference for showing up late doesn't seem to make much of a difference. □

Example 38. Evaluate the integral

$$\oint_\gamma \mathrm{d}x + xy \,\mathrm{d}z,$$

where γ is the closed curve forming the boundary of the Pringles surface given by the equation $z = x^2 - y^2$, for $x^2 + y^2 \leq 1$, by

a) calculating the line integral directly;

b) using Stokes' theorem.

Solution: The surface in question is shown in Fig. 45.
Part a) We can parametrize the curve in terms of the plane-polar angle θ as

$$\begin{aligned}
x &= \cos\theta \\
y &= \sin\theta \\
z &= \cos^2\theta - \sin^2\theta; \qquad \theta \in [0, 2\pi].
\end{aligned}$$

The integral then becomes

$$\int_0^{2\pi} [-\sin\theta + \sin\theta\cos\theta(-2\cos\theta\sin\theta - 2\sin\theta\cos\theta)] \,\mathrm{d}\theta = -4\int_0^{2\pi} \sin^2\theta\cos^2\theta \,\mathrm{d}\theta = -\pi.$$

Part b) We note that the surface is well-defined everywhere, and that the given integral can be written in the form $\oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, with $\mathbf{F} = (1, 0, xy)$. To use Stokes' theorem we then write

$$\oint_\gamma \mathrm{d}x + xy \,\mathrm{d}z = \oint_\gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathrm{d}\mathbf{S},$$

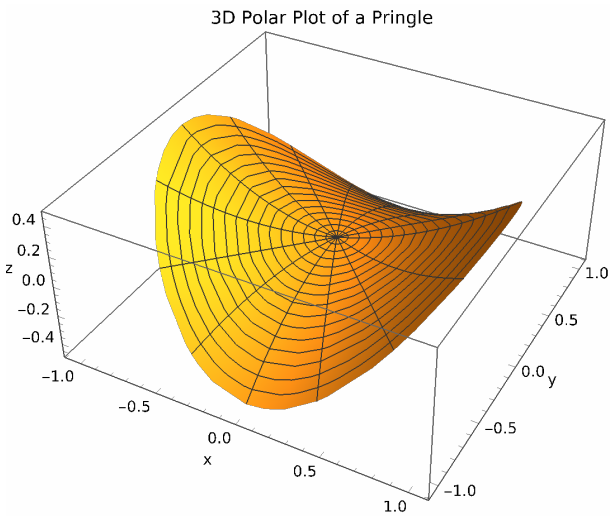


Figure 45: The Pringles crisp, a hyperbolic paraboloid.

and so we need to find the curl of $\mathbf{F}$ and an expression for the vector area $\mathrm{d}\mathbf{S}$. The curl is

$$\nabla \times \mathbf{F} = (x, -y, 0).$$

For the area we can write $g = z - f(x, y) = z - x^2 + y^2$, the gradient of which will provide the normal vector to the surface

$$\nabla g(x, y, z) = (-2x, 2y, 1),$$

so that

$$\mathrm{d}\mathbf{S} = \nabla g \, \mathrm{d}x \, \mathrm{d}y = (-2x, 2y, 1) \, \mathrm{d}x \, \mathrm{d}y.$$

This is oriented in the right direction. Inserting these expressions into the integral we have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iint_A (x, -y, 0) \cdot (-2x, 2y, 1) \, \mathrm{d}x \, \mathrm{d}y \\
&= -2 \iint_A (x^2 + y^2) \, \mathrm{d}x \, \mathrm{d}y \\
&= -2 \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r(r^2 \cos^2\theta + r^2 \sin^2\theta) \, \mathrm{d}r \\
&= -4\pi \left(\frac{r^4}{4} \right)_0^1 = -\pi.
\end{aligned}$$

□