CP1 special relativity: problem sheet 1

Main questions

1. Lorentz transformations.

(a) The space and time coordinates of two events as measured in a frame S are as follows: Event 1: $ct_1 = x_0$, $x_1 = x_0$, $y_1 = 0$, $z_1 = 0$; Event 2: $ct_2 = x_0/2$, $x_2 = 2x_0$, $y_2 = 0$, $z_2 = 0$.

Find an inertial frame in which these events occur at the same time. Take the new frame to have spacetime origin that coincides with that of S. What time in the new frame do both events occur?

(b) Prove that for any two space-like separated events there exists an inertial frame in which the two events occur simultaneously.

(c) Prove that for any two space-like separated events there exists a frame in which the time-ordering of the two events is reversed.

2. A small, powerful laser is placed on a turntable that rotates at 1200 rev/s. The laser, whose beam makes a 30° angle with the horizontal, shines on clouds 50 km away. Calculate the speed with which the light spot on the clouds moves. Does this speed violate the limitation of the speed of light? Explain.

3. Two observers A and B, in inertial frames, have identical clocks. B passes close to A with a relative velocity $0.5c$. t seconds after B has passed (as measured on A's clock), A sends her a light signal. $2t$ seconds after receiving this signal (as measured on her own clock), B acknowledges by a return light signal to A. When does A receive it?

4. The Galaxy is about 10^5 light years across and the most energetic cosmic rays known have energies of the order of 10^{19} eV (the rest-mass of the proton is $938\,\mathrm{MeV}/c^2$, and in case it hasn't been covered yet in the lectures the energy of a particle $E = \gamma mc^2$).
(a) How long would it take a proton with this energy to cross the Galaxy as measured in the rest frame of (i) the Galaxy and (ii) the proton.
(b) In the inertial frame of the proton, what is the size of the Galaxy?

5. A burst of muons is produced by a cosmic ray interacting in the upper atmosphere. They travel towards the Earth's surface with an average speed of 0.99c. If 1% of the burst survive to reach ground level, estimate the height of the burst and calculate the distance as measured in the muon's frame. The muon mean lifetime is 2.2×10^{-6} s.

6. Velocity transformations.

(a) Show that two successive Lorentz transformations with velocities v_1 and v_2 in the same direction are equivalent to a single Lorentz transformation with velocity $(v_1 + v_2)/(1 + v_1 v_2/c^2)$.

(b) Suppose that an object has speed $0.5c$ in the x-direction in a frame S. What is the object's velocity in a frame S' that is moving relative to S with velocity $(0.8c, 0, 0)$?

(c) *(*) optional* Suppose that an object has velocity $0.5c$ in the x-direction in a frame S. What is the object's velocity in a frame S' that is moving relative to S with velocity $(0.8c/\sqrt{2}, 0.8c/\sqrt{2}, 0)$ with coordinate axis aligned with those of S?

7. The twin paradox (C. Darwin, Nature 180, 976 (1957)). On New Year's day 2030 an astronaut (A) sets out from Earth at speed 0.8c to travel to the nearest star -Centauri, about 4 light-years away (as measured in the Earth frame of reference). On reaching the star, A immediately turns around and returns to Earth arriving back on New Year's day 2040 (Earth time). A has a sibling B who remains on Earth and they agree to send each other greetings by radio every New Year's day until A returns.

(a) Convince yourself that A sends only 6 messages (including the one on the last day of the trip), whereas B sends 10.

(b) Draw a space-time diagram of A's journey in the Earth frame (mark the scales of x and ct in light years). Draw the world lines of all the radio signals that B transmits. Use the diagram to verify that A (astronaut) has received only 1 signal up to the moment of reaching the star and turning back, and receives the other 9 during the return half of the trip.

(c) On a second space-time diagram (again in the Earth frame) show the world lines of the astronaut (A) and all the signals that A sends. Verify that B (on Earth) receives one message each 3 years of Earth time for the first 9 years of the journey, then receives 3 more messages in the last year to make the total of 6 - which is the expected total since according to A's time the trip takes 3 years out and 3 years back.

8. Past Prelim Question: State the Lorentz transformations which relate the space and time coordinates in two inertial frames, S and S', travelling with relative speed v along the x (x') axis. Use the transformations to derive expressions for the phenomena of length contraction and time dilation.

A rocket passes close to the earth with relative velocity $0.8c$, at which time observers on the earth and the rocket reset their clocks ($t_E = t_R = 0$).

(a) After 30 minutes as measured by the rocket's clock, the rocket passes a space- station which is fixed relative to the earth. What time is recorded by the observer on the earth?

(b) How far is the space-station relative to the earth (i) in earth coordinates, (ii) in rocket coordinates?

(c) When the rocket passes the space-station, the rocket sends a radio signal back to earth. At what time (by earth time) does the signal arrive?

(d) The station on earth replies immediately. At what time (by rocket time) is the reply received by the rocket?

Interesting questions to think about

These extra questions vary from a bit harder than the other questions (*) to much harder (***).

Unless told otherwise, you do not need to do these questions for tutorials (it might make sense to try them over the vacation as a challenge). They are beyond the syllabus for the present course, so don't worry if you can't do them.

1. (*) Consider the wave equation (where E_y is the y component of the electric field)

$$\left(\frac{1}{c^2}\left.\frac{\partial^2}{\partial t^2}\right|_x - \left.\frac{\partial^2}{\partial x^2}\right|_t\right) E_y = 0 \ . \tag{0.1}$$

(a) Verify that

$$E_y = f(x-ct) + g(x+ct) \ ,$$

where f, g are any twice differentiable functions, is a solution of Eq. (0.1).

(b) Consider the Galilean transformation $x' = x - vt$ and $t' = t$. Assuming $E'_y = E_y$ (the electric field itself is unchanged under the transformation), show that the wave equation Eq. (0.1) transforms to

$$\left(\frac{1}{c^2}\left.\frac{\partial^2}{\partial t'^2}\right|_{x'} - \left(1-\frac{v^2}{c^2}\right)\left.\frac{\partial^2}{\partial x'^2}\right|_{t'} - \frac{2v}{c^2}\left.\frac{\partial}{\partial x'}\right|_{t'}\left.\frac{\partial}{\partial t'}\right|_{x'}\right) E'_y = 0 \ . \tag{0.2}$$

Verify that

$$E'_y = f(x' - (c-v)t') + g(x' + (c+v)t') \ ,$$

where f, g are any twice differentiable functions, is a solution of Eq. (0.2).

(c) Show that, in contrast, Eq. (0.1), is invariant under Lorentz boosts in the x-direction. Is the same true for the wave equation for a wave of speed $v < c$?

(d) In year 2 you will meