Lectures on Ordinary Differential Equations

Bence Kocsis

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Computational software and further reading

Differential equations are arguably among the most important mathematical tools in modern science and engineering. They are used to model scientific phenomena across virtually all disciplines including natural sciences, computer science, social science, etc. In this course we will mostly focus on analytic methodologies for solving differential equations and will aim at a general understanding of the different types of behaviours these equations may describe. However it is also **strongly encouraged** for the students to simulataneously familiarise themselves with computer software capable of solving differential equations, in particular **MATHEMATICA**[1], which not only produces numerical solutions but also fully analytic parameterised solutions. This tool may allow the student to verify the results of exercises including the problems posed for homework. More importantly these tools are incredibly helpful in real life to get a quick solution on the expected outcome. However we emphasize that MATHEMATICA *cannot be used on the exam* for a good reason as these computer tools do not replace the need for human insight. Indeed, without a basic understanding of *how* to derive the solution, you may not even understand the main properties of the solutions and the possible complexities that may arise.[2] The comprehensive understanding of ordinary differential equations will provide insight into the range of possibilities various equations may represent, and guide your future endeavours to construct your own scientific models to explain observations and to make predictions.

These lecture notes are based on previous Oxford lecture notes by Julia Yeomanns, James Binney, Alex Schekochihin and the text book *Ordinary Differential Equations with Applications* by Rice and Strange. However, given the shortness of the term, these lectures on ordinary differential equations are far from being comprehensive, they should be regarded only as a brief introduction to the subject. I strongly encourage not only to study these lecture notes very thoroughly, including the optional sections marked by an asterisk, but to read additional references on the subject. In particular, Alex Schekochihin's lecture notes include a more rigorous discussion of the mathematical framework and it gives a glimpse into the history of the development of this mathematical discipline including hilarious anecdotes of its pioneers, a must read. I also highly recommend looking at the references therein in detail for a deeper mathematical background and important alternative approaches not included in the core syllabus here (e.g. numerical methods, asymptotics, etc). Studying these extensions in depth is an excellent time investment that will pay dividends far beyond your exam. [3]

1 Terminology and important concepts

A differential equation is an equation that contains a derivative or a differential of one or more functions. Examples

1. $\dfrac{dy}{dx} = ye^x$
2. $y' = ye^x$
3. $dy - ye^x dx = 0\,,$
4. $x^2 dy + 3y dx = 25 dx$
5. $yy' + x = 3$
6. $y'' + 4y = (x^2+1)^3$
7. $\dfrac{d^2y}{dx^2} + 4y = (x^2+1)^3$
8. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$
9. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y} = t + x$

[1] Students can download a registered copy of MATHEMATICA from the University of Oxford website.
[2] also, MATHEMATICA sometimes makes mistakes and/or returns an incomplete solution
[3] Feedback on how to improve the text is most welcome: bence.kocsis@physics.ox.ac.uk

Here Examples 1–3 and 6–7 are equivalent ways of writing the same equation.[4] In Examples 1–6 the unknown function is $y(x)$, where x is called the **independent variable** and y is the **dependent variable**. In Example 8, the unknown function is $u(x, y, t)$, the dependent variable is u, the independent variables are x, y, and t. To simplify the language we will often label the indepedent variable with y and the dependent variable by x or t, but of course you may use any notation for the same quantity. For instance, "$dx/dy = xe^y$ and $x(y) =$?" is obviously exactly equivalent to the problem "$dy/dx = ye^x$ and $y(x) =$?" by swapping the labels of the dependent and independent variables.

Example 3 is written in **differential form** in which the equation specifies how infinitesimal changes in y and in x, i.e. the differentials, dy and dx, are related.[5] The differentials dx and dy denote infinitesimal changes in x and in y, respectively. Note that dy/dx may be treated literally as a fraction to obtain the differential form of a differential equation[6], $dy = (dy/dx)dx = y'(x)dx$.

Examples 1 and 2 are **resolved with respect to the derivative**.

1.1 Classifications of differential equations

Differential equations may be classified in multiple ways.

- **Ordinary**: if the unknown function depends on only one independent variable. Examples 1–6
- **Partial**: if the unknown function depends on more than one independent variable. Examples 7-8
- The **order** of a differential equation is the order of the highest derivative. Most generally an n^{th} order ordinary differential equation may be written as

$$F(x, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}) = 0 \tag{1}$$

  where F is an arbitrary given function.

- **Linear** ordinary differential equations (ODEs) are differential equations linear in the dependent variable, which may be written as a linear combination of the independent variable and its derivatives:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x) \tag{2}$$

  where $a_i(x)$ for $i = 1, \ldots, n$ are arbitrary (either linear or nonlinear) functions of x and $a_n(x) \neq 0$.

  Exercise show that all examples above are linear except for 5. Example 5 is nonlinear because of the term yy'.

- A linear ODE is **homogeneous** if it may be written in the form of Equation (2) where $f(x) = 0$ for all x, otherwise it is **inhomogeneous**.

Some famous examples of ordinary differential equations in physics:

1. $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t)$, Newton's second law of motion for a point mass's position in time $\mathbf{r}(t) = (x(t), y(t), z(t))$ in a given force field $\mathbf{F}(x, y, z, t)$
2. $\frac{1}{2}m\dot{r}^2 + \frac{L^2}{mr^2} + U(r) = E$, orbit of a body in a central potential in classical mechanics where E and L are the energy and angular momentum and $U(r)$ is the potential energy, a given function of radius, r
3. $m\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} - k\mathbf{r} = F(t)$, driven harmonic oscillator describing the motion around an equilibrium where m, γ, and k are constants, $F(t)$ is a given external forcing function in time
4. $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t) + q\mathbf{E}(\mathbf{r}, t)$, Lorentz force in electromagnetism for the motion of a point charge in a given electromagnetic field given by $\mathbf{E}$ and $\mathbf{B}$ everywhere in space and time
5. $\dot{N} = -\lambda N$, radioactive decay in atomic physics; the rate of change is proportional to the number of particles N where λ is the decay constant

[4] Here $y' = dy/dx$, $y'' = d^2y/dx^2$, and generally $y^{(n)} = d^ny/dx^n$. Another convention often used for time derivatives are overdots: $\dot{y} = dy/dt$, $\ddot{y} = d^2y/dt^2$.

[5] Given the symmetry, either y or x may be considered to be the independent variable, so that the unknown function to be found is either $y(x)$ or $x(y)$. This freedom may prove useful in cases where $y(x)$ may not be obtained analytically.

[6] This is useful only for first order differential equations.

6. $C\dot{T} = -\sigma A T^4$, black body radiation cooling in thermodynamics expressing the change of temperature T of a body of area A and heat capacity C, where σ is the Stephan-Boltzmann constant

7. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x) = E\psi$, Schrödinger equation in quantum mechanics which describes the wave function $\psi(x)$ that determines the probability of finding a particle at some location having energy E in a given one dimensional potential well $V(x)$

8. $\ddot{X}^\mu + \Gamma^\mu_{\alpha\beta}\dot{X}^\alpha\dot{X}^\beta = 0$, geodesic equation in general relativity – motion of a body in spacetime $X^\alpha(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$, where τ is the time as experienced by the body, the derivatives are with respect to τ, and $\Gamma^\mu_{\alpha\beta}(t, x, y, z)$ are given functions of spacetime, μ, α, β are 4-dimensional (i.e spacetime) indices and summation is assumed over α, and β,

Exercise Identify the order of these ODEs, which are linear/nonlinear?

1.2 Solution of a differential equation

Definition

A **solution** of an n^{th} order differential equation on an interval $a \le x \le b$ is any function possessing all the necessary derivatives, which when substituted for y, y', y'', $\ldots$, $y^{(n)}$, reduces the differential equation to an identity.

We introduce the main properties of the solutions through simple examples next.

Example 1

Find the solution of $y' = x$.

SOLUTION

The equation to be solved is $dy/dx = x$ or in differential form

$$dy = xdx \tag{3}$$

Integrating both sides

$$\int dy = \int xdx\,. \tag{4}$$

simplifies to

$$y(x) = \frac{1}{2}x^2 + C \tag{5}$$

where C is an integration constant which may be any real number. This clearly defines infinitely many solutions all of which satisfy $y' = x$ everywhere along the interval $x \in (-\infty, \infty)$.

More generally, the same method may be used to solve any differential equation of the type $y' = f(x)$, where the solution is clearly

$$y(x) = \int f(x)dx + C\,. \tag{6}$$

Example 2

Find the solution of $y'' = x$.

SOLUTION

Introducing $z(x) = y'(x)$, the equation to be solved is

$$z' = x\,. \tag{7}$$

which is the same equation as in the previous example. This gives

$$z(x) = \int x dx = \frac{1}{2}x^2 + C_1 \tag{8}$$

where C_1 is an integration constant, an arbitrary real number. Now substituting the definition $z(x) = y'$, we get

$$y'(x) = \frac{1}{2}x^2 + C_1 \,. \tag{9}$$

Again integrating both sides w.r.t. x gives

$$y(x) = \int \left(\frac{1}{2}x^2 + C_1\right) dx = \frac{1}{6}x^3 + C_1 x + C_2 \,. \tag{10}$$

Here we have two integration constants C_1 and C_2, both may assume any real number. This also defines infinitely many solutions[7] for $y'' = x$ on the interval $(-\infty, \infty)$.

Example 3

Verify that $x^2 + y^2 = 4$ defines a function that is a solution of $x + yy' = 0$ and indicate any limitations of the solution.

SOLUTION

If we differentiate $x^2 + y^2 = 4$ with respect to x, we obtain

$$2x + 2yy' = 0 \tag{11}$$

which is the given differential equation. Furthermore solving $x^2 + y^2 = 4$ for y gives $y(x) = \pm\sqrt{4 - x^2}$. These are two solution functions limited to the interval $[-2, 2]$.

These examples highlight two typical types of solutions that we encounter in solving ordinary differential equations. We may formalise them as follows.

Definition

1. A real function $y = f(x)$ is called an **explicit solution** of the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ on $[a, b]$ if a direct substitution leads to an identity, i.e.

$$F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0 \text{ on } [a, b] \tag{12}$$

2. A relation $g(x, y) = 0$ is called an **implicit solution** of the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ on $[a, b]$ if $g(x, y) = 0$ represents at least one real function f on $[a, b]$ such that $y = f(x)$ is an explicit solution on this interval.

In Examples 1 and 2, $y(x) = \frac{1}{2}x^2 + C$ and $y(x) = \frac{1}{6}x^3 + C_1 x + C_2$ were the explicit solutions, respectively, and in Example 3, $x^2 + y^2 - 4 = 0$ was the implicit solution and $y(x) = \pm\sqrt{4 - x^2}$ was the explicit solution. To understand the relevance of the phrase "represents at least one real function" in the above definition of the implicit solution consider $x^2 + y^2 + 1 = 0$. This satisfies $yy' + x = 0$, which can be verified by taking the derivative with respect to x, but since $x^2 + y^2 < 0$ is not possible, there are no real x and y which solve $x^2 + y^2 + 1 = 0$. Thus, it is not an implicit solution since it does not represent any real valued function. The relation $x^2 + y^2 + 1 = 0$ is called a **formal solution** of $yy' + x = 0$; which means that it appears to be a solution of $yy' + x = 0$ since it reduces the differential equation to an identity.

[7] in fact $\infty \times \infty$ solutions

Families of solutions

Differential equations usually have infinitely many solutions. In Example 1, $y(x) = \frac{1}{2}x^2 + C$ was a solution to $y' = x$ where C was an integration constant, an arbitrary real number. Such a solution is called a **family of solutions**; one solution is obtained for each value of the parameter C. We label families of solutions by the number of parameters in the family. Thus $\frac{1}{2}x^2 + C$ is a **one-parameter family**. Similarly, in Example 2 the solution of $y'' = x$ had two integration constants, i.e. $y = \frac{1}{3}x^3 + C_1x + C_2$, which is called a **two-parameter family**. The same procedure solves the n^{th} order equation $y^{(n)} = f(x)$, by integrating n times in turn over x, which yields n independent integration constants and an n-parameter family of solutions.

More generally, the solutions of all first order ODEs represent a one-parameter family of curves, since the solutions of all first order ODEs inevitably have one integration constant. The integration constant enters the solution when the derivative is eliminated by integrating the equation. Similarly, a general theorem states that the solutions of all second order differential equations have exactly two indepedent integration constants as two integrals are needed to eliminate the second derivative in the equation. Finally, the **general solution** of all n^{th} order differential equation contains n integration constants, which leads to an n-parameter family. Any instance of the general solution with a specific choice of integration constants is called a **particular solution** or **particular integral** of the differential equation.

One must be cautious when identifying the number of parameters in a family of solutions. For instance note that $y = C_1e^x + C_2e^x = (C_1 + C_2)e^x$ is actually a one-parameter family by letting $C = C_1 + C_2$. In an n-parameter family there are n essential constants that cannot be reduced by algebraic manipulations to fewer than n constants.

Example 4

Concentric circles around the origin are defined implicitly by the one parameter family of curves $x^2 + y^2 = C^2$ where C is a constant parameter. Find the corresponding differential equation.

SOLUTION Differentiating both sides with respect to x gives, $2x + 2yy' = 0$. This is the differential equation of the given family of circles.

Example 5

Find the general solution of the differential equation $yy' + x = 0$.

SOLUTION The solution is obviously given by Example 4, but it is instructive to see how. Integrate both sides w.r.t. x

$$\int yy'dx + \int xdx + C = 0 \tag{13}$$

Since the derivative of y^2 with respect to x is $2yy'$, we get

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 + C = 0\,. \tag{14}$$

Alternatively, the differential equation may be written in the symmetric form $ydy + xdx = 0$, where integration gives immediately $\frac{1}{2}y^2 + \frac{1}{2}x^2 + C = 0$. Since $x^2 + y^2 > 0$, C must be negative, so we may redefine the constant as

$$y^2 + x^2 = C^2\,. \tag{15}$$

This is the implicit general solution of the equation. We may obtain the explicit general solution by solving for y:

$$y = \pm\sqrt{C^2 - x^2}. \tag{16}$$

Integral curves, parametric solutions

Generally, the graph of the solutions of a differential equation in the (x, y) plane are called **integral curves** of the differential equation. These curves may be parameterised not only by x as in $y(x)$, but more generally as $(x(s), y(s))$ where s is a parameter, which runs through all the different points along the curves given by the functions $x(s)$ and $y(s)$. Such parameterised curves define **parametric solutions** to the differential equation,

which is an alternative way of defining the general solution or a particular solution of the equation. For Example 5, the integral curves are $(x, y) = (x, \pm\sqrt{C^2 - x^2})$ for any fixed C in the domain $-C < x < C$ or alternatively $(x, y) = (C\cos\varphi, C\sin\varphi)$ for $0 \le \varphi \le 2\pi$. This parametric solution can be verified to satisfy the implicit solution, Eq. (15), since $y^2 + x^2 = C^2\sin^2\varphi + C^2\cos^2\varphi = C^2(\sin^2\varphi + \cos^2\varphi) = C^2$, an identity for all φ. Here C selects the circle using the radius and φ identifies the point along the circle using the polar angle. This parameteric solution highlights the fact that the two distinct solutions $y(x) = \pm\sqrt{C^2 - x^2}$ for a given C combine to describe a single smooth curve, a circle. The points where they combine $x = \pm C$ have $y' = \pm\infty$ where $y(x)$ is not differentiable, but the integral curve is nevertheless completely well behaved everywhere, as it is infinitely continuously differentiable with respect to φ.

1.3 Initial conditions and boundary conditions

We have seen that the general solution of an n^{th} order differential equation contains n arbitrary integration constants. The differential equation is typically subject to the same number of auxiliary conditions which are used to specify the values of these constants.

Definition

1. **Initial conditions** are auxiliary conditions for a given differential equation which specify the value of the independent variable and its derivatives $(y, y', y'', \ldots y^{(n-1)})$ at a single $x = x_0$. The differential equation with its initial conditions is called an **initial value problem**.

2. **Boundary conditions** are auxiliary conditions for a given differential equation which specify the independent variable y and/or its derivatives at two or more x values. The differential equation with its initial conditions is called a **boundary value problem**.

An initial condition for Example 1 may be for instance $y(2) = 3$. Substituting into the general solution Eq. (5), we get $C = 1$, so the solution of this initial value problem is $y(x) = \frac{1}{2}x^2 + 1$. If the initial condition for Example 2 is $y(1) = 0$, $y'(1) = 1$, then from Eqs. (9)–(10) we get $C_1 = \frac{1}{2}$, $C_2 = \frac{2}{3}$, so the solution of this initial value problem is $y(x) = \frac{1}{6}x^3 + \frac{1}{2}x + \frac{2}{3}$.

1.4 Existence and Uniqueness*

Given the initial conditions at some point x_0, is there always a unique solution to a differential equation or can there be multiple solutions or no solution at all? To see what to expect, a useful concept in calculus is the Taylor-series expansion of a function around the point x_0. For well behaved functions, the value of the function at $x = x_0 + \Delta x$ may be written as

$$y(x_0 + \Delta x) = y(x_0) + y'(x_0)\Delta x + \frac{1}{2!}y''(x_0)\Delta x^2 + \cdots + \frac{1}{n!}y^{(n-1)}(x_0)\Delta x^n + \ldots \tag{17}$$

Here the sum of the first two terms in this expansion $y(x_0) + y'(x_0)\Delta x$ approximates $y(x)$ with a linear function; this is the tangent line to the function $y(x)$ at x_0. The first three terms fit a quadratic function, a parabola, at x_0, which is usually a better approximation sufficiently close to x_0 as it also accounts for how the function curves away from the local tangent. The more terms we account for in the expansion, the better the approximation becomes.

Now imagine that you are given an initial value problem for a differential equation in the form

$$y^{(n)} = f(x, y, y'', \ldots, y^{(n-1)})\,. \tag{18}$$

with some given f function and given initial conditions at x_0, i.e. with given $y(x_0)$, $y'(x_0)$, $\ldots$, $y^{(n-1)}(x_0)$. The first n terms in Eq. (17) up to order Δx^{n-1} may be obtained by substituting the initial conditions. The next term, $\frac{1}{n!}y^{(n)}(x_0)\Delta x^n$, is given by the differential equation itself (18). Higher order terms may be obtained by differentiating the differential equation with respect to x and using the initial conditions. Thus, if all goes well, we may expect to obtain arbitrarily high order approximations in the Taylor series from the conditions at x_0 which may describe the function arbitrarily accurately, in many cases up to arbitrarily large Δx. In other cases, this gives an arbitrarily accurate approximate solution only within a limited distance to the initial point. In this case one can construct a numerical algorithm to solve the differential equation iteratively by calculating the

approximate value of $y(x)$ and its derivatives at a nearby point $x + \Delta x$ substituting the differential equation in the Taylor expansion for given initial conditions, and use this as the initial condition for the next iteration step.

So what may ever go wrong? In some cases, the equation is not given in the form of Eq. (18), and there are multiple or no solutions when expressing $y^{(n)}(x)$ with y and its $n-1$ derivatives. In this case the integral curves may either intersect or they may avoid a given region in the (x, y) plane altogether. Also if $y(x)$ or its derivatives tend to $\pm\infty$ at some x, so the "integral curve becomes vertical", the Taylor series approximation Eq. (17) will break, and the solution of the equation typically cannot be extended beyond that point. We will see examples of these cases throughout the course.

Theorem on existence and uniqueness

For first order differential equations given by

$$y' = f(x, y) \tag{19}$$

and initial condition $y(x_0) = y_0$, a **sufficient condition for a unique solution to exist** in a rectangle around (x_0, y_0) is to require that $f(x, y)$ and $\partial f/\partial y$ exist and be continuous within that rectangle.

Identifying where this condition fails is often useful for understanding where one might expect complications to arise or additional solutions for a given differential equation. The integral curves cannot be uniquely extended beyond the point where the curves intersect or where $y(x) \to \pm\infty$. (See e.g. Example 2 in Section 2.1 below.)

1.5 Summary

The classification of differential equations introduced in this chapter will allow us to construct targeted methods to solve the various types of equations in a systematic way.

- We have seen that the solutions of a general n^{th} order differential equations represent an **n-parameter family of curves**, where each curve is identified by n **independent integration constants**. These constants are often arbitrary real numbers. Thus differential equations typically have continuously infinitely many solutions.
- The fact that a single differential equation describes a full family of curves completely in a compact form makes differential equations powerful for building simple scientific models to describe complex phenomena.
- For sufficiently well behaved differential equations, a given **initial condition** selects a **unique** member curve of this family, also known as an **integral curve**.

Procedure for solving differential equations

Step I Manipulate the equation to bring it in a form in which it can be integrated to undo the derivates in the equation. We will discuss strategies on how to do this in practice for different types of ODEs in the rest of this course.

Step II Carry out the integrals. For an n^{th} order differential equation, n integrals need to be evaluated to eliminate the n^{th} derivate, and the result has exactly n **integration constants**.

Step III If there are no more derivatives in the equation, the differential equation has been reduced to an algebraic equation called the **implicit general solution** of the differential equation.

Step IV Solve this equation for the independent variable, i.e. express the independent variable explicitly as a function of the dependent variable. This defines the **explicit general solution**. Any instance of the explicit general solution for a specific choise of integration constant is called a **particular solution** or particular integral of the differential equation. In some cases we also get additional solutions, which are not instances of the general solution, these are called **singular solutions**.

Step IV' The general solution of a differential equation may also be specified without an explicit solution $y(x)$ by specifying the integral curve in a **parametric solution**, e.g. defining functions $x(s)$ and $y(s)$ which track the integral curve in the (x, y) plane when changing s.

Step V The values of the integration constants are found by satisfying the initial conditions or boundary conditions. For an n^{th} order differential equation, n independent constraints are necessary (but sometimes not sufficient) to obtain a unique solution.

2 First order ODEs

First order ordinary differential equations may be generally written as

$$F(x, y, y') = 0 \tag{20}$$

where F is a given function of x, y, and y' on some domain $x \in [a, b]$. To solve the equation, we almost always start by first expressing the derivative y' as a function of x and y, writing it as[8]

$$y' = f(x, y) \tag{21}$$

where f is a given function of x and y. In this form, the equation is said to be **resolved by** y'. Since $y' = dy/dx$ by definition, this equation may also be written in differential form as $dy = f(x, y)dx$ or in **symmetric form**

$$P(x, y)dy + Q(x, y)dx = 0 \tag{22}$$

where $f(x, y) = -Q(x, y)/P(x, y)$ if $P(x, y) \neq 0$. The initial condition is usually given by specifying the value $y(x) = y_0$ when $x = x_0$ for some x_0.

For first order ODEs, the integral curves may be visualized even without solving the equations. Substitute Eq. (21) in the leading order approximation of the Taylor series Eq. (17) near $x = x_0$:

$$y(x_0 + \Delta x) = y_0 + y'(x_0)\Delta x = y(x_0) + f(x_0, y_0)\Delta x \tag{23}$$

In the second equality we substituted y' from Eq. (21). Plotting these small straight line segments for a grid of (x_0, y_0) gives the **direction field**, showing which way the system flows from any point on the grid. Stitching together these segments forms the integral curves. In MATHEMATICA use STREAMPLOT to visualize the direction field and DSOLVE to obtain the general solution analytically with a computer. It is advised to review the examples and exercises below both with and without MATHEMATICA in turn.

While first order ODEs are not always soluble analytically, there are several classes of first order ODEs that may be solved analytically as discussed in turn next.

2.1 Separable differential equations

A first order differential equation is **separable** if it may be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \text{or} \quad f(x)dx = g(y)dy \tag{24}$$

Integrate both sides

$$\int f(x)dx = \int g(y)dy + C\,. \tag{25}$$

This is the implicit solution of the differential equation. The integration constant C may be obtained from the initial condition, $y(x_0) = y_0$ for given x_0 and y_0. In this case

$$\int_{x_0}^{x} f(x)dx = \int_{y_0}^{y} g(y)dy\,. \tag{26}$$

The explicit solution is obtained by solving for y.

Note that more generally

$$\frac{dy}{dx} = \frac{f_1(x)g_1(y)}{f_2(x)g_2(y)} \tag{27}$$

is obviously also separable since in this case we may define $f(x) = f_1(x)/f_2(x)$ and $g(y) = g_2(y)/g_1(y)$ to recover Eq. (24). Thus,

$$\frac{f_1(x)}{f_2(x)}dx = \frac{g_2(y)}{g_1(y)}dy \tag{28}$$

which may be readily integrated. Note that if $g_1(y) = 0$ for any $y = y_0$ then the constant function $y(x) = y_0$ is also a solution to the equation. Note also that no matter how ugly nonlinear the $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(y)$ functions are the x and y integrals nicely separate, and the problem is reduced to a (possibly complicated) integration exercise.

[8] Note that it is not always possible to manipulate the equations analytically to express y', for instance $y' + \exp(y') = y$ cannot be resolved by y'. In this case taking the derivative of the equation with respect to x, and substituting $z = y'$ may help simplify the equation, see Sec. 2.8.3.

Example 1

Consider the radioactive decay equation $\dot{y} = -\lambda y$ for fixed $\lambda > 0$ decay constant.
a) Obtain the general solution $y(t)$. Plot the integral curves.
b) Find the solution which satisfies the initial condition $y = y_0$ at $t = 0$.

SOLUTION

First note that the constant $y(t) = 0$ for all t is a solution. If $y \neq 0$ then we may divide by y and multiply by dt to obtain the differential form and then integrate

$$\frac{dy}{dt} = -\lambda y \quad \Rightarrow \quad \int \frac{dy}{y} = -\int \lambda dt + C_1 \quad \Rightarrow \quad \ln|y| = -\lambda t + C_1$$
$$\Rightarrow \quad y(t) = C_2 e^{\lambda t} \qquad \left(C_2 = \pm e^{C_1}\right) \tag{29}$$

The integral curves are monotonically decaying exponentials as a function of t asymptotically approaching $y \to 0$ for large positive t and $y \to \pm\infty$ for large negative t, and another solution is the constant $y(t) = 0$ for all t. Figure 1 shows the integral curves for different C_2. Note that only a finite number of curves is shown for clarity, but in reality these curves cover the (t, y) space completely.

Given the initial condition $y = y_0$ at $t = 0$, we get $C_2 = y_0$, and $y(t) = y_0 e^{-\lambda t}$. Here the original integration constant C_1 was used to define the constant $C_2 = \pm e^{C_1}$. Thus, the general solution on $(-\infty, \infty)$ is $y(t) = C_2 e^{\lambda t}$ where C_2 may be any real number, i.e. negative, 0, or positive. Obviously solutions with $y_0 > 0$ will have $y(t) > 0$ at all times.

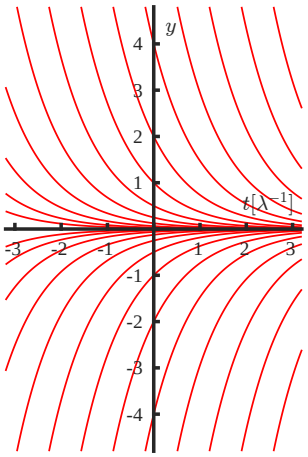


Figure 1: Integral curves of Eq. (29).

