Georg Viehhauser

CP3 Ordinary differential equations

Lecture notes MT2025

Introduction

These notes are for the 1st year CP3 Ordinary Differential Equation lecture course in MT2025. This course aspires to satisfy several goals:

- It covers the syllabus as outlined in the student handbook.

  > Ordinary differential equations; integrating factors. Second-order linear differential equations with constant coefficients; complementary functions and particular integrals. Application to forced vibrations of mechanical or electrical resonant systems, including the use of a complex displacement variable; critical damping; quality factor (Q), bandwidth, rms, peak and average values. *[Physical interpretation of complex impedance and power factor is not assumed]*

- It should cover the material required in exams (I did work through the exam papers of the past 10 years to make sure that this is at least retroactively correct).

These two items will be covered mostly in the chapters on Chapters 1, 2 and 4.

- I have also included material that has been taught in past years. This applies for example for Chapter 5.
- I would like to teach you skills that you find will useful in your future career. In this spirit I plan therefore to devote some time in this course to computational methods for solving differential equations. This is covered in the Chapter 3. Note that this is currently strictly off-syllabus, but I still hope that you find this useful and entertaining.

I have relied heavily on past lecture notes by Bence Kocsis and Alex Schekochihin. I also have used WIKIPEDIA and CHATGPT to develop some of the ideas in this course.

Contents

1 Definitions **2**

Solutions to differential equations . . . 3

Existence and uniqueness of solutions . . . 6

When do we get more, or less, than one solution for a differential equation? . . . 6

2 First-order ordinary differential equations **9**

Separable first order ODEs . . . 10

Singular solutions . . . 11

Almost separable first order ODEs . . . 12

Homogeneous first order differential equations . . . 12

Almost homogeneous differential equations . . . 14

Exact differentials . . . 16

Integrating factor . . . 17

Linear first order ODEs . . . 20

Solving linear first order ODEs with an integrating factor . . . 20

Particular integral and complementary function . . . 21

3 Computational methods to solve differential equations **25**

Symbolic ODE solutions in software . . . 25

Symbolic solutions in mathematical software packages . . . 25

Large language models (LLMs) . . . 26

Numeric integration . . . 26

Euler method . . . 27

Multistep methods . . . 27

Analysis of numeric methods . . . 28

Runge-Kutta methods . . . 32

Stiffness and A-stability . . . 34

A-stability of Runge-Kutta methods . . . 36

A-stability of multistep methods . . . 40

Numeric solutions of ODEs in mathematical software packages . . . 41

4 Higher order differential equations **43**

Phase space . . . 44

Conservative systems . . . 46

Non-conservative systems . . . 47

Perturbative solutions . . . 49

Linear ODEs . . . 51

Homogeneous linear ODEs . . . 51

Fundamental systems of solutions of homogeneous linear ODEs . . . 52

Finding the linear combination coefficients 53
Inhomogeneous linear ODEs 56
Method of variation of constants 57
Finding solutions for a general second order linear ODE 58
Linear ODEs with constant coefficients 60
Linear homogeneous ODEs with constant coefficients 60
The damped harmonic oscillator revisited 63
Method of undetermined coefficients 64
The driven harmonic oscillator 65
Power budget of the driven harmonic oscillator 67
The quality factor 69
Filtering 69
Universality 71

5 Systems of ordinary differential equations equations 73
Homogeneous linear systems with constant coefficients 73
Hermitian systems 74
Non-Hermitian systems 76

Appendix A: Examples for symbolic software integration 80
A.1 MATHEMATICA 80
A.2 MATLAB 83
A.3 `python` 85

Appendix B: The indefinite integral of $1/x$ 89

Index 93

Chapter 1
Definitions

Differential equations are equations that contain differentials (e.g. $\mathrm{d}x$, $\mathrm{d}y$ or $\mathrm{d}t$) and/or derivatives (eg. $\mathrm{d}y/\mathrm{d}x$). Our task will be to solve these differential equations, which means we will express one of the variables in the differential equation (sometimes called the dependent variable) as a function of the other variables (the independent variables), in a way that is compatible with the differential equation. As we will see in many cases this will not be sufficient to determine the solution unambiguously, and we will need to use additional constraints if we want to resolve these ambiguities.

We will use the following notations for derivatives:

$\frac{\mathrm{d}y}{\mathrm{d}x}$	Leibniz notation
$y' = y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$	Lagrange notation
$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$	Newton notation (for derivatives with respect to time)

In this course we will not always write the dependence explcitly ($y(x)$). Where we do not (y), the dependence is still implied.

Differential equations can be classified in several ways:

Identifying the type of differential equation will guide you in how to start the process of solving it.

- In *ordinary* differential equations (ODEs) the dependent variable depends only on one independent variable. Differential equations in which the dependent variable depend on more than one independent variables are called *partial differential equations*. The former will be the topic of this course, the latter will be discussed in CP4 and in the 2$^{\text{nd}}$ year Mathematical Methods course.

Both the independent and the dependent variable can be real or complex.

- The *order of the differential equation* is the highest order of derivative of the dependent variable that appears in the differential equation. Most generally an n^{th} order ordinary differential equation can be written as

$$F(x, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}) = 0, \tag{1.1}$$

  where F is an arbitrary given function.

  Differential equations in physics are typically of first or second order, but there are several equations that are fourth order.

- *Linear differential equations* are differential equations that are linear in the dependent variable. They can be written as a linear combination of the dependent variable and its derivatives,

Linear means that the dependent variable and its derivatives only appear in the first power and are not multiplied or composed with each other.

$$a_0(x)y(x) + a_1(x)y'(x) + a_2y''(x) + \ldots + a_{n-1}(x)y^{(n-1)} + a_n(x)y^{(n)} = f(x), \tag{1.2}$$

  where x is the independent variable, and the $a_i(x)$ and $f(x)$ are coefficients generally dependent on x.

  The opposite are *non-linear differential equations*. Non-linear differential equations can exhibit very complex solutions.

This complex behaviour is the subject of chaos theory.

- A linear ODE is *homogeneous* if it is a homogeneous linear equation in the unknown function and its derivatives. this means that, if $y(x)$ is a solution, so is $\lambda y(x)$, for any (non-zero) constant λ. Linear differential equations that are not homogenous are called *inhomogenous*.

  Linear ODEs of the form given in eq. (1.2), for which $f(x) = 0$, are homogenous linear ODEs.

The term "homogeneous" refers to a scaling property: A function f is homogeneous if $f(\lambda x) = \lambda^\alpha f(x)$ for all $\lambda > 0$ and some real number α (the degree). More precisely this should be called a linear homogeneous ODE. See also sidenote on p. 12.

Example 1:
Let's look at some differential equations in physics and identify their type

Equation	ordinary/ partial	linear/ non-linear	homogenous/ inhomogeneous	order	name
$\frac{\mathrm{d}V}{\mathrm{d}p} = \frac{V}{p}$	ordinary	linear	homogenous	1^{st}	Boyle's law (thermodynamics)
$V = R\dot{Q}$	ordinary	linear	inhomogeneous	1^{st}	Ohm's law (electronics)
$\dot{N} = -\lambda N$	ordinary	linear	homogenous	1^{st}	Radioactive decay (nuclear physics)
$m\ddot{x} + 2\gamma\dot{x} - kx = F(t)$	ordinary	linear	inhomogeneous	2^{nd}	Driven harmonic oscillator (mechanics)
$\ddot{\theta} - \frac{g}{l}\sin\theta = 0$	ordinary	non-linear	homogenous	2^{nd}	Pendulum (mechanics)
$m\dot{v} = -kv^2$	ordinary	non-linear	homogenous	1^{st}	Drag force (mechanics)
$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = V_0\cos(\omega t)$	ordinary	linear	inhomogeneous	2^{nd}	Driven LCR circuit (electronics)
$\ddot{x} + a\dot{x} + bx + cx^3 = C\cos(\omega t)$	ordinary	non-linear	inhomogeneous	2^{nd}	Duffing equation (mechanics)
$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V(x)\Psi = E\Psi$	ordinary	linear	homogenous	2^{nd}	1D time-independent Schrödinger equation (quantum mechanics)
$\frac{\partial\rho}{\partial t} = -\left(\frac{\partial(u_x\rho)}{\partial x} + \frac{\partial(u_y\rho)}{\partial y} + \frac{\partial(u_z\rho)}{\partial z}\right)$	partial	linear	homogeneous	1^{st}	Continuity equation (fluid dynamics)
$\frac{\partial^2 T}{\partial x^2} = \frac{C}{\kappa}\frac{\partial T}{\partial t}$	partial	linear	homogeneous	2^{nd}	1D heat diffusion equation (thermodynamics)
$\lambda\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + EI\frac{\partial^4 y}{\partial x^4} = \Phi(x,t)$	partial	linear	inhomogeneous	4^{th}	Bernoulli beam (mechanics)
$\frac{\partial\rho}{\partial t} = c\left(\frac{\partial^2\rho^{\gamma+1}}{\partial x^2} + \frac{\partial^2\rho^{\gamma+1}}{\partial y^2} + \frac{\partial^2\rho^{\gamma+1}}{\partial z^2}\right)$	partial	non-linear	inhomogeneous	2^{nd}	Porous medium equation (fluid mechanics)

There are several important fundamental observations to make here. First, differential equations appear everywhere in physics – almost every physical law can be expressed in differential form. That's not surprising, we want the laws of physics to be universally valid, no matter what the specific point in space or time is, for which they should apply.

Second, equations arising from very different areas of physics often look strikingly similar. Conceptually, this suggests the presence of underlying principles that transcend specific applications. Practically, it means that understanding the behaviour of a system in one domain can provide direct insight into systems in entirely different domains.

In Example 1 for instance the damped oscillator and the LCR circuit.

Often we can adapt a physics problem by adding or omitting terms in the differential equations to reflect additional or absent dependencies.

For example, if we do not have a time-dependence in the 1D thermal diffusion equation, we are left with a simple homogenous 2^{nd} order ODE in space.

Solutions to differential equations

From now on we will focus on ODEs. Nevertheless, the concepts discussed here apply generally.

Solving differential equations is fundamentally different from solving algebraic equations. Although there are established methods for handling specific types of differential equations (many of which we will explore in this course), there is no universal strategy that works for all possible cases.

For a function to be a solution to a differential equation within an interval of the independent variable, all the required derivatives must exist in this region, and when they are used, the equation must be satisfied within the domain. The solution can be given in two ways.

- An *explicit* solution is a solution that gives the dependent variable as a function of the independent variable, $y = f(x)$.
- An *implicit* solution is a relation between the dependent and independent variables without any derivatives, which satisfies the ODE and is not an explicit solution. An implicit solution can not always be converted into an explicit solution made up of elementary functions.

"Solution" here means ideally a combination of elementary functions. In many cases, solutions to ODEs cannot be expressed in this way. However, we can always define a new function as the solution to a differential equation, even if it can't be expressed by elementary functions.

Elementary functions are built from polynomials, exponentials/logarithms, and trigonometric functions (and their inverses), using the usual algebraic operations and composition.

Sometimes the terms "reduction to quadratures" or "solving by quadratures" are used to describe the process of finding solutions of differential equations in terms of integrals. If these integrals can be resolved you are done, but if not, they can be used to define a new function to describe the solution to the differential equation, and they can still be useful to place bounds, look at various limits, find series approximations, etc.

Example 2:
Find the solution for $y(x)$ in $y' = \sin x / x$.

Solution This looks quite straightforward,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{x},$$
$$\int \mathrm{d}y = y = \int \frac{\sin x}{x}\,\mathrm{d}x + C.$$

Unfortunately, there is no solution for this integral that can be expressed using elementary functions, and we simply declare

$$y = \int \frac{\sin x}{x}\,\mathrm{d}x + C \equiv \mathrm{Si}(x) + C,$$

which defines a new function $\mathrm{Si}(x)$, called the sine integral function,

If you cannot find satisfying solutions analytically, solutions can in many cases be found using computational means (we will study this in more detail in chapter 3.

Example 3:
Find the solution for $y(x)$ in $y' = x$.

Solution

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = x,$$
$$\mathrm{d}y = x\,\mathrm{d}x,$$
$$\int \mathrm{d}y = \int x\,\mathrm{d}x,$$
$$y = \frac{x^2}{2} + C.$$

Note that we found the solution using indefinite integrals, which introduces an integration constant C, an arbitrary real number.

Indefinite integrals are sometimes called antiderivatives.

Example 4:
Find the solution for $y(x)$ in $y'' = x$.

Solution Start with $z = y'$.

$$\begin{aligned} y'' = z' &= \frac{\mathrm{d}z}{\mathrm{d}x} = x, \\ \int \mathrm{d}z &= \int x\,\mathrm{d}x, \\ y' = \frac{\mathrm{d}y}{\mathrm{d}x} &= z = \frac{x^2}{2} + C_1, \\ \int \mathrm{d}y &= \int \left(\frac{x^2}{2} + C_1\right)\mathrm{d}x, \\ y &= \frac{x^3}{6} + C_1 x + C_2. \end{aligned}$$

In this case we have two consecutive indefinite integrals, and thus we get two undetermined integration constants. Generally, we find that the solution of an n^{th} order ODE will involve n indefinite integrals giving rise to n integration constants. Because the integration constants can take on any value the solution of the ODE is not just one function, but a whole family of solutions. For a 1^{st}order ODE, this will be a one-parameter family, for a 2^{nd}order ODE a two-parameter family, and so on.

When solving higher order differential equations check that these are really independent constants, i.e. they cannot be reduced to one constant by algebraic manipulations (e.g. $C_1 x + C_2 x = Cx$ with $C = C_1 + C_2$).

Example 5:
Find the solution for $y(t)$ in $\dot{y} = -\lambda y$, with $\lambda > 0$.

Solution

$$\begin{aligned} \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= -\lambda y(t), \\ \frac{\mathrm{d}y(t)}{y(t)} &= -\lambda\,\mathrm{d}t, \\ \int \frac{\mathrm{d}y}{y(t)} &= -\lambda \int \mathrm{d}t, \\ \ln y(t) &= -\lambda t + C, \\ y(t) &= C\exp(-\lambda t). \end{aligned}$$

Note that in the last step we have renamed the integration constant $e^C \to C$. We can do that as a this point it is still only an undetermined constant. We will use the symbol C and equivalent symbols with this flexibility throughout this course.

The value of the integration constants cannot be determined from the differential equation itself, we need to use additional information to constrain them. It is common to use two designations to describe such constraints

In principle any set of n constraints would be sufficient to constrain the integration constants of an n^{th} order ODE.

- *Initial conditions* constrain the dependent variable and all its required derivatives at a single value of the independent variable. Often this will be for a time-dependent problem a point in time at the start of the problem, hence the name. The differential equation with its initial conditions is called an *initial value problem.*

- *Boundary conditions* constrain the dependent variable and some of its derivatives at two or more values of the independent variable, typically at values where changes to the ODE will occur, or at the ends of the allowed parameter space for the independent variable.

An example would be a vibrating string, for which the amplitude has to be zero at each end.

Solutions to an ODE, where the integration constants are not resolved are called *general solutions*. A solution where a specific value for the integration constants has been chosen is called a *particular* solution or *particular integral* of the differential equation.

Finding a solution for a differential equation, subject to initial or boundary conditions, is sometimes called a *Cauchy problem*.

Existence and uniqueness of solutions

Can we always find a solution to the Cauchy problem? And if yes, is it unique, i.e. for a given x_0, will there only be one solution at this point?

For a first order differential equation that is linear in y', the Cauchy problem can be written as

$$y' = f(x, y(x)), \text{ and } y(x_0) = y_0. \tag{1.3}$$

In the more general case that $F(x, y(x), y'(x)) = 0$, which allows for problems that are not linear in y', there can be more than one value for the slope in each point (e.g. for $(y')^2 = a$ the slope can be $\pm\sqrt{a}$). In that case you need to pick a branch for the Cauchy theorem to apply.

It can now be shown that if f is continuous in x and Lipschitz continuous in y (its derivative has an upper bound) in a region around (x_0, y_0), then a solution exists and is unique in an interval $[x_0 - \epsilon, x_0 + \epsilon]$ with $\epsilon > 0$.

This theorem is variously known as the Picard–Lindelöf theorem, Picard's existence theorem, the Cauchy–Lipschitz theorem, or the existence and uniqueness theorem..

Example 6:
For the ODE in Example 5, $y(t)$ in $\dot{y} = -\lambda y$, with $\lambda > 0$, find the solution for $y(0) = y_0$.

Solution Previously we found for the general solution $y(t) = C\exp(-\lambda t)$. We can now plug in the initial condition to get

$$y_0 = Ce^0 = C,$$

and thus the particular solution is

$$y(t) = y_0 \exp(-\lambda t).$$

This differential equation for example describes radioactive decay. There the integration constant can either be interpreted as the number of radioactive nuclei y_0 at the time $t = 0$, $y(t) = y_0 \exp(-t/\tau)$, or as a shift in time, $y(t) = \exp(-(t - t_0)/\tau)$. The latter implies that no matter how many nuclei have already decayed, the remainder will still decay with an exponential with the same time constant.

While the existence theorem is reassuring, we have to remember that the fact that a solution exists does not guarantee that this solution can expressed by elementary functions, or found by analytical methods at all.

On the other hand, once we have found a solution to the Cauchy problem (and if it is only by guessing), and the conditions for the existence theorem are satisfied, we know that we have *the* solution.

When do we get more, or less, than one solution for a differential equation?

For initial value problems for which we cannot find one unique solution, we just need to think about cases where the conditions of the existence theorem are violated.

- If the right-hand side $f(x, y)$ of a 1st order ODE, $y' = f(x, y)$, is not continuous at the initial point, a solution may not exist.

Example 7:
Find the solution for $y'(x) = 1/x$ for the initial condition $y(0) = 0$.

Solution This has no solution for $x_0 = y_0 = 0$, since $f(x, y) = 1/x$ is undefined at $x = 0$. The initial value problem has no solution passing through (0,0).

- A local solution to a differential equation may exist in a domain close to the initial condition, but it can fail to extend globally.

Example 8:
Find the solution for $y' = y^2$ for the initial condition $y(0) = 1$.

Solution For the general solution,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= y^2, \\ \int \frac{\mathrm{d}y}{y^2} &= \int \mathrm{d}x + C, \\ -\frac{1}{y} &= x + C, \end{aligned}$$

And from the initial condition $C = -1$ and thus

$$y(x) = \tfrac{1}{1-x},$$

which blows up at $x = 1$.

- If $f(x, y)$ is not Lipschitz-continuous in y, uniqueness can fail.

Example 9:
Solve the Cauchy problem $y' = y^{2/3}$, with $y(0) = 0$.

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= y^{2/3}, \\ \int \mathrm{d}x &= \int y^{-2/3}\mathrm{d}y + C, \\ y &= \frac{x^3}{3} + C. \end{aligned}$$

and to satisfy the initial condition $C = 0$. Hence, $y(x) = x^3/3$ is a solution.

However, $y(x) = 0$ is also a solution. In fact any function of the form

$$y(x) = \begin{cases} y(x) = \frac{(x-x_2)^3}{3} & \text{for } x > x_2, \\ y(x) = 0 & \text{for } x_1 \le x \le x_2, \\ y(x) = \frac{(x-x_1)^3}{3} & \text{for } x < x_1, \end{cases}$$

with two constants $x_1 \le 0 \le x_2$ is a solution. The integral curves representing these solution all intersect at $x = 0$.

The reason why the existence theorem does not apply is that $f(x, y) = y^{2/3}$, and thus $\mathrm{d}f/\mathrm{d}x = (2/3)\,y^{-1/3}$, which diverges for $y = 0$ and thus we can have infinitely many solutions to this Cauchy problem.

- Finally, for boundary value problems, the boundary conditions may be incompatible with the ODE.

Example 10:
Find the solution for $y'' + y = 0$ for $y(0) = 0$ and $y(\pi) = 1$.

Solution The solution for this ODE can be written as

$$y(x) = C_1 \cos x + C_2 \sin x.$$

You can verify this by plugging this solution into the ODE. We will discuss how to find solutions for higher order linear ODEs in all its glory in Chapter 4.

Plugging in the conditions for the two boundaries then gives

$$0 = C_1 \times 1 + C_2 \times 0,$$
$$1 = C_1 \times (-1) + C_2 \times 0,$$

and the two equations are obviously not compatible.

Key lessons:

- In ordinary differential equations (ODEs) the dependent variable depends only on one independent variable.
- An ODE is of order n if it can be written in the form $F(x, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}) = 0$.
- Linear ODEs have the form

$$a_0(x)y(x) + a_1(x)y'(x) + a_2y''(x) + \ldots + a_{n-1}(x)y^{(n-1)} + a_n(x)y^{(n)} = f(x).$$

- A linear homogeneous ODE has the form

$$a_0(x)y(x) + a_1(x)y'(x) + a_2y''(x) + \ldots + a_{n-1}(x)y^{(n-1)} + a_n(x)y^{(n)} = 0.$$

- For a solution to a differential equation all the required derivatives must exist, and when they are used, the equation must be satisfied (within a region).
  - An explicit solution is of the form $y = f(x)$.
  - An implicit solution is a relation between the dependent and independent variables without any derivatives, which satisfies the ODE and is not an explicit solution.
- The solution of an n^{th} order ODE will involve n indefinite integrals giving rise to n integration constants.
  - To find the particular solution the integration constants must be determined from initial or boundary conditions. The differential equation with its initial conditions is called an initial value problem.
  - The initial value problem $y' = f(x, y(x))$ and $y(x_0) = y_0$ has a unique solution in an interval $[x_0 - \epsilon, x_0 + \epsilon]$ with $\epsilon > 0$, if f is continuous in x and Lipschitz continuous in y in a region around (x_0, y_0).

Chapter 2
First-order ordinary differential equations

The general form for a first order differential equation is

$$F(x, y(x), y'(x)) = 0, \tag{2.1}$$

where F is a function of the independent variable, and the dependent variable and its first-order derivative. If the differential equation is linear in y' we can write it as

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = 0, \tag{2.2}$$

which is referred to as the *symmetric form*.

If $Q(x, y) \neq 0$ we can also write the differential equation as

$$y' = \frac{P(x, y)}{Q(x, y)} = f(x, y). \tag{2.3}$$

This form is called the *explicit form* of the differential equation and in this form is is said *to be resolved by the derivative*.

This form can be used to interpret the ODE geometrically, even without solving it. In the proximity of any given point (x, y) in the domain where $f(x, y)$ is defined, we can approximate the function $y(x)$ as

$$y(x + \Delta x) = y(x) + y'(x)\Delta x = y(x) + f(x, y)\Delta x. \tag{2.4}$$

Hence plotting a line with a gradient $f(x, y)$ in each point of the domain shows the tangent of the function $y(x)$ in each point, or the direction how it will develop from this point. We call this the *direction field*.

Once we have solved a first-order ODE, its solutions can be visualized as a family of curves in the (x, y) (or (t, y)) plane, each corresponding to a different value of the integration constant(s). These curves are known as *integral curves*. Connecting the stubs in the direction field gives the integral curves for the ODE. Like any curve in the $x-y$ plane, an integral curve can be described in various ways: as an explicit function $y = y(x)$, implicitly as $x = x(y)$, or more generally through a parametric representation $(x(s), y(s))$, where s is a parameter. When the solution of an ODE is expressed in this parametric form, it is called a *parametric solution*.

There is a wide range of approaches that can be useful to solve first order ODEs. Sometimes more than one method will work, sometimes, unfortunately, none. For the remainder of this chapter we will walk through some of the most useful methods for solving first-order ODEs.

An example where this is not possible is $y' + \exp(y') = y$, which is still a perfectly valid and solvable (albeit not using elementary functions) 1st order ODE.

$Q(x, y) = 0$ would correspond to solutions $\mathrm{d}x = 0$, which implies x constant.

Where this is not possible the equation can sometimes be resolved by y as $y = f(x, y')$, and be solved with a suitable substitution.

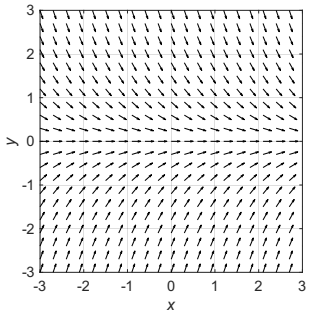


Figure 2.1: Direction field for $y' = -y$.

This is what numeric methods to solve ODEs do. We will investigate these in Chapter 3.

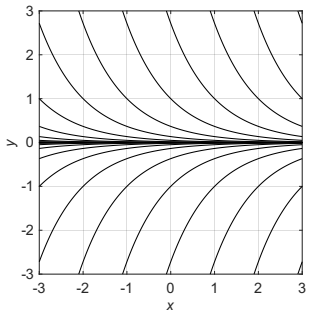


Figure 2.2: Integral curves for $y' = -y$.

Separable first order ODEs

A first order differential equation is separable if it may be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \frac{f(x)}{g(y)} \quad \text{or} \quad f(x)\mathrm{d}x = g(y)\mathrm{d}y, \tag{2.5}$$

which can be expressed in the integral form,

$$\int f(x)\mathrm{d}x = \int g(y)\mathrm{d}y + C. \tag{2.6}$$

Hopefully the integrals are solvable, but if not this still provides an implicit solution to the problem.

Example 11:
Solve the equation $y' = xy$.

Solution We start with rewriting the equation in separable form:

$$\underbrace{\frac{1}{y}}_{g(y)}\,\mathrm{d}y = \underbrace{x}_{f(x)}\,\mathrm{d}x.$$

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y}\,\mathrm{d}y &= \int x\,\mathrm{d}x,\\ \ln|y| &= \frac{x^2}{2} + C,\\ |y| &= e^{\frac{x^2}{2}+C},\\ y &= Ce^{\frac{x^2}{2}},\end{aligned}$$

which also includes the special solution $y(x) = 0$, corresponding to $C = 0$.

More generally,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f_1(x)g_1(y)}{f_2(x)g_2(y)} \tag{2.7}$$

can be written as

$$\frac{f_1(x)}{f_2(x)}\mathrm{d}x = \frac{g_2(y)}{g_1(y)}\mathrm{d}y, \tag{2.8}$$

and subsequentially integrated.

We just have to be careful about what happens if $g_1(y) = 0$ for some $y = y_0$ and/or $f_2(x) = 0$ for some $x = x_0$. In that case, $y = y_0$ in the former case or $x = x_0$ in the latter are also solutions of the separable ODE. There can be a number of such lines, one for each of the zeros of $g_1(y)$ and $f_2(x)$.

If the different integral curves come together in one point, they must have the same derivative by definition, and we can patch together these solutions with the solutions of eq. (2.8) for a *composite solution*.

An example for this is Example 6 above.

Example 12:
Find the solution(s) of $xy' + y = y^2$.

Solution We can write this in the symmetric form

$$x\mathrm{d}y = y(y-1)\mathrm{d}x,$$

which is separable.

$$\int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}y}{y(y-1)} + C = \int \frac{\mathrm{d}y}{y-1} - \int \frac{\mathrm{d}y}{y} + C,$$

$$\ln|x| = \ln\left|\frac{y-1}{y}\right| + C,$$

$$Cx = \frac{y-1}{y},$$

$$y = \frac{1}{1+Cx}.$$

But this does not include all the possible solutions (you might have seen that $y(x) = 0$ is also a solution). Where did we miss this? The answer is that for the separation we had to divide by $xy(y-1)$ and we need to be careful that in this process we do not exclude the solutions that make this divisor zero. The first condition is $x = 0$, which solves the symmetric form, but obviously not our original ODE. The remaining conditions are $y(x) = 0$, the solution we could have guessed from inspection, and $y(0) = 1$, which is covered by our original solution, if $C = 0$.

Changing to the symmetric form made the ODE more general, it allows now for integral curves with a tangent that is parallel to the y axis, which is not allowed in the original form, as $y' \to \infty$.

But what about the Cauchy theorem? From Figure 2.3 we can see that there is a unique solution for every point in the (x, y) plane, except for $x_0 = 0$ where there is only a solution for $y_0 = 0$ and for $y_0 = 1$, but not for any other values of y. Here the Cauchy problem can be expressed as

$$y' = f(x, y) = \frac{y(y-1)}{x},$$

and $f(x, y) \to \infty$ for $x = 0$, allowing for the exception. We can also see that our original solution is not able to reach any point on the x-axis. The separate solution $y(x) = 0$ is required to provide a solution there so that there is one solution for every point in the (x, y) plane where the criteria of the Cauchy theorem are met.

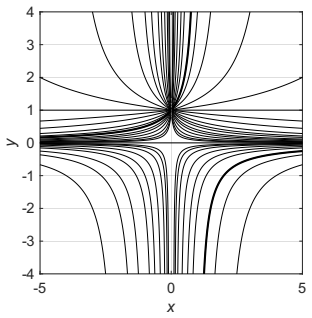


Figure 2.3: Integral curves for $xy' + y = y^2$. The thick curve is for $C = 1$.

Singular solutions

In the last example we have encountered a *singular solution* to a 1st order ODE. A singular solution is a solution curve that cannot be obtained by choosing any value of the constant in the one-parameter general solution. Singular solutions fall into two categories: envelope-type solutions, and other "stray" solutions (e.g., equilibria of autonomous ODEs, or solutions lost when we divide by a factor during separation, as in the previous example).

Autonomous ODEs do not depend explicitly on the independent variable. They can thus be written in the form $y' = f(y)$. One solution for this equation is a constant value $y(x) = C$ such that $f(C) = 0$. Such a solution is called an equilibrium solution.

Envelope-type singular satisfy the ODE and the tangency (discriminant) condition. If the general solution is expressed implicitly as

$$F(x, y, C) = 0, \tag{2.9}$$

where C is the parameter, then the singular solution is obtained by eliminating C between this equation and

$$\frac{\mathrm{d}F}{\mathrm{d}C}(x, y, C) = 0. \tag{2.10}$$

Then verify that this envelope solves the ODE and is not contained in the one-parameter family to ensure that this solution is singular.

Almost separable first order ODEs

Consider a first order ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax+by), \tag{2.11}$$

with real constants a and b. Such an ODE becomes separable after a substitution $z(x) = ax + by(x)$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x} = a + bf(z), \tag{2.12}$$

and thus

$$\int \frac{\mathrm{d}z}{a+bf(z)} = \int \mathrm{d}x + C = x + C. \tag{2.13}$$

If we obtain an explicit solution for $z(x)$, we can find $y(x)$ after substituting back $y(x) = (z(x) - ax)/b$.

Example 13:
Find the general solution for $(x+2y)y' = 1$.

Solution For $x + 2y \neq 0$ we can write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+2y}.$$

We substitute $z(x) = x + 2y$, and have $a = 1$, $b = 2$ and $f(z) = 1/z$. Thus

$$x + C = \int \frac{\mathrm{d}z}{1+2/z} = \int \frac{z\mathrm{d}z}{z+2} = \int \left(1 - \frac{2}{z+2}\right)\mathrm{d}z = z - 2\ln|z+2|,$$

which cannot be resolved explicitly for $z(x)$. But we can back-substitute

$$(x+2y) - 2\ln|(x+2y)+2| = x + C,$$

which gives an implicit solution for $y(x)$,

$$x + 2y + 2 = Ce^y.$$

Let's look again at the divisions. First we divided by $x+2y$. If $x+2y = 0$, then $\mathrm{d}x/\mathrm{d}y = 0$, and thus $x = C$, which describes tangents to the integral lines that are parallel to the y-axis, for which y can take any value, so this is not a general solution. The other issue could be with $z = x + 2y = -2$, but this is already part of our general solution for the case $C = 0$.

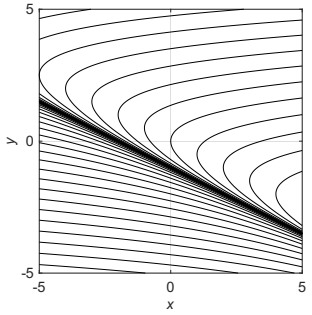


Figure 2.4: Integral lines for $(x+2y)y' = 1$.

Homogeneous first order differential equations

A first order ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right), \tag{2.14}$$

where $f(y.x)$ is a function that depends on x and y only through y/x, is called a *homogenous* differential equation. Again, we can use a substitution, $z = y/x$, to convert this to a separable differential equation. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{x^2} = \frac{f(z) - z}{x}. \tag{2.15}$$

Note that the same term "homogeneous differential equation" is used already on p. 3. There we used it to describe a linear differential equation $L(y, y', y'', \ldots y^{(n)}) = f(x)$ with $f(x) = 0$. Here it is used for a first order differential equation that does not change under the uniform scaling $x \to \lambda x$, $y \to \lambda y$. More precisely this would be called a uniform homogeneous ODE.

