Ordinary Differential Equations

Problem Set 1: First-Order ODEs

For each topic, a set of "Standard problems" of exercises is offered first, followed by "Supplementary Problems". The latter are not necessarily any more difficult than the former: "supplementary" just means that you can get by without them if you choose to do as little as possible. If you are struggling with time, you might want to do some of these during the vacation. Problems that are slightly more challenging are marked with a star. Finally, there are "Extracurricular Problems", which are extracurricular (but not necessarily hard). Some of these will require independent study of material that may or may not be covered in the lecture notes.

Standard problems

1.1 Determine the order of each equation, and whether it is linear or nonlinear.

(i) $y'' + k^2 y = f(x)$,

(ii) $y''' + 2yy' = \sin x$,

(iii) $y' + y^2 = yx$.

1.2 Solve using the stated method (implicit solutions are sufficient).

(a) Exact differential

$$y' = \frac{(3x^2 + 2xy + y^2)\sin x - 2(3x + y)\cos x}{2(x + y)\cos x}.$$

(b) Integrating factor

$$y^2\,\mathrm{d}x - (xy + x^3)\,\mathrm{d}y = 0.$$

(c) Separable

(i) $y' = \frac{xe^y}{1+x^2}, \quad y(0) = 0,$

(ii) $y' = \frac{2xy^2+x}{x^2y-y}.$

(d) Reducible to separable

$$y' = 2(2x + y)^2.$$

(e) Homogeneous

$$2y' = \frac{xy + y^2}{x^2}.$$

(f) Reducible to homogeneous

$$y' = \frac{x + y - 1}{x - y - 2}.$$

(g) Linear

$$\text{(i)}\ \ y' + \tfrac{y}{x} = 3, \quad y(1) = \tfrac{3}{2},$$
$$\text{(ii)}\ \ y' + y\cos x = \sin 2x.$$

1.3. Solve (implicit solutions sufficient):

$$\begin{aligned}
(i)\quad & y' = \frac{x - y\cos x}{\sin x}, \\
(ii)\quad & (3x + x^2)y' = 5y - 8, \\
(iii)\quad & y' + \frac{2x}{y} = 3, \\
(iv)\quad & 2y' = \frac{y}{x} + \frac{y^3}{x^3}, \\
(v)\quad & xyy' - y^2 = (x+y)^2 e^{-y/x}, \\
(vi)\quad & x(x-1)y' + y = x(x-1)^2, \\
(vii)\quad & 2xy' - y = x^2, \\
(viii)\quad & y' = \cos(y+x),\ y(0) = \pi/2, \\
(ix)\quad & y' = \frac{x-y}{x-y+1}, \\
(x)\quad & y' = \cos 2x - y\cot x,\ y\left(\frac{\pi}{2}\right) = \frac{1}{2}.
\end{aligned}$$

Supplementary Problems

1.4 By introducing $z = 4y - x$ or otherwise, show that the general solution of

$$y' - 16y^2 + 8xy = x^2.$$

is

$$y = \frac{x}{4} - \frac{1}{8}\tanh(2x + C),$$

Are there any other solutions?

1.5 Integrating factor.

Solve the following equations, which can be reduced to equations in exact differentials.

(a)
$$2(x - y^4)\,\mathrm{d}y = y\,\mathrm{d}x.$$

Hint: look for an integrating factor in the form $\Lambda(y)$.

(b*)
$$y(2y\,\mathrm{d}x - x\,\mathrm{d}y) + x^2(y\,\mathrm{d}x + 2x\,\mathrm{d}y) = 0.$$

Hint: You will need an integrating factor, which you will need to guess. In the lectures we discussed promising forms.

1.6 Sometimes an ODE can be turned into a homogeneous one by the change of variables $y = z^n$. Solve the following system by this method:

$$y\,\mathrm{d}x + x(2xy + 1)\,\mathrm{d}y = 0.$$

Hint: you will need to find n such that the equation does become homogeneous; once you have done this, remember that sometimes it is more convenient to look for a solution in the form $x = x(y)$, rather than $y = y(x)$.

1.7 Quasi-homogeneous equations

(a) An ODE $y' = f(x, y)$ is called *quasi-homogeneous* if it satisfies $f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta-\alpha} f(x, y)$, for all λ and some $\alpha \neq 0$, $\beta \neq 0$. Show that the change of variables $y = x^{\beta/\alpha} z$ reduces a quasi-homogeneous equation to a separable one.

(b) Solve the resulting equation in quadratures. Show that if z_0 exists satisfying $f(1, z_0) = \beta z_0/\alpha$, then $y = z_0 x^{\beta/\alpha}$ is also a solution of the equation. Under what condition are composite solutions possible?

Extracurricular Problems

1.8 Integral and Orthogonal Curves.

(a) If a family of curves on the plane (x, y) is specified in the form

$$f(x, y, C) = 0,$$

where C is a parameter, then one can find a first-order ODE for which these curves are integral curves by taking the differential of this curve and then substituting $C = C(x, y)$ extracted from this equation.

By this method, find the ODE whose integral curves are

$$y = \tan\left[\ln(Cx)\right].$$

(b) Show that the curves that are orthogonal to the integral curves of the ODE $F(x, y, y') = 0$ are integral curves of the ODE

$$F\left(x, y, -\frac{1}{y'}\right) = 0.$$

The explicit expression for these orthogonal curves can then be found by integrating this equation.

(c) Show that the curves orthogonal to the family of curves in (a) are given by $3x^2 + 2y^3 + 6y = C$.

1.9 Equations unresolved with respect to derivative. Solve the following ODEs by resolving it with respect to y and the introduction of parameter $p = y'$:

$$\text{(i)}\ \ (y')^2 - 2xy' = x^2 - 4y,$$

$$\text{(ii)}\ \ y' = e^{xy'/y},$$

$$\text{(iii)}\ \ y' = y\,e^{-xy'/y},$$

$$\text{(iv)}\ \ y' = y^2 e^{-xy'/y}$$

1.10* Parametric Solutions

Consider an equation formed by y' and x lying on the "folium of Descartes":

$$(y')^3 + x^3 = xy'.$$

Try $y' = p = xu$, work out a parametrisation of the curve (x, y'), and then find a parametric solution in the form $x = x(u)$ and $y = y(u)$. In what range of values of u (and, therefore, of x and y) is your solution valid?