Example 2

Solve the equation $\dot{y} = y^{2/3}$. Find the general solution, plot the integral curves.

SOLUTION First note that $y(t) = 0$ for all t is a solution, this is going to be a *singular solution* as we shall see, since it will not be a special case of the general solution (Eq. 30) for any choice of C. As long as $y \neq 0$ we may divide by $y^{2/3}$ and multiply by dt to obtain the differential form and then integrate

$$\frac{dy}{dt} = y^{2/3} \quad \Rightarrow \quad \int y^{-2/3} dy = \int dt + C \quad \Rightarrow \quad 3y^{1/3} = t + C \quad \Rightarrow \quad y = \frac{1}{27}(t + C)^3 \tag{30}$$

If the initial condition satisfies $y(t_0) = y_0 < 0$ then the solution increases and intersects the singular solution $y(t) = 0$ at a later time $t \to t_1 = -C = t_0 + 3|y_0|^{1/3}$. At that point the solution becomes invalid, as the first step to obtain these solutions dividing by $y^{2/3}$ was illegal. Hence, beyond $t \geq t_1$ the solution is the constant $y(t) = 0$. Thus, we get the composite general solutions over the full time domain $t \in (-\infty, \infty)$

$$y(t) = \begin{cases} -\frac{1}{27}(t_1 - t)^3 & \text{if } t_1 \geq t \\ 0 & \text{if } t \geq t_1 \end{cases} \tag{31}$$

where $t_1 = -C$ is an arbitrary real number.[9]

For initial conditions with $y(t_0) = y_0 > 0$, the singular solution is avoided at $t > t_0$. Thus, we get additionally the solutions defined over half lines $t > t_1$

$$y(t) = \frac{1}{27}(t - t_1)^3 \ \text{ if } \ t > t_1\,. \tag{32}$$

where $t_1 = -C$ is any real number. In summary, the integral curves are given by (i) the singular solution $y(t) = 0$ for all t, (ii) Eq. (31) for all t, and Eq. (32) for $t \geq t_1$. These curves cover $y = 0$, $y \leq 0$, and $y > 0$, respectively, for all t; hence the full (t, y) space.[10]

[9] Note that this solution is continuously differentiable everywhere. The second derivative does not exist at t_1, but the existence of the second derivative is generally not necessary for a solution of a first order differential equation.

[10] They cover the space fully but here not uniquely. Here (i) and (ii) intersect at $y = 0$. We may generally check if/where such peculiarity is to be expected for an ODE, by taking the $\partial/\partial y$ derivative of the right hand side of the differential equation resolved by y': $\partial y^{2/3}/\partial y = -\frac{2}{3} y^{-1/3} \to \pm\infty$ at $y \to 0$ violating the condition for uniqueness stated below Eq. (19). Note in comparison that Example 1 was well behaved since in that case $\partial(-\lambda y)/\partial y = -\lambda$ is continuous everywhere.

Finally, observe that the integral curves in Eq. (32) emerge from an infinitesimally close vicinity of $y = 0$ shown with an empty circle in Figure 2. If this differential equation models a physical system, we may expect any infinitesimal positive perturbation at $y = 0$ (i.e. 'noise') to kick the system onto these solutions with $y > 0$. This leads to additional composite solutions of the form

$$y(t) = \begin{cases} -\frac{1}{27}(t_1 - t)^3 & \text{if } t \le t_1 \\ 0 & \text{if } t_1 \le t \le t_2 \\ \frac{1}{27}(t - t_2)^3 & \text{if } t \ge t_2 \end{cases} \tag{33}$$

for any t_1 and t_2. Here t_1 and t_2 are related to the integration constant (cf. Eq. 30). Surprisingly, the integration constant for these composite solutions (33) appear not to be globally constant as $C = -t_1$ for $t \le t_1$, undefined for $t_1 \le t \le t_2$ where the solution follows the singular solution, and it is $C = -t_2$ for $t > t_2$. Nevertheless it is piecewise constant and $dC/dt = 0$ for all t where it is defined.

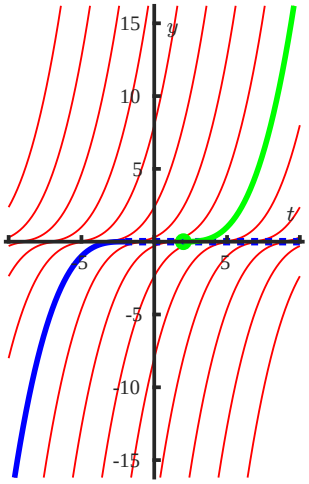


Figure 2: Integral curves are composite solutions of Eq. (30) (red line) and $y(t) = 0$, as in Eq. (31) (thick blue) or Eq. (32) (thick green).

2.2 Almost separable differential equations

Consider the ODE in the following form

$$\frac{dy}{dx} = f(ax + by) \quad \text{for } a \text{ and } b \text{ real constants} \tag{34}$$

where $f(ax + by)$ indicates any function that depends on x and y only through the combination $ax + by$. This ODE becomes separable by a change of the dependent variable. Let $z(x) = ax + by(x)$, then

$$\frac{dz}{dx} = a + b\frac{dy}{dx} = a + bf(z) \quad \Rightarrow \quad \int \frac{dz}{a + bf(z)} = \int dx + C = x + C \tag{35}$$

This specifies the implicit general solution for $z(x)$. We may continue by substituting back $y(x) = (z(x) - ax)/b$ and solve for $y(x)$.[11]

Example 3

Solve the equation $(x + 2y)y' = 1$. Find the general solution.

SOLUTION For $x + 2y \neq 0$,

$$(x + 2y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + 2y} \equiv f(x + 2y) \tag{37}$$

Let $z(x) = x + 2y(x)$,

$$\frac{dz}{dx} = 1 + 2\frac{dy}{dx} = 1 + \frac{2}{z} = \frac{z + 2}{z} \quad \Rightarrow \quad \int \frac{z\,dz}{z + 2} = \int dx + C \tag{38}$$

Here we assumed $z + 2/z \neq 0$, i.e. $z \neq -2$. The trick to evaluate the integral of any fraction of two polynomials is to decompose it in partial fractions, i.e write it as a sum of fractions. In this case

$$\frac{z}{z + 2} = \frac{z + 2 - 2}{z + 2} = 1 - \frac{2}{z + 2} \tag{39}$$

[11] Note that if we consider z to be a parameter, and the integral is evaluated to give some function $I(z)$, this already gives a parameteric solution. Indeed, the points of the integral curves may be parameterised by z according to Eq. (35) as

$$x(z) = I(z) - C\,, \quad y(z) = \frac{z - ax(z)}{b} = \frac{1}{b}[z - aI(z) + C], \quad \text{where} \quad I(z) = \int \frac{dz}{a + bf(z)} \tag{36}$$

which may be easily integrated

$$\int \frac{z\,dz}{z+2} = 2\int dz - 2\int \frac{dz}{z+2} = z - 2\ln|z+2| = x + C \quad \Rightarrow \quad (x+2y) - 2\ln|(x+2y)+2| = x + C_1 \tag{40}$$

Introducing $C_1 = -\ln|2C|$, simplifying and taking the exponential of both sides yields

$$x + 2y + 2 = C_2 e^y \quad \Rightarrow \quad x = Ce^y - 2y - 2 \tag{41}$$

Here it is not possible to obtain the explicit solution $y(x)$ analytically with elementary functions. Furthermore note that the initial manipulations were invalid when $z = x + 2y = 0$. Then $dx/dy = 0$, so $x = C$, but in this case $y(x)$ is multivalued and cannot be defined. Furthermore, if $z(x) = -2$ for all x then $dz/dx = 0$ which is a valid solution. But in this case $x + 2y = -2$, so $x = -2y - 2$, which is included in the general solution Eq. (41) for $C = 0$.

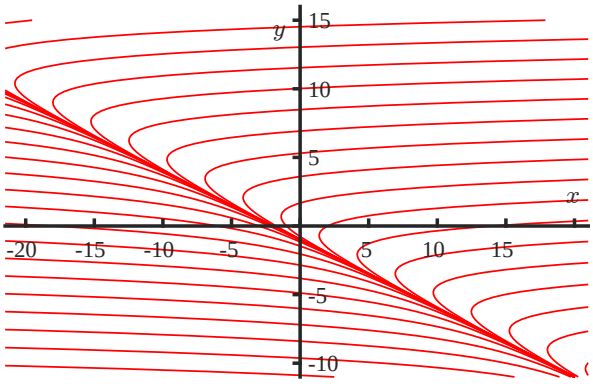


Figure 3: Integral curves of Eq. (41) for different values of C.

2.3 Homogeneous differential equations

A first order differential equation is **homogeneous** if it may be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{42}$$

where $f(y/x)$ is any function that depends on y and x only through y/x. This reduces to a separable differential equation by introducing $z(x) = y(x)/x$.

$$\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y(x)}{x^2} = \frac{f(z) - z}{x}\,. \tag{43}$$

Thus either $z(x) = z_0$ for all x where z_0 satisfies $z_0 = f(z_0)$ or else

$$\int \frac{dz}{f(z) - z} = \int dx + C\,. \tag{44}$$

We may continue by substituting back $y(x) = z(x)x$ and solve for $y(x)$.[12]

Note that

$$\frac{dy}{dx} = f\left(\frac{a_0x^k + a_1x^{k-1}y + \cdots + a_ky^k}{b_0x^k + b_1x^{k-1}y + \cdots + b_ky^k}\right) \tag{46}$$

is clearly a homogeneous equation for arbitrary (a_i, b_i) constants for $i = (1, \ldots, k)$, as long as the sums are not 0 since we may simplify the numerator and denominator by x^k, which leads to the fraction of two polynomials of y/x. If $k = 0$ then it is a separable equation. Further, even more generally the right hand side in Eq. (46) may be the sum of an arbitrary number of such functions with different (a_1, a_2, b_1, b_2, k) and the equation is still homogeneous.

[12] If we consider z to be a parameter, the integral curves may be readily plotted once the integral is evaluated

$$x(z) = I(z) + C\,, \quad y(z) = x(z)z = Cz + zI(z) \quad \text{where} \quad I(z) = \int \frac{dz}{f(z) - z} \tag{45}$$

Example 4

Solve the equation $xy' = \sqrt{x^2 - y^2} + y$.

SOLUTION The equation is clearly defined only for $|y| \le |x|$. If $x \neq 0$

$$\frac{dy}{dx} = \frac{\sqrt{x^2 - y^2} + y}{x} = \sqrt{1 - \left(\frac{y}{x}\right)^2} + \frac{y}{x}\,. \tag{47}$$

Introduce $z(x) = y(x)/x$ which satisfies $|z| \le 1$,

$$\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{z}{x} = \frac{\sqrt{1 - z^2}}{x}\,. \tag{48}$$

Clearly $z(x) = \pm 1$ for all x are solutions (i.e. singular solutions as we shall see) or otherwise this is a separable equation which we may integrate

$$\int \frac{dz}{\sqrt{1 - z^2}} = \int \frac{dx}{x} \quad \Rightarrow \quad \arcsin z = \ln|x| + C_1 \quad \Rightarrow \quad z = \sin\ln(C|x|) \quad \Rightarrow \quad y = x\sin(\ln C|x|)\,. \tag{49}$$

This is the general solution for any $C > 0$. In addition there are two singular solutions $z(x) = \pm 1$ or $y(x) = \pm x$ for all x, these solutions satisfy the differential equation but do not correspond to an instance of the general solution for any value of C. Since the range of $\arcsin z$ is between $\pm\pi/2$, this implies that $|\ln C|x|| \le \pi/2$, so the solution Eq. (49) is valid only in the finite domain $C^{-1}e^{-\pi/2} \le |x| \le C^{-1}e^{\pi/2}$. At the boundaries, the curve intersects one of the singular solutions $y(x) = \pm x$, and outside of this finite domain the curve must follow the singular solution. Thus we get composite solutions just like in Eqs. (31) or (33) as shown in Figure 4. The red lines show several examples of curves defined by Eq. (49) for different C, thick blue and thick green lines show two examples of composite solutions.[13] These curves start from just above the singular solution marked with an empty circle, follow the general solution shown with solid lines, and merge onto the singular solution shown with dotted lines.

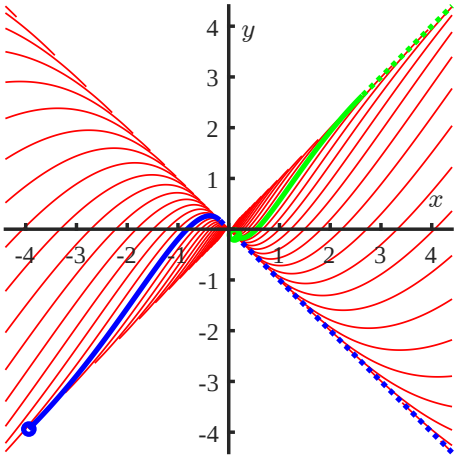


Figure 4: Integral curves of Eq. (49) for different values of C and examples of composite solutions.

2.4 Almost homogeneous differential equations

The following equation is not a homogeneous equation but it can be turned into a homogeneous equation by a change of variables

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{50}$$

where f is a given function and $(a_1, a_2, b_1, b_2, c_1, c_2)$ are given constants. Change of variables $X = x - u$ and $Y = y - v$ with appropriately chosen u and v eliminates the constants c_1 and c_2, if they satisfy

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\,. \tag{51}$$

Substituting leads to a homogeneous equation

$$\frac{dY}{dX} = f\left(\frac{a_1X + b_1Y}{a_2X + b_2Y}\right) = f\left(\frac{a_1 + b_1(Y/X)}{a_2 + b_2(Y/X)}\right) \tag{52}$$

[13] Note again that the only a few examples are shown to avoid clutter, but the solutions cover completely the region $|y| < |x|$.

Now we may proceed with the substitution $Z = Y/X$ to turn this into a separable equation, which may then be integrated to obtain a solution for $Z(X)$. Changing back to the original variables we may obtain $Y(X)$ and ultimately $y(x)$. Note that if the matrix in Eq. (51) has zero determinant, $a_1b_2 - a_2b_1 = 0$, there are no solutions for u and v but in this case, $a_2x + b_2y = k(a_1x + b_1y)$ for all x and y where $k = a_2/a_1 = b_2/b_1$. In this case the equation is in the form

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2}\right) = g(a_1x + b_1y) \tag{53}$$

so then the substitution $z = a_1x + b_1y$ will immediately turn this into a separable equation, see Sec. 2.2.

More generally

$$\frac{dy}{dx} = f\left(\frac{a_0x^k + a_1x^{k-1}y + \cdots + a_ky^k + c_1}{b_0x^k + b_1x^{k-1}y + \cdots + b_ky^k + c_2}\right) \tag{54}$$

may also be turned into a homogeneous equation by the same type of shift of variables $X = x + u$ and $Y = y + u$, where $(a_0, a_1, \ldots, a_k)$ and $(b_0, b_1, \ldots, b_k)$ are given constants.

Example 5

Solve the equation

$$\frac{dy}{dx} = \frac{y + 2}{2x + y - 4} \tag{55}$$

SOLUTION This equation is not homogeneous but it would be homogeneous if the constants 2 and -4 were not present. These constants may be eliminated by a variable change $x = X + a$ and $y = Y + b$ for carefully chosen values of a and b:

$$\frac{dY}{dX} = \frac{dy}{dx} = \frac{y + 2}{2x + y - 4} = \frac{Y + (b + 2)}{2X + Y + (2a + b - 4)} = \frac{Y}{2X + Y} = \frac{Y/X}{2 + (Y/X)} \tag{56}$$

The constants are eliminated if $b + 2 = 0$ and $2a + b - 4 = 0$. These are two equations for two unknowns, a and b. The solution is $(a, b) = (3, -2)$. Thus $X = x - 3$, $Y = y + 2$. Now we have succeeded in turning the equation into a homogeneous equation. This may be turned into a separable equations by another change of variable.

If $X = 0$, then $dY/dX = \frac{1}{2}$ for any Y. Assuming that $X \neq 0$, let $Z = Y/X$. Then its derivative is

$$\frac{dZ}{dX} = \frac{1}{X}\left(\frac{dY}{dX} - Z\right) = \frac{1}{X}\left(\frac{Z}{2 + Z} - Z\right) = \frac{-Z(Z + 1)}{X(2 + Z)} \quad \Rightarrow \quad -\int \frac{2 + Z}{Z(Z + 1)} = \int \left(\frac{1}{1 + Z} - \frac{2}{Z}\right) dZ = \int \frac{dX}{X}$$

$$\Rightarrow \quad \ln|1 + Z| - 2\ln|Z| = \ln\left|\frac{X}{C}\right| \quad \Rightarrow \quad \frac{1 + Z}{Z^2} = \frac{X}{C} \tag{57}$$

Here we assumed $Z \neq -1$ and $Z \neq 0$. This is the implicit general solution along with the singular solutions $Z(X) = 0$ and $Z(X) = -1$ for all X. Substitute $Z = Y/X$ and multiply both sides by CY^2/X,

$$Y^2 = C(X + Y) \quad \Rightarrow \quad (y + 2)^2 = C(x + y - 1) \quad \Rightarrow \quad x = 1 - y + \frac{1}{C}(y + 2)^2 \tag{58}$$

$$\Rightarrow \quad y(x) = 4 - 2C_1 \pm 2\sqrt{(x - C_1)(3 - C_1)} \tag{59}$$

This is the explicit general solution along with the singular solution $y(x) = 1 - x$. Note that the singular solution satisfies $x = 1 - y$ which intersects the general solution at $y = -2$ (as shown by Eq. 59). Here the term with C drops out. All integral curves intersect at this point, $(x, y) = (3, -2)$, and we get composite solutions as in Examples 2 and 4, where integral curve given by the general solution Eq. (59) may merge onto the singular solution $y(x) = 1 - x$.

Alternatively, we may construct a parametric solution immediately from the implicit general solution Eq. (57) by expressing $X(Z)$ and using $Y(Z) = ZX(Z)$:

$$X = C\frac{1 + Z}{Z^2} \quad \Rightarrow \quad Y = ZX = C\frac{1 + Z}{Z}$$

$$(X(Z), Y(Z)) = C\left(\frac{1 + Z}{Z^2}, \frac{1 + Z}{Z}\right) \quad \Rightarrow \quad (x(Z), y(Z)) = \pm C\left(\frac{1 + Z}{Z^2}, \frac{1 + Z}{Z}\right) + (3, -2) \tag{60}$$

This also shows that all curves intersect for $Z = -1$ at $(x, y) = (3, -2)$, the solutions may transition to the singular solution $y(x) = 1 - x$.

2.5 Exact Differentials

A first order differential equation is defined to be an **exact differential** or full differential if there exists a function $f(x, y)$ whose full differential vanishes as x and y change:

$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0\,. \tag{61}$$

Note that for a function of two variables, $f(x, y)$, the differential df denotes the change of the output of the function when changing both of its input arguments by an infinitesimal amount, $x \to x + dx$ and $y \to y + dy$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\,. \tag{62}$$

where $\partial f/\partial x$ and $\partial f/\partial y$ are the partial derivatives with respect to x and with respect to y, respectively, which means evaluate the derivative by holding the other variable fixed.

If this is zero, the function output does not change, it is a constant.

$$df(x, y) = 0 \quad \Rightarrow \quad f(x, y) = \int df = C\,. \tag{63}$$

For each C this defines a level curve in the (x, y) plane. This is the implicit general solution of this differential equation. The explicit solution follows from solving for y for all fixed x and C.

To test whether or not the equation is an exact differential we may use the general theorem stating the symmetry of mixed partial derivatives in Eq. (61)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}\,. \tag{64}$$

Thus, the equation is an exact differential if and only if it may be written as

$$\underbrace{P(x, y)}_{\frac{\partial f}{\partial x}}\,dx + \underbrace{Q(x, y)}_{\frac{\partial f}{\partial y}}\,dy = 0 \;\text{ where } P(x, y) \text{ and } Q(x, y) \text{ must satisfy } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\,. \tag{65}$$

or equivalently if it may be written as

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} \;\text{ where } P(x, y) \text{ and } Q(x, y) \text{ must satisfy } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{66}$$

If this condition is satisfied, the equation must be an exact differential, and we will be able to find the function $f(x, y)$ by integration with respect to x and y.

Example 6

Solve the exact differential equation

$$2xydx + (x^2 - y^2)dy = 0 \tag{67}$$

SOLUTION Let us check if the equation is an exact differential by taking the relevant partial derivatives

$$\frac{\partial}{\partial y}2xy = 2x, \quad \frac{\partial}{\partial x}(x^2 - y^2) = 2x\,. \tag{68}$$

Since these are equal, this is an exact differential. Therefore, there must exist a function $f(x, y)$ which satisfies

$$df = \underbrace{2xy}_{\frac{\partial f}{\partial x}}\,dx + \underbrace{(x^2 - y^2)}_{\frac{\partial f}{\partial y}}\,dy = 0\,. \tag{69}$$

To find this function let us integrate the first condition: $\partial f/\partial x = 2xy$ with respect to x to get

$$f(x,y) = \int \frac{\partial f}{\partial x} dx + C(y) = \int 2xy dx + C(y) = x^2 y + C(y) \tag{70}$$

Note that the integration constant for the integral over x is written as $C(y)$ since it be any function of y. To convince ourselves, let us substitute back

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial (x^2 y)}{\partial x} + \frac{\partial C(y)}{\partial x} = 2xy + 0 \tag{71}$$

as required. To find $C(y)$, plug back the result of Eq. (70) into the definition of $\partial f/\partial y$, and match this to the second term in Eq. (70)

$$\frac{\partial f}{\partial y} = x^2 - y^2 \quad \Rightarrow \quad x^2 - y^2 = x^2 + \frac{dC(y)}{dy} \,. \tag{72}$$

Thus we are left with a separable ODE for $C(y)$

$$\frac{dC(y)}{dy} = -y^2 \quad \Rightarrow \quad C(y) = -\int y^2 dy = -\frac{1}{3} y^3 + C_1 \tag{73}$$

where C_1 is any constant real number. Plugging back

$$f(x,y) = x^2 y - \frac{1}{3} y^3 + C_1 \,. \tag{74}$$

The full differential of this expression is the one posed in the problem indeed. Thus $f(x,y)$ must be constant. The implicit general solution is

$$x^2 y - \frac{1}{3} y^3 = C_2 \,. \tag{75}$$

where we have redefined the integration constant as C_2.

2.6 Integrating Factor

If an ODE $P(x,y)dx + Q(x,y)dy = 0$ is not an exact differential, i.e. $\partial P/\partial y \neq \partial Q/\partial y$, we can still try to turn it into an exact differential by multiplying the equation by some function $\Lambda(x,y)$, known as an integrating factor:

$$df(x,y) = \underbrace{\frac{\partial f}{\partial x}}_{\Lambda(x,y)P(x,y)} dx + \underbrace{\frac{\partial f}{\partial y}}_{\Lambda(x,y)Q(x,y)} dy = \Lambda(x,y)\left[P(x,y)dx + Q(x,y)dy\right] = 0 \,. \tag{76}$$

So in this case the integrating factor $\Lambda(x,y)$ must satisfy

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \Rightarrow \quad \frac{\partial}{\partial y}(\Lambda P) = \frac{\partial}{\partial x}(\Lambda Q) \quad \Rightarrow \quad Q\frac{\partial \Lambda}{\partial x} - P\frac{\partial \Lambda}{\partial y} = \Lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) . \tag{77}$$

For any given differential equation, how do we know if such an analytic function $\Lambda(x,y)$ exists? Unfortunately there is no way to know. There is no general algorithm to find the integrating factor for an arbitrary ODE. However for special types of ODEs, it is known that an integrating factor exists and how to find it. We shall see examples below. Generally, it is useful to start by trying functions of single variables, $\Lambda(x)$ or $\Lambda(y)$. Sometimes, more complicated expressions, polynomials or rational functions, e.g. $\Lambda(x^\alpha \pm y^\alpha)$, $\Lambda(x^\alpha y^\beta)$, etc. are needed, which may often be identified by inspection. In these cases, it may be useful to first change the dependent variable from y to $z = x^\alpha \pm y^\alpha$, $z = x^\alpha y^\beta$, etc., and then examine if the integrating factor may be a function of this new variable $\Lambda(z)$. You need practice to build an intuition to recognise patterns which may lead you to a viable integrating factor for complicated cases. But if nothing works it is certainly also possible that no integrating factor exist for the given equation!

Example 7

Find the integrating factor for $(x^2 + y^2 + y)dx - xdy = 0$.

SOLUTION I: Scrutinize the symmetries in the equation, it may be written as $(x^2 + y^2)dx + (ydx - xdy) = 0$. Notice that $xdy - ydx = x^2 d(y/x)$, so the equation may be written as

$$(x^2 + y^2)dx - x^2 d(y/x) = 0 \tag{78}$$

The integrating factor is $1/(x^2 + y^2)$, since multiplying by it leads to a separable differential equation. Separable equations are always exact differentials, since $P(x)dx + Q(y)dy = 0$ implies that $\partial P/\partial y = 0 = \partial Q/\partial x$. Thus,

$$\begin{aligned} 0 = dx - \frac{x^2}{x^2+y^2} d(y/x) = dx - \frac{d(y/x)}{1+(y/x)^2} = dx - \frac{dz}{1+z^2} \quad \left(z = \frac{y}{x}\right) \quad \Rightarrow \quad x - \arctan z = C \\ \Rightarrow \quad x - \arctan(y/x) = C \quad \Rightarrow \quad \arctan(y/x) = x - C \quad \Rightarrow \quad \frac{y}{x} = \tan(x - C) \\ \Rightarrow \quad y = x \tan(x - C) \end{aligned} \tag{79}$$

For given initial conditions we get $C = x_0 - \arctan(y_0/x_0) + n\pi$, or equivalently $C = x_0 - \arctan(y_0/x_0) + n\pi$ where n is an arbitrary integer.[14] Without loss of generality we may set $n = 0$.

Example 8

Find the solution to the most general first order linear differential equation

$$a_1(t)\dot{y} + a_0(t)y = g(t) \tag{80}$$

for any $a_0(t)$, $a_1(t)$, and $f(t)$, where $a_1(t) \neq 0$.