An alternative, but equivalent, definition is that of a symmetric ODE (eq. (2.2)) with

$$\frac{P(\lambda x, \lambda y)}{P(x,y)} = \frac{Q(\lambda x, \lambda y)}{Q(x,y)},$$

i.e. P and Q scale in the same way when their arguments are scaled.

and

$$\int \frac{\mathrm{d}z}{f(z)-z} = \int \frac{\mathrm{d}x}{x} + C, \tag{2.16}$$

and the solution for $y(x)$ can be found by back-substituting $y(x) = xz(x)$.

We again have to be careful with the division by $f(z) - z$. If $f(z) = z$ for any $z = z_0$, we get additional solutions, $z(x) = z_0$, which correspond to $y(x) = xz_0$.

Example 14:

Solve the equation $xy' = \sqrt{x^2 - y^2} + y$.

Solution Let $y = zx$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}.$$

Substituting into the differential equation gives

$$x\left(z + x\frac{\mathrm{d}z}{\mathrm{d}x}\right) = \sqrt{x^2 - z^2x^2} + zx = |x|\sqrt{1-z^2} + zx.$$

Cancelling xz on both sides yields

$$x^2\frac{\mathrm{d}z}{\mathrm{d}x} = |x|\sqrt{1-z^2}.$$

Note that the only explicit dependence is on the absolute value of x, and thus the solution must be left-right symmetric around the y-axis.

To solve this differential equation we define an angle $\varphi(x)$ by $z = \sin\varphi$, which gives

$$\sqrt{1-z^2} = |\cos\varphi|.$$

Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \cos\varphi\,\frac{\mathrm{d}\varphi}{\mathrm{d}x},$$

and substituting into the previous equation yields

$$x^2\cos\varphi\,\frac{\mathrm{d}\varphi}{\mathrm{d}x} = |x|\,|\cos\varphi|.$$

Provided $\cos\varphi \neq 0$ and $x \neq 0$, division gives

$$x\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \operatorname{sgn}(x)\operatorname{sgn}(\cos\varphi).$$

Hence the sign of φ' depends on the signs of both x and $\cos\varphi$:

$$\operatorname{sgn}(\varphi') = \operatorname{sgn}(x)\operatorname{sgn}(\cos\varphi).$$

On any interval where $\cos\varphi$ keeps a fixed sign, the differential relation $x\varphi' = \operatorname{sgn}(x)\operatorname{sgn}(\cos\varphi)$ reduces to either $\varphi' = 1/x$ or $\varphi' = -1/x$. Integrating gives

$$\varphi = \pm\ln|x| + C, \implies z = \sin(\pm\ln|x| + C), \implies y = x\sin(\pm\ln|x| + C).$$

We again have to be careful, as we have divided by $\sqrt{1-z^2}$. If $\sqrt{1-z^2} = 0$, then $z = \pm 1$, and thus $y = \pm x$. These satisfy the original equation directly:

$$x\frac{\mathrm{d}(\pm x)}{\mathrm{d}x} = \pm x, \qquad \sqrt{x^2 - (\pm x)^2} + (\pm x) = 0 \pm x = \pm x.$$

These are not included in the general solution and therefore $y = \pm x$ are singluar solutions of this ODE. These are envelope curves, as can be seen from the general condition for the general solutions

$$F(x, y, z) = y - x\sin(\pm\ln|x| + C) = 0,$$

and the tangent condition

$$\frac{\mathrm{d}F}{\mathrm{d}C} = 0 = -x\cos(\pm\ln|x| + C),$$

which must be both satisfied for an envelope curve, and thus

$$\pm\ln|x| + C = \frac{\pi}{2} \implies y = x\sin\left(\frac{\pi}{2}\right) = \pm x.$$

For the general solutions we must ensure that the sign of $\cos(\pm\ln|x| + C)$ remains consistent with the chosen branch. Specifically,

$$\operatorname{sgn}(\cos(\pm\ln|x| + C)) = \pm\operatorname{sgn}(x).$$

Thus each curve $y = x\sin(\pm\ln|x|+C)$ is a valid solution only on intervals where this sign condition holds. The sign changes whenever $\cos(\pm\ln|x|+C) = 0$, which applies for $|y| = |x|$, corresponding to the singular envelope lines $y = \pm x$. Outside of this finite domain the curve follows the singular solution. General solutions therefore must be composite solutions of our general solution and the singular solutions, glued together at $|x_k| = \exp(\pm(2\pi + k\pi - C))$, $k \in \mathbb{Z}$.

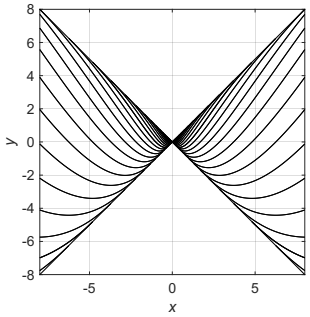


Figure 2.5: Integral lines for

$$xy' = \sqrt{x^2 - y^2} + y.$$

More generally,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_0x^k + a_1x^{k-1}y + a_2x^{k-2}y^2 + \ldots + a_ky^k}{b_0x^k + b_1x^{k-1}y + b_2x^{k-2}y^2 + \ldots + b_ky^k}\right) \tag{2.17}$$

is a homogenous differential equation, as can be seen after division by x^k. Even further, the equation will remain homogenous after the addition of any other function of that form, even with different sets of k, a_i and b_1.

Almost homogeneous differential equations

An equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{2.18}$$

can be made homogenous, if we can get rid of the constants c_1 and c_2. We achieve this by substituting $X = x + u$ and $Y = y + v$ with the constants u and v chosen so that

$$a_1u + b_1v = c_1 \text{ and } a_2u + b_2v = c_2. \tag{2.19}$$

This can be achieved if

$$(a_1b_2 - a_2b_1)u = b_2c_1 - b_1c_2 \text{ and } (a_1b_2 - a_2b_1)v = a_1c_2 - a_2c_1, \tag{2.20}$$

which can be solved for u and v as long as $a_1b_2 \neq a_2b_1$.

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X} = f\left(\frac{a_1X + b_1Y}{a_2X + b_2Y}\right) = f\left(\frac{a_1 + b_1\frac{Y}{X}}{a_2 + b_2\frac{Y}{X}}\right), \tag{2.21}$$

which is homogenous.

More generally,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_0x^k + a_1x^{k-1}y + a_2x^{k-2}y^2 + \ldots + a_ky^k + c_1}{b_0x^k + b_1x^{k-1}y + b_2x^{k-2}y^2 + \ldots + b_ky^k + c_2}\right)$$

can be solved by a substitution $X = x + u$ and $Y = y + v$ with similar requirements for the constant u and v.

$a_1b_2 = a_2b_1$ implies $a_2x + 2b_2y = k(a_1x + b_1y)$ for all x and y, where $k = a_2/a_1 = b_2/b_1$. In this case the equation can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2}\right) = g(a_1x + b_1y),$$

which we have encountered already (eq. (2.11)), and can be solved by the substitution $z = a_1x + b_1y$.

Example 15:
Solve the equation $y' = \frac{y+2}{2x+y-4}$.

Solution We use $X = x+u$ and $Y = y+v$, so that the equation becomes homogenous with $u = (4+2)/(-2) = -3$ and $v = (-4)/(-2) = 2$. The equation then is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{Y}{2X+Y} = \frac{\frac{Y}{X}}{2+\frac{Y}{X}}.$$

We continue along a well-known path, $Z = Y/X$ and $f(Z) = Z/(2+Z)$, so that

$$\int \frac{\mathrm{d}X}{X} + C = \ln|X| + C = \int \frac{\mathrm{d}Z}{f(Z)-Z} = \int \frac{\mathrm{d}Z}{\frac{Z}{2+Z} - Z} =$$
$$= -\int \frac{(2+Z)\mathrm{d}Z}{Z(Z+1)} = \int \frac{\mathrm{d}Z}{Z+1} - 2\int \frac{\mathrm{d}Z}{Z} = \ln|Z+1| - 2\ln|Z|,$$

and thus

$$X = C\frac{Z+1}{Z^2} = C\frac{\frac{Y}{X}+1}{\left(\frac{Y}{X}\right)^2} = C\frac{XY+X^2}{Y^2},$$

which after division by X yields

$$Y^2 = C(X+Y),$$
$$(y+2)^2 = C(x+y-1).$$

In this process, we have used several divisions, first by $X = x-3$. For $x = 3$, $\mathrm{d}y/\mathrm{d}x = 1$, the function can take any value, but the slope is always 1, except for when $y = -2$, more on that later .

We then divided by $f(Z)-Z$, and thus we need to be careful to not lose the solution for $f(Z) = Z$, which is

$$\frac{Z}{2+Z} = Z,$$
$$1 = 2+Z = 2+\frac{Y}{X}$$
$$-1 = \frac{y+2}{x-3},$$
$$y = 1-x,$$

which is a singular solution not included in our general solution.

The general and the singular solution intersect at $(3,-2)$, which is again the point where a composite solution can be stitched together. At this point many solutions can exist, because the criterion for the Cauchy problem is not satisfied (the derivatives of $f(x,y)$ are not defined).

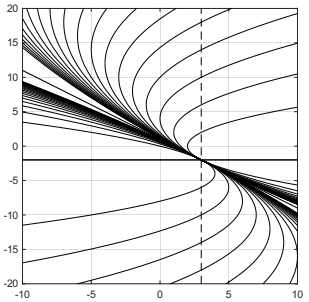


Figure 2.6: Integral lines for $y' = \frac{y+2}{2x+y-4}$. The dashed line indicates $x = 3$.

Example 16:
Solve the equation $y' = \frac{2x+y+2}{2x+y-4}$.

Solution In this case $a_1 b_2 = a_2 b_1$, so we can write this equation as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+y+2}{2x+y-4} = f(2x+y),$$

with $z = 2x + y$, $\mathrm{d}z/\mathrm{d}x = 2 + \mathrm{d}y/\mathrm{d}z$, and $f(z) = (z+2)/(z-4)$. Thus

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{z+2}{z-4} + 2 = 3\frac{z-2}{z-4}, \\
x + C &= \frac{1}{3}\int \frac{z-4}{z-2}\mathrm{d}z = \frac{1}{3}\int \frac{(u+2)\mathrm{d}u}{u} = \\
&= \frac{1}{3}(z - 2 + 2\ln|z-2|) = \frac{1}{3}(2x + y - 2 + 2\ln|2x+y-2|), \\
3x + C &= 2x + y - 2 + 2\ln|2x+y-2|, \\
x - y + C &= 2\ln|2x+y-2|, \\
Ce^{x-y} &= (2x+y-2)^2.
\end{aligned}$$

Again, we need to check for solutions that got lost in divisions. Here this is the division by $z-2 = 2x+y-2 = 0$, but this is included in our general solution, for $C = 0$.

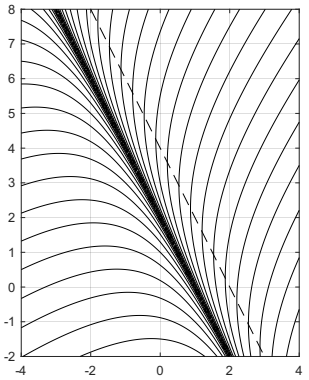


Figure 2.7: Integral lines for $y' = \frac{2x+y+2}{2x+y-4}$. The dashed line indicates $y = 4 - 2x$, where the denominator in the ODE is zero. and thus the tangent is parallel to the y-axis.

Exact differentials

Let's write the symmetric first order ODE from eq. (2.2) in the form

$$P(x,y) + Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{2.22}$$

If we can find a function $\Phi(x,y)$, with the derivatives

$$\frac{\partial \Phi}{\partial x} = P(x,y) \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = Q(x,y), \tag{2.23}$$

then we call the ODE *exact*, and Φ is called the *potential function*. We can then write the ODE as

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{2.24}$$

but this is nothing but the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(x, y(x)) = 0, \tag{2.25}$$

and the (implicit) solution to an exact differential equation is then

$$\Phi(x,y) = C. \tag{2.26}$$

The level sets of $\Phi(x,y)$ are then the integral curves of eq. (2.2).

To test whether or not the ODE is an exact differential we can use a general theorem that the order of differentiation in mixed second order derivatives does not matter

$$\frac{\partial^2 \Phi(x,y)}{\partial x \partial y} = \frac{\partial^2 \Phi(x,y)}{\partial y \partial x}. \tag{2.27}$$

If the relations hold, then it must be that

$$\frac{\partial}{\partial y}\left(\frac{\partial \Phi(x,y)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \Phi(x,y)}{\partial y}\right) \quad \text{or} \quad \frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} \tag{2.28}$$

which is a necessary requirement for a potential function to exist, and for the ODE to be exact.

In multivariable calculus, we use partial derivatives (∂) instead of ordinary derivatives (d) when dealing with functions of multiple variables. The partial derivative indicates the rate of change of a function with respect to one variable, while holding all other variables constant.

This is true if $\Phi(x,y)$ and its first order derivatives are continuous.

Separable differential equations $\mathrm{d}y/\mathrm{d}x = f(x)/g(y)$ are always exact, as $\partial f/\partial y = 0 = \partial g/\partial x$.

Homogenous differential equations are not necessarily exact (and vice versa).

It is sufficient if it is valid for all points in a simply connected region. In a simply connected region any pair of points can be connected by a curve that is contained in the region.

Example 17:
Solve the equation $2xy\mathrm{d}x + (x^2 - y^2)\mathrm{d}y = 0$.

Solution We start with checking whether the ODE is an exact differential,

$$\frac{\partial}{\partial y} 2xy = 2x \quad \text{and} \quad \frac{\partial}{\partial x}\left(x^2 - y^2\right) = 2x,$$

which it is. We therefore should find a potential function with

$$\frac{\partial \Phi}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = x^2 - y^2,$$

which we should be able to find from integrating these equations. We just need to be careful with the integration constants when integrating a multivariable function. From the first equation

$$\Phi(x,y) = \int \frac{\partial \Phi(x,y)}{\partial x}\mathrm{d}x + C(y) = \int 2xy\mathrm{d}x + C(y) = x^2 y + C(y),$$

where we used $C(y)$ to allow for the fact that, while not depending on x, it generally can be a function of y. We get this factor from inserting this result into the second condition and

$$\frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y + C(y)\right) = x^2 + \frac{\mathrm{d}C(y)}{\mathrm{d}y} = x^2 - y^2,$$

which gives

$$\frac{\mathrm{d}C(y)}{\mathrm{d}y} = -y^2,$$

and, after integration,

$$C(y) = -\frac{y^3}{3} + C,$$

and therefore for the potential function

$$\Phi(x,y) = x^2 y - \frac{y^3}{3} + C,$$

which, as discussed must be a constant, and thus an implicit solution for the original ODE is

$$x^2 y = \frac{y^3}{3} + C.$$

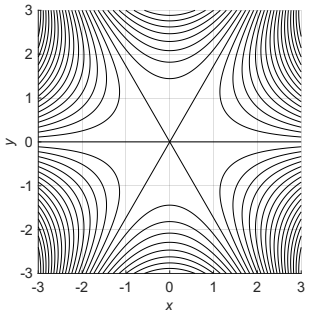


Figure 2.8: Integral lines for $2xy\mathrm{d}x + (x^2 - y^2)\mathrm{d}y = 0$.

.

Integrating factor

If a first order ODE is not an exact differential it sometimes is possible to change it into an exact differential by multiplying it with a function $\Lambda(x,y)$,

$$\Lambda(x,y)\left[P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y\right] = 0, \tag{2.29}$$

with a potential function given by

$$\frac{\partial \Phi}{\partial x} = \Lambda(x,y)P(x,y) \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = \Lambda(x,y)Q(x,y). \tag{2.30}$$

The criterion whether such a potential function exists is now

$$\frac{\partial}{\partial y}\Lambda(x,y)P(x,y) = \frac{\partial}{\partial x}\Lambda(x,y)Q(x,y), \tag{2.31}$$

or

$$Q(x,y)\frac{\partial \Lambda(x,y)}{\partial x} - P(x,y)\frac{\partial \Lambda(x,y)}{\partial y} = \Lambda(x,y)\left(\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right). \quad (2.32)$$

Generally, $\Lambda(x,y)$ can depend on x and y. If it does, there exists no general strategy to find it. However, there are cases where it depends only either on x or on y, and in that case there is. For example, if we assume that $\Lambda = \Lambda(x)$, a function of x only then

$$Q(x,y)\frac{\mathrm{d}\Lambda(x)}{\mathrm{d}x} = \Lambda(x)\left(\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right), \quad (2.33)$$

or

$$\frac{\mathrm{d}\Lambda(x)}{\Lambda(x)} = \frac{1}{Q(x,y)}\left(\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right)\mathrm{d}x. \quad (2.34)$$

The left side only depends on x, so the right side also must be a function of x only, if our assumption $\Lambda = \Lambda(x)$ is correct. The equation can then be integrated to yield

$$\Lambda(x) = \exp\left(\int \frac{1}{Q(x,y)}\left(\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right)\mathrm{d}x\right). \quad (2.35)$$

We can omit any integration constant in the calculation of the integrating factor, because it only results in a constant factor that drops out from the equation.

Similarly, if $\Lambda = \Lambda(y)$,

$$\Lambda(y) = \exp\left(\int \frac{1}{P(x,y)}\left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right)\mathrm{d}y\right), \quad (2.36)$$

if the integrand is a function of y only.

Example 18:
Solve the equation $y' = -\frac{2}{y} - \frac{3y}{2x}$.

Solution Rearranging gives

$$\left(4x + 3y^2\right)\mathrm{d}x - 2xy\mathrm{d}y = 0,$$

thus $P(x,y) = 4x + 3y^2$ and $Q(x,y) = 2xy$ and

$$\frac{\partial P(x,y)}{\partial y} = 6y \neq \frac{\partial Q(x,y)}{\partial x} = 2y,$$

and thus the ODE is not exact. However,

$$\frac{1}{Q(x,y)}\left(\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right) = \frac{2}{x},$$

a function of x only. The integrating factor is then

$$\Lambda(x) = \exp\left(2\int \frac{\mathrm{d}x}{x}\right) = x^2.$$

To check

$$\frac{\partial\left(\Lambda(x)P(x,y)\right)}{\partial y} = 6x^2y = \frac{\partial\left(\Lambda(x)Q(x,y)\right)}{\partial x},$$

as intended.

From here on we are on known ground.

$$\frac{\partial \Phi(x,y)}{\partial x} = \Lambda(x)P(x,y) = 4x^3 + 3x^2y^2 \quad \text{and} \quad \frac{\partial \Phi(x,y)}{\partial y} = \Lambda(x)Q(x,y) = 2x^3y.$$

$$\Phi = \int (4x^3 + 3x^2y^2)\mathrm{d}x + C(y) = x^4 + x^3y^2 + C(y).$$

$$\frac{\partial\Phi}{\partial y} = 2x^3 y + \frac{\mathrm{d}C(y)}{\mathrm{d}y} \stackrel{!}{=} 2x^3 y.$$

Hence $C(y) = C$ and the implicit solution is $x^4 + x^3 y^2 = C$, and the explicit solution is $y = \pm\sqrt{(C - x^4)/x^3}$.

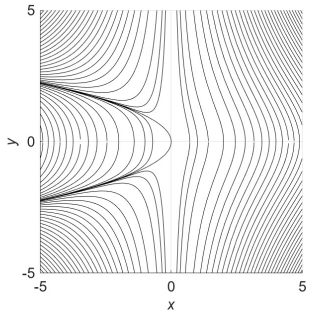


Figure 2.9: Integral lines for

$$y' = -\frac{2}{y} - \frac{3y}{2x}.$$

If $\Lambda(x, y)$ does depend on both x and y, then there is no general algorithm, and we need to make an educated guess on the integrating factor. One approach is to try polynomials or rational functions, e.g. $\Lambda(x^\alpha \pm y^\beta)$ or $\Lambda(x^\alpha y^\beta)$, etc., which may sometimes be identified by inspection. In these cases, it may be useful to first change the dependent variable from y to $z = x^\alpha \pm y^\beta$, or $z = x^\alpha y^\beta$, etc., and then examine if the integrating factor may be a function of this new variable $\Lambda(z)$.

Example 19:
Solve the equation $(x^2 + y^2 + y)\mathrm{d}x - x\mathrm{d}y = 0$.

Solution

$$P(x, y) = x^2 + y^2 + y \quad \text{and} \quad Q(x, y) = -x,$$

$$\frac{\partial P}{\partial y} = 2y + 1 \quad \text{and} \quad \frac{\partial Q}{\partial x} = -1, \text{ so no exact differential.}$$

Furthermore,

$$\frac{1}{Q(x, y)}\left(\frac{\partial P(x, y)}{\partial y} - \frac{\partial Q(x, y)}{\partial x}\right) = -\frac{2(y+1)}{x},$$
$$\frac{1}{P(x, y)}\left(\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y}\right) = -\frac{2(y+1)}{x^2 + y^2 + y},$$

and thus $\Lambda = \Lambda(x, y)$, and we are down to guessing.

We try $\Lambda = \Lambda(z)$ with $z = x^2 + y^2$. With this we want to satisfy the exactness condition

$$\frac{\partial}{\partial y}(\Lambda P) = \frac{\partial}{\partial x}(\Lambda Q),$$
$$2y\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z}P + \Lambda(z)\frac{\mathrm{d}P}{\mathrm{d}y} = 2x\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z}Q + \Lambda(z)\frac{\mathrm{d}Q}{\mathrm{d}x},$$

We use product rule and

$$\frac{\mathrm{d}\Lambda\left(x^2 + y^2\right)}{\mathrm{d}x} = 2x\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z},$$
$$\frac{\mathrm{d}\Lambda\left(x^2 + y^2\right)}{\mathrm{d}y} = 2y\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z}.$$

which gives

$$2\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z}\left(yP - xQ\right) + \Lambda(z)\left(\frac{\mathrm{d}P}{\mathrm{d}y} - \frac{\mathrm{d}Q}{\mathrm{d}x}\right) = 0.$$

The brackets evaluate to

$$yP - xQ = y(x^2 + y^2 + y) - x(-x) = (y+1)(x^2 + y^2) = (y+1)z,$$

$$\frac{\mathrm{d}P}{\mathrm{d}y} - \frac{\mathrm{d}Q}{\mathrm{d}x} = (2y+1) - (-1) = 2(y+1).$$

Hence

$$2\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z}\cancel{(y+1)}z + \Lambda(z)2\cancel{(y+1)} = 0,$$

$$\frac{1}{\Lambda(z)}\frac{\mathrm{d}\Lambda(z)}{\mathrm{d}z} = -\frac{1}{z} \implies \ln\Lambda = -\ln z + C \implies \Lambda(z) = \frac{C}{z}.$$

We choose $C = 1$ to get the integrating factor

$$\Lambda(x, y) = \frac{1}{x^2 + y^2}.$$

We now multiply the differential equation with our integrating factor

$$\frac{x^2+y^2+y}{x^2+y^2}\,\mathrm{d}x - \frac{x}{x^2+y^2}\,\mathrm{d}y = 0,$$

which results in an exact ODE. The potential function is given by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{x^2+y^2+y}{x^2+y^2} = 1 + \frac{y}{x^2+y^2}, \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}y} = -\frac{x}{x^2+y^2}.$$

We integrate $\mathrm{d}\Phi/\mathrm{d}y$ (with x constant),

$$\Phi(x,y) = -\int \frac{x}{x^2+y^2}\,\mathrm{d}y = -\arctan\left(\frac{y}{x}\right) + C(x) \quad (x \neq 0),$$

and then differentiate with respect to x and match $\mathrm{d}\Phi/\mathrm{d}x$:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{y}{x^2+y^2} + C'(x) = 1 + \frac{y}{x^2+y^2} \implies C'(x) = 1 \implies C(x) = x.$$

Thus our potential function is

$$\Phi(x,y) = -\arctan\left(\frac{y}{x}\right) + x.$$

Exactness implies $\Phi(x,y) = C$. Therefore, we get the implicit solution

$$\arctan\left(\frac{y}{x}\right) = x + C,$$

and the explicit solution

$$y(x) = x\tan(x+C), \qquad (x \neq 0).$$

The line $x = 0$ is also an integral curve of the original differential form (both terms vanish for $x = 0$) and cannot be produced by the general solution, so it is a singular solution we need to add. The integrating factor $\Lambda = 1/(x^2+y^2)$ is undefined only at $(0,0)$, so solutions are understood on domains avoiding the origin.

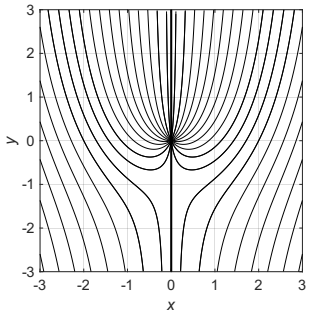


Figure 2.10: Integral lines for $(x^2+y^2+y)\mathrm{d}x - x\mathrm{d}y = 0$. The thick line is the singular solution $x = 0$.

Linear first order ODEs

Solving linear first order ODEs with an integrating factor

A linear first order ODE can be written in the form

$$y' + a(t)y = f(t). \tag{2.37}$$

For this ODE we can find an integrating factor for this ODE. We start with recasting it as

$$\mathrm{d}y + (a(t)y - f(t))\mathrm{d}t = 0. \tag{2.38}$$

Comparing this with the symmetric form, eq. (2.2), we find

$$P(t,y) = a(t)y - f(t) \quad \text{and} \quad Q(t,y) = 1. \tag{2.39}$$

We then can try (cf. eq. (2.35))

$$\frac{1}{Q(t,y)}\left(\frac{\partial P(t,y)}{\partial y} - \frac{\partial Q(t,y)}{\partial x}\right) = a(t), \tag{2.40}$$

which turns out to be a function of t only, and thus the integrating factor can be found from

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = \Lambda(t)a(t),$$
$$\Lambda(t) = \exp\left(\int a(t)\,\mathrm{d}t\right). \tag{2.41}$$

As noted before, we do not need to worry about an integration constant here, as it only provides a constant factor that cancels out from the equation.

With the integrating factor we should now be able to get an exact differential, and we can find the solution for y from

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Lambda(t)y(t)) &= \Lambda(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{\mathrm{d}\Lambda}{\mathrm{d}t}y(t) = \\ &= \Lambda(t)[f(t) - \cancel{a(t)y(t)}] + \cancel{\Lambda(t)a(t)y(t)} = \\ &= \Lambda f(t). \end{aligned} \tag{2.42}$$

and

$$y(t) = \frac{1}{\Lambda(t)}\left(\int \Lambda(t)f(t)\mathrm{d}t + C\right). \tag{2.43}$$

Note that this gives us a straightforward way to deal with a broad class of differential equations of type eq. (2.37), using first eq. (2.41) and then eq. (2.43), and no matter how complicated $a(t)$ and $f(t)$ are.

Example 20:
Solve the equation $x^2y' + 3xy = \frac{\cos x}{x}$.

Solution Let's first write this in the standard form

$$y' + \frac{3}{x}y = \frac{\cos x}{x^3}.$$

Then $a(x) = \frac{3}{x}$ and $f(x) = \frac{\cos x}{x^3}$. Thus

$$\Lambda(x) = \exp\left(\int \frac{3}{x}\mathrm{d}x\right) = x^3.$$

and

$$y(x) = \frac{1}{x^3}\left[\int \cancel{x^3}\frac{\cos x}{\cancel{x^3}}\mathrm{d}x + C\right] = \frac{\sin x + C}{x^3}.$$

If you think about $x = 0$ as possible solution, the equation itself is undefined at $x = 0$. That means $x = 0$ cannot even be part of the domain of the differential equation, hence it can't be a solution.

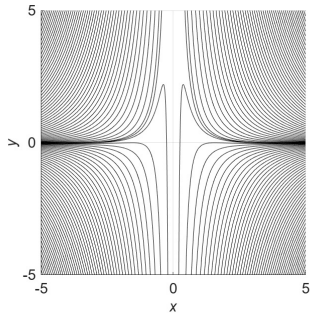


Figure 2.11: Integral lines for

$$x^2y' + 3xy = \frac{\cos x}{x}$$

.

Particular integral and complementary function

There is a different approach to solving 1st order linear differential equations, that has the benefit that we will be able to use it also on higher order differential equations. Let's start with our linear first order ODE in standard form again.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t). \tag{2.44}$$

Now let's assume that we are lucky and by some magic we can find one *particular solution* y_p for this equation. Our problem is then how we can restore the degree of freedom (the integration constant), that allows us to adapt the solution to an arbitrary starting condition. This is the point where we can exploit the linearity of the equation, which we can use to add a *complementary function* y_h, which is a general solution to the same differential equation but in its homogeneous form, i.e. with $f(t) = 0$.

Also called the particular integral.

$$\left.\begin{aligned}\frac{\mathrm{d}y_\mathrm{p}}{\mathrm{d}t} + a(t)\,y_\mathrm{p} &= f(t)\\ \frac{\mathrm{d}y_\mathrm{h}}{\mathrm{d}t} + a(t)\,y_\mathrm{h} &= 0\end{aligned}\right) +$$

$$\frac{\mathrm{d}y_\mathrm{p}}{\mathrm{d}t} + \frac{\mathrm{d}y_\mathrm{h}}{\mathrm{d}t} + a(t)\,y_\mathrm{p} + a(t)\,y_\mathrm{h} = f(t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y_\mathrm{p} + y_\mathrm{h}\right) + a(t)\left(y_\mathrm{p} + y_\mathrm{h}\right) = f(t), \tag{2.45}$$

and the general solution to the original problem is given by $y(t) = y_\mathrm{p}(t) + y_\mathrm{h}(t)$.

In practice, we often perform this approach the other way round: we find first a solution to the homogeneous problem, and then search for a particular solution, which after addition gives us the solution to the inhomogeneous problem.

For a first order linear homogeneous ODE the solution can be found from

$$\frac{\mathrm{d}y_\mathrm{h}}{\mathrm{d}t} = -a(t)\,y_\mathrm{h}, \tag{2.46}$$

which is clearly separable,

$$\int \frac{\mathrm{d}y_\mathrm{h}}{y_\mathrm{h}} = -\int a(t)\mathrm{d}t + C,$$

$$\ln|y_\mathrm{h}| = -\int a(t)\mathrm{d}t + C,$$

$$y_\mathrm{h} = C\exp\left(-\int a(t)\mathrm{d}t\right). \tag{2.47}$$

But how can we find a particular solution? One way to is to use the *method of variation of constants*. For this we replace the integration constant in y_h with an arbitrary unknown function of t, $\psi(t)$, and substitute this ansatz into the differential equation to determine this unknown function. So,

$$y_\mathrm{p} = \psi(t)\exp\left(-\int a(t)\mathrm{d}t\right). \tag{2.48}$$

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}\exp\left(-\int a(t)\mathrm{d}t\right) - \cancel{\psi(t)a(t)\exp\left(-\int a(t)\mathrm{d}t\right)} + \cancel{+\,a(t)\psi(t)\exp\left(-\int a(t)\mathrm{d}t\right)} = f(t). \tag{2.49}$$

We see two terms conveniently cancelling out, and we are left with

We have omitted the integration constant, because we only need one particular solution.

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = f(t)\exp\left(\int a(t)\mathrm{d}t\right),$$

$$\psi(t) = \int f(t)\exp\left(\int a(t')\mathrm{d}t'\right)\mathrm{d}t, \tag{2.50}$$

and thus

$$y_\mathrm{p} = \exp\left(-\int a(t)\mathrm{d}t\right)\int f(t)\exp\left(\int a(t')\mathrm{d}t'\right)\mathrm{d}t, \tag{2.51}$$

and finally we get

$$\begin{aligned} y(t) &= y_\mathrm{h} + y_\mathrm{p} = \\ &= C\exp\left(-\int a(t)\mathrm{d}t\right) + \exp\left(-\int a(t)\mathrm{d}t\right)\int f(t)\exp\left(\int a(t')\mathrm{d}t'\right)\mathrm{d}t = \\ &= \exp\left(-\int a(t)\mathrm{d}t\right)\left[\int f(t)\exp\left(\int a(t')\mathrm{d}t'\right)\mathrm{d}t + C\right], \end{aligned} \tag{2.52}$$

the identical result to eq. (2.43).

Example 21:
As in Example 16, solve the equation $x^2y' + 3xy = \frac{\cos x}{x}$, but now using a particular integral and complementary function.

