Ordinary Differential Equations

Problem Set 2: Second-Order ODEs, Part I

For each topic, a set of "Standard problems" of exercises is offered first, followed by "Supplementary Problems". The latter are not necessarily any more difficult than the former: "supplementary" just means that you can get by without them if you choose to do as little as possible. If you are struggling with time, you might want to do some of these during the vacation. Problems that are slightly more challenging are marked with a star. Finally, there are "Extracurricular Problems", which are extracurricular (but not necessarily hard). Some of these will require independent study of material that may or may not be covered in the lecture notes.

Standard problems

2.1 Homogeneous linear ODEs. Solve these equations:

(i) $y'' + 2y' - 15y = 0$,

(ii) $y'' - 6y' + 9y = 0$, $y(0) = 0$ and $y'(0) = 1$,

(iii) $y'' - 4y' + 13y = 0$,

(iv) $y''' + 7y'' + 7y' - 15y = 0$.

In (iii), write the solution both in terms of complex exponentials and in terms of sines and cosines.

2.2 Damped Oscillator. A damped harmonic oscillator is displaced by a distance y_0 and released at time $t = 0$. Show that the subsequent motion is described by the differential equation

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 = 0, \text{ with } y = y_0, y' = 0 \text{ at } t = 0,$$

explaining the physical meaning of the parameters m, γ and ω_0.

(a) Find and sketch solutions for (i) overdamping, (ii) critical damping, and (iii) underdamping. (iv) What happens for $\gamma = 0$?

(b) For a lightly damped oscillator ($\gamma \ll \omega_0$), the quality factor, or Q-factor, is defined as

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per period of oscillation}},$$

for $\omega = \omega_0$. Show that $Q = \omega_0/\gamma$.

2.3. Verify that $y = x + 1$ is a solution of

$$(x^2 - 1)y'' + (x + 1)y' - y = 0.$$

Hence find the general solution of this equation. Check your answer:

$$y = C_1(x+1) + C_2\left(\frac{x+1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\right).$$

2.4 Nonlinear ODEs. All of this problem set so far has been a drill in solving linear, second-order ODEs. There are few general methods for solving nonlinear ones (and most of them cannot be integrated in quadratures anyway). Still, there are a few tricks, which you now have an opportunity to practice. All of them, naturally, are based on methods for lowering the order of the equation down to first.

(a) If bits of the equation can be manipulated into full derivatives of some expressions, the equation's order can sometimes be lowered by direct integration. Practice this by solving the ODE

$$1 + yy'' + (y')^2 = 0.$$

(b) If the equation does not contain y, only its derivatives, the order is lowered by treating the lowest-order derivative that does appear as the new function. Practice this by solving the ODE

$$xy'' = y' + (y')^3.$$

(c) If the equation does not contain x, you can lower its order by letting $p = y'$ and looking for solutions in the form $p = p(y)$. Note that then, by the chain rule, $y'' = p\frac{\mathrm{d}p}{\mathrm{d}y}$. Solve by this method (plus other appropriate changes of variable) the ODE

$$y''(y-1) + y'(y-1)^2 = (y')^2.$$

Supplementary Problems

2.5 Homogeneous Nonlinear ODEs. Continuing from Q2.4, here are some more methods for lowering the order of nonlinear ODEs.

(a) An equation is called homogeneous with respect to y if it does not change under the rescaling $y \to \lambda y$ for all $\lambda \neq 0$ (i.e. every term contains the same number of powers of y or its derivatives). For such an equation, one can lower the order by introducing a new function $z(x)$ via $y' = yz$. Then $y'' = y(z' + z^2)$, so the second derivative is expressible in terms of the first; powers of y will cancel because the equation is homogeneous.

Solve by this method the following ODE:

$$xyy'' + x(y')^2 = 3yy'.$$

Check your answer: $y^2 = C_1x^4 + C_2$.

(b*) This equation can also be solved by the method that you practiced in Q2.4(a), if you can spot the full derivatives. Do it.