SOLUTION Let us divide both sides by $a_1(t)$ and let $a(t) = a_0(t)/a_1(t)$ and $f(t) = g(t)/a_1(t)$:

$$\dot{y} + a(t)y = f(t)\,. \tag{81}$$

or in differential form

$$dy + [a(t)y - f(t)]dt = 0\,. \tag{82}$$

The equation is separable if either $a(t)$ or $f(t)$ is a constant for all t, but it may not separable for general $a(t)$ and $f(t)$ functions. In the general case, let us look for an integrating factor in the form $\Lambda(t)$. Multiplying the equation gives

$$dF = \underbrace{\Lambda(t)}_{\frac{\partial F}{\partial y}}\, dy + \underbrace{\Lambda(t)[a(t)y - f(t)]}_{\frac{\partial F}{\partial t}}\, dt = 0\,. \tag{83}$$

Therefore

$$\frac{\partial^2 F}{\partial t \partial y} = \frac{\partial^2 F}{\partial y \partial t} \quad \Rightarrow \quad \frac{\partial}{\partial t}[\Lambda(t)] = \frac{\partial}{\partial y}\{\Lambda(t)[a(t)y - f(t)]\} \quad \Rightarrow \quad \frac{d\Lambda}{dt} = \Lambda(t)a(t)\,. \tag{84}$$

This is clearly a separable equation for Λ. Here either $\Lambda(t) = 0$ for all t, which is not a useful integrating factor, or else

$$\int \frac{d\Lambda}{\Lambda} = \int a(t)dt \quad \Rightarrow \quad \ln\Lambda = \int a(t)dt \quad \Rightarrow \quad \Lambda(t) = \exp\left(\int_{t_0}^{t} a(t')dt'\right) \tag{85}$$

Here the freedom of choosing t_0 is equivalent to adding an integration constant. Any choice of this parameter leads to a viable integrating factor. Given the integrating factor we ought to have an exact equation. Let us proceed in finding F from Eq. (83) following the same procedure as in Example 6. First use the $\partial F/\partial y$ term,

$$\frac{\partial F}{\partial y} = \Lambda(t) \quad \Rightarrow \quad F(t,y) = y\Lambda(t) + C(t) \quad \Rightarrow \quad \frac{\partial F}{\partial t} = y\frac{d\Lambda}{dt} + \frac{dC}{dt} = y\Lambda(t)a(t) + \frac{dC}{dt}\,. \tag{86}$$

[14] This is due to the fact that $\tan(x - C)$ is periodic with period $n\pi$.

Here we have used Eq. (84) to write $d\Lambda/dt = \Lambda a$. Let us combine this with the $\partial F/\partial t$ term in Eq. (83)

$$\frac{\partial F}{\partial t} = \Lambda(t)[a(t)y - f(t)] \quad \Rightarrow \quad y\Lambda(t)a(t) + \frac{dC}{dt} = \Lambda(t)a(t)y - \Lambda(t)f(t) \quad \Rightarrow \quad \frac{dC}{dt} = -\Lambda(t)f(t) \tag{87}$$

$$\Rightarrow \quad C(t) = -\int \Lambda(t)f(t)dt + C_1 \tag{88}$$

This can now be substituted into the defintion of $F(t, y)$ in Eq. (86) which must be constant by virtue of the exact differential $dF = 0$:

$$F(t, y) = y\Lambda(t) - \int \Lambda(t)f(t)dt + C_1 = C_2\,. \tag{89}$$

Let us rearrange this to solve for y,

$$y(t) = \frac{1}{\Lambda(t)}\left[\int \Lambda(t)f(t)dt + C_3\right]\,. \tag{90}$$

Given the inital condition $y(t_0) = y_0$, we get

$$y(t) = y_0 + \frac{1}{\Lambda(t)}\int_{t_0}^{t} \Lambda(t')f(t')dt' \;\text{ where }\; \Lambda(t) = \exp\left(\int_{t_0}^{t} a(t')dt'\right)\,. \tag{91}$$

Remarkably, the fact that the differential equation was linear in y allowed us to reduce the equation to calculating integrals, no matter how complicated nonlinear the functions $a(t)$ and $f(t)$ may be.

Solving first order linear differential equations via an integrating factor

Step I	Put the equation into standard form $\dot{y} + a(t)y = f(t)$. (Note: The coefficient of $\dot{y}$ must be 1.)
Step II	Identify $a(t)$ and compute $\Lambda(t) = \exp\left(\int a(t)dt\right)$
Step III	Multiply the standard form of the equation by $\Lambda(t)$
Step IV	Integrate both sides of the modified equation and solve for y.

Example 9

Solve $x^2y' + 3xy = \dfrac{\cos x}{x}$.

SOLUTION Writing the equation in standard form

$$y' + \frac{3}{x}y = \frac{\cos x}{x^3} \tag{92}$$

Thus $a(x) = 3/x$ and so we define

$$\Lambda(x) = \exp\left(\int \frac{3}{x}dx\right) = \exp\left(3\ln x\right) = \exp\left(\ln(x^3)\right) = x^3\,. \tag{93}$$

Multiplying the differential equation with this expression gives

$$x^3y' + 3x^2y = \cos x \quad \Rightarrow \quad x^3dy + 3x^2ydx = \cos x dx \tag{94}$$

The left hand side is an exact differential $d(x^3y) = x^3dy + 3x^2ydx$, thus we can simply integrate

$$d(x^3y) = \cos x dx \quad \Rightarrow \quad x^3y = \sin x + C \quad \Rightarrow \quad y(x) = \frac{\sin x}{x^3} + \frac{C}{x^3} \tag{95}$$

2.7 Linear 1st Order Differential Equations

The most general linear first order ODE may be written as

$$\dot{y} + a(t)y = f(t)\,. \tag{96}$$

As we have already seen this is an equation which can be made into an exact differential by the use of an integrating factor for arbitrary $a(t)$ and $f(t)$. Here we present an important alternative method to solve the equation. This method generalises to solving higher order linear differential equations.

A general theorem states that the general solution of inhomogeneous linear differential equations may always be expressed as the sum

$$y(t) = y_{\rm H}(t) + y_{\rm P}(t)\,. \tag{97}$$

where $y_{\rm H}(t)$ labels the **general solution of the corresponding homogeneous equation**, i.e Eq. (96) with $f(t) = 0$, and $y_{\rm P}(t)$ is any **particular solution of the inhomogeneous equation** (96).

Let us start with finding the general solution of the homogeneous equation, which is clearly separable:

$$\frac{dy_{\rm H}}{dt} + a(t)y_{\rm H} = 0 \quad \Rightarrow \quad \int \frac{dy_{\rm H}}{y_{\rm H}} = -\int a(t)dt + C_1 \quad \Rightarrow \quad \ln y_{\rm H} = -\int a(t)dt + C_1 \tag{98}$$

$$\Rightarrow \quad y_{\rm H}(t) = C_2 e^{-\int a(t)dt}\,. \tag{99}$$

In particular, in the simple case where $a(t)$ is a constant and $f(t)$ is a simple elementary function, or a polynomial or a polynomial times an elementary function, we may often simply guess the particular solution as we will see when discussing linear differential equations with constant coefficients. But for arbitrary $a(t)$, we may always find the particular solution $y_{\rm P}(t)$ using the **method of variation of constants**. This means that we replace the integration constant in $y_{\rm H}(t)$ with an arbitrary unknown function of t and substitute this ansatz into the differential equation to determine this unknown function. In this case let us replace C_2 with $\Phi(t)$ in Eq. (99), and look for a particular solution to the inhomogeneous equation in the form

$$y_{\rm P}(t) = \Phi(t)e^{-\int a(t)dt}\,. \tag{100}$$

Plug back into Eq. (96)

$$\underbrace{\left[\frac{d\Phi}{dt}e^{-\int a(t)dt} - a(t)\Phi(t)e^{-\int a(t)dt}\right]}_{\dot{y}_{\rm P}} + \underbrace{a(t)\Phi(t)e^{-\int a(t)dt}}_{a(t)y_{\rm P}} = f(t) \tag{101}$$

Notice the cancellation of the terms proportional to $\Phi(t)$, i.e. the 2nd and 3rd terms. This leads to a separable equation for $\Phi(t)$:

$$\frac{d\Phi}{dt}e^{-\int a(t)dt} = f(t) \quad \Rightarrow \quad \frac{d\Phi}{dt} = f(t)e^{\int a(t)dt} \quad \Rightarrow \quad \Phi = \int f(t)e^{\int a(t')dt'}dt + C \tag{102}$$

Here we may drop the integration constant as we need any particular solution.

Substitute back into Eq. (100), we get

$$y_{\rm P}(t) = e^{-\int a(t)dt}\int f(t)e^{\int a(t')dt'}dt\,. \tag{103}$$

Finally, substitute back in Eq. (97)

$$y(t) = Ce^{-\int a(t)dt} + e^{-\int a(t)dt}\int f(t)e^{\int a(t')dt'}dt = e^{-\int a(t)dt}\left[C + \int f(t)e^{\int a(t')dt'}dt\right] \tag{104}$$

This is clearly the same result as in Eq. (90).

Solving first order linear differential equations via variation of constants

Step I	Put the equation into standard form $\dot{y} + a(t)y = f(t)$. (Note: The coefficient of $\dot{y}$ must be 1.)
Step II	Write down the corresponding homogeneous equation $\dot{y}_H + a(t)y_H = 0$.
Step III	Integrate this separable differential equation to find $y_H(t)$.
Step IV	Now assume that the integration constant found in $y_H(t)$ is not a constant but may be a function $C(t)$, and substitute this ansatz in the inhomogeneous equation for $y(t)$.
Step V	Integrate the equation to solve for $C(t)$. The general solution is this modified version of $y_H(t)$ with this $C(t)$.

Example 9 revisited

Solve $x^2y' + 3xy = \dfrac{\cos x}{x}$.

SOLUTION Writing the equation in standard form

$$y' + \frac{3}{x}y = \frac{\cos x}{x^3} \tag{105}$$

Let us start with the homogeneous equation

$$y'_H + \frac{3}{x}y_H = 0 \quad \Rightarrow \quad \frac{dy_H}{y_H} = -\int \frac{3}{x}dx \quad \Rightarrow \quad \ln|y_H| = -3\ln|x| + C_1 \quad \Rightarrow \quad y_H = \pm e^{C_1}e^{-3\ln|x|} = C_2x^{-3} \tag{106}$$

Now let us look for the solution by replacing the C_2 constant with an arbitrary function $g(x)$ which we will determine, i.e.

$$y(x) = g(x)x^{-3} \tag{107}$$

Substitute back in equation (105)

$$y' + \frac{3}{x}y = g'(x)x^{-3} - 3g(x)x^{-4} + \frac{3}{x}g(x)x^{-3} = \frac{\cos x}{x^3} \quad \Rightarrow \tag{108}$$

The terms proportional to $g(x)$ cancel, as they always should (cf. Eq. 101) leaving us with a separable equation

$$g'(x)x^{-3} = \frac{\cos x}{x^3} \quad \Rightarrow \quad \int dg = \int \cos x dx + C \quad \Rightarrow \quad g(x) = \sin x + C \tag{109}$$

Plug back into Eq. (107)

$$y(x) = \frac{\sin x + C}{x^3}\,. \tag{110}$$

2.8 Other equations*

2.8.1 Bernoulli equation

$$y' + a(x)y = b(x)y^n \qquad (n \neq 0, 1) \tag{111}$$

Here $y(t) = 0$ is a solution. Otherwise if $y \neq 0$, then divide by y^n

$$\frac{y'}{y^n} = b(x) - a(x)y^{1-n} \quad \Rightarrow \quad \frac{1}{1-n}\frac{d}{dt}y^{1-n} = b(x) - a(x)y^{1-n} \quad \Rightarrow \tag{112}$$

Let $z = y^{1-n}$

$$\frac{1}{1-n}z' = b(x) - a(x)z \quad \Rightarrow \quad z' + (n-1)a(x)z = (n-1)b(x) \tag{113}$$

This is a linear 1st order ODE, which may be solved as discussed above.

2.8.2 Ricatti equation

$$y' = a(x)y^2 + b(x)y + c(x) \tag{114}$$

There is no general solution to this equation for arbitrary $a(x)$, $b(x)$, and $c(x)$, but if one particular solution may be guessed $y_P(x)$ then the general solution may be obtained in the form[15]

$$y(x) = y_H(x) + y_P(x) \tag{115}$$

Substuting this ansatz back into the equation, after some straightforward algebra, we get that $y_H(x)$ must satisfy

$$y_H' - [2a(x)y_P(x) + b(x)]y_H = a(x)y_H^2 \tag{116}$$

which is a Bernoulli equation with $n = 2$.

2.8.3 Equation resolved by y

In some cases the equation is not resolved by y' but it may be resolved by y and written in the form

$$y = f(x, y') \tag{117}$$

where $f(x, y')$ is a given function of x and y'. In case where this equation is difficult to solve for y' we can proceed as follows. Introduce $z = y'$, so that $y = f(x, z)$ according to Eq. (117). Taking the differential of both sides

$$z dx = dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial z} dz \tag{118}$$

Thus,

$$\left(\frac{\partial f}{\partial x} - z\right) dx + \frac{\partial f}{\partial z} dz = 0 \tag{119}$$

so

$$\frac{dz}{dx} = \frac{\left(z - \dfrac{\partial f}{\partial x}\right)}{\dfrac{\partial f}{\partial z}} \tag{120}$$

We have turned the equation into the form

$$z' = F(x, z) \tag{121}$$

which is an equation resolved by z'. If $F(x, z)$ is of the types discussed above in this chapter we may solve this for $z(x)$ or $x(z)$. Given that $y = f(x, z)$, this already represents a parameteric solution to plot the integral curves in the (x, y) plane as $(x, f(x, z(x)))$ or $(x(z), f(x(z), z))$. If needed, we may proceed by solving for $y(x)$

$$y = \int y' dx + C = \int z(x) dx + C \,. \tag{122}$$

2.9 Summary

This chapter was a collection of methods to solve different types of first order ODEs. We have seen examples where the integral curves spanned the full (x, y) space without ever intersecting and others where there were a finite number of singular solutions in addition to the general solution making the solution not smooth and not unique when they crossed the singular solution. In some cases the integral curves were closed loops, in others they were unbounded.

[15] Here y_H is similar to that arising in the corresponding homogeneous equation of a linear differential equation (Eq.98) as it does not have a zeroth order term, independent of y like $c(x)$ in Eq. (116). However the y_H term has a modified coefficient here.

Solving first order differential equations

Step I Manipulate the equation to bring it into the form $y' = F(x, y)$ or $y = F(x, y')$. In the latter case introduce $z = y'$ to bring it into the former form.

Step II A straightforward "graphical solution" is to plot the **direction field**, i.e. the line segments $y(x + \Delta x) = y_0 + y'\Delta x$ on a grid of (x_0, y_0) for small Δx .

Step III Identify if the equation is of one of the following types to obtain the solution:

- $y' = \dfrac{f(x)}{g(y)}$ is **separable** $\Rightarrow$ integrate
- $y' = f(ax + by)$ $\Rightarrow$ change to $z = ax + by$ $\Rightarrow$ separable
- $y' = f(x/y)$ is **homogeneous** $\Rightarrow$ change to $z = x/y$ $\Rightarrow$ separable
- $y' = f\left(\frac{a_1x+b_1y+c_1}{a_1x+b_1y+c_1}\right)$ $\Rightarrow$ change to $X = x + u$ and $Y = y + v$ $\Rightarrow$ find (u, v) s.t. homogeneous
- $y' = -\dfrac{P(x, y)}{Q(x, y)}$ is **exact** if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ $\Rightarrow$ find $f(x, y)$ such that $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = 0$ is the same equation $\Rightarrow$ $f(x, y) = C$
- $y' = -\dfrac{P(x, y)}{Q(x, y)}$ is exact if $\dfrac{\partial(\Lambda P)}{\partial y} = \dfrac{\partial(\Lambda Q)}{\partial x}$ where $\Lambda(x, y)$ is an **integrating factor** we must find. In some cases $\Lambda(x)$ or $\Lambda(y)$ make the equation exact
- $y' + a(x)y = f(x)$ is **linear**
  - $\Lambda(x) = e^{\int a(x)dx}$ is an integrating factor
  - or use **variation of constants** by solving the corresponding homogeneous equtaion, $y'_{\mathrm{H}} + a(x)y_{\mathrm{H}} = 0$, then make the integration constant there into a function of x and plug back into the differential equation to determine this function
- $y' + a(x)y = b(x)y^n$ **Bernoulli** $\Rightarrow$ change to $z = y^{1-n}$ $\Rightarrow$ linear

NOTE In MATHEMATICA use DSOLVE to obtain the general solution analytically and STREAMPLOT to visualize the direction field.

3 Second Order Differential Equations, Autonomous Systems

We start our analysis of second order differential equation with a special case, the autonomous systems, which is a special type of possibly nonlinear second order ODEs which can be reduced to a first order ODE that may be solved analytically, as we shall see.

Given that the solutions of generally nonlinear second order ODEs represent nonlinear two-parameter family of curves, these typically cannot be visualised as clearly as first-order systems in Sec. 2, as the integral curves inevitably intersect. An important exception is the case of **autonomous systems**, which are described by a differential equation that does not explicitly depend on the independent variable, and may be written as

$$\ddot{r} = f(r, \dot{r}), \quad r(t) = ? \tag{123}$$

where $f(r, \dot{r})$ is a given function which does not depend explicitly on t. Newton's second law is an example if the force depends on time only through the instantenous position and velocity. This equation can be turned into a first order equation as follows. Divide both sides by $v = \dot{r}$

$$\frac{\ddot{r}}{\dot{r}} = \frac{dv/dt}{dr/dt} = \frac{dv}{dr} = \frac{f(r, v)}{v} \tag{124}$$

This is now a first order differential equation which specifies how v changes as a function of r, regardless of time. The (r, v) space is referred to as the **phase space**. Since it is governed by a first order differential equation, the phase space evolution is characterised by a one-parameter family of curves, which gives an understanding of the behaviour of the system, as we will see through examples below. For a given initial condition, the evolution is restricted to any instance of this family, a single curve in phase space.

3.1 Conservative systems

Conservative systems are common in physics, where $f(r, v)$ depends explicitly only on r but not on v. In this case the differential equation is separable, since after multiplying by v we get

$$v\frac{dv}{dr} = f(r) \quad \Rightarrow \quad \boxed{\frac{1}{2}v^2 + V(r) = E} \text{ where } V(r) = -\int f(r)dr \text{ and } E \text{ is an integration constant} \tag{125}$$

Here the integration constant E is often referred to as the specific energy, and $V(r)$ is called the potential.

Having done one integral, the original second order equation for $r(t)$ is now a first order differential equation for $r(t)$. After solving for $v = dr/dt$, it becomes a separable differential equation

$$\frac{dr}{dt} = \pm\sqrt{2E - 2V(r)} \quad \Rightarrow \quad \int \frac{dr}{\sqrt{2E - 2V(r)}} = \pm \int dt = \pm t + C \tag{126}$$

Once this integral is evaluated, we have derived the implicit general solution for $r(t)$, where C and E are the two integration constants.

The harmonic oscillator and the mathematical pendulum are important examples which turn out to describe a large variety of problems in nature. Harmonic oscillators, i.e. systems where the resoring force is proportional to the displacement, describe the motion of nonlinear systems near their stable equilibria in an approximate way. The mathematical pendulum where the restoring force is proportional to the sine of the angular displacement describes situations with oscillatory bounded motion and circulating motion as we will see. This mathematical model also describes the behaviour of planetary systems or motion of stars within galaxies, etc.[16] Let us first examine the basic properties of these autonomous systems by solving these ordinary differential equations, and then show how one can obtain an understanding of the structure of the integral curves in phase space without actually solving the equations but by investigating the properties near the equilibria.

Example 1

Plot the integral curves in phase space and explain the motion in phase space for

(a) the harmonic oscillator: $\ddot{r} = f(r) = -kr$, where k is the spring constant.

[16] For planetary systems the direction of the eccentricity vector for example can oscillate back and forth or it can go around perpetually, depending on the initial conditions.

(b) the mathematical pendulum: $\ddot{\theta} = f(\theta) = -k\sin\theta$, where $k = g/L$, where L is the length, g is the gravitational acceleration.

SOLUTION

(a) For the harmonic oscillator $\ddot{r} = f(r) = -kr \quad \Rightarrow \quad v\dfrac{dv}{dr} = f(r) \quad \Rightarrow V(r) = -\int f(r)dr = \int krdr = \frac{1}{2}kr^2$. Thus,

$$\frac{1}{2}v^2 + \frac{1}{2}kr^2 = E \tag{127}$$

Here E is the integration constant which must clearly satisfy $E > 0$ in order for this equation to hold for real v and r values. This equation describes an ellipse in phase space centered on $(r, v) = (0, 0)$ with axes of $\sqrt{2E}$ and $\sqrt{2E/k}$; and it is a circle in phase space if measuring v in units of $\sqrt{k}$ shown in Figure 5. The evolution is periodic with an amplitude that is set by E. Clearly, the potential $V(r)$ has a minimum at $r = 0$. All curves in phase space oscillate around this point perpetually.

(b) For the pendulum $\ddot{\theta} = f(\theta) = -k\sin\theta \quad \Rightarrow \quad \dfrac{\ddot{\theta}}{\dot{\theta}} = \dfrac{d\omega/dt}{d\theta/dt} = \dfrac{d\omega}{d\theta} = -\dfrac{k\sin\theta}{\omega}$ where we defined the phase velocity $\omega = \dot{\theta}$. Thus $V(\theta) = -\int f(\theta)d\theta = \int k\sin\theta d\theta = -k\cos\theta$ and

$$\frac{1}{2}\omega^2 - k\cos\theta = E \quad \Rightarrow \quad \omega(\theta) = \pm\sqrt{2E + 2k\cos\theta}\,. \tag{128}$$

Figure 5 shows the integral curves. Here solutions exist for all $E > -k$. The potential $V(\theta)$ has a minimum at $\theta = 2n\pi$ and a maximum at $\theta = (2n+1)\pi$, where n is an integer. The phase space separates into distinct regions with qualitatively different behaviour

- At $E = -k$ the pendulum is at rest at the potential minimum $\theta = 0$, a stable fixed point.
- For slightly larger $E \gtrsim -k$, we have $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ so that Eq. (128) shows that $\frac{1}{2}\omega^2 + \frac{1}{2}k\theta^2 = E + k$, which is equivalent to Eq. (127) and we get harmonic oscillations around the minimum.
- If $-k < E < k$, the pendulum goes back and forth both in θ and in ω around $(\theta, \omega) = (0, 0)$, but now the shape of the curve in phase space is not an ellipse but the curve given by Eq. (128).
- If $E = k$, the pendulum is fixed at $(\theta, \omega) = (\pi, 0)$, this is an unstable fixed point, because in the vicinity of this point at $\theta = \pi + \phi$, we have $\cos\theta = -1 + \frac{1}{2}\phi^2$ so that Eq. (128) shows that $\frac{1}{2}\omega^2 - \frac{1}{2}k\phi^2 = E + k$, which is the equation of a hyperbola.
- If $E > k$, the pendulum goes around perpetually respectively either with $\omega > 0$ or with $\omega < 0$ throughout the evolution, such that ω does not change sign.

Thus, the fixed point at $E = k$ separates the phase space into oscillatory solutions (sometimes called librating solutions) for $E < k$ and circulating solutions for $E > k$. The boundary between these regions, where E is arbitrarily close to k is called the **separatrix**. In this case, $\omega/(2\sqrt{k}) = \pm(1+\cos\theta)^{1/2}/2^{1/2} = \pm\cos(\theta/2)$. Note that the integral curves in phase space are distinct except for the separatrix, where the integral curves self-intersect at the unstable fixed points $(\theta, \omega) = ((2n+1)\pi, 0)$, where n is an integer.[17]

The time evolution may be obtained from Eq. (126). For the harmonic oscillator, this yields

$$(r(t), v(t)) = (A\cos(\omega t + \phi_0), -A\omega\sin(\omega t + \phi_0)) \quad \text{where } A = \sqrt{\frac{2E}{k}}\,,\ \omega = \sqrt{k} \tag{129}$$

where E and ϕ_0 are the two integration constants. We will examine this in more detail when studying constant coefficient linear differential equations. For the pendulum, the time evolution cannot be expressed in a closed form with simple functions, it is given by the elliptic integral.

[17] The fact that integral curves do not intersect anywhere else may also be expected from the theorem on existence and uniqueness quoted below Eq. (19), since $d\omega/d\theta = -(k\sin\theta)/\omega$, the derivative of the right hand side with respect to ω is discontinuous at these points.

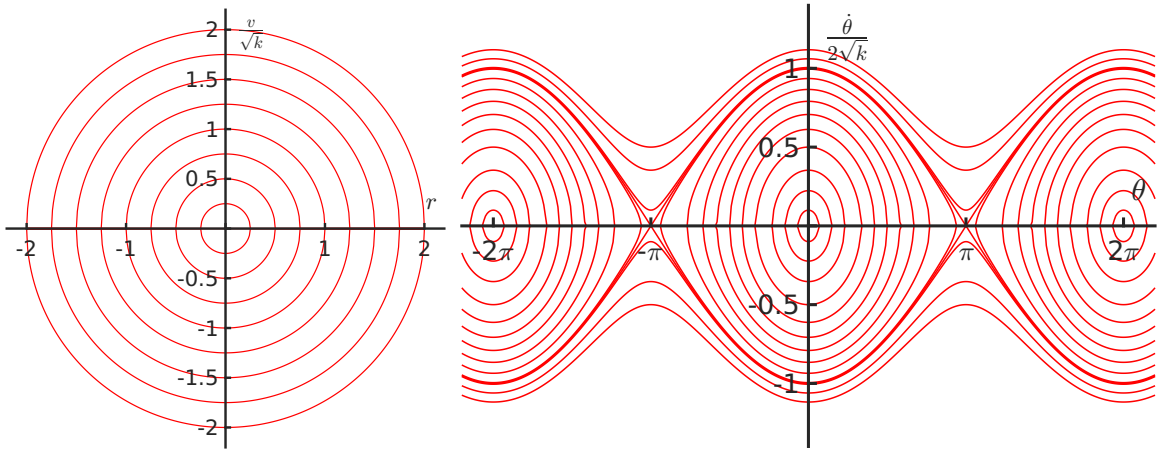


Figure 5: Phase space portrait of the oscillator (Eq. 127, left panel) and the pendulum (Eq. 128, right panel) for different values of E. The harmonic oscillator exhibits a circular motion in the plane of $(r, v/\sqrt{k})$. The pendulum behaves as a harmonic oscillator near $(\theta, \omega) = (0, 0)$ or equivalently near $(\pm 2\pi, 0)$, but it has an unstable equilibrium at $(\theta, \omega) = (\pm\pi, 0)$. For $\omega/(2\sqrt{k}) > 1$, the motion is not oscillatory in the sense that θ increases monotonically without bounds, and similarly for $\omega/(2\sqrt{k}) < -1$.

3.2 Nonconservative systems – the general case

In Sec. 3.1, we have found the solution of second order differential equations analytically, which was possible because the force depended only on the instantaneous position but not on time nor the velocity. In the more general non-conservative case, we still get a first order ODE for the phase space evolution (Eq. 124) whose solution is always a one-parameter family of curves, which in the general case may not be solved analytically in a closed form. In this case we may still obtain the qualitative structure of phase space by looking at the directon field, see below Eq. (23). Another useful technique is to study the motion as a **perturbation** around a known solution, in particular near that of a conservative system or near **fixed points**, where $\dot{v} = 0$ and $v = 0$. We Taylor-expand the equations of motion for small deviations around the known solution to first order to understand how the system behaves in their vicinity, and this often gives insight into the behaviour of the system also in the general case, far from the equilibrium, as we will see. We present the analysis through the examples of a damped harmonic oscillator and the damped mathematical pendulum.