Solution We again put the differential equation into standard form,

$$y' + \frac{3}{x}y = \frac{\cos x}{x^3}.$$

The we solve the homogeneous equation.

$$\begin{aligned} y_\mathrm{h}' + \frac{3}{x}y_\mathrm{h} &= 0, \\ \frac{\mathrm{d}y_\mathrm{h}}{y_\mathrm{h}} &= -\frac{3}{x}\mathrm{d}x, \\ \ln|y_\mathrm{h}| &= -3\ln|x| + C, \\ y_\mathrm{h} &= Cx^{-3}. \end{aligned}$$

Then we use the method of variation of constants to find a particular solution,

$$y_\mathrm{p}(x) = \psi(x)x^{-3},$$

plugging it into the original differential equation,

$$\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}x^{-3} - \cancel{3\psi(x)x^{-4}} + \cancel{\frac{3}{x}\psi(x)x^{-3}} = \frac{\cos x}{x^3},$$

to give

$$\psi(x) = \int \cos x \mathrm{d}x = \sin x.$$

Then $y_\mathrm{p}(x) = \sin x/x^3$, and finally

$$y(t) = y_\mathrm{h} + y_\mathrm{p} = \frac{C}{x^3} + \frac{\sin}{x^3},$$

as before.

Key lessons:

- A summary of the methods to solve 1st order ODEs:
  - Can you write the ODE as $P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = 0$? If yes,
    - Can you find a potential function Φ: Is $\partial P/\partial y = \partial Q/\partial x$? If yes, the ODE is exact. Find Φ from integration and set it constant. If no, find integrating factor Λ,
    - Is $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ a function of x only? If yes $\Lambda = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x\right)$, if no,
    - Is $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ a function of y only? If yes $\Lambda = \exp\left(\int \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}y\right)$, if no,
    - Guess Λ.
    - Potential function can be found from $\partial\Phi/\partial x = \Lambda P$ and $\partial\Phi/\partial y = \Lambda Q$. Find Φ and set constant.
  - Can you write the ODE as $y' = f(x,y)$? If yes,
    - Can you write it as $y' = f(x)/g(y)$? If yes, the equation is separable, solve by integration $\int f(x)\mathrm{d}x = \int g(y)\mathrm{d}y + C$.
    - Can you write it as $y' = f(ax+by)$? If yes, the equation is almost separable. Substitute $z(x) = ax + by(x)$, use $(\mathrm{d}z/\mathrm{d}x) = a + b(\mathrm{d}y/\mathrm{d}x) = a + bf(z)$ and solve for z, substitute back.
    - Can you write the ODE as $y' = f(x/y)$? If yes, the equation is uniform homogeneous. Substitute $z(x) = y/x$. Then $\mathrm{d}z/\mathrm{d}x = (f(z) - z)/x$ and integrate.
    - Can you write the ODE as $y' = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)$? If yes, the equation is almost homogeneous. Substitute $X = x + u$ and $Y = y + v$ and solve $a_1 u + b_1 v = c_1$ and $a_2 u + b_2 v = c_2$. Integrate $\frac{\mathrm{d}Y}{\mathrm{d}X} = f\left(\left(a_1 + b_1\frac{Y}{X}\right)/\left(a_2 + b_2\frac{Y}{X}\right)\right)$, which is homogeneous.
  - Is the ODE linear, $y' + a(t)y = f(t)$?
    - Either solve using an integrating factor. The integrating factor is $\Lambda(t) = \exp\left(\int a(t)\,\mathrm{d}t\right)$ and the solution $y(t) = (1/\Lambda(t))\left(\int \Lambda(t) f(t)\mathrm{d}t + C\right)$.
    - Or use $y(t) = y_\mathrm{p}(t) + y_\mathrm{h}(t)$, with the solution to the linear homogenous equation $y_\mathrm{h} = C\exp\left(-\int a(t)\mathrm{d}t\right)$. Find a particular solution using the method of variation of constants. which yields $y_\mathrm{p} = \exp\left(-\int a(t)\mathrm{d}t\right)\int f(t)\exp\left(\int a(t')\mathrm{d}t'\right)\mathrm{d}t$.
  - Are there any singular solutions?
    - Is the differential equation autonomous ($y' = f(y)$ only)? If yes, there will be a special solution $y(x) = C$.
    - Have we divided by a function of the variables? If yes, the divisor equal zero can correspond to a solution. Check whether it is.
    - If the special solution is not already included in the general solution, it is a singular solution.

Chapter 3
Computational methods to solve differential equations

The material in this chapter is not on syllabus.

Today, there is very powerful computer software available that can help us solve differential equations. However, to understand whether the solutions coming out of the software make sense, we still need to understand the theory and have experience ourselves in solving these differential equations. That's why you still have to pass this course.

These mathematical software packages typically give you two different approaches to solving differential equations, either symbolic (basically following the techniques we have discussed so far, although they have a much deeper library of non-elementary functions and of solutions to differential equations that cannot be solved with the basic techniques we have discussed so far), or numeric, which is particularly useful for problems that do not have solutions that can be expressed through elementary functions.

In this chapter I will give you first an introduction in symbolic solving using some commonly used software packages, and then discuss techniques to solve differential equations numerically, and finally how these are implemented in these software packages.

I am certainly not an expert in any of the used packages, so I am convinced that you will be able to create better/more efficient code. This discussion is just to give you a start.

Symbolic ODE solutions in software

Symbolic solutions in mathematical software packages

Many mathematical software packages, for example MATHEMATICA, MATLAB, or `python`, offer capabilities to handle symbolic functions, and to solve differential equations symbolically. They also have more or less accessible capabilities to plot streamline plots. In Appendix A examples for ODE solutions using all three packages are discussed. These packages will be able to apply all the methods we discussed in the previous chapter, but they also know a few more tricks that we have left out in our discussion of general approaches in the previsou chapter.

Another mathematical software package that is available through the University is MAPLE, but I know much less about that.

While all these packages will find symbolic solutions, the ODE solvers in different packages have different scope, and to fully exploit their capability, one needs to know how they work, and how their optional features can be enabled. For example, some packages give singular solutions immediately, while in other packages the search for singular solutions must be specifically requested. To understand and interpret the solutions correctly, you also need to have an educated expectation of what the

solutions should look like.

The packages will usually push for explicit solutions, even if they can't be expressed through elementary function, so that the results, while correct, are not always intuitive. Some packages give the option to provide implicit solutions, which can help in this case.

You also need to understand the exact way how the packages define functions and variables. For example, all packages use a symbolic logarithm function `log` that is defined for a complex argument. Thus, the result of $\int \mathrm{d}x/x$ is given as $\ln x$ rather than $\ln|x|$, which is the form which is actually more relevant for the real ODEs we have been looking at. Features like this, if not understood properly, can lead to interpretation of the results that miss part of the solutions. The examples in Appendix A do explore some of these issues.

See Appendix B for a more detailed discussion.

Large language models (LLMs)

AI large language models are now easily accessible and quite capable in solving differential equations. Large language models do approach the problem very much like a human would do. For example, you will see that CHATGPT will document the steps used in the derivation, and they will be very similar to our approaches in the previous chapter. For more complicated problems you will need to guide the AI through the process. For example, you will need to ask specifically for singular solutions or implicit solutions, but it usually is quite capable of providing these when prompted. Interestingly, CHATGPT's answer to the integral of $1/x$ will include the absolute value.

You can also use LLMs to create code for the software packages discussed earlier, although again, CHATGPT will make mistakes in interpreting the capabilities and limitations of the functions it employs, so you need to use the results critically.

Here I am looking at CHATGPT as the most prominent example.

They will also make mistakes, akin to human mistakes, so you will need to screen any results very carefully.

This of course makes it an attractive tool to look for guidance in solving your homework problems. There is nothing wrong with this when you are stuck, but use it wisely: learn from it, but don't let it do the work for you. It won't be around in your exam.

Numeric integration

The symbolic approaches discussed so far to some extend mimic the analytic approaches we have discussed in the previous chapter. However, where computers are irreplaceable, is in finding solutions to differential equations that cannot be solved satisfactorily symbolically,

There is a large number of different approaches to numerical solving of differential equations. This is just a very short introduction in this vast field, that it supposed to give you some first ideas about the issues, in particular around convergence, that are relevant.

The starting point will again be the 1st order Cauchy problem.

As we will discuss later, this is not a strong restriction, as we can convert higher order ODEs into systems of first order ODEs.

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = f(y(t), t) \quad \text{with} \quad y(t_0) = y_0. \tag{3.1}$$

If we want to solve this equation numerically on a digital computer this will involve some kind of a prediction of the value of the function based on discrete values of the function and its derivative (starting from the initial value). To do the calculation on a digital computer the times at which these are evaluated will be discrete.

A completely different approach to solving differential equations using electronic means is analogue computing, which has been widely used between the 1940s to the 1970s, for, among other things, to solve the equation of motions for the Apollo missions. Analogue computing avoids the issues associated with discretisation (at the cost of many other drawbacks).

Euler method

The simplest approach to numerical integration is the (forward) Euler method. We start with replacing the derivative y' in eq. (3.1) by the finite difference approximation

This method has been developed first by Leonhard Euler in 1768.

$$y'(t) \simeq \frac{y(t+h) - y(t)}{h}, \tag{3.2}$$

which can be rearranged to

$$y(t+h) \simeq y(t) + hy'(t) = y(t) + hf(y(t), t). \tag{3.3}$$

We finally discretize the independent variable, $t_0, t_1 = t_0 + h, t_2 = t_0 + 2h, \ldots$ to get

$$y_{n+1} = y_n + hf(y_n, t_n). \tag{3.4}$$

Example 22:
Use the Euler method for the initial value problem $y' = y$, $y(0) = 1$ using a step size $h = 1/2$.

Solution We have $f(y(t), t) = y$, or $f(y_n, t_n) = y_n$, so that

$$\begin{aligned}
y_1 &= y_0 + hy_0 = 1 + 0.5 = 1.5, \text{ exact: } 1.648721271,\\
y_2 &= y_1 + hy_1 = 1.5 + 0.75 = 2.25, \text{ exact: } 2.718281828,\\
y_3 &= y_2 + hy_2 = 2.25 + 1.125 = 3.375, \text{ exact: } 4.48168907,\\
y_4 &= y_3 + hy_3 = 3.375 + 1.6875 = 5.0625, \text{ exact: } 7.389056099,\\
&\vdots
\end{aligned}$$

with the exact result ($y = \exp(t)$) for comparison.

It is obvious that the Euler method is not terribly accurate for this step size. There are two main approaches to improve the accuracy. The first is to use not only one previously computed value, but to include additional points. Such methods are called *multistep methods*.

Alternatively, one can use more points in the interval $[t_n, t_{n+1}]$. This leads to the family of *Runge–Kutta methods*.

While these two are the most commonly used classes of numeric methods, other methods do exist, and are available in mathematical software packages.

Multistep methods

Almost all practical multistep methods fall within the family of linear multistep methods, which have the form

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} b_i \underbrace{f(t_{n+i}, y_{n+i})}_{f_{n+i}}, \tag{3.5}$$

where the coefficients a_i and b_i determine the specific method. This relation allows you to find y_{n+k} from the previous $y_n, y_{n+1}, \ldots, y_{n+k-1}$.

Another classification is between *explicit* and *implicit* methods. An explicit method does rely entirely on past information. Here this is achieved for $b_k = 0$. In that case the equation can directly be resolved for y_{n+k}. In implicit methods the value of y_{n+k} depends on the value of $f(t_{n+k}, y_{n+k})$, and the equation must be solved for y_{n+k}. The simplest

Solving this equation is generally not trivial, and can require significant computational resources. Nevertheless, implicit methods have advantages, as we will see below.

implicit method is the backward Euler method

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1}) \equiv y_n + hf_{n+1}. \tag{3.6}$$

Sometimes an explicit multistep method is used to "predict" the value of y_{n+k}. That value is then used in an implicit formula to "correct" the value. The result is a *predictor–corrector method*.

Example 23:
Use the two-step Adams–Bashforth method

$$y_{n+2} = y_{n+1} + \frac{3}{2}hf(t_{n+1}, y_{n+1}) - \frac{1}{2}hf(t_n, y_n)$$

for the initial value problem $y' = y$, $y(0) = 1$ using a step size $h = 1/2$.

Solution Here we have the problem that we only have one initial value, but to start the method we need values for y_0 and y_1. One way to deal with this is to use the result for y_1 from a one-step method, for example from the Euler method as used before.

$$y_2 = y_1 + \frac{3}{2}hy_1 - \frac{1}{2}hy_0 = 1.5 + \frac{3}{2} \times \frac{1}{2} \times 1.5 - \frac{1}{2} \times \frac{1}{2} \times 1 = 2.375, \text{ exact: } 2.718281828,$$
$$y_3 = y_2 + \frac{3}{2}hy_2 - \frac{1}{2}hy_1 = 2.375 + \frac{3}{2} \times \frac{1}{2} \times 2.375 - \frac{1}{2} \times \frac{1}{2} \times 1.5 = 3.7812, \text{ exact: } 4.48168907,$$
$$y_4 = y_3 + \frac{3}{2}hy_3 - \frac{1}{2}hy_2 = 3.7812 + \frac{3}{2} \times \frac{1}{2} \times 3.7812 - \frac{1}{2} \times \frac{1}{2} \times 2.375 = 6.0234, \text{ exact: } 7.389056099.$$

$\vdots$

Analysis of numeric methods

To evaluate the performance of a numeric method three criteria need to be studied:

- Consistency: How well does the method approximate the solution locally?
- Stability: Do any small flaws in the approximation (including errors in the initial conditions) blow up over time?
- Convergence: Does the outcome of the application of the method approximate the solution?

A numerical method is said to be *convergent* if the numerical solution approaches the exact solution as the step size $h \to 0$. More formally,

$$\lim_{h \to 0} \left(\max_n \left| y_n - y(t_n) \right| \right) = 0, \tag{3.7}$$

where $n = 0, 1, 2, \ldots, t_{\max}/h$ is the index of the discrete values of the independent variable for which the numeric solution is evaluated with a step size h, within the interval over which the ODE is defined, $y_{n,h}$ are the results of the numeric method at these points, and $y(t)$ are the corresponding exact solutions to the ODE at the corresponding values of the independent variable.

A numerical method is said to be *consistent* if the local truncation error tends to 0 as the step size $h \to 0$. The *local truncation error* is the

A more quantitative measure of the convergence is the *order of accuracy*, which is defined by

$$\lim_{h \to 0} \left(\max_n \left| y_n - y(t_n) \right| \right) = \mathcal{O}(h^p), \text{ for } h \to 0,$$

with p the order of accuracy. If a method is of order of accuracy p, reducing the step size by half will typically reduce the error by a factor of 2^p.

error in one step of the method, assuming that no error has been made in earlier steps,

$$\delta_n^h = y(t_n) - y_n. \tag{3.8}$$

The truncation error is thus a local property. Formally, for consistency

$$\lim_{h\to 0} \delta_n^h = 0, \text{ for all } n \text{ in the domain of the ODE.} \tag{3.9}$$

For a numerical method that has order p the local truncation error typically goes as

$$\delta_n^h = \mathcal{O}\left(h^{p+1}\right) \text{ as } h \to 0. \tag{3.10}$$

Typically here means for cases where the method is stable and the exact solution is smooth (Lipschitz-continuous).

Hence a method is consistent if it has an order greater than 0.

A numerical method is said to be *stable* if errors (from initial conditions, previous steps or numerical issues like round-off or truncation) do not grow uncontrollably as the computation proceeds, or, in other words, small perturbations in the data or computations lead to only small changes in the numerical solution. Formally,

$$\left|y_{n+1} - y(t_{n+1})\right| \le C\left|y_n - y(t_n)\right| \text{ for some constant } C. \tag{3.11}$$

A numerical method can be consistent but not stable. If a method is consistent and stable, it is convergent.

The statement that convergence is equivalent to consistency plus stability is called the *Dahlquist equivalence theorem* (sometimes also called the *Fundamental Theorem of Numerical Analysis*) after Germund Dahlquist (1925 - 2005) who developed many of the early concepts for the theory of numerical analysis as applied to differential equations.

Example 24:
Show that the (forward) Euler method is consistent and find its order. You can assume that y is twice continuously differentiable on $[a, b]$ so y'' is continuous (hence bounded) on that interval.

Solution The truncation error for step $n+1$ is given by

$$\delta_{n+1}^h = y(t_{n+1}) - y_{n+1}.$$

We use the definition of the Euler method,

$$y_{n+1} = y_n + hf(y_n, t_n),$$

and approximate the value of the function by the first elements of a Taylor expansion

$$y(t_{n+1}) \simeq y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n).$$

And thus,

$$\delta_{n+1}^h \simeq y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) - y_n - hf(y_n, t_n).$$

We use $y(t_n) = y_n$ and $y'(t_n) = f(y_n, t_n)$, to get

$$\delta_{n+1}^h = \frac{h^2}{2}y''(t_n).$$

Because y'' is continuous on the compact interval $[a, b]$ it is bounded, and thus there exists a constant M with $|y''(t)| \le M$ for all $t \in [a, b]$. Hence

$$\delta_{n+1}^h \le \frac{Mh^2}{2},$$

and

$$\lim_{h\to 0} \delta_{n+1}^h = \mathcal{O}(h^2),$$

so the Euler method is consistent and has order 1.

Similar considerations using Taylor series show that a linear multistep method of the form eq. (3.5) is consistent if and only if

$$\sum_{i=0}^{k-1} a_i = -1 \quad \text{and} \quad \sum_{i=0}^{k} b_i = k + \sum_{i=0}^{k-1} i a_i.$$

Example 25:
Show that the (forward) Euler method is stable, and thus convergent. You can assume that f is Lipschitz-continuous in y, i.e. there exists $L > 0$ with $|f(t,y) - f(t,z)| \leq L|y - z|$ for all $t \in [t_0, T], y, z$.

Solution We start with the exact relation

$$y(t_{n+1}) = y(t_n) + hf\left(t_n, y(t_n)\right) + \delta_{n+1}^h.$$

The difference of this and the result from the numeric method is

$$\begin{aligned}\Delta_{n+1} &= y(t_{n+1}) - y_{n+1} = \\ &= \left[y(t_n) + hf\left(t_n, y(t_n)\right) + \delta_{n+1}^h\right] - \left[y_n + hf(t_n, y_n)\right] = \\ &= \Delta_n + h\left[f\left(t_n, y(t_n)\right) - f(t_n, y_n)\right] + \delta_{n+1}^h.\end{aligned}$$

Now we take the norm and apply the Lipschitz continuity,

$$\begin{aligned}|\Delta_{n+1}| &= |\Delta_n| + h\left|f\left(t_n, y(t_n)\right) - f(t_n, y_n)\right| + \left|\delta_{n+1}^h\right| \leq \\ &\leq |\Delta_n| + hL|\Delta_n| + \left|\delta_{n+1}^h\right| = (1 + hL)|\Delta_n| + \left|\delta_{n+1}^h\right|.\end{aligned}$$

This is the fundamental stability inequality for the Euler equation. It shows that errors are propagated multiplicatively by at most $(1 + hL)$ in each step.

We can now iterate this result,

$$|\Delta_n| \leq (1 + hL)^n |\Delta_0| + \sum_{i=1}^{n} (1 + hL)^{n-i} \left|\delta_i^h\right|.$$

If $\Delta_0 = 0$ we can omit the first term.

From Example 24 we can use the consistency condition, $|\delta_n^h| \leq Mh^2$ for some constant M (uniform in i). We can use this for

$$\begin{aligned}|\Delta_n| &\leq Mh^2 \sum_{i=1}^{n} (1 + hL)^{n-i} = Mh^2 \frac{(1 + hL)^n - 1}{(1 + hL) - 1} = \\ &= Mh^2 \frac{(1 + hL)^n - 1}{hL} = \frac{Mh}{L}\left[(1 + hL)^n - 1\right] \leq \\ &\leq \frac{Mh}{L}\left[\lim_{n\to\infty}\left(1 + \frac{nhL}{n}\right)^n - 1\right] = \frac{M}{L}\left(e^{Lt_n} - 1\right)h,\end{aligned}$$

where we used $nh = t_n$. Thus for t_n fixed (or $t_n \leq T$) we get the global error bound

$$|\Delta_n| = \mathcal{O}(h).$$

showing that the Euler method is convergent. The critical step that allowed this is the stability inequality which prevented amplification of the local errors beyond the multiplicative factor $(1 + hL)^{n-i}$.

To understand the stability of a general multistep method we can

investigate eq. (3.5) for $y' = f = 0$. This leaves us with the homogeneous recurrence relation

$$\sum_{i=0}^{k} a_i y_{n+i} = 0. \tag{3.12}$$

By assuming $f = 0$ we focus on the part that drives the stability, and we disconnect ourselves from the specific ODE. The ODE only contributes bounded forcing terms (the right-hand side in eq. (3.5) is scaled by h, and so it goes to zero as $h \to 0$) and does not contribute to any unbounded growth. The real challenge to stability comes not from f but from how the recurrence amplifies perturbations. Thus stability established in this way is also referred to as *zero-stability*.

After applying the method m times, $y_m = \phi^m y_0$, and we get

$$\sum_{i=0}^{k} a_i \phi^n \phi^i y_0 = 0, \tag{3.13}$$

where we can divide by $\phi^n y_0$ to obtain the characteristic polynomial

$$\sum_{i=0}^{k} a_i \phi^i = 0, \tag{3.14}$$

for which we can find the roots. In the general case some of the roots might be degenerate. We only count the l distinct roots, which we label ϕ_r with $1 \le r \le l$, $l \le k$. The multiplicities for each distinct root are m_r with $\sum_i^l m_i = k$. We can then re-write the n-step solution of the recurrence relation as

$$y_n = \sum_{i=1}^{l} P_i(n) \left(\phi_i\right)^n, \tag{3.15}$$

For single (simple) roots ϕ_i (multiplicity $m_i = 1$), the contribution to the solution is $y_n^{(i)} = C_i \phi^n$, where C_i is a constant determined by initial conditions. We then sum these, exploiting linear independence of the solutions. To maintain linear independence we need to multiply degenerate solutions with a polynomial of order $m_i - 1$.

where the $P_i(n)$ are polynomials in n of order $m_i - 1$. The method is zero-stable if, and only if, the roots of the characteristic polynomial lie within the unit disk in the complex plane, with any root on the unit circle simple. This condition is known as the *root condition*.

Example 26:
Show that the two-step Adams–Bashforth method is zero-stable.

Solution The two-step Adams–Bashforth method is given by

$$y_{n+2} - y_{n+1} = \frac{3}{2} h f(t_{n+1}, y_{n+1}) - \frac{1}{2} h f(t_n, y_n).$$

Thus, the characteristic polynomial is $\phi^2 - \phi = 0$, with two solutions $\phi_1 = 0$ and $\phi_2 = 1$. The first is within the unit disk, and the second is a simple root on the unit circle. The general solution of the homogeneous recurrence is thus

$$y_n = C_1 \times 0^n + C_2 \times 1^n = C_2,$$

so solutions are bounded (indeed constant) and perturbations do not grow. The two-step Adams–Bashforth method is zero-stable.

A linear multistep method of the form eq. (3.5) has the order p if the following condition holds for the coefficients of the methods

$$\sum_{i=0}^{k-1} a_i = -1 \quad \text{and} \quad q \sum_{i=0}^{k} i^{q-1} b_i = k^q + \sum_{i=0}^{k-1} i^q a_i \quad \text{for} \quad q = 1, 2, \ldots, p. \tag{3.16}$$

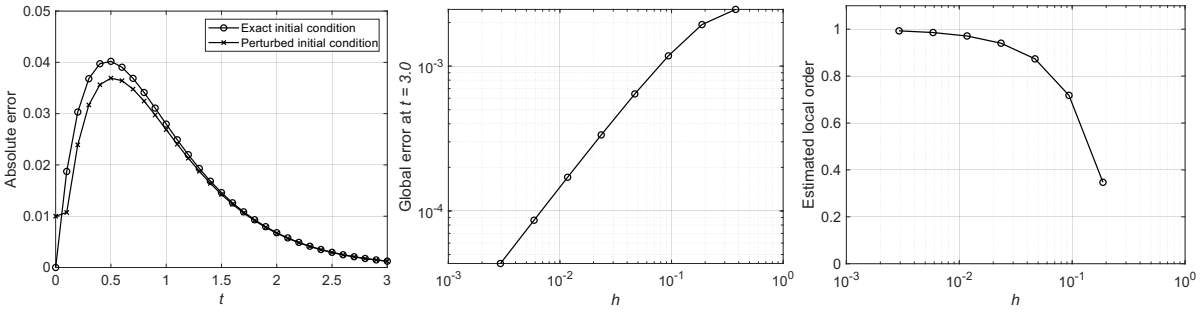


Figure 3.1: Convergence of Euler method for the initial value problem $y' = -2y$, $y(0) = 1$. Left: Absolute error vs time for $h = 0.1$, for the exact initial condition and for an initial condition biased by 0.01. Middle: Global error $y(3) - y_{3/h}$ as a function of the step size. Right: Estimate of order from slope of global error vs step size between two consecutive step sizes.

Runge-Kutta methods

These methods were developed around 1900 by the German mathematicians Carl Runge and Wilhelm Kutta.

In Runge-Kutta methods we obtain the solution at a new time t_{n+1} from a weighted average of derivatives calculated at several points in the interval from the last point at t_n. Typically this includes the slope at the beginning of the interval (k_1), and possibly some intermediate slopes (k_2, k_3,...) and an estimate of the slope at the end of the interval, which then get combined in a weighted average.

Explicit Runge-Kutta methods have the form

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i, \tag{3.17}$$

where

$$\begin{aligned}
k_1 &= f(t_n, y_n), \\
k_2 &= f(t_n + c_2 h, y_n + (a_{21} k_1) h), \\
k_3 &= f(t_n + c_3 h, y_n + (a_{31} k_1 + a_{32} k_2) h), \\
&\vdots \\
k_s &= f(t_n + c_s h, y_n + (a_{s1} k_1 + a_{s2} k_2 + \ldots + a_{s,s-1} k_{s-1}) h).
\end{aligned} \tag{3.18}$$

To specify a particular method, one needs to provide the integer s (the number of stages), and the coefficients a_{ij} (for $1 \le j < i \le s$), b_i (for $i = 1, 2, \ldots, s$) and c_i (for $i = 2, 3, \ldots, s$). These coefficients are often displayed in a 'Butcher tableau',

The matrix a_{ij} is called the Runge–Kutta matrix, while the b_i and c_i are known as the weights and the nodes.

$$\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}$$

The forward Euler method is the simplest Runge–Kutta method. It is the only consistent explicit Runge–Kutta method with one stage. The corresponding tableau is

$$\begin{array}{c|c}
0 & \\
\hline
 & 1
\end{array}$$

A widely used explicit method is called "RK4". Its tableau is

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	1/3	1/3	1/6

This method is sometimes referred to as the 'classic Runge–Kutta method' or simply as '*the* Runge–Kutta method'.

RK4 is a fourth-order numerical method.

Example 27:
Use RK4 to find the solution for the initial value problem $y' = x + y$, $y(0) = 1$ at $x = 0.1$ and compare with the exact solution.

Solution We have $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$, and we will use $h = 0.1$. The RK4 update is

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

with

$$k_1 = hf(x_n, y_n) = 0.1 \times (0 + 1) = 0.1,$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) = 0.1 \times (0.05 + (1 + 0.05)) = 0.1 \times 1.1 = 0.11,$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) = 0.1 \times (0.05 + (1 + 0.055)) = 0.1 \times 1.105 = 0.1105,$$
$$k_4 = hf(x_n + h, y_n + k_3) = 0.1 \times (0.1 + (1 + 0.1105)) = 0.1 \times 1.2105 = 0.12105.$$

and thus

$$y_1 = 1 + \frac{1}{6}[0.1 + 2 \times 0.11 + 2 \times 0.1105 + 0.12105] = 1 + \frac{1}{6} \times 0.66205 = 1.1103416\dot{6}.$$

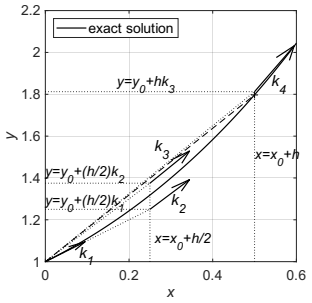


Figure 3.2: Stages for RK4 for the initial value problem $y' = x + y$, $y(0) = 1$ for $h = 0.5$.

For comparison, the exact solution can be found using the integrating factor e^{-x}:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}y\right) = xe^{-x},$$
$$e^{-x}y = \int xe^{-x}\,\mathrm{d}x = -(x+1)e^{-x} + C,$$
$$y(x) = -(x+1) + Ce^{x}.$$

We apply the initial condition $y(0) = 1 = -(0 + 1) + C$ to get $C = 2$, and thus

$$y_{\text{exact}}(0.1) = 2e^{0.1} - 0.1 - 1 \simeq 1.1103418361512953.$$

The magnitude of the difference to the RK4 result is 1.6948×10^{-7}.

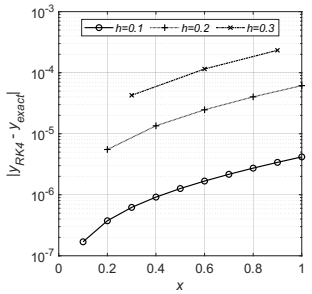


Figure 3.3: $|y_{\text{RK4}} - y_{\text{exact}}|$ for the initial value problem $y' = x + y$, $y(0) = 1$ for different step sizes.

Implicit Runge-Kutta methods have the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i, \tag{3.19}$$

where

$$k_i = f\left(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij}k_j\right), i = 1, 2, \ldots, s. \tag{3.20}$$

In an implicit Runge-Kutta method, the sum over j goes up to s and not only to i and thus the coefficient matrix is square and not triangular, yielding a Butcher tableau of the form

Note that Runga-Kutta methods are trivially zero-stable, as they are single-step methods by definition. Setting $f = h\sum_{i=1}^{s} b_i k_i = 0$ leaves $y_{n+1} = y_n$. The homogeneous recurrence is the identity map, so errors do not grow. That satisfies the root-condition notion of zero-stability.

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array} = \begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^{\mathrm{T}} \end{array}$$

The simplest implicit Runge-Kutta method is the backward Euler method. Its tableau is

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

Example 28:
The trapezoidal rule (also called implicit RK2) is a 2-stage, 2nd order implicit Runge-Kutta method with the tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

Use it to find the solution for the initial value problem $y' = x + y$, $y(0) = 1$ at $x = 0.1$ and compare with the exact solution.

Solution We have $f(x, y) = x + y$ and

$$y_{n+1} = y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2,$$

with

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f\left(x_n + h, y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2\right) = f(x_{n+1}, y_{n+1}), \end{aligned}$$

where we have used the update equation for the last equation. Thus, the trapezoidal rule can be written as

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]. \tag{3.21}$$

For our problem

$$\begin{aligned} y_1 = 1 + \frac{h}{2}\left[(x_0 + y_0) + (x_1 + y_1)\right] = \\ = 1 + 0.05 \times \left[(0 + 1) + (0.1 + y_1)\right] = 1.055 + 0.05\,y_1. \end{aligned}$$

We can solve this implicit equation to get $y_1 = 1.055/0.95 \simeq 1.11052631578947$, and the difference to the exact result is 1.8447×10^{-4}.

Stiffness and A-stability

For some differential equations standard explicit methods such as the Euler method, explicit Runge-Kutta methods and other methods discussed above, despite being fundamentally convergent, exhibit instability in the solutions, unless the step size is taken to be extremely small, even though the true solution itself might be varying slowly. Such an equation

Convergence applies in the limit $h \to 0$.

is called a *stiff* equation. A stiff equation does not need to be very complicated, and other methods may produce stable solutions for the same step size. There is no exact definition and thus no quantitative criterion for stiffness, but typically it involves processes that evolve on very different timescales (some fast, some slow). Standard explicit methods "chase" the fast, decaying dynamics and thus require tiny steps to remain stable, even if we're mostly interested in the slow behaviour. Implicit methods (both linear multistep and Runge-Kutta methods) are often more robust, and achieve convergence with larger step sizes.

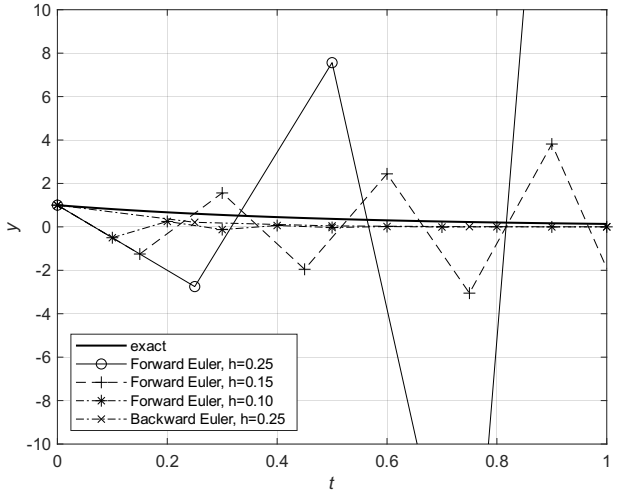


Figure 3.4: Numerical solution to the initial value problem $y' = -15y$, $y(0) = 1$ for different methods and step sizes.

