Geometric Optics Lecture Notes

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Contents

1 Early history **2**

2 The nature of light **3**
2.1 The wave nature of light . . . 3
2.2 Refractive index . . . 4
2.3 The particle nature of light . . . 5
2.4 The electromagnetic spectrum . . . 5
2.5 What is a ray of light ? . . . 6

3 Propagation through a discontinuity **6**
3.1 Reflection . . . 6
3.2 The law of refraction . . . 8
3.2.1 Total internal reflection . . . 8

4 Fermat and Huygens principles **9**
4.1 Fermat principle . . . 9
4.1.1 Direct propagation . . . 10
4.1.2 Demonstrating the law of reflection . . . 10
4.1.3 Demonstrating the law of refraction . . . 11
4.1.4 A case with maxima . . . 11
4.1.5 Justification of Fermat's principle. . . . 12
4.2 Huygens principle . . . 13
4.2.1 Direct propagation . . . 13
4.2.2 Demonstrating the law of reflection . . . 13
4.2.3 Demonstrating the law of refraction . . . 14

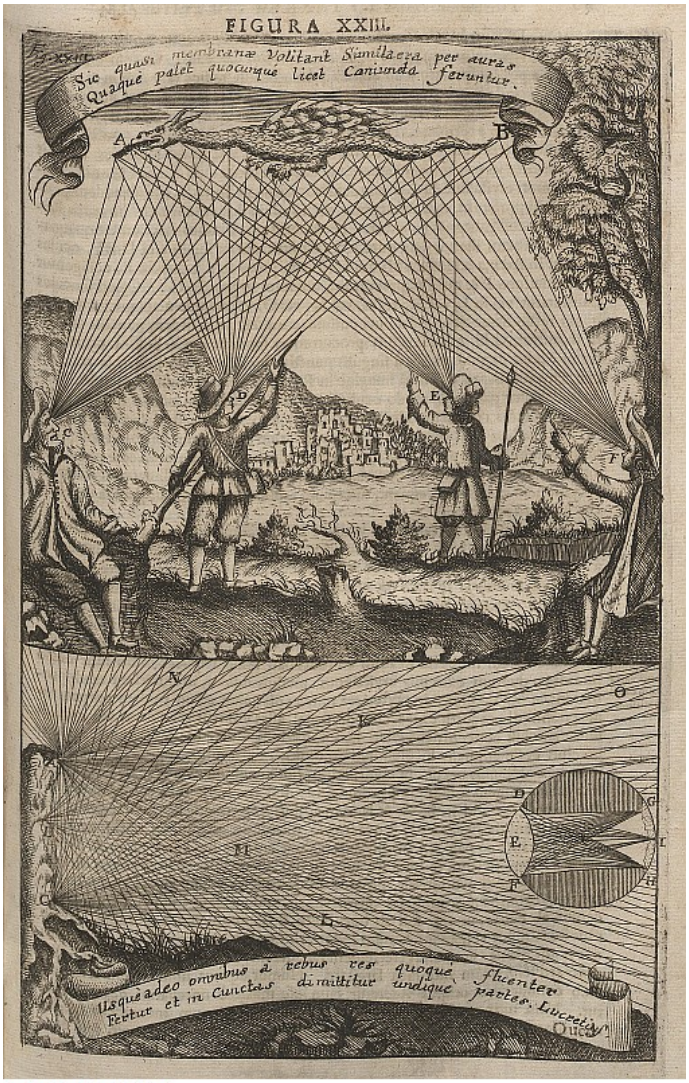
5 The prism **16**
5.1 Deviation angle. . . . 16
5.2 Minimum deviation angle. . . . 17
5.3 Application to crystals and halos . . . 18

6 Lenses **19**
6.1 General case . . . 19
6.1.1 Condition for an image formation . . . 19
6.1.2 Convex lens . . . 20
6.1.3 Concave lens . . . 22
6.1.4 Sign convention . . . 23
6.2 Ray tracing . . . 24
6.3 Simple lens systems . . . 25
6.3.1 The eye . . . 25
6.3.2 The magnifying lens . . . 26
6.4 Complex lens systems . . . 27
6.4.1 The refracting telescope. . . . 27
6.4.2 The coumpound microscope. . . . 27

7 Mirrors **29**
7.1 Parabolic mirror . . . 29
7.2 Spherical mirror . . . 30
7.3 Concave mirror . . . 31
7.4 Convex mirror . . . 32
7.5 Newtonian telescope . . . 32

8 Optical abberations **32**
8.1 Spherical abberation . . . 33
8.2 Coma . . . 33
8.3 Chromatic aberration . . . 34

1 Early history



Quelle: Deutsche Fotothek

Figure 1: Illustration from System der visuellen Wahrnehmung beim Menschen (1687) depicting emission theory

The study of optics starts very early, with people needing practical devices to, e.g. light fire using sunlight. Potential lenses dating back to -750 have been found and records of lenses used to start fires have appeared in the literature as early as -423 (The Clouds by Aristophane)[1].

[1]Despite having been part of popular culture, the idea that Archimedes used mirrors to concentrate light rays and set fire to the Roman fleet

The first theoretical understanding of optics we know of come from Hero of Alexandria (1st century). Hero noticed that the law of reflection can be understood as a principle of least line: that in the case of reflection light was taking the shortest, most economical path. The first know treaty about optics was written by Ptolemy during the 2nd century. It describes vision as originating from particles emitted by the eyes and reflected back by objects (see Fig. 1), very similar to the way a radar works, but far from the way our eyes actually work! Ptolemy measured the angles of refraction between air water and glass but seem to have adjusted his measurement to fit a parabolic law and, possibly convinced by the weakness of his method, never fully expressed it as a law.

In 985 the mathematician and engineer Ibn Sahl derived what became several hundreds years later, Snell's law of refraction. He used the law of refraction to make the first theoretical study of biconvex lenses (see Fig. 2). The aim being to know a-priori which shape of lens is needed to heat an object by a lens.

Ibn al-Haytham (965-1040) wrote a 7 tomes *Book of Optics* where he refuted the idea that vision was due to our eyes sending out rays but that some bodies emitted lights and others reflected it and that light rays arriving in our eyes where the source of vision. Unfortunately on refraction he discards the work of Ibn Sahl and rather comes back to the erroneous tables of Ptolemy. His text, rather than the one of Ibn Sahl, was later translated in Latin and reached Europe. That allowed Descartes and Snell to rediscover the law of refraction by themselves.

At the end of the 13th century, the persian Kamāl al-Dīn al-Fārisī and the german Theodoric of Freiberg pierce at the same time and without communicating the secret of the rainbow. They both base their work on the work on refraction from Ibn al-Haytham and they both use an experimental setup consisting of a glass ball filled with water as a model of a droplet.

This was a short early history of optics. The rest of the course is based on what was discovered or re-discovered in Europe by the such as Descartes, Snell, Fermat, Huygens etc.

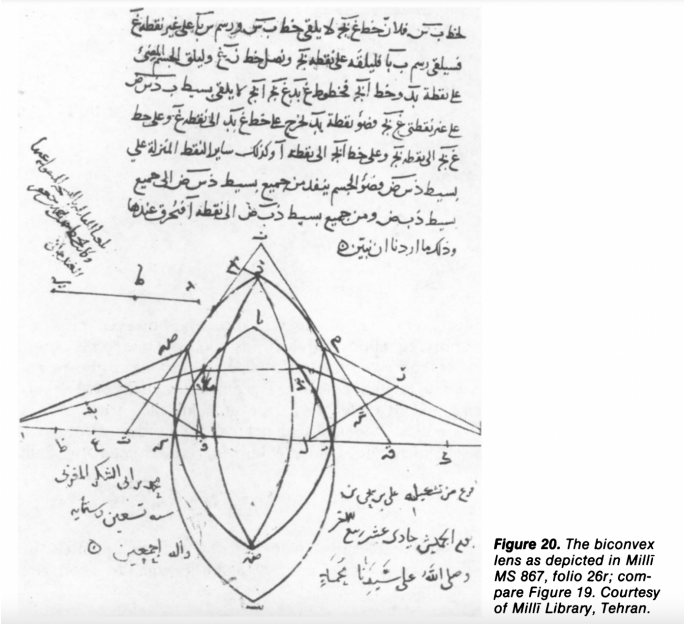


Figure 20. *The biconvex lens as depicted in Millī MS 867, folio 26r; compare Figure 19. Courtesy of Millī Library, Tehran.*

Figure 2: Drawing of a bi-convex lens and ray paths from Ibn Sahl' burning mirrors and lenses treaty from 985. Reproduced by Rashed et al. 1990 "A Pioneer in Anaclastics: Ibn Sahl on Burning Mirrors and Lenses".

2 The nature of light

2.1 The wave nature of light

As you will see later in the 1st year Electromagnetic course, light can be represented as the propagation of the joint oscillation of an electric and a magnetic field, also called an electromagnetic wave and we will discuss this further in the wave optics part of the course. However, some of the wave properties of light must be understood in order to understand geometric optics. The simplest wave equation we can use to describe light is the plane-parallel, simply

in the battle of Syracuse was most probably made up 400 years after the battle by Anthemius of Tralles.

polarised, propagating, monochromatic wave:

$$\Psi(x,t) = A\cos\left[\omega\left(t - \frac{x}{v}\right)\right]. \tag{1}$$

In this expression A is the amplitude of the wave, ω is the angular frequency (in rad/s) and v is the propagation speed of the wave. The wave equation is often written as :

$$\Psi(x,t) = A\cos\left(\omega t - kx\right), \tag{2}$$

where k is called the wave vector. For a fixed time, the wave is periodic in x and the period, λ, is called the wavelength. It is measured in meters and defined such as:

$$k = \frac{2\pi}{\lambda}. \tag{3}$$

Similarly, at a fixed position the wave has a periodicity of T called the period, measured in seconds and defined such as:

$$\omega = \frac{2\pi}{T}. \tag{4}$$

We can also define the frequency of the wave, measured in inverse seconds, also known as Hertz:

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{5}$$

The phase speed of light is the speed at which the oscillations of the electro-magnetic field propagates and is defined as :

$$v = \frac{\omega}{k} = \nu\lambda. \tag{6}$$

2.2 Refractive index

In vacuum the light propagates at a constant speed $c = 299792458 m/s$. When light propagates in a medium, however, the electric field interacts with the electrons present in molecules and atoms the light is slowed down[2]. The speed of light in a given medium is usually defined through the refractive index of the medium. The refractive index is the ratio of the phase velocity of light in vacuum to the phase velocity of light in a medium:

$$n = \frac{c}{v}. \tag{7}$$

In general the speed of light is given by the permittivity ϵ and the permeability μ of a medium so that:

$$v = \frac{1}{\sqrt{\epsilon\mu}}, c = \frac{1}{\sqrt{\epsilon_0\mu_0}}, \tag{8}$$

where ϵ_0 and μ_0 are the permittivity and the permeability of the vacuum. The refractive index can therefore directly be linked to the properties of the medium:

$$n = \sqrt{\epsilon_r\mu_r}, \tag{9}$$

where the relative permittivity and permeability are defined relative to the ones in the vacuum.

The refractive index is 1 in the vacuum and larger than 1 in any material with a relative permittivity or a relative permeability larger than 1[3].Table 1 shows refractive indexes for common material. The refractive index directly scales with the density of the medium because denser medium have more atoms to interact with the light. As a consequence it is usually small in gases but can become close to 3 in liquids and solids. The scaling with density leads to a scaling with pressure and temperature in Earth atmosphere, which is the reason for mirages. The refractive index also varies with wavelength, which causes the dispersion of light.. As shown by equation 13, when the speed of light is reduced,

[2]What happens is that the electromagnetic wave induces an electric field that moves the center of the electronic cloud of the material. This displacement of the electronic cloud leads to a polarisation of the medium. Because the electric field of light at a given position varies with time, the polarisation will similarly vary with time, which will give rise to an electric current and to a new electro-magnetic field that will combine with the propagating wave. Following Lenz's law the electromagnetic wave created by the polarised material will oppose the wave creating the polarisation. Light is therefore slowed down and, sometimes, attenuated when crossing a medium.

[3]In reality the refractive index can be a complex number, which accounts for absorption. It can also be smaller than 1 leading to apparent wave speed larger than c ! This concerns only the phase speed though and not the group speed, which is the one that can never be larger than c.

Table 1: Refractive index of common materials at 589nm from https://refractiveindex.info/

Species	State	T,P	n
Helium	gas	0° C 1bar	1.000036
Air	gas	0° , 1bar	1.000293
Water	gas	100° , 1bar	1.000188
Water	liquid	20° , 1bar	1.333
Borosilicate glass	solid	20° , 1bar	1.47
Polycarbonate glass	solid	20° , 1bar	1.58
Titaniate glass	solid	20° , 1bar	1.84
Diamond	solid	20° , 1bar	2.41

either its wavelength or its frequency must decrease. In reality the frequency always stays the same and only the wavelength changes, which leads to:

$$n = \frac{c}{v} = \frac{\lambda_0}{\lambda}, \tag{10}$$

where λ_0 is the wavelength of the wave in the vacuum.

2.3 The particle nature of light

Although the wave description of light can explain numerous phenomenon, some of them can only be understood when light is a considered as a collection of individual particle, called photons. For example, the photoelectric effect, where the absorption of a single photon leads to the ejection of a single electron. More generally the quantified nature of the energy levels in atoms usually means that processes like absorption and emission of light are quantised. Each one of these particles called photons have an energy proportional to the frequency of light it corresponds to:

$$E = h\nu. \tag{11}$$

One way to picture photons is shown in Figure 3. You can think of them as wave packets, well defined in space. The wave packet moves at the group speed, whereas inside each packet the wave itself moves at the phase speed. Only the group speed carries information and thus only the group speed must always be smaller or equal to the speed of light in the vacuum. The group speed is defined by:

$$v_g = \frac{\partial \omega}{\partial k}. \tag{12}$$

Within some approximations it can be shown that group and phase speed are linked by the relationship:

$$v_g = v_p \left(1 + \frac{\lambda}{n}\frac{\partial n}{\partial \lambda}\right). \tag{13}$$

When the optical index is constant with wavelength, the group and the phase velocity are the same. When the refractive index varies with wavelength then the phase and group speed can be different and the medium is called *dispersive*. In all further application of this course, we will always assume that the group and phase velocities are the same, even when the refractive index depends on wavelength. This approximation is valid as long as the change in phase velocity with wavelength is larger than the change of the difference between group and phase velocity with wavelength. This condition is naturally met when the refractive index is a linear function of wavelength. For the remaining of this course we will stay in that regime, assume that phase and group velocities are always equal and call them both the velocity of the wave or photon[4].

