Electromagnetism (CP2)

Revision Lectures TT2025

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1 Electrostatics

1.1. State Coulomb's Law for the force between two charges, Q_1 and Q_2. Hence show how the electric field $\mathbf{E}$ at a point $\mathbf{r}$ may be defined. What is meant by the statement that $\mathbf{E}$ is a conservative field? [4]

1.2. A thundercloud and the ground below can be modelled as a charge of +40 As at a height of 10 km and a charge of −40 As at a height of 6 km above an infinite conducting plane. A person with an electrometer stands immediately below the thundercloud. What value of electric field do they measure, and what is its direction? [5]

1.3. An array of localised charges q_i experience potentials V_i as a result of their mutual interaction. Show that their mutual electrostatic energy, W, is given by $W = \frac{1}{2}\sum_i q_i V_i$. [6]

1.4. A sphere of radius a is located at a large distance from its surroundings which define the zero of potential. It carries a total charge q. Determine the potential of its surface and the electrostatic energy of its charges in two separate situations:

(a) with the charge spread uniformly on its surface, [4]

(b) with the charge distributed uniformly within its volume. [10]

1.5. Calculate the electric field strength E and the electrostatic potential V, as functions of radial distance r, for a sphere of uniform positive charge density ρ_0, of radius R, centred at the origin. Sketch graphs of E and V against r. [9]

1.6. The electron charge density of a hydrogen atom in its ground state is given by

$$\rho(r) = -\frac{e}{\pi a_0^3}\exp[-2r/a_0],$$

where a_0 is the Bohr radius (5.3×10^{-11} m). Show that the electric field due to the electron cloud is given by

$$E(r) = \frac{e}{4\pi\epsilon_0}\left\{\frac{(e^{-2r/a_0}-1)}{r^2} + \frac{2e^{-2r/a_0}}{a_0 r} + \frac{2e^{-2r/a_0}}{a_0^2}\right\}. \quad [9]$$

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Explain qualitatively what happens when such an atom is placed in a steady, uniform electric field, of strength E_0. [4]

$$\left[\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx\right]$$

1.7. The space between two concentric spheres, of radii a and b ($b > a$), is filled with air which has a relative permittivity of 1. Show that the capacitance C of the combination is given by

$$C = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right).$$ [7]

The inner sphere is raised to a potential V and then isolated, the outer sphere being earthed. The outer sphere is then removed. Show that the resulting potential V' of the remaining sphere is given by

$$V' = \frac{bV}{b-a}.$$ [5]

If the values of a and b are 0.9 m and 1.0 m respectively, and given that air cannot sustain an electric field greater than 3000 V mm^{-1}, calculate the maximum potential to which the inner sphere can be initially charged. [8]

1.8. An *electric dipole* consists of charges $-q$ and $+q$ separated by a distance $2l$, the resulting dipole moment $\mathbf{p}$ being of magnitude $2ql$ and with direction from $-q$ to $+q$. At a point (r, θ) relative to the centre and the direction of the dipole axis, derive from first principles, in the case where $r \gg l$,

(a) the electrostatic potential, [4]

(b) the radial and tangential components of the electric field, [4]

(c) the torque exerted on such a dipole by a uniform electric field $\mathbf{E}$. [6]

Using these results find the angle θ for which the resultant electric field $\mathbf{E}$ at the point (r, θ) is in a direction *normal* to the axis of the dipole. [11]

1.9. Show that the work W done in bringing a dipole of equal magnitude from infinity to a point at distance r from the first, along the normal to its axis, is given by

$$W = \frac{4q^2 l^2}{4\pi\epsilon_0 r^3} \cos\theta,$$

where θ is the angle between the axes of the dipoles. [12]

1.10. If a second dipole, free to rotate, is placed firstly along the line $\theta = 0$, and secondly in the plane $\theta = \pi/2$, in what direction will it point relative to the first? [4]

2 Magnetostatics

2.1. State the Biot-Savart Law which describes the magnetic flux density d$\mathbf{B}$ at a distance $\mathbf{r}$ from a current element $I\mathrm{d}\mathbf{l}$. [4]

Find the magnitude B of the magnetic flux density on the axis of a plane coil of n turns and radius a for a current I in the coil and at a distance z from the plane of the coil. [7]

Two such coils are placed a distance d apart on the same axis. They are connected in series in such a way as to produce fields on the axis in the same direction. Write down an expression for the magnitude of the net field B' on the axis at a distance x from the point midway between the coils. [3]

Show that the derivative of B' with respect to x is zero when $x = 0$. Find the value of d for which the second derivative of B' with respect to x is also zero at $x = 0$. Under these conditions, show that the variation of B' between $x = 0$ and $x = d/2$ is less than 6 percent. [11]

2.2. A very long cylindrical solenoid has radius R and is wound with N turns of wire per unit length. If the winding carries a current I, show that the magnetic induction B inside the coil is radially uniform and give an expression for its value. [8]