Example 2 – damped harmonic oscillator

Plot the integral curves in phase space and explain the motion in phase space for the damped harmonic oscillator:

$$\ddot{r} = f(r, v) = -kr - \gamma v \tag{130}$$

where $k > 0$ and $\gamma > 0$ are given parameters describing the spring constant and the damping constant respectively where $\gamma < \sqrt{k}$.

SOLUTION

In this case we can derive the solution directly without any approximation. Using Eq. (124) for the damped harmonic oscillator

$$\frac{dv}{dr} = \frac{f(r, v)}{v} = \frac{-\gamma v - kr}{v} \quad \Rightarrow \quad \frac{dv}{dr} = -\gamma - k\frac{r}{v} \tag{131}$$

This is a homogeneous equation which can be solved using the usual substitution for homogeneous equations: $z = v/r$, $v = zr$, so that $z' = (v'r - v)/r^2 = (v' - z)/r$. Thus,

$$\frac{dz}{dr} = \frac{v' - z}{r} = \frac{-\gamma - k/z - z}{r} = -\frac{z^2 + \gamma z + k}{rz} \quad \Rightarrow \quad \int \frac{z\,dz}{z^2 + \gamma z + k} = -\int \frac{dr}{r} + C\,. \tag{132}$$

We may simplify the integral over z by noting that

$$z^2 + \gamma z + k = \left(z + \frac{\gamma}{2}\right)^2 + \left(k - \frac{\gamma^2}{4}\right), \tag{133}$$

Hence let us change integration variables as $u = z + \frac{1}{2}\gamma$, $dz = du$, and denote the constant with $D = k - \frac{1}{4}\gamma^2$,

$$\int \frac{z dz}{z^2 + \gamma z + k} = \int \frac{u du}{u^2 + D} - \frac{\gamma}{2} \int \frac{du}{u^2 + D} = \frac{1}{2} \ln |u^2 + D| - \frac{\gamma}{2\sqrt{D}} \arctan\left(\frac{u}{\sqrt{D}}\right) . \tag{134}$$

Plugging back into Eq. (132), we get the implicit solution

$$\frac{1}{2} \ln |u^2 + D| - \frac{\gamma}{2\sqrt{D}} \arctan\left(\frac{u}{\sqrt{D}}\right) = -\ln |r| + C . \tag{135}$$

Substituting $u = z + \frac{1}{2}\gamma$, and rearranging gives

$$\ln\left(v^2 + \gamma r + k r^2\right) - \frac{\gamma}{\sqrt{D}} \arctan\left(\frac{v/r + \frac{1}{2}\gamma}{\sqrt{D}}\right) = 2C \quad \Rightarrow \quad v^2 + \gamma r + k r^2 = C_1 \exp\left(\frac{\gamma}{\sqrt{D}} \arctan\left(\frac{v/r + \frac{1}{2}\gamma}{\sqrt{D}}\right)\right) \tag{136}$$

In the limit $\gamma \to 0$, we clearly recover the solution for the harmonic oscillator where $E = \frac{1}{2}C_1$, since $\exp(0) = 1$. A parameteric solution $(v(u), r(u))$ may be derived directly from (135) by solving for r, i.e.[18]

$$r(u) = \pm \frac{C_2}{\sqrt{u^2 + D}} \exp\left[\frac{\gamma}{2\sqrt{D}} \arctan\left(\frac{u}{\sqrt{D}}\right)\right] , \quad v(u) = z(u) r(u) = \left(u - \frac{1}{2}\gamma\right) r(u) . \tag{137}$$

The integral curves in phase space are spirals towards the fixed point $(r, v) = (0, 0)$.

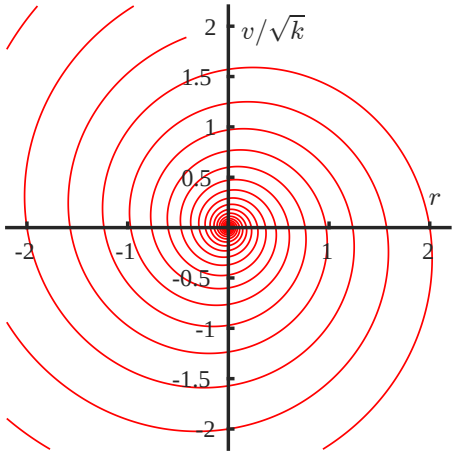


Figure 6: Phase space portrait of the damped harmonic oscillator for $\gamma = 0.3\sqrt{k}$ (cf. Figure 5 left panel for the case with $\gamma = 0$).

Perturbative solution

For equations more general than Eq. (130) where $f(r, v) = f(r) - \gamma v$, an analytical solution typically does not exist for the phase space evolution. In this case the following technique is often useful. Let us proceed as in the conservative case to derive the energy. Multiply both sides of Eq. (131) by v and rearrange

$$v \frac{dv}{dr} - f(r) = -\gamma v \quad \Rightarrow \quad \underbrace{\frac{d}{dr}\left(\frac{1}{2}v^2 + V(r)\right)}_{E(r,v)} = -\gamma v \tag{138}$$

Here we introduced the potential $V(r) = -\int f(r) dr$, where $f(r)$ corresponds to the part conservative part of the force, which does not depend on v. We may identify the left hand side as being the derivative of the energy,

[18] The integral curves may be stitched together from several of these arc segments in the form $C_2 = q^n C_3$, where $q = -\exp(\frac{\gamma\pi}{2\sqrt{D}})$ and n is an arbitrary integer, where u goes from $-\infty$ to ∞.

which is now not a constant as it changes in proportion to the instantaneous velocity. If γ is sufficiently small[19], we may obtain the solution perturbatively around a known solution, assuming that it may be written as

$$v(r) = v_0(r) + \delta v(r) \tag{139}$$

where $v_0(r)$ is the known solution for the case of $\gamma = 0$, and our goal is to determine the correction δv which we assume satisfies $\delta v \ll v_0$. We may substitute this ansatz into Eq. (138), drop terms of order δv^2 assuming that they are small, and solve the corresponding linear first order differential equation for δv. We leave this type of linear perturbative analysis to the end of this section, and first consider the following physically more insightful approach.

For the harmonic oscillator the solution for $\gamma = 0$ is oscillatory which defines the energy

$$E \approx \frac{1}{2}v_0^2 + \frac{1}{2}kr^2 \quad \Rightarrow \quad \frac{dE}{dr} \approx -\gamma v_0 \tag{140}$$

where we used Eq. (138) to express the derivative, and the average rate of change of energy from cycle to cycle is

$$\Delta E \approx \int_{\text{one cycle}} \frac{dE}{dr} dr \approx -\gamma \oint v_0(r) dr = -\gamma\sqrt{k} \oint \frac{v_0(r)}{\sqrt{k}} dr = -\gamma\sqrt{k}\,\pi \frac{2E}{k} = -\frac{2\pi\gamma}{\sqrt{k}} E \tag{141}$$

where in the second equality we substituted $dE/dr \approx -\gamma v_0$ and in the fourth we evaluated the area of the phase space curve by noting for the harmonic oscillator with no damping $(r, v_0/\sqrt{k})$ is a circle with radius $\sqrt{2E/k}$ since $2E/k = (v_0/\sqrt{k})^2 + r^2$, as shown by Eq. (140) and Fig. 5. Thus, in the case with damping the evolution is not restricted to the closed phase space curves with fixed E, but the system slowly spirals inward by decreasing its E. The energy changes during each oscillation according to Eq. (141). We may describe the phase space evolution in polar coordinates in phase space[20], since the oscillation amplitude or "radius" in polar coordinates is $R = \sqrt{2E/k}$ which slowly changes with the polar angle, θ, on average as

$$\left\langle \frac{dE}{d\theta} \right\rangle = \frac{\Delta E}{2\pi} \approx \frac{\gamma}{\sqrt{k}} E \quad \Rightarrow \quad E \approx E_0 e^{-(\gamma/\sqrt{k})\theta}\,. \tag{142}$$

Alternatively, we could use the time itself to parameterise the orbit instead of θ by noting that the unperturbed oscillation angular frequency is $\omega = \sqrt{k}$ given by Eq. (129), so that $\theta = \omega t = \sqrt{k}t$. Thus,

$$E \approx E_0 e^{-(\gamma/\sqrt{k})(\sqrt{k}t)} = E_0 e^{-\gamma t}\,. \tag{143}$$

An alternative derivation of this result is to note that

$$E = \frac{1}{2}v_0^2 + \frac{1}{2}kr^2 \quad \Rightarrow \quad \frac{dE}{dt} = \frac{dE}{dr}\frac{dr}{dt} \approx \frac{dE}{dr} v_0 = -\gamma v_0(t)^2 \quad \Rightarrow \quad \frac{dE}{dt} \approx -\gamma \langle v_0(t)^2 \rangle = -\gamma E \tag{144}$$

where in the last equality we averaged the small energy change over an oscillation cycle and argue as follows. The unperturbed solution has $r_0(t) = A\cos(\omega t + \phi)$ and $v_0(t) = -A\omega \sin(\omega t + \phi)$, so $\langle v_0^2 \rangle = \frac{1}{2}A^2\omega^2$ since $\sin^2 + \cos^2 = 1$ and so the average of $\sin^2$ over a period is a half. Furthermore, substituting the leading order solution, $E \approx \frac{1}{2}v_0(t)^2 + \frac{1}{2}kr_0(t)^2 = \frac{1}{2}A^2\omega^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) = \frac{1}{2}A^2\omega^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] = \frac{1}{2}A^2\omega^2$. Combining with the previous expression shows that $\langle v_0^2 \rangle = E$. Integrating Eq. (144) leads to Eq. (143).

This approximate solution gives more insight into the evolution of the system than the exact solution Eq. (137). The system follows a spiral in phase space, where the radius of the spiral shrinks exponentially to zero either in phase space polar angle or in time. We discuss the time evolution of the damped harmonic oscillator in more detail when we discuss the n^{th} order linear homogeneous differential equations with constant coefficients.

[19] i.e. if $\gamma \ll \sqrt{k}$, note that in the opposite case $\gamma \gg \sqrt{k}$, we may neglect the term proportional to k and treat the solution to that equation as $v_0(r)$ in a similar perturbation analysis

[20] Polar coordinates are usually defined through $(x, y) = R(\cos\theta, \sin\theta)$ or $(R, \theta) = (\sqrt{x^2 + y^2}, \arctan(y, x))$. Here we set $x \equiv r$ and $y = v/\sqrt{k}$, so that $(r, v/\sqrt{k}) = R(\cos\theta, \sin\theta)$. The solution with no damping ($\gamma = 0$) has $R = \sqrt{2E/k}$ constant for all θ and in the case of small damping $R(\theta) = \sqrt{2E(\theta)/k}$, given by Eq. (142).

Example 3 – the damped pendulum

Sketch the integral curves in phase space for the damped pendulum: $\ddot{\theta} = f(\theta, \dot{\theta}) = -\gamma\dot{\theta} - k\sin\theta$, where $k = gL$, where L is the length, g is the gravitational acceleration, and $\gamma \ll \sqrt{k}$.

SOLUTION

Since the damped pendulum is autonomous, we may use Eq. (124) to describe the phase space evolution:

$$\ddot{\theta} = f(\theta) = -\gamma\dot{\theta} - k\sin\theta \quad \Rightarrow \quad \frac{d\omega}{d\theta} = \frac{\dot{\omega}}{\dot{\theta}} = \frac{\ddot{\theta}}{\omega} = -\gamma - \frac{k\sin\theta}{\omega} \tag{145}$$

where we defined the phase velocity $\omega = \dot{\theta}$. This first order differential equation does not admit a solution in a closed form. Nevertheless we may sketch the integral curves either by plotting the direction field (see below Eq. 23) or by finding the fixed points and investigating the motion near the fixed points.

The fixed points are defined by $\ddot{\theta} = \dot{\theta} = 0$, i.e. $\theta = n\pi$ where n is an arbitrary integer. We may distinguish cases where n is even and when n is odd. To examine the motion near these fixed points, substitute $\theta = n\pi + \delta\theta$:

$$\Rightarrow \begin{cases} \delta\ddot{\theta} = -\gamma\,\delta\dot{\theta} - k\,\delta\theta\,, & (\theta \approx 2n\pi) \\ \delta\ddot{\theta} = -\gamma\,\delta\dot{\theta} + k\,\delta\theta\,, & (\theta \approx (2n+1)\pi) \end{cases} \tag{146}$$

The top equation describes a damped harmonic oscillator which we have examined previously. The phase space evolution is a spiral relaxing towards the equilibrium point. The bottom equation is similar but with a negative spring constant. In this case the solution with $\gamma = 0$ is an exponential in time $\exp(\sqrt{k}\delta\theta)$, showing that these equilibria are unstable, where θ will move away from $\theta = (2n+1)\pi$ exponentially in time.

Generally we may derive the phase space evolution perturbatively $\omega(\theta) = \omega_0(\theta) + \delta\omega(\theta)$ where ω_0 represents the known solution for no damping, Eq. (128), and $\delta\omega(\theta)$ is an unknown function we wish to determine. Then

$$\underbrace{\frac{d}{dr}\left(\frac{1}{2}\omega^2 + k\cos\theta\right)}_{E(\theta,\omega)} = -\gamma\omega \quad \Rightarrow \quad E \approx \frac{1}{2}\omega_0^2 + k\cos\theta\,, \quad \frac{dE}{dt} = \frac{dE}{dr}\frac{dr}{dt} = -\gamma\omega^2 \approx -\gamma\langle\omega_0^2\rangle \tag{147}$$

The phase evolution on short timescales of order the oscillation period is approximately that of the unperturbed pendulum ω_0 where the system is confined to a curve set by the integration constant E. On much longer timescales E slowly decreases. Thus, the integral curves in phase space are reminiscent of the case with no damping described by given E contours, showing oscillations for low energy and circulation for high energy (Figure 5), but now the phase space trajectory slowly transitions between different level curves of E towards lower and lower E. The high energy circulating solutions ultimately reduce their energy to the point that the phase space trajectory crosses the separatrix and the system starts to execute damped oscillations, which manifests as a spiral towards an equilibrium point.

Perturbative solution in an arbitrary potential*

For completeness, note that an approximate solution for the equation

$$\ddot{r} = f(r) - \gamma v \quad \Rightarrow \quad \frac{dv}{dr} = \frac{\ddot{r}}{\dot{r}} = \frac{f(r)}{v} - \gamma \tag{148}$$

in the small damping limit may be obtained from $v(r) = v_0(r) + \delta v(r)$ for an arbitrary force where v_0 is defined by requiring that it solves equation (138) with no damping term

$$\frac{d}{dr}\left(\frac{1}{2}v_0^2 + V(r)\right) = 0 \quad \Rightarrow \quad \frac{1}{2}v_0^2 + V(r) = E \approx \text{constant during an oscillation}\,, \quad V(r) = -\int f(r)dr\,, \tag{149}$$

$$\Rightarrow \quad v_0(r) = \pm\sqrt{2E - 2V(r)}\,. \tag{150}$$

Assuming that δv is small, let us substitute in Eq. (138) and drop the small terms second order or higher in δv:

$$\frac{d}{dr}\left(\frac{1}{2}(v_0+\delta v)^2 + V(r)\right) = -\gamma(v_0+\delta v) \quad \Rightarrow \quad \underbrace{\frac{d}{dr}\left(\frac{1}{2}v_0^2 + V(r)\right)}_{0} + \frac{d}{dr}(v_0\,\delta v) + \underbrace{O(\delta v^2)}_{\approx 0} = -\gamma(v_0+\delta v)\,. \tag{151}$$

Here the first term must vanish due to the definition of v_0 and the third term is approximately zero given the assumption that δv is small. Thus we get a first order linear differential equation

$$v_0' \, \delta v + v_0 \, \delta v' = -\gamma v_0 - \gamma \delta v \quad \Rightarrow \quad \delta v' + \left(\frac{v_0'}{v_0} + \gamma \right) \delta v = -\gamma \quad \Rightarrow \quad \delta v' + [f(r) + \gamma] \, \delta v = -\gamma \, . \tag{152}$$

In the last equality, we used the definition of $v_0(r)$, Eq. $f(r)$. which may be solved using the integrating factor Sec. 2.6 or variation of constants Sec. 2.7.

3.3 Summary

Second order differential equations may be simplified to first order differential equations when the equation does not depend explicitly on time (i.e. the independent variable). The trick is to first solve the equation first in **phase space**, i.e. determine how the velocity depends on position (i.e the dependent variable), then solve the resulting first order differential equations connecting the velocity to position.

Fixed points, the points in phase space where both the velocity and acceleration vanish, are important features of differential equations. In the immediate vicinity of fixed points, the integral curves in phase space may form closed loops (e.g. harmonic oscillator), spirals (e.g. damped harmonic oscillator), or ×-shaped intersecting curves (e.g. unstable fixed point of a pendulum). The nature of the solution may be identified by sketching the integral curves in the vicinity of the fixed points and connecting the curves in the intermediate regions. We have seen how this works for a pendulum. At low energies there are oscillations around the stable fixed points where the phase space trajectories are closed loops or spirals. This structure is maintained away from these stable fixed points until the energy reaches the unstable fixed points, beyond which we get "unbounded" phase space curves where the angular velocity cannot change sign. The boundary between these regions is called the separatrix, which are curves connecting the unstable fixed points.

Solving autonomous second order differential equations

Step I Manipulate the equation to bring it into the form $y'' = F(y', y), \quad y(x) = ?$

Step II Introduce $v = y'$, and derive $\dfrac{dv}{dy} = \dfrac{y''}{y'} = \dfrac{F(v, y)}{v}$.

Step III This is a first order differential equation for $v(y)$, independent of x, which may be solved by the techniques in Sec. 2. The solutions are the one-parameter family of curves in **phase space** (y, v).

Step IV To obtain the solutions in phase space proceed depending on $F(v, y)$ as follows

- For **conservative** systems, if $F(v, y)$ does not depend on v, the equation is separable

  $\Rightarrow \quad \int v dv = \int F(y) dy + E$, where E is an integration constant.

- Find the fixed points of the system y where $y'' = y' = 0$, i.e. $F(0, y) = 0$ and $v = 0$.

  1. Examine small perturbations around the fixed points. Stable fixed points show closed loops in phase space or inward spirals, unstable fixed points manifest as outward spirals or ×-shaped intersections.
  2. The integral curves may often be sketched by connecting the curves, noting that the qualitative behaviour can only change on the boundary curves which cross unstable fixed points, i.e. ×-shaped intersections.

- If $F(v, y)$ weakly depends on v, the solution may be obtained perturbatively as $v(y) = v_0(y) + \delta v(y)$.

  1. Find $v_0(y)$, which satisfies $\dfrac{dv_0}{dy} = \dfrac{F(0, y)}{v_0} \quad \Rightarrow$ "conservative" equation for $v_0(y)$
  2. Substitute this solution in the original equation $\Rightarrow$ find how E changes in phase space

- At the very least, one can alway plot the direction field on a grid in phase space.

4 Linear Differential Equations

Returning to some of the famous examples of ODEs listed in Section1, we may notice that many of the equations are *second order linear ODEs*:

1. $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t)$, Newton's second law of motion for a point mass's position in time $\mathbf{r}(t) = (x(t), y(t), z(t))$ in a given force field $\mathbf{F}(x, y, z, t)$

2. $m\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} - k\mathbf{r} = F(t)$, driven harmonic oscillator specifying the motion around an equilibrium where m, γ, and k are constants, $F(t)$ is a given external forcing function in time

3. $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t) + q\mathbf{E}(\mathbf{r}, t)$, Lorentz force in electromagnetism for the motion of a point charge in a given electromagnetic field given by $\mathbf{E}$ and $\mathbf{B}$ everywhere in space and time

4. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x) = E\psi$, Schrödinger equation in quantum mechanics which describes the wave function $\psi(x)$ that determines the probability of finding a particle at some location having energy E in a given one dimensional potential well $V(x)$

Examples 1 and 3 are generally not linear, unless $F(r, t)$, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are linear functions of $\mathbf{r}$. In this chapter we will examine linear systems.

We examine n^{th} order linear differential equations in y:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)\,, \qquad y(x) =? \tag{153}$$

Linear equations commonly arise in physics in perturbation theory when studying the evolution close to a known solution, such as the motion around an equilibrium. Indeed, the first approximation in a Taylor series is the linear approximation.[21]

As for all ODEs there are typically infinitely many solutions to Eq. (161), as discussed in Sec. 1.2. Given that these are n^{th} order ODEs,

- there are n essentially independent integration constants which enter when the derivatives are undone by evaluating integrals;
- the solutions represent n-parameter family of curves in the (x, y) plane; and
- there must be n initial conditions or boundary conditions to fix the values of these integration constants. In particular for initial conditions, $(y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)) = (y_0, y_0', \ldots, y_0^{(n-1)})$ are given.

We shall start by observing the following interesting property of linear ODEs.

THEOREM If y_1 and y_2 are two solutions then their difference $y_1 - y_2$ solves the **corresponding homogeneous equation** which is defined by setting the term indepedent of y and its derivaties to 0, i.e.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0\,. \tag{154}$$

PROOF If y_1 and y_2 are solutions, they satisfy Eq. (161). Let us take the difference of these equations.

$$a_n(x)y_2^{(n)} + \cdots + a_1(x)y_2' + a_0(x)y_2 = f(t) \tag{155}$$

$$-a_n(x)y_1^{(n)} - \cdots - a_1(x)y_1' - a_0(x)y_1 = -f(t) \tag{156}$$

$$\Rightarrow a_n(x)(y_2^{(n)} - y_1^{(n)}) + \cdots + a_1(x)(y_2' - y_1') + a_0(x)(y_2 - y_1) = 0 \tag{157}$$

$$\Rightarrow a_n(x)(y_2 - y_1)^{(n)} + \cdots + a_1(x)(y_2 - y_1)' + a_0(x)(y_2 - y_1) = 0 \text{ as required.} \tag{158}$$

How is it possible that there are infinitely many solutions, but the difference between any two of them solves the corresponding homogeneous equation? This suggests that the general solution must be written in the following form

$$y(x) = y_{\text{H}}(x) + y_{\text{P}}(x) \tag{159}$$

[21] We get a linear differential equation if the originally nonlinear equation describing the system is expanded to first order in y and the derivatives of y.

where $y_H(x)$ is the **general solution of the corresponding homogeneous equation** (154), which comprise an n-parameter family of curves, and $y_P(x)$ is **one particular solution of the inhomogeneous equation** (161), i.e the same for all solutions. If so, for any two solutions $y_1(x)$ and $y_2(x)$ having the same $y_P(x)$ but different $y_{H1}(x)$ and $y_{H2}(x)$, their difference will always satisfy the homogeneous equation. Therefore the n-parameter family of curves of the general inhomogeneous equation may be written as the n-parameter family of curves of the corresponding homogeneous equation plus a single particular solution $y_P(x)$. We shall investigate how to obtain $y_H(x)$ and $y_P(x)$ in turn below.

The solution of the corresponding homogeneous equation is sometimes called the **complementary function** or **complementary solution**.

4.1 Linear homogeneous differential equations

Let us start by understanding the general solution of linear homogeneous differential equations

$$\boxed{a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0}. \tag{160}$$

Homogeneous differential equations satisfy the following important property.

4.1.1 Linear combinations of solutions are also solutions

THEOREM If $y_1, y_2, \ldots, y_n$ are solutions to an n^{th} order homogeneous differential equation, then any of their linear combinations $C_1y_1 + C_2y_2 + \cdots + C_ny_n$ with arbitrary $C_1, C_2, \ldots, C_n$ complex numbers is also a solution of the same equation.

PROOF We provide the proof for $n = 2$. If y_1 and y_2 are solutions, they satisfy Eq. (257). Let us multiply the first equation by C_1 and the second by C_2 and add

$$C_1a_n(x)y_2^{(n)} + \cdots + C_1a_1(x)y_2' + C_1a_0(x)y_2 = 0 \tag{161}$$

$$C_2a_n(x)y_1^{(n)} + \cdots + C_2a_1(x)y_1' + C_2a_0(x)y_1 = -0 \tag{162}$$

$$\Rightarrow a_n(x)(C_1y_1^{(n)} + C_2y_2^{(n)}) + \cdots + a_1(x)(C_1y_1' + C_2y_2') + a_0(x)(C_1y_1 + C_2y_2) = 0 \tag{163}$$

$$\Rightarrow a_n(x)(C_1y_1 + C_2y_2)^{(n)} + \cdots + a_1(x)(C_1y_1 + C_2y_2)' + a_0(x)(C_1y_1 + C_2y_2) = 0 \text{ as required.} \tag{164}$$

In the last equality we used the linear property of the differential operator, i.e. for any two functions $f(x)$ and $g(x)$ the derivative of the linear combination is the linear combination of the derivative

$$\frac{d}{dt}\left[af(x) + bg(x)\right] = a\frac{df}{dx} + b\frac{dg}{dx} \tag{165}$$

The same argument shows that the linear combination of n different solutions $C_1y_1 + C_2y_2 + \cdots + C_ny_n$ is also a solution.

4.1.2 The family of curves for linear differential equations

The superposition theorem (4.1.1) clearly indicates the type of solutions homogeneous linear differential equations may have.

$$\boxed{y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)} \qquad (C_1, C_2, \ldots, C_n \text{ are arbitrary generally complex constants}) \tag{166}$$

The n-parameter family of curves which represents the general solution with n independent integration constants arises simply as an arbitrary **linear combination of n linearly independent functions**, $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. This is very different from the beautifully complex geometry for the family of curves obtained for the nonlinear first order equations discussed in Section 2. This implies that the solution of linear equations may not have the types of complexities that we have observed for first order nonlinear equations, such as singular solutions or non-unique composite solutions.

4.1.3 Linearly independent functions and the Wronskian

Suppose we found n functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ which solve an n^{th} order differential equation. To verify that these span the complete set of solutions, we must make sure that they are indeed **linearly independent**, meaning that

$$C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) = 0 \text{ for all } x \text{ only if } C_1 = C_2 = \cdots = C_n = 0 \tag{167}$$

Otherwise if there is at least some $C_j \neq 0$ for which the linear combination is zero, then we may rearrange the equation and express $y_j(x)$ as the linear combination of the other functions. This means that $y_j(x)$ does not add anything to the set of solutions given by the other functions.

Given a set of solutions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ how can we determine whether or not they are linearly independent? Recall the analogous problem in linear algebra: given vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, how do we know if they are linearly independent? For vectors the definition of being linearly independent is similar as for functions

$$C_1 \mathbf{v}_1 + C_2 \mathbf{v}_2 + \cdots + C_n \mathbf{v}_n = 0 \text{ only if } C_1 = C_2 = \cdots = C_n = 0 \tag{168}$$

but here the sum on the left hand side may also be written in matrix form by inserting the vectors as column vectors into an $n \times n$ matrix

$$\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ only if } C_1 = C_2 = \cdots = C_n = 0 \quad \Leftrightarrow \quad \det \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \neq 0 \tag{169}$$

Clearly, this holds if the matrix formed by the column vectors is invertible, as the inverse matrix acting on the zero vector is then the zero vector. The matrix is invertible if and only if its determinant is not zero; this is a straightforward way to test linear independence for vectors.