2.4 The electromagnetic spectrum

Light and matter interact very differently as a function of the wavelength of the light. On the short wavelength side, gamma rays can break atoms and X-rays and UVs are energetic enough to ionise atoms and break molecular bonds. UV

[4]A more general condition is $\frac{\partial n}{\partial \lambda} > \sqrt{\frac{n}{2}\frac{\partial^2 n}{\partial^2 \lambda}}$

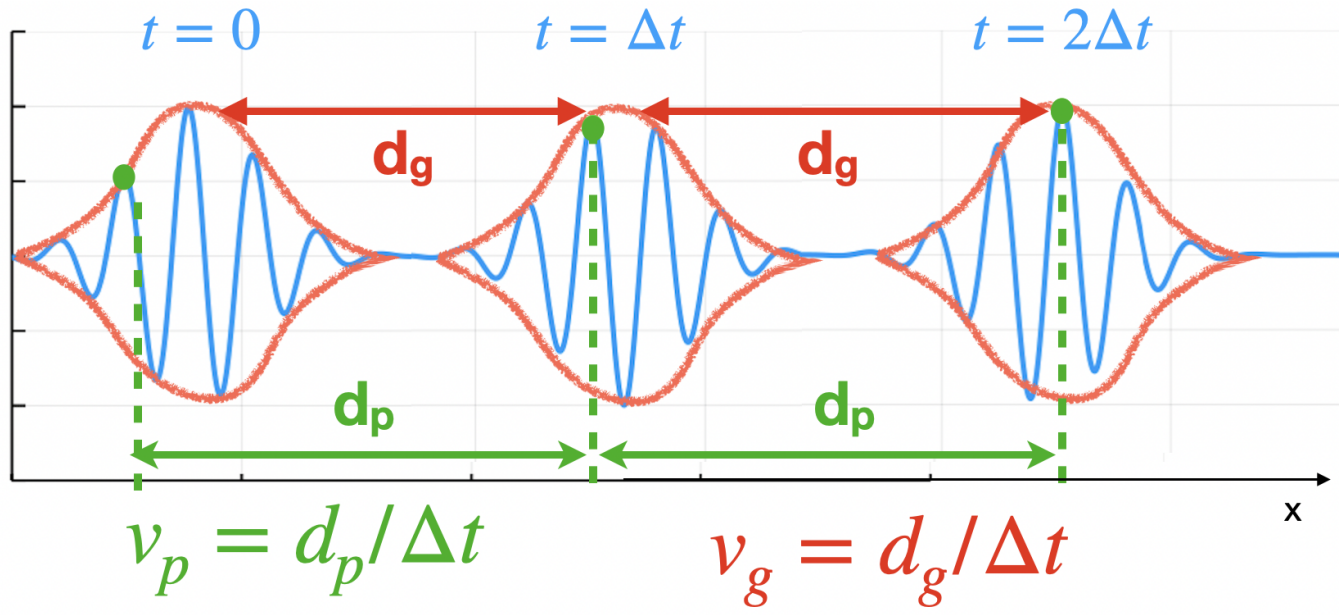


Figure 3: A wave packet propagates in the x-direction. The group speed is the speed of the envelope shown in red whereas the phase speed is the speed of the phase within the envelope shown by the green lines. In our example the phase speed is larger than the group speed, so the green dot propagates to the right within the wavepacket.

and Optical wavelengths have energies similar to the difference between electron levels in atoms and molecules and can thus be absorbed and emitted at specific wavelengths corresponding to these transitions. In the infrared photons carry energies similar to the vibrational modes of molecules and are therefore strongly absorbed by molecules such as water in our atmosphere. At radio and micro-wave wavelengths, the low energetic photons can excite the rotational degrees of freedom of molecules.

2.5 What is a ray of light ?

At wavelengths comparable to the atomic or molecular sizes, the quantum nature of light cannot be ignored and photons must me treated as individual particles. This is true for gamma rays, X rays and UVs. At longer wavelengths the wave nature of light is more useful to understand the processes that are happening. If the wavelength of light is larger or comparable to the objects that are used, a full wave treatment is necessary, which we will discuss in the wave optics chapter next term.

In between these two extremes, when the wavelength of light is larger than the atomic size but smaller than the objects used to study it, is the realm of geometric optics. In that case we can simplify electro-magnetic waves into light rays. The light ray is defined as the line that is always orthogonal to the wavefront and which points towards the direction of energy propagation (see Fig.5).

3 Propagation through a discontinuity

3.1 Reflection

The law of reflection was found early on and the oldest formulation that is known is the one of Hero of Alexandria during the 1st century. It says that a light ray arriving on a surface with an angle θ_1 with respect to the normal of the surface is reflected with an angle θ_2 with respect to the normal such that $\theta_1 = \theta_2$. As shown in Figure 3.2, both angles are defined with respect to the normal to the surface and counted positively from the normal towards the light ray. This is the convention that we will use for the remainder of this course.

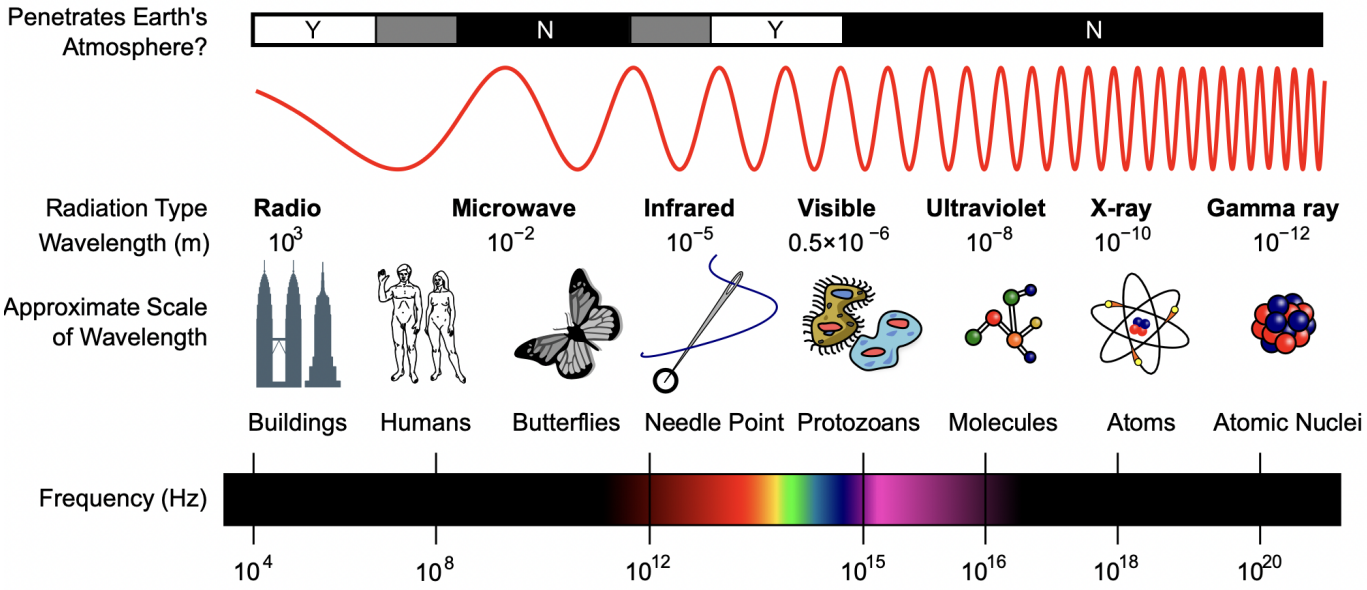


Figure 4: The electromagnetic spectrum. Light at different frequencies interact in different ways with matter.

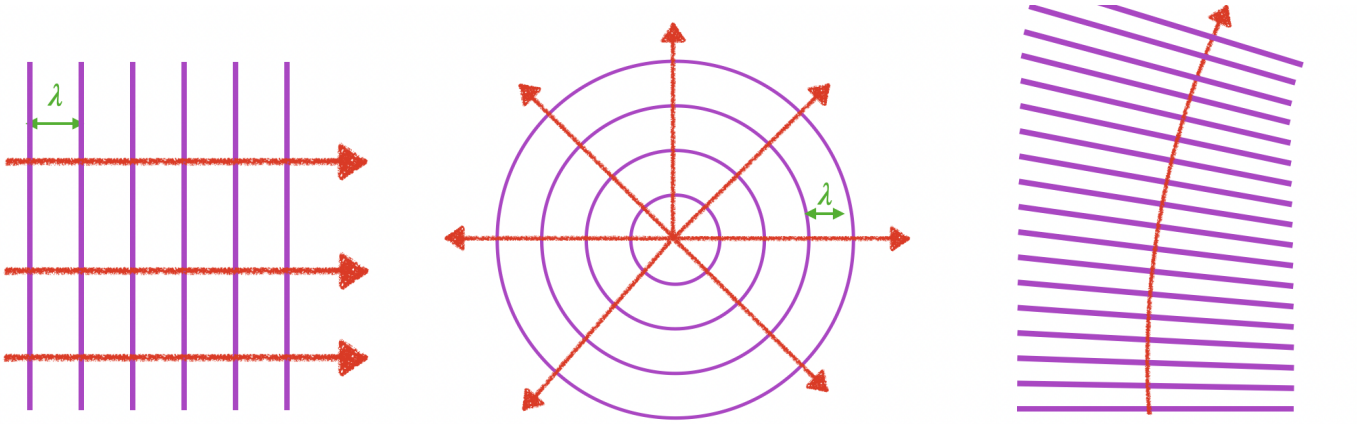


Figure 5: The light rays (red arrows) are always perpendicular to the wavefronts (purple).

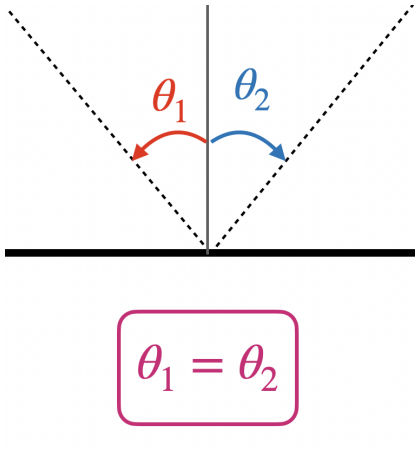


Figure 6: The law of reflection

3.2 The law of refraction

Snell and Descartes derived independently the law of refraction, also know as the sine law, Snell's law or the Snell-Descartes law[5]. The law can be expressed as : A ray travelling in a medium of optical index n_1 and arriving at a surface with an angle θ_1 compared to the normal at the surface emerges into a medium of optical index n_2 with an angle θ_2 with respect to the normal such as :

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{14}$$

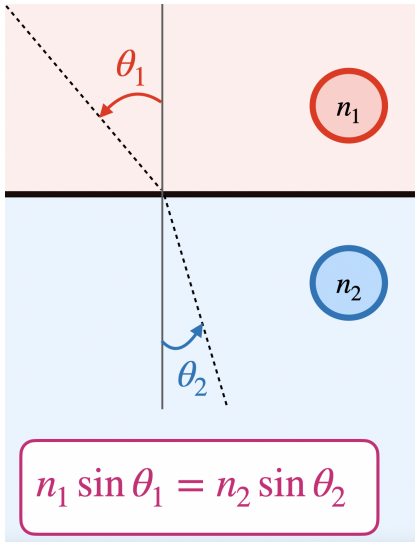


Figure 7: The law of refraction, also know as Snell's law, Snell-Descartes law or the sine law.

3.2.1 Total internal reflection

Using the law of refraction, one can find an expression for the angle θ_1 as a function of θ_2:

$$\theta_2 = \arcsin\left(\frac{n_1}{n_2}\sin\theta_1\right). \tag{15}$$

Given that that arcsin function is defined only in the $[-1:1]$ interval, if $n_1/n_2 \sin\theta_1$ is larger than 1 then the law of refraction predicts that there is no θ_2 so that the ray of light can go from A to B. Otherwise said, if a light is shone at

[5]Neither Snell nor Descartes knew that Ibn Sahl had discovered the low 600 years before them in Baghdad. One could argue that the law should be called Sahl's law.

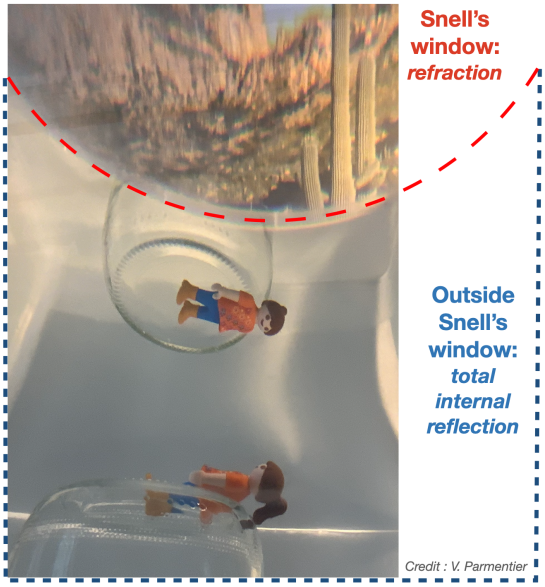


Figure 8: Example of a Snell's window. The Cactus is outside and can only be seen inside a circle where the incident angle is smaller than the critical angle. Away from Snell's window total internal reflection occur.

at angle θ_1 larger than some value there will be no light going through the boundary. The maxima angle is called the *critical angle* and can be written :

$$\theta_c = \arcsin\left(\frac{n_2}{n_1}\right). \tag{16}$$

For $\theta_1 > \theta_c$ the light does not cross the boundary and is instead reflected following the law of reflection.

The critical angle has many practical consequences and can be easily noticed. For example if you sit at the bottom of a pool and looking upward, you see a bright circle. That is where your viewing angle is smaller than the critical angle and so you can see the outside of the pool, usually a bright sky. Around this bright circle, the water surface acts like a mirror and you can usually see the bottom of the pool itself. The bright circle is known as *Snell's window*. I illustrate this in Figure 3.2.1 where a playmobile is laying down in the water. The cactus outside the pool can only be seen when the angle between the camera and the water is smaller than the critical angle, close to the vertical. Away from Snell's window the light is reflected and I can see the reflection of the playmobile on the surface of the water.

4 Fermat and Huygens principles

4.1 Fermat principle

The reflection and refraction laws where both determined first experimentally. But is there a general principle that can explain why the light is behaving that way ? Why a ray of light takes a given path when going from a point A to a point B ? For reflection, it was determined already by the ancient Greeks that it works as if the light take the shortest path. However, this principle was not working for refraction. When asked about it by his friend Marin Cureau de la Chambre, Pierre de Fermat conjectures that the principle should be the one of least time rather than the one of shortest length. This can be written as : *The actual path between two points taken by a beam of light is the one which is traversed in the least time*. As we will see later in more details, this version of Fermat's principle is not completely satisfactory, particularly because it doesn't allow for multiple path to be taken by the light. Indeed, when I look at an object in a mirror I can see the object twice: once through direct vision and once through its reflection in the mirror, although only the direct vision one corresponds to the shortest path taken ! A better version of Fermat's principle allows the light to also take the paths that correspond to a local minima in the time travelled. Thus one can say : *"The paths between two points taken by a beam of light are the ones corresponding to stationary travel times"*.

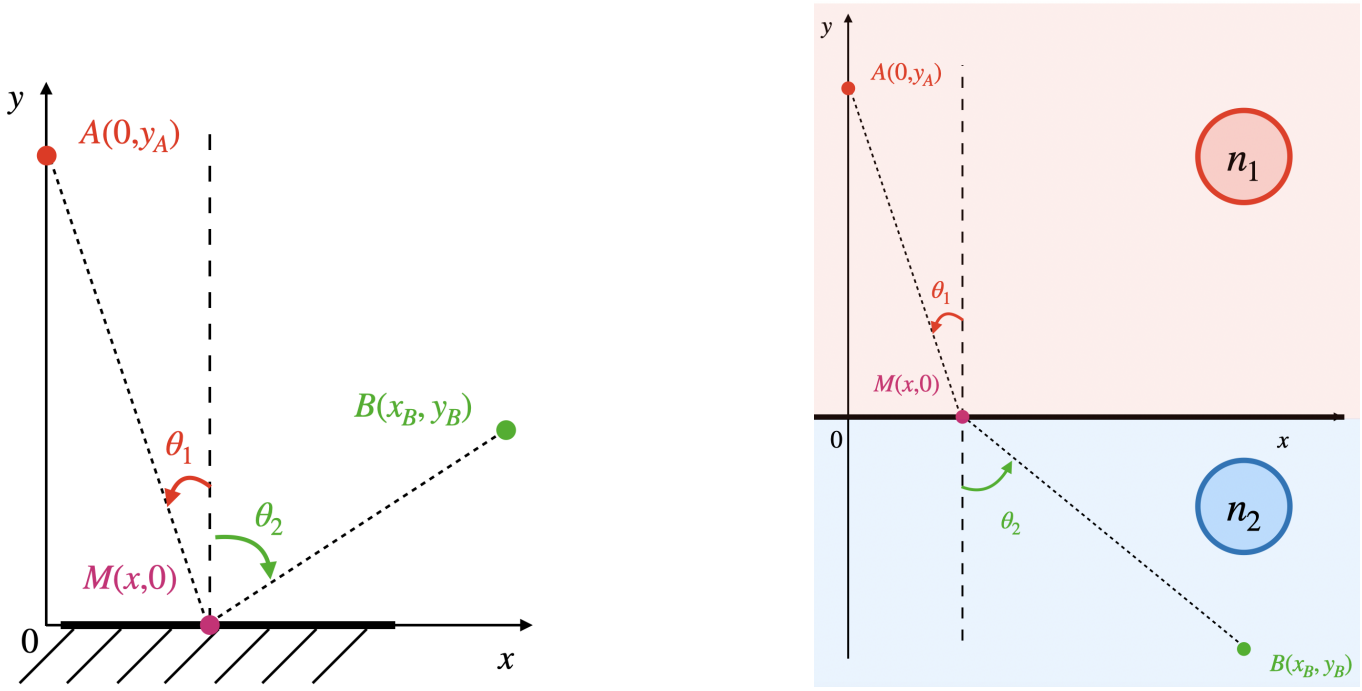


Figure 9: Notations used in the demonstration of the law of reflection (left) and the law of refraction (right) using Fermat's principle.

