Ordinary Differential Equations

Problem Set 4: Systems of Linear ODEs

For each topic, a set of "Standard problems" of exercises is offered first, followed by "Supplementary Problems". The latter are not necessarily any more difficult than the former: "supplementary" just means that you can get by without them if you choose to do as little as possible. If you are struggling with time, you might want to do some of these during the vacation. Problems that are slightly more challenging are marked with a star. Finally, there are "Extracurricular Problems", which are extracurricular (but not necessarily hard). Some of these will require independent study of material that may or may not be covered in the lecture notes.

Standard problems

4.1 Solve the coupled differential equations

$$\begin{aligned} \dot{x} + ax - by &= f, \\ \dot{y} + ay - bx &= 0, \end{aligned}$$

where a, b, and f are constants.

4.2 Solve the coupled differential equations

$$\begin{aligned} y' + 2z' + 4y + 10z - 2 &= 0, \\ y' + z' + y - z + 3 &= 0, \end{aligned}$$

where $y = 0$ and $z = -2$ at $x = 0$.

4.3 Find the general, real solutions of the following homogeneous systems of ODEs

$$\text{(i)} \begin{cases} \dot{x} = -2y + 2z, \\ \dot{y} = x - y + z, \\ \dot{z} = y - z, \end{cases}$$

$$\text{(ii)} \begin{cases} \dot{x} = 4x - y - z, \\ \dot{y} = x + 2y - z, \\ \dot{z} = x - y + 2z. \end{cases}$$

4.4 Find the general, real solutions of the following inhomogeneous systems of ODEs

$$\text{(i)} \begin{cases} \dot{x} = 4x + 3y - 3z, \\ \dot{y} = -3x - 2y + 3z, \\ \dot{z} = 3x + 3y - 2z + 2e^{-t}, \end{cases}$$

$$\text{(ii)} \begin{cases} \dot{x} = -5x + y - 2z + \cosh t, \\ \dot{y} = -x - y + 2\sinh t + \cosh t, \\ \dot{z} = 6x - 2y + 2z - 2\cosh t. \end{cases}$$

Supplementary Problems

4.5 Solve the coupled differential equations

$$2y'' - 3y' + 2z' + 3y + z = e^{2x},$$
$$y'' - 3y' + z' + 2y - z = 0.$$

Is it possible to have a solution to these equations for which $y = z = 0$ at $x = 0$? What is wrong with this system?

4.6 Charged Particle in Electromagnetic Field.

A particle of mass m and charge q is placed, initially at rest, in straight, constant electric E and magnetic B fields, which are neither perpendicular nor parallel to each other. It will experience the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right).$$

where v is the particle's velocity. Find $v(t)$ and sketch the particle's motion.

Extracurricular Problems

4.7 Non-diagonalisable systems.

Find the general solutions of the following homogeneous systems of ODEs

$$\text{(i)} \begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 4y - x, \end{cases}$$

$$\text{(ii)} \begin{cases} \dot{x} = x - 2y, \\ \dot{y} = -x - y - 2z, \\ \dot{z} = y + z, \end{cases}$$

$$\text{(iii)} \begin{cases} \dot{x} = 2x - 5y - 8z, \\ \dot{y} = 7x - 11y - 17z, \\ \dot{z} = -3x + 4y + 6z. \end{cases}$$

You can find solutions by triangulation or via Jordan's basis, whichever you prefer.