Let us now derive a similar condition for a set of n functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. Taking the derivatives of both sides in Eq. (167), linearly independent functions also satisfy all of the following n conditions:

$$\left.\begin{aligned} C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) &= 0 \\ C_1 y_1'(x) + C_2 y_2'(x) + \cdots + C_n y_n'(x) &= 0 \\ C_1 y_1''(x) + C_2 y_2''(x) + \cdots + C_n y_n''(x) &= 0 \\ &\vdots \\ C_1 y_1^{(n-1)}(x) + C_2 y_2^{(n-1)}(x) + \cdots + C_n y_n^{(n-1)}(x) &= 0 \end{aligned}\right\} \text{ for all } x \text{ only if } C_1 = C_2 = \cdots = C_n = 0 \tag{170}$$

or equivalently

$$\begin{pmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ y_1''(x) & y_2''(x) & \ldots & y_n''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for all } x \text{ only if } C_1 = \cdots = C_n = 0 \tag{171}$$

$$\Leftrightarrow \quad W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ y_1''(x) & y_2''(x) & \ldots & y_n''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0 \tag{172}$$

Thus, the determinant formed from the functions and their derivatives in the above way must be nonzero for the functions to be linearly independent. This determinant is called the **Wronskian**, labelled $W(x)$. It is sufficient to check if $W(x_0) \neq 0$ for any x_0 for which not all $y_i(x_0) = 0$ then $W(x) \neq 0$ for all x.

Example 1

Show that the functions $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent.

Solution Here $y_1'(x) = 1$, $y_2'(x) = 2x$, so

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \tag{173}$$

This is not zero except at $x = 0$, hence the functions are linearly independent.

Example 2

Are the functions e^{ix}, $\sin x$, and $\cos x$ linearly independent?

Solution Generally we have $e^{ix} = \cos x + i \sin x$, so $e^{ix} - i \sin x - \cos x = 0$. Thus, there exists a linear combination of these functions with not all zero coefficients which returns zero for all x, hence these functions are not linearly independent.

4.1.4 Going from one set of basis functions to another

The n linearly independent functions which solve the homogeneous linear differential equation are however not unique just as how the basis vectors in position space are not unique. For instance, the linear combinations of $\mathbf{e_x} = (1, 0, 0)$ and $\mathbf{e_y} = (0, 1, 0)$ span the x-y space but any two linearly independent combinations $\mathbf{e_x}$ and $\mathbf{e_y}$ may also be chosen as the basis vectors and their linear combinations also span the same space completely.

Given n linearly independent solutions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$, the general solution of the differential equation is the linear combination of any n basis functions which are arbitrary linearly independent linear combinations of $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. Indeed, we may write

$$\begin{aligned} y(x) &= C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) = \sum_i C_i y_i(x) = \mathbf{C} \cdot \mathbf{y}(x) = \begin{pmatrix} C_1 \\ C_2 \\ \cdots \\ C_n \end{pmatrix} \cdot \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} \\ &= \underbrace{(C_1 \;\; C_2 \;\; \cdots \;\; C_n)}_{\mathbf{C}^{\mathrm{T}}} \underbrace{\begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}}_{\mathbf{y}(x)} = \mathbf{C}^{\mathrm{T}} \mathbf{y}(x) = \underbrace{\mathbf{C}^{\mathrm{T}} \mathbf{S}^{-1}}_{\mathbf{B}^{\mathrm{T}}} \underbrace{\mathbf{S}\,\mathbf{y}(x)}_{\mathbf{Y}(x)} = \underbrace{(B_1 \;\; B_2 \;\; \cdots \;\; B_n)}_{\mathbf{B}^{\mathrm{T}}} \underbrace{\begin{pmatrix} Y_1(x) \\ Y_2(x) \\ \vdots \\ Y_n(x) \end{pmatrix}}_{\mathbf{Y}(x)} = \mathbf{B} \cdot \mathbf{Y}(x) \\ &= B_1 Y_1(x) + B_2 Y_2(x) + \cdots + B_n Y_n(x)\,, \quad \text{where } \mathbf{Y}(x) = \mathbf{S}\mathbf{y}(x), \quad \mathbf{B} = \left(\mathbf{C}^{\mathrm{T}} \mathbf{S}^{-1}\right)^{\mathrm{T}} = (\mathbf{S}^{-1})^{\mathrm{T}} \mathbf{C} \end{aligned} \tag{174}$$

Here we introduced combined the linear combination coefficients into a vector $\mathbf{C}$, wrote the sum as a scalar product, then as a matrix multiplication of a row vector or an $1 \times n$ matrix ($\mathbf{C}^{\mathrm{T}}$) times a column vector or an $n \times 1$ matrix of functions $\mathbf{y}(x)$, then inserted an identity matrix as $\mathbf{S}^{-1}\mathbf{S}$, and used the associativity of matrix multiplication (i.e. $ABC = (AB)C = A(BC)$) to introduce the new set of basis functions and their linear combination coefficients. We have shown that if $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent solutions, then $Y_i(x) = \sum_j S_{ij} y_j(x)$ are also linearly independent solution functions where S_{ij} is a matrix of arbitrary constant elements (i.e. independent of x) whose determinant is nonzero[22]. Conversely, $y_j(x) = \sum_j (S^{-1})_{ij} y_j(x)$. The linear combinations of $\{Y_1(x), \ldots, Y_n(x)\}$ span the same set of functions as the linear combinations of $\{y_1(x), \ldots, y_n(x)\}$.

Thus the general solution of an n^{th} order differential equation is spanned by any n linearly independent functions for which $W \neq 0$ (see Eq. 172) and each of which solves the differential equation.

4.1.5 Finding the linear combination coefficients

Suppose we are given initial conditions at x_0, i.e.

$$\left(y(x_0),\; y'(x_0),\; y''(x_0), \cdots, y^{(n-1)}(x_0)\right) = \left(y_0,\; y_0',\; y_0'',\; \cdots,\; y_0^{(n-1)}\right) \tag{175}$$

[22] Sometimes it is more convenient to work with orthogonal matrices $\mathbf{O}^{-1} = \mathbf{O}^{\mathrm{T}}$ to switch between bases. The column vectors of orthogonal matrices are unit vectors which are orthogonal to each other, and similarly for row vectors.

are given numbers. How do we find the linear combination coefficients in the general solution? Simply substitute in the general solution (Eq. 166)

$$\left.\begin{array}{rcl} y_0 &=& C_1 y_1(x_0) + C_2 y_2(x_0) + \cdots + C_n y_n(x_0) \\ y_0' &=& C_1 y_1'(x_0) + C_2 y_2'(x_0) + \cdots + C_n y_n'(x_0) \\ y_0'' &=& C_1 y_1''(x_0) + C_2 y_2''(x_0) + \cdots + C_n y_n''(x_0) \\ &\vdots& \\ y_0^{(n-1)} &=& C_1 y_1^{(n-1)}(x_0) + C_2 y_2^{(n-1)}(x_0) + \cdots + C_n y_n^{(n-1)}(x_0) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix} = \begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ y_1''(x_0) & y_2'' & \cdots & y_n''(x_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{pmatrix} \tag{176}$$

$$\Rightarrow \quad \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ y_1''(x_0) & y_2''(x_0) & \cdots & y_n''(x_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix} \tag{177}$$

In the second step we have expressed the system of equations as a single vector equation, where the matrix is the same as in the Wronskian, Eq. (172). In the third step we multiplied both sides by the inverse matrix, which exists for linearly independent solutions as shown by Eq. (172). Having determined the constants, let us substitute back in Eq. (166):

$$y(x) = \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \end{pmatrix} \begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ y_1''(x_0) & y_2''(x_0) & \cdots & y_n''(x_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix} \tag{178}$$

Given n linearly independent solutions and the initial conditions, this is clearly a **unique solution** to the initial value problem Eqs. (257) and (175) for all x.

4.1.6 Linear homogeneous differential equation with constant coefficients

An important special case of linear differential equations, where the solution may be obtained analytically is when the coefficients are constant numbers

$$\boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0}\,. \tag{179}$$

Since for first order differential equations, $n = 1$, the solution is proportional to $e^{\lambda x}$, let us look for solutions in the same form

$$y(x) = e^{\lambda x}\,. \tag{180}$$

Let us substitute in the differential equation to find λ. In this case $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$, and so on, so

$$a_n \lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0\,. \quad \Rightarrow \quad \left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + \lambda + a_0\right) e^{\lambda x} = 0\,. \tag{181}$$

Therefore $e^{\lambda x}$ is a solution if and only if λ satisfies

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + \lambda + a_0 = 0\,. \tag{182}$$

This is called the **characteristic equation** of the differential equation, the left hand side being the **characteristic polynomial**. The *fundamental theorem of algebra* (Gauss 1799) states that every n^{th} order polynomial has

exactly n generally complex roots (but not necessarily distinct) so that the characteristic polynomial may be written as

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + \lambda + a_0 = a_n(\lambda - \lambda_1)(\lambda - \lambda_2) \times \cdots \times (\lambda - \lambda_n) = 0 \tag{183}$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are generally complex numbers, the roots of the characteristic polynomial. If the $(a_1, \ldots, a_n)$ coefficients are real then complex solutions come in pairs; if λ is a root, its complex conjugate λ^* must also be a root. Each root may be substituted in Eq. (180) all of which solve the ODE (179). But since any linear combination of solutions is a solution, we get that the solution of the differential equation (179) is

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}} \qquad (\lambda_1, \lambda_2, \ldots, \lambda_n \text{ all distinct}) \tag{184}$$

If $(\lambda_1, \ldots, \lambda_n)$ are all distinct, this is an n-parameter family of curves, i.e. the complete general solution. However if not all λ_i are distinct but some are equal then their C_i coefficients can be combined into a single coefficient, and the general solution must have additional functions not of the form Eq. (180). In this case exponential times a polynomial is going to be a solution as stated next, which may be confirmed by substituting back into the equation.

Example 3

Find the solution to the following initial value problem:

$$y'' + 3y' + 2y = 0\,, \qquad y(0) = 1\,,\ y'(0) = 2\,,\ y(x) = ? \tag{185}$$

Solution Susbstitute the ansatz $y = e^{\lambda x}$ into the equation, where $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$.

$$(\lambda^2 + 3\lambda + 2)e^{\lambda x} = 0\,, \tag{186}$$

The characteristic polynomial, which is now a quadratic polynomial, must clearly be zero for this equation to hold. We find

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \times 2}}{2} = \begin{cases} -1 \\ -2 \end{cases} \tag{187}$$

Thus the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}\,. \tag{188}$$

Given the boundary conditions,

$$y(0) = C_1 + C_2 = 1 \tag{189}$$

$$y'(0) = -C_1 - 2C_2 = 2 \tag{190}$$

we find that $C_1 = 4$ and $C_2 = -3$, so the solution is

$$y(x) = 4e^{-x} - 3e^{-2x}\,. \tag{191}$$

Example 4

Find the general solution to the following differential equation

$$y'' - 6y' + 9y = 0\,, \quad y(x) = ? \tag{192}$$

Solution Susbstituting the ansatz $y = e^{\lambda x}$ into the equation, leads to the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0\,. \tag{193}$$

The roots are

$$\lambda_{1,2} = \frac{6 \pm \sqrt{6^2 - 4 \times 9}}{2} = \begin{cases} 3 \\ 3 \end{cases} \tag{194}$$

Indeed the characteristic polynomial is now $(\lambda-3)^2$, so here the two roots coincide. Now we cannot conclude that the general solution is the linear combination of these solutions $C_1e^{3x}+C_2e^{3x}$ because this is really a one-parameter family of curves $(C_1+C_2)e^{3x}$, where the parameter is $C=C_1+C_2$. This is a problem because we cannot satisfy the initial conditions $y(x_0)=a$ and $y'(x_0)=b$ for any given a and b by setting the value of a single number C. Indeed, the general solution of any second order ordinary differential equations is a two parameter family of curves, so we are clearly missing a solution.

To find the missing solution, we may use the method of variation of constants. In Ce^{3x} we set the value of C to an unknown function of x and look for a solution in the form

$$y(x)=\psi(x)e^{3x} \tag{195}$$

where $\psi(x)$ is an unknown function of x that we wish to determine. Let us substitute back in the differential equation,

$$\underbrace{(\psi''+6\psi'+9\psi)e^{-3x}}_{y''}-6\underbrace{(\psi'+3\psi)e^{-3x}}_{y'}+9\underbrace{\psi e^{3x}}_{y}=0 \tag{196}$$

There are several cancellations in this equations, it simplifies to

$$\psi''e^{3x}=0\,. \tag{197}$$

Thus we must have $\psi''=0$. We can integrate this equation twice

$$\psi=C_1+C_2x\,. \tag{198}$$

Therefore the solution follows from Eq. (195),

$$y(x)=(C_1+C_2x)e^{3x}\,. \tag{199}$$

This is now indeed a two-parameter family of curves, the solution is now the linear combination of two linearly independent functions of x.

Example 5

Find the general solution to the following differential equation

$$y'''=0\,,\quad y(x)=? \tag{200}$$

Solution We may simply integrate three times to get immediately $C_1+C_2x+C_3x^2$. Alternatively, in this case the characteristic polynomial is $\lambda^3=(\lambda-0)^3$, so $\lambda=0$ is a repeated root with multiplicity 3. So we recover the solution as $y(x)=(C_1+C_2x+C_3x^2)e^{0x}=C_1+C_2x+C_3x^2$.

The previous two examples may be generalised as follows for repeated roots with arbitrary multiplicity.

Solutions for repeated roots

THEOREM Let $p(\lambda)=\sum_{i=1}^n a_i\lambda^i$ be the characteristic polynomial of a constant coefficient linear ODE. If λ_r is a repeated root of the characteristic equation with multiplicity k, i.e. $p(\lambda_\mathrm{r})$ may be expressed as $(\lambda-\lambda_\mathrm{r})^k$ times a polynomial of order $n-k$, or equivalently, $p(\lambda_\mathrm{r})=0$, $p'(\lambda_\mathrm{r})=0$, $p''(\lambda_\mathrm{r})=0,\ldots,p^{(k-1)}(\lambda_\mathrm{r})=0$, then all $(k-1)^{\text{th}}$ order polynomial times $e^{\lambda_\mathrm{r}x}$ are solutions to the differential equation

$$\boxed{y(x)=(C_1+C_2x+\ldots C_kx^{k-1})e^{\lambda_\mathrm{r}x}}\qquad(\lambda_\mathrm{r}\text{ is a root with multiplicity }k>1) \tag{201}$$

The general solution is the linear combination of all distinct e^{λ_ix} with the polynomial times exponentials Eq. (203) for repeated roots.

For example, if the characteristic polynomial of some 7th order constant coefficient linear ODE is

$$(\lambda-a)^3(\lambda-b)^2(\lambda-c)(\lambda-d) \tag{202}$$

where $\{a,b,c,d\}$ are 4 different real numbers, these are the roots of the characteristic equation with multiplicity 3, 2, 1, 1, respectively, and the general solution of the ODE is

$$y(x)=(C_1+C_2x+C_3x^2)e^{ax}+(C_4+C_5x)e^{bx}+C_6e^{cx}+C_7e^{dx}\,. \tag{203}$$

λ_1	λ_2	$y(x)$	$y(x)$ (alternative form)	$y(x)$ (alternative form 2)
0	0	$C_1 + C_2 x$		
κ	κ	$(C_1 + C_2 x)e^{\kappa x}$		
κ_1	κ_2	$C_1 e^{\kappa_1 x} + C_2 e^{\kappa_2 x}$	$C_1 \sinh(\kappa_1 x) + C_2 \cosh(\kappa_1 x)$	
$i\omega$	$-i\omega$	$C_1 \sin(\omega x) + C_2 \cos(\omega x)$	$C \cos(\omega x + \delta)$	$\mathrm{Re}\left(Ce^{i\delta}e^{i\omega x}\right)$
$\kappa + i\omega$	$\kappa - i\omega$	$[C_1 \sin(\omega x) + C_2 \cos(\omega x)]e^{\kappa x}$	$C \cos(\omega x + \delta)e^{\kappa x}$	$\mathrm{Re}\left(Ce^{i\delta}e^{\kappa x + i\omega x}\right)$

Table 1: The possible solutions of second order linear constant coefficient homogeneous differential equations depending on the roots of the characteristic polynomial (Eq. 183) λ_1 and λ_2. Here κ, κ_1, κ_2, and ω denote nonzero real numbers, $\kappa_1 \neq \kappa_2$, and C_1, C_2, C, and δ are arbitrary integration constants, and $\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$ denotes the real part of a complex number.

Complex roots

As noted above, the fundamental theorem of algebra states that the characteristic equation generally has complex roots even if the $(a_1, \ldots, a_n)$ polynomial coefficients in Eq. (183) are real, but in this case the solutions come in pairs; if λ is a root, its complex conjugate λ^* is also a root. Complex (non-real) distinct roots $\lambda_{1,2}$ can always be expressed as $\kappa \pm i\omega$ where κ and ω are real, and the coefficients C_1 and C_2 are also complex in a way that the linear combination $C_1 e^{\kappa x + i\omega x} + C_2 e^{\kappa x - i\omega x}$ is real. Substituting the identity $e^{i\varphi} = \cos\varphi + i \sin\varphi$ for $\varphi = \lambda x$, the linear combination of these solutions is equivalently written as

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} = C_1 e^{(\kappa + i\omega)x} + C_2 e^{(\kappa - i\omega)x} = [A_1 \cos(\omega x) + A_2 \sin(\omega x)]\, e^{\kappa x} = A e^{\kappa x} \cos(\omega x + \delta) \tag{204}$$

where $A_1 = C_1 + C_2$ and $A_2 = i(C_1 - C_2)$. But since C_1 and C_2 were arbitrary complex numbers, A_1 and A_2 are also generally arbitrary complex numbers. If we want the solution to be real, A_1 and A_2 must be real. In the last equality we used $(A_1, A_2) = (A\cos\delta, A\sin\delta)$ and $A = \sqrt{A_1^2 + A_2^2}$ and $\tan\delta = A_2/A_1$, where A and δ are real. Thus even if some or all roots of the characteristic equation λ are complex, we may get real function solutions which are then expressed with trigonometric functions in this way.

An even smarter way to deal with complex numbers popping up in the solutions of real valued linear differential equations is as follows. First note that generally if the differential equation has real coefficients, then the complex conjugate of the solution must also be a solution[23]. Further, all linear combinations of solutions are also solutions. This implies that the real part of the solution, which is defined to be the average of the function and its complex conjugate, $\mathrm{Re}(y) = \frac{1}{2}(y + y^*)$, must also be a solution to the differential equation. Therefore if we encounter a complex number or a complex function, we may just take the real part of the solution, and treat that as the general solution of the equation.

For second order constant coefficient differential equations, Table 1 summarizes the possible solutions depending on the roots of the characteristic equation, which in this case is quadratic. Purely real nonzero roots imply exponential or linear times exponential solutions, purely imaginary roots imply sinusoidal oscillations, and complex roots lead to sinusoidal oscillations with exponentially damped or exponentially increasing amplitude. Finally, if both roots are zero, then the solution is linear.

Example 6

Find the solution to the initial value problem

$$y'' + 2y' + 4y = 0\,, \quad y(0) = 1,\ y'(0) = -1 + 2\sqrt{2}. \tag{205}$$

Solution Let us substitute the ansatz $y = e^{\lambda x}$,

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 4e^{\lambda x} = 0 \quad \Rightarrow \quad (\lambda^2 + 2\lambda + 4)e^{\lambda x} = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm i\sqrt{3}. \tag{206}$$

Therefore the general solution is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} = \left(C_1 e^{i\sqrt{3}\,x} + C_2 e^{-i\sqrt{3}\,x}\right) e^{-x} = \left(A \sin(\sqrt{3}\,x) + B\cos(\sqrt{3}\,x)\right) e^{-x} \tag{207}$$

[23] to prove this take the complex conjugate of the equation

To set the values of the integration constants A and B we use the initial conditions. First note that

$$y'(x) = -\left(A\sin(\sqrt{3}\,x) + B\cos(\sqrt{3}\,x)\right)e^{-x} + \left(A\sqrt{3}\cos(\sqrt{3}\,x) - B\sqrt{3}\sin(\sqrt{3}\,x)\right)e^{-x} \tag{208}$$

Thus the initial values satisfy

$$\begin{cases} y(0) = 1 = B \\ y'(0) = -1 + 2\sqrt{2} = -B + A\sqrt{3} \end{cases} \quad \Rightarrow \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad y(x) = \left(2\sin(\sqrt{3}\,x) + \cos(\sqrt{3}\,x)\right)e^{-x}\,. \tag{209}$$

Example 7 – damped oscillator

Find the general solution for the position of the damped harmonic oscillator as a function of time where the force per unit mass of the oscillator is $-\gamma\dot{r} - kr$. Thus

$$\ddot{r} + \gamma\dot{r} + kr = 0\,, \quad r(t) = ? \tag{210}$$

Determine how $r(t)$ and the specific energy $E(t) = \frac{1}{2}\dot{r}^2(t) + \frac{1}{2}kr^2(t)$ change for very large t.

Solution This is an inhomogeneous linear differential equation, so the solution must be of the form

$$\lambda^2 + \gamma\lambda + k = 0 \quad \Rightarrow \quad \lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - k} \quad \Rightarrow \quad r(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\,. \tag{211}$$

The term $D = \frac{1}{4}\gamma^2 - k$ may either be positive, zero, or negative, which leads to different outcomes.

(a) $D > 0$ overdamped oscillator: In this case we have two real solutions, which may be written as

$$r(t) = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} = \left(C_1 e^{-\sqrt{D}\,t} + C_2 e^{\sqrt{D}\,t}\right)e^{-\frac{1}{2}\gamma t} \longrightarrow C_2 e^{\sqrt{D}\,t} e^{-\frac{1}{2}\gamma t} \tag{212}$$

Note that as long as $\gamma > 0$, both λ are negative, so we get exponential damping. At very late times, $t \gg D^{-1/2}$, the first term in the parenthesis is negligible[24], so we get that $r(t)$ tends to $C_2 e^{-(\frac{1}{2}\gamma - \sqrt{D})t}$ and $\dot{r}(t)$ tends to $-C_2(\frac{1}{2}\gamma - \sqrt{D})e^{-(\frac{1}{2}\gamma - \sqrt{D})t}$. The specific energy changes asymptotically in proportion to $e^{-(\gamma - 2\sqrt{D})t}$ for large t.

(b) $D = 0$: In this case we have a repeated root $\lambda_1 = \lambda_2 = -\frac{1}{2}\gamma$ with multiplicity two, so the solution is

$$r(t) = (C_1 + C_2 t)e^{-\lambda t} = (C_1 + C_2 t)e^{-\frac{1}{2}\gamma t} \longrightarrow C_2 t e^{-\frac{1}{2}\gamma t}\,. \tag{213}$$

where we have indicated the late time behaviour for $t \gg C_1/C_2$. The energy changes asymptotically proportional to $t^2 e^{-\gamma t}$.

(c) $D < 0$ underdamped oscillator: In this case we have two complex solutions, which may be written as

$$r(t) = \left[C_1\sin(\omega t) + C_2\cos(\omega t)\right]e^{-\frac{1}{2}\gamma t} = Ce^{-\frac{1}{2}\gamma t}\cos(\omega t + \delta) \tag{214}$$

where $\omega = |D|^{1/2}$. The late time behaviour cannot be further simplified. The energy changes asymptotically in portion to $\cos^2(\omega t + \delta)e^{-\gamma t}$.

Note that this solution gives a simple parameteric solution in phase space (cf. Figure 6), since

$$\dot{r}(t) = \left[\left(C_1\omega - \frac{C_2\gamma}{2}\right)\cos(\omega t) + \left(C_2\omega - \frac{C_1\gamma}{2}\right)\sin(\omega t)\right]e^{-\frac{1}{2}\gamma t} = -C'e^{-\frac{1}{2}\gamma t}\sin(\omega t + \delta')\,. \tag{215}$$

where

$$C' = \sqrt{\left(C_2\omega - \frac{C_1\gamma}{2}\right)^2 + \left(C_1\omega - \frac{C_2\gamma}{2}\right)^2} \quad \delta' = \arctan\left(\frac{C_1\omega - \frac{C_2\gamma}{2}}{C_2\omega - \frac{C_1\gamma}{2}}\right) \tag{216}$$

For $\gamma \ll \omega$, we have $C' \approx C$ and we get a spiral which is almost a circle in phase space $(r, \dot{r}/\omega)$ whose radius changes as $Ce^{-\frac{1}{2}\gamma t}$.