Calculate the self-inductance per unit length of the solenoid. [7]

A superconducting solenoid has radius 0.5 m, length 7 m and consists of 1000 turns. Calculate the magnetic induction in the solenoid, and the energy stored in it when it carries a current of 5000 A. You may approximate its behaviour to that of a very long solenoid. [7]

2.3. A long coaxial cable consists of two thin-walled coaxial cylinders of radii a and b. The space between the cylinders is maintained as a vacuum and a current I flows down the inner and returns along the outer cylinder. Calculate the magnetic field at a distance r from the axis when

(i) $b > r > a$, (ii) $r > b$ and (iii) $r < a$. [10]

Hence show that the *self inductance* of a length l of this cable is $L = \dfrac{\mu_0 l}{2\pi} \ln(b/a)$. [8]

For the case where the inner cylinder is replaced by a solid wire, also of radius a, throughout which the current is uniformly distributed, sketch the variation of the magnitude of magnetic field with r over the range $r = 0$ to $r = 2b$. [7]

3 Electromagnetic Induction & Electrodynamics

3.1. State the laws of *electromagnetic induction*. [3]

A thin circular annular disc of outer radius a, inner radius $a/2$ and thickness d is made from a metal of resistivity ρ. Electrical contact is made to the disc through two stationary brushes of negligible impedance. The first of these extends around the entire outer periphery at radius a, while the second makes contact around the entire inner edge at radius $a/2$.

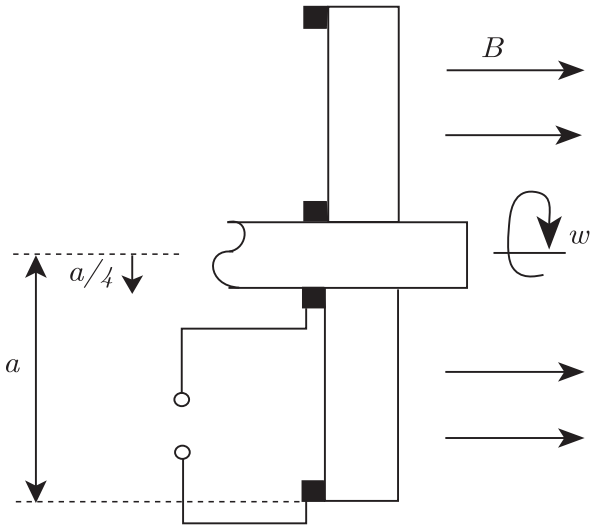
(a) Calculate the electrical resistance of this arrangement. [10]

(b) Derive an expression for the potential difference between the brushes if the disc rotates at angular velocity ω in a uniform magnetic induction $\mathbf{B}$ parallel to the rotation axis. [8]

(c) What value of load resistor R connected between the brushes would allow maximum power to be delivered to the load while the disc rotates, as in (b)? [4]

3.2. A homopolar generator consists of a metal disc of radius a and a central axle which has radius $a/4$. The disc has resistivity ρ and thickness t. It is rotated in a uniform magnetic field B about an axis through the centre, which is parallel to B and perpendicular to the plane containing the disc, at an angular frequency ω. Thin ring brushes make good electrical contact with the disc near the axle and near the outer rim of the disc as shown.

(a) Calculate the resistance of the disc R_D measured between the brushes. [7]



(b) Show that the potential difference between the brushes is $(15/32)\omega Ba^2$. [7]

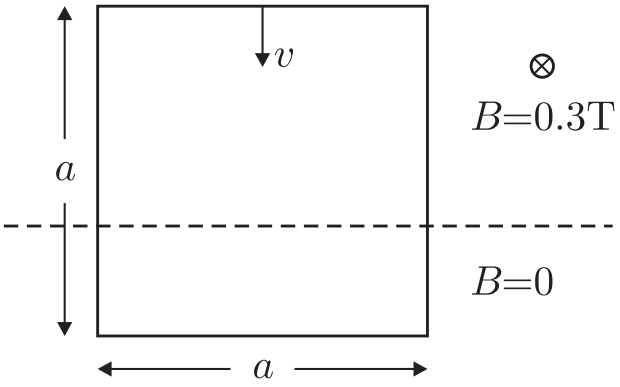
(c) A load resistance R_L is connected across the generator and the drive is removed. Show that, in the absence of mechanical friction, the time τ taken for the disc to slow down to half its initial angular speed is

$$\tau = \left(\frac{32}{15}\right)^2 \times \left[\frac{m(R_L + R_D)\ln 2}{2a^2B^2}\right] . \qquad [8]$$

[Take the moment of inertia of the disc and axle to be $\frac{1}{2}ma^2$ where m is the total mass.]