Example 29:
Find a limit for the step size for the forward Euler method for the initial value problem $y' = -15y$, $y(0) = 1$. Show that no such limit applies for the backward Euler method for this ODE.

Solution Applying the forward Euler method for this ODE yields

$$y_{n+1} = y_n + h(-15y_n) = (1 - 15h)\, y_n.$$

After n steps, $y_n = (1-15h)^n y_0$, so for stability we need $|1 - 15h| < 1$, or $h < 2/15 \simeq 0.133$.

For backward Euler, on the other hand, $y_{n+1} = y_n + h(-15y_{n+1})$, and thus

$$y_{n+1} = \frac{1}{1+15h} y_n,$$

which makes y_{n+1} smaller than y_n for all $h > 0$, and the method converges for all h.

One way to analyse stiffness is to study the behaviour of the numerical method using the test ODE $y' = ky$ with $k \in \mathbb{C}$ and the initial condition $y(0) = 1$. The solution of this equation is $y(t) = e^{kt}$. In the case that $\Re(k) < 0$, this solution approaches zero as $t \to \infty$. If the numerical method also exhibits this behaviour (for any given step size), then the method is said to be *A-stable*.

For scalar 1st order ODEs with real coefficients k will always be real, but if we apply this apparatus to higher order ODEs, which equivalently can be written as a system of several 1st order ODEs, then complex k will occur. (A lot) more on this in the next chapter. The relevant part in any case is $\Re(k)$.

A-stability is stronger than stability. It implies that a stable method will produce bounded solutions independent of the step size.

The choice of test function is not arbitrary. Let's assume we have a more general (and complicated) ODE $y' = f(t, y(t))$ with a solution $y(t)$

for a given set of initial conditions. We can then look at a slightly different solution $\eta(t) = y(t) + \delta(t)$, where $\delta(t)$ is a small perturbation. For this we can write

$$\eta'(t) = f(\eta(t)) \simeq f(y(t)) + \frac{\partial f}{\partial y}(y(t))\,\delta(t), \tag{3.22}$$

where we used a Taylor expansion that we truncated after the linear term. Subtracting the equation for $y'(t) = f(y(t))$, we obtain an approximate evolution equation for the perturbation:

$$\delta'(t) \simeq \frac{\partial f}{\partial y}(y(t))\,\delta(t). \tag{3.23}$$

If the derivative is constant (or varies slowly), then this behaves like

$$\delta'(t) = k\,\delta(t), \tag{3.24}$$

which corresponds to our test ODE. k describes the local growth or decay rates of perturbations. A large (and negative) $\Re(k)$ will result in a stiff problem.

If we are studying a system of ODEs $y_i' = f_i(y_1, y_2, \ldots, y_n)$, then the derivative is replaced by the Jacobian

$$J(y) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(y) & \frac{\partial f_1}{\partial y_2}(y) & \cdots & \frac{\partial f_1}{\partial y_n}(y) \\ \frac{\partial f_2}{\partial y_1}(y) & \frac{\partial f_2}{\partial y_2}(y) & \cdots & \frac{\partial f_2}{\partial y_n}(y) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(y) & \frac{\partial f_n}{\partial y_2}(y) & \cdots & \frac{\partial f_n}{\partial y_n}(y) \end{pmatrix},$$

and the eigenvalues λ_i (which can be complex even for real components) of this matrix determine the local stability of the equilibrium: if $\Re(\lambda_i) < 0$ the solution moves towards the equilibrium, whereas for $\Re(\lambda_i) > 0$ it moves away. For the analysis of numerical methods, each eigenvalue λ_i corresponds to one scalar test equation $y' = \lambda_i y$. These are also called the *modes* of the system.

A-stability of Runge-Kutta methods

Using the test ODE $y' = ky$ on Runge-Kutta methods as defined in eqns. (3.19) and (3.20) yields for the stages

$$k_i = k\left(y_n + h\sum_{j=1}^{s} a_{ij}k_j\right). \tag{3.25}$$

We can write this equation as a matrix equation

$$\mathbf{k} = ky_n\mathbf{1} + hk\mathbf{A}\mathbf{k}, \tag{3.26}$$

where $\mathbf{k}$ is a vector of dimension s containing the k_i as components, and $\mathbf{1}$ a vector of the same dimension with every component set to 1. We can rearrange to

$$(\mathbf{I} - hk\mathbf{A})\mathbf{k} = ky_n\mathbf{1}, \tag{3.27}$$

with the $s \times s$ unit matrix $\mathbf{I}$. This equation can be solved for $\mathbf{k}$,

$$\mathbf{k} = (\mathbf{I} - hk\mathbf{A})^{-1}k\mathbf{1}\,y_n \equiv \tilde{\mathbf{k}}y_n, \tag{3.28}$$

where we have introduced $\tilde{\mathbf{k}}$, which only depends on h and k (and the Runge-Kutta coefficients a_{ij}). We can use this now in the update equation

$$\begin{aligned} y_{n+1} = y_n + h\mathbf{b}\cdot\mathbf{k} = y_n + h\mathbf{b}\cdot\tilde{\mathbf{k}}y_n = \\ = y_n\left[1 + \mathbf{b}^{\mathrm{T}}(\mathbf{I} - hk\mathbf{A})^{-1}\mathbf{1}\,hk\right] \equiv \Phi(hk)\,y_n, \end{aligned} \tag{3.29}$$

where we have introduced the *stability function* $\Phi(hk)$, which depends on the product of h and k (and the coefficients describing the specific Runge-Kutta method). Acknowledging this connection we introduce the complex variable $z = hk$ (sometimes called the *stability parameter* or the *amplification argument*).

This makes sense, because h controls the time step size (bigger h means bigger change per step), whereas k controls the growth rate of the test function (bigger k means steeper slope). For the linear test equation, the total effect in one step is thus proportional to the product hk.
z is complex because we assumed $k \in \mathbb{C}$, the step size h is real.

Repeated application of the update function gives

$$y_n = (\Phi(z))^n y_0. \tag{3.30}$$

$y_n \to 0$ for $n \to \infty$ is thus achieved if $|\Phi(z)| < 1$. The *region of absolute stability* contains all points $z \in \mathbb{C}$, for which $|\Phi(z)| < 1$. If, for a given numerical method, this region of absolute stability contains the whole region $\Re(z) < 0$, i.e. the complex left half-plane, then the method is A-stable.

Example 30:
Find the region of absolute stability for the forward Euler method and hence show that the forward Euler method is not A-stable.

Solution The forward Euler method applied to the test equation $y' = ky$ gives

$$y_{n+1} = y_n + hf(t_n, y_n) = y_n + hky_n = (1+hk)y_n.$$

Hence, $y_n = (1+hk)^n y_0$ and the stability function $\Phi(z) = 1+z$ with $z \in \mathbb{C}$. The region of absolute stability is thus given by $|\Phi(z)| = |1+z| < 1$, which is a disk in the complex plane with the centre at $z = -1$ and a radius of 1. This does not contain the whole of $\Re(z) < 0$, and thus the forward Euler method is not A-stable.

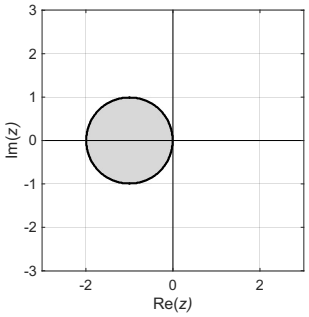


Figure 3.5: Region of absolute stability for the forward Euler method.

Example 31:
Find the region of absolute stability for the trapezoidal method and hence show that the trapezoidal method is A-stable.

Solution When we apply the update equation for the trapezoidal method, eq. (3.21), to the test equation $y' = ky$ we get

$$y_{n+1} = y_n + \frac{h}{2}\left(ky_n + ky_{n+1}\right).$$

Solving this implicit equation for y_{n+1} yields

$$y_{n+1} = \frac{1+\frac{hk}{2}}{1-\frac{hk}{2}} y_n.$$

Thus the stability function is

$$\Phi(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}.$$

The region of absolute stability is given by $|\Phi(z) < 1|$, which leads to the condition

$$\begin{aligned}
\left|1+\frac{z}{2}\right| &< \left|1-\frac{z}{2}\right|, \\
\left(1+\frac{z}{2}\right)\left(1+\frac{z^*}{2}\right) &< \left(1-\frac{z}{2}\right)\left(1-\frac{z^*}{2}\right), \\
\cancel{1} + \frac{z+z^*}{2} + \cancel{\frac{|z|^2}{4}} &< \cancel{1} - \frac{z+z^*}{2} + \cancel{\frac{|z|^2}{4}}, \\
\Re(z) &< -\Re(z),
\end{aligned}$$

which results in the condition $\Re(z) < 0$ for the region of absolute stability. This covers the requirement for A-stability and thus the trapezoidal method is A-stable.

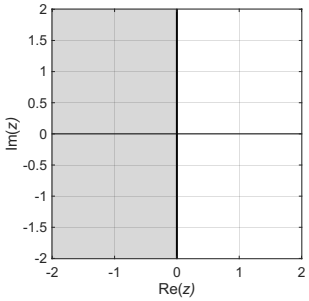


Figure 3.6: Region of absolute stability for the trapezoidal method.

While the trapezoidal method is A-stable, and does not blow up for large (and negative) k, it will not converge quickly if $z \to -\infty$, as the stability function $|\Phi(z)| = |(1/z+1/2)/(1/z-1/2)| \to 1$ in this case. A more stringent requirement would be

Remember that $y_n = (\Phi(z))^n y_0$.

$$\lim_{\Re(z)\to-\infty} \Phi(z) = 0. \tag{3.31}$$

This is called L-stability. The trapezoidal method is A-stable, but not L-stable.

Example 32:
Find the region of absolute stability for the backward Euler method and hence show that the backward Euler method is both A-stable and L-stable.

Solution For the usual test function we get $y_{n+1} = y_n + hky_{n+1}$, and thus

$$y_{n+1} = \frac{1}{1+hk} y_n,$$

and thus $\Phi(z) = (1+z)^{-1}$, and for the region of absolute stability $|1-z| > 1$, which is the whole of the complex plane, with the exception of a disk centred around $z = 1$ with radius 1. This includes $\Re(z) < 0$ for the region of stability, and thus the backward Euler method is A-stable.

It is also L-stable as

$$\lim_{\Re(z)\to-\infty} \Phi(z) = \lim_{\Re(z)\to-\infty} \frac{1}{1+z} = 0.$$

It actually satisfies the stronger condition

$$\lim_{|z|\to\infty} \frac{1}{1+z} = 0.$$

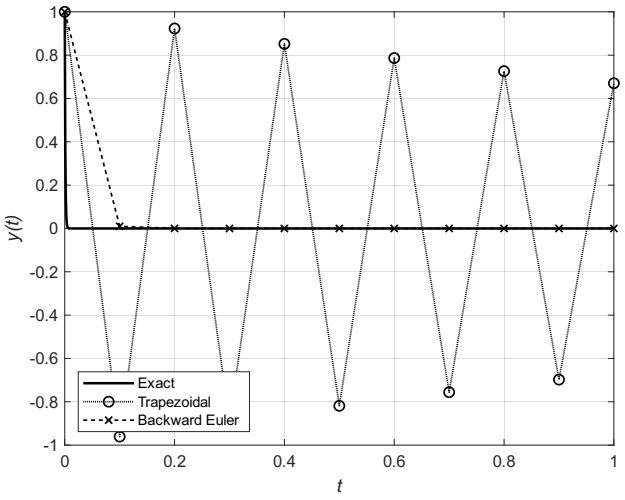


Figure 3.7: Numerical solutions to the initial value problem $y' = -1000y$, $y(0) = 1$ for $h = 0.1$.

With the numbers given $hk = z = -100$ and the stability functions are

trapezoidal: $\Phi(z) = -0.96078$,

backward Euler: $\Phi(z) = 0.00990$.

We can use eq. (3.29) to find an expression for the stability function for a general Runge-Kutta method. There we had

$$\Phi(z) = 1 + \mathbf{b}^{\mathrm{T}}(\mathbf{I} - z\mathbf{A})^{-1}\mathbf{1}z. \tag{3.32}$$

Note that $(\mathbf{I} - z\mathbf{A})$ must be invertible for this, or $\det(\mathbf{I} - z\mathbf{A}) \neq 0$. z values where the determinant is 0 are isolated points in the complex z plane, where the stability function $\Phi(z)$ has a pole.

We can now use the matrix-determinant lemma that states that

$$\det(\mathbf{M} + \mathbf{u}\mathbf{v}^{\mathrm{T}}) = \det(\mathbf{M})(1 + \mathbf{v}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{u}). \tag{3.33}$$

In our case we can associate $\mathbf{M} = \mathbf{I} - z\mathbf{A}$, $\mathbf{u} = z\mathbf{1}$, and $\mathbf{v} = \mathbf{b}$ to get

$$\Phi(z) = \frac{\det(\mathbf{I} - z(\mathbf{A} - \mathbf{1}\mathbf{b}^{\mathrm{T}}))}{\det(\mathbf{I} - z\mathbf{A})}. \tag{3.34}$$

The determinant in the numerator can also be expressed in block form

$$\det\begin{pmatrix} \mathbf{I} - z\mathbf{A} & \mathbf{1} \\ -z\mathbf{b}^{\mathrm{T}} & 1 \end{pmatrix}.$$

From this expression we can see, that $\Phi(z)$ is a rational function whose numerator and denominator are determinants (polynomials) of degree at most s.

For explicit Runge-Kutta methods the matrix $\mathbf{A}$ is strictly lower triangular and thus

$$\det(\mathbf{I} - z\mathbf{A}) = \prod_{i=1}^{s}(1 - z a_{ii}) = 1, \tag{3.35}$$

as all the coefficients $a_{ii} = 0$.

For explicit Runge-Kutta methods

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & a_{s3} & \cdots & 0 \end{pmatrix}.$$

In eq. (3.32) we can expand $(\mathbf{I} - z\mathbf{A})^{-1}$ in a series

$$(\mathbf{I} - z\mathbf{A})^{-1} = \mathbf{I} + z\mathbf{A} + z^2\mathbf{A}^2 + \cdots + z^{s-1}\mathbf{A}^{s-1}, \tag{3.36}$$

which breaks off as all terms proportional to $\mathbf{A}^s$ and higher are 0. Thus

If you multiply two strictly lower triangular matrices you get again a triangular matrix, but the number of non-zero diagonals below the main diagonal increases by at least 1 with each multiplication, so if you do this s times the matrix becomes 0.

$$\begin{aligned}\Phi(z) = 1 + z\mathbf{b}^{\mathrm{T}}\left(\mathbf{I} + z\mathbf{A} + z^2\mathbf{A}^2 + \cdots + z^{s-1}\mathbf{A}^{s-1}\right)\mathbf{1} = \\ = 1 + z\mathbf{b}^{\mathrm{T}}\mathbf{1} + z^2\mathbf{b}^{\mathrm{T}}\mathbf{A}\mathbf{1} + z^3\mathbf{b}^{\mathrm{T}}\mathbf{A}^2\mathbf{1} + \cdots + z^s\mathbf{b}^{\mathrm{T}}\mathbf{A}^{s-1}\mathbf{1}.\end{aligned} \tag{3.37}$$

This is a polynomial in z of maximum order s, thus explicit Runge-Kutta methods cannot be A-stable. Implicit methods give a rational stability function, enabling A- and possibly L-stability.

Example 33:
Find the stability function for RK4 from eq. (3.37).

Solution As a reminder, the tableau for RK4 is

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	1/3	1/3	1/6

$$\begin{aligned}\Phi(z) = 1 + z(b_1 + b_2 + b_3 + b_4) + \\ + z^2(b_1, b_2, b_3, b_4)\begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \\ + z^3(b_1, b_2, b_3, b_4)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{32}a_{21} & 0 & 0 & 0 \\ a_{42}a_{21} + a_{43}a_{31} & a_{43}a_{32} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \\ + z^4(b_1, b_2, b_3, b_4)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{43}a_{32}a_{21} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \\ = 1 + z(b_1 + b_2 + b_3 + b_4) + \\ + z^2(b_2a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43})) + \\ + z^3(b_3a_{32}a_{21} + b_4(a_{42}a_{21} + a_{43}(a_{31} + a_{32}))) + \\ + z^4 b_4 a_{43}a_{32}a_{21} =\end{aligned}$$

$$
\begin{aligned}
&= 1 + z\left(\frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6}\right) + \\
&\quad + z^2\left(\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \left(0 + \frac{1}{2}\right) + \frac{1}{6} \times (0 + 0 + 1)\right) + \\
&\quad + z^3\left(\frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{6} \times \left(0 + 1 \times \left(0 + \frac{1}{2}\right)\right)\right) + \\
&\quad + z^4 \times \frac{1}{6} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \\
&= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}.
\end{aligned}
$$

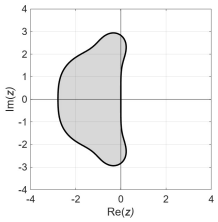


Figure 3.8: Region of absolute stability for RK4.

Note that these are the first five terms of the Taylor expansion of $\exp(z)$.

A-stability of multistep methods

We write the defining equation for a k-step linear multistep method (eq. (3.5)) as

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} b_i f(t_{n+i}, y_{n+i}). \tag{3.38}$$

We can now insert our test function $f(t_n, y_n) = k y_n$ to get

$$\sum_{i=0}^{k} a_i y_{n+i} = hk \sum_{i=0}^{k} b_i y_{n+i}. \tag{3.39}$$

Next, we assume a solution of the form $y_n = \phi^n y_0$, so that

$$\sum_{i=0}^{k} a_i \phi^n \phi^i y_0 = hk \sum_{i=0}^{k} b_i \phi^n \phi^i y_0. \tag{3.40}$$

We can cancel $\phi^n y_0$, and are left with

$$\sum_{i=0}^{k} a_i \phi^i - z \sum_{i=0}^{k} b_i \phi^i = 0, \tag{3.41}$$

We also use again $z = hk$.

which is a polynomial in ϕ of order k. We can now find the k roots ϕ_i of this equation, which represent candidates for the stability function. For stability, we require that all roots are less than 1, $|\phi_i| < 1$, and the region of absolute stability is where these conditions are met. As before, if this region contains the left half-plane $\Re(z) < 0$ the method is A-stable. Unfortunately, there is usually no analytic expression for the shape of the region of absolute stability for $k > 1$.

Example 34:
Find the roots for the stability polynomial for the second-order Adams–Bashforth method

$$y_{n+2} = y_{n+1} + \frac{3}{2} h f(t_{n+1}, y_{n+1}) - \frac{1}{2} h f(t_n, y_n),$$

and thus find the maximum step size h as a function of k, for which the method is stable.

Solution We have $k = 2$, $a_0 = 0$, $a_1 = -1$, $a_2 = 1$, $b_0 = -1/2$, and $b_1 = 3/2$. The characteristic polynomial is thus

$$-\phi + \phi^2 - z\left(-\frac{1}{2} + \frac{3}{2}\phi\right) = 0,$$

or

$$\phi^2 - \left(\frac{3z}{2} + 1\right)\phi + z\frac{1}{2} = 0,$$

which has the roots

$$\phi_{1,2} = \frac{1}{2}\left(\frac{3z}{2}+1 \pm \sqrt{\left(\frac{3z}{2}+1\right)^2 - 2z}\right) = \frac{1}{2}\left(\frac{3z}{2}+1 \pm \sqrt{\frac{9}{4}z^2+z+1}\right).$$

The region of absolute stability contains all z for which $|\phi_1(z)| < 1$ and $|\phi_2(z)| < 1$.

The largest step size can be found from the boundary of the region of absolute stability,

$$\begin{aligned}
-1 &= \frac{1}{2}\left(\frac{3z}{2}+1 \pm \sqrt{\frac{9}{4}z^2+z+1}\right),\\
-2 &= \frac{3z}{2}+1 \pm \sqrt{\frac{9}{4}z^2+z+1},\\
\frac{3z}{2}+3 &= \pm\sqrt{\frac{9}{4}z^2+z+1},\\
\left(\frac{3z}{2}+3\right)^2 &= \frac{9}{4}z^2+z+1,\\
\cancel{\frac{9}{4}z^2}+9z+9 &= \cancel{\frac{9}{4}z^2}+z+1,\\
z &= -1.
\end{aligned}$$

Thus the maximum step size is given by $h = -k^{-1}$.

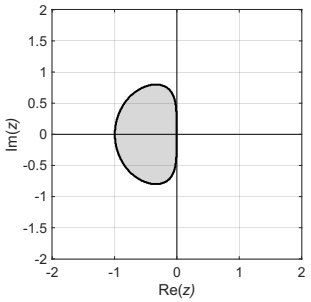


Figure 3.9: Region of absolute stability for the second-order Adams–Bashforth method.

The other condition, $1 = \phi_{1,2}$, leads to $z = 0$.

Numeric solutions of ODEs in mathematical software packages

All mathematical software packages discussed above include powerful functions for numeric integration that are capable of solving systems of differential equations. They all provide a range of different methods (explicit and implicit) and orders, and combinations of them in predictor-corrector methods, that can be selected by the user. In addition, all of them provide options to specify computational parameter like size and number of steps, precision and accuracy goals, etc.

MATHEMATICA offers the `NDSolve` function which uses adaptive Runge–Kutta methods by default, but a full range of other methods can be selected. The result is an interpolating function, which allows you to evaluate the solution at any point within the specified range and can be plotted or used for further analysis.

MATLAB includes a whole ODE solver family, like `ode45`, `ode23`, `ode15s`, etc. , which differ in the numerical method used and in numerical efficiency and stability depending on whether your problem is stiff or non-stiff, and whether you need speed or accuracy, and thus what kind of ODEs they are best suited for. The most common one is `ode45`, which uses a Runge–Kutta method. The output in all cases is a column vector of time points (chosen adaptively by the solver) and a matrix of solution values, where each row corresponds to one of the time points, and each column corresponds to one component of your system.

In `python` the modern function is `scipy.integrate.solve_ivp`. Its output is an object with fields for time points and solution values at those times. It also includes a field giving a continuous solution you can evaluate at any t, if requested (`dense_output=True`).

All three packages solve core differential problems effectively both symbolically and numerically, but there are differences in accessibility and in some aspects in performance. However, at our level it is more important that you pick the software package that you know and are comfortable with (and learn all its quirks), it will give you what you need.

Key lessons:

- Numerical methods on a digital computer calculate the solution at discrete points in time by developing the derivative along finite steps of given size of the independent variable.
- Numerical methods fall into two groups of methods:
  - Linear multistep methods have the form

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} b_i \underbrace{f(t_{n+i}, y_{n+i})}_{f_{n+i}}.$$

  - Runge-Kutta methods have the form

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i, \text{ where } k_i = f\left(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j\right), i = 1, 2, \ldots, s.$$

- Numerical methods can be
  - Explicit methods, which rely only on previous information for the calculation of a new solution point.
  - Implicit methods, where the value of y_{n+k} depends on the value of $f(t_{n+k}, y_{n+k})$. Implicit methods are computationally more demanding as they require solving for y_{n+k}.
- A convergent method equals the exact solution as the step size $h \to 0$. For a numeric method to be convergent it must be
  - Consistent, i.e. the local truncation error tends to 0 as the step size $h \to 0$.
  - (Zero-)Stable, i.e. errors from previous steps do not grow uncontrollably, formally $\left|y_{n+1} - y(t_{n+1})\right| \leq C\left|y_n - y(t_n)\right|$ for some constant C. For multistep methods to be zero-stable they need to satisfy the root condition, Runge-Kutta methods are always zero-stable.
  - The order of accuracy p is defined by $\lim_{h\to 0}\left(\max_n \left|y_n - y(t_n)\right|\right) = \mathcal{O}(h^p)$ for $h \to 0$. If a method is of order of accuracy p, reducing the step size by half will typically reduce the error by a factor of 2^p. For a numerical method of p the local truncation error typically goes as $\delta_n^h = \mathcal{O}\left(h^{p+1}\right)$ as $h \to 0$.
- Stiff differential equations are equations for which some convergent numeric methods only converge for small step sizes. For a numeric method to converge for a stiff problem independent of the step size it has to be A-stable.
- Explicit methods are usually not A-stable, implicit methods are.
- Mathematical software packages provide functions to perform symbolical or numerical solving of differential equations.

Chapter 4
Higher order differential equations

An n^{th} order differential equation

$$F\left(t, y, y', y'', ..., y^{(n)}\right) = 0 \tag{4.1}$$

can be converted into a system of n coupled first order ODEs

This system of differential equations acquires additional equations, if $y(t)$ has additional dimensions, so, for example, if it depends on three independent variables, e.g. (x, y, z), the system will encompass $3n$ first-order ODEs.

$$\begin{aligned} y' &= p_1, \\ p_1' &= p_2. \\ &\vdots \\ p_{n-2}' &= p_{n-1} \\ F\left(t, y, p_1, p_2, ...p_{n-1}, p_{n-1}'\right) &= 0, \end{aligned} \tag{4.2}$$

which need to be solved simultaneously.

If we restrict our attention to ODEs that are resolved with respect to the highest derivative,

$$y^{(n)} = f\left(t, y, y', y'', ..., y^{(n-1)}\right), \tag{4.3}$$

this, and many other systems of ODEs that do not necessarily reduce to a single higher-order equation, can be written in the general form

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}). \tag{4.4}$$

$\mathbf{y}'$ is called the 'phase velocity', while $\mathbf{y}$ is the 'phase' of the physical system. The word 'phase' comes from an older usage in physics meaning the 'state' or 'condition' of a system. In the context of Newton's 2$^{\text{nd}}$ law these are actual velocities, but for other 2$^{\text{nd}}$ order ODEs they can have other dimensions. Similar considerations apply for the 'force' terms.

where $\mathbf{y}'$ is an n-dimensional vector function, and $\mathbf{f}$ is an n-dimensional velocity field.

The initial-value problem is then to find the solution that satisfies

$$\mathbf{y}(t_0) = \mathbf{y}_0, \tag{4.5}$$

i.e. we need n initial conditions, as we noted before, to solve the initial-value problem. As each of the n first-order ODEs requires the determination of one integration constant, solving the initial value there are n integration constants, which need to be found from the n initial conditions. Similarly to the case of 1$^{\text{st}}$ order ODEs as discussed in Chapter 2, it can be shown that, with similar conditions on the ODE, for a set of initial conditions the initial value problem has one and only one solution.

While a priori $y(t)$ is a function of t that can assume any value at any t, once it is constrained by the differential equation and we know the value of the function and its $n-1$ derivatives at one time t_0, the value of the function is fixed for all values of t.

Or we find an alternative way to determine the n integration constants associated with the solution of the ODE.

Example 35:
Find the solution for the 2nd order ODE $y'' = (y')^2$.

Solution We can write the 2nd order ODE as a system of two 1st order ODEs,

$$y' = v,$$
$$v' = (y')^2 = v^2.$$

To solve this system of differential equation we can divide the second equation by the first

$$\frac{\mathrm{d}v/\cancel{\mathrm{d}x}}{\mathrm{d}y/\cancel{\mathrm{d}x}} = \frac{v^{\cancel{2}}}{\cancel{v}},$$

which is separable,

$$\frac{\mathrm{d}v}{v} = \mathrm{d}y,$$

with the solution $\ln|v| = y + C_1$ or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = C_1 e^y,$$

which is again separable,

$$e^{-y}\mathrm{d}y = C_1\,\mathrm{d}x,$$

and we get

$$-e^{-y} = C_1 x + C_2, \text{ or } y = -\ln(-C_1 x - C_2).$$

However, we need to be careful with the cancellation at the beginning. By doing this we have eliminated solutions $v = y' = 0$, or y constant. These constitute another branch of solutions (similar to singular solutions in 1st order ODEs).

Phase space

In principle, we could graphically represent the solutions to a higher-order differential equation using integral curves, as we have done for first-order ODEs. However, these would now have to be representations in $\mathbb{R}^{n+1}$. Any projection to a lower dimension means that more than one integral line goes through any point.

The "+1" is for the independent variable.

However, there is a special class of ODEs (eq. (4.1)), where f does not explicitly depend on the independent variable,

$$y^{(n)} = f\left(y, y', y'', ..., y^{(n-1)}\right), \tag{4.6}$$

In that case, the system of first-order ODEs, eq. (4.4) becomes

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \tag{4.7}$$

i.e. the change of the state ('phase') of the system for any phase only depends on the state itself. Since it is governed by a first-order differential equation, this evolution is characterised by a one-parameter family of n functions in $\mathbb{R}^n$. This sub-space is called the *phase space*, and the curves describe *phase trajectories*. A *phase portrait* is a geometric representation of the orbits of a dynamical system in phase space. For a given initial condition eq. (4.5), one phase trajectory is selected.

This is called an 'autonomous' system.

This, of course, works most conveniently for $n = 2$.

Non-linear ODEs can result in very complex. quasi-periodic trajectories in phase space ('attractors'). This is one of the topics studied in chaos theory.

A very fundamental 2nd order ODE in mechanics is

$$\ddot{x} = a(t, x, \dot{x}), \tag{4.8}$$

In its most general form this would be a 3-dimensional vector equation, but this does not make a fundamental difference for the discussion here.

which is Newton's 2nd law in the form of a differential equation. It can be split into the 1st order equations

$$\begin{aligned} \dot{x} &= \frac{p}{m}, \\ \dot{p} &= ma(t, x, \dot{x}), \end{aligned} \tag{4.9}$$

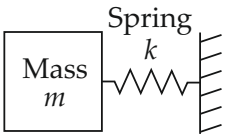
We have used $p = v/m = \dot{x}/m$, as p transforms more cleanly under coordinate changes in both classical and relativistic mechanics. x and p are canonical partners.

and ma is the force F on the object. In this case, the phase space would be spanned by the position x and the momentum p of the object.

At this point we will introduce a very important 2nd order differential equation in physics: the differential equation describing the harmonic oscillator,

$$\ddot{x} = -\omega^2 x, \tag{4.10}$$

where ω is a constant. It describes a displacement x that is forced back to the equilibrium position by a term that is linear in the displacement. This corresponds to a potential that is quadratic around the equilibrium point. This is a situation with wide applicability, as any arbitrary well behaved (continuous) potential can at least locally around minima be approximated by a parabolic potential. This differential equation therefore shows up in many different fields of physics. Here, we will introduce the equation in the context of a mass m that is connected to a base through a spring with spring constant k. In this case, from Newton's 2nd law, $ma = -kx$, and $\omega = \sqrt{k/m}$.



$\omega = \sqrt{k/m}$

Figure 4.1: Mass-spring system.

Example 36:
Find the phase portrait for the harmonic oscillator.

Solution We get

$$\begin{aligned} \dot{x} &= \frac{p}{m}, \\ \dot{p} &= -m\omega^2 x. \end{aligned}$$

If we now redefine $t \to \omega t$, $X = x$, and $P = p/(m\omega)$, we find

$$\begin{aligned} \mathrm{d}X &= P\mathrm{d}t, \\ \mathrm{d}P &= -X\mathrm{d}t. \end{aligned}$$

This equation shows the development of the phase trajectories at any given point (X, P) in phase space. It is given by the vector $(P, -X)\mathrm{d}t$, which is everywhere orthogonal to the radius vector (X, P). Thus, the phase portrait consists of a series of concentric circles in the space spanned by (X, P), or implicitly $X^2 + P^2 = C^2$. An explicit solution describing this is

$$\begin{aligned} X &= C\sin(\theta + \theta_0), \\ P &= C\cos(\theta + \theta_0), \end{aligned}$$

with two constants C and θ_0, and a variable θ.

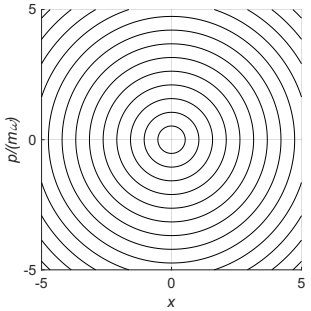


Figure 4.2: Phase portrait for $\ddot{x} = -\omega^2 x$.

We will come back to understanding the relation between θ and the independent variable t later.

The solution for the harmonic oscillator is oscillatory. Oscillatory solutions are only possible for ODEs with order larger than 1 (if we restrict ourselves to real coefficients in the ODE).

Conservative systems

Systems of the form eq. (4.6), where the only explicit dependence of f is on y, $f = f(y)$, are called *conservative systems*. Such equations are separable. For example, for Newton's 2nd law we get

$$\begin{aligned}
\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} &= f(x), \quad \Big| : (\mathrm{d}x/\mathrm{d}t) \\
\frac{\mathrm{d}v/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{f(x)}{\mathrm{d}x/\mathrm{d}t}, \quad \Big| \times m^2(\mathrm{d}x/\mathrm{d}t) \\
p\frac{\mathrm{d}p}{\mathrm{d}x} &= m^2 f(x), \quad \Big| \int \mathrm{d}x
\end{aligned} \tag{4.11}$$

$$\frac{p^2}{2m} - \int m f(x')\,\mathrm{d}x' = C_1. \tag{4.12}$$

We can, of course, associate $p^2/(2m)$ with the kinetic energy, $-\int m f(x)\,\mathrm{d}x$ with the potential energy, and then relabel the integration constant as E, the total energy of the system.