4.2 Linear inhomogeneous differential equations

Now consider the general inhomogeneous case for linear differential equations

$$\boxed{a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)}\,. \tag{217}$$

[24] More preceisely if $t \gg D^{-1/2}\ln\sqrt{C_1/C_2}$

4.2.1 Method of variation of constants

The $y_P(x)$ particular solution may be found from the method of variation of constants just as for the 1st order equations (Section 2.7). Let us see how this works for second order equations. Assume that we have solved the homogeneous equation, and we have two linearly independent solutions at our disposal. Let us look for solutions of the inhomogeneous equation as a linear combination of the two solutions of the homogeneous equation but with two unknown functions instead of constants, $\alpha(x)$ and $\beta(x)$

$$y_P(x) = \alpha(x) y_{H1}(x) + \beta(x) y_{H2}(x) \tag{218}$$

and without loss of generality we require this linear combination also satisfies the relation[25]

$$\alpha' y_{H1} + \beta' y_{H2} = 0\,. \tag{219}$$

We will obtain these functions by substituting back in the differential equation. First calculate the derivatives

$$y_P' = \alpha y_{H1}' + \beta y_{H2}' + \underbrace{\alpha' y_{H1} + \beta' y_{H2}}_{0} = \alpha y_{H1}' + \beta y_{H2}'\,, \quad y_P'' = \alpha y_{H1}'' + \beta y_{H1}'' + \alpha' y_{H1}' + \beta' y_{H2}' \tag{220}$$

In the first equation two terms drop out because of Eq. (219). Substitute back in the equation

$$\begin{aligned} & a_2 y_P'' + a_1 y_P' + a_0 y_P = f(x) \\ \Rightarrow \quad & a_2\left(\alpha y_{H1}'' + \beta y_{H1}'' + \alpha' y_{H1}' + \beta' y_{H2}'\right) + a_1\left(\alpha y_{H1}' + \beta y_{H2}'\right) + a_0\left(\alpha y_{H1} + \beta y_{H2}\right) = f(x) \quad \Rightarrow \\ \Rightarrow \quad & a_2(\alpha' y_{H1}' + \beta' y_{H2}') + \underbrace{\left(a_2 y_{H1}'' + a_1 y_{H1}' + a_0 y_{H1}\right)}_{0}\alpha + \underbrace{\left(a_2 y_{H2}'' + a_1 y_{H2}' + a_0 y_{H2}\right)}_{0}\beta = f(x) \end{aligned} \tag{221}$$

Note that the terms linear in α and β drop out, as they should, since the linear combination of y_{H1} and y_{H2} solves the homogeneous equation. Let us divide both sides by a_2 and combine this with Eq. (219). We are left with two equations for the two unknowns α' and β', which we can write in matrix form

$$\begin{pmatrix} y_{H1} & y_{H2} \\ y_{H1}' & y_{H2}' \end{pmatrix}\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x)/a_2(x) \end{pmatrix} \tag{222}$$

One way to solve this is using Cramer's rule

$$\begin{aligned} \alpha' &= \frac{\begin{vmatrix} 0 & y_{H2} \\ f(x)/a_2(x) & y_{H2}' \end{vmatrix}}{\begin{vmatrix} y_{H1} & y_{H2} \\ y_{H1}' & y_{H2}' \end{vmatrix}} = \frac{-y_2 f(x)/a_2(x)}{y_{H1}y_{H2}' - y_{H2}y_{H1}'}\,, \quad \beta' = \frac{\begin{vmatrix} y_{H1} & 0 \\ y_{H1}' & f(x)/a_2(x) \end{vmatrix}}{\begin{vmatrix} y_{H1} & y_{H2} \\ y_{H1}' & y_{H2}' \end{vmatrix}} = \frac{y_{H1} f(x)/a_2(x)}{y_{H1}y_{H2}' - y_{H2}y_{H1}'} \\ \alpha &= \int \frac{-y_{H2} f(x)/a_2(x)}{y_{H1}y_{H2}' - y_{H2}y_{H1}'}dx\,, \quad \beta = \int \frac{y_{H1} f(x)/a_2(x)}{y_{H1}y_{H2}' - y_{H2}y_{H1}'}dx \end{aligned} \tag{223}$$

Note that the denominator is the Wronskian (Eq. 172). For constant coefficient differential equations we may simplify the numerator and the denominator by $y_{H1}y_{H2}$, and substitute $y_{H1} = e^{\lambda_1 x}$ and $y_{H2} = e^{\lambda_2 x}$ to get[26]

$$\alpha(x) = -\int \frac{e^{-\lambda_1 x} f(x)}{a_2(\lambda_2 - \lambda_1)}dx\,, \quad \beta(x) = \int \frac{e^{-\lambda_2 x} f(x)/a_2}{a_2(\lambda_2 - \lambda_1)}dx\,. \tag{224}$$

Thus

$$y_P = \alpha(x)e^{\lambda_1 x} + \beta(x)e^{\lambda_2 x} = \frac{\int^x e^{\lambda_2(x-u)} f(u)du - \int^x e^{\lambda_1(x-u)} f(u)du}{a_2(\lambda_2 - \lambda_1)}\,. \tag{225}$$

[25] in Sec. 4.2.2 we show that this does not restrict the solution

[26] The particular integral is proportional to a linear combination of the convolution of the corresponding homogeneous solutions and the driving function. This is not accidental, it is related to the Green's function and Fourier transforms, to be discussed next year in Mathematical Methods.

4.2.2 Direct calculation of variation of constants*

You may wonder where does the constraint Eq. (219) come from or is it even necessary? It is not necessary but it does simplify the derivation. Let us now find the particular solution without that assumption using variation of constants for any single solution of the homogeneous equation, $y_{\rm H}(x)$. For simplicity we shall assume a constant coefficient differential equation here. Let us introduce an unknown function $\psi(x)$ as

$$y_{\rm P}(x) = \psi(x) y_{\rm H}(x) \quad \Rightarrow \quad y'_{\rm P} = \psi y'_{\rm H} + \psi' y_{\rm H}\,, \quad y''_{\rm P} = \psi y''_{\rm H} + 2\psi' y'_{\rm H} + \psi'' y_{\rm H} \tag{226}$$

Substitute back in the equation

$$\begin{aligned} &a_2 y''_{\rm P} + a_1 y'_{\rm P} + a_0 y_{\rm P} = f(x) \quad \Rightarrow \quad a_2\left(\psi y''_{\rm H} + 2\psi' y'_{\rm H} + \psi'' y_{\rm H}\right) + a_1\left(\psi y'_{\rm H} + \psi' y_{\rm H}\right) + a_0 \psi y_{\rm H} = f(x) \quad \Rightarrow \\ &\Rightarrow \quad a_2 y_{\rm H}\psi'' + (2a_2 y'_{\rm H} + a_1 y_{\rm H})\psi' + \underbrace{\left(a_2 y''_{\rm H} + a_1 y'_{\rm H} + a_0 y_{\rm H}\right)}_{=0 \text{ (definition of } y_{\rm H})}\psi = f(x) \end{aligned} \tag{227}$$

This is now a first order linear differential equation for ψ'. Here we may use any linear combination of the homogeneous equation for $y_{\rm H}$, here let us just take one of them $y_{\rm H} = e^{\lambda_1 x}$. Divide the equation by $a_2 y_{\rm H}$ and substitute $y_{\rm H} = e^{\lambda_1 x}$

$$\psi'' + \left(2\lambda_1 + \frac{a_1}{a_2}\right)\psi' = \frac{1}{a_2} e^{-\lambda x} f(x) \quad \Rightarrow \quad \psi'' + (\lambda_1 - \lambda_2)\psi' = \frac{1}{a_2} e^{-\lambda_1 x} f(x) \tag{228}$$

Here we used the fact that $\lambda_1 + \lambda_2 = -a_1/a_2$, since $\lambda_{1,2} = \left(-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}\right)/(2a_2)$. Let us now substitute this into the solution of first order differential equations using the integrating factor and Eq. (90)

$$\begin{aligned} \Lambda(x) &= e^{\int (\lambda_1 - \lambda_2) dx} = e^{(\lambda_1 - \lambda_2)x} \\ \psi'(x) &= \frac{1}{\Lambda(x)} \int^x \Lambda(u) \left[\frac{1}{a_2} e^{-\lambda_1 u} f(u)\right] du = \frac{e^{(\lambda_2 - \lambda_1)x}}{a_2} \int^x e^{-\lambda_2 u} f(u) du\,, \\ \psi(x) &= \int^x \psi'(v) dv = \int^x \frac{e^{-(\lambda_2 - \lambda_1)v}}{a_2} \int^v e^{-\lambda_2 u} f(u) du dv \end{aligned} \tag{229}$$

Note that we have used our freedom to label the integration variable arbitrarily as u and v to avoid confusion with the integration bound x. Since we are after any particular solution, we set the integration constant to 0 and omitted it in the equation. Also note that the integration over v may be evaluated independently of the driving function $f(u)$. Thus, we change the order of the $du\ dv$ integrals. Since the integraton domain is clearly the triangle $0 < u < v < x$, so we now integrate first over v between u and x, and then over u from 0 to x

$$\begin{aligned} \psi(x) &= \int^x \left(\int_u^x \frac{e^{(\lambda_2 - \lambda_1)v}}{a_2} dv\right) e^{-\lambda_2 u} f(u) du = \int^x \frac{\left[e^{(\lambda_2 - \lambda_1)x} - e^{(\lambda_2 - \lambda_1)u}\right] e^{-\lambda_2 u} f(u)}{a_2(\lambda_2 - \lambda_1)} du \\ &= \frac{e^{(\lambda_2 - \lambda_1)x} \int^x e^{-\lambda_2 u} f(u) du - \int^x e^{-\lambda_1 u} f(u) du}{a_2(\lambda_2 - \lambda_1)} \end{aligned} \tag{230}$$

Thus the particular solution is obtained by substitution into the ansatz Eq. (226):

$$y_{\rm P}(x) = \psi(x) e^{\lambda_1 x} = \frac{\int^x e^{\lambda_2 (x-u)} f(u) du - \int^x e^{\lambda_1 (x-u)} f(u) du}{a_2(\lambda_2 - \lambda_1)}\,. \tag{231}$$

which is the same as Eq. (225) which was obtained using the constraint Eq. (219). That solution was simpler as it did not require us to evaluate a double integral.

Note that the particular solution may be found using the method of variation of constants in exactly the same way, even if the

$f(x)$	$y_{\rm P}(x)$
b = constant	b/a_0
bx^k	$c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$
$be^{\alpha x}$ $\quad (\alpha \notin \{\lambda_1, \ldots \lambda_n\})$	$ce^{\alpha x}$
$bx^k e^{\alpha x}$ $\quad (\alpha = \lambda_i$ of multiplicity $k)$	$(c_0 + c_1 x + \cdots + c_k x^k)e^{\alpha x}$
$b\sin(\alpha x)$ $\quad (\alpha \notin \{i\lambda_1, \ldots, i\lambda_n\})$	$c\sin(\alpha x) + d\cos(\alpha x)$
$b\cos(\alpha x)$ $\quad (\alpha \notin \{i\lambda_1, \ldots, i\lambda_n\})$	$c\sin(\alpha x) + d\cos(\alpha x)$
$bx^k e^{\alpha x}$ $(\alpha \notin \{\lambda_i\})$	$(c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k)e^{\alpha x}$
$bx^k \sin(\alpha x)$ $(\alpha \notin \{\lambda_i\})$	$\begin{cases} (c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k)\sin(\alpha x) + \\ +(d_0 + d_1 x + d_2 x^2 + \cdots + d_k x^k)\cos(\alpha x) \end{cases}$
$bx^k \cos(\alpha x)$ $(\alpha \notin \{\lambda_i\})$	$\begin{cases} (c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k)\sin(\alpha x) + \\ +(d_0 + d_1 x + d_2 x^2 + \cdots + d_k x^k)\cos(\alpha x) \end{cases}$
$\sum_{i=1}^k f_i(x)$ $\quad (f_i(x)$ are the functions above)	$\sum_{i=1}^k y_{{\rm P}i}(x)$ $\quad (y_{{\rm P}i}(x)$ are the functions above)

Table 2: Method of undetermined coefficients: particular solutions for various driving functions $f(x)$ for linear constant coefficient differential equations.

4.2.3 Method of undetermined coefficients

In many cases, the particular solution may be guessed immediately as we will now show. This provides a shortcut, to finding the particular solution without the need of going through the lengthy derivation using the variation of constants.

Often, the particular solution is an instance of the same "family of functions" as the $f(x)$ driving function on the right hand side of the differential equation. For instance, if the driving function is a polynomial, the particular solution $y_{\rm P}(x)$ is also a polynomial of the same order, and if it is a polynomial times and exponential then $y_{\rm P}(x)$ is a polynomial of the same order times and exponential with the same decay constant. Note that since $\sin\alpha x = \frac{1}{2i}(e^{i\alpha x} - e^{-i\alpha x})$ and $\cos\alpha x = \frac{1}{2}(e^{i\alpha x} + e^{-i\alpha x})$, and $e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$, the expressions involving a polynomial times a sin or cos for $f(x)$ follow immediately. These relations for sin and cos also imply that if $f(x)$ is any product of sin and cos, or the product of exponentials times, sin, and cos or any product thereof, also simplifies to a sum of exponentials, so we again get polynomials times exponentials (or polynomials times sin or cos for purely imaginary roots).

Similar to Eq. (203), if any of the terms in $y_{\rm P}$ duplicates any of the terms in $y_{\rm H}$ then the duplicated terms in $y_{\rm P}$ must be multiplied by the minimum positive integer power of x required to make all terms in $y_{\rm H} + y_{\rm P}$ linearly independent, including the particular solution.

To find the constant coefficients, we proceed as in Sec. 4.1.5, one must substitute back into the differential equation (217) and require the equation to hold for all x.

Note that $y_{\rm P}(x)$ must always be linearly independent of $y_{\rm H}(x)$, the solution of the corresponding homogeneous equation, i.e. it cannot be proportional to the linear combination of $e^{\lambda_i x}$, where λ_i are the roots of the characteristic polynomial.

Example 8 – forced oscillator

Find the general solution for the position of the forced damped harmonic oscillator as a function of time where the force per unit mass of the oscillator is $-\gamma\dot{r} - \omega_0^2 r$ and the forcing function is $F\cos(\omega_f t)$ where F and ω_f are the amplitude and the angular frequency of the forcing function. Here ω_0 is the normal mode frequency of the oscillator in the absense of an external forcing and damping. The forced oscillator satifies the linear second order ODE

$$\ddot{r} + \gamma\dot{r} + \omega_0^2 r = F\cos(\omega_f t)\,, \quad r(t) = ? \tag{232}$$

Determine the asymptotic behaviour for very large t.

SOLUTION This is an inhomogeneous linear differential equation, so the solution must be of the form

$$r(t) = r_{\rm H}(t) + r_{\rm P}(t) \tag{233}$$

where $r_{\rm H}(t)$ is the solution of the homogeneous equation and $r_{\rm P}(t)$ is any particular solution of the inhomogeneous equation. The homogeneous solution was derived in Example 4. Using the method of undetermined coefficients it is clear that the forcing function is a member of the family of functions $\{\sin(\omega_f t), \cos(\omega_f t)\}$, so we may look for a particular solution as a linear combination of these functions. A more elegant solution is to use complex numbers such that the solution is the real part of the equation. Thus, let us consider the equation

$$\ddot{r} + \gamma\dot{r} + \omega_0^2 r = F e^{i\omega_f t}\,, \quad r(t) = ? \tag{234}$$

In this case the family of functions is $e^{\pm i\omega_f t}$, which are complex conjugates of each other, so we may simply look for a solution in the form $y_{\rm P}(t) = Ae^{i\omega_f t}$, understanding that the final solution will be the real part of the result. Substitute this in the equation

$$(-\omega_f^2 + i\gamma\omega_f + \omega_0^2)Ae^{i\omega_f t} = Fe^{i\omega_f t} \quad \Rightarrow \quad [(-\omega_f^2 + i\gamma\omega_f + \omega_0^2)A - F]e^{i\omega_f t} = 0 \quad \Rightarrow \quad A = \frac{F}{(-\omega_f^2 + i\gamma\omega_f + k)} \tag{235}$$

Thus the solution is

$$\begin{aligned} y_{\rm P} &= \mathrm{Re}\left(\frac{Fe^{i\omega_f t}}{(-\omega_f^2 + i\gamma\omega_f + \omega_0^2)}\right) = F\,\mathrm{Re}\left(\frac{e^{i\omega_f t}}{(\omega_0^2 - \omega_f^2) + i\gamma\omega_f}\frac{(\omega_0^2 - \omega_f^2) - i\gamma\omega_f}{(\omega_0^2 - \omega_f^2) - i\gamma\omega_f}\right) = F\,\mathrm{Re}\left(\frac{[(\omega_0^2 - \omega_f^2) - i\gamma\omega_f]e^{i\omega_f t}}{(\omega_0^2 - \omega_f^2)^2 + (\gamma\omega_f)^2}\right) \\ &= \frac{F\,\mathrm{Re}\left((\cos\delta + i\sin\delta)e^{i\omega_f t + \delta}\right)}{\sqrt{(\omega_f^2 - \omega_0^2)^2 + (\gamma\omega_f)^2}} = A\,\mathrm{Re}\left(e^{i\omega_f t + \delta}\right)\,, \quad A = \frac{F}{\sqrt{(\omega_f^2 - \omega_0^2)^2 + (\gamma\omega_f)^2}}\,, \quad \delta = \arctan\left(\frac{\gamma\omega_f}{\omega_f^2 - \omega_0^2}\right) \\ &\Rightarrow \quad y_{\rm P} = A\cos(\omega_f t + \delta)\,, \end{aligned} \tag{236}$$

This must be added to the homogeneous solution obtained in Example 4. Denoting

$$D = \frac{1}{4}\gamma^2 - \omega_0^2\,, \tag{237}$$

we get the following possibilities for the general solution

$$y(t) = \begin{cases} \left(C_1 e^{-\sqrt{D}\,t} + C_2 e^{\sqrt{D}\,t}\right)e^{-\frac{1}{2}\gamma t} + A\cos(\omega_f t + \delta) & D > 0 \\ (C_1 + C_2 t)e^{-\frac{1}{2}\gamma t} + A\cos(\omega_f t + \delta) & D = 0\,, \\ C\cos(\sqrt{-D}\,t + \varphi)e^{-\frac{1}{2}\gamma t} + A\cos(\omega_f t + \delta) & D < 0\,,\ \gamma \neq 0\,, \\ C\cos(\omega_0 t + \varphi) + \dfrac{F\cos(\omega_f t)}{|\omega_f^2 - \omega_0^2|} & D < 0\,,\ \gamma = 0\,,\ \omega_{\rm res} = \omega_0 \neq \omega_f\,, \\ C\cos(\omega_0 t + \varphi) + \dfrac{Ft\sin(\omega_0 t)}{2\omega_0} & D < 0\,, \gamma = 0\,,\ \omega_{\rm res} = \omega_0 = \omega_f\,. \end{cases} \tag{238}$$

It is remarkable that for $\gamma \neq 0$, which is the case for most realistic systems, the particular solution dominates at late times. The system forgets its initial condition! The asymptotic phase space evolution is given by the parametric curve

$$(y, \dot{y}/\omega_f) = A(\cos(\omega_f t + \delta), -\sin(\omega_f t + \delta))\,. \tag{239}$$

which is clearly a circle if measuring the velocity in units of ω_f. If the initial condition in phase space is outside of this circle, the system spirals inwards to reach this circle, and if it is initially inside this circle the system spirals outwards to reach this circle. This type of behaviour is different from the stable and outstable fixed points discussed earlier for autonomous systems, it is called a **limit cycle**.

The A amplitude of the oscillation at late times is given by Eq. (236), which depends on the system parameters ω_0, γ, and the forcing frequency ω_f. The amplitude has a maximum as a function of the forcing frequency, which we may denote as the resonance frequency and amplitude at resonance

$$\left.\frac{dA}{d\omega_f}\right|_{\omega_f = \omega_{\rm res}} = 0 \quad \Rightarrow \quad \omega_{\rm res} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}\,, \quad A_{\rm res} = \frac{F}{\sqrt{\gamma^2\omega_0^2 - \frac{1}{4}\gamma^4}}\,. \tag{240}$$

The width of the peak is characterised by the full width half maximum $\Delta\omega$, defined as $A(\omega_{\rm res} \pm \frac{1}{2}\Delta\omega) = \frac{1}{2}A_{\rm res}$. For small γ, we get to leading order in γ: $\delta \approx -\pi/2$, so that $y(t) \to y_p(t) \approx \dfrac{F\sin(\omega_0 t)}{\gamma\omega_0}$, and $\omega_{\rm res} \approx \omega_0$, $\Delta\omega = 2\gamma$.

4.3 Second order linear differential equations – the general case

The most general second order linear ODE does not have constant coefficients but may always be written as

$$\boxed{y'' + a(x)y' + b(x)y = f(x)} \tag{241}$$

There is no analytic solution for this equation for arbitrary $a(x)$ and $b(x)$ functions, but the knowledge of a single solution of the homogeneous equation is sufficient to obtain the general solution of the inhomogeneous equation. The other linearly independent solution may be obtained via the method of variations of constants.

The general solution if one solution is known

Indeed, let us label the known solution of the homogeneous equation by $Cy_1(x)$, and find both the second homogeneous solution and the particular solution of the inhomogeneous equation using the variation of constants by replacing C by $\psi(x)$ looking for a solution in the form

$$y_2(x) = \psi(x)y_1(x) \tag{242}$$

and determine $\psi(x)$ by plugging back in the differential equation (241). The derivatives are

$$y_2' = \psi' y_1 + \psi y_1', \quad y_2'' = \psi'' y_1 + 2\psi' y_1' + \psi y_1'' \tag{243}$$

Thus, Eq. (241) becomes

$$\underbrace{[\psi'' y_1 + 2\psi' y_1' + \psi y_1'']}_{y_2''(x)} + a(x)\underbrace{[\psi' y_1 + \psi y_1']}_{y_2'(x)} + b(x)\underbrace{[\psi y_1]}_{y_2(x)} = f(x) \tag{244}$$

$$\Rightarrow \quad \psi'' y_1 + \psi'\left[2y_1' + a(x)y_1\right] + \psi\underbrace{[y_1'' + a(x)y_1' + b(x)y_1]}_{0} = f(x) \quad \Rightarrow \quad \psi'' + \psi'\left[2\frac{y_1'}{y_1} + a(x)\right] = f(x)$$

This is a first order linear homogeneous equation for ψ', which we already solved in Eq. (90). The solution is

$$\Lambda(x) = e^{\int[2(y_1'/y_1)+a(x)]dx} = e^{2\int y_1'/y_1\,dx}e^{\int a(x)dx} = e^{2\ln y_1}e^{\int a(x)dx} = y_1(x)^2 e^{\int^x a(u)du} \tag{245}$$

$$\psi'(x) = \frac{1}{\Lambda(x)}\left[\int^x \Lambda(u)f(u)du + C\right] \quad \Rightarrow \quad \psi(x) = C\int^x \frac{dv}{\Lambda(v)} + \int^x \frac{1}{\Lambda(v)}\int^v \Lambda(u)f(u)dudv \tag{246}$$

Substituting back in Eq. (242), $y_2 = \psi(x)y_1(x)$ we get the general solution of Eq. (241):

$$y(x) = C_1 y_1(x) + y_2(x) = C_1 y_1(x) + C_2 y_1(x)\int^x \frac{dv}{\Lambda(v)} + y_1(x)\int^x \frac{1}{\Lambda(v)}\int^v \Lambda(u)f(u)dudv \tag{247}$$

The first two terms on the right hand side represent the homogeneous solution and the third term is a particular solution of the inhomogeneous equation.[27]

[27] The double integral may be simplified by changing the order of the integrals. Labelling the second term in Eq. (247) with $C_2 y_2(x)$, the integral over v simplifies as

$$y_P(x) = y_2(x)\int^x \Lambda(u)f(u)du - y_1(x)\int^x\int^u \frac{dv}{\Lambda(v)}\Lambda(u)f(u)du = \int^x \frac{y_2(x)y_1(u) - y_1(x)y_2(u)}{y_1(u)}\Lambda(u)f(u)du \tag{248}$$

Finding the first solution

The general second order equation (241) can only be solved if at least one solution of the corresponding homogeneous equation is known. How can we guess one solution? Although there is no general algorithm here, we can try the following functions if the coefficient functions $a(x)$ and $b(x)$ are polynomials:

- $e^{\lambda x}$
- x^n
- $c_1 x + c_2 x^2 + \dots c_{n-1} x^{n-1} + x^n$
- $(c_1 x + c_2 x^2 + \dots c_{n-1} x^{n-1} + x^n) e^{\lambda x}$.

Example 9 – Hermite equation

Find the general solution of the linear differential equation

$$y'' - xy' + 2y = 0 \tag{249}$$

Solution This is a linear second order equation with non-constant coefficients. We need to guess the first solution and obtain the second solution with the variation of constants. We proceed with the guesses listed above. Let us start with trying $y = e^{\lambda x}$

$$(\lambda^2 - \lambda x + 2\lambda) e^{\lambda x} = 0 \quad \Rightarrow \quad \text{this does not work} \tag{250}$$

Now try $y = x^n$. Substituting back in the differential equation:

$$\underbrace{n(n-1)x^{n-2}}_{y''} - x \underbrace{n x^{n-1}}_{y'} + 2 \underbrace{x^n}_{y} = n(n-1)x^{n-2} + (2-n)x^n = 0 \quad \Rightarrow \quad \text{does not work} \tag{251}$$

Althought this does not work, it does eliminate the x^n term if we set $n = 2$, although there is an extra x^{n-2} term left which is nonzero here. So let us try a general 2nd order polynomial, $y = ax^2 + bx + c$, where we may hope that the lower order terms may be adjusted to cancel out the extra remaining term. Indeed after substituting, we get

$$2a - x(2ax + b) + 2(ax^2 + bx + c) = 0 \quad \Rightarrow \quad 2(a + c) + bx = 0 \quad \Rightarrow \quad a + c = 0 \text{ and } b = 0\,. \tag{252}$$

This shows that $a = 1$, $b = 0$, $c = -1$, is a solution, i.e. $y = x^2 - 1$. Indeed then $y'' = 2$, $y' = 2x$, which when substituted into the solution satisfies the equation for all x.

Given one solution we may proceed to find the second linearly independent solution using the variation of constants. Introducing an unknown coefficient $\psi(x)$, let us try

$$y = (x^2 - 1)\psi \quad \Rightarrow \quad y' = 2x\psi + (x^2 - 1)\psi' \quad \Rightarrow \quad y'' = 2\psi + 4x\psi' + (x^2 - 1)\psi'' \tag{253}$$

Substitute into the differential equation

$$\underbrace{(2\psi + 4x\psi' + (x^2 - 1)\psi'')}_{y''} - x \underbrace{(2x\psi + (x^2 - 1)\psi')}_{y'} + 2 \underbrace{(x^2 - 1)\psi}_{y} = 0 \quad \Rightarrow \quad (x^2 - 1)(\psi'' - x\psi') = 0 \tag{254}$$

This is indeed a separable first order ODE for $z = \psi'$ which satisfies

$$\frac{dz}{dx} - xz = 0 \quad \Rightarrow \quad \int \frac{dz}{z} = \int x dx \quad \Rightarrow \quad \ln|z| = \frac{1}{2}x^2 \quad \Rightarrow \quad z = e^{\frac{1}{2}x^2} \quad \Rightarrow \quad \psi = \int e^{\frac{1}{2}x^2} dx\,. \tag{255}$$

Here we omitted the integration constants as we only need a second linearly independent solution. We may find the second solution as $(x^2 - 1)\psi(x)$. The general solution is then the linear combination of these two solutions:

$$y(x) = C_1(x^2 - 1) + C_2(x^2 - 1) \int e^{\frac{1}{2}x^2} dx \tag{256}$$

Note that the integral that defines the second linearly independent function cannot be simplified in a closed form with elementary functions.[28]

[28] This is called the imagnary error function.

What if we cannot find any analytical solutions?*

Generally, not all second order differential equations have closed form analytic solutions with simple functions. Many of these equations which come up in physics define **special functions** usually given by an infinite Taylor series or recursion relations.[29] These include the normal modes of physical systems with cylindrical or spherical symmetry like the Bessel functions (describes the radial shape of a vibrating circular drumhead after being struck or the shape of a rope where the suspension rotates uniformly), the spherical Bessel functions (describes spherical waves or spherical diffusion), the Hankel functions (describes waves in cyllindrical symmetry), spherical harmonics (e.g. the shapes of electron shells around an atom in quantum mechanics or chemistry, or the normal modes of a perturbed elastic spherical body such as a star or a perturbed water-balloon), etc. If nothing else works, the equations may often be solved solved by an infinite series expansion[30].

For initial value problems, iterative numerical algorithms can give a solution such as the Runge Kutta method, available in MATHEMATICA and PYTHON.

