CP4 Mathematical Methods 2 - Cheat Sheet 高频公式 / 题型模板 / 易错点

0. 高频优先级

Tier 1: Vector calculus, waves/string/dispersion, normal modes.
Tier 2: Taylor/Hessian, Jacobians/coordinates, probability.
Tier 3: exact differentials, implicit derivatives, geometry and asymptotic limits.

1. Series and limits

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \cdots$$

Power limit: $L = \lim f(x)^{g(x)}$ means

$$\ln L = \lim g(x) \ln f(x).$$

Rationalise root differences:

$$\sqrt{A} - \sqrt{B} = \frac{A - B}{\sqrt{A} + \sqrt{B}}.$$

2. Multivariable Taylor

$$f(a+h, b+k) = f_0 + hf_x + kf_y + \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) + O(3).$$

Stationary point: $f_x = f_y = 0$. Let

$$D = f_{xx} f_{yy} - f_{xy}^2.$$

$D > 0, f_{xx} > 0$: min. $D > 0, f_{xx} < 0$: max. $D < 0$: saddle. $D = 0$: inconclusive.

3. Exact differentials and chain rule

$$\mathrm{d}f = M\,\mathrm{d}x + N\,\mathrm{d}y \iff M_y = N_x$$

inside a simply connected region. Then integrate one component and fit the other.
If $u = u(s,t), v = v(s,t)$,

$$J = \frac{\partial(u,v)}{\partial(s,t)} = u_s v_t - u_t v_s,$$

$$\mathrm{d}s = \frac{v_t\,\mathrm{d}u - u_t\,\mathrm{d}v}{J}, \quad \mathrm{d}t = \frac{-v_s\,\mathrm{d}u + u_s\,\mathrm{d}v}{J}.$$

Thus

$$\left(\frac{\partial s}{\partial v}\right)_u = -\frac{u_t}{u_s v_t - u_t v_s}.$$

Implicit functions: write total differentials, collect unknown derivatives, solve a linear system.

4. Coordinates and Jacobians

$$\iint f\,\mathrm{d}x\,\mathrm{d}y = \iint f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

Polar: $x = r\cos\theta, y = r\sin\theta,\ \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.
Cylindrical: $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.
Spherical: $\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.
Toroidal:

$$x = (R_0 + r\cos\theta)\cos\phi, \quad y = (R_0 + r\cos\theta)\sin\phi, \quad z = r\sin\theta,$$

$$\mathrm{d}V = r(R_0 + r\cos\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi, \quad A = 4\pi^2 R_0 a, \quad V = 2\pi^2 R_0 a^2.$$

Surface $z = f(x,y)$:

$$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Parametric surface:

$$\mathrm{d}\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v.$$

Curve length: $L = \int |\mathbf{r}'(t)|\,\mathrm{d}t$.

5. Vector calculus

$$\nabla \cdot \mathbf{F} = F_{x,x} + F_{y,y} + F_{z,z}, \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}.$$

Gauss:

$$\iiint_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S.$$

Stokes:

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\,\mathrm{d}S.$$

Green:

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R (N_x - M_y)\,\mathrm{d}A.$$

Area:

$$A = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Conservative field:

$$\mathbf{F} = \nabla\phi \Rightarrow \int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A).$$

In simply connected regions, $\nabla \times \mathbf{F} = 0$ is enough. Check holes/singularities.
Identities:

$$\nabla \cdot (\phi\mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{A}$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

6. Probability

PDF: $\int f_X(x)\,\mathrm{d}x = 1$. CDF:

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(s)\,\mathrm{d}s, \quad f_X = F_X'.$$

Marginals:

$$f_X(x) = \int f_{X,Y}(x,y)\,\mathrm{d}y, \quad f_Y(y) = \int f_{X,Y}(x,y)\,\mathrm{d}x.$$

Expectation:

$$\mathbb{E}[g] = \iint g(x,y) f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Transformation $Y = g(X)$:

$$f_Y(y) = \sum_{x_i : g(x_i)=y} f_X(x_i) \left| \frac{\mathrm{d}x_i}{\mathrm{d}y} \right|.$$

Main trap: draw support first; marginal limits often change at boundary curves.