An implicit solution for x we can then get from the first-order equation

$$\begin{aligned}
p &= \pm\left[2m\left(C_1 + \int m f(x')\,\mathrm{d}x'\right)\right]^{1/2}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \pm\left[2\left(C_1 + \int f(x')\,\mathrm{d}x'\right)\right]^{1/2}, \\
\int \mathrm{d}t = t &= \pm\int\left[2\left(C_1 + \int f(x')\,\mathrm{d}x'\right)\right]^{-1/2}\mathrm{d}x + C_2.
\end{aligned} \tag{4.13}$$

Note that the integration here adds another integration constant for a total of two, as expected from a second-order ODE.

We can use this to find $x(t)$ for Example 18,

$$\begin{aligned}
t + C_2 &= \pm\int \frac{\mathrm{d}x}{2C_1 - 2\int \omega^2 x'\mathrm{d}x'} = \\
&= \pm\frac{1}{\sqrt{2C_1}}\int \frac{\mathrm{d}x}{\sqrt{1 - \frac{\omega^2 x^2}{2C_1}}} = \\
&= \pm\frac{1}{\omega}\int \frac{\mathrm{d}u}{\sqrt{1+u^2}} = \\
&= \pm\frac{1}{\omega}\arcsin u = \\
&= \pm\frac{1}{\omega}\arcsin\left(\frac{\omega x}{\sqrt{2C_1}}\right).
\end{aligned} \tag{4.14}$$

which can be recast into the explicit form

$$x(t) = C_1 \sin(\omega t + C_2). \tag{4.15}$$

Example 37:
Find the phase portrait for the mathematical pendulum, which is described by $\ddot{\theta} = -k\sin\theta$.

A mathematical pendulum is a frictionless (undamped) pendulum. From Newton's 2nd law we can find that $k = g/l$, where l is the length of the pendulum, and g the gravitational acceleration.

Solution

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -k\sin\theta, \quad \Big| : (\mathrm{d}\theta/\mathrm{d}t)$$
$$\frac{\mathrm{d}\omega/\cancel{\mathrm{d}t}}{\mathrm{d}\theta/\cancel{\mathrm{d}t}} = \frac{\mathrm{d}\omega}{\mathrm{d}\theta} = -\frac{k\sin\theta}{\omega}, \quad \Big| \int \mathrm{d}\theta$$
$$\frac{\omega^2}{2} = -\int k\sin\theta\,\mathrm{d}\theta + C = k\cos\theta + C,$$
$$\omega = \pm\sqrt{2(k\cos\theta + C)}.$$

We can again write this as an equation for conserved energy

$$\frac{m\omega^2 l^2}{2} - mgl\cos\theta = E.$$

Note that this is not a solution of the movement of the pendulum in the time domain.

The solution in the time domain can only be given in quadratures, involving elliptical integrals, but not by elementary functions.

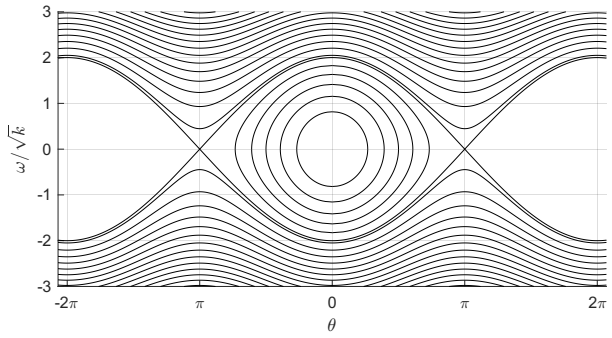


Figure 4.3: Phase portrait for $\ddot{\theta} = -k\sin\theta$. The thick lines correspond to $C = k$.

The lowest allowed value for C is $-k$, which corresponds to the pendulum being stationary at the nadir. As C increases the motion first resembles the harmonic oscillator from Example 18. As it approaches $C = k$ the maximum velocity does not linearly increase with the deflection, and for $C = k$ there are unstable equilibrium points at $\omega = \pm\pi$. These are the only points where phase trajectories intersect, the pendulum can drop either way. For $C > k$ the pendulum will rotate around and maintain this motion with slight variations of the angular velocity, and θ will increase without bounds.

The solution for $C = k$, which separates oscillatory motion from circular motion is called the *separatrix*.

Non-conservative systems

In second-order non-conservative systems, the ODE will depend not only depend on the dependent variable, but also its derivative. Eq. (4.11) will still apply (with the modification that $p(\mathrm{d}p/\mathrm{d}x) = f(x,p)$), which is still a first-order differential equation for the phase space evolution, the solution of which is still a one-parameter family of curves in phase space. However, the energy of the system will not correspond to an integration constant any more.

Example 38:
Find the phase portrait for the damped harmonic oscillator $\ddot{x} = -kx - \gamma\dot{x}$, where k and γ are constants describing the restoring and damping forces, respectively, and $\gamma < \sqrt{k}$ (weak damping).

Solution We divide the ODE by $v = \mathrm{d}x/\mathrm{d}t$ to get

$$\frac{\mathrm{d}v/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-kx-\gamma v}{v} = -k\frac{x}{v} - \gamma,$$

which is a homogeneous first-order ODE, so we substitute $z = v/x$ for which

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\frac{\mathrm{d}v}{\mathrm{d}x} - z}{x} = -\frac{k/z + \gamma + z}{x}.$$

Thus

$$\int \frac{z\,\mathrm{d}z}{z^2+\gamma z+k} = -\int \frac{\mathrm{d}x}{x} + C.$$

The left-side integral requires a bit of work,

$$\int \frac{z\,\mathrm{d}z}{z^2+\gamma z+k} = \int \frac{z\,\mathrm{d}z}{\left(z+\frac{\gamma}{2}\right)^2 + k - \frac{\gamma^2}{4}} =$$
$$= \int \frac{u\,\mathrm{d}u}{u^2+D} - \frac{\gamma}{2}\int \frac{\mathrm{d}u}{u^2+D} = \frac{1}{2}\ln\left|u^2+D\right| - \frac{\gamma}{2\sqrt{D}}\arctan\left(\frac{u}{\sqrt{D}}\right),$$

where we used the substitutions $u = z + \gamma/2$, and $D = k - \gamma^2/4$, so that

$$\frac{1}{2}\ln\left|u^2+D\right| - \frac{\gamma}{2\sqrt{D}}\arctan\left(\frac{u}{\sqrt{D}}\right) = -\ln|x| + C,$$

and after re-substitution

$$\ln\left|v^2+\gamma x+kx^2\right| = \frac{\gamma}{2\sqrt{k-\frac{\gamma^2}{4}}}\arctan\frac{\frac{v}{x}+\frac{\gamma}{2}}{\sqrt{k-\frac{\gamma^2}{4}}} + C.$$

$$v^2+\gamma x+kx^2 = C\exp\left(\frac{\gamma}{\sqrt{k-\frac{\gamma^2}{4}}}\arctan\frac{\frac{v}{x}+\frac{\gamma}{2}}{\sqrt{k-\frac{\gamma^2}{4}}}\right). \qquad (4.16)$$

Note that v^2+kx^2 is not a constant but will decline, mostly due to the $-\gamma x$ term. In the limit $\gamma \to 0$ we recover the result for the undamped oscillator, $v^2 + kx^2 = C$.

A parametric solution with the parameter u can be written as

$$x(u) = \frac{C}{\sqrt{u^2+D}}\exp\left[\frac{\gamma}{2\sqrt{D}}\arctan\left(\frac{u}{\sqrt{D}}\right)\right],$$
$$v(u) = z(u)x(u) = \left(u - \frac{\gamma}{2}\right)x(u),$$

and the complete integral curve can be stitched together (because of the periodicity of the arctan function), from segments with changing $C \to qC$, with $q = -\exp\left(\pi\gamma/(2\sqrt{D})\right)$.

A damped harmonic oscillator includes a velocity-dependent damping term. In practice, this can be achieved using a viscous damper, which converts kinetic energy into heat through the flow of a viscous fluid across orifices or through shear action. For moderate speeds the damping force is proportional to the velocity.

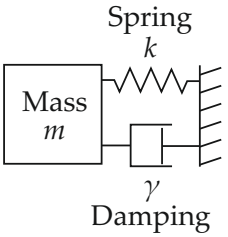


Figure 4.4: Damped mass-spring system.

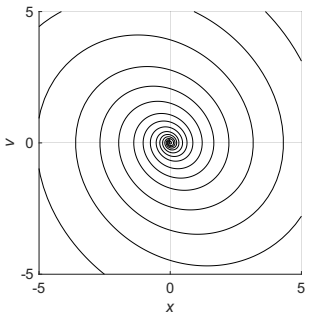


Figure 4.5: Phase portrait for the damped harmonic oscillator $\ddot{x} = -kr - \gamma\dot{x}$, with $k = 1$ and $\gamma = 0.3$.

The phase trajectories are not closed circles any more, but spiral towards the point (0,0). This is a *stationary point* ($\ddot{x} = \dot{x} = 0$). This point has not changed with damping. In the damped case all the phase trajectories end in this point. The damped oscillator is an example of a *dissipative system*.

Stationary points are sometimes also called *fixed points*.

Perturbative solutions

While we have been able to find an implicit analytic solution for the phase space trajectories for the damped oscillator that solution is not really very intuitive and for many non-conservative problems we will not even be able to find such an analytic solution. However, if the non-conservative contributions are relatively small, then treating them as a perturbation is sometimes a useful approach. In that case we write the ODE $\ddot{x} = f(x, \dot{x})$ as

$$v\frac{\mathrm{d}v}{\mathrm{d}x} - f(x) = f(v), \tag{4.17}$$

where we have split the function $f(x, \dot{x})$, in the part leading to a conservative solution, $f(x) = -kx$, and the non-conservative part $f(v) = -\gamma v$. Then we can write

$$\frac{\mathrm{d}}{\mathrm{d}x}\underbrace{\left(\frac{p^2}{2m} + V(x)\right)}_{E} = mf(v), \tag{4.18}$$

again with the potential energy $V(x) = -\int mf(x)\,\mathrm{d}x$. If $f(v) \ll f(x)$, then we should be able to compose the solution as

$$v(x) = v_0(x) + \delta v(x), \tag{4.19}$$

where $v_0(x)$ is the conservative solution which we hopefully can obtain analytically, and $\delta v(x)$ a correction, which we hope will be small, $\delta v(x) \ll v_0(x)$.

For the harmonic oscillator the energy in the conservative case is $E = mv_0^2/2 + mkx^2/2$, and with damping $\mathrm{d}E/\mathrm{d}x \simeq -m\gamma v_0$. We can then calculate the energy loss over one cycle of the oscillator as

$$\begin{aligned}\Delta E \simeq \int_{\text{one cycle}} \frac{\mathrm{d}E}{\mathrm{d}x}\,\mathrm{d}x \simeq -m\gamma \oint v_0(x)\,\mathrm{d}x = \\ = -m\gamma\sqrt{k}\oint \frac{v_0(x)\,\mathrm{d}x}{\sqrt{k}} = -m\gamma\pi\sqrt{k}\,\frac{2E}{km} = -\frac{2\pi\gamma}{\sqrt{k}}E,\end{aligned} \tag{4.20}$$

where we used the fact that $\oint (v_0(x)/\sqrt{k})\,\mathrm{d}x$ gives the area of the path in phase space, which for the conservative system is a circle in the $(x, v/\sqrt{k})$ plane (see Figure 4.2) with a radius of $\sqrt{2E/(mk)}$.

In terms of the polar angle θ in phase space, we can write the energy loss as

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}\theta} \simeq \frac{\Delta E}{2\pi} = \frac{\gamma}{\sqrt{k}}\langle E\rangle, \tag{4.21}$$

or

$$\langle E\rangle = E_0 \exp\left(-\frac{\gamma}{\sqrt{k}}\theta\right) = E_0\exp\left(-\gamma t\right), \tag{4.22}$$

where we obtained the result in the time domain, from eq. (4.15), where $\omega = \sqrt{k}$ and thus $\theta = \omega t = \sqrt{k}\,t$.

This approximation gives a more intuitive result than eq. (4.16): the system follows a spiral in phase space, contracting exponentially in time to the stationary point of the system, with a time constant that equals the damping constant γ.

The symbol $\oint$ denotes a closed loop integral. Such an integration is performed along a closed curve, meaning the start and end points of the curve are identical (after one cycle the system reaches the same point in phase space).

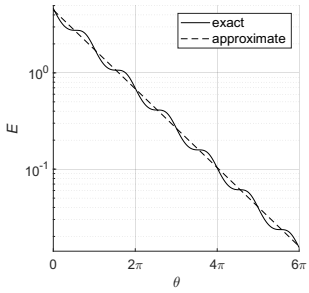


Figure 4.6: Exact solution for $v^2/2 + kx^2/2$ from eq. (4.16) with $k = 1$ and $\gamma = 0.3$, and the approximative perturbative solution eq. (4.22) .

An alternative approach builds on the solution for the undamped oscillator in the time domain, eq. (4.15). We start again with

$$E = \frac{mv_0^2}{2} + \frac{mkx^2}{2}, \tag{4.23}$$

and use

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \simeq \frac{\mathrm{d}E}{\mathrm{d}x} v_0 = -m\gamma v_0^2. \tag{4.24}$$

Next thing we average again over a period, to get

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} \simeq -m\gamma\langle v_0^2\rangle. \tag{4.25}$$

From eq. (4.15) for the undamped system $x_0 = C_1\sin(\omega t + C_2)$, and thus $v_0 = C_1\omega\cos(\omega t + C_2)$, and

$$\langle v_0^2\rangle = C_1^2\omega^2\langle\cos^2(\omega t)\rangle = \frac{C_1^2\omega^2}{2}. \tag{4.26}$$

$$\langle\cos^2 x\rangle = \frac{1}{2}\langle 1-\cos(2x)\rangle = \\ = \frac{1}{2T}\int_0^T (1-\cos(2x))\,\mathrm{d}t = \\ = \frac{1}{2}(1+0) = \frac{1}{2}.$$

But at the same time,

$$\langle E\rangle \simeq \frac{1}{2}mv_0^2 + \frac{1}{2}mkr_0^2 = \frac{C_1^2 m}{2}\left[\omega^2\cos^2(\omega t + C_2) + k\sin^2(\omega t + C_2)\right] = \\ = \frac{C_1^2 m\omega^2}{2}\left[\cos^2(\omega t + C_2) + \sin^2(\omega t + C_2)\right] = \frac{C_1^2 m\omega^2}{2}, \tag{4.27}$$

and thus

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} \simeq -\gamma\langle E\rangle, \tag{4.28}$$

and the previous result follows again.

More general second-order ODEs usually cannot be solved analytically. Depending on your needs and means there are then three different approaches: First, one can gain a qualitative insight in the behaviour of the solutions by identifying the stationary points and approximating the differential equation there. Once the behaviour around the stationary points is known, one can interpolate between them, often using perturbative arguments. Second, one can plot the direction field without solving the ODE. Third, we can use numeric methods as discussed in the previous chapter.

Example 39:
Sketch the phase trajectories for the damped pendulum: $\ddot\theta = -\gamma\dot\theta - k\sin\theta$, where $k = g/l$, with l the length of the pendulum, g the gravitational acceleration, and $\gamma \ll \sqrt{k}$.

Solution This is an autonomous ODE, but

$$\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = -\gamma - \frac{k\sin\theta}{\omega}$$

does not have a closed-form solution.

The ODE has stationary points ($\theta = \dot\theta = 0$) at $\omega = 0$ and $\theta = n\pi$ with n an arbitrary integer, as in the undamped case. To examine the trajectories near these stationary points, we substitute $\theta = n\pi + \delta\theta$ to get

$$\delta\ddot\theta = \begin{cases} -\gamma\delta\dot\theta - k\delta\theta, & \text{close to } \theta = 2n\pi, \\ -\gamma\delta\dot\theta + k\delta\theta, & \text{close to } \theta = (2n+1)\pi, \end{cases}$$

We are using the small-angle approximation $\sin\delta\theta \simeq \delta\theta$ for small $\delta\theta$.

where n is an arbitrary integer.

The first of these equations is the equation for a damped harmonic oscillator, as in the previous section: close to the stationary points at $\theta = 2n\pi$, the phase trajectories will behave as in the previous section.

The second equation has a different sign for the restoring force term, which results in the trajectory being driven away exponentially from the stationary point. It is an unstable equilibrium point.

In both regions the perturbative reduction of the energy applies,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\underbrace{\left(\frac{m\omega^2 l^2}{2} + mkl^2\cos\theta\right)}_{E(\theta,\omega)} = -\gamma\omega. \tag{4.29}$$

and thus again,

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \frac{\mathrm{d}\langle E\rangle}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\gamma\langle\omega^2\rangle \simeq -\gamma\langle\omega_0^2\rangle. \tag{4.30}$$

In the weak damping case, the phase trajectories will follow the unperturbed trajectories on short time scales, but will lose energy on longer time scales and thus spiral towards one of the stable stationary points.

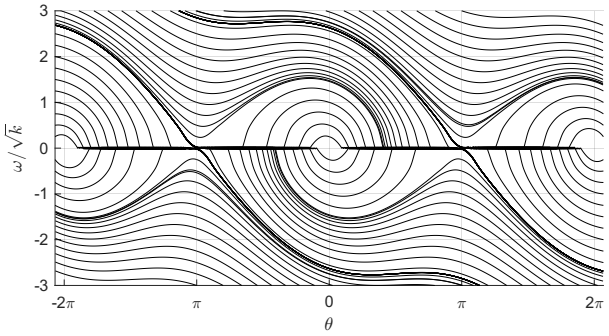


Figure 4.7: Phase portrait for $\ddot{\theta} = -k\sin\theta$. The line at $\omega = 0$ and the change of density of the trajectories at this line is an artefact of the way the streamline plot works in MATLAB.

Linear ODEs

As we have seen at the beginning of this chapter, to allow us to solve the initial value problem for an n^{th} order ODE we need n integration constants. This implies that there need to be n independent solutions to the ODE.

If there would be, let's say, only one solution $f(x)$, then all solutions would look like $C_i f(x)$, and the overall solution $(C_1 + C_2 + ...C_n)f(x) = Cf(x)$ would not give us enough degrees of freedom.

Homogeneous linear ODEs

We will first focus on homogeneous linear ODEs,

$$a_n(x)y^{(n)}(x) + ... + a_1(x)y'(x) + a_0(x)y(x) = 0. \tag{4.31}$$

If we have two solutions y_1 and y_2 for such a linear homogeneous ODE, then due to the linearity of the ODE the linear combination $y = C_1 y_1 + C_2 y_2$ with two arbitrary constants C_1 and C_2 that can be complex, is also a solution,

$$\left.\begin{array}{l|l} a_n(x)y_1^{(n)}(x) + ... + a_1(x)y_1'(x) + a_0(x)y_1(x) = 0 & \times C_1 \\ a_n(x)y_2^{(n)}(x) + ... + a_1(x)y_2'(x) + a_0(x)y_2(x) = 0 & \times C_2 \end{array}\right\} +$$

$$a_n(x)\left(C_1 y_1^{(n)}(x) + C_2 y_2^{(n)}(x)\right) + ... + a_1(x)\left(C_1 y_1'(x) + C_2 y_2'(x)\right) + \\ + a_0(x)\left(C_1 y_1(x) + C_2 y_2(x)\right) = 0, \tag{4.32}$$

$$a_n(x)\left(C_1y_1(x)+C_2y_2(x)\right)^{(n)}+\ldots+a_1(x)\left(C_1y_1(x)+C_2y_2(x)\right)'+ \\ +a_0(x)\left(C_1y_1(x)+C_2y_2(x)\right)=0, \quad (4.33)$$

where we have made use of the linearity of the differential operation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(af(x)+bg(x)\right)=a\frac{\mathrm{d}f(x)}{\mathrm{d}x}+b\frac{\mathrm{d}g(x)}{\mathrm{d}x}. \quad (4.34)$$

Fundamental systems of solutions of homogeneous linear ODEs

The solutions to an n^{th} order linear homogeneous ODE therefore constitute an n-dimensional vector space, and any solution can be expressed by a linear combination of a basis of n linearly independent solutions to the ODE.

Such a basis is sometimes called a *fundamental system.*

How can we verify that a set of solution is indeed linearly independent? A set of solutions is linearly independent if

$$C_1y_1(x)+C_2y_2(x)+\ldots+C_ny_n(x)=0, \\ \text{for all } x \text{ only if } C_1=C_2=\ldots=C_n=0. \quad (4.35)$$

If this condition is not fulfilled, we can write for example

$$y_n=-\frac{C_1}{C_n}y_1-\frac{C_2}{C_n}y_2-\ldots-\frac{C_{n-1}}{C_n}y_{n-1}$$

and one solution can be expressed as a linear combination of the others.

For $n=2$ this is a trivial question, but for higher n this can be less obvious, and we will use a more formal approach, which will have additional benefits. We did already note that the solutions to a linear ODE behave like vectors in a vector space of functions. In linear algebra a set of vectors $\mathbf{v}_i$ is linearly independent if

$$C_1\mathbf{v}_1+C_2\mathbf{v}_2+\ldots+C_n\mathbf{v}_n=0, \text{ only if } C_1=C_2=\ldots=C_n=0. \quad (4.36)$$

In matrix notation

$$(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)\,\mathbf{c}=\mathbf{Y}\mathbf{c}=\mathbf{0}, \text{ only if } \mathbf{c}=\mathbf{0}, \quad (4.37)$$

where $\mathbf{c}$ is the column vector with the C_i coefficients and $\mathbf{Y}$ an $n\times n$ matrix with the vectors $\mathbf{v}_i$ as columns. In that case linear independence holds if $\det\mathbf{Y}\neq 0$.

$\det\mathbf{Y}\neq 0$ implies that the matrix is invertible.

We can construct a similar criterion for linearly independent functions. We have seen in eqns. (4.4) and (4.5) that a solution to the ODE for all values of the independent variable can be written as a row vector of the function and its $n-1$ derivatives. If we combine the n row vectors for the n solutions we obtain again an $n\times n$ matrix

$$\mathbf{Y}=\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}.$$

As before, the solutions are linearly independent if the Wronskian $W=\det\mathbf{Y}\neq 0$. This implies that if the Wronskian is non-zero somewhere within an interval the functions are linearly independent somewhere in this interval.

This determinant is called the *Wronskian* after the Polish mathematician Józef Maria Hoene-Wroński (1776-1853).

The inverse statement is not sufficient. For example, the Wronskian for x^2 and $|x|x$ disappears, but the functions are linearly independent

for a region around 0. To prove linear dependence the most robust tactic is to demonstrate a non-trivial linear relation of the functions. If $W = 0$, we need to fulfil additional requirements, for example that the functions are analytic on a connected domain, and are solutions of the same linear homogeneous ODE with continuous coefficients (which x^2 and $|x|x$ are not).

Example 40:
Show that the functions $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent.

Solution

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2.$$

This is not zero except at $x = 0$, hence the functions are linearly independent.

Example 41:
Show that $\sin\theta$, $\cos\theta$, and $\exp(\mathrm{i}\theta)$ are linearly dependent.

Solution We know, of course, that $\cos\theta + \mathrm{i}\sin\theta = \exp(\mathrm{i}\theta)$, so the functions are linearly dependent.

Alternatively, we can investigate the Wronskian

$$W = \begin{vmatrix} \sin\theta & \cos\theta & \exp(\mathrm{i}\theta) \\ \cos\theta & -\sin\theta & \mathrm{i}\exp(\mathrm{i}\theta) \\ -\sin\theta & -\cos\theta & -\exp(\mathrm{i}\theta) \end{vmatrix} =$$

$$= \exp(\mathrm{i}\theta)\left[\left(\sin^2\theta + \mathrm{i}\sin\theta\cos\theta\right) + \left(-\mathrm{i}\sin\theta\cos\theta + \cos^2\theta\right) + \right.$$
$$\left. + \left(-\sin^2\theta - \cos^2\theta\right)\right] = 0.$$

In addition, $\sin\theta$, $\cos\theta$, and $\exp(\mathrm{i}\theta)$ are all solutions to the linear homogeneous ODE $y''(\theta) + y(\theta) = 0$, and thus we can conclude that the functions are linearly dependent.

Finding the linear combination coefficients

Any solution to the ODE can be expressed in terms of the functions constituting the fundamental system with the coefficients C_i as

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t) = \mathbf{y}^{\mathrm{T}}(t)\mathbf{c}. \tag{4.38}$$

To find these coefficients for a set of starting conditions y_0 we can write eq. (4.5) as

$$\mathbf{y}_0 = \mathbf{y}(t_0) = \mathbf{Y}(t_0)\mathbf{c}. \tag{4.39}$$

We thus can obtain the coefficients from

$$\mathbf{c} = \mathbf{Y}^{-1}(t_0)\mathbf{y}_0. \tag{4.40}$$

and the solution can be written as

$$\mathbf{y}(t) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\mathbf{y}_0 \tag{4.41}$$

and

$$y(t) = \mathbf{y}^{\mathrm{T}}(t)\mathbf{Y}^{-1}(t_0)\mathbf{y}_0. \tag{4.42}$$

The inverse of $\mathbf{Y}$ exists if $W = \det\mathbf{Y} \neq 0$.

For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and thus for a 2^{nd} order linear homogeneous ODE with two solutions $y_1(t)$ and $y_2(t)$

$$\mathbf{Y}^{-1}(t) = \frac{1}{\mathbf{W}} \begin{pmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{pmatrix}.$$

The interpretation of this formula is that the initial condition $\mathbf{y}_0$ gets decomposed – by multiplication by $\mathbf{Y}^{-1}(t_0)$ into components corresponding to each member of the fundamental system (projected onto the basis vectors of the vector space of solutions) – and then by multiplication with $\mathbf{Y}(t)$ each member is forwarded in time along its respective basis vector $y_i(t)$.

We could, of course, also have avoided all this matrix algebra, and simply written the n initial conditions required to determine the n constants C_i,

$$\begin{aligned} y(t_0) = y_0 &= C_1 y_1(t_0) + C_2 y_2(t_0) + \cdots + C_n y_n(t_0), \\ y'(t_0) = y'_0 &= C_1 y'_1(t_0) + C_2 y'_2(t_0) + \cdots + C_n y'_n(t_0), \\ &\vdots \\ y^{(n)}(t_0) = y^{(n)}_0 &= C_1 y^{(n)}_1(t_0) + C_2 y^{(n)}_2(t_0) + \cdots + C_n y^{(n)}_n(t_0), \end{aligned} \tag{4.43}$$

but this is just the system eq. (4.39) describe.

Example 42:
Find the solution for the harmonic oscillator $y'' + ky = 0$ with the initial conditions $y(0) = y_0$ and $y'(0) = 0$.

There are many ways to solve the ODE for the harmonic oscillator, and this is not the easiest/most efficient, but it is here to illustrate the formalism developed above.

Solution We can write the 2nd order ODE as a system of two coupled 1st order ODEs

$$\begin{aligned} y'(t) &= v(t), \\ v'(t) &= -ky(t), \end{aligned}$$

which we have to solve simultaneously. We can see that $y'(t) \propto v(t)$ and $p'(t) \propto y(t)$. It should thus be possible to find a linear combination $z(t) = v(t) + \alpha y(t)$ for which $z'(t) \propto z(t)$ or $z'(t) = \lambda z(t)$, an ODE which we know how to solve, with two suitable constants α and λ. We find these from

$$z'(t) = v'(t) + \alpha y'(t) = -ky(t) + \alpha v(t) = \alpha\left(v(t) - \frac{k}{\alpha} y(t)\right),$$

which we want to satisfy the first order ODE

$$z'(t) = \lambda z(t) = \lambda\left(v(t) + \alpha y(t)\right),$$

which leads to $\lambda = \alpha$ and $\alpha = -k/\alpha$, where the latter gives $\alpha_{1,2} = \pm \mathrm{i}\sqrt{k} \equiv \pm \mathrm{i}\omega$.

We get two decoupled 1st order ODEs

$$\begin{aligned} z'_1(t) &= \mathrm{i}\omega z_1(t), \\ z'_2(t) &= -\mathrm{i}\omega z_2(t), \end{aligned}$$

which are separable, with the solutions $z_1(t) = C_1 \exp(\mathrm{i}\omega t)$ and $z_2(t) = C_2 \exp(-\mathrm{i}\omega t)$, with the complex integration constants C_1 and C_2. As $\alpha_1 = -\alpha_2$, the $z_{1,2}(t)$ are complex conjugates, and $C_2 = C_1^*$ with two real degrees of freedoms (the real and the imaginary part). We can now find the general solution for $y(t)$ from

$$\begin{aligned} y(t) = \frac{z_1(t) - z_2(t)}{2\mathrm{i}\omega} &= \frac{C_1 \exp(\mathrm{i}\omega t) - C_1^* \exp(-\mathrm{i}\omega t)}{2\mathrm{i}\omega} = \\ &= \frac{\Re(C_1)}{\omega} \sin(\omega t) + \frac{\Im(C_1)}{\omega} \cos(\omega t) = \\ &= C'_1 \cos(\omega t) + C'_2 \sin(\omega t), \end{aligned}$$

with two new real constants C_1' and C_2', which we find from the initial conditions. We have found two linearly independent solutions for the ODE, $y_1(t) = \cos(\omega t)$ and $y_2(t) = \sin(\omega t)$ and

$$\mathbf{C} = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}, Y(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\omega \sin(\omega t) & \omega \cos(\omega t) \end{pmatrix}, \mathbf{y}_0 = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}.$$

Thus $W(t) = \det Y(t) = \omega$, and

$$Y^{-1}(t) = \begin{pmatrix} \cos(\omega t) & -\frac{1}{\omega}\sin(\omega t) \\ \sin(\omega t) & \frac{1}{\omega}\cos(\omega t) \end{pmatrix},$$

so that

$$Y^{-1}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}.$$

With that

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix} \begin{pmatrix} y_0 \\ 0 \end{pmatrix} = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{y}(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\omega \sin(\omega t) & \omega \cos(\omega t) \end{pmatrix} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

or

$$y(t) = y_0 \cos(\omega t).$$

Example 43:
Find the solution for the linear homogeneous ODE $x^2 y'' + xy' - 2y = 0$ with the initial conditions $y(1) = y_0$ and $y'(1) = 0$.

Solution We notice that the terms in the ODE are proportional to $x^n y^{(n)}$, so we start with the substitution $t = \ln x$ (and thus $x = e^t$). Then the derivatives become

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}, \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \\
&= \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \\
&= \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{x}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right) = \\
&= \frac{1}{x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right),
\end{aligned}$$

and the ODE becomes

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \cancel{\frac{\mathrm{d}y}{\mathrm{d}t}}\right) + \cancel{\frac{\mathrm{d}y}{\mathrm{d}t}} - 2y = 0,$$

which we can write as a system of two 1st order ODEs

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= v, \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= 2y.
\end{aligned}$$

This ODE is an example for a class of differential equations called Euler-Cauchy equations. They have the form

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1}(x) + \\ + \cdots + a_1 x y'(x) + a_0 y(x) = 0.$$

They can all be transformed to linear ODEs by the substitution used here. Alternatively, you can use a trial solution $y = x^m$.