4.1.1 Direct propagation

We now want to test if our new principle can reproduce what we experience about light propagation, mainly that objects are usually where we see them. This is clearly true if in most cases the rays of light from an object propagates in straight line to our eyes. In an homogeneous medium, e.g. a medium with a constant optical index, the speed of light is constant. As a consequence, the fastest path from one point to another is always a straight line. Thus for the direct vision of objects in an homogeneous medium Fermat principle is validated.

4.1.2 Demonstrating the law of reflection

What about the law of reflection ? The problem posed is Figure 9 is the following: what is the fastest path from A to B that goes through the surface of the mirror ? Since all the light is propagating in the same medium, the travel time will be:

$$t_{AMB} = \frac{n}{c}(AM + MB). \tag{17}$$

If we define the horizontal axis as the x axis and the vertical one as the y axis, A is situated at $(0, y_A)$, B at (x_B, y_B) and M at $(x, 0)$. We can write the lengths as:

$$AM = \sqrt{x^2 + y_A^2} \tag{18}$$

$$MB = \sqrt{(x_B - x)^2 + (y_B)^2} \tag{19}$$

Now Fermat's principle tells us that over all the possible paths corresponding to all the positions x of M on the mirror, the light will take the stationary one. Thus we are looking for the derivative of the travel time with respect to x to be zero:

$$\frac{dt_{AMB}}{dx} = 0. \tag{20}$$

Substituting the expressions for t_{AMB}, $\overline{AM}$ and $\overline{MB}$, taking the derivative an cancelling the constant terms we get:

$$\frac{x}{\sqrt{x^2 + y_A^2}} - \frac{(x_B - x)}{\sqrt{(x_B - x)^2 + (y_B)^2}} = 0. \tag{21}$$

We can now recognise the sinus of the angles θ_1 and θ_2 and the equation becomes:

$$\sin\theta_1 = \sin\theta_2, \tag{22}$$

taking the sinus out we obtain the law of reflection :

$$\theta_1 = \theta_2. \tag{23}$$

The other solution, $\theta_1 = \pi - \theta_2$, can be discarded since it would not allow the ray of light to reach point B.

4.1.3 Demonstrating the law of refraction

We will now us Fermat's principle to demonstrate the law of refraction. The situation depicted in the right panel of Figure 9 is very similar to the one for the law of reflection, apart from the fact that we now have two medium of different refractive indexes. We will therefore proceed similarly than for the law of reflection with only minor changes. The time to travel from A to B through M is now :

$$t_{AMB} = \frac{n_1}{c}AM + \frac{n_2}{c}MB. \tag{24}$$

And the expressions of AM and MB are unchanged:

$$AM = \sqrt{x^2 + y_A^2} \tag{25}$$

$$MB = \sqrt{(x_B - x)^2 + (y_B)^2}. \tag{26}$$

We still want to find the stationary time (which is the minimum time in this case) which can be written:

$$\frac{dt_{AMB}}{dx} = 0. \tag{27}$$

Replacing t_{AMB}, $\overline{AM}$ and $\overline{MB}$ by their expressions we get that the shortest time is reached for:

$$\frac{n_1}{c}\frac{x}{\sqrt{x^2 + y_A^2}} - \frac{n_2}{c}\frac{(x_B - x)}{\sqrt{(x_B - x)^2 + (y_B)^2}} = 0. \tag{28}$$

This equation is equivalent to equation 21 but with factors n_i/c that do not cancel. Recognising the sinus of the angles θ_1 and θ_2 we obtain Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{29}$$

4.1.4 A case with maxima

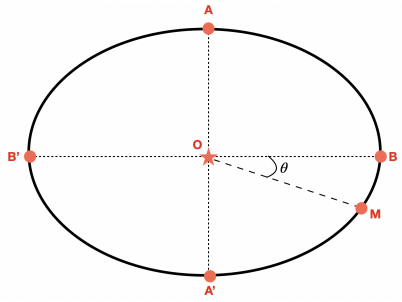


Figure 10: Ellipsoidal mirror. The observer situated in point O sees its reflection in points where the travel time is minimal (A and A') and where the travel time is maximal (B and B').

Fermat's principle as discussed above predicts that light will go through not only paths corresponding to a local minimum of travel time but also go through paths corresponding to local maximum of travel time. This can explain what happens, for example, inside an ellipsoidal mirror. As shown in figure 10, let's imagine an observer standing at

the centre of an ellipsoidal mirror, where does he see his own reflection ? Since we are only interested into the rays of light that bounce once on the mirror we will assume that the reflectance of the mirror is small (e.g. 10%), so that subsequent reflections won't be visible to us. Since light must go as a straight line from O to M and from M to O, the length of the path OMO will be $|OM|$. Because we are in an homogeneous media, the travel time will be stationary if the distance is stationary, which will happen if $|OM|$ if minimal or maximal. As the point M goes through the ellipse, the travel time reaches two minima (at A and A') and two maxima (at B and B'). As a consequence, Fermat's principle predicts that I can see four images of myself. This can also be intuitively understood by assuming that each part of the ellipsoidal mirror can be approximated by a plane mirror tangent to the ellipsoidal mirror surface and using the law of reflection for the plane mirror.

Now we can push the problem forward and wonder what happens if I am standing in the middle of a spherical mirror ? In the case of a spherical mirror, all the possible paths from O to M and back are exactly equivalent: the whole mirror acts as a stationary point and there are therefore an infinite number of images !

4.1.5 Justification of Fermat's principle.

We saw that Fermat's principle works very well to explain the laws of reflection and refraction. However we have not yet understood *why* it works so well. More that this, Fermat's principle seem to imply that light knows *a-priori* which path is the one of least time and consciously chooses it. Fermat had to wait until after his death and the development of the wave theory of light by e.g. Robert Hooke[6] or Christian Huygeens to have the physical basis to demonstrate his principle.

In order to understand what is happening, we have to imagine that the observer emits photons in all directions. As seen before, each photons that is emitted can be represented by an electromagnetic wave:

$$\Psi(x,t) = A\cos(\omega t - kx). \tag{30}$$

What we are interested to know is what is the phase difference between the emitted photon at point A and the received one at point B. If our photon leaves point A with a phase ϕ_0, it will arrive at B a time t after with the same phase, ϕ_0. However after a time t, the phase of the photon that is now at point A is different from ϕ_0 by ωt and: $\phi = \phi_0 + \omega_t$. At a time t the phase difference between point A and B is therefore $\phi - \phi_0 = \omega t$.

As we will see in much more details in Hillary term, waves with similar phases[7] sum up coherently, whereas waves with phases unrelated to each other cancel out. If two photons originating from A at different times arrive at a point B at the same time they will have a phase difference equal to :

$$\Delta\phi = \phi_2 - \phi_1 = \omega t_2 - \omega t_1. \tag{31}$$

Where t_1 and t_2 are the time it takes for photons 1 and 2 to go from point a to point B. This time is path dependent.

Let us now choose a path that goes through a point M that is situated on the x axis. We can imagine, for example that the x-axis is separating two medium of different optical index. What we are interested to know is the phase of photons arriving in B, coming from A and passing *through* M. This can be written :

$$\phi(x) = \phi_0 + \omega t(x). \tag{32}$$

Now we will look what happens for the photons that pass close to M. Do all the photons passing by a point M' situated at $x + dx$ arrive in B with a similar phase ? The phase of these photons can be written with a Taylor expansion:

$$\phi(x+\delta x) = \phi(x) + \frac{d\phi}{dx}\delta x + \frac{d^2\phi}{d^2x}(\delta x)^2 + o((\delta x)^3). \tag{33}$$

We can replace ϕ by its expression:

$$\phi(x+\delta x) = \phi(x) + \omega\frac{dt}{dx}\delta x + \omega\frac{d^2t}{d^2x}(\delta x)^2 + o((\delta x)^3). \tag{34}$$

[6]Robert Hooke 1635-1703 studied at ChristChurch college in Oxford. The Robert Hooke building in the physics department is named after him

[7]As we will see later, by similar phases we mean phases that differ by less than a wavelength.

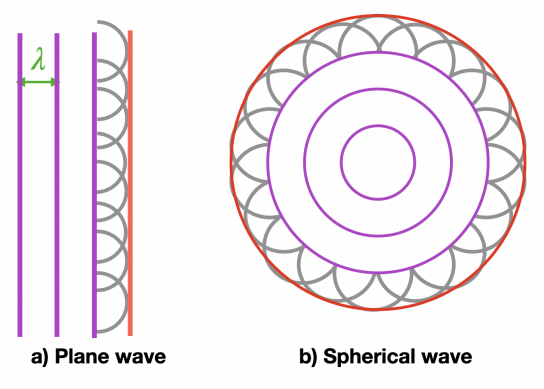


Figure 11: Huygens principle applied to the propagation of plane and spherical wave. The wavefront (purple) is source of an infinite number of secondary circular waves propagating in the same direction as the original wave (grey). The envelope of these waves creates the new wavefront (red).

We now see that if $\frac{dt}{dx} = 0$, then the light passing by M and by M' arrive at B with the same phase within the first order approximation in δx. That is Fermat's principle: light indeed takes all the possible path to go from A to B, but only the stationary ones, e.g. the ones with $\frac{dt}{dx} = 0$, interfere constructively. On all the other paths, the waves cancel each other and thus do not carry any energy.

4.2 Huygens principle

16 years after Fermat's principle was published Christiaan Huygens and much later Fresnel started thinking about the wave nature of light and derived what is now known as the Huygens-Fresnel principle. It states that when a lightwave is propagating, *every point on a wavefront can be considered the source of spherical wavelets with the same wavelength adn propagating in the same direction of the original wavefront. The envelope of these wavelength then forms the new wavefront*.

4.2.1 Direct propagation

As for Fermat's principle we must first verify that Huygens principle works in the simple case of direct propagation. As shown in figure 11, if we separate the wavefront of a plane wave into an infinite number of circular wavelengths propagating in the same direction then the envelope of these wavelengths is indeed a plane wave propagating in the same direction. Similarly, for a spherical wave, applying Huygens principle also leads to a spherical waves with the right properties.

4.2.2 Demonstrating the law of reflection

In order to demonstrate the law of reflection using Huygens' principle we need to go through a slightly complicated geometrical construction described in figure 12.

In the first step we draw the wavefront of an incoming plane hitting the mirror with an angle θ_1. The ray of light corresponding to the wave is shown has a red arrow. Following Huygens principle, each wavefront produces a new spherical wave when it hits the mirror. What we want is to find all the secondary waves produced by a specific wavefront when it hits the mirror and we will colour each of them in orange. The wavefront will first hit point P_0 at time t_0, then point P_1 at $t_0 + \lambda/c$, point P_2 at $t_0 + 3\lambda/c$ etc.. Each time a spherical wave, shown in orange, will be produced and propagate upward because of the reflection. Figure 12 shows the situation at $t_0 + 4\lambda/c$: the wave at P_0 has travelled 4 wavelengths away from the mirror, whereas the wave at P_1 has travelled only 3 wavelengths etc. We can do the same construction with the second wavefront, which will produce similar spheric waves late by a factor λ/c (shown in green), by the third wavefront (shown in turquoise) etc.

In the second step we look at the envelope of the waves created by each wavefront (e.g. the envelope of the waves drawn in orange, the one of the waves drawn in green etc.). The envelope of each of these waves will create the new wavefront. Although we have chosen wavefronts separated by a given value (chosen arbitrarily to be the wavelength), Huygens principle mentions an infinite number of secondary waves. So we can extrapolate from the four orange waves drawn that the envelope will be a plane wave. We draw the same envelope for the waves pictured in each colour and obtain three wavefronts of the reflected wave. We can now draw the ray of light corresponding to these wavefronts by

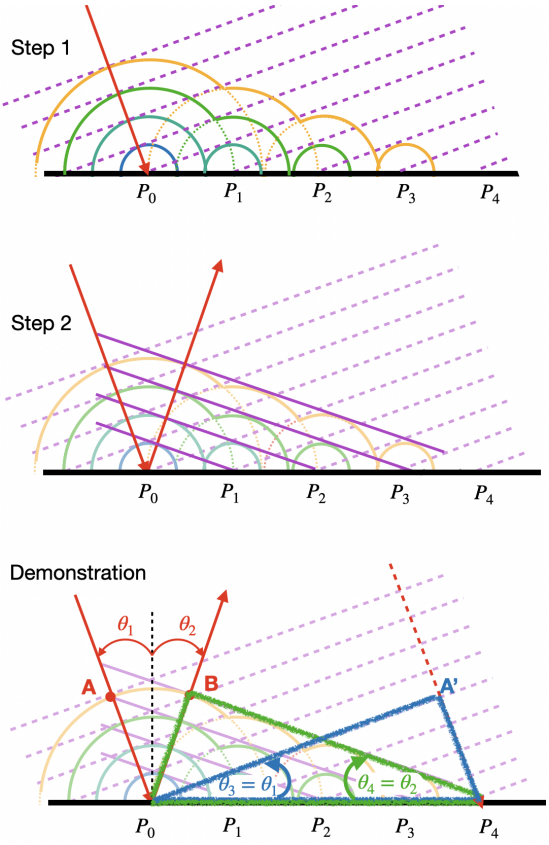


Figure 12: The law of reflection derived with Huygens' principle. In step 1 the incoming wave (purple) hits the surface of the mirror. Each wavefront is the source of a new semi-spherical wave propagating upwards with the same speed and same wavelengths. Since each wavefront hit the surface at time intervals λ/c, each create a wavelet shown as different colours. In step 2 we take the envelope of the wavelets to recover three different wavefronts of the reflected wave. The perpendicular to these wavefronts is the ray of light. The bottom panel defines important quantities to demonstrate the law of reflection (see text).

drawing the orthogonal of the wavefronts. If the original ray of light was hitting the mirror on point P_0, the reflected ray of light needs to also go through point P_0.