4.4 Summary

In this chapter we have examined the solutions of linear differential equations

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)\,. \tag{257}$$

The general solution may always be written in the form

$$y(x) = C_1 y_{\mathrm{H}1}(x) + C_2 y_{\mathrm{H}2}(x) + \cdots + C_n y_{\mathrm{H}n}(x) + y_{\mathrm{P}}(x) \tag{258}$$

where $\{y_{\mathrm{H}1}(x), \cdots, y_{\mathrm{H}n}(x)\}$ are linearly independent functions which solve the corresponding homogeneous equation (i.e. $f(x) = 0$), and $y_{\mathrm{P}}(x)$ is a particular solution of the full inhomogeneous equation. Given the solutions of the corresponding homogeneous equation, $y_{\mathrm{P}}(x)$ may always be obtained by the method of variation of constants, i.e. expressed as a linear combination of the homogogeneous solutions with coefficient functions of x, i.e. as

$$y_{\mathrm{P}}(x) = \psi_1(x)y_{\mathrm{H}1}(x) + \psi_2(x)y_{\mathrm{H}2}(x) + \cdots + \psi_n(x)y_{\mathrm{H}n}(x)\,. \tag{259}$$

This ansatz can be subtituted in the differential equation to find a differential equations for $\{\psi_1(x), \cdots, \psi_n(x)\}$. For second order linear differential equations, it is sufficient to have a single solution $y_{\mathrm{H}1}(x)$, and $y_{\mathrm{P}}(x) = \psi(x)y_{\mathrm{H}1}(x)$ is sufficient to find both the particular solution and the second solution $y_{\mathrm{H}2}(x)$. In that case, after substituting in the original equation, we get a first order linear inhomogeneous differential equation for $\psi(x)$, whose homogeneous solution is $y_{\mathrm{H}2}(x)/y_{\mathrm{H}1}(x)$ and the inhomogeneous solution is $y_{\mathrm{P}}(x)/y_{\mathrm{H}1}(x)$. We have also seen that in many cases the particular solution has a similar form as the driving function $f(x)$. This fact offers another technique to search for $y_{\mathrm{P}}(x)$, see Table 2.

We have seen that for constant coefficient differential equations, where $\{a_0(x), \cdots, a_n(x)\}$ are constants that do not depend on x, solutions may be obtained in the form $e^{\lambda x}$. Substituting this ansatz leads to a "characteristic" polynomial of λ, whose roots give the exponents in the solution, which is called the "characteristic equation". These roots may be either real or complex. In the complex case, the roots come in pairs, and we get a linear combination of $\sin(\omega x)$ and $\cos(\omega x)$ times an exponential. If there are repeated roots, then we get solutions of the form $xe^{\lambda x}$, $x^2e^{\lambda x}$, and so on. The solutions $\{y_{\mathrm{H}1}(x), \cdots, y_{\mathrm{H}n}(x)\}$ and $y_{\mathrm{P}}(x)$ must all be linearly independent. For second order equations, the general solution of the homogeneous equation may be $C_1 + C_2x$, $C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$ or $(C_1 + xC_2)e^{\lambda x}$, depending on the roots of the characteristic equation, both 0, distinct, or nonzero non-distinct. In the distinct case if λ is complex, it may be written as $\kappa \pm i\omega$, and the solution can also be written as $[C_1\sin(\omega x) + C_2\cos(\omega x)]e^{\kappa x}$ or as $C\cos(\omega x + \phi])e^{\kappa x}$ where C and ϕ are arbitrary constants.

[29] You will encounter these in Mathematical Methods next year.

[30] Plug in a polynomial of arbitrarily high order into the differential equation, and expand $a(x)$, $b(x)$, and $f(x)$ to a Taylor-series. Since the derivative and products of polynomials is still a polynomial, you can determine the coefficients of the solution order-by-order such that the equation holds for all powers of x.

5 Systems of Linear Differential Equations

Let us now examine how to solve systems of linear differential equations. Interestingly, all systems of linear differential equations can be turned into a system of first order differential equation, as we will show below. So the most general linear systems of differential equations may always be written as

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{pmatrix} = \begin{pmatrix} A_{11}(x) & A_{12}(x) & \cdots & A_{1n}(x) \\ A_{21}(x) & A_{22}(x) & \cdots & A_{2n}(x) \\ \vdots & \vdots & & \vdots \\ A_{n1}(x) & A_{n2}(x) & \cdots & A_{nn}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} + \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ \vdots \\ ? \end{pmatrix} \tag{260}$$

or more briefly as

$$\boxed{\mathbf{y}'(x) = \mathbf{A}(x)\mathbf{y}(x) + \mathbf{f}(x)}, \qquad \mathbf{y}(x) = ? \tag{261}$$

Here $\mathbf{y}(x)$ represents n unknown functions, $\mathbf{y}'(x)$ are the derivatives of these functions and $\mathbf{A}(x)$ is a given $n \times n$ matrix whose elements are given functions of x, and $\mathbf{f}(x)$ denotes n given functions of x. The solution of the equation is $\mathbf{y}(x)$ whose derivative equals the right hand side of this equation for every x.

5.1 n^{th} order differential equations as systems of 1^{st} order differential equations

Consider a single n^{th} order linear differential equation

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x) \tag{262}$$

Let us introduce n new variables, $\{z_0, \ldots, z_{n-1}\}$ as follows, which we shall treat as n independent functions of x:

$$\mathbf{z}(x) = \begin{pmatrix} z_0(x) \\ z_1(x) \\ z_2(x) \\ \vdots \\ z_{n-1}(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}. \tag{263}$$

We labelled the first element with the 0 index just for convenience, so that the k^{th} derivative has the matching k index.[31] Further, note that by construction $dz_i/dx = z_{i+1}(x)$ for $i = (0, 1, \ldots, n-2)$. The derivative of z_{n-1} may be calculated using the differential equation

$$\frac{dz_{n-1}}{dx} = \frac{d}{dx}y^{(n-1)} = y^{(n)}(x) = f(x) - \left[a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y\right] \tag{264}$$

where we used Eq. (262) to express $y^{(n)}$ with the lower order derivatives of y. Substituting Eq. (263), we get the following system of differential equations

$$\underbrace{\frac{d}{dx}\begin{pmatrix} z_0(x) \\ z_1(x) \\ z_2(x) \\ \vdots \\ z_{n-2}(x) \\ z_{n-1}(x) \end{pmatrix}}_{\mathbf{z}'(x)} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0(x) & -a_1(x) & -a_2(x) & \ldots & -a_{n-2}(x) & -a_{n-1}(x) \end{pmatrix}}_{\mathbf{A}(x)} \underbrace{\begin{pmatrix} z_0(x) \\ z_1(x) \\ z_2(x) \\ \vdots \\ z_{n-2}(x) \\ z_{n-1}(x) \end{pmatrix}}_{\mathbf{z}(x)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}}_{\mathbf{f}(x)} \tag{265}$$

Let us label the coefficient matrix $\mathbf{A}(x)$ as shown whose elements are zero other than the superdiagonal elements and the bottom row as shown, and label the unknown functions $(z_0(x), \ldots, z_{n-1}(x))$ as a vector of functions $\mathbf{z}(x)$, and the terms independent of $\mathbf{z}(x)$ with $\mathbf{f}(x)$. Thus Eq. (262) is equivalently written as a system of n first order linear differential equations

$$\mathbf{z}' = \mathbf{A}(x)\mathbf{z} + \mathbf{f}(x) \tag{266}$$

[31] This is not essential in any way; in many of our examples below we will use a notation where the index runs from 1 to n.

This shows that any n^{th} order differential equation may always be written as a system of n first order differential equations. Similarly given a system of n^{th} order differential equations, say k of them, this may be written as a system of $n \times k$ first order differential equations.[32]

Example 1

A mass m is suspended from the ceiling by a spring of equilibrium length L and spring constant k. Newton's equations of motion are given as 3 second order linear ODEs:

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \\ m\ddot{z} \end{pmatrix} = \begin{pmatrix} -kx \\ -ky \\ -k(z-L) - mg \end{pmatrix} \tag{267}$$

where (x, y, z) denote the position of the body, and overdots denote time derivatives. Express this system of second order differential equation as 6 first order linear differential equations.

Solution The second time derivative of the position is

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} -\frac{k}{m}x \\ -\frac{k}{m}y \\ -\frac{k}{m}(z-L) - g \end{pmatrix} \tag{268}$$

Let us define 6D vector with components as $\mathbf{X} = (X_1, X_2, X_3, X_4, X_5, X_6) = (x, y, z, \dot{x}, \dot{y}, \dot{z})$. The derivative of the components of this vector are $\mathbf{X} = (\dot{X}_1, \dot{X}_2, \dot{X}_3, \dot{X}_4, \dot{X}_5, \dot{X}_6) = (\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z})$, which are

$$\underbrace{\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \end{pmatrix}}_{\frac{d}{dt}\mathbf{X}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} X_4 \\ X_5 \\ X_6 \\ -\frac{k}{m}X_1 \\ -\frac{k}{m}X_2 \\ -\frac{k}{m}(X_3 - L) - g \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k}{m} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{k}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{k}{m} & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix}}_{\mathbf{X}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{kL}{m} - g \end{pmatrix}}_{\mathbf{f}} \tag{269}$$

Defining $\mathbf{A}$ and $\mathbf{f}$ as above we have

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{f}\,. \tag{270}$$

Note that the solution of the original 3 second order equations is the three functions of time $(x(t), y(t), z(t))$ which represents a curve in the 3D position space, and the solution to the 6 first order equations is $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t))$ which represents a curve in the 6D space of position and velocities, i.e. the phase space.

Example 2

Consider the system of two coupled harmonic oscillators in 1D, where the displacement of the oscillators are $y_1(t)$ and $y_2(t)$, respectively, and the system evolves according to the following force per unit mass:

$$\left\{ \begin{array}{l} \ddot{y}_1 = -\gamma_1 \dot{y}_1 - k_1 y_1 - k_2(y_1 - y_2) \\ \ddot{y}_2 = -\gamma_2 \dot{y}_2 - k_2(y_2 - y_1) \end{array} \right\} \tag{271}$$

Express these differential equations as a system of first order differential equations.

Solution Let us define the vector $\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t), Z_4(t)) = (y_1(t), \dot{y}_1(x), y_2(t), \dot{y}_2(t))$.

$$\left\{ \begin{array}{l} \dot{Z}_2 = -\gamma_1 Z_2 - k_1 Z_1 - k_2(Z_1 - Z_3) \\ \dot{Z}_4 = -\gamma_2 Z_4 - k_2(Z_3 - Z_1) \end{array} \right\}, \quad \text{where} \quad \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{pmatrix} \tag{272}$$

[32] Note that the converse is not true: an arbitrary system of n first order differential equations, may not be written as a single n^{th} order differential equation, only in case the coefficient matrix has the particular form as in Eq. (265).

Its differential equation satisfies

$$\frac{d}{dt}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1 - k_2 & -\gamma_1 & k_2 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & 0 & -k_2 & -\gamma_2 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix} \tag{273}$$

Note that this solution is not unique, we could have labeled the variables in a different order.[33] For instance if defining $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t)) = (y_1(t), y_2(t), \dot{y}_1(x), \dot{y}_2(t))$ we get

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 - k_2 & k_2 & -\gamma_1 & 0 \\ k_2 & -k_2 & 0 & -\gamma_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}. \tag{274}$$

The solution of the original second order equations for a set of initial conditions (i.e. given initial positions and velocities) is the position of the two bodies in time, which represents a curve in 2D. The solution to the first order equations are the positions and velocities of the two bodies as a function of time, which represents a curve in 4D.

5.2 General properties of linear systems of differential equations

Since the solution of a single first order differential equation is a one-parameter family of curves, where the parameter is the integration constant, the solution of a system of n first order differential equations is a set of n one-parameter family of curves in n dimensions. Each solution is an n-element vector, whose elements are functions of x.

For linear systems of differential equations (261), all theorems in Section 4 derived for n^{th} order linear differential equations apply.

- The difference between two solutions solves the **corresponding homogeneous equation** i.e. the one where $\mathbf{f}(x) = \mathbf{0}$, i.e. all n elements of $\mathbf{f}(x)$ are zero:

$$\mathbf{y}'_{\mathrm{H}}(x) = \mathbf{A}(x)\mathbf{y}_{\mathrm{H}}(x) \tag{275}$$

- The general solution of the inhomogeneous equation Eq. (261) may be written as

$$\mathbf{y}(x) = \mathbf{y}_{\mathrm{H}}(x) + \mathbf{y}_{\mathrm{P}}(x) \tag{276}$$

  where $\mathbf{y}_{\mathrm{H}}(x)$ is the general solution of the corresponding homogeneous equation and $\mathbf{y}_{\mathrm{P}}(x)$ is any single particular solution of the inhomogeneous equation.

- The linear combination of any two solutions of a *homogeneous* linear system of differential equations is also a solution of the same equation.

- The **general solution** of the *homogeneous* linear system of differential equations (275) may be written as a linear combination of n *linearly independent* solutions

$$\mathbf{y}_{\mathrm{H}}(x) = C_1\mathbf{y}_{\mathrm{H}1}(x) + C_2\mathbf{y}_{\mathrm{H}2}(x) + \cdots + C_n\mathbf{y}_{\mathrm{H}n}(x) = \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{y}_{\mathrm{H}1}(x) & \mathbf{y}_{\mathrm{H}2}(x) & \dots & \mathbf{y}_{\mathrm{H}n}(x) \\ | & | & & | \end{pmatrix}}_{\mathbf{Y}(x)}\underbrace{\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}}_{\mathbf{C}}. \tag{277}$$

  The general solution is clearly a linear combination of n vectors of functions, where each element of these vectors is a function of x. We may pack these functions into an $n \times n$ matrix as shown above, labelled $\mathbf{Y}(x)$. This is called the **fundamental system**.

[33] We could have also defined the 4 new variables as any linear combinations of these 4 variables, and the corresponding system of equations will still be linear 1st order. The eigenvectors of the matrix define the most useful set of variables as we will see below, because in that case the equations are not coupled and can be integrated immediately.

Here "linearly independent" may be verified similarly to that for vectors (Eq. 169) by observing if the Wronskian determinant is non-zero, which is now defined using the individual solution vectors as the column vectors of the matrix (i.e. the fundamental system):

$$W = \det\begin{pmatrix} | & | & & | \\ \mathbf{y}_{\mathrm{H}1}(x) & \mathbf{y}_{\mathrm{H}2}(x) & \dots & \mathbf{y}_{\mathrm{H}n}(x) \\ | & | & & | \end{pmatrix} \neq 0 \Leftrightarrow \text{linearly independent} \tag{278}$$

- The particular solution may be derived using the variation of constants from the corresponding *homogeneous* solution, i.e. making the constants into functions.

$$\mathbf{y}_{\mathrm{P}}(x) = \psi_1(x)\mathbf{y}_{\mathrm{H}1}(x) + \psi_2(x)\mathbf{y}_{\mathrm{H}2}(x) + \cdots + \psi_n(x)\mathbf{y}_{\mathrm{H}n}(x)\,. \tag{279}$$

- The linear combination coefficients are obtained from the initial conditions given at some x_0, i.e. given $\mathbf{y}(x_0)$. Substitute in the solution

$$\begin{aligned} &\mathbf{y}(x_0) = C_1\mathbf{y}_{\mathrm{H}1}(x_0) + C_2\mathbf{y}_{\mathrm{H}2}(x_0) + \cdots + C_n\mathbf{y}_{\mathrm{H}n}(x_0) + \mathbf{y}_{\mathrm{P}}(x_0) = \\ &= \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{y}_{\mathrm{H}1}(x_0) & \mathbf{y}_{\mathrm{H}2}(x_0) & \dots & \mathbf{y}_{\mathrm{H}n}(x_0) \\ | & | & & | \end{pmatrix}}_{\mathbf{Y}(x_0)} \underbrace{\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}}_{\mathbf{C}} + \mathbf{y}_{\mathrm{P}}(x_0) \quad \Rightarrow \quad \mathbf{C} = \mathbf{Y}(x_0)^{-1}\left[\mathbf{y}(x_0) - \mathbf{y}_{\mathrm{P}}(x_0)\right] \end{aligned} \tag{280}$$

- Substituting back these linear combination coefficients in the general solution Eq. (276)–(277) we get that the solution may always be written as

$$\mathbf{y}(x) = \mathbf{Y}(x)\underbrace{\mathbf{Y}(x_0)^{-1}\left[\mathbf{y}(x_0) - \mathbf{y}_{\mathrm{P}}(x_0)\right]}_{\mathbf{C}} + \mathbf{y}_{\mathrm{P}}(x) \tag{281}$$

We will use these properties to obtain the solution for linear systems of equations with constant coefficients next.

5.3 Homogeneous linear systems with constant coefficients

Let us consider a homogeneous system of equations with constant coefficients:

$$\boxed{\mathbf{y}'_{\mathrm{H}}(x) = \mathbf{A}\,\mathbf{y}_{\mathrm{H}}(x)} \tag{282}$$

where $\mathbf{A}$ is an $n \times n$ matrix of given numbers (they do not depend on x), and $\mathbf{y}_{\mathrm{H}}(x)$ is a vector of n unknown functions, to be determined by solving this equations.

5.3.1 Diagonal coefficient matrix

First note that if the coefficient matrix $\mathbf{A}$ is diagonal with elements $A_{ii} = a_i$, then Eq. (282) represents n independent first order equations, each of which may be integrated immediately as in Eq. (29).

$$y'_{\mathrm{H}i}(x) = a_i y_{\mathrm{H}i}(x) \quad \Rightarrow \quad y_{\mathrm{H}i}(x) = C_i e^{a_i x}\,. \tag{283}$$

Therefore, the general solution for $a_i \neq 0$ for all i is

$$\mathbf{y}_{\mathrm{H}}(x) = C_1\underbrace{\begin{pmatrix} e^{a_1 x} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_{\mathrm{H}1}(x)} + C_2\underbrace{\begin{pmatrix} 0 \\ e^{a_2 x} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_{\mathrm{H}2}(x)} + \cdots + C_n\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ e^{a_n x} \end{pmatrix}}_{\mathbf{y}_{\mathrm{H}n}(x)} = C_1\underbrace{e^{a_1 x}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_{\mathrm{H}1}(x)} + C_2\underbrace{e^{a_2 x}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{y}_{\mathrm{H}2}(x)} + \cdots + C_n\underbrace{e^{a_n x}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{y}_{\mathrm{H}n}(x)}. \tag{284}$$

The individual solution vectors $\mathbf{y}_{\mathrm{H}i}(x)$ are $e^{\lambda_i x}\mathbf{v}_i$ where $\mathbf{v}_i$ are the unit basis vectors along the i^{th} coordinate. Note that this is the solution for any value of $(a_1, a_2, \cdots, a_n)$, even in the case $a_1 = a_2 = \cdots = a_n \neq 0$:

$$\mathbf{y}_{\mathrm{H}}(x) = C_1\, e^{ax}\underbrace{\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}}_{\mathbf{y}_{\mathrm{H}1}(x)} + C_2\, e^{ax}\underbrace{\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}}_{\mathbf{y}_{\mathrm{H}2}(x)} + \cdots + C_n\, e^{ax}\underbrace{\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}}_{\mathbf{y}_{\mathrm{H}n}(x)}. \qquad (a_1 = a_2 = \cdots = a_n = a) \tag{285}$$

Are these solution vectors linearly independent? Surely yes, since only $C_1 = 0$ can make the first component of the solution zero, only $C_2 = 0$ can make the second component zero, and so on. The result will be the zero vector only if $C_1 = C_2 = \cdots = 0$. Thus, Eq. (285) is the general solution of Eq. (282) if the A matrix is diagonal with identical diagonal elements, a.[34]

So while these solutions are somewhat reminiscent of the solutions for single n^{th} order equations, consisting of linear combinations of exponentials of x, they are very different in important ways. In contrast, the solution of a single n^{th} order equation casted as a system of first order ODEs, may be written in the non-degenerate case as

$$\mathbf{z}_{\mathrm{H}}(x) = \begin{pmatrix} y(x)\\ y'(x)\\ y''(x)\\ \vdots\\ y^{(n-1)}(x)\end{pmatrix} = C_1 e^{\lambda_1 x}\begin{pmatrix}1\\ \lambda_1\\ \lambda_1^2\\ \vdots\\ \lambda_1^{n-1}\end{pmatrix} + C_2 e^{\lambda_2 x}\begin{pmatrix}1\\ \lambda_2\\ \lambda_2^2\\ \vdots\\ \lambda_2^{n-1}\end{pmatrix} + \cdots C_n e^{\lambda_n x}\begin{pmatrix}1\\ \lambda_n\\ \lambda_n^2\\ \vdots\\ \lambda_n^{n-1}\end{pmatrix} \quad (\text{if } \{\lambda_i\} \text{ all distinct}) \tag{286}$$

These vectors are linearly dependent if $\lambda_i = \lambda_j$ for some $i \neq j$, and in that case we have found previously that in $y(x)$, the $(C_i + C_j)e^{\lambda_i x}$ were replaced by $(C_i + xC_j)e^{\lambda_i x}$ if all other $\{\lambda_k\}$ were distinct. For repeated roots with multiplicity k, the first component $y(x)$ in the corresponding eigensubspace is $(C_r + C_{r+1}x + \ldots C_{r+k-1}x^k)e^{\lambda_r x}$, and the other components may be found from taking derivatives.

It is not surprising that the solutions of n^{th} order linear ODEs are qualitatively different from n independent first order ODEs, since the $\mathbf{A}$ coefficient matrix of the first order system corresponding to the n^{th} order equation (see Eq. 265) is not diagonal while that of independent first order equations is diagonal..

5.3.2 Non-diagonal constant coefficient matrix

In the general case, where the coefficient matrix $\mathbf{A}$ is not diagonal, we can obtain a solution by diagonalising $\mathbf{A}$. This is always possible if $\mathbf{A}$ is symmetric real, i.e. $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ or more generally if it is Hermitian $\mathbf{A} = \mathbf{A}^{*\mathrm{T}}$.[35]

The *right* eigenvectors of the matrix $\mathbf{A}$ are defined as[36]

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \text{ for } i = 1, 2, \ldots, n, \tag{287}$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues which satisfy[37]

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{288}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Note that the left hand side is generally an n^{th} order polynomial which has n complex roots (not necessarily distinct, see discussion below Eq. 182), so it may be written as the product $\det(\mathbf{A} - \lambda_i\mathbf{I}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$. This is the same characteristic polynomial that we encountered for a single n^{th} order linear ODEs.

[34] i.e. if $\mathbf{A} = a\mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.

[35] Here $\mathbf{A}^{\mathrm{T}}$ denotes the transpose, and $\mathbf{A}^*$ is the complex conjugate of the matrix elements

[36] The *left* eigenvectors of the matrix may be generally different $\mathbf{u}_i\mathbf{A} = \lambda_i\mathbf{u}_i$.

[37] This follows from subtracting $\lambda_i\mathbf{v}_i$ from both sides, which may be written as $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$, where $\mathbf{I}$ is the identity matrix.

Let us express $\mathbf{y}_{\rm H}(x)$ as a linear combination of the eigenvectors and write the expression in matrix form.

$$\mathbf{y}_{\rm H}(x) = c_1(x)\mathbf{v}_1 + c_2(x)\mathbf{v}_2 + \cdots + c_n(x)\mathbf{v}_n = \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix}}_{\mathbf{c}(x)} \tag{289}$$

Substitute back in the equation (282),

$$\frac{d}{dx}\underbrace{[\mathbf{R}\,\mathbf{c}(x)]}_{\mathbf{y}_{\rm H}(x)} = \mathbf{A}\,\underbrace{\mathbf{R}\,\mathbf{c}(x)}_{\mathbf{y}_{\rm H}(x)} \tag{290}$$

Given that the $\mathbf{R}$ matrix is comprised of the eigenvectors of $\mathbf{A}$, we can express the product with a diagonal matrix of eigenvalues (let us label this with $\mathbf{D}$) times the $\mathbf{R}$ matrix, since

$$\mathbf{AR} = \mathbf{A}\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \\ | & | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}}_{\mathbf{D}} \tag{291}$$

Substituting in Eq. (290), and introducing the vector $\mathbf{y}_{\rm H}(x) = \mathbf{R}\mathbf{c}(x)$, we get

$$\frac{d}{dx}\underbrace{(\mathbf{R}\,\mathbf{c}(x))}_{\mathbf{y}_{\rm H}(x)} = \mathbf{ARc}(x) = \mathbf{RDc}(x) \quad \Rightarrow \quad \mathbf{Rc}'(x) = \mathbf{RDc}(x)\,. \quad \Rightarrow \quad \mathbf{c}'(x) = \mathbf{Dc}(x)\,. \tag{292}$$

This is exactly what we wanted, a system of homogeneous first order differential equations, where the coefficient matrix is diagonal. This is just n independent equations for the components of $\mathbf{A}$, which we can integrate independently

$$\frac{dc_i}{dx} = \lambda_i c_i \quad \Rightarrow \quad \int \frac{dc_i}{c_i} = \int \lambda_i dx \quad \Rightarrow \quad \ln|c_i| = \lambda_i x + C_i \quad \Rightarrow \quad c_i(x) = C_i e^{\lambda_i x}\,. \tag{293}$$

In the last step we relabelled the integration constant $\pm e^{-C_i}$ with C_i. Substituting this general solution for the components of $\mathbf{y}_{\rm H}(x)$, we may obtain the solution in the original basis

$$\mathbf{y}_{\rm H}(x) = \mathbf{Rc}(x) = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} C_1 e^{\lambda_1 x} \\ C_2 e^{\lambda_2 x} \\ \vdots \\ C_n e^{\lambda_n x} \end{pmatrix} = \boxed{C_1 e^{\lambda_1 x}\mathbf{v}_1 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_n} \tag{294}$$

It is useful to express this result in a matrix form, as follows

$$\mathbf{y}_{\rm H}(x) = \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} e^{\lambda_1 x} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 x} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n x} \end{pmatrix}}_{\mathbf{E}(x)} \underbrace{\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}}_{\mathbf{C}} = \mathbf{RE}(x)\mathbf{C} \tag{295}$$

Here we introduced the diagonal matrix $\mathbf{E}$ with diagonal elements $E_{ii} = e^{\lambda_i x}$ and the vector of arbitrary integration constants $\mathbf{C}$.

Given initial conditions at x_0, i.e. $\mathbf{y}_{\rm H}(\mathbf{x_0})$, we may obtain the vector $\mathbf{C}$ by multiplying both sides with $\mathbf{E}^{-1}(x_0)\mathbf{R}^{-1}$.[38]

$$\mathbf{y}_{\rm H}(x_0) = \mathbf{RE}(x_0)\mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \mathbf{E}^{-1}(x_0)\mathbf{R}^{-1}\mathbf{y}_{\rm H}(x_0) \tag{296}$$

[38] Recall that $\mathbf{R}^{-1}$ is the inverse of the matrix composed of right eigenvector columns of the matrix $\mathbf{A}$. The row vectors of $\mathbf{R}^{-1}$ are left eigenvectors of $\mathbf{A}$.