We divide the second by the first equation to get

$$\frac{\mathrm{d}v/\cancel{\mathrm{d}t}}{\mathrm{d}y/\cancel{\mathrm{d}t}} = 2\frac{y}{v},$$

which can be written as $v\,\mathrm{d}v = 2y\,\mathrm{d}y$, which yields $v = \pm\sqrt{2}\,y$, so that we have two 1st order differential equations

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}t} &= \sqrt{2}\,y_1,\\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -\sqrt{2}\,y_2,\end{aligned}$$

with the solutions $y_1(t) = \exp(\sqrt{2}\,t)$ and $y_2(t) = \exp(-\sqrt{2}\,t)$, which become $y_1(x) = x^{\sqrt{2}}$ and $y_2(x) = x^{-\sqrt{2}}$ when we return to the original variable, so that the general solution is

$$y(t) = C_1 x^{\sqrt{2}} + C_2 x^{-\sqrt{2}}.$$

We then have

$$\mathbf{C} = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}, Y(t) = \begin{pmatrix} x^{\sqrt{2}} & x^{-\sqrt{2}} \\ \sqrt{2}\,x^{(1-\sqrt{2})} & -\sqrt{2}\,x^{(1+\sqrt{2})} \end{pmatrix}, \mathbf{y}_0 = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}.$$

Thus $W(t) = \det Y(t) = -\sqrt{2}\left(x^{(1+2\sqrt{2})} + x^{(1-2\sqrt{2})}\right)$, and

$$Y^{-1}(t) = -\frac{1}{\sqrt{2}\left(x^{(1+2\sqrt{2})} + x^{(1-2\sqrt{2})}\right)} \begin{pmatrix} -\sqrt{2}\,x^{(1+\sqrt{2})} & -x^{-\sqrt{2}} \\ -\sqrt{2}\,x^{(1-\sqrt{2})} & x^{\sqrt{2}} \end{pmatrix},$$

so that

$$Y^{-1}(1) = \frac{1}{4}\begin{pmatrix} 2 & \sqrt{2} \\ 2 & -\sqrt{2} \end{pmatrix}.$$

With that

$$\mathbf{C} = \frac{1}{4}\begin{pmatrix} 2 & \sqrt{2} \\ 2 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} y_0 \\ 0 \end{pmatrix} = \frac{y_0}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{y}(t) = \frac{y_0}{2}\begin{pmatrix} x^{\sqrt{2}} & x^{-\sqrt{2}} \\ \sqrt{2}\,x^{(1-\sqrt{2})} & -\sqrt{2}\,x^{(1+\sqrt{2})} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$y(t) = \frac{y_0}{2}\left(x^{\sqrt{2}} + x^{-\sqrt{2}}\right).$$

Inhomogeneous linear ODEs

The approach outlined in the previous section allows us also to find a general solution for the inhomogeneous problem

$$a_n(x)\,y^{(n)}(x) + \ldots + a_1(x)\,y'(x) + a_0(x)\,y(x) = f(x). \tag{4.44}$$

In that case we just need to find one particular solution for the inhomogeneous problem y_p and add it to the general solution y_h of the same ODE, but without the inhomogeneity.

The general solution is a linear superposition of the basis functions of the fundamental system of the homogeneous ODE, $y_\mathrm{h}(x) = \sum_n C_i\,y_i(x)$.

$$\left.\begin{aligned} a_n(x)\,y_\mathrm{p}^{(n)}(x) + \ldots + a_1(x)\,y_\mathrm{p}'(x) + a_0(x)\,y_\mathrm{p}(x) &= f(x) \\ a_n(x)\,y_\mathrm{h}^{(n)}(x) + \ldots + a_1(x)\,y_\mathrm{h}'(x) + a_0(x)\,y_\mathrm{h}(x) &= 0 \end{aligned}\right) +$$

$$a_n(x)\left(y_\mathrm{p}(x)+y_\mathrm{h}(x)\right)^{(n)}+\ldots+a_1(x)\left(y_\mathrm{p}(x)+y_\mathrm{h}(x)\right)'+ \\ +a_0(x)\left(y_\mathrm{p}(x)+y_\mathrm{h}(x)\right)=f(x). \quad (4.45)$$

Note that the particular solution must be linearly independent from the homogeneous solution (including any of its basis functions) as otherwise $y_\mathrm{p}(x)=\sum_n C_i y_i(x)$, which, when we insert it into the ODE, would give zero, but by definition the result of the ODE for the particular solution must be $f(x)$. In general, the homogeneous and the particular solutions will look quite different.

Method of variation of constants

To find the particular solution we can use the same strategy we have already discussed in the context of 1st order ODEs, the *method of variation of constants*. The trick is again to turn the coefficients C_i into functions of the independent variable, $\psi_i(t)$. Then

$$\mathbf{y}(t)=\mathbf{Y}(t)\boldsymbol{\psi}(t), \quad (4.46)$$

where $\boldsymbol{\psi}(t)$ is now a vector with the new functions as components, $\boldsymbol{\psi}^\mathrm{T}(t)=(\psi_1(t),\psi_2(t),\ldots,\psi_n(t))$.

A general form to write eq. (4.4)

$$\mathbf{y}'(t)=\mathbf{A}(t)\mathbf{y}(t)+\mathbf{f}(t), \quad (4.47)$$

with a suitable $n\times n$ matrix $\mathbf{A}(t)$ and a vector of inhomogeneities $\mathbf{f}(t)$. We can then substitute our solution to get

$$\mathbf{y}'(t)=\mathbf{Y}'(t)\boldsymbol{\psi}(t)+\mathbf{Y}(t)\boldsymbol{\psi}'(t)=\mathbf{A}(t)\mathbf{Y}(t)\boldsymbol{\psi}(t)+\mathbf{f}(t). \quad (4.48)$$

As $\mathbf{Y}(t)$ is composed of the solutions to the homogeneous ODE, $\mathbf{Y}'(t)\overset{!}{=}\mathbf{A}(t)\mathbf{Y}(t)$, and we are left with

$$\mathbf{Y}(t)\boldsymbol{\psi}'(t)=\mathbf{f}(t). \quad (4.49)$$

As $W(t)=\det\mathbf{Y}(t)\neq 0$, we can rearrange this to

$$\boldsymbol{\psi}'(t)=\mathbf{Y}^{-1}(t)\mathbf{f}(t), \quad (4.50)$$

which we can integrate

$$\boldsymbol{\psi}(t)=\int_{t_0}^{t}\mathbf{Y}^{-1}(t')\mathbf{f}(t')\,\mathrm{d}t'+\mathbf{c}, \quad (4.51)$$

with a vector $\mathbf{c}$ of integration constants, which we can now use in our ansatz,

$$\mathbf{y}(t)=\mathbf{Y}(t)\mathbf{c}+\mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(t')\mathbf{f}(t')\,\mathrm{d}t'= \\ =\mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\mathbf{y}_0+\mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(t')\mathbf{f}(t')\,\mathrm{d}t', \quad (4.52)$$

The first term corresponds to the homogeneous solution $\mathbf{y}_\mathrm{h}(t)$, where we have used our earlier result, eq. (4.40), and the second to the particular solution $\mathbf{y}_\mathrm{p}(t)$. Note that by the choice of the lower integration boundary, t_0, the initial conditions $\mathbf{y}(t_0)=\mathbf{y}_0$ are maintained.

Example 44:
Find a particular solution for $y'' + ky = F$, where F is a constant.

Solution We can guess a particular solution by setting $y'' = 0$, which gives $y_\mathrm{p}(t) = F/k$. But let's see whether we can find the same result formally: We start again with writing the inhomogeneous 2nd order ODE as a system of the two coupled 1st order ODEs

$$\begin{aligned} y'(t) &= v(t), \\ v'(t) &= -ky(t) + F. \end{aligned}$$

Thus,

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ F \end{pmatrix},$$

and

$$\begin{aligned} \mathbf{y}_\mathrm{p}(t) &= Y(t) \int_{t_0}^{t} Y^{-1}(t') \mathbf{f}(t') \, \mathrm{d}t' = \\ &= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\omega \sin(\omega t) & \omega \cos(\omega t) \end{pmatrix} \int \begin{pmatrix} \cos(\omega t') & -\frac{1}{\omega} \sin(\omega t') \\ \sin(\omega t') & \frac{1}{\omega} \cos(\omega t') \end{pmatrix} \begin{pmatrix} 0 \\ F \end{pmatrix} \mathrm{d}t' = \\ &= \frac{F}{\omega} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\omega \sin(\omega t) & \omega \cos(\omega t) \end{pmatrix} \int \begin{pmatrix} -\sin(\omega t') \\ \cos(\omega t') \end{pmatrix} \mathrm{d}t' = \\ &= \frac{F}{\omega^2} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\omega \sin(\omega t) & \omega \cos(\omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} = \begin{pmatrix} F/\omega^2 \\ 0 \end{pmatrix}, \end{aligned}$$

so that

$$y_\mathrm{p}(t) = F/\omega^2 = F/k,$$

as expected.

Finding solutions for a general second order linear ODE

We know that for a second order differential equation

$$y'' + a(x)y' + b(x)y = f(x) \tag{4.53}$$

we need to have two linearly independent solutions to the homogeneous equation, in addition to a particular solution for the inhomogeneity. In case that we managed to guess one solution, we can use variation of constants to find not only a particular solution, but also the second solution to the homogeneous equation. Assuming that we know $y_1(x)$ we start with the ansatz $y_a(x) = \psi(x) y_1(x)$, which yields

Some things to try to guess the first solution, if the coefficient functions $a(x)$ and $b(x)$ are polynomials:

- $y_1(x) = e^{\lambda x}$,
- $y_1(x) = x^n$,
- $y_1(x) = \sum_{i=1}^{n} c_i x^i$,
- $y_1(x) = \left(\sum_{i=1}^{n} c_i x^i \right) e^{\lambda x}$.

$$\begin{aligned} y_a'(x) &= \psi'(x) y_1(x) + \psi(x) y_1'(x), \\ y_a''(x) &= \psi''(x) y_1(x) + 2\psi'(x) y_1'(x) + \psi(x) y_1''(x), \end{aligned} \tag{4.54}$$

which we can insert into the ODE,

$$\begin{aligned} \psi''(x) y_1(x) + \psi'(x) \left[2y_1'(x) + a(x) y_1(x) \right] + \\ + \psi \underbrace{\left[y_1''(x) + a(x) y_1'(x) + b(x) y_1(x) \right]}_{0} = f(x), \end{aligned} \tag{4.55}$$

which we can write as

$$\psi''(x)+\psi'\left(2\frac{y_1'(x)}{y_1(x)}+a(x)\right)=\frac{f(x)}{y_1(x)}, \tag{4.56}$$

which is a 1st order differential equation for $\psi'(x)$, which can be solved with the integrating factor (see eq. (2.43)),

$$\begin{aligned}\Lambda(x)&=\exp\left(\int^x\left[2\frac{y_1'(u)}{y_1(u)}+a(u)\right]\mathrm{d}u\right)=\\&=\exp\left(2\int^x\frac{y_1'(u)}{y_1(u)}\,\mathrm{d}u\right)\exp\left(\int^x a(u)\,\mathrm{d}u\right)=\\&=y_1^2(x)\exp\left(\int^x a(u)\,\mathrm{d}u\right),\end{aligned} \tag{4.57}$$

and

$$\psi'(x)=\frac{1}{\Lambda(x)}\left(\int^x\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u+C\right), \tag{4.58}$$

which integrates to

$$\psi(x)=\int^x\psi'(v)\,\mathrm{d}v=\int^x\frac{1}{\Lambda(v)}\int^v\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u\,\mathrm{d}v+C\int^x\frac{\mathrm{d}v}{\Lambda(v)}, \tag{4.59}$$

which we can substitute into the ansatz to get

$$y_a(x)=y_1(x)\int^x\frac{1}{\Lambda(v)}\int^v\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u\,\mathrm{d}v+Cy_1(x)\int^x\frac{\mathrm{d}v}{\Lambda(v)}, \tag{4.60}$$

and the general solution is

$$y(x)=C_1y_1(x)+C_2\underbrace{\left(\int^x\frac{\mathrm{d}v}{\Lambda(v)}\right)y_1(x)}_{y_2(x)}+\underbrace{y_1(x)\int^x\frac{1}{\Lambda(v)}\int^v\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u\,\mathrm{d}v}_{y_\mathrm{p}(x)}, \tag{4.61}$$

where we have indicated the two solutions to the homogeneous equation and the particular solution.

The particular solution can be further written as

$$\begin{aligned}y_\mathrm{p}(x)&=\underbrace{y_1(x)\left(\int^x\frac{\mathrm{d}v}{\Lambda(v)}\right)}_{y_2(x)}\left(\int^x\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u\right)-y_1(x)\int^x\int^u\frac{\mathrm{d}v}{\Lambda(v)}\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u=\\&=\int^x\frac{y_2(x)y_1(u)}{y_1(u)}\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u-\int^x\frac{y_1(x)y_1(u)\int^u\frac{\mathrm{d}v}{\Lambda(v)}}{y_1(u)}\frac{\Lambda(u)f(u)}{y_1(u)}\,\mathrm{d}u=\\&=\int^x\frac{y_2(x)y_1(u)-y_1(x)y_2(u)}{y_1^2(u)}\Lambda(u)f(u)\,\mathrm{d}u.\end{aligned} \tag{4.62}$$

We use

$$\begin{aligned}\left(\int^x a(u)\,\mathrm{d}u\right)\left(\int^x b(v)\,\mathrm{d}v\right)&=\\&=\int^x a(u)\int^u b(v)\,\mathrm{d}v\,\mathrm{d}u+\\&\quad+\int^x b(v)\int^v a(u)\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Example 45:
Find the general solution for $x^2y''-2xy'+2y=x^2\ln x$.

Solution First, we can guess a homogeneous solution. Here, a simple power will do it: $y_1=x$. To verify,

$$0-2x+2x=0.$$

Next, we write the ODE in the standard form

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=\ln x,$$

and we find the integrating factor

$$\Lambda(x) = y_1^2(x) \exp\left(\int^x a(u)\,\mathrm{d}u\right) = x^2 \exp\left(\int^x \left(-\frac{2}{u}\right)\mathrm{d}u\right) = x^2 \exp\left(-2\ln x\right) = 1,$$

from which we can find the second homogeneous solution

$$y_2(x) = y_1(x) \int^x \frac{\mathrm{d}v}{\Lambda(v)} = x \int^x \mathrm{d}v = x^2,$$

and a particular solution

$$\begin{aligned} y_\mathrm{p}(x) &= \int^x \frac{y_2(x)y_1(u) - y_1(x)y_2(u)}{y_1^2(u)} \Lambda(u) f(u)\,\mathrm{d}u = \\ &= \int^x \frac{x^2 u - x u^2}{u^2} \ln(u)\,\mathrm{d}u = \\ &= x^2 \int^x \frac{\ln u}{u}\,\mathrm{d}u - x \int^x \ln u\,\mathrm{d}u = \\ &= x^2 \frac{(\ln x)^2}{2} - x(x\ln x - x) = \\ &= \frac{1}{2}(x\ln x)^2 - x^2(\ln x - 1). \end{aligned}$$

We can omit integration constants as we only need one particular solution.

So, the complete general solution is

$$y(x) = C_1 x + C_2 x^2 + \frac{1}{2}(x\ln x)^2 - x^2(\ln x - 1).$$

Linear ODEs with constant coefficients

Linear homogeneous ODEs with constant coefficients

A class of ODEs for which solutions can be found in a straightforward way are linear homogeneous ODEs with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0. \tag{4.63}$$

We can use the ansatz $y(x) = \exp(\lambda x)$, which yields

$$\begin{aligned} a_n \lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_2 \lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = \\ = e^{\lambda x}\left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0\right) = 0. \end{aligned} \tag{4.64}$$

Thus $y(x) = \exp(\lambda x)$ is a solution if

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{4.65}$$

This is called the *characteristic equation* of the differential equation, the left hand side being the *characteristic polynomial*.

This is an n^{th} order polynomial in λ, which can be factorized using n, generally complex, roots λ_i,

$$\begin{aligned} a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = \\ = a_n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{n-1})(\lambda - \lambda_n) = 0. \end{aligned} \tag{4.66}$$

If the roots are unique, n unique solutions can be constructed by plugging them into the ansatz, which we can use to create a general solution

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_{n-1} e^{\lambda_{n-1} x} + C_n e^{\lambda_n x}. \tag{4.67}$$

Example 46:
Find the solution to the initial value problem $y'' + 3y' + 2y = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 2$.

Solution Using the ansatz $y(x) = \exp(\lambda x)$ we get the characteristic polynomial

$$\lambda^2 + 3\lambda + 2 = 0$$

with the solutions

$$\lambda_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \begin{cases} -1, \\ -2. \end{cases}$$

The general solution is then

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

Using the initial conditions

$$y(0) = 1 = C_1 + C_2,$$
$$y'(0) = 2 = -C_1 - 2C_2,$$

gives the coefficients $C_1 = 4$ and $C_2 = -3$, so that the solution to the initial value problem is

$$y(x) = 4e^{-x} - 3e^{-2x}.$$

If some of the roots are identical (degenerate), the exponential ansatz would give identical solutions and the set of solutions would not constitute a complete fundamental system. We therefore need to find other, linearly independent, solutions to substitute for the degeneracies.

The coefficients for the degenerate solutions would simply add up, resulting in a single coefficient, which would be insufficient to adapt the general solution to any given set of initial conditions.

One way to find these solutions is by using the *method of variation of constants*. For this, we replace the constant with a function of the independent variable, which results in a differential equation, the solution of which provides the necessary linearly independent solutions.

Example 47:
Find the solution to the initial value problem $y'' - 6y' + 9y = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 2$.

Solution The characteristic polynomial is $\lambda^2 - 6\lambda + 9 = 0$ with the two degenerate solutions

$$\lambda_{1,2} = 3 \pm \sqrt{9 - 9} = 3.$$

The general solution ansatz $y(x) = C_1 \exp(3x) + C_2 \exp(3x)$ can be written as $y(x) = (C_1 + C_2)\exp(3x) = C\exp(3x)$. This cannot satisfy the initial conditions

$$y(0) = 1 = C,$$
$$y'(0) = 2 = 3C.$$

The complete fundamental system can be found using the method of variation of constants,

$$y(x) = \Psi(x)\exp(3x).$$

Inserting this into the differential equation gives

$$\underbrace{(\Psi'' + 6\Psi' + 9\Psi)e^{3x}}_{y''} - 6\underbrace{(\Psi' + 3\Psi)e^{3x}}_{y'} + 9\underbrace{\Psi e^{3x}}_{y} = 0,$$

which reduces to $\Psi'' e^{3x} = 0$, so we need to solve $\Psi'' = 0$. After two integrations we get $\Psi = C_1 + C_2 x$ and the complete general solution is

$$y(x) = C_1 e^{3x} + C_2 x e^{3x},$$

which now provides enough degrees of freedom to solve the initial value problem

$$\begin{aligned} y(0) &= 1 = C_1, \\ y'(0) &= 2 = 3C_1 + C_2, \end{aligned}$$

which gives the coefficients $C_1 = 1$ and $C_2 = -1$, and the solution to the initial value problem

$$y(x) = (1 - x)e^{3x}.$$

It is possible that some or all the roots of the characteristic equation are complex, even if the coefficients of the ODE are real, making the solutions complex. What can we do about this?

Many problems in physics require as results real observables, which need to satisfy real initial values.

Any polynomial with real coefficients can be factorised into a product of linear $(\lambda - \lambda_i)$ and/or 2^{nd} order factors $(\lambda^2 + a\lambda + b)$ with real coefficients a and b. The latter correspond to complex roots $\lambda_{i,j} = a/2 \pm \mathrm{i}\sqrt{b - a^2/4} = \lambda \pm \mathrm{i}\omega$, with λ and ω real.

The first thing to note is that complex solutions for the roots always occur in pairs of complex conjugates, $\lambda_i = \lambda + \mathrm{i}\omega$ and $\lambda_j = \lambda_i^* = \lambda - \mathrm{i}\omega$, where λ and ω are the real and imaginary parts of $\lambda_{i,j}$. The corresponding solutions are $y_i(x) = \exp(\lambda_i x)$ and $y_j(x) = \exp(\lambda_i^* x)$. We can use these to create an alternative fundamental system using orthogonal linear combinations with the new, real basis functions

$$\begin{aligned} Y_i(x) = \frac{y_i(x) + y_j(x)}{2} = \frac{e^{\lambda_i x} + e^{\lambda_i^* x}}{2} = \frac{e^{(\lambda+\mathrm{i}\omega)x} + e^{(\lambda-\mathrm{i}\omega)x}}{2} = \\ = e^{\lambda x}\left(\frac{e^{\mathrm{i}\omega x} + e^{-\mathrm{i}\omega x}}{2}\right) = e^{\lambda x}\cos(\omega x), \end{aligned} \tag{4.68}$$

and the orthogonal combination

$$Y_j(x) = \frac{y_i(x) - y_j(x)}{2} = e^{\lambda x}\sin(\omega x). \tag{4.69}$$

Note that even though these basis functions are real, they can still be used to satisfy complex initial conditions if needed, with complex coefficients C_i.

Example 48:
Find the general solution for the harmonic oscillator $\ddot{y} + \omega^2 y = 0$.

Solution The characteristic polynomial for the usual ansatz $y(t) = \exp(\lambda t)$ is $\lambda^2 = -\omega^2$, Thus $\lambda_{1,2} = \pm\mathrm{i}\omega$, and

$$y(t) = C_+ e^{\mathrm{i}\omega t} + C_- e^{-\mathrm{i}\omega t},$$

or

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t),$$

with the integration constants C_+, C_-, C_1 and C_2 to be determined from the initial conditions.

Alternatively, we can write the latter result as

$$y(x) = C\cos(\omega x + \delta) = C\left[\cos\delta\cos(\omega x) - \sin\delta\sin(\omega x)\right],$$

with the integration constants $C = \sqrt{C_1^2 + C_2^2}$ and $\delta = \arctan(-C_2/C_1)$.

For a 2nd order linear homogeneous ODE with constant coefficients the characteristic polynomial is of order 2, with two roots λ_1 and λ_2. All the possible solutions, depending on the character of λ_1 and λ_2, which in turn depend on the coefficients in the ODE, are listed in Table 4.1.

λ_1	λ_2	Solution $y(x)$
κ	κ	$(C_1 + C_2 x)e^{\kappa x}$
κ_1	κ_2	$C_1 e^{\kappa_1 x} + C_2 e^{\kappa_2 x}$
$i\omega$	$-i\omega$	$C_1 \cos(\omega x) + C_2 \sin(\omega x)$
$\kappa + i\omega$	$\kappa - i\omega$	$e^{\kappa x}[C_1 \cos(\omega x) + C_2 \sin(\omega x)]$

Table 4.1: Possible solutions of a 2nd order linear homogeneous ODE with constant coefficients, depending on the roots of the characteristic polynomial. κ, κ_1, κ_2 and ω are real parameters from the solution of the characteristic polynomial, and C_1 and C_2 are integration constants that can be complex.

The damped harmonic oscillator revisited

We will now return to the damped harmonic oscillator

$$\ddot{y} + \gamma \dot{y} + k y = 0, \tag{4.70}$$

with constant coefficients γ and k.

We now know how to deal with this: With the usual ansatz $y(t) = \exp(\lambda t)$ the characteristic polynomial is

$$\lambda^2 + \gamma \lambda + k = 0 \tag{4.71}$$

with the solutions

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - k} = -\frac{\gamma}{2} \pm \omega_0 \sqrt{\left(\frac{\gamma}{2\omega_0}\right)^2 - 1}, \tag{4.72}$$

where we have introduced $\omega_0 = \sqrt{k}$ in the last equation. We call this frequency the *natural frequency* of the oscillator.

We have now three different cases:

- $\frac{\gamma}{2\omega_0} < 1$ (weak or undercritical damping):
  The general solution is
  $$y(t) = e^{-\frac{\gamma}{2}t}\left[C_1 \cos(\omega' t) + C_2 \sin(\omega' t)\right], \tag{4.73}$$
  with $\omega' = \omega_0 \sqrt{1 - \left(\frac{\gamma}{2\omega_0}\right)^2} \simeq \omega_0$, where the last approximation applies if $\frac{\gamma}{2\omega_0} \ll 1$. This is an oscillatory solution that decays exponentially.

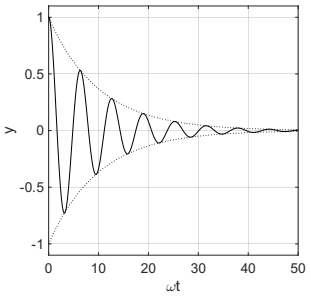


Figure 4.8: Underdamped ($\gamma = \omega/10$) harmonic oscillator with the initial condition $y(0) = 1$, $y'(0) = 0$.

- $\frac{\gamma}{2\omega_0} = 1$ (critical damping):
  The root at $\lambda = \gamma/2$ is degenerate, and the general solution is
  $$y(t) = (C_1 + C_2 t)e^{-\frac{\gamma}{2}t}. \tag{4.74}$$
  This solution decays exponentially, and for initial conditions $y_0 \geq 0$ and $y_0' \geq 0$ never becomes negative.

- $\frac{\gamma}{2\omega} > 1$ (strong or overcritical damping):

  The general solution is

$$y(t) = C_1 e^{-\frac{\gamma}{2}t+\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\,t} + C_2 e^{-\frac{\gamma}{2}t-\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\,t} = \\ = C_1 e^{-t/\tau_1} + C_2 e^{-t/\tau_2}, \quad (4.75)$$

  which is a superposition of two decaying exponentials with the two time constants

$$\tau_1 = \frac{2}{\gamma}\left(1-\sqrt{1-\left(\frac{2\omega_0}{\gamma}\right)^2}\right)^{-1}, \\ \tau_2 = \frac{2}{\gamma}\left(1+\sqrt{1-\left(\frac{2\omega_0}{\gamma}\right)^2}\right)^{-1}, \quad (4.76)$$

  with $\tau_1 > \tau_2$.

  For large t, the second term in the solution becomes negligible, and the long-term behaviour is dominated by

$$y(t) \xrightarrow[t\to\infty]{} C_1 e^{-\left(1-\sqrt{1-\left(\frac{2\omega_0}{\gamma}\right)^2}\right)\frac{\gamma}{2}t}. \quad (4.77)$$

The exact time development will depend on the initial conditions, but generally, critical damping gets to zero fastest in a monotonic sense (no oscillations, minimum settling time). Underdamped systems may cross zero earlier (because of oscillations), but they keep ringing and take longer overall to fully die out.

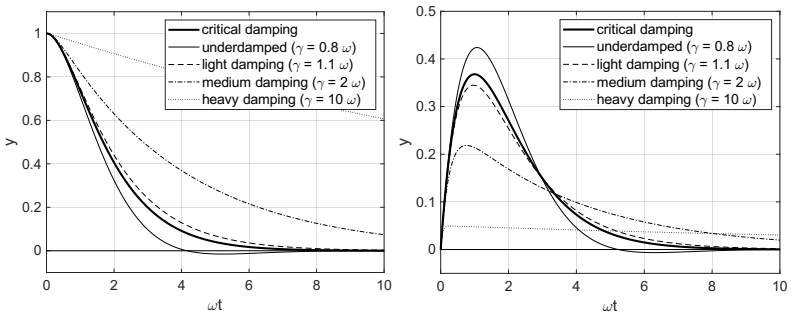


Figure 4.9: Response of a damped harmonic oscillator in the time domain for two different sets of boundary conditions: $y(0) = 1$, $y'(0) = 0$ (left); $y(0) = 0$, $y'(0) = 1$ (right).

Method of undetermined coefficients

To find a particular solution for an inhomogeneous linear ODE with constant coefficients one can play through the method of variation of constants. From this it becomes clear that for certain classes of inhomogeneities, the particular function will have a specific form. This can be used for an educated guess of the particular function, which can then be proven and adjusted to the specific ODE. This is called the *method of undetermined coefficients*. A list of these particular functions is given in table 4.2.

Inhomogeneity $f(x)$	Particular function ansatz $y_p(x)$
b	b/a_0
bx^k	$c_0 + c_1 x + \cdots + c_k x^k$
$bx^k e^{\alpha x}$	$(c_0 + c_1 x + \cdots + c_k x^k) e^{\alpha x}$
$be^{\beta x}$	$cx^m e^{\beta x}$
$b\sin(\alpha x)$ or $b\cos(\alpha x)$	$c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$
$bx^k \sin(\alpha x)$ or $bx^k \cos(\alpha x)$	$(c_0 + c_1 x + \cdots + c_k x^k)\sin(\alpha x) +$ $+(d_0 + d_1 x + \cdots + d_k x^k)\cos(\alpha x)$
$\sum f_i(x)$, ($f_i(x)$ the functions above)	$\sum y_{p,i}(x)$, ($y_{p,i}(x)$ the functions above)

Table 4.2: Method of undetermined coefficients: particular solutions for various inhomogeneities $f(x)$ for the ODE $a_0 y + a_1 y' + \cdots + a_n y^{(n)} = f(x)$ with constant coefficients a_i. b, the c, c_i and d_i are constants $\in \mathbb{C}$, k is a constant $\in \mathbb{Z}^+$, α is a constant $\in \mathbb{C}$, that is not equal to any of the roots of the characteristic polynomial (λ_i), β is a constant $\in \mathbb{C}$, that is equal to a root of the characteristic polynomial with the degeneracy m.

Example 49:
Find a particular solution for the ODE $y'' - 3y' + 2y = e^t$.

Solution The characteristic polynomial is

$$\lambda^2 - 3\lambda + 2 = 0,$$

which is solved by $\lambda_1 = 1$ and $\lambda_2 = 2$. Thus the homogeneous solution is

$$y_\mathrm{h}(t) = C_1 e^t + C_2 e^{2t}.$$

Next we can guess a particular solution using the method of undetermined coefficients. First, we note that the exponential in the homogeneous solution corresponds to one of the solutions of the homogeneous equation. Thus we use the ansatz $y_\mathrm{p}(t) = cte^t$ and plug it into the ODE,

$$[(2c + ct) - 3(c + ct) + 2ct]\, e^t = -ce^t,$$

which needs to equal the inhomogeneity in the ODE, which is satisfied for $c = -1$, and the particular solution is

$$y_\mathrm{p}(t) = -te^t.$$

The driven harmonic oscillator

We will now come back to the damped harmonic oscillator, but add an inhomogeneity that constitutes a periodic driving force

$$y'' + \gamma y' + ky = f(t) = \frac{F}{m}\cos(\omega t). \tag{4.78}$$

As you will learn next year, any excitation in the time domain can be shown to be a superposition of excitations with fixed frequency with varying amplitude (the transition from the time to the frequency domain is called a Fourier transformation). Due to the linearity of the ODE, we can thus find the response of the system as the superposition of the solutions of the ODE for these different frequencies, weighted with the frequency-dependent amplitudes.

As we have seen before, the homogenous solution tends exponentially to zero for large t, so that after a sufficiently long time only the particular solution remains. The particular solution we can find using the method of undetermined coefficients, using the trial function

$$y_\mathrm{p} = c_1 \cos(\omega t) + c_2 \sin(\omega t). \tag{4.79}$$

This leads to a system of two equations to solve for the two coefficients c_1 and c_2, which can be solved with some effort. However, a commonly used alternative approach is to find a particular solution for the closely related inhomogenous ODE

$$z'' + \gamma z' + \frac{k}{m} z = \frac{F}{m} e^{\mathrm{i}\omega t}, \tag{4.80}$$

Due to the linearity of the ODE we can recover the original equation using $y = \Re(z) = (z + z^*)/2$ and $\cos\alpha = (e^{\mathrm{i}\alpha} + e^{-\mathrm{i}\alpha})/2$.

where $z(t) = z_0 e^{i\theta(t)}$ is now a complex function with z_0 a real constant. As the only frequency available once the homogeneous solution has decayed is the driving frequency, the phase of this function will be given by $\theta(t) = \omega t + \delta$, where δ is a constant phase. We can insert this into the ODE to get

$$z_0 e^{i(\cancel{\omega t} + \delta)} \left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \frac{F}{m} \cancel{e^{i\omega t}}, \tag{4.81}$$

where we have used $\omega = \sqrt{k/m}$. We find

$$\begin{aligned} z_0(\cos\delta + i\sin\delta) &= \frac{F/m}{\omega_0^2 - \omega^2 + i\gamma\omega} = \frac{F/m}{\omega_0^2 - \omega^2 + i\gamma\omega} \frac{\omega_0^2 - \omega^2 - i\gamma\omega}{\omega_0^2 - \omega^2 - i\gamma\omega} = \\ &= \frac{F/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} \frac{\omega_0^2 - \omega^2 - i\gamma\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}. \end{aligned} \tag{4.82}$$

Comparison of the two sides gives

$$z_0(\omega) = \frac{F/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}, \tag{4.83}$$

$$\delta(\omega) = \arctan\frac{\gamma\omega}{\omega^2 - \omega_0^2}, \tag{4.84}$$

and

$$y_p(t) = \frac{z + z^*}{2} = z_0 \cos(\omega t + \delta). \tag{4.85}$$

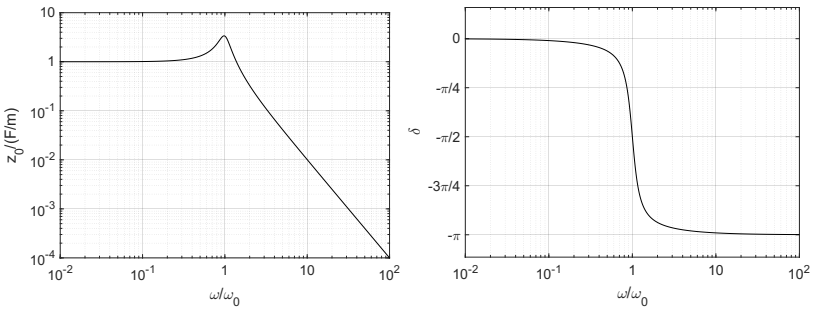


Figure 4.10: Response of a forced damped harmonic oscillator as a function of the excitation frequency for $\gamma/\omega_0 = 0.3$. Amplitude (left) and phase shift (right). Note that the plots use a log-log (left) and log-x scale (right).