Now that we have drawn the incoming and reflected waves, can we demonstrate the law of reflection? Looking at the bottom panel of figure 12, what we need to show is that the angle θ_1 between the incident ray of light and the normal and θ_2 between the reflected ray of light and the normal are equal. We first define point A as being on the ray of light hitting P_0 and four wavefronts away from P_0. Similarly we define A' on the hypothetical ray of light hitting point P_4 and four wavefronts away from P_4. Finally let us define point B as being on the reflected ray of light, four wavelengths away from P_0. By construction we have $|A'P_4| = |AP_0| = |BP_0|$. We now look at right-angled triangle $P_0A'P_4$ and notice that since $\overline{AP_0}$ is orthogonal to $\overline{P_0A'}$ and that the normal is orthogonal to the mirror, we have that the angle $\theta_3 \equiv \angle P_4P_0A'$ is equal to θ_1. We can also the triangle P_0BP_4 to determine that the angle $\theta_4 \equiv \angle BP_4P_0$ is complementary to the angle $\angle BP_0P_4$ which is itself complementary of the angle θ_2 meaning that $\theta_2 = \theta_4$. Finally, using both triangles we can write:

$$\sin\theta_2 = \sin\theta_4 = \frac{|P_0B|}{|P_0P_4|} \tag{35}$$

and

$$\sin\theta_1 = \sin\theta_3 = \frac{|A'P_4|}{|P_0P_4|}. \tag{36}$$

Given that $|A'P_4| = |P_0B|$, the equation above implies that $\sin\theta_1 = \sin\theta_2$ and so that $\theta_1 = \theta_2$.

4.2.3 Demonstrating the law of refraction

We will now use Huygens' principle to find the ray of light refracted by a surface separated a medium of index n_1 and a medium of index n_2. The construction shown in Figure 13 is very similar to the one in Figure 12 and so we now show the first two steps at once. We consider an incoming plane wave on a surface. The wavefront hits first point P_0 then point P_1 etc. and every time it creates a wavelet that propagates downward. However, contrary to the reflection case,

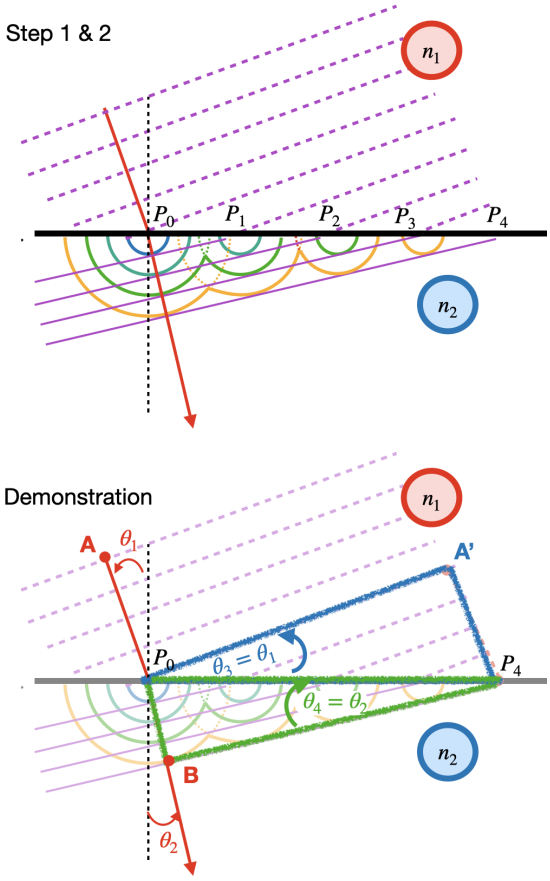


Figure 13: The law of refraction derived with Huygens' principle. In step 1 and 2 the incoming wave (purple) hits the surface separating the two medium of different optical index. Each wavefront is the source of a new semi-spherical wave propagating downward with a speed slower by n_1/n_2. Each wavefront hit the surface at time intervals λ/c and create a wavelet shown as different colours. We can take the envelope of the wavelets to recover three different wavefronts of the refracted wave. The perpendicular to these wavefronts is the ray of light. The bottom panel defines the quantities use to demonstrate the law of refraction (see text).

the new wave propagates at a smaller speed, because the medium has a larger optical index. Each wavelet at the points P_0, P_1, P_2 and P_3 are created with a time delay of $\lambda_0/(n_1 c)$, where λ_0 is the wavelength of the wave in the vacuum .

what we now need is to find a relationship between the angle θ_1 between the incident ray of light and the normal and the angle θ_2 between the reflected ray of light and the normal. Following the same demonstration as for the case of reflection, we define a point A on the incoming ray of light 4 wavefront away from O. Thus by definition we have:

$$|AP_0| = \frac{4\lambda_0}{n_1}. \tag{37}$$

In medium 2 the wavelet emitted by O propagates more slowly and thus in the same time it takes for the wave to travel from A to O in medium 1, the wave in medium 2 only propagates from O to B. Since B sits on the fourth wavefront away from O we have that:

$$|BP_0| = \frac{4\lambda_0}{n_2}. \tag{38}$$

As for the case of reflection we construct point A' on the hypothetical ray of light hitting point P_4 and four wavefronts away from P_4. By construction we now have $|A'P_4| = |AP_0|$ but, because the refractive index are different, we have $|A'P_4| = \frac{n_2}{n_1}|BP_0|$. We now look at right-angled triangle $P_0A'P_4$ and notice that since $\overline{AP_0}$ is orthogonal to $\overline{P_0A'}$ and that the normal is orthogonal to the mirror, we have that the angle $\theta_3 \equiv \angle P_4P_0A'$ is equal to θ_1. We can also the triangle P_0BP_4 to determine that the angle $\theta_4 \equiv \angle BP_4P_0$ is complementary to the angle $\angle BP_0P_4$ which is itself complementary of the angle θ_2 meaning that $\theta_2 = \theta_4$. Finally, using both triangles we can write:

$$\sin\theta_2 = \sin\theta_4 = \frac{|P_0B|}{|P_0P_4|} \tag{39}$$

and

$$\sin\theta_1 = \sin\theta_3 = \frac{|A'P_4|}{|P_0P_4|}. \tag{40}$$

Given that $|A'P_4| = \frac{n_2}{n_1}|BP_0|$, the equation above implies that $n_1 \sin\theta_1 = n_1 \sin\theta_2$, the law of refraction.

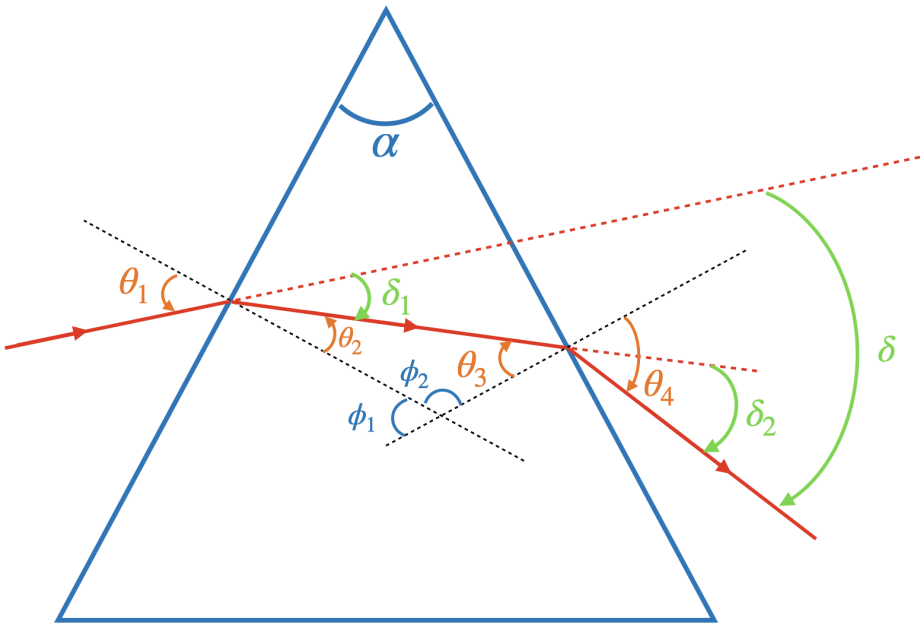


Figure 14: Ray diagram to calculate the deviation angle δ as a function of the prism angle α and the incidence angle θ_1.

5 The prism

5.1 Deviation angle.

A prism is a triangular object of angle at the top α and of optical index n usually used inside the air for which the optical index will be taken as unity. As shown in Figure 14, a ray of light entering the prism with an angle θ_1 is deflected and emerges with an angle θ_2. It then travels in straight line inside the prism and hits the other boundary with an angle θ_3 and emerges with an angle θ_4.

The total deviation that happens to a ray of light is called the deviation angle, δ. It is the angle from the direction of the incident ray and the direction of the outgoing ray of light. In order to correctly define it on a diagram, it is necessary to prolongate artificially the incident ray of light. Our goal is to find an expression for the deviation angle that only depends on the prism characteristics (e.g. α and n) and on the incident angle. For this we will follow an incident ray of light and count the deviation that happens at each boundaries. We define δ_1 and δ_2 the deviation that happen at the two boundaries. The total deviation angle will simply be the sum of the two $\delta = \delta_1 + \delta_2$.

δ_1 is the angle between the dashed red line and the normal minus θ_2. So :

$$\delta_1 = \theta_1 - \theta_2. \tag{41}$$

Similarly, δ_2 is equal to θ_4 minus the angle between the normal and the red dashed lines emerging from the prism:

$$\delta_2 = \theta_4 - \theta_3. \tag{42}$$

The total deviation angle is now:

$$\delta = \theta_4 + \theta_1 - (\theta_2 + \theta_3). \tag{43}$$

We can easily link the pairs of θ angles by Snell's law:

$$\sin\theta_1 = n \sin\theta_2, \tag{44}$$

$$\sin\theta_4 = n \sin\theta_3, \tag{45}$$

which give :

$$\delta = \theta_1 + \theta_4 - \left(\arcsin\frac{\sin\theta_1}{n} + \arcsin\frac{\sin\theta_4}{n}\right). \tag{46}$$

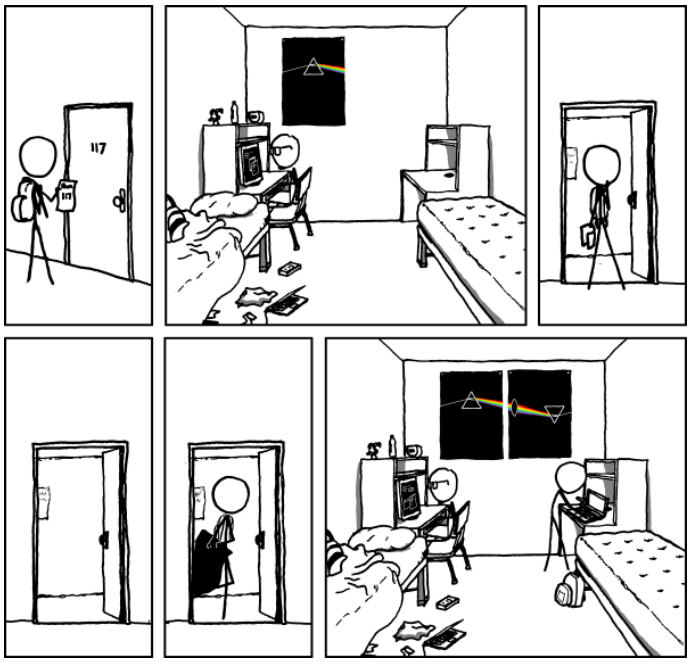


Figure 15: Prism can be used to split white light into many different colours. From XKCD (https://xkcd.com/964/)

but we need one more relationship to link θ_3 and θ_2. This one can be found by showing that the angle ϕ_1 is equal to α. Indeed ϕ_1 is the angle between the normal of the prism sides, so is the same angle as the angle between the prism sides. As a consequence we obtain that $\phi_1 = pi - \alpha$. Finally using the fact that the sum of angles in a triangle is always π we get that $\theta_2 + \theta_3 + \pi - \alpha = \pi$ which can be written as:

$$\theta_2 + \theta_3 = \alpha. \tag{47}$$

Putting it all together we obtain an expression for the deviation angle:

$$\delta = \theta_1 - \alpha + \arcsin\left(n \sin\left(\alpha - \arcsin\left(\frac{1}{n}\sin\theta_1\right)\right)\right). \tag{48}$$

The deviation angle is an increasing function of the optical index. If the material has an optical index that depends on wavelength, the deviation angle will be different for different colours and the prism can be used to separate a white light beam into a colorful rainbow.

5.2 Minimum deviation angle.

We now want to know how the deviation angle varies with the incident angle. We plotted equation 48 in in Figure 16 for the case of a prism of optical index $n = 1.33$ and angle $\alpha = 28$, 60 and 90°. The equation is undefined for small and large incident angles, this corresponds to the case of an internally reflected ray for which a more complicated calculation would be needed to calculate the deviation angle.

We now want to find an expression for the minimum deviation angle. For this we could take the derivative of equation 48 and set it equal to zero, but that would be a tedious task. Instead, we will use the fact that all optical path are reversible.

We first note from equation 46 that δ can be written as a symmetric function of θ_1 and θ_4 (meaning that $\delta(\theta_1, \theta_4) = \delta(\theta_4, \theta_1)$. Let us assume that the minimum of deviation δ_m is reached for $(\theta_{1m}, \theta_{4m})$. Given that the path of light are reversible, we can consider a ray of light incident with an angle θ_{4m} and know that it would would emerge with an angle θ_{1m}. If θ_{4m} were different than θ_{1m}, the function δ would have a minimum for both θ_{4m} and θ_{1m}. Since we know from Figure 16 that the function has only one minimum we conclude that at the minimum we have $\theta_4 = \theta_1$.

If $\theta_4 = \theta_1$ then the ray of light must be parallel to the base of the prism while travelling inside the prism. This means that $\theta_2 = \alpha/2$ and thus that $\theta_1 = \arcsin(n \sin(\alpha/2))$ and so that the deviation angle is given by :

$$\delta_m = 2\arcsin(n \sin(\alpha/2)) - \alpha \tag{49}$$

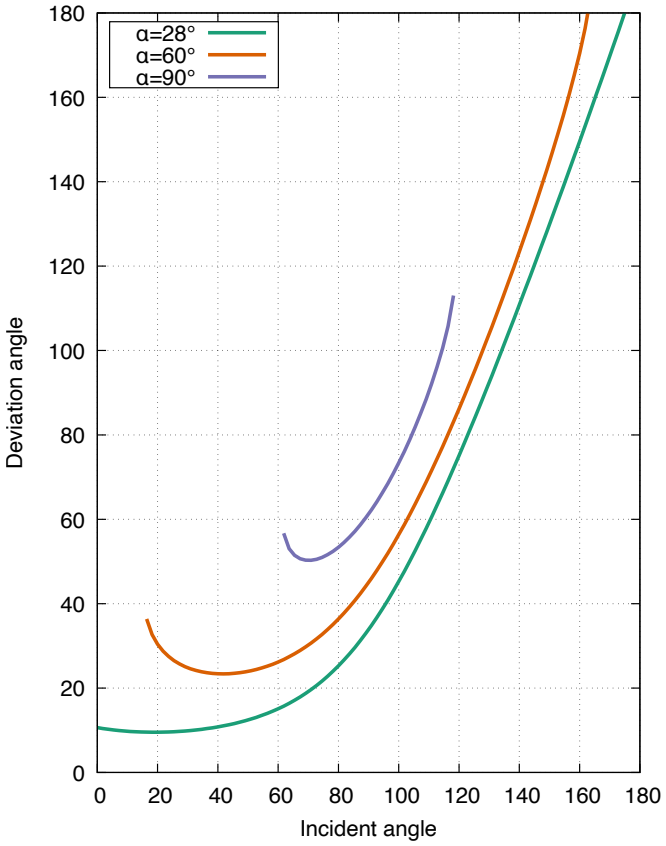


Figure 16: Minimum deviation angle of a prism as a function of the incident angle calculated with equation 48 for three different values of the apex angle α. The optical index of the prism is $n = 1.33$

5.3 Application to crystals and halos

Ice crystals are responsible for most of the halos that can be seen around the sun or the moon. In order to calculate the location of these halos a useful trick is to approximate the crystal as a prism like we do in Fig. 17. For the case of hexagonal crystals the apex angle of the prism subtended by two alternative sides is 60°.