7. Waves on strings

Small displacement:

$$\rho y_{tt} = T y_{xx}, \qquad c = \sqrt{T/\rho}.$$

D'Alembert:

$$y = f(x - ct) + g(x + ct).$$

Fixed-fixed:

$$k_n = \frac{n\pi}{L}, \quad \omega_n = ck_n, \quad y = \sum_{n\geq 1} (A_n \cos\omega_n t + B_n \sin\omega_n t) \sin\frac{n\pi x}{L}.$$

Fixed-free:

$$k_n = \frac{(n+1/2)\pi}{L}.$$

Power of $y = A\cos(kx - \omega t)$:

$$\langle P \rangle = \frac{1}{2} T\omega k A^2 = \frac{T}{2c} A^2\omega^2 = \frac{1}{2}\rho c A^2 \omega^2.$$

Interface under same tension, incident from medium 1:

$$r = \frac{A_r}{A_i} = \frac{k_1 - k_2}{k_1 + k_2}, \quad t = \frac{A_t}{A_i} = \frac{2k_1}{k_1 + k_2}.$$

With $k_i = \omega\sqrt{\rho_i/T}$,

$$r = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}, \quad T_{\text{pow}} = \frac{4k_1k_2}{(k_1+k_2)^2}, \quad R_{\text{pow}} = r^2.$$

Point mass at junction:

$$T[y_x(0^+,t) - y_x(0^-,t)] = M y_{tt}(0,t).$$

8. Dispersion

$$v_p = \frac{\omega}{k}, \qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

Packet envelope travels at v_g; peaks/troughs at v_p.
Cyclotron/waveguide type:

$$\omega^2 = \Omega^2 + c_s^2 k^2,$$
$$v_p = \frac{\sqrt{\Omega^2 + c_s^2k^2}}{k}, \quad v_g = \frac{c_s^2 k}{\sqrt{\Omega^2 + c_s^2k^2}},$$
$$v_p v_g = c_s^2.$$

Water waves:

$$\omega^2 = gk\tanh(kh).$$

$kh \ll 1$: $v_p = v_g = \sqrt{gh}$.
$kh \gg 1$: $v_p = \sqrt{g/k}$, $v_g = v_p/2$.
Two nearby waves:

$$\cos[(k+\delta k)x - (\omega+\delta\omega)t] + \cos[(k-\delta k)x - (\omega - \delta\omega)t]$$
$$= 2\cos(\delta k x - \delta\omega t)\cos(kx - \omega t).$$

Envelope speed $= \delta\omega/\delta k \to v_g$.

9. Normal modes

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0, \quad \mathbf{x} = \mathbf{a}e^{i\omega t}, \quad (K - \omega^2 M)\mathbf{a} = 0.$$

Frequencies:

$$\det(K - \omega^2 M) = 0.$$

General solution:

$$\mathbf{x}(t) = \sum_j (A_j \cos\omega_j t + B_j \sin\omega_j t)\mathbf{a}_j.$$

Initial velocities: $\dot{\mathbf{x}}(0) = \sum_j B_j\omega_j\mathbf{a}_j$. Mass-weighted orthogonality:

$$\mathbf{a}_i^T M \mathbf{a}_j = 0 \quad (i \neq j).$$

Normal coordinates diagonalise energy:

$$E = \sum_j \frac{1}{2}(\dot{q}_j^2 + \omega_j^2 q_j^2).$$

Pendulum small-angle: $x \simeq \ell\theta$, restoring acceleration $-(g/\ell)x$. Coupling spring gives terms proportional to displacement differences.

10. PDE / material derivative

Nearest-neighbour model:

$$\rho(x,t) = \frac{\rho(x+\lambda, t-\tau) + \rho(x-\lambda, t-\tau)}{2}$$

Taylor expands to

$$\rho_t = \frac{\lambda^2}{2\tau}\rho_{xx}.$$

Material derivative:

$$\frac{D\rho}{Dt} = \rho_t + \mathbf{u}\cdot\nabla\rho, \quad \mathbf{u} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Continuity form:

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0.$$

Use Gauss to turn volume rate of change into boundary flux.

11. Exam templates

Flux: closed? use Gauss; if not closed, close and subtract.
Stokes: choose easiest surface; state orientation.
Work: conservative test first; otherwise parameterise.
Surface: write $\mathrm{d}S$ before integrating.
Normal modes: EOM → determinant → eigenvectors → initial data.
String interface: displacement continuity + force/slope condition.
Probability: sketch support; marginal limits; normalise.
Stationary: solve all branches; Hessian; if $D = 0$, path test.