Ordinary Differential Equations

Problem Set 3: Second-Order ODEs, Part II

For each topic, a set of "Standard problems" of exercises is offered first, followed by "Supplementary Problems". The latter are not necessarily any more difficult than the former: "supplementary" just means that you can get by without them if you choose to do as little as possible. If you are struggling with time, you might want to do some of these during the vacation. Problems that are slightly more challenging are marked with a star. Finally, there are "Extracurricular Problems", which are extracurricular (but not necessarily hard). Some of these will require independent study of material that may or may not be covered in the lecture notes.

Standard problems

3.1 Inhomogeneous ODEs. Consider the equation

$$y'' - 3y' + 2y = f(x).$$

What is the general form (in terms of functions and constants) of its particular solution for

(i) $f(x) = x^2$,

(ii) $f(x) = e^{4x}$,

(iii) $f(x) = e^x$,

(iv) $f(x) = \sinh x$,

(v) $f(x) = \sin x$,

(vi) $f(x) = x \sin x$,

(vii) $f(x) = e^{2x} + \cos^2 x$.

If you dislike unfinished business, work out the constants.

3.2 Inhomogeneous ODEs. Solve these equations:

(i) $5y'' + 2y' + y = 2x + 3$, $y = -1$, $y' = 0$ at $x = 0$,

(ii) $y'' - y' - 2y = e^{2x}$,

(iii) $4y'' - 4y' + y = 8e^{x/2}$, $y = 0$, $y' = 1$ at $x = 0$,

(iv) $y'' + 3y' + 2y = xe^{-x}$,

(v) $y'' - 4y' + 3y = 10 \cos x$,

(vi) $y'' + 4y = x + \cos 2x$, $y = 0$ at $x = 0$,

(vii) $y'' - 2y' + 2y = e^x(1 + \sin x)$, $y = 0$ at $x = 0$ and at $x = \frac{\pi}{2}$,

(viii) $y'' + 2y' + y = 2e^{-x} + x^3 + 2\cos x$,

(ix) $y'' - 2y' + y = 3e^x$, $y = 3$, $y' = 0$ at $x = 0$,

(x) $x^2y'' + xy' + y = x$.

3.3 Forced Oscillator.

When a varying couple $I\cos\omega t$ is applied to a torsional pendulum with natural period $2\pi/\omega_0$ and moment of inertia I, the angle of the pendulum obeys the equation of motion

$$\ddot{\theta} + \omega_0^2\theta = \cos\omega t.$$

The couple is first applied at time $t = 0$ when the pendulum is at rest in equilibrium.

(a) Show that, in the subsequent motion, the root-mean-square angular displacement is

$$\langle\theta^2\rangle^{1/2} = \frac{1}{|\omega_0^2 - \omega^2|},$$

where the average is taken over a time large compared to $1/|\omega_0 - \omega|$.

(b) Discuss the motion as $|\omega_0 - \omega| \to 0$.

3.4 Forced and Damped Oscillator.

Consider the damped oscillator subject to an oscillatory driving force:

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = F\cos(\omega t).$$

(i) Explain what is meant by the stationary solution of this equation, and calculate this solution for the displacement $y(t)$ and the velocity $\dot{y}(t)$.

(ii) Sketch the amplitude and phase of $y(t)$ and $\dot{y}(t)$ as a function of ω.

(iii) Determine the resonant frequency for both the displacement and the velocity.

(iv) Defining $\Delta\omega$ as the full width at half maximum of the resonance peak calculate $\Delta\omega/\omega_0$ to leading order in γ/ω_0.

(v) For a lightly damped, driven oscillator near resonance, calculate the energy stored and the power supplied to the system. Hence confirm that $Q = \omega_0/\gamma$ as in the damped oscillator without a driving term (Q2.2). How is Q related to the width of the resonance peak?

3.5 Consider the differential equation

$$x(x+1)y'' + (2 - x^2)y' - (2 + x)y = (x+1)^2.$$

(a) One of its homogeneous solutions is $y_1(x) = 1/x$. Find the general solution.

(b) Now pretend that you do not know that $1/x$ is a homogeneous solution, but know the second homogeneous solution, $y_2(x)$, that you found in (a) (in fact, if you stare at the equation for a few seconds, or minutes, you will see that you could have guessed that solution). Use the knowledge of $y_2(x)$ to find both $y_1(x)$ and the general solution of the equation.

Supplementary Problems

3.6 Solve the differential equation

$$y'' - 2y' + (k^2 + 1)y = e^x \sin^2 x$$

for general values of the real parameter k. Identify any special values of k for which your solution fails and solve the equation also for those values.

3.7 Find the continuous solution with continuous first derivative of the equation

$$y'' + 2y' + 2y = \sin x + f(x), \text{ where } f(x) = \begin{cases} 0, & x \le 0, \\ x^2, & x > 0, \end{cases}$$

subject to $y(-\pi/2) = y(\pi) = 0$. Hint: obtain a general solution for each of the cases $x \le 0$ and $x > 0$ and then determine any loose constants by making these solutions agree at $x = 0$.

3.8 Oscillator with Modulated Force.

A mass m is constrained to move in a straight line and is attached to a spring of strength $m\omega_0^2$ and a dashpot which produces a retarding force $m\gamma v$, where v is the velocity of the mass and $\gamma \ll \omega_0$. An amplitude-modulated periodic force $mA\cos(\sigma t)\sin(\omega t)$ with $\sigma \ll \omega$ and $\omega = \omega_0$ is applied to the mass.

Show that, in the long-time limit, the displacement is an amplitude-modulated wave

$$y(t) = -\frac{A\sin(\sigma t + \phi)\cos(\omega t)}{2\omega\sqrt{\sigma^2 + \gamma^2/4}}, \quad \tan\phi = \frac{\gamma}{2\sigma}.$$