Hexagonal ice crystals that stay suspended in the atmosphere often orient themselves so that their flat face is facing the ground. However, all possible orientations along this axis are possible. If we imagine a collection of such crystals illuminated by the parallel rays from the sun, then we can see that the incident angle is homogeneously sampled. by the solar rays. Otherwise said, there are as many crystals oriented with an angle between $\theta_0 - d\theta/2$ and $\theta_0 + d\theta/2$, whatever the value of θ_0. All the rays entering crystals with an angle within $[\theta_0 - d\theta/2, \theta_0 + d\theta/2]$ will emerge with a deviation angle between $[\delta_0 - d\delta/2, \delta_0 + d\delta/2]$. The width of the interval $d\delta$ is given by:

$$d\delta = \left.\frac{\partial \delta}{\partial \theta}\right|_{\theta=\theta_0} d\theta. \tag{50}$$

For the minimum deviation angle, θ_m, the deviation angle $\delta(\theta)$ has a minimum and so $d\delta = 0$. This means that all the crystals oriented with an angle θ_m within $d\theta$ deflect the sunlight with a deviation angle very close to δ_m. This means that there will be more crystals that deflect the light with an angle equal or close to δ_m than any other angle.

If the observer is looking some crystals near the sun, the deflection angle is equal to the angle between the sun and the crystals. Because we just demonstrated that more light will cross the crystals with a deflection angle close to the minimum value, the observer sees an increase of light at this angle. For the most common type of crystals, hexagonal crystals, the minimum deviation angle is 22° and gives rise to one of the most common halo, the 22° halo. This one can often be seen by hiding the sun with your hand (or somebody's head as in Fig. 17). The deviation angle can never be smaller than the minimum deviation angle, meaning that the inner edge of the halo has always a sharp edge while the outer edge is a continuous light distribution. Furthermore, the optical index of the crystal varies with wavelength, meaning that the minimum deviation angle that depends on wavelength, which gives a slight coloration to the halo. Different types of crystals lead to different equivalent prism angles which have different minimum for the deviation

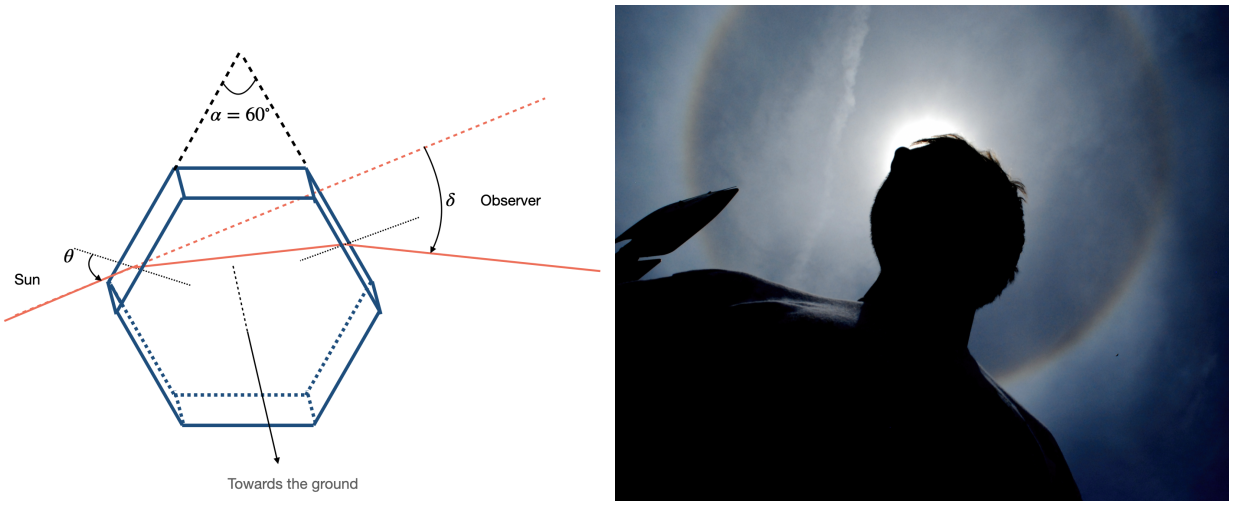


Figure 17: Left: An hexagonal ice crystal can be considered as prism of apex angle 60°. Right: a 22° halo around the sun (credit: V. Parmentier).

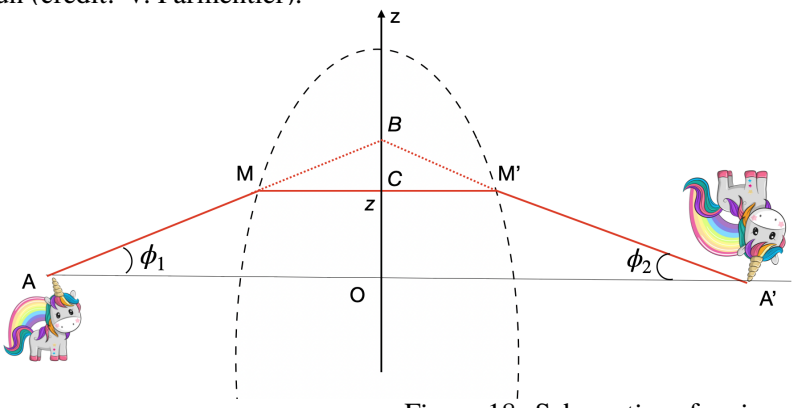


Figure 18: Schematics of an imaging device.

angle leading to multiple type of halos.

A similar reasoning can be applied to understand the formation of rainbow, which is one of the problems of the problem set.

6 Lenses

6.1 General case

Imaging systems are devices that aim to produce an image of an object. As shown in Figure 18, for this to happen we need that *all* the rays of light leaving from a single point A in the object arrive at a single point A' in the image, *whatever their point of entry M in the device*. Since Fermat's principle tells us that light always takes the fastest path from A to A', this means that the optical path of all the rays emerging from A and arriving to A' must be the same and equal to an extrema. It is clear from Fig 18 that usually the optical path out of the imaging system, $L_{AM} + L_{M'A'}$, are not independent of M. Our goal is therefore to build an imaging system so that the optical path $L_{MM'}$ inside the imaging system compensates the change of optical path with M outside of the imaging system.

6.1.1 Condition for an image formation

We now aim to build one of the simplest imaging system. We are going to assume that our imaging system has a thickness $d(z)$ that depends on z, an homogeneous optical index n and that it is in air where the optical index is $n = 1$. We further make the *paraxial approximation*. This means that we only consider rays that make a small angle compared to the optical axis. This means that inside the lens, the angles will also remain small and that we can assume that MM' is parallel to the optical axis, so that $|MM'| \approx d(z)$. Looking at Figure 18, we see that optical path from A to A' of a

ray crossing the lens at a heigh z will therefore be:

$$L(z) = |AM| + n|MM'| + |M'A'|. \tag{51}$$

Fermat principle tells us that in order to make an image, we need all the rays emerging from A, crossing the lens and arriving into the image point A' to have the same optical path, whatever the height z of the point M where they enter the lens. So our goal is to express the optical path as a function of z and find at which condition the optical path is independent of z. We can first write $|AM|$ and $|M'A'|$ as a function of z:

$$|AM| = \sqrt{\left(|AO| - \frac{d(z)}{2}\right)^2 + z^2} \tag{52}$$

$$|M'A'| = \sqrt{\left(|OA'| - \frac{d(z)}{2}\right)^2 + z^2}. \tag{53}$$

We now assume that our lens is thin, meaning that we can neglect $d(z)^2$ compared to z^2 and that $|AO| >> d(z)$. Additionally the paraxial approximation tells us that $z << |AO|$, which leads to :

$$|AM| = |AO|\sqrt{1 - \frac{d(z)}{|AO|} + \left(\frac{z}{|AO|}\right)^2} \approx |AO| - \frac{d(z)}{2} + \frac{z^2}{2|AO|} \tag{54}$$

$$|M'A'| = |OA'|\sqrt{1 - \frac{d(z)}{|OA'|} + \left(\frac{z}{|OA'|}\right)^2} \approx |OA'| - \frac{d(z)}{2} + \frac{z^2}{2|OA'|}. \tag{55}$$

We can now calculate the full path length by replacing each terms in eq. 51 by their value:

$$L(z) = |AO| + |OA'| + (n-1)d(z) + \frac{z^2}{2}\left(\frac{1}{|AO|} + \frac{1}{|OA'|}\right). \tag{56}$$

We now look for the shortest optical path, meaning the path such that the derivative is zero:

$$\frac{\partial d(z)}{\partial z} = -\frac{z}{n-1}\left(\frac{1}{|AO|} + \frac{1}{|OA'|}\right). \tag{57}$$

This is satisfied for :

$$d(z) = d(0) - \frac{z^2}{2(n-1)}\left(\frac{1}{|AO|} + \frac{1}{|OA'|}\right). \tag{58}$$

This demonstrate that a piece of glass shaped as a parabola should be able to image an object situated in A. However, what we want is a device that works for all potential objects and images, no matter what is their position. From the above equation, this would work only if :

$$\frac{1}{|AO|} + \frac{1}{|OA'|} = \frac{1}{f}, \tag{59}$$

with f a constant that we will call the focal length for reasons we will explain later. We will now find specific systems where this condition is valid.

6.1.2 Convex lens

We now try to create an imaging device by making a lens shape defined by the intersection of two spheres (see Figure 19). This is the easiest kind of lens to make. Let us call O_1, R_1 and O_2, R_2 the centre and radius of the two spheres. Let us assume for now that the lens is convex, meaning that $|O_1O_2| < R_1 + R_2$.

What we want is to find the point A' where all the rays originating from A converge after hitting the lens, whatever the heigh z where they enter the lens. In order to solve this problem with the tools we have we will define an equivalent prism for each possible ray entering the lens at each possible point M situated at a heigh z. The equivalent prism is

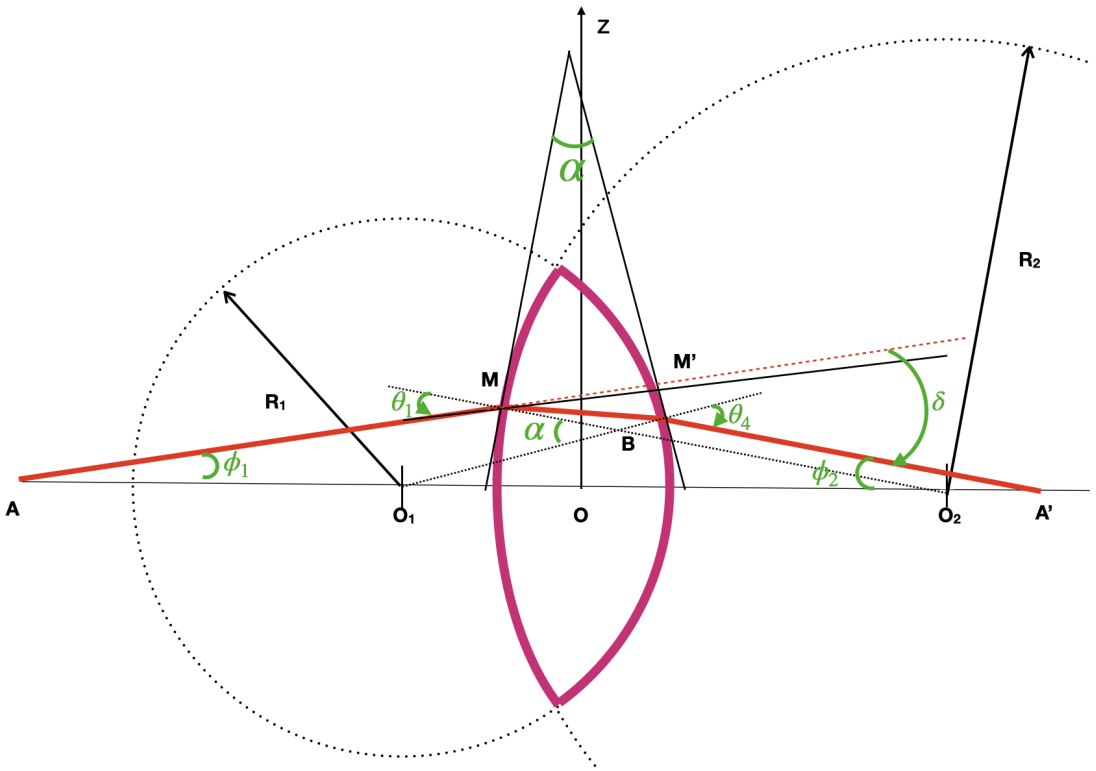


Figure 19: Convex lens schematic.

defined by the tangent at the lens where the ray of light enters and exits the lens. From Figure 19, we can consider the triangle made by O_1O_2 and the normal to the lens sides. We use the *paraxial approximation*, which assumes that all the angles between the rays and the optical axis are small. We further use the *thin lens approximation*, which assumes that the width of the lens is small compared to the object and the image distances. If these two approximations are valid we can assume that M and M' are both situated at heigh z and that $\angle BO_1O_2 \approx z/R_1$ and $\angle BO_2O_1 \approx z/R_2$, leading to an expression for the apex angle,α, of our equivalent prism:

$$\alpha = \frac{z}{R_1} + \frac{z}{R_2}. \tag{60}$$

In the paraxial approximation and looking at triangles AMO_2 and $O_1M'A'$, the incident and emergent angles, θ_1 and θ_4 can be written as :

$$\theta_1 \approx \phi_1 + \frac{z}{R_2} \approx \frac{z}{AO} + \frac{z}{R_2} \tag{61}$$

$$\theta_4 \approx \phi_2 + \frac{z}{R_1} \approx \frac{z}{OA'} + \frac{z}{R_1}. \tag{62}$$

We can now use our equation for the deviation angle of the prism in the limit of small apex and small incident angles. In that limit equation 46 and 48 become:

$$\delta = (n-1)\alpha \tag{63}$$

$$\delta = \frac{n-1}{n}\left(\theta_1 + \theta_4\right). \tag{64}$$

Combining these two equations and using our derived expressions for θ_1 and θ_4 we obtain:

$$\frac{1}{|AO|} + \frac{1}{|OA'|} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{65}$$

This equation shows that the quantity $\frac{1}{OA'} + \frac{1}{AO}$ is indeed constant whatever A for a lens made of two spherical sides ! Given that the equation also shows that $\frac{1}{OA'} + \frac{1}{AO}$ is independent of z we conclude that the convex lens is an imaging

device. In equation 58 we argued that $d(z)$ must decrease as z^2 for an imaging device. This is indeed true for a lens made of two sphere but only for small angles. Indeed, the width of the lens can be written:

$$d(z) = (R_2 \cos\phi_2 + R_1 \cos\phi_1) - O_1O_2 \tag{66}$$

$$\approx R1 + R2 - O_1O_2 - \frac{\phi_2^2 + \phi_1^2}{2} \tag{67}$$

$$\approx d(0) - z^2\left(\frac{1}{2R_1} + \frac{1}{2R_2}\right). \tag{68}$$

As we will see in a future chapter, departure from the parabolic shape at large angles often leads to blurring in the resulting image called aberations.

Equation 65 was derived under both the thin lens approximation and the paraxial approximation. In the second order in $d(z)$ it becomes:

$$\frac{1}{|AO|} + \frac{1}{|OA'|} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{n-1}{n}\frac{d(0)}{R_1R_2}\right). \tag{69}$$

6.1.3 Concave lens

We now try to again to create an imaging device by making a lens shape defined by the intersection of two spheres but this time we are assuming that $|O_1O_2|$>$R_1 + R_2$, leading to a concave lens (see Figure 20).