For low frequencies ($\omega \to 0$), the inertia and the damping terms (the derivatives) are irrelevant, and the response of the harmonic oscillator is entirely given by the driving force against the restoring term, $z_0(0) \to F/(\omega_0^2 m)$ with no phase shift $\delta(0) \to 0$.

As the frequency approaches the natural frequency of the oscillator, ω_0, the amplitude increases, and a peak emerges. We call this peak a *resonance*. We can find the *resonance frequency* from $\mathrm{d}z_0(\omega_{\max})/\mathrm{d}\omega = 0$, which yields

$$\omega_{\max} = \omega_0\sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}. \tag{4.86}$$

For weak damping ($\gamma/\omega_0 \to 0$) the resonance frequency $\omega_{\max} \simeq \omega_0$, and it decreases for increased damping. If $\gamma/\omega_0 > \sqrt{2}$ the resonance peak disappears and the amplitude falls monotonically with frequency.

The amplitude at the resonance frequency is then

$$z_{0,\max} = \frac{F/m}{\gamma\omega_0\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}} \simeq \frac{F/m}{\gamma\omega_0} = \frac{z_0(0)}{\gamma/\omega_0}, \tag{4.87}$$

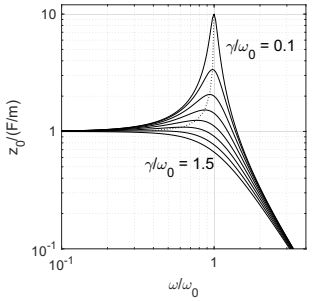


Figure 4.11: Displacement of forced damped oscillator for different values of γ/γ_0. γ/ω_0 has been varied from 0.1 to 1.5 in steps of 0.2. The dashed line connects the maxima of the resonant peaks.

Remember that critical damping is $\gamma/\omega_0 = 2$, so resonance peaks will only appear for strongly underdamped oscillators.

where the approximation applies again in the low damping limit ($\gamma/\omega_0 \to 0$). For $\gamma = 0$ the amplitude would explode.

We can describe the width of the resonance by the *full width at half maximum* (FWHM), which is given as the frequency interval between the points where the amplitude has dropped to half its peak value. In the low damping limit we can use $\omega_{\max} \simeq \omega_0$ and, as the resonance is narrow in this case, we can approximate

This width is also called the *bandwidth* of the oscillator.

$$\left(\omega_0^2 - \omega^2\right)^2 = (\omega_0 + \omega)^2(\omega_0 - \omega)^2 \simeq 4\omega_0^2(\omega_0 - \omega)^2. \tag{4.88}$$

With this the amplitude becomes

$$z_0(\omega) \simeq \frac{F/m}{\sqrt{4\omega_0^2(\omega_0 - \omega)^2 + \omega_0^2\gamma^2}}, \tag{4.89}$$

and the half-amplitude points satisfy

$$\frac{F/m}{\sqrt{4\omega_0^2(\omega_0 - \omega_{1/2})^2 + \omega_0^2\gamma^2}} = \frac{1}{2}\frac{F/m}{\omega_0\gamma}, \tag{4.90}$$

which results in $(\omega_0 - \omega_{1/2})^2 = 3\gamma^2/4$ or $\omega_{1/2} = \omega_0 \pm \sqrt{3}\gamma/2$, so that the amplitude FWHM is $\Delta\omega = \sqrt{3}\gamma$.

The FWHM is often given for the distribution of absorbed power, $P(\omega) = z_0^2(\omega)$, which gives a power FWHM of γ.

The resonance occurs as the extrema of the excitation hit the oscillator always at the right time, which is when the oscillator passes through the equilibrium point. This corresponds to a phase lag of $\delta \simeq -\pi/2$.

For large frequencies ($\omega \to \infty$), the amplitude drops $\propto \omega^{-2}$ and the phase shift increases to $\delta \to -\pi$.

Power budget of the driven harmonic oscillator

The total energy of the oscillator is the sum of the kinetic and potential energies,

$$\begin{aligned} E(t) = T + V &= \frac{m y'^2(t)}{2} + \frac{m\omega_0^2 y^2(t)}{2} = \\ &= \frac{m}{2}\left(\omega_0^2 z_0^2(\omega)\sin^2(\omega t + \delta) + \omega^2 z_0^2(\omega)\cos^2(\omega t + \delta)\right) = \\ &= \frac{m\omega^2 z_0^2(\omega)}{2}. \end{aligned} \tag{4.91}$$

This is constant in time, as it should be for a steady state solution, but lets look at this in a bit more detail.

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m y'^2(t)}{2} + \frac{m\omega_0^2 y^2(t)}{2}\right) = m\left(y''(t)\,y'(t) + \omega_0^2 y'(t)\,y(t)\right) = \\ &= m y'(t)\left(y''(t) + \omega_0^2 y(t)\right) = m y'(t)\left(-\gamma y'(t) + \frac{F}{m}\cos(\omega t)\right), \end{aligned} \tag{4.92}$$

where we can associate the first term with the power dissipated and the second term with the power supplied to the oscillator.

The power supplied is oscillating, but we can find the average,

$$\begin{aligned} \langle P_{\text{in}}\rangle &= \frac{1}{T}\int_0^T y'(t) F\cos(\omega t)\,\mathrm{d}t = \\ &= -\frac{\omega F z_0(\omega)}{T}\int_0^T \cos(\omega t)\sin(\omega t + \delta)\,\mathrm{d}t = -\frac{\omega F z_0(\omega)}{2}\sin\delta(\omega). \end{aligned} \tag{4.93}$$

The sign in the derivation is a consequence of the definition of the sign of the phase shift. Independent of this choice the final result $\langle P_{\text{in}}\rangle > 0$.

We can now replace the sine in this equation using eq. (4.82)

$$\sin\delta(\omega) = -\frac{\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = -\frac{\gamma\omega z_0(\omega)}{F/m}, \tag{4.94}$$

so that

$$\langle P_{\text{in}}\rangle = \frac{\gamma\omega^2 m}{2} z_0^2(\omega). \tag{4.95}$$

The average power dissipated in the oscillator in the damping is

$$\langle P_{\text{d}}\rangle = m\gamma\langle y'^2\rangle = m\gamma\omega^2 z_0^2(\omega)\frac{1}{T}\int_0^T \sin^2(\omega t+\delta)\,\mathrm{d}t = \frac{\gamma\omega^2 m}{2} z_0^2(\omega), \tag{4.96}$$

and thus

$$\langle P_{\text{in}}\rangle - \langle P_{\text{d}}\rangle = 0, \tag{4.97}$$

as expected.

During each cycle of the oscillation some fraction of the energy is used to uphold the oscillation, countering the energy loss due to the damping, but there is also a fraction of the energy, which gets stored temporarily in the spring or inertia and then is returned to the driving force every cycle.

The *apparent power* describes the total exchange capacity between the source and the oscillator. It can be found by multiplying the root-mean-square (rms) values of the driving force and the velocity. The rms value for a function $f(t)$ is found from

$$f_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t}, \tag{4.98}$$

which for the motion of the harmonic oscillator gives

$$\begin{aligned} F_{\text{rms}} &= \frac{F}{\sqrt{2}}, \\ y_{\text{rms}}(\omega) &= \frac{z_0(\omega)}{\sqrt{2}}, \\ y'_{\text{rms}}(\omega) &= \frac{\omega z_0(\omega)}{\sqrt{2}}, \end{aligned} \tag{4.99}$$

and thus the apparent power

$$S = F_{\text{rms}} y'_{\text{rms}} = \frac{F\omega z_0(\omega)}{2}. \tag{4.100}$$

By multiplying the rms values we remove the effect of the phase shift between the force and the velocity.

The *real power* is the power that actually gets dissipated irreversibly in the damper. We have calculated this before in eq. (4.93),

$$\langle P_{\text{in}}\rangle = -\frac{\omega F z_0(\omega)}{2}\sin\delta(\omega). \tag{4.101}$$

The ratio of the two, $\langle P_{\text{in}}\rangle/S$, is called the *power factor*. This ratio is

$$\frac{\langle P_{\text{in}}\rangle}{S} = -\sin\delta(\omega) \tag{4.102}$$

for the harmonic oscillator.

The minus sign is again a consequence of the choice of our definition of the phase difference. The power factor is positive.

The remainder is the *reactive power* Q, which can be found from $S^2 = Q^2 + P^2$.

The duplicate use of the letter Q for both the Q-factor and the reactive power is unfortunate. It comes from the different communities that introduced the terms, physicists and control engineers for the former, and electrical engineers for the latter.

For small frequencies the power factor is zero, almost all power is reactive, stored in the spring. At the resonance the power factor is 1, all apparent power is going into upholding the oscillation against the dissipation. For very high frequencies, the power becomes mostly reactive again, this time stored in the inertia of the mass. The driving force spends most of its effort accelerating the mass in one half-cycle, storing energy in kinetic form. Then, in the next half-cycle, the mass gives that energy back as it decelerates and reverses direction.

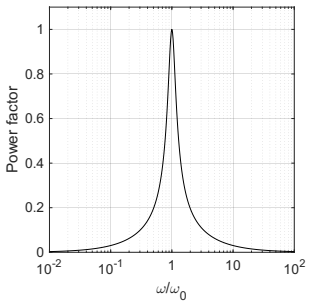


Figure 4.12: Power factor as a function of the excitation frequency for a forced harmonic oscillator with $\gamma/\omega_0 = 0.3$.

The quality factor

A parameter often used to describe the damping properties of a weakly damped oscillator is the *quality-factor (Q-factor)*. It can be defined/interpreted in different ways:

- It describes the ratio of the stored energy over the energy lost per period on the resonance, times 2π.

$$\begin{aligned} Q &= 2\pi \left.\frac{E}{\langle P_\mathrm{d}\rangle / T}\right|_{\text{on resonance}} = \\ &= \cancel{2\pi} \left.\frac{\cancel{m}\cancel{\omega^2}\, z_0^2(\omega)/\cancel{2}}{(\cancel{2\pi}/\omega)\gamma\cancel{\omega^2}\,\cancel{m}\, z_0^2(\omega)/\cancel{2}}\right|_{\text{on resonance}} = \\ &= \omega/\gamma\big|_{\text{on resonance}} \simeq \omega_0/\gamma, \end{aligned} \tag{4.103}$$

  This definition emphasizes how many oscillations the system makes before the energy significantly decays (if Q is large it will oscillate many times before it decays).

- When we discussed the free (undriven) damped oscillator we noticed that the energy decays exponentially with a rate γ. The Q-factor can therefore be interpreted as the ratio of oscillation frequency to energy decay rate of the free oscillator.

  The rate is half that for the amplitude.

- For the driven oscillator the ratio of the amplitude on resonance over the amplitude for low excitation frequencies is $z_{0,\max}/z_0(0) = Q$.

- We did show that the FWHM of the resonance peak for a weakly damped oscillator is given by $\Delta\omega = \omega_0/Q$, and thus the Q-factor can be interpreted as $Q = \omega_0/\Delta\omega$.

  Both the latter interpretations relate the Q-factor to the sharpness of the resonance. The higher the Q-factor, the sharper the resonance.

Filtering

In physics, a filter is a system that transmits a signal with a frequency-dependent attenuation. As the harmonic oscillator has a frequency-dependent response given by eq. (4.83) it performs as a filter:

$$\frac{z_0(\omega)}{F} = \frac{1}{m}\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}. \tag{4.104}$$

We call this the *transfer function* of the filter.

At low frequencies the response of the oscillator is frequency-independent, but for an overdamped oscillator the response will start to drop around

the natural frequency and the response will fall from then on $\propto \omega^{-2}$. The overdamped harmonic oscillator operates as a low-pass filter.

As it is described by a 2nd order differential equation, it is classified as a 2nd order low-pass filter.

If the oscillator is underdamped, a resonance peak will emerge, and the response aound the peak will selectively be enhanced around the natural frequency, the underdamped oscillator performs as a bandpass filter. High frequencies will be attenuated in the same way as for the overdamped oscillator.

When defining the filter response we need to be careful about the normalisation. The above statements are true for constant F, or, correspondingly, constant excitation acceleration amplitude $a_0 = F/m$.

Example 50:
A mass m is connected to a surface that is oscillating with a position given by $x(t) = x_0 \cos(\omega t)$ by a spring with spring constant k and a damping element with a damping factor γ. Find the transfer function $z_0(\omega)/x_0$.

Solution Newton's 2nd law gives

$$y''(t) = \frac{1}{m}\left[-k(y(t) - x(t)) - \gamma(y'(t) - x'(t)\right].$$

Thus,

$$y''(t) + \omega_0^y(t) + \gamma y'(t) = \omega_0^2 x(t) + \gamma x'(t) = \\ = \omega_0^2 x_0 \cos(\omega t) - \gamma x_0 \omega \sin(\omega t) = x_0 \Re\left((\omega_0^2 + \mathrm{i}\gamma\omega) e^{\mathrm{i}\omega t}\right).$$

Again, we can use the ansatz $y(t) = y_0 e^{\mathrm{i}\omega t}$, which yields this time

$$z_0(-\omega^2 - \mathrm{i}\gamma\omega + \omega_0^2) e^{\mathrm{i}(\cancel{\omega t} + \delta)} = x_0(\omega_0^2 + \mathrm{i}\gamma\omega) \cancel{e^{\mathrm{i}\omega t}}.$$

which gives for the transfer function

$$\frac{z_0(\omega)}{x_0} = \sqrt{\frac{\omega_0^4 + \gamma^2\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

which is 1 for $\omega \to 0$, for underdamped oscillators goes through the usual resonance and for $\omega \to \infty$ drops like ω^{-2}.

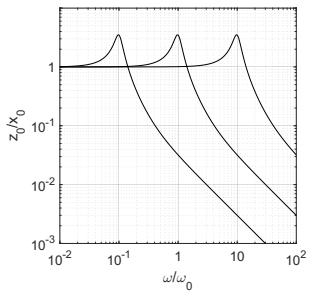


Figure 4.13: Filter response functions $z_0(\omega)/x_0$ for harmonic oscillators with resonances at $0.1\omega_0$, ω_0 and $10\omega_0$, with $1/Q = 0.3$ in all cases.

If we want to filter excitations at finite frequencies the strategy would be to chose a filter with as low a natural frequency as possible. For example, a bicycle tyre can be modelled as a damped harmonic oscillator. The rubber of the tyre is the source of heavy damping (it will deform, and the deformation will end ultimately in heat), with typical Q-factors between 2 to 5. The compressed air in the tyre acts like a spring, with a resonance frequency between 30 and 100 Hz. The higher the air pressure, the stiffer the tyre, and thus the higher the resonance frequency of the tyre becomes. Thus, a tyre at lower pressure will result in a more comfortable ride, at the cost of increased friction.

Acceleration spectra from seismic ground vibrations are typically uniform, and even increase for very low frequencies.

In case of a uniform acceleration spectrum, the optimisation is different. Because of the many different frequencies it does not make sense to look at a displacement amplitude at a single frequency, but instead we

look at the RMS displacement

$$\frac{z_{\rm rms}}{F/m} = \sqrt{\int_0^\infty \left(\frac{z_0(\omega)}{F/m}\right)^2 \mathrm{d}\omega} = \sqrt{\int_0^\infty \frac{\mathrm{d}\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = \\ = \sqrt{\frac{\pi}{2\omega_0^2\gamma}} = \sqrt{\frac{\pi Q}{2\omega_0^3}}. \quad (4.105)$$

To make this equation dimensionally correct, we would need to replace the acceleration F/m with an acceleration spectral density with units $\mathrm{ms}^{-2}/\sqrt{\mathrm{Hz}}$, but we will defer this subtlety to year 2, when we will discuss the frequency domain and spectral densities in more detail.

To achieve a small rms displacement we want a small Q-factor (strong damping), and a large resonance frequency (a stiff system). the reason is that the response of the oscillator at low frequencies is proportional to ω_0^{-2}, so, if the resonance frequency increases by a factor α the bandwidth of the filter increases by the factor α, while the magnitude of the response over this bandwidth drops by α^2, resulting in a net drop by $1/\alpha$.

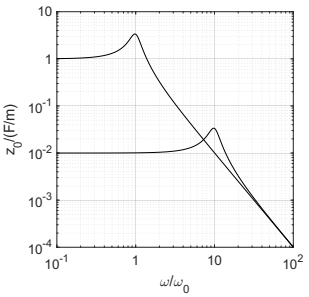


Figure 4.14: Filter response functions $z_0(\omega)/(F/m)$ for harmonic oscillators with resonances at ω_0 and $10\omega_0$, with $1/Q = 0.3$ in both cases.

Universality

Equations of the form eq. (4.78) are very common in physics,

$$\underbrace{my''}_{\text{inertia}} + \underbrace{\gamma y'}_{\text{damping}} + \underbrace{ky}_{\text{restoring}} = f(t) = \underbrace{F_0\cos(\omega t)}_{\text{periodic driving}}. \quad (4.106)$$

Table 4.3: Some undamped, damped, and driven oscillators across mechanical and electrical systems.

Type	Undamped oscillator	Damped oscillator	Driven oscillator
Mass–spring • Variable: Displacement y • Independent: t	$m\ddot{y}+ky=0$ • Inertia: m • Restoring: k • $\omega_0=\sqrt{\frac{k}{m}}$	$m\ddot{y}+\gamma\dot{y}+ky=0$ • Damping: γ (drag)	$m\ddot{y}+\gamma\dot{y}+ky=F_0\cos(\omega t)$ • Driving: external force
Torsional spring • Variable: Angle θ • Independent: t	$I\ddot{\theta}+k\theta=0$ • Inertia: I (moment of inertia) • Restoring: k • $\omega_0=\sqrt{\frac{k}{I}}$	$I\ddot{\theta}+\gamma\dot{\theta}+k\theta=0$ • Damping: γ (drag)	$I\ddot{\theta}+\gamma\dot{\theta}+k\theta=M_0\cos(\omega t)$ • Driving: external torque
Pendulum (for small angles) • Variable: Angle θ • Independent: t	$I\ddot{\theta}+mgl\theta=0$ • Inertia: $I=ml^2$ • Restoring: mgl (for small angles) • $\omega_0=\sqrt{\frac{g}{l}}$	$I\ddot{\theta}+\gamma\dot{\theta}+mgl\theta=0$ • Damping: γ (drag)	Not common
Acoustic (Helmholtz) resonator (Air plug oscillating against cavity compressibility) • Variable: Displacement x • Independent: t	$M_{\rm a}\ddot{x}+K_{\rm a}x=0$ • Inertia: $M_{\rm a}=\rho L_{\rm eff}A$ • Restoring: $K_{\rm a}=\frac{\rho c^2A^2}{V}$ • $\omega_0=\sqrt{\frac{K_{\rm a}}{M_{\rm a}}}=c\sqrt{\frac{A}{VL_{\rm eff}}}$	$M_{\rm a}\ddot{x}+\gamma\dot{x}+K_{\rm a}x=0$ • Damping: γ, due to viscous, thermal, and radiation losses	$M_{\rm a}\ddot{x}+\gamma\dot{x}+K_{\rm a}x=F_0\cos(\omega t)$ • Driving: External force (sound wave) • Equation can also be written as $\frac{L_{\rm eff}V}{c^2A}\ddot{p}+\frac{RV}{\rho c^2A^2}\dot{p}+p=-p_{\rm eff}(t)$
RLC circuit • Variable: Charge Q • Independent: t	$L\ddot{Q}+\frac{1}{C}Q=0$ • Inertia: L • Restoring: $\frac{1}{C}$ • $\omega_0=1/\sqrt{LC}$	$L\ddot{Q}+R\dot{Q}+\frac{1}{C}Q=0$ • Damping: R	$L\ddot{Q}+R\dot{Q}+\frac{1}{C}Q=V_0\cos(\omega t)$ • Driving: AC voltage source
Plasma oscillations (Electrons in a plasma) • Variable: Displacement x • Independent: t	$m_e\ddot{x}+kx=0$ • Inertia: m_e • Restoring: $k=\frac{ne^2}{\varepsilon_0}$ • $\omega_0=\sqrt{\frac{ne^2}{m_e\varepsilon_0}}$	$m_e\ddot{x}+R\dot{x}+kx=0$ • Damping: R due to collisions	$m_e\ddot{x}+R\dot{x}+kx=-eE_0\cos(\omega t)$ • Driving: external E field

The deeper reason for the general application is that close to an equilibrium point any potential can be approximated by

$$V(x) \simeq V(x_0) + \frac{1}{2}k(x-x_0)^2.$$

This approximation can be justified by a Taylor expansion of the potential around the equilibrium point.

The restoring force is then the derivative of the potential, and thus linear with the displacement.

The final ingredient then is a damping term that is proportional to the 1st derivative of the displacement function.

The practical implication of the similarity of the different equations is that if an approach to deal with the problem in one system works, the same will work in a different system. This has been used for example in analogue computing, which was a technique to solve differential equations based on *RLC* elements together with operational amplifiers, for example for the control of air or space craft, before the advent of digital computers.

If you want to explore this more, there will be a 2nd year practical, EL24, which explores this.

Key lessons:

- Any n^{th} order ODE can be expressed as a system of n 1st order ODEs, with n independent solutions..
- Conservative systems are described by differential equations of the form $\ddot{x} = F(x)$. Conservative systems are separable, and the constant of the separation can often be associated with the energy of the system.
- Dissipative systems are non-conservative. The energy in these system decays exponentially with a time constant given by the inverse of the linear damping term.
- For a linear homogeneous n^{th} order ODE, the n solutions form an n-dimensional vector space, and any solution can be expressed by a linear combination of these solutions.
- The linear independence of different solutions can be verified using the Wronskian.
- Linear homogeneous differential equations with constant coefficients can be solved with an ansatz $y(x) = \exp(\lambda x)$. This results in a characteristic polynomial, the n roots of which determine the n solutions for the ODE. Complex roots result in solutions that can be expressed by trigonometric functions.
- The harmonic oscillator is a 2nd order ODE. It describes a system with a restoring force that is proportional to the displacement, which is the case close to any equilibrium point in an arbitrary potential. The equation for a damped harmonic oscillator is $\ddot{x} + \gamma\dot{x} + kx = 0$.
- For weak damping (undercritical damping, $\frac{\gamma}{2\omega_0} < 1$) the displacement is an exponentially decaying sine (cosine) curve. For supercritical and critical damping the displacement decays exponentially. Critical damping gets to zero fastest in a monotonic sense.
- A particular solution for a linear ODE can be found using the method of undetermined coefficients.
- A driven harmonic oscillator displays a resonance at $\omega = \omega_0$. For small ω $z_0(0) \to F/(\omega_0^2 m)$ with no phase shift $\delta(0) \to 0$. For large ω the amplitude drops $\propto \omega^{-2}$ and the phase shift increases to $\delta \to -\pi$. The resonance is described by the Q value, which describes the ratio of the stored energy over the energy lost per period on the resonance, times 2π. The Q value for an underdamped harmonic oscillator is given by $Q = \omega_0/\gamma$.
- The high-frequency suppression of the amplitude can be used as a low-pass filter.

Chapter 5
Systems of ordinary differential equations equations

In the previous chapter we have seen how any n^{th} order ODE can be written and solved as a system of n 1^{st} order ODEs. However, at this point there is nothing that prevents us expand this formalism now to systems of ODEs with more than on dependent variables. As before, we can express this now as a system of 1^{st} order ODEs, just more of them.

We have seen that the general form of such a system is given by

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{f}(t), \tag{5.1}$$

where $\mathbf{y}(t)$ now is the vector of functions that solve the system of 1^{st} order ODEs, $\mathbf{A}$ is the matrix of coefficients that characterize this set of ODEs, and $\mathbf{f}(t)$ is a vector of inhomogeneities (forcing terms) for the ODEs.

Homogeneous linear systems with constant coefficients

A special case for which, as usual, the solutions can be found in a straightforward way, is the case of constant coefficients, which we will study here for the homogeneous case.

We have seen how we can deal with an additional inhomogeneity already (see p. 57).

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \tag{5.2}$$

where the coefficient matrix $\mathbf{A}$ is constant.

Let's start with assuming further that $\mathbf{A}$ is diagonal with elements $A_{ii} = a_i$ and $A_{ij} = 0$ for $i \neq j$. In that case, the system of differential equations splits into n independent 1^{st} order ODEs, each of which may be integrated immediately,

$$y_i'(t) = a_i y_i(t) \implies y_i(t) = C_i e^{a_i t}.$$

Therefore the general solution for $a_i \neq 0$ is

$$\mathbf{y}(t) = C_1 \underbrace{\begin{pmatrix} e^{a_1 t} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_1(t)} + C_2 \underbrace{\begin{pmatrix} 0 \\ e^{a_2 t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_2(t)} + \cdots + C_n \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ e^{a_n t} \end{pmatrix}}_{\mathbf{y}_n(t)} = C_1 e^{a_1 t} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_1(t)} + C_2 e^{a_2 t} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_2(t)} + \cdots + C_n e^{a_n t} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{\mathbf{y}_n(t)}$$

Note that these solutions are orthogonal even when the a_i are degenerate.

Hermitian systems

If the coefficient matrix $\mathbf{A}$ is not diagonal, we can diagonalize it, if it is Hermitian, $\mathbf{A} = \mathbf{A}^{\mathrm{T}*}$. The right eigenvectors are defined by

"Right" just reminds you the vector multiplies $\mathbf{A}$ on the right.

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \text{ for } i = 1, 2, \ldots, n, \tag{5.3}$$

and the λ_i are the eigenvalues which satisfy

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0, \tag{5.4}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. The left side in this equation is generally an n^{th} order polynomial with n roots that can be complex and degenerate. Note that because $\mathbf{A}$ constant, also the λ_i and $\mathbf{v}_i$ are constant.

This is similar to eq. (4.66).

We can collect the eigenvectors in the matrix

$$\mathbf{R} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}. \tag{5.5}$$

for which

$$\mathbf{A}\mathbf{R} = \mathbf{R}\boldsymbol{\lambda} = \mathbf{R}\mathbf{D}, \tag{5.6}$$

where we have introduced the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{5.7}$$

Then, for a general solution $\mathbf{y}(t) = \mathbf{R}\mathbf{c}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}\mathbf{c}(t)) = \mathbf{R}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}(t) = \mathbf{A}\mathbf{R}\mathbf{c}(t) = \mathbf{R}\mathbf{D}\mathbf{c}(t), \tag{5.8}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}(t) = \mathbf{D}\mathbf{c}(t), \tag{5.9}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_n(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_n(t) \end{pmatrix}. \tag{5.10}$$

This is a system of homogeneous first order differential equations with a diagonal coefficient matrix with the solution

$$\mathbf{c}(t) = \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}. \tag{5.11}$$

Substituting this general solution for the components of $y(t)$, we may obtain the solution in the original basis

$$\mathbf{y}(t) = \mathbf{R}\mathbf{c}(t) = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_3 e^{\lambda_3 t} \end{pmatrix} = \\ = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n t}\mathbf{v}_n, \quad (5.12)$$

which we can also write as

$$\mathbf{y}(t) = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \underbrace{\mathbf{R}\mathbf{E}(t)}_{\mathbf{Y}(t)}\mathbf{c}. \quad (5.13)$$

Here we have introduced an $n \times n$ *evolution matrix* $\mathbf{E}(t)$, that depends on the independent variable (here t).

This relation also has to hold for the initial condition, $\mathbf{y}(t_0) = \mathbf{R}\mathbf{E}(t_0)\mathbf{c}$, and thus we can find the coefficients from

$$\mathbf{c} = \underbrace{(\mathbf{E}(t_0))^{-1}\mathbf{R}^{-1}}_{(\mathbf{Y}(t))^{-1}}\mathbf{y}(t_0), \quad (5.14)$$

which we can now use in the general solution,

$$\mathbf{y}(t) = \mathbf{R}\mathbf{E}(t)\,(\mathbf{E}(t_0))^{-1}\mathbf{R}^{-1}\mathbf{y}(t_0) = \\ = \mathbf{R}\begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} e^{-\lambda_1 t_0} & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2 t_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda_n t_0} \end{pmatrix} \mathbf{R}^{-1}\mathbf{y}(t_0) = \\ = \mathbf{R}\mathbf{E}(t - t_0)\mathbf{R}^{-1}\mathbf{y}(t_0). \quad (5.15)$$

If the problem is inhomogeneous, we can then use the method of variation of constants, eq. (4.52),

$$\mathbf{y}(t) = \underbrace{\mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\mathbf{y}(t_0)}_{y_\mathrm{h}(t)} + \underbrace{\mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(t')\mathbf{f}(t')\,\mathrm{d}t'}_{y_\mathrm{p}(t)} = \\ = \mathbf{R}\mathbf{E}(t)\,[\mathbf{R}\mathbf{E}(t_0)]]^{-1}\mathbf{y}(t_0) + \mathbf{R}\mathbf{E}(t)\int_{t_0}^{t}\left[\mathbf{R}\mathbf{E}(t')\right]]^{-1}\mathbf{f}(t')\,\mathrm{d}t' = \\ = \mathbf{R}\mathbf{E}(t - t_0)\mathbf{R}^{-1}\mathbf{y}(t_0) + \int_{t_0}^{t}\mathbf{R}\mathbf{E}(t - t')\mathbf{R}^{-1}\mathbf{f}(t')\,\mathrm{d}t'. \quad (5.16)$$

Example 51:
Solve the inhomogeneous system of differential equations

$$u' = 3u + 2v + 4e^{5t},$$
$$v' = u + 2v,$$

for the two functions $u(t)$ and $v(t)$.

Solution Using the notation from the previous chapter we have

$$\mathbf{y}(t) = \begin{pmatrix} u \\ v \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \text{ and } \mathbf{f}(t) = \begin{pmatrix} 4e^{5t} \\ 0 \end{pmatrix}.$$

We are first focussing on the homogeneous equation and look for the eigenvalues,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - 5\lambda + 4 = 0,$$

which yields the two eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$. The two corresponding eigenvectors are

$$(\mathbf{A} - 4 \times \mathbf{I}) y_1 = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies a = 2b \implies y_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$(\mathbf{A} - 1 \times \mathbf{I}) y_2 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies a = -b \implies y_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus the general solution of the homogeneous equation is

$$y(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}.$$

To find the particular solution using the method of variation of constants we have

$$\mathbf{Y} = \begin{pmatrix} 2e^{4t} & e^{t} \\ e^{4t} & -e^{t} \end{pmatrix} \text{ and } \mathbf{Y}^{-1} = \frac{1}{-3e^{5t}} \begin{pmatrix} -e^{t} & -e^{t} \\ -e^{4t} & 2e^{4t} \end{pmatrix},$$

so that we get

$$\begin{aligned} \mathbf{y}_\text{p}(t) &= -\frac{1}{3} \begin{pmatrix} 2e^{4t} & e^{t} \\ e^{4t} & -e^{t} \end{pmatrix} \int^t \begin{pmatrix} -e^{t'} & -e^{t'} \\ -e^{4t'} & 2e^{4t'} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \mathrm{d}t' = \\ &= \frac{1}{3} \begin{pmatrix} 2e^{4t} & e^{t} \\ e^{4t} & -e^{t} \end{pmatrix} \int^t \begin{pmatrix} 4e^{t'} \\ 4e^{4t'} \end{pmatrix} \mathrm{d}t' = \\ &= \frac{1}{3} \begin{pmatrix} 2e^{4t} & e^{t} \\ e^{4t} & -e^{t} \end{pmatrix} \begin{pmatrix} 4e^{t} \\ e^{4t} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t}, \end{aligned}$$

and the complete general solution

$$y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t}.$$

Alternatively, we could have guessed an ansatz

$$y_\text{p}(t) = \begin{pmatrix} r \\ s \end{pmatrix} e^{5t},$$

and plugged this into the inhomogeneous system of ODEs, to find the constants r and s.

Non-Hermitian systems

If the $n \times n$ matrix $\mathbf{A}$ does not have n linearly independent eigenvectors, it is not diagonalizable. This happens when the multiplicity of at least one of the eigenvalues λ_i is larger than the number of associated independent eigenvectors. A normal matrix is never defective. A complete basis for a defective matrix can be formed by augmenting the eigenvectors with *generalized eigenvectors*.

Such a matrix is called a *defective* matrix.

Normal matrices are matrices for which $\mathbf{A} = \mathbf{A}^*$, which includes Hermitian and unitary matrices as special cases.