Plugging back in the solution, Eq. (294)

$$\mathbf{y}_{\mathrm{H}}(x) = \mathbf{R}\mathbf{E}(x)\underbrace{\mathbf{E}^{-1}(x_0)\mathbf{R}^{-1}\mathbf{y}_{\mathrm{H}}(x_0)}_{\mathbf{C}} = \underbrace{\mathbf{R}\begin{pmatrix} e^{\lambda_1 x} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 x} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n x} \end{pmatrix}}_{\mathbf{E}(x)} \underbrace{\begin{pmatrix} e^{-\lambda_1 x_0} & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2 x_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda_n x_0} \end{pmatrix}}_{\mathbf{E}^{-1}(x_0)} \mathbf{R}^{-1}\mathbf{y}_{\mathrm{H}}$$

$$= \underbrace{\mathbf{R}\begin{pmatrix} e^{\lambda_1 (x-x_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 (x-x_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n (x-x_0)} \end{pmatrix}}_{\mathbf{E}(x-x_0)} \mathbf{R}^{-1}\mathbf{y}_{\mathrm{H}} = \mathbf{R}\mathbf{E}(x-x_0)\mathbf{R}^{-1}\mathbf{y}_{\mathrm{H}}(x_0) \tag{297}$$

In the second line we multiplied together the two diagonal matrices and noticed that this is of the same form but evaluated at $x - x_0$. This is the general solution of the initial value problem Eq. (282) with given initial conditions $\mathbf{y}_{\mathrm{H}}(x_0)$. The result has a clear interpretation: first transform the initial condition $\mathbf{y}_{\mathrm{H}}(x_0)$ to the eigenbasis of the differential equation by multiplying by the matrix $\mathbf{R}^{-1}$, then advance the system with the diagonal matrix $\mathbf{E}$ forward by $x - x_0$, finally transform back to the original basis by multiplying by the matrix $\mathbf{R}$. Note that generally the eigenbasis consists of the normal modes of the system which exhibit independent exponentially decaying or growing oscillations. Hence, the initial conditions is first decomposed into normal modes, then these are evolved by $x - x_0$, finally the original components are obtained from the linear combinations of the final state of the normal modes.

5.4 Inhomogeneous linear systems

5.4.1 Method of variation of constants

Let us apply the variation of constants to get the inhomogeneous solutions

$$\mathbf{y}'(x) = \mathbf{A}(x)\mathbf{y}(x) + \mathbf{f}(x) \tag{298}$$

where

$$\mathbf{y}_{\mathrm{H}}(x) = \mathbf{Y}(x)\mathbf{C} \tag{299}$$

is the general solution of the corresponding homogeneous equation $\mathbf{y'}_{\mathrm{H}}(x) = \mathbf{A}(x)\mathbf{y}_{\mathrm{H}}(x)$. We look for the particular solution in the form

$$\mathbf{y}_{\mathrm{P}}(x) = \mathbf{Y}(x)\boldsymbol{\psi}(x) \tag{300}$$

where we turned the vector of constants $\mathbf{C}$ into a vector of functions, which we denoted by $\boldsymbol{\psi}(x)$.

Let us plug back in the solution

$$\mathbf{y}'(x) = \mathbf{Y}'(x)\boldsymbol{\psi}(x) + \mathbf{Y}(x)\boldsymbol{\psi}'(x) = \mathbf{A}(x)\underbrace{\mathbf{Y}(x)\boldsymbol{\psi}(x)}_{\mathbf{y}(x)} + \mathbf{f}(x) \quad \Rightarrow \quad \underbrace{[\mathbf{Y}'(x) - \mathbf{A}(x)\mathbf{Y}(x)]}_{0}\boldsymbol{\psi}(x) + \mathbf{Y}(x)\boldsymbol{\psi}'(x) = \mathbf{f}(x) \tag{301}$$

In the second step we have rearranged the expression to collect all terms proportional to $\boldsymbol{\psi}(x)$. The coefficient is the matrix $\mathbf{Y}'(x) - \mathbf{A}(x)\mathbf{Y}(x)$ which is zero due to the definition of $\mathbf{y}_{\mathrm{H}}(x)$ and $\mathbf{Y}(x)$:

$$\left.\begin{aligned} \mathbf{y}_{\mathrm{H}}(x) &= \mathbf{Y}(x)\mathbf{C} \\ \mathbf{y}'_{\mathrm{H}}(x) &= \mathbf{Y}'(x)\mathbf{C} \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{y}'_{\mathrm{H}}(x) = \mathbf{A}(x)\underbrace{\mathbf{Y}(x)\mathbf{C}}_{\mathbf{y}_{\mathrm{H}}(x)} = \mathbf{Y}'(x)\mathbf{C} \quad \Rightarrow \quad [\mathbf{Y}'(x) - \mathbf{A}(x)\mathbf{Y}(x)]\,\mathbf{C} = \mathbf{0} \tag{302}$$

Since this must be true for all $\mathbf{C}$ we find that $\mathbf{Y}'(x) - \mathbf{A}(x)\mathbf{Y}(x) = 0$. Therefore, the coefficient of $\boldsymbol{\psi}(x)$ is zero and we are left with the equation which only depends on $\boldsymbol{\psi}'(x)$

$$\mathbf{Y}(x)\boldsymbol{\psi}'(x) = \mathbf{f}(x) \quad \Rightarrow \quad \boldsymbol{\psi}'(x) = \mathbf{Y}(x)^{-1}\mathbf{f}(x) \quad \Rightarrow \quad \boldsymbol{\psi}(x) = \int \mathbf{Y}(x)^{-1}\mathbf{f}(x)dx \tag{303}$$

Here we set the integration constant to zero, because we only need any particular solution.[39] Substitute back in Eq. (300)

$$\mathbf{y}_{\mathrm{P}}(x) = \mathbf{Y}(x) \int \mathbf{Y}(x)^{-1}\mathbf{f}(x)dx \tag{304}$$

and so

$$\mathbf{y}(x) = \mathbf{y}_{\mathrm{H}}(x) + \mathbf{y}_{\mathrm{P}}(x) = \mathbf{Y}(x)\mathbf{C} + \mathbf{Y}(x) \int \mathbf{Y}(x)^{-1}\mathbf{f}(x)dx \tag{305}$$

For given initial conditions $\mathbf{y}(x_0)$, we get

$$\mathbf{y}(x) = \mathbf{Y}(x)\mathbf{Y}^{-1}(x_0)\mathbf{y}(x_0) + \mathbf{Y}(x) \int_{x_0}^{x} \mathbf{Y}(x')^{-1}\mathbf{f}(x')dx' \tag{306}$$

This is the general solution to linear systems of differential equations where the coefficient matrix $\mathbf{A}$ depends on x. In the special case of constant coefficient linear systems where $\mathbf{A}$ does not depend on x but which has n eigenvalues, we have shown previously that $\mathbf{Y}(x) = \mathbf{R}\mathbf{E}(x)$ where $\mathbf{R}$ is the matrix of right eigenvalues of $\mathbf{A}$ and $\mathbf{E}(x)$ is a diagonal matrix with elements $e^{\lambda x}$. In this case

$$\begin{aligned}\mathbf{y}(x) &= \mathbf{R}\mathbf{E}(x)\left[\mathbf{R}\mathbf{E}(x_0)\right]^{-1}\mathbf{y}(x_0) + \mathbf{R}\mathbf{E}(x) \int_{x_0}^{x} \left[\mathbf{R}\mathbf{E}(x_0)\right]^{-1}\mathbf{f}(x')dx' \\ &= \mathbf{R}\mathbf{E}(x-x_0)\mathbf{R}^{-1}\mathbf{y}(x_0) + \int_{x_0}^{x} \mathbf{R}\mathbf{E}(x-x_0)\mathbf{R}^{-1}\mathbf{f}(x')dx'\end{aligned} \tag{307}$$

5.4.2 Method of undetermined coefficients

If the forcing function is sufficiently simple, i.e. polynomials, exponentials, etc., we can try to guess the particular solution as in Table 2 for n^{th} order equations.

Example 3

Solve the equations

$$\left\{ \begin{array}{l} \dot{x} = 3x + 2y + 4e^{5t} \\ \dot{y} = x + 2y \end{array} \right\} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ? \tag{308}$$

Solution The system of differential equations may be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix} + e^{5t} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{309}$$

Consider the homogeneous equations. The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = 4\,,\ \lambda_2 = 1\,. \tag{310}$$

The corresponding eigenvectors are

$$(\mathbf{A} - 4\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x = 2y \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{311}$$

$$(\mathbf{A} - 1\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x = -y \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{312}$$

Thus, the general solution of the homogeneous equation is

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathrm{H}} = C_1 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{313}$$

Given that the forcing function is proportional to an exponential, let us look for a particular solution in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathrm{P}} = e^{5t} \begin{pmatrix} A \\ B \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 5e^{5t} \begin{pmatrix} A \\ B \end{pmatrix} = e^{5t} \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + e^{5t} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{314}$$

Thus the general solution is the homogeneous plus the particular solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}\,. \tag{315}$$

[39] Show that if we included an integration constant $\mathbf{C}$ this term would contribute to the homogeneous solution.

5.5 Constant coefficient non-Hermitian linear systems – Jordan Form

Consider again the case of constant coefficient linear systems but in which the coefficient matrix $\mathbf{A}$ does not depend on x but which does not have n eigenvectors. Such a matrix is called a **defective matrix**. This may happen if the matrix is not Hermitian, i.e. $\mathbf{A} \neq \mathbf{A}^{*T}$.

Suppose λ_1 repeats k times, but there is only 1 eigenvector which satisfies $\mathbf{A}\mathbf{v} = \lambda_1\mathbf{v}$, denote this by $\mathbf{v}_1$. In this case, a theorem states that we can always define a **Jordan chain**, as follows:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \tag{316}$$
$$\mathbf{A}\mathbf{v}_2 = \lambda_1\mathbf{v}_2 + \mathbf{v}_1 \tag{317}$$
$$\vdots \tag{318}$$
$$\mathbf{A}\mathbf{v}_k = \lambda_1\mathbf{v}_k + \mathbf{v}_{k-1} \tag{319}$$

Here $\mathbf{v}$ is called the generator of the chain and $\{\mathbf{v}_2, \cdots, \mathbf{v}_k\}$ are the **associated eigenvectors** or **generalised eigenvectors**. Note that these equations may be written as

$$\left\{\begin{array}{l} (\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}_1 = \mathbf{0} \\ (\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}_2 = \mathbf{v}_1 \\ \quad\vdots \\ (\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}_k = \mathbf{v}_{k-1} \end{array}\right\} \quad\Rightarrow\quad \left\{\begin{array}{l} (\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}_1 = \mathbf{0} \\ (\mathbf{A} - \lambda_1\mathbf{I})^2\,\mathbf{v}_2 = 0 \\ \quad\vdots \\ (\mathbf{A} - \lambda_1\mathbf{I})^k\,\mathbf{v}_k = 0 \end{array}\right\} \tag{320}$$

To find a solution we can solve the last equation first to find $\mathbf{v}_k$ for which $(\mathbf{A}-\lambda_1\mathbf{I})^k\mathbf{v}_k = 0$ but $(\mathbf{A}-\lambda_1\mathbf{I})^{k-1}\mathbf{v}_k \neq 0$. Then we can use the recursion to find all other vectors $\mathbf{v}_{j-1} = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_j$, where $j = (k, k-1, \cdots, 2)$.

Let us pack all eigenvectors and all associated eigenvectors constructed in this way but for all distinct λ_i into a matrix.

$$\mathbf{R} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \tag{321}$$

Similar to how a matrix of eigenvectors can be used to diagonalise a matrix, this matrix can be used to block-diagonalise the $\mathbf{A}$ matrix. Given s number of distinct eigenvalues, we get

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \begin{pmatrix} \begin{pmatrix} \mathbf{J}_1 \end{pmatrix} & 0 & & 0 \\ 0 & \begin{pmatrix} \mathbf{J}_2 \end{pmatrix} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \begin{pmatrix} \mathbf{J}_s \end{pmatrix} \end{pmatrix} \quad \text{where} \begin{pmatrix} \mathbf{J}_i \end{pmatrix} = \begin{pmatrix} \lambda_i & 1 & 0 & & 0 \\ 0 & \lambda_i & 1 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & \lambda_i & 1 \\ 0 & & & 0 & \lambda_i \end{pmatrix} \tag{322}$$

where the first block of nonzero elements whose eigenvalue has multiplicity k, denoted by $\mathbf{J}_1$ here is a $k \times k$ matrix, whose diagonal elements are λ_1 and the superdiagonal elements are 1, all other elements being zero. Other Jordan blocks corresponding to the other distinct eigenvalues are square matrices with a size corresponding to the multiplicity of the given eigenvalues.

Within a Jordan block we have

$$\mathbf{J} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R} \quad\Rightarrow\quad \mathbf{A} = \mathbf{R}\mathbf{J}\mathbf{R}^{-1} \quad\Rightarrow\quad \frac{d}{dx}\mathbf{y}(x) = \mathbf{A}\mathbf{y}(x) = \mathbf{R}\mathbf{J}\mathbf{R}^{-1}\mathbf{y}(x) \quad\Rightarrow\quad \frac{d}{dx}\mathbf{R}^{-1}\mathbf{y}(x) = \mathbf{J}\mathbf{R}^{-1}\mathbf{y}(x) \tag{323}$$

Let us denote $\mathbf{R}^{-1}\mathbf{y} = \boldsymbol{\xi}$ then this block becomes

$$\boldsymbol{\xi}' = \mathbf{J}\boldsymbol{\xi} \quad\Rightarrow\quad \frac{d}{dx}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix} = \begin{pmatrix} \lambda_i & 1 & 0 & & 0 \\ 0 & \lambda_i & 1 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & \lambda_i & 1 \\ 0 & & & 0 & \lambda_i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix} \tag{324}$$

or in components

$$\xi_1' = \lambda_1 \xi_1 + \xi_2 \tag{325}$$

$$\vdots \tag{326}$$

$$\xi_{k-1}' = \lambda_1 \xi_{k-1} + \xi_k \tag{327}$$

$$\xi_k' = \lambda_1 \xi_k \tag{328}$$

The last equation may be integrated immediately and then substituted in the one before last which becomes a first order linear equation that we may solve and so on:

$$\xi_k = C_k e^{\lambda_1 x} \tag{329}$$

$$\xi_{k-1} = (C_{k-1} + C_k x) e^{\lambda_1 x} \tag{330}$$

$$\vdots \tag{331}$$

$$\xi_1 = \left(C_1 + C_2 x + \cdots + C_k \frac{x^{k-1}}{(k-1)!} \right) e^{\lambda_1 x} . \tag{332}$$

Now we can substitute back to find the solution. For the first k components of the solution we have

$$\begin{aligned}
\mathbf{y} = \mathbf{R}\boldsymbol{\xi} &= \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_1 & \cdots & \mathbf{v}_k \\ | & | & & | \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_1 & \cdots & \mathbf{v}_k \\ | & | & & | \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} e^{\lambda_1 x} & x e^{\lambda_1 x} & & \cdots & \frac{x^{k-1}}{(k-1)!} e^{\lambda_1 x} \\ 0 & e^{\lambda_1 x} & x e^{\lambda_1 x} & & \\ \vdots & & \ddots & \ddots & \vdots \\ & & & e^{\lambda_1 x} & x e^{\lambda_1 x} \\ 0 & \cdots & & 0 & e^{\lambda_1 x} \end{pmatrix}}_{\mathbf{E}(x)} \underbrace{\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{k-1} \\ C_k \end{pmatrix}}_{\mathbf{C}} \\
&= e^{\lambda_1 x} \left[C_1 \mathbf{v}_1 + C_2 (x\mathbf{v}_1 + \mathbf{v}_2) + C_3 \left(\frac{x^2}{2!} \mathbf{v}_1 + x\mathbf{v}_2 + \mathbf{v}_3 \right) + \cdots + C_k \left(\frac{x^{k-1}}{(k-1)!} \mathbf{v}_1 + \cdots + x\mathbf{v}_{k-1} + \mathbf{v}_k \right) \right] \\
&= C_1 \underbrace{e^{\lambda_1 x} \mathbf{v}_1}_{\mathbf{y}_1(x)} + C_2 \underbrace{e^{\lambda_1 x} (x\mathbf{v}_2 + \mathbf{v}_1)}_{\mathbf{y}_2(x)} + \cdots + C_k \underbrace{e^{\lambda_1 x} \left(\frac{x^{k-1}}{(k-1)!} \mathbf{v}_1 + \cdots + x\mathbf{v}_{k-1} + \mathbf{v}_k \right)}_{\mathbf{y}_k(x)} .
\end{aligned} \tag{333}$$

Note that in the first line we have expressed the part of the solution corresponding to the λ_1 eigensubspace in the same form as previously Eq. (294) for the case where the matrix was not defective and had distinct eigenvalues, $\mathbf{R}\mathbf{E}(x)\mathbf{C}$, where we are just showing the elements within this block where $\mathbf{R}$ and $\mathbf{E}(x)$ are $k \times k$ matrices, and $\mathbf{C}$ is $k \times 1$. The solution has a similar form where the eigenvectors are replaced by generalised eigenvectors and the "diagonal evolution matrix" $\mathbf{E}(x)$ within this $k \times k$ block is replaced by an upper triangular evolution matrix whose diagonal elements are the same but the upper triangular elements are nonzero, $E_{ij}(x) \propto x^{j-i} e^{\lambda_1 x}$. In the last equality we have defined the linearly independent vectors which correspond to the individual integration constants, these are the linearly indepenent solutions of the differential equation corresponding to eigenvalue λ_1.

All Jordan blocks with different eigenvalues have the solution in this form. In particular, if all other eigenvalues are distinct then $\mathbf{y}_i(x) = e^{\lambda_i x} \mathbf{v}_i$ for $i > k$ then the complete general solution is

$$\begin{aligned}
\mathbf{y}(x) =& C_1 \underbrace{e^{\lambda_1 x} \mathbf{v}_1}_{\mathbf{y}_1(x)} + C_2 \underbrace{e^{\lambda_1 x} (x\mathbf{v}_2 + \mathbf{v}_1)}_{\mathbf{y}_2(x)} + \cdots + C_k \underbrace{e^{\lambda_1 x} \left(\frac{x^{k-1}}{(k-1)!} \mathbf{v}_1 + \cdots + x\mathbf{v}_{k-1} + \mathbf{v}_k \right)}_{\mathbf{y}_k(x)} + \\
&+ C_{k+1} \underbrace{e^{\lambda_{k+1} x} \mathbf{v}_{k+1}}_{\mathbf{y}_{k+1}(x)} + \cdots + C_n \underbrace{e^{\lambda_n x} \mathbf{v}_{k+1}}_{\mathbf{y}_n(x)} . \qquad (\lambda_1 = \lambda_2 = \cdots = \lambda_k,\ \{\lambda_i\}_{i>k} \text{ distinct})
\end{aligned} \tag{334}$$

The solution may also be written as $\mathbf{R}\mathbf{E}(x)\mathbf{C}$ as in Eqs. (294) and (333), where $E(x)$ is "block diagonal", with $E_{ij}(x) \propto x^{j-i} e^{\lambda_1 x}$ within the $k \times k$ block with $1 \le i \le k$ and $1 \le j \le k$ and $E_{ii}(x) = e^{\lambda_i x}$ for $k+1 \le i \le n$ and zero everywhere else. In this case we have $n - k + 1$ number of blocks with the first block being $k \times k$ and all

$n-k$ others are 1×1. In general, the solution is a sum of the the solution for all Jordan blocks, where $\mathbf{E}(x)$ is block diagonal with possibly multiple blocks larger than 1×1.

Note that this solution is consistent with that found for single n^{th} order equations with repeated roots, see e.g. Eq. (203). In that case we must construct the n-dimensional vector defining our system using Eq. (263). For repeated indices of multiplicity k, the derivatives of polynomials times exponentials will generate the Jordan blocks of size $k\times k$.

Example 4

Solve the equations

$$\left\{\begin{array}{l} \dot{x} = -x + y - 2z \\ \dot{y} = 4x + y \\ \dot{z} = 2x + y - z \end{array}\right\} \tag{335}$$

Solution The system of differential equations may be written in matrix form as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \underbrace{\begin{pmatrix}-1 & 1 & -2\\ 4 & 1 & 0\\ 2 & 1 & -1\end{pmatrix}}_{\mathbf{A}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}. \tag{336}$$

First find the eigenvalues of $\mathbf{A}$

$$\det(\mathbf{A}-\lambda\mathbf{I}) = (1+\lambda)^2(1-\lambda) = 0 \quad\Rightarrow\quad \lambda_1 = \lambda_2 = -1\,,\ \lambda_3 = 1. \tag{337}$$

Find the eigenvector corresponding to $\lambda_3 = 1$

$$(\mathbf{A}-1\mathbf{I})\mathbf{v}_3 = \begin{pmatrix}-2 & 1 & -2\\ 4 & 0 & 0\\ 2 & 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad\Rightarrow\quad x = 0\,,\ y = 2z \quad\Rightarrow\quad \mathbf{v}_3 = \begin{pmatrix}0\\ 2\\ 1\end{pmatrix}. \tag{338}$$

Find eigenvectors corresponding to $\lambda_1 = \lambda_2 = -1$:

$$(\mathbf{A}-(-1)\mathbf{I})\mathbf{v}_1 = \begin{pmatrix}0 & 1 & -2\\ 4 & 2 & 0\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad\Rightarrow\quad y = 2z\,,\ x = -\frac{1}{2}y = -z \quad\Rightarrow\quad \mathbf{v}_1 = \begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}. \tag{339}$$

Here z may be chosen arbitrarily, we set it to $z = 1$. There are no other linearly independent solutions to this equation, so no other eigenvectors exist for $\lambda_1 = \lambda_2 = 1$. Let us search for a generalised eigenvector

$$(\mathbf{A}-(-1)\mathbf{I})^2\mathbf{v}_2 = \begin{pmatrix}0 & 0 & 0\\ 8 & 8 & -8\\ 4 & 4 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad\Rightarrow\quad z = x + y \quad\Rightarrow\quad \mathbf{v}_2 = \begin{pmatrix}0\\ 1\\ 1\end{pmatrix}. \tag{340}$$

Here x and y may be chosen arbitrarily, we set $x = 0$ and $y = 1$. We can verify that $(\mathbf{A}-(-1)\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$.

Given the eigenvalues, eigenvectors and associated eigenvectors we can substitute to find the solution:

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = e^{-t}\left\{C_1\begin{pmatrix}-1\\ 2\\ 1\end{pmatrix} + C_2\left[\begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}t + \begin{pmatrix}0\\ 1\\ 1\end{pmatrix}\right]\right\} + C_3e^{t}\begin{pmatrix}0\\ 2\\ 1\end{pmatrix}. \tag{341}$$

6 Summary

The key results of this chapter are as follows.

- All systems of arbitrary order linear differential equations can be expressed as a system of first order differential equations.

- The structure of the solutions is reminiscent of the one for n^{th} order single equations, it may be written as

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} = C_1\mathbf{y}_{\text{H}1}(x) + C_2\mathbf{y}_{\text{H}2}(x) + \cdots + C_n\mathbf{y}_{\text{H}n}(x) + \mathbf{y}_{\text{P}}(x) \tag{342}$$

  where $\{\mathbf{y}_{\text{H}1}(x), \mathbf{y}_{\text{H}2}(x), \cdots, \mathbf{y}_{\text{H}n}(x)\}$ are linearly independent solutions of the homogeneous equation, $\mathbf{y}_{\text{P}}(x)$ is the particular solution of the homogogeneous equation.

- All of these solutions, $\mathbf{y}_{\text{Hi}}(x)$, are n element vectors, each component being a function of x (at least one component must be nonzero for each of these vectors respectively).

- For constant coefficient linear systems of differential equations, with a real-symmetric $\mathbf{A} = \mathbf{A}^{\text{T}}$ or complex-Hermitian coefficient matrix $\mathbf{A} = \mathbf{A}^{*\text{T}}$, the homogeneous solution may always be written as

$$\mathbf{y}_{\text{H}}(x) = C_1 e^{\lambda_1 x}\mathbf{v}_1 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_n \,. \tag{343}$$

- If the matrix $\mathbf{A}$ has repeated roots for any eigenvalue, and the matrix does not have as many eigenvectors as the multiplicity of the eigenvalue, the matrix is defective. The solution may be expressed with the Jordan form by constructing the associated eigenvectors (or generalised eigenvectors), see Eq. (333).

- For simple driving functions, sums of exponentials, sines, cosines, polynomials, or their products, the particular solution may be guessed just like for the case of n^{th} order single equations. In the more general case, you have to apply the variation of constants using the homogeneous solution.

Procedure for solving linear systems of differential equations with constant coefficients

Step I Find all eigenvalues $\{\lambda_i\}$ of $\mathbf{A}$, i.e. solve $\det(A - \lambda\mathbf{I}) = 0$. This is an n^{th} order polynomial of λ, so there are always n generally complex roots (not necessarily distinct).

Step II Find all eigenvectors for each λ_i.

Step III For repeated eigenvalues, find all eigenvectors or if the corresponding eigensubspace is defective find all associated eigenvectors by constructing the Jordan chain.

Step IV If $\mathbf{A}$ has a complete set of n eigenvectors, then the solution is

$$\mathbf{y}_{\mathrm{H}}(x) = C_1 e^{\lambda_1 x}\mathbf{v}_1 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_n \,. \tag{344}$$

Step V Otherwise, if $\mathbf{A}$ has λ_r repeated eigenvalues with fewer corresponding eigenvectors, then the general solution for each of these defective eigensubspaces of dimension k may be constructed using the associated eigenvectors as

$$e^{\lambda_r x}\left[C_1\mathbf{v}_1 + C_2(x\mathbf{v}_2 + \mathbf{v}_1) + \cdots + C_k\left(\frac{x^{k-1}}{(k-1)!}\mathbf{v}_1 + \cdots + x\mathbf{v}_{k-1} + \mathbf{v}_k\right)\right] \tag{345}$$

where $\{C_1, C_2, \ldots, C_k\}$ are integration constants.

Step VI Add up the contributions of all eigensubspaces.

Step VII Collect all terms corresponding to different integration constants C_i, label these $\mathbf{y}_i(x)$ and pack them into a matrix, $\mathbf{Y}(x)$, called the fundamental system.

$$\mathbf{Y}(x) = \begin{pmatrix} | & | & & | \\ \mathbf{y}_{\mathrm{H}1}(x) & \mathbf{y}_{\mathrm{H}2}(x) & \ldots & \mathbf{y}_{\mathrm{H}n}(x) \\ | & | & & | \end{pmatrix} \tag{346}$$

Step VIII Obtain the inhomogeneous solution by guessing the solution, or the variation of constants, or use

$$\mathbf{y}_{\mathrm{P}}(x) = \mathbf{Y}(x)\int^{x}\mathbf{Y}(x')^{-1}\mathbf{f}(x')dx' \tag{347}$$

Step IX The general solution is $\mathbf{y}(x) = \mathbf{Y}(x)\mathbf{C} + \mathbf{y}_{\mathrm{P}}(x)$

Step X Find the integration constants using the initial conditions

$$\mathbf{y}(x) = \mathbf{Y}(x)\mathbf{Y}^{-1}(x_0)\mathbf{y}(x_0) + \mathbf{Y}(x)\int_{x_0}^{x}\mathbf{Y}(x')^{-1}\mathbf{f}(x')dx' \tag{348}$$