We will proceed as for the case of the convex lens. What we want is to find the point A' where all the rays originating from A converge after hitting the lens, whatever the heigh z where they enter the lens. In order to solve this problem with the tools we have we will define an equivalent prism for each possible ray entering the lens at each possible point M situated at a heigh z. The equivalent prism is defined by the tangent at the lens where the ray of light enters and exits the lens. From Figure 20, we can consider the triangle made by O_1O_2 and the normal to the lens sides. Using the paraxial approximation and the thin lens approximation we can assume that M and M' are both situated at heigh z and that $\angle BO_1O_2 \approx z/R_1$ and $\angle BO_2O_1 \approx z/R_2$, leading to an expression for the apex angle α:

$$\alpha = \frac{z}{R_1} + \frac{z}{R_2}. \tag{70}$$

The expression for the incident and emergent angles, θ_1 and θ_4, is now a bit different from the convex case:

$$\theta_1 \approx -\phi_1 + \frac{z}{R_1} \approx -\frac{z}{AO} + \frac{z}{R_1} \tag{71}$$

$$\theta_4 \approx \phi_2 + \frac{z}{R_2} \approx \frac{z}{OA'} + \frac{z}{R_2}. \tag{72}$$

We can now use our equation for the deviation angle of the prism in the limit of small apex and small incident angles. In that limit equation 46 and 48 become:

$$\delta = (n-1)\alpha \tag{73}$$

$$\delta = \frac{n-1}{n}(\theta_1 + \theta_4). \tag{74}$$

Combining these two equations and using our derived expressions for θ_1 and θ_4 we obtain:

$$\frac{1}{|AO|} - \frac{1}{|OA'|} = -(n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{75}$$

This formula for the concave lens is very similar than the one for the convex lens, apart from two negative signs.

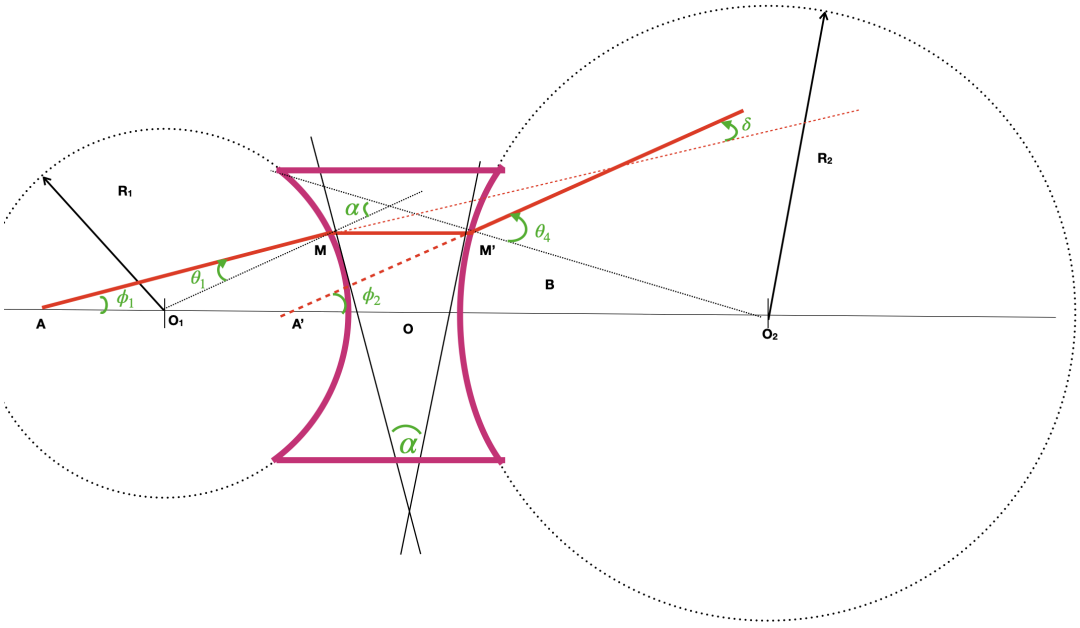


Figure 20: Concave lens schematic.

6.1.4 Sign convention

Because having different formulas for the concave and the convex case is not convenient, we can try to merge them into one formula. But for this we need to define new quantities that will change sign depending on the convex or concave nature of the lens. Because there are two minus signs in the convex formula compared to the concave formula, there can be multiple ways to define quantities that work for both concave and convex lenses. These different definitions are known as *sign conventions*.

In this course we will use the "real is positive" convention. For this we define a direction to our optical axis and use the centre of the lens as the zero point. We define the left hand side of the lens has the *object land* and the right hand side of the lens as the *image land*. An object is said *real* if it lives in object land, which is on the left side of the lens. In that case we call u the distance $|OA|$ and we have $u > 0$. If the object is on image land, which is on the right side of the lens then it is a virtual object and we define u as $u = -|OA|$ and thus have $u < 0$.

Similarly we note the distance between the lens and the image as v. Real images are situated on the right side of the lens and in that case $v = |OA'| > 0$. Virtual images are situated on the left side of the lens and in that case $v = -|OA'| < 0$.

We can now merge equation 75 and 65 into a single equation:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \tag{76}$$

where f is called the *focal distance*. The focal length is *positive* for *convex* lenses and *negative* for *concave*. It is defined by the lensmaker formula derived earlier:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{77}$$

The convention we use is that $R_i > 0$ for a convex interface and $R_i < 0$ for a concave interface. For a lens made of two convex interface, $f = +(n-1)(1/|R_1| + 1/|R_2|)$ is positive. For a lens made of two convex interfaces, $f = -(n-1)(1/|R_1| + 1/|R_2|)$ is negative.

If consider a thick lens rather than a thin lens, the lensmaker equation becomes:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{n-1}{n}\frac{d(0)}{R_1 R_2}\right). \tag{78}$$

We also need a convention in the vertical axis. For this the usual convention is to define as positive any distance measured from the optical axis up and negative any distance measured from the optical axis down.

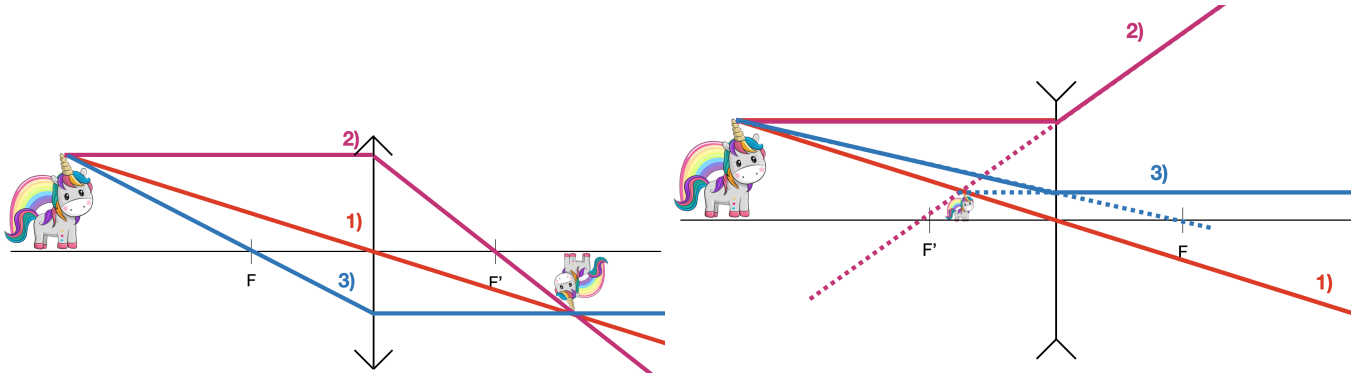


Figure 21: Ray tracing examples for a concave lens with a positive focal length (top) and a convex lens with a negative focal lens (bottom).

The lens *power* is defined as the inverse of the focal lens and noted $\delta = 1/f$. It is measured in $m^{-}1$ also called *dioptre*. As shown by the lens-maker formula, when combining multiple curved surface the power of the lenses sum up together. So two lenses of power δ_1 and δ_2 next to each other will have a power $\delta_1 + \delta_2$.

Other sign convention exist, which makes the whole optics a bit confusing, particularly if you try to look for formulas on the internet, always verify which sign convention is used. Note that the sign convention for the distances u and v and for the sign of the radius of curvatures of the lens are independent from each other.

6.2 Ray tracing

We can now use the lens formula to calculate the position of an image or an object in specific cases. A real object situated at a distance of f from the lens on the object side has $u = f$ and the solution to the thin lens formula becomes $v = \infty$. We call that point F, the object focal point. So an object on the object focal plane will have an image at infinity. This also means that all rays emitted from a point situated on the object focal plane emerge from the lens paralel to each other. The angle between these rays and the optical axis depends on the vertical position of the object in the focal plane. For an object at the focal point, the rays all emerge parallel to the optical axis.

Similarly, if an object is at infinity then $u = \infty$ then the image is at $v = f$. We call this point the image focal point and name it F'. For an object at infinity, all rays arrive parallel to each other. For an object at infinity on the optical axis all rays arrive parallel to the optical axis.

Using these two characteristics, we can very simply draw the image of an object. For this we will first simplify our drawing of a lens and will represent a concave (or convergent) lens with ↕. A convex (or divergent) will be the same symbol but with inverted arrow tips. For a concave lens the focal length is positive and we define F as the object focal point. It is the point on the optical axis that is on the left side of the lens and situated at a distance f from the centre of the lens. We define F' as the image focal point. It is the point on the optical axis that is on the right side of the lens and situated at a distance f from the centre of the lens.

In order to draw the image we can use three special rays.

- The first ray is the one going through the centre of the lens and it is always undeflected.
- The second ray is the ray going through the object focal point F, it comes out parallel to the optical axis.
- The third ray is the ray arriving parallel to the optical axis it comes out passing through the image focal point F'.

The image of a point is found at the intersection fo these three rays.

There is a problem if our object is on the optical axis, since in that case all three rays are the same one. To solve this we need to invoke the aplanetism of the lens. The aplanetism is the approximation that if an object is moved orthogonally to the optical axis its images also moves orthogonally to the optical axis. So in order to find the image of a point situated on the optical axis we can first find the image of a point just above or just below the optical axis.

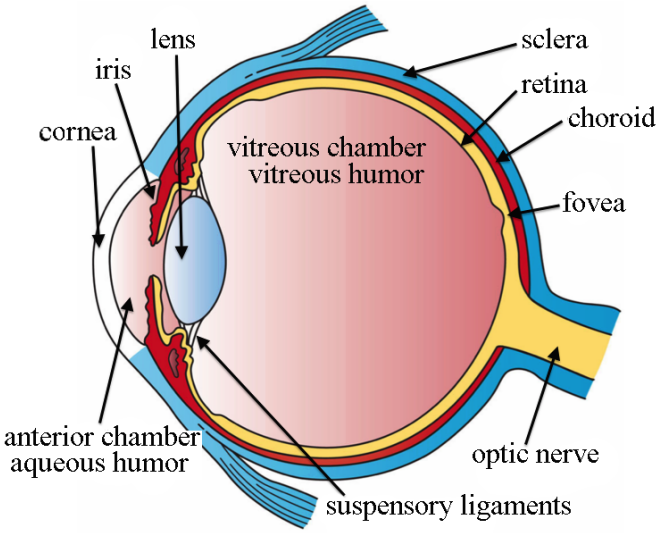


Figure 22: Schematic representation of the eye. Source wikipedia.

6.3 Simple lens systems

6.3.1 The eye

The eye is a complex optical system whose purpose is to image an object at a given distance in the retina. For this the eye has two lenses. The fist one is the cornea which is fixed, has a refractive index of 1.376 and provides 2/3 of the converging power of the eye. The second one is the lens, which is made of a transparent proteins with a refractive index varying from 1.386 and 1.406, it separates the vitreous humour and the aqueous humour, two fluids with a refractive index close to 1.336. The lens is holded by the ciliary muscles. By contracting they can change the shape of the lens and thus its focal length, this is how the eye can see a well-defined image of objects situated at different distances. This process is known as accommodation.

We will represent this whole complex system as a single lens with a varying focal length. A normal eye is at rest (meaning the ciliary muscles are not working) when imaging an object situated at infinity. For the adult the size of the eye is roughly 24mm, and so this is the focal lens of the eye at rest and generally the longest focal length. When accommodating the eye can reduce its focal length, allowing it to make sharp images of objects closer than infinity. For the adult eye the closest point that can be imaged is called the *Puntum Proximum* and is often taken as $PP = 25cm$. Using the thin lens formula with an object at the Puntum Proximum ($25cm$) and an image on the retina ($24mm$) we see that the maximum focal length of the eye is roughly $22mm$. Overall that human eye can therefore change the power of its lens by 4 to 5 dioptre. This changes considerably with age, ranging from 15 dioptre at 10 years old to 1 dioptre at 70 years old.

When the eye at rest is focuses an object at infinity before the retina, then it becomes blury. Because the eye can only reduce its focal length by accommodating, this can't be corrected by accommodation. The person is then nearsighted or myopic. In order to correct this one needs to increase the focal length of the eye, which means decreasing the power of the eye. It therefore needs a divergent lens, with a negative dioptre power. If the furthest sharp object I can see is situated 1m from me, I need glasses that will make an object positioned at infinity appear to be at 1cm from me. using the thin lens equation with $u = \infty$ and $v = -1m$ I get that I need a lens of focal lens $f = -1m$ and so a power of $\delta = -1\,\mathrm{D}$

When the eye at maximum contraction cannot see a sharp image of an object that is close by, the person is long-sighted or presbyopic. A comfortable reading distance is roughly 25cm, which is often taken as the standard Puntum Proximum. If somebody cannot see a sharp image of objects closer than 70cm his Puntum Proximum is then at 70cm. To bring it back at 25cm I need glasses that will make an object positioned at 25cm appear to be at 70cm. Using the thin lens equation with $u = 25cm$, $v = -75cm$ I obtain that I need glasses with a positive focal length $f = 38cm$, corresponding to $\delta = +2.5\,\mathrm{D}$.

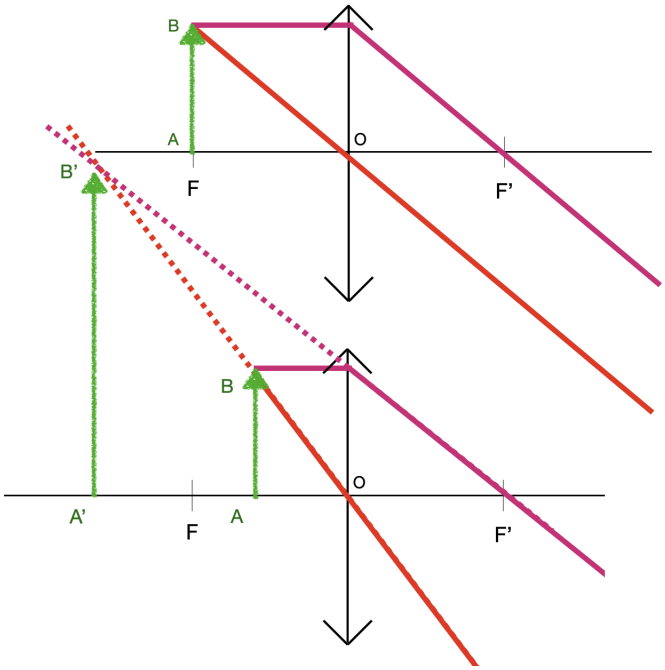


Figure 23: Two ways of using a magnifying glass either by placing the object at the focal point (top), or by placing the image at the Puntum Proximum (bottom).

6.3.2 The magnifying lens

A magnifying lens is a converging lens whose purpose is to produce an image that has an angular size bigger than the angular size the same object would have if situated at the Puntum Proximum and looked at without a lens.