(c) A generalised version of (a) is an equation that does not change under the rescaling $x \to \lambda x$, $y \to \lambda^\alpha y$ for all $\lambda \neq 0$ and for some α (the linear version of this with $\alpha = 0$ is Euler's equation, you encountered the first-order case in Q1.7). This is solved by letting $x = e^t$ when $x > 0$, $x = -e^t$ when $x < 0$, and $y = z(t)e^{-\alpha t}$. The result will be an equation for z in which t does not appear explicitly, so you can use the method practiced in Q2.4(c).

Consider the following ODE:

$$x^2(y'' + 2yy') + 2xy^2 - 2y = 0, x > 0$$

and establish for what value of α it has the invariance property described above. Then solve the equation by the proposed method.

Extracurricular Problems

2.6 Boundary-Value Problem.

(a) Solve the ODE

$$y'' + k^2 y = 0, k > 0.$$

(b) Consider the above equation on the interval $x \in [0, L]$. Are there solutions that are not identically zero everywhere but have $y(0) = y(L) = 0$? For which values of k do such solutions exist? Find these solutions and explain in terms of linear algebra the meaning of what you have done.

(c*) Can any function that is defined in the interval $[0, L]$ and vanishes at its ends be represented as a linear combination of solutions of the ODE with different ks? Work out how to do it.

The full theory of boundary-value problems is not covered in this course. Read up on the topic or wait till the Mathematical Methods course in your 2nd year – although you will, in fact, need to solve this ODE many times in CP4.

2.7 Higher-Order Linear ODEs with Constant Coefficients.

Solve the following ODEs

$$\text{(i)}\ \ y'''' - 6y''' + 8y'' + 6y' - 9y = 0,$$

$$\text{(ii)}\ \ y'''' + 4y'' + 4y = 0,$$

Hint: in (i), you will need to guess some roots of a 4th-order polynomial and then factorise it.

2.8* Series Solution of Linear ODEs.

ODEs (usually second order) that cannot be solved in quadratures can sometimes be solved in terms of series. One can then give these solutions names and establish all their properties, thus expanding the library of functions that we know how to handle. Such functions are called special functions.

Consider Airy's equation

$$y'' + xy = 0.$$

This equation cannot, alas, be solved in quadratures. Seek its solution in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

(a) Without worrying about convergence of this series or about the legitimacy of differentiating it term by term, work out what equations the coefficients c_n must satisfy in order for the power series to be a solution of Airy's equation.

(b) Find two linearly independent such series solutions (remember that if they are linearly independent at one point, they are linearly independent everywhere). Check that they are indeed linearly independent. Thus give a general solution of Airy's equation.

If you are having a difficulty with these derivations, you will find the solution in, e.g. Sec. 3.8 in the book by Tikhonov, Vasil'eva, & Sveshnikov, "Differential Equations", 1985.

Certain linear combinations of the solutions that you have found are called Airy functions, an example of special functions. Entire books are written about them (Vall´ee & Soares, 2004, "Airy Functions and Applications to Physics").

(c) Find the general solution of the equation $y'' + y = 0$ by the same method and convince yourself that the series that you have obtained are sines and cosines. These functions too were once special.

Note that the method presented above is a generalisation of the method of looking for polynomial solutions, where the series was able to be truncated at the n^{th} coefficient. But even a Taylor series is not a universal panacea. It is not always possible to find solutions in this form.

For example, the equation

$$x^2y'' + xp(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are given by a convergent Taylor series, is only guaranteed to have one solution in the form of a Frobenius series

$$y(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n,$$

where α is not, in general, a (positive) integer; you can always get the second solution as discussed in the lecture notes. You will encounter the Frobenius method in the Mathematical Methods course. However, no one stops you from getting intrigued now and at least convincing yourself that the Frobenius series works for this general ODE and that a Taylor series in general does not. A good example of such a general ODE to play with is Bessel's equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a real number (interesting things happen depending on whether it is an integer). On Bessel functions too, there are entire books in the library (short and elementary: Bowman, "Introduction to Bessel Functions", 2003; long and comprehensive: Watson, "A Treatise on the Theory of Bessel Functions", 1944).