Let's assume that we have an eigenvalue λ with a multiplicity of m, but only one eigenvector $\mathbf{v}_1$ that satisfies the condition $\mathbf{A}\mathbf{v}_1 = \lambda \mathbf{v}_1$. We can

then find the generalised eigenvectors from a *Jordan chain*.

$$\begin{aligned}
\mathbf{A}\mathbf{v}_1 &= \lambda\mathbf{v}_1,\\
\mathbf{A}\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1,\\
&\vdots\\
\mathbf{A}\mathbf{v}_m &= \lambda\mathbf{v}_m + \mathbf{v}_{m-1}.
\end{aligned} \tag{5.17}$$

These can be rewritten as

$$\left\{\begin{array}{r} (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}\\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1\\ \vdots\\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_m = \mathbf{v}_{m-1}\end{array}\right\} \Longrightarrow \left\{\begin{array}{r} (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}\\ (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}\\ \vdots\\ (\mathbf{A}-\lambda\mathbf{I})^m\mathbf{v}_m = \mathbf{0}\end{array}\right\} \tag{5.18}$$

Combining all the eigenvectors and the generalised eigenvectors for all the eigenvalues of the system gives a set of n components that we can combine again into a matrix $\mathbf{R}$ that we can use to block-diagonalize $\mathbf{A}$,

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \begin{pmatrix} \left(\mathbf{J}_1\right) & 0 & \cdots & 0\\ 0 & \left(\mathbf{J}_2\right) & & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & \left(\mathbf{J}_s\right)\end{pmatrix}, \text{ where } \left(\mathbf{J}_i\right) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0\\ 0 & \lambda_i & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \lambda_i & 1\\ 0 & \cdots & \cdots & 0 & \lambda_i\end{pmatrix}, \tag{5.19}$$

Note that for a non-degenerate eigenvalue $m_i = 1$, and thus the Jordan matrix is just the 1×1 matrix λ_i, consistent with the previous section.

where s is the number of distinct eigenvalues, and each Jordan block $\left(\mathbf{J}_i\right)$ is a $m_i \times m_i$ square matrix with λ_i in the diagonal and 1 in each element of the superdiagonal.

To find the solutions for the system, we will first focus on a single Jordan block. For the block $\mathbf{J} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R}$, which can be recast as $\mathbf{A} = \mathbf{R}\mathbf{J}\mathbf{R}^{-1}$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_i(t) = \mathbf{A}\mathbf{y}_i(t) = \mathbf{R}\mathbf{J}\mathbf{R}^{-1}\mathbf{y}_i(t) \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(\underbrace{\mathbf{R}^{-1}\mathbf{y}_i(t)}_{\boldsymbol{\xi}(t)}) = \mathbf{J}\underbrace{\mathbf{R}^{-1}\mathbf{y}_i(t)}_{\boldsymbol{\xi}(t)}, \tag{5.20}$$

for the vector $\mathbf{y}_i(t)$ of m_i solutions associated with the Jordan block $\left(\mathbf{J}_i\right)$. This can be written in matrix form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{m_i}\end{pmatrix} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0\\ 0 & \lambda_i & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \lambda_i & 1\\ 0 & \cdots & \cdots & 0 & \lambda_i\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{m_i}\end{pmatrix}, \tag{5.21}$$

or in components,

$$\begin{aligned}
\xi_1' &= \lambda_i\xi_1 + \xi_2,\\
&\vdots\\
\xi_{m_i-1}' &= \lambda_i\xi_{m_i-1} + \xi_{m_i},\\
\xi_{m_i}' &= \lambda_i\xi_{m_i}.
\end{aligned} \tag{5.22}$$

We can solve this system of 1st order ODEs from the tail, and then plugging in the solution at every stage into the previous equation to get

$$\begin{aligned}
\xi_{m_i} &= C_{m_i} e^{\lambda_i t}, \\
\xi_{m_i-1} &= \left(C_{m_i-1} + C_{m_i} t\right) e^{\lambda_i t}, \\
&\vdots \\
\xi_1 &= \left(C_1 + C_2 x + \cdots + C_{m_i} \frac{t^{m_i-1}}{(m_i-1)!}\right) e^{\lambda_i t},
\end{aligned} \tag{5.23}$$

We can substitute this back to get the general solution

$$\mathbf{y}_i(t) = \mathbf{R}\boldsymbol{\xi} = \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{m_i} \\ | & | & & | \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{m_i-2}}{(m_i-2)!}e^{\lambda_i t} & \frac{t^{m_i-1}}{(m_i-1)!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{m_i-2}}{(m_i-2)!}e^{\lambda_i t} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & \cdots & \cdots & 0 & e^{\lambda_i t} \end{pmatrix}}_{\mathbf{E}(t)} \underbrace{\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{m_i-1} \\ C_{m_i} \end{pmatrix}}_{\mathbf{c}} =$$

$$\begin{aligned}
&= C_1 \underbrace{\mathbf{v}_1 e^{\lambda_i t}}_{\mathbf{y}_1(t)} + C_2 \underbrace{(t\mathbf{v}_2 + \mathbf{v}_1) e^{\lambda_i t}}_{\mathbf{y}_2(t)} + \cdots \\
&\qquad \cdots + C_{m_i} \underbrace{\left(\frac{t^{m_i-1}}{(m_i-1)!}\mathbf{v}_1 + \cdots + t\mathbf{v}_{m_i-1} + \mathbf{v}_{m_i}\right) e^{\lambda_i t}}_{\mathbf{y}_{m_i}(t)}.
\end{aligned} \tag{5.24}$$

Note that for a non-degenerate eigenvalue, this equation collapses to $\mathbf{y}(t) = C\mathbf{v}e^{\lambda t}$, as in the last section.

Each Jordan block will have a similar set of solutions, so that the general solution is

$$\mathbf{y}(t) = \sum_{i=1}^{s} \mathbf{y}_i(t), \tag{5.25}$$

with a total of $\sum_{i=1}^{s} m_i = n$ contributing solutions.

Example 52:
Solve the homogeneous system of differential equations

$$\begin{aligned}
u' &= -u + v - 2w, \\
v' &= 4u + v, \\
w' &= 2u + v - w,
\end{aligned}$$

for the functions $u(t)$, $v(t)$ and $w(t)$.

Solution We have

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}.$$

First, we find the eigenvalues of $\mathbf{A}$ from

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1+\lambda)(1-\lambda)(1+\lambda) + 1 \times [4 \times (1+\lambda)] - 2 \times [4 - 2 \times (1-\lambda)] = \\
&= (1+\lambda)^2(1-\lambda) = 0,
\end{aligned}$$

which gives the eigenvalues $\lambda_1 = 1$ and $\lambda_{2,3} = -1$.

The eigenvector associated with $\lambda_1 = 1$ is given by

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{v}_1 = \begin{pmatrix} -1-1 & 1 & -2 \\ 4 & 1-1 & 0 \\ 2 & 1 & -1-1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \quad a = 0 \wedge b = 2c \quad \implies \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},$$

and the eigenvector for $\lambda_{2,3} = -1$ by

$$(\mathbf{A} - \lambda_{2,3} \mathbf{I})\mathbf{v}_2 = \begin{pmatrix} -1+1 & 1 & -2 \\ 4 & 1+1 & 0 \\ 2 & 1 & -1+1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \quad b = 2z \wedge a = -\frac{b}{2} = -c \quad \implies \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

For the third solution required for a system of three 1^{st} order ODEs we need an additional generalized eigenvector, which we get from

$$(\mathbf{A} - \lambda_{2,3} \mathbf{I})^2 \mathbf{v}_3 = \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -8 \\ 4 & 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\implies \quad a + b = c \quad \implies \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = C_1 \mathbf{v}_1 e^{\lambda_1 t} + [C_2 \mathbf{v}_2 + C_3(\mathbf{v}_2 t + \mathbf{v}_3)]\, e^{\lambda_2 t} =$$

$$= C_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^t + \left[C_2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + C_3 \left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \right] e^{-t}.$$

Appendix A
Examples for symbolic software integration

A.1 MATHEMATICA

You can get a licensed version of MATHEMATICA through the university. Go to `register.it.ox.ac.uk/self/index>Software`. Choose the Home/Personal/Student option and use the instructions to create an account and install it on your computer.

We wil start with something simple; let's solve $x^2y' + 3xy = \frac{\cos x}{x}$ (Example 20).

```
ClearAll;
eqn = x^2*y'[x] + 3 x*y[x] == Cos[x]/x;
solution = DSolve[eqn, y[x], x]
```

which gives

$$\text{Out[3]=}\ \left\{\left\{y[x] \to \frac{C_1}{x^3} + \frac{\text{Sin}[x]}{x^3}\right\}\right\}$$

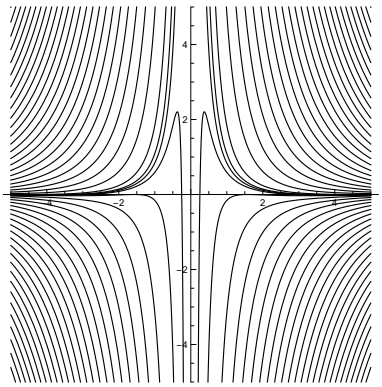
similar to what we found before.

Now let's draw the integral lines. First we will prepare a table of aesthetically suitable integration constants

```
Cval = Table[index^2, {index, 1, 50, 1}];
Cval = Join[Reverse[-Cval], {-0.25}, {0}, {0.25}, Cval];
```

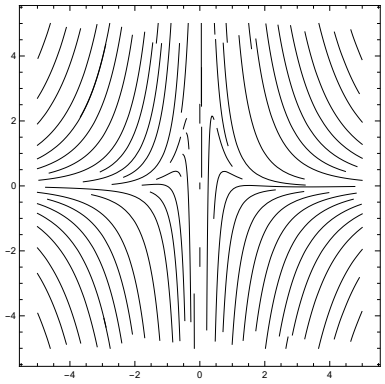
with this we can now plot the corresponding integral lines, using a table of plots

```
sol = ySol = (y[x] /.  solution) /.  C[1] -> Cval;
Plot[sol, {x, -5, 5}, PlotRange -> {-5, 5},
  AspectRatio -> 1,
  PlotStyle -> {{Black, Thickness[0.003]}}]
```



Alternatively, we can plot the streamlines using STREAMPLOT

```
plot = StreamPlot[{1, Cos[x]/x^3 - 3 y/x}, {x, -5, 5},
  {y, -5, 5}, StreamScale -> None, StreamStyle -> {Black},
  StreamColorFunction -> None]
```



Note that for this we do not need to solve the differential equation.

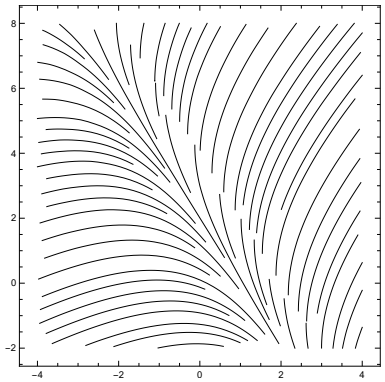
If we want to solve the ODE for a given starting condition, we need to enclose the equation and the initial conditions in a list. For example

```
ClearAll;
eqn = x^2*y'[x] + 3 x*y[x] == Cos[x]/x;
cond = y[1] == 1;
solution = DSolve[{eqn, cond}, y[x], x]
```

$$\left\{\left\{y[x] \to \frac{1 - \text{Sin}[1] + \text{Sin}[x]}{x^3}\right\}\right\}$$

Now to something more tricky: Solve $y' = \frac{2x+y+2}{2x+y-4}$ (Example 16). Let's start with the STREAMLINE plot,

```
StreamPlot[{1, (2x + y + 2)/(2x + y - 4)}, {x, -4, 4},
  {y, -2, 8},StreamScale -> None, StreamStyle -> {Black},
  StreamColorFunction -> None]
```



So, what do we get when we solve the equation?

```
ClearAll;
eqn = y'[x] == (2x + y[x] + 2)/(2x + y[x] - 4);
solution = DSolve[eqn, y[x], x,
  IncludeSingularSolutions -> True]
```

gives

DSolve: Inverse functions are being used by DSolve, so some solutions may not be found.

DSolve: Inverse functions are being used by DSolve, so some solutions may not be found.

$$\left\{\left\{y[x] \to -2\,(-2+x) - 2\left(1 + \text{ProductLog}\left[-e^{-1-\frac{3x}{2}+c_1}\right]\right)\right\},\right.$$
$$\left.\{y[x] \to -2\,(-1+x)\}\right\}$$

Note three things. First, there are warning messages. We will come back to these in a moment. Second, we had to explicitly request to in-

PRODUCTLOG denotes the Lambert W function, also called the omega function or product logarithm. It gives the solution for w in $z = we^w$, where $z \in \mathbb{C}$. This function is multivalued but the default for MATHEMATICA is to use the principal value.

clude the singular solutions (of which there is one). Finally, the general solution is explicit, at the cost of using the `ProductLog` function. In fact, there is no direct way to get `DSolve` in MATHEMATICA to produce implicit solutions. However, here the explicit solution can be solved for the integration constant to give an implicit form.

```
ClearAll;
eqn = y'[x] == (2x + y[x] + 2)/(2x + y[x] - 4);
solution = First@DSolve[eqn, y[x], x,
  GeneratedParameters -> c]
impl = Solve[solution /.  Rule -> Equal, c[1]][[1, 1]]
  /.  Rule -> Equal
```

Most of the complications in this code are to select the right elements in output arrays of the solutions, and due to the fact that MATHEMATICA distinguishes between $a \to b$, which is a transformation rule, and $a == b$, which represents a symbolic equation. Welcome to the somewhat byzantine syntax of MATHEMATICA.

which gives

DSolve: Inverse functions are being used by DSolve, so some solutions may not be found.

$$\left\{y[x] \to -2\,(-2+x) - 2\left(1 + \text{ProductLog}\left[-e^{-1-\frac{3x}{2}+c[1]}\right]\right)\right\}$$

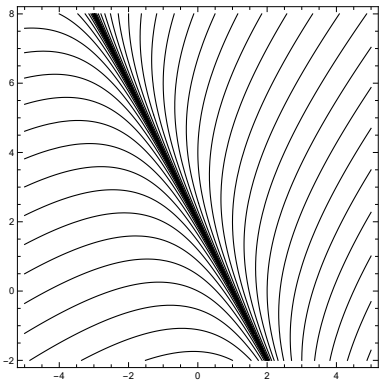
Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$c[1] == \frac{1}{2}\left(2 + 3x + 2\,\text{Log}\left[-\frac{1}{2}\,e^{1+\frac{1}{2}\,(-2x-y[x])}\,(2 - 2x - y[x])\right]\right)$$

which agrees with our analytical solution after some transformation of the integration constant. We can now plot this as a `ContourPlot` with different contours for different values of the integration constant

```
ContourPlot[
  Evaluate@
    Table[impl /.  Log[z_] :> Log[Abs@z] /.  y[x] -> y,
    {c[1], -10, 10, 1/2}], {x, -5, 5}, {y, -2, 8},]
    MaxRecursion -> 3]
```

Note that for this the logarithm (which in MATHEMATICA is defined for complex arguments), is replaced with $\ln|z|$ for a real z. See also Appendix B.



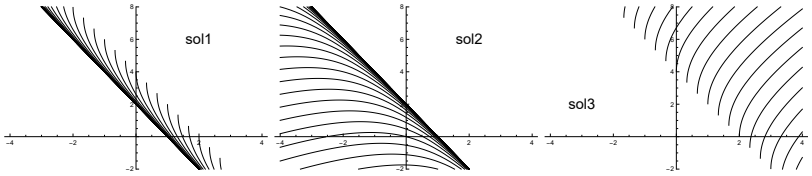
Alternatively, we can plot the explicit solution, but we must include all branches of the PRODUCTLOG function

See also Appendix B.

```
sol1 = Table[(y[x] /.  solution) /.  C[1] -> Cval,
  {Cval, -20, 20, 0.5}];
sol2 = Table[(y[x] /.  solution) /.  C[1] -> Cval - Pi*I,
  {Cval, -20, 20, 0.5}];
sol3 = Table[(y[x] /.  solution) /.  C[1] -> Cval /.
  ProductLog -> (ProductLog[-1, #] &),
  {Cval, -20, 20, 0.5}];
sol = Join[sol1, sol2, sol3];
Plot[sol, {x, -4, 4}, PlotRange -> {-2, 8},
  PlotStyle -> {{Black, Thickness[0.003]}}]
```

`ProductLog -> (ProductLog[-1, #] &)` replaces `ProductLog(x)` with a function that returns `ProductLog(-1,x)`, i.e. the -1 branch of the Lambert W function evaluated at x.

with the different branches which need to get joined



A.2 MATLAB

The University of Oxford has a campus-wide license for MATLAB, which also applies for students. Access through `https://uk.mathworks.com/academia/tah-portal/university-of-oxford-30972477.html`.

We will illustrate the use of MATLAB in solving the equation $y' = \frac{2x+y+2}{2x+y-4}$ (Example 12) and plotting its integral lines.

Again it is good practise to start each MATLAB programme with clearing all variables

```
clear all
```

To use symbolic functions in MATLAB we first need to define the symbols used in the functions using the syms command

```
syms y(x)
```

Next we need to define the function

```
eqn = diff(y,x) == (2*x+y(x)+2)/(2*x+y(x)-4);
```

In this line `diff(x,y)` stands for the derivative dy/dx and note that the equality in the equation needs to be written using ==. Then we can solve the equation using

```
solutions = dsolve(eqn,Implicit=true)
```

where we used `Implicit=true`, because the explicit solution cannot be expressed through elementary functions. (See what do you get when you leave it out.)

We get the solutions

```
solutions =
```

$$\begin{pmatrix} 2x - 2\log(2x + y(x) - 2) + y(x) + 2 = C_1 + 3x \\ \frac{2x + y(x) - 2}{2x + y(x) - 4} = 0 \end{pmatrix}$$

which is what we expect, it even includes the singular solution.

Hoewever, the general solution differs from the solution we obtained analytically, as it is missing the absolute value within the logarithm. This is due to the way symbolic integration in MATLAB is set up. The consequence is that if we use real integration constants to draw the integral lines, we are missing half the solutions, as we can see when we plot the result. For this we need to rearrange the implicit solution found before into a plotable MATLAB function.

The result of a symbolic integration of $1/x$ gives

```
syms x; int(1/x)
```

```
ans =
```

$\log(x)$

MATLAB's `syms` command, by default, declares symbolic variables as complex. For complex variables, $\ln(z)$ is defined for all $z \neq 0$. The absolute value is not needed when considering the complex logarithm. Practically, for the solution to our differential equation the solutions with the negative argument can be restored by addding a complex integration constant -2πi. The exact justification requires complex analysis, beyond our current level (see also Appendix B).

The lesson is that when using mathematics software you need to first, always look critically at the result, and second, understand exactly the definitions used by the software.

```
syms u %new variable u needed to replace y
C1val = -20:20;
figure
hold on
for i = 1:length(C1val)
   % rearrange function
   impl = subs(lhs(solutions(1))-rhs(solutions(1)), ...
      [sym("C1"),y],[C1val(i),u])
```

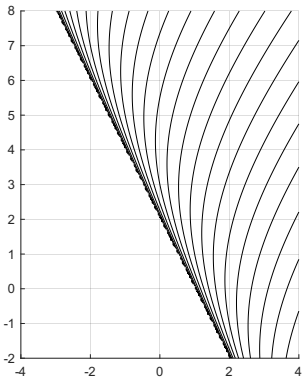
```
impl =
```

$u - x - 2\log(u + 2x - 2) + 22$

```
   % convert symbolic function into Matlab function
   impl = matlabFunction(impl,'Vars',[x,u]);
```

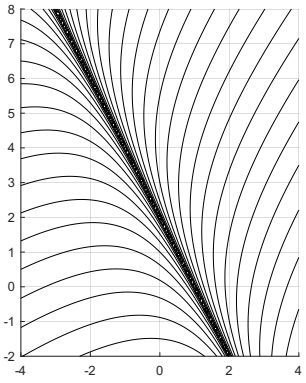
```
impl = function_handle with value:
    @(x,u)u-x-log(u+x.*2.0-2.0).*2.0+2.2e+1
```

```
    % plot
    fimplicit(impl,"Color","black")
end
hold off
```



For the complete solution we need to use the solution with the absolute value in the logarithm

```
C1val = -20:20;
figure
hold on
for i = 1:length(C1val)
    f2=@(x,u)u-x-log(abs(u+x*2.0-2.0))*2.0 + 2.0 - ...
       C1val(i);
    fimplicit(f2,"Color","black")
end
hold off
```



Alternatively, we can use complex integration constants as outlined in the marginnote.

To solve the Cauchy problem we can add a starting condition

```
clear all
syms y(x)
eqn = diff(y,x) == (2*x+y(x)+2)/(2*x+y(x)-4);
cond = y(1) == 1;
solutions = dsolve(eqn,cond,Implicit=true)
```

solution = $y(x) - 2\log(2x + y(x) - 2) = x$

A.3 python

python compilers are widely available as open source. I use the Spyder IDE (https://www.spyder-ide.org/).

Again, let's start with the simple example. First we have to import the required functions, and also the symbolic variable x,

```
from sympy import Function, dsolve, Eq, Derivative,
   sin, cos, symbols
from sympy.abc import x
```

Next we need to define y as a symbolic function

```
y = Function('y')
```

Finally we set up the differential equation and solve it

```
eqn = Eq(x**2*Derivative(y(x), x)+3*x*y(x),cos(x)/x)
solution = dsolve(eqn, y(x))
print(solution)
```

and the result is

`Eq(y(x), (C1 + sin(x))/x**3)` as expected.

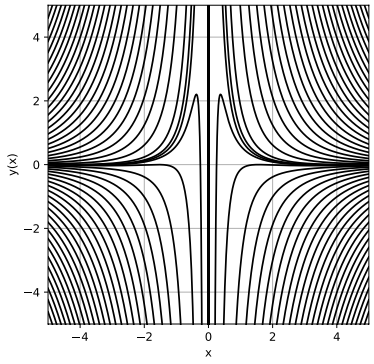
We can plot this with

```
import numpy as np
import matplotlib.pyplot as plt
from sympy import lambdify

C1 = symbols('C1')

C1arr = (np.arange(1,50))**2
C1arr = np.append(np.append(-C1arr[::-1],
     np.array([-0.25,0.,0.25])),C1arr)

plt.figure(figsize=(5, 5))
for index in range(len(C1arr)):
# Substitute integration constant
  rhs_specific = solution.rhs.subs(C1, C1arr[index])
# Convert symbolic expression to numerical function
  f = lambdify(x, rhs_specific, modules='numpy')
  x_vals = np.linspace(-5, 5, 500)
  y_vals = f(x_vals)
  plt.plot(x_vals, y_vals, color='black')
plt.xlabel('x')
plt.ylabel('y(x)')
plt.grid(True)
plt.xlim(-5, 5)
plt.ylim(-5, 5)
plt.show()
```



What do we get when we try the more tricky example, $y' = \frac{2x+y+2}{2x+y-4}$ (Example 12)?

```
from sympy import Function, dsolve, Eq, Derivative
from sympy.abc import x

y = Function('y')
eqn = Eq(Derivative(y(x),x),
     (2*x + y(x) + 2)/(2*x + y(x) - 4))
solution = dsolve(eqn, y(x))

print(solution)
```

gives

```
[Eq(y(x), -2*x - 2*LambertW(-E*sqrt(C1*exp(-3*x))/2) + 2),
Eq(y(x), -2*x - 2*LambertW(E*sqrt(C1*exp(-3*x))/2) + 2)]
```

There is no option to force `python` to give an implicit solution. It will only supply an implicit solution if it is not able to find an explicit solution. Here it finds two, corresponding to a real argument for Lambert's W function.

Interestingly, a symbolic integration of $1/x$ in `python` does give $\ln x$, so it also operates in the complex plane. The two solutions to our differential equation are convenient, but not quite consistent.

Plotting the result is complicated by the different branches of the Lambert W function.

```
import numpy as np
import matplotlib.pyplot as plt
from sympy import Function, dsolve, Eq, Derivative,
     symbols, lambdify
from sympy.abc import x
from scipy.special import lambertw

# constant symbol and C1 array
C1 = symbols('C1')
C1arr = (np.arange(0, 20, 0.5))**5

# plotting grid
x_min, x_max = -5.0, 5.0
x_vals = np.linspace(x_min, x_max, 500)

# tolerance for treating values as real
IMAG_TOL = 1e-8

plt.figure(figsize=(6, 6))

# helper wrapper to map SymPy's LambertW to scipy's
# lambertw for chosen branch
def lambert_wrapper_k(k):
  def _w(z):
    return lambertw(z, k)
  return _w

# helper to convert evaluated y (possibly complex) into
# real-or-nan array
def real_if_ok(arr):
  z = np.asarray(arr, dtype=complex)
  ok = np.isfinite(z.real) & (np.abs(z.imag) <= IMAG_TOL)
  out = np.full(z.shape, np.nan, dtype=float)
  out[ok] = z.real[ok]
  return out, ok
```

```
# ensure solution is iterable
sols = solution if isinstance(solution,
     (list, tuple)) else [solution]

# plot each C1, each solution returned by dsolve,
# and two LambertW branches
for Cval in C1arr:
  for sol in sols:
    expr = sol.rhs if hasattr(sol, "rhs") else sol
    for branch in (0, -1):
      # lambdify mapping:
      #LambertW -> scipy.special.lambertw with branch k
      lam_map = {"LambertW":  lambert_wrapper_k(branch)}
      f = lambdify(x, expr.subs(C1, Cval),
          modules=[lam_map, "numpy"])

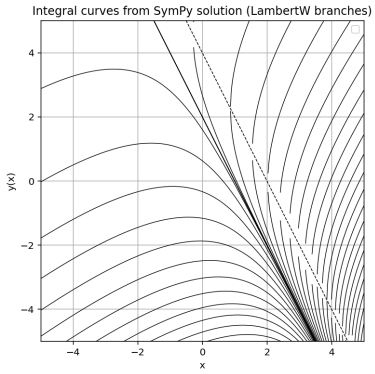
      # evaluate on grid
      try:
        y_vals = f(x_vals)
      except Exception as e:
        print(f"Evaluation failed for C1={Cval},
          branch={branch}:  {e}")
        continue

      # convert to real where appropriate, mask others
      y_real, ok_mask = real_if_ok(y_vals)

      # restrict to indices where ok_mask is True
      idxs = np.nonzero(ok_mask)[0]
      if idxs.size == 0:
        continue
      runs = np.split(idxs,
          np.where(np.diff(idxs) != 1)[0] + 1)
      for run in runs:
        if run.size >= 2:  # at least two points to plot
        plt.plot(x_vals[run], y_real[run], color='black',
          linewidth=0.7)

# draw asymptote 2x+y-4=0 for reference
xs = np.linspace(x_min, x_max, 400)
plt.plot(xs, 4 - 2*xs, '-', color='black', linewidth=0.8)

plt.xlabel('x')
plt.ylabel('y(x)')
plt.grid(True)
plt.xlim(x_min, x_max)
plt.ylim(-5, 5)
plt.legend()
plt.title('Integral curves from SymPy solution
     (LambertW branches)')
plt.show()
```



`python` also has a cute function that will classify an ODE, which can help find an approach to solving an ODE, and can be used to give as a hint to `dsolve`,

```
from sympy import classify_ode
print(classify_ode(eqn, y(x)))
```

Appendix B
The indefinite integral of $1/x$

The indefinite integral of $1/x$ is a surprisingly tricky thing, in particular in the context of symbolic integrations performed by different MATHEMATICAl software packages. The actual result depends crucially on the domains for which the logarithm is defined.

First, the easy thing, is to show that $\mathrm{d}\ln(x)/\mathrm{d}x$ is $1/x$:

$$y = \ln(x),$$
$$\exp(y) = x.$$

This derivation does rely on the Fundamental Theorem of Calculus that establishes a crucial link between differentiation and integration, as inverse operations. There are more elaborate derivations that do not rely on this.

After differentiating both sides,

$$\exp(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\exp(y)} = \frac{1}{x}.$$

The difficulties arise when we look at the integral version of this. The first thing to notice is that (for real x) an indefinite integral of $1/x$ cannot cover the whole domain of $\mathbb{R}$; because of the singularity at $x = 0$ we cannot integrate from negative to positive x. The solution falls therefore into two parts

$$\int \frac{1}{x}\,\mathrm{d}x = \begin{cases} \ln(-x) + C_- & , x < 0 \\ \ln(x) + C_+ & , x > 0 \end{cases},$$

with $C_-, C_+ \in \mathbb{R}$ are integration constants, which in principle can be different, as the domain is not path-connected. In practice we can combine this to

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C, \tag{B.1}$$

which is the form that I will use to integrate $1/x$ when we solve differential equations analytically in this course.

A more general approach is to allow for a complex argument of the logarithm. You have seen in your complex numbers course that in that case $z = |z|\exp(\mathrm{i}\theta) = |z|\exp(\mathrm{i}(\theta + 2n\pi)$ with n an integer number, and

$$\ln z = \ln\left(|z|e^{\mathrm{i}(\theta+2n\pi)}\right) = \ln|z| + \mathrm{i}(\theta + 2n\pi).$$

This is how common mathematics software packages like MATHEMATICA or MATLAB define their symbolic logarithm function. With that definition

$$\int \frac{1}{z}\mathrm{d}z = \ln z + C,$$

which is more general than eq. (B.1), but now requires a complex constant for negative values of x,

$$\ln(x) = \ln(-x) + \mathrm{i}\pi.$$

Similar considerations do apply to the Lambert W function, which gives the solution for w in $z = we^w$. Alternatively, it can be defined through the differential equation

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{1}{z}\frac{W(z)}{W(z)+1}.$$

It is required, for example, for explicit solutions of Example 12.

Compare to the natural logarithm $\ln(z)$, which gives the solution for y in $z = e^y$.

Compare again to the natural logarithm, which can be defined by

$$\frac{\mathrm{d}(\ln z)}{\mathrm{d}z} = \frac{1}{z}.$$

.

Index

A-stability, 35
  of multistep methods, 40
  of Runge-Kutta methods, 37
Almost homogeneous ODEs, 14
Amplification argument, 36
Antiderivative, 4
Apparent power, 68
Autonomous ODEs, 11, 44

Bandwidth, 67
Boundary condition, 6
Butcher tableau, 32

Cauchy–Lipschitz theorem, 6
Characteristic equation, 60
Characteristic polynomial, 60
Complementary function, 21
Composite solution, 10
Conservative systems, 46
Consistency of numerical method, 28
Convergence of numerical method, 28

Dahlquist equivalence theorem, 29
Direction field, 9
Dissipative system, 48

Euler method, 27
Evolution matrix, 75
Exact differentials, 16
Existence theorem, 6
Explicit form (1st order ODE), 9
Explicit solution, 4

Filter, 69
Fixed points, 48
Full width at half maximum, 67
Fundamental system, 52
Fundamental theorem of numerical analysis, 29

General solution, 6

Harmonic oscillator
  Damped, 48, 63
  Driven, 65
  Undamped, 45, 62
Homogeneous differential equations
  Linear homogeneous ODEs, 3
  Uniform homogeneous ODEs, 12

Implicit RK2 method, 34
Implicit solution, 4
Inhomogeneous ODEs, 3
Initial condition, 5
Initial value problem, 5
Integral curves, 9
Integrating factor, 17

Jordan chain, 77

L-stability, 38
Lambert W function, 81, 90
Linear differential equations, 2

Mathematical pendulum, 47
Method of variation of constants, 22, 57, 61
Modes of a system of ODEs, 36
Multistep methods, 27

Natural frequency, 63
Newton's 2nd law, 45
Non-linear differential equations, 2
Numerical methods
  Explicit methods, 27
  Implicit methods, 27
  Multistep methods, 27
  Predictor-corrector methods, 28

Order of a differential equation, 2
Order of accuracy, 28
Ordinary differential equations, 2

Parametric solution, 9
Partial derivative, 16
Partial differential equations, 2
Particular integral, 6, 21
Particular solution, 6, 21
Perturbative solutions, 49
Phase portrait, 44
Phase trajectories, 44
Picard's existence theorem, 6
Picard–Lindelöf theorem, 6
Potential function, 16
Power factor, 68
Predictor-corrector method, 28
Product logarithm, 81

Quality factor (Q-factor), 69

Real power, 68
Reduction to quadratures, 4
Region of absolute stability, 37
Resonance, 66
Resonance frequency, 66

RK4, 33
Root condition, 31
Runge-Kutta methods, 32

Separable ODEs, 10
Separatrix, 47
Singular solution, 11
Solving by quadratures, 4
Space space, 44
Stability function, 36
Stability of numerical methods, 29
Stability parameter, 36
Stationary point, 48
Stiff differential equation, 35
Symmetric form (1^{st} order ODE), 9

Transfer function, 69