It can be used in two different configuration. In the first configuration the object is placed at the focal point of the lens so that the image is created at infinity. This has the advantage that the eye does not have to make any effort to look at the image. In that case the angular size of the object through the lens is equal to L/f with L being the vertical size of the object, whereas the angular size of the object seen without any lens at the Puntum Proximum would be L/PP with PP the Puntum Proximum distance (usually taken as 25cm). The magnification of the lens is then defined as the ratio of the two quantities:

$$M = \frac{L/f}{L/PP} = \frac{PP}{f}. \tag{79}$$

The lens can also be used to get a larger magnification. For this the magnifying glass is adjusted so that a virtual image of the object is formed at the Puntum Proximum. In that case we can use the lens equation to find the position of the object:

$$\frac{1}{u} - \frac{1}{PP} = \frac{1}{f} \tag{80}$$

leading to:

$$u = \frac{1}{\frac{1}{f} + \frac{1}{PP}}. \tag{81}$$

Because the image is at the Puntum Proximum the magnification is directly the ratio of the size of the image to the size of the object. As can be seen in Fig. we get that:

$$M = \frac{|A'B'|}{|AB|} = -\frac{v}{u} = PP\left(\frac{1}{f} + \frac{1}{PP}\right) = 1 + \frac{PP}{f}. \tag{82}$$

In that second case the magnification is larger.

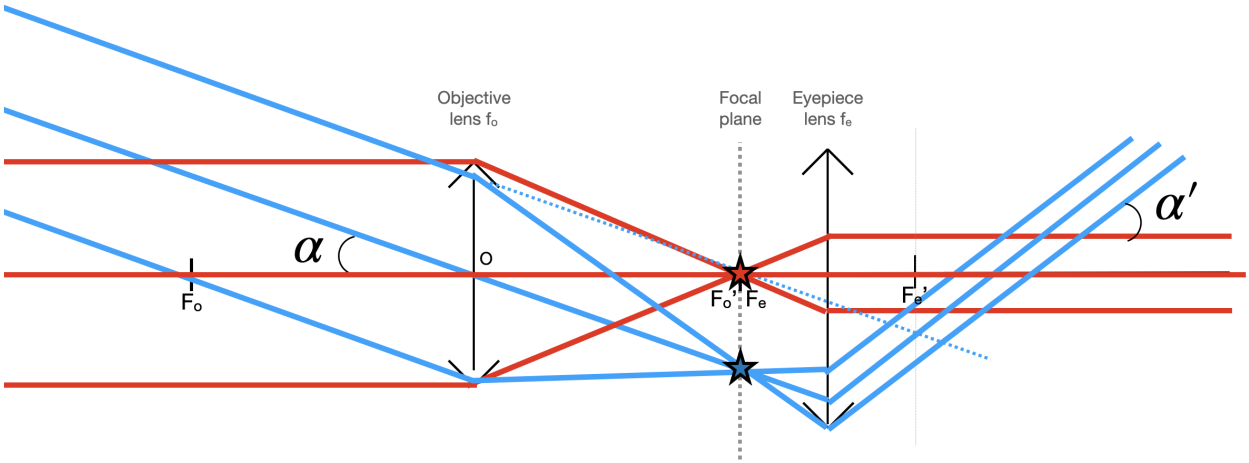


Figure 24: A Gallilean telescope composed of an Objective and an Eyepiece looks at two stars situated at infinity.

6.4 Complex lens systems

6.4.1 The refracting telescope.

The purpose of a telescope is to take a real object situated at infinity and make a real image of it situated at infinity. Ideally the telescope works so that the image has a larger angular size as the object. The telescope is composed of two converging lenses, an objective of focal lens f_o and an eyepiece of focal lens f_e. The two lenses are arranged so that the object focal point of the eyepiece coincides with the image focal point of the objective.

The Figure 25 shows how a telescope increases the angular distance between two stars separated by an angle α. First the objective makes an image of the two stars on its image focal plane. The size of this image is $f_o\alpha$. Then this image serves as an object for the eyepiece lens, which makes an image at infinity. As a consequence we can write the angular separation between the images of the two stars as $\alpha' = f_o\alpha/f_e$. The magnification of the telescope is therefore given by:

$$M = \frac{\alpha'}{\alpha} = \frac{f_o}{f_e}. \tag{83}$$

The larger the objective focal lens the larger the magnification. However larger objective focal lens means longer telescopes, since the eyepiece must be very close to the objective image focal point. In order to increase magnification one could also decrease the eyepiece focal length. This is however not a perfect solution because lenses with very small focal lengths are lenses with strongly curved surfaces which can lead to strong optical aberrations.

In the Objective focal plane one can easily place a device like a mask that can block the light of one of the sources but not the other. This technique, called chronography, is used for exemple to image planets around other stars. Because a planet is much fainter than its host star one need to block the light of the star in order to see the planet.

6.4.2 The coumpound microscope.

The goal of a microscope is to make a large image at the Puntum Proximum of an object that is very small. Similarly to the telescope it uses an objective lens of focal f_o and an eyepiece of focal f_e, but contrary to the telescope the object that we are trying to image is not at infinity but close to the first lens. Let us assume our object is AB situated at a distance $u > 0$ from the objective lens. The image $A'B'$ is situated at a distance v on the image plane of the objective lens. This real image will serve as real object for the eyepiece lens that will make a virtual image $A'B'$. We call u and v the object and image through the first lens and u' v' the object and image through the second lens.

We now adjust the length L of the microscope so that the final image is at the Puntum Proximum of the observer. So $v' = -PP$. The magnification of the microscope is the ratio of the angular size of the object if it were at the Puntum Proximum and the angular shape of the image. Since our image is situated at the Puntum Proximum, the magnification becomes simply:

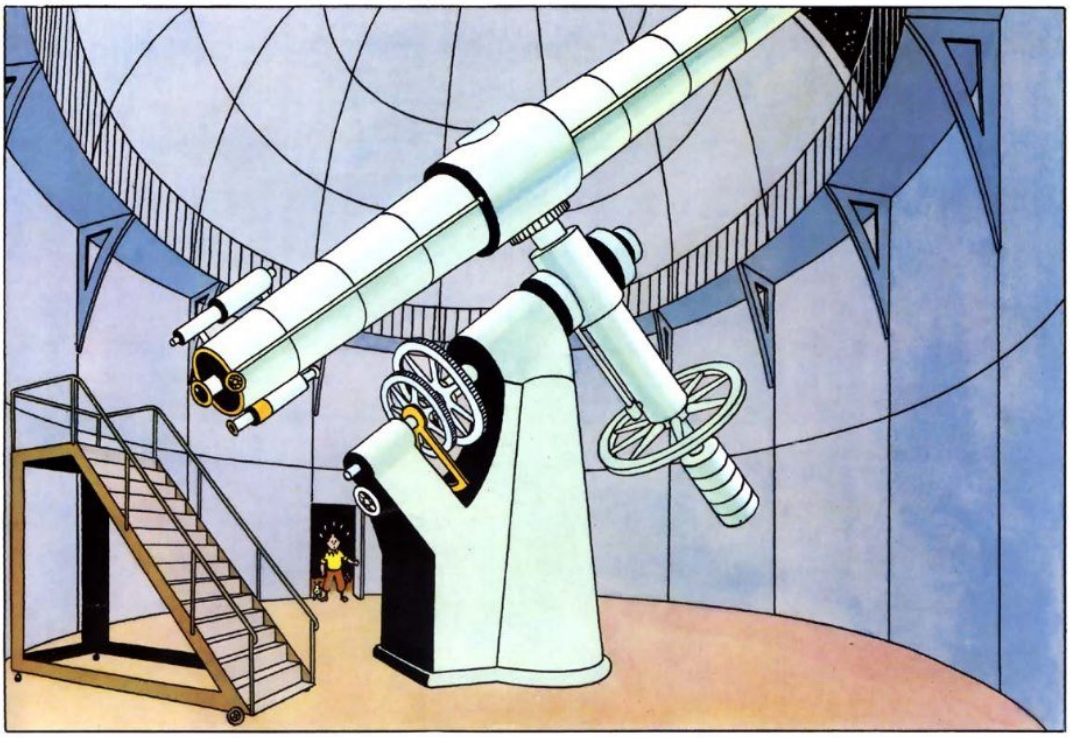


Figure 25: The refracting telescope from the Yerkes observatory near Chicago depicted in the comic book "Tintin et l'étoile mysterieuse". The refracting telescope had a focal length of 20m, one of the largest refractor ever built.

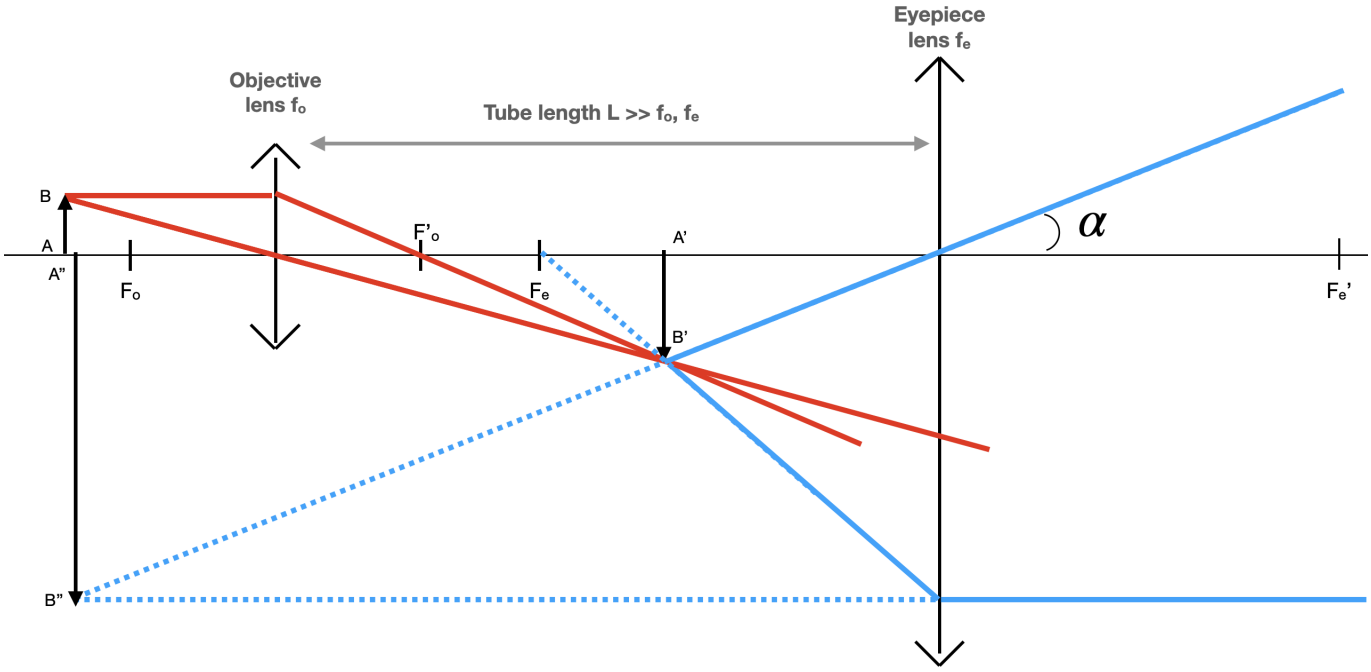


Figure 26: A compound microscope composed of an objective and an eyepiece makes a bigger image of a small object and places that image at the puntum proximum.

$$M = \frac{A''B''}{AB}. \tag{84}$$

In order to calculate the total magnification we will calculate the magnification from the objective and from the eyepiece separately. The magnification from the objective is:

$$M_o = \frac{A'B'}{AB} = \frac{v}{u} = \frac{v}{f_o} - 1, \tag{85}$$

where the last equality results from the thin lens equation applied to the objective. Similarly we can write for the eyepiece that:

$$M_e = \frac{A''B''}{A'B'} = \frac{v'}{u'} = \frac{v'}{f_e} - 1. \tag{86}$$

The total magnification is then:

$$M = M_o M_e = \left(\frac{v}{f_o} - 1\right)\left(\frac{v'}{f_e} - 1\right). \tag{87}$$

In this equation we know $v' = PP$ but we do not know v. To determine v we first find a relationship between v, u' and tube length by looking at Figure 26:

$$v + u' = L. \tag{88}$$

Using the thin lens equation for the eyepiece we can express u' as a function of v':

$$u' = \frac{v' f_e}{v' - f_e}. \tag{89}$$

Replacing in the previous equation we get an expression for v as a function of v':

$$v = L - u' = L - \frac{v' f_e}{v' - f_e} = \frac{v'(L - f_e) - L f_e}{v' - f_e} \approx \frac{L(v' - f_e)}{v' - f_e} \approx L \tag{90}$$

Finally, we replace v by its expression in equation 87 and set $v' PP$ to obtain the final equation:

$$M = \frac{A''B''}{AB} = \frac{(PP - f_e)(L - f_o)}{f_e f_o}. \tag{91}$$

Usually the eyepiece and the objective focal lengths are no larger than a few centimeters, meaning that we can simplify the above equation by assume $f_e << PP$ and $f_o << L$, leading to:

$$M \approx \frac{L \times PP}{f_e f_o}. \tag{92}$$

7 Mirrors

7.1 Parabolic mirror

Mirrors are often used to image a source situated at infinity into its focal plane. This is the case, for example, of satellite dishes and most telescopes. We will now use Fermat's principle to determine what shape should be used for this to work.

For this we turn to Figure 27. We want all the rays that arrive parallel to the optical axis to converge on a single point that we will call F. We define F' as the symmetric point of F by O so that $|OF'| = |OF|$. Applying Fermat's principle, we know that the ray of light will take the fastest path. Because the optical index is the same everywhere, the fastest path is here the shortest. If we furthermore want that all the rays converge on F whatever the point M where

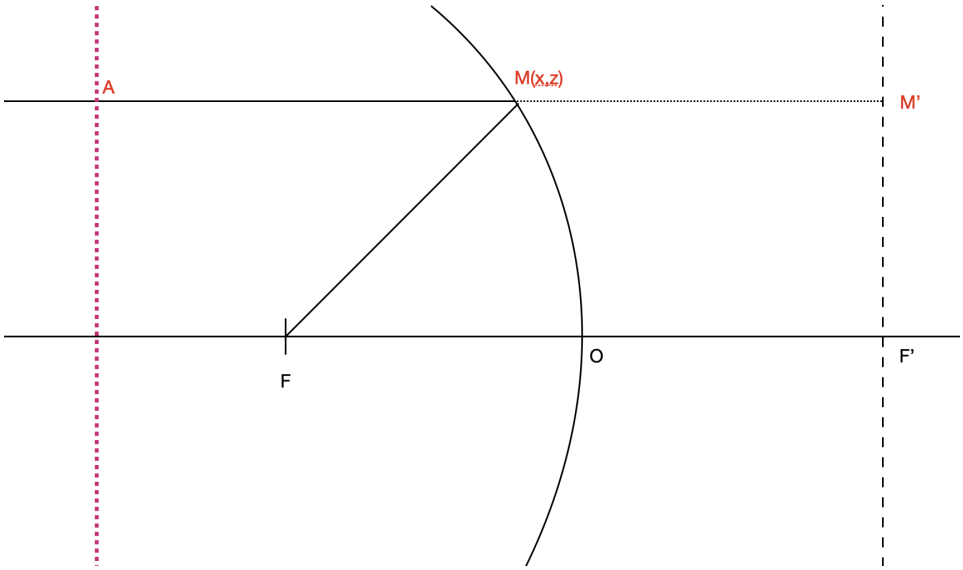


Figure 27: A parabolic mirror focuses all rays arriving parallel to the optical axis exactly on its focal point. It is a rare case of perfect stigmatism.

they hit the mirror. We know that the path from infinity to a given wavefront is the same for all rays. So we consider a given wavefront and a point A on this wavefront. Overall we want to find the mirror shape so what we want is that for all points M we have :

$$|AM| + |MF| = Constant. \tag{93}$$

We now consider M' which is in the line orthogonal to the optical axis and passing by F'. We can write that :

$$|AM| = |AM'| - |MM'|. \tag{94}$$

Because the rays arrive parallel to the optical axis, AM' is constant. So our condition becomes that:

$$|MF| - |MM'| = Constant. \tag{95}$$

If we consider the ray passing by F we find that the constant must be zero. So for all the rays to converge in F, we need all the points M on the mirror to respect the condition:

$$|MF| = |MM'|. \tag{96}$$

This condition can also be described as the ensemble of point M that are at equal distance between a point (F) and a line (F'M'). This is exactly the definition of a parabola of focus F and of vertex $F'M'$.

What we demonstrated is that, *without any approximation*, a parabolic mirror will focus all rays arriving parallel to the optical axis on the focal point of the parabola. The parabolic mirror is therefore a rare case of perfect stigmatism for the specific case of a source at infinity on the optical axis. This is different from the lens equation, where the approximation of small angles was necessary to obtain the stigmatism condition.

Due to this useful property, parabolic mirrors have been in wide use. Particularly it has been very useful used as "off-axis" telescopes, where only one part of the parabola is used. This is the case for satelite dish receiver. The "off-axis" system is possible only because of the exact stigmatism of the mirror (e.g. it does not have to be used near the optical axis) and allows to avoid shadowing effects of the receiver.

7.2 Spherical mirror

Parabolic mirrors are complex to build. Obtaining a spherical shape is much easier than a parabolic shape when polishing mirrors. That is because a spherical mirror has the same curvature at all points. In the paraxial approximation, however, a mirror has a shape so that:

$$y = \sqrt{R^2 - x^2}. \tag{97}$$

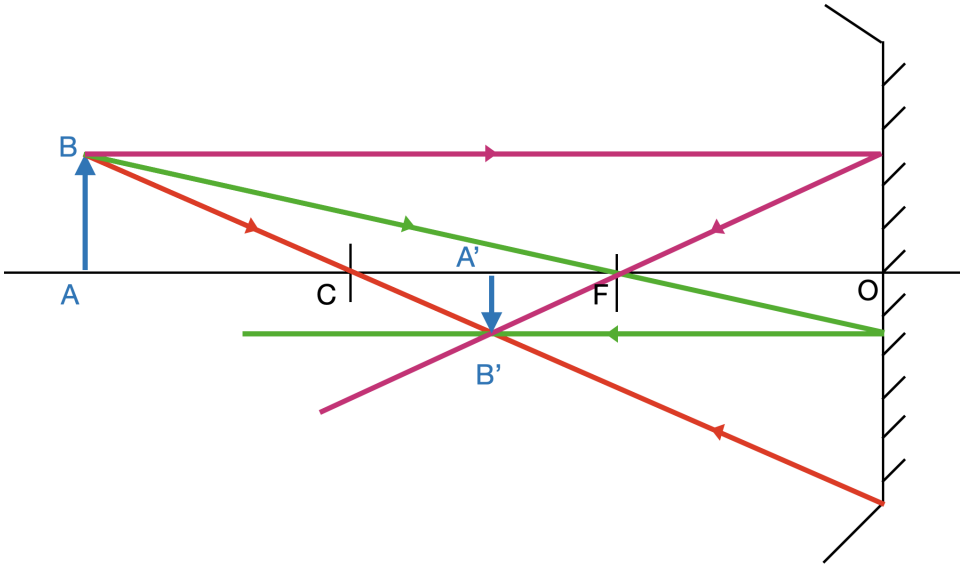


Figure 28: Image formation by a concave mirror using the three special rays. The image is real and upside down.

In the paraxial approximation $x^2 << R^2$ and the Taylor expansion of the mirror equation becomes:

$$y = R - \frac{x^2}{2R} + o\left(\frac{x^4}{R^3}\right). \tag{98}$$

This shows that in the paraxial approximation, and up to order 4 a spherical mirror looks very similar than a parabolic mirror of focus $R/2$. We will now use spherical mirrors in the paraxial approximation for the rest of this chapter.

Both concave and convex spherical mirrors follow the same equation as the thin lens equation, although we won't see the demonstration here:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}. \tag{99}$$

This equation is written for the *real is positive* sign convention. As for the case of lenses, an object will have a positive distance if it is situated on the left hand side of the mirror. However what is a real image is not an image that is formed on the same side as the object, which is different than for the lens case. So for mirrors an image will have a positive distance if it is on the left hand side of the mirror.

7.3 Concave mirror

For a concave spherical mirror we know that:

- Any ray passing by the centre of the mirror will be reflected in the same direction. That can be understood by considering the plane mirror tangent in all points of the spherical mirror.
- Any ray arriving parallel to the optical axis will hit be reflected in the direction of the focal point, as was shown for the parabolic mirror case.
- Any ray passing by the focal point will be reflected parallel to the optical axis.

Overall, a concave mirror acts very much like a converging lens, except that the image focal point is the same as the object focal point.

7.4 Convex mirror

Image formation for a convex mirror is similar to the case of the concave mirror but the center and the focal point of the mirror are now situated on the non-reflective part of the mirror. As shown in Figure 29 finding the image by a convex mirror always necessitate the use of non-existing rays, shown as dashed line.

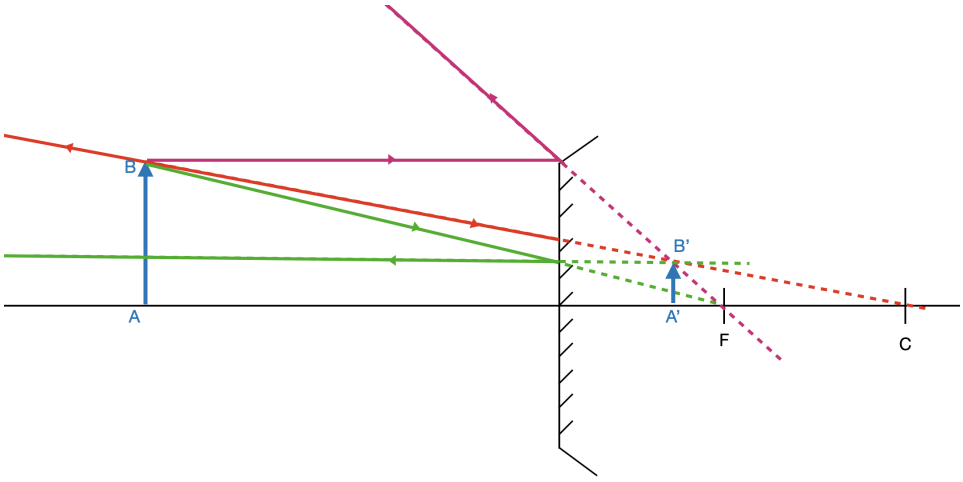


Figure 29: Image formation by a convex mirror. Here the image is smaller than the original and virtual. In order to find the image one needs to use fake rays of light, represented by dashed lines.

7.5 Newtonian telescope

A Newtonian telescope is using a concave primary mirror and a converging lens for the eyepiece. The advantage compared to a refracting telescope is that large mirrors are much easier to make than large lenses. Furthermore mirrors suffers less from optical aberations than lenses, leading to higher quality images. However, the disadvantage is that the image from the primary forms *inside the telescope tube*. This is not convenient as looking at this image with an eyepiece would force the observer to be in front of the primary mirror and impede light to reach the telescope. The solution is to add a secondary mirror that will move outside the telescope the image originally formed by the primary mirror.

As shown in Figure 30, two stars with and angular separation α will form an image of size :

$$|A'B'| = f_o\alpha. \tag{100}$$

The secondary mirror makes an image |A"B"| that is the same size as |A'B'| but is localized on the object focal point of the eyepiece lens. The eyepiece forms an image if |A"B"| at infinity. The angular separation between the two rays are then:

$$\alpha' = |A''B''|/f_e. \tag{101}$$

Finally the telescope magnification is given by

$$M = \frac{\alpha'}{\alpha} = \frac{f_o}{f_e}. \tag{102}$$

This formula is the same as the one for the refracting telescope ! Other telescope design use a divergent mirror as the secondary mirror. This allows to fold the focal length of the primary mirror and obtain a telescope with a large focal length that can fit in a small tube.

8 Optical abberations

By using the paraxial approximation we placed ourself into ideal conditions. However real lenses and mirrors are often used away from the paraxial approximation, leading to rays of light converging away from where we would like them to converge.

8.1 Spherical abberation

When using a spherical mirror or a spherical lens away from the center, the approximation made to derive the thin lens equation and the mirror equation becomes innacurate. For a mirror this will happen when the spherical shape becomes

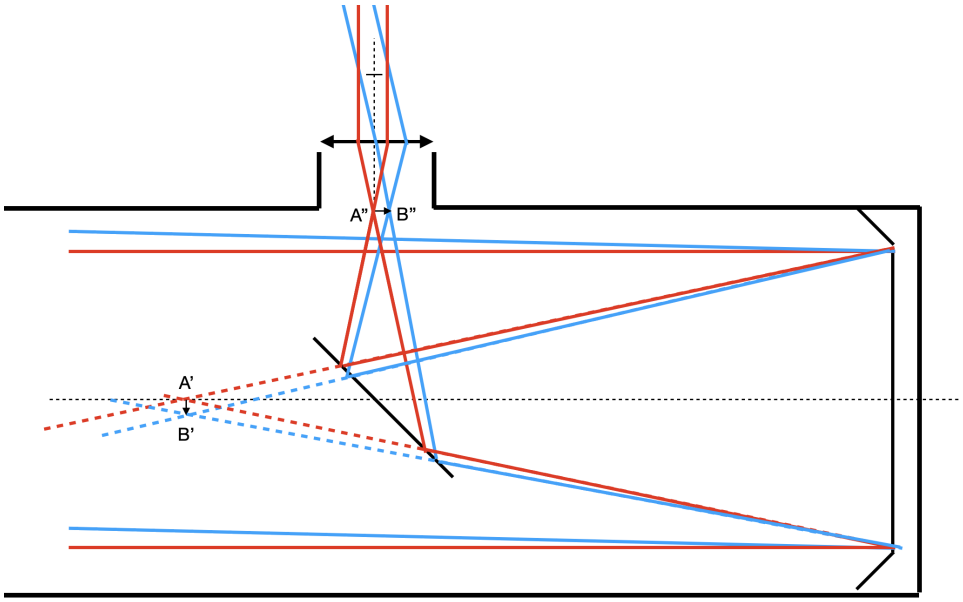


Figure 30: Image formation by a convex mirror. Here the image is smaller than the original and virtual. In order to find the image one needs to use fake rays of light, represented by dashed lines.

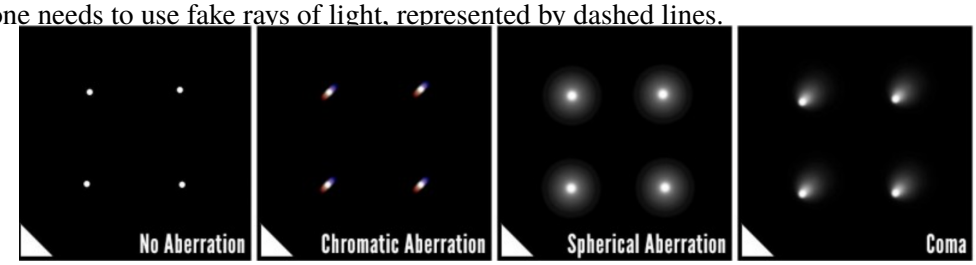


Figure 31: Different types of aberrations taken from https://www.lonelyspeck.com/

significantly different than a parabolic shape. The difference between a sphere and a parabola can be written:

$$\Delta x = \frac{y^4}{3R^3}. \tag{103}$$

The typical mirrors used in the optics lab have a radius of $y =\approx 1cm$ and a focal length of $f = 10cm$ so a radius of curvature of $R = 20cm$. This leads to a departure from a paraboloid shape of $\Delta x = 400nm$ which is similar to the the wavelength we use. For the case of a source at infinity in the optical axis, we saw that the parabolic mirror was perfectly stigmatic. For our optics lab mirror, light that arrive from infinity and that hit the edges of the mirror will have an optical path that is roughly one to two wavelengths larger than light hitting the mirror on the centre. As a consequence lightrays from the side of the mirror won't converge exactly on the focal point. If a screen is placed at the focal point, the resulting image will therefore be a larger spot than expected from perfect conditions. This departure from parabolic shape also happens for lenses and thus they too suffer from spherical aberation.

8.2 Coma

The coma is due to the non-stigmatic properties of the parabolic mirror (or the parabolic lens). As shown in Fig. 33, rays hitting the parabolic mirror far from the optical axis do not converge on the same plane nor on the same distance to the optical axis as rays that arrive close to the optical axis. As a consequence a coma-shaped feature instead of a point source is seen in the image.

8.3 Chromatic aberration

Chromatic aberration are specific to lenses because they rely on refraction rather than reflection. The optical index of lens material is always wavelength dependent to some degrees. As a consequence rays of different colours coming from a single point will converge in slightly different points. If the position of the focus is shifted along the optical

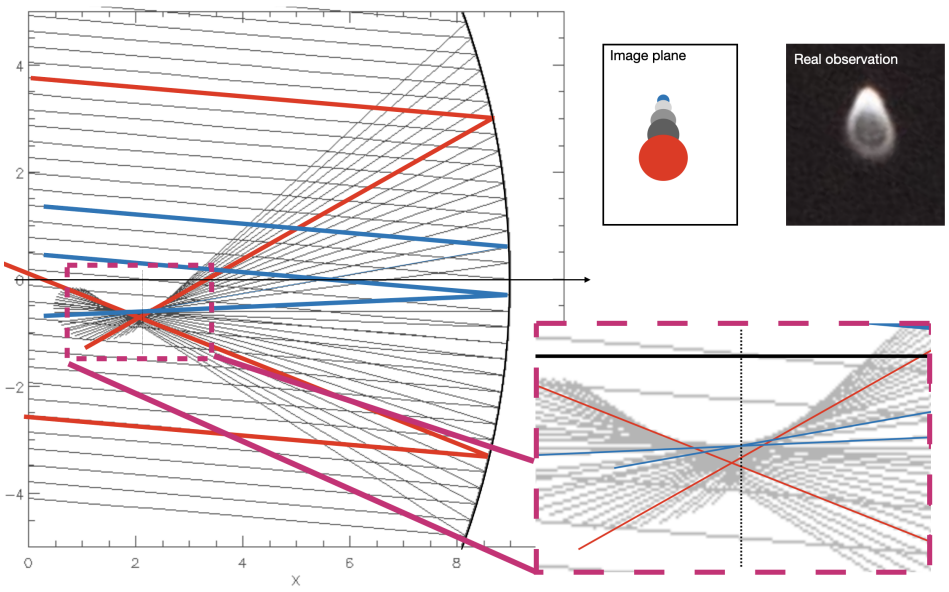


Figure 32: Formation of a coma aberration for off-axis sources in a parabolic mirror. The rays that hit the parabola far from the optical axis (here shown in red) converge on a different position than the rays hitting the parabolic mirror close to the optical axis. If a screen is placed near the focal point (zoomed-in inset) then the image is not a point source, but has a characteristic coma shape.

axis, the image of a point will be more or less wide depending of its colour. This will Chromatic aberration can often be seen in pictures taken with a poor quality camera lens: the edge of objects appear coloured.



Figure 33: This picture of a horse shows a purple fringing above its mane, sign of a chromatic aberration of its lens.

Appendix

Notations		
Quantity	Notation	Comments
Segment	$\overline{AB}$	
Length of $\overline{AB}$	$\|AB\|$	$\|AB\| = \|BA\|$
Oriented length of $\overline{AB}$	AB	$AB = -BA$
Vector	$\overrightarrow{AB}$	$\overrightarrow{AB} = -\overrightarrow{BA}$
Vector's modulus	$\|\overrightarrow{AB}\|$	$\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\|$
Angle	$\angle ABC$	$\angle ABC = -\angle CBA$