3.3. A vertical square loop of wire with sides a is falling with velocity v as shown in the figure from a region of horizontal magnetic induction B into a region where $B = 0$. If the resistance of the loop is R, show that the magnitude of the current in the loop is

$$I = \frac{Bav}{R} . \qquad [8]$$



Describe the forces acting on the loop due to the magnetic field, and indicate their directions. [7]

If $a = 10\,\text{cm}$ and the wire has a diameter of 1 mm and is made of copper (resistivity $= 1.7 \times 10^{-8}\,\Omega\text{m}$, density $= 8960\,\text{kg m}^{-3}$), and $B = 0.3\,\text{Tesla}$, calculate the steady state velocity, if this is reached while the upper arm of the loop is still in the magnetic field. [7]

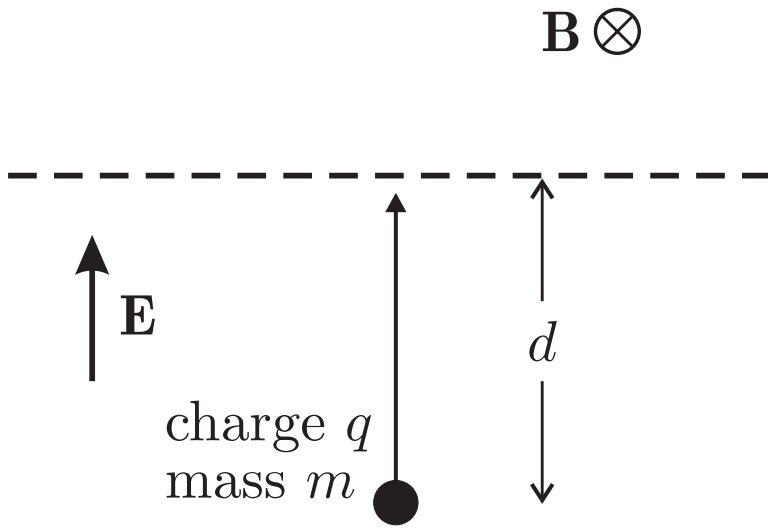
3.4. Two parallel rails separated by a distance d lie along the direction of greatest slope on an incline making an angle θ with the horizontal. A flat bar of mass m rests horizontally across the rails at the top of the incline. Both the bar and the rails are good conductors and the rails are joined by a large resistance R at the bottom of the incline. A uniform, vertical magnetic field of flux density B exists throughout the region.

The bar is released from rest and slides freely down the rails, remaining always horizontal (i.e. perpendicular to the rails). Find an expression for the induced current and hence find the equation of motion of the bar. [10]

Show that the bar will approach a constant speed and find an expression for this speed. [4]

3.5. In a particular experiment, a particle of mass m and charge $+q$ moves with speed v along the x-axis towards increasing x. Between $x = 0$ and $x = b$, there is a region of uniform magnetic field $\mathbf{B}$ in the y-direction. Deduce the conditions under which the particle will reach the region $x > b$. In the event that it does reach this region, find an expression for the angle to the x-axis at which it will enter it. [10]

In a second experiment, the same particle is accelerated from rest by a constant electric field $\mathbf{E}$ acting over a length d. The particle then encounters a region of constant magnetic field $\mathbf{B}$ perpendicular to its velocity, as shown in the figure below. Deduce the magnitude $|\mathbf{B}|$ such that the particle will re-enter the region of constant electric field at a distance d from the point at which it left. Assuming this value of $|\mathbf{B}|$, sketch the particle's trajectory in the region of constant magnetic field and derive an expression for the time spent there. [10]



4 Electromagnetic Waves

4.1. By considering Ampere's law applied to a parallel plate capacitor being charged by a current I_{C}, explain why it is necessary to introduce a displacement current I_{d} given by

$$I_{\mathrm{d}} = \varepsilon_0 A \frac{\mathrm{d}E}{\mathrm{d}t} \ ,$$

where A is the area of the plates and E is the electric field strength between the plates of the capacitor. [7]

Show how Ampere's law with the addition of the displacement current can be written in differential form as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} \ ,$$

where $\mathbf{B}$ is the magnetic flux density and $\mathbf{J}$ is the current density. [5]

Write down Maxwell's equations for fields in a vacuum devoid of charges and currents. Deduce from these equations the wave equation for $\mathbf{E}$ and the speed of these waves. [8]

4.2. State Maxwell's equations appropriate to fields in a vacuum where there are charges and currents. [4]

Show that Maxwell's equations, in a vacuum devoid of charges and currents, lead to wave equations for the electric and magnetic fields and deduce the speed of propagation of the waves. [5]

Show that a plane wave solution may be obtained with field components E_y and B_x, with all other components zero. Deduce the direction of propagation and find the relation between the magnitudes of E_y and B_x. Draw a sketch showing the relative orientation of the field components and the direction of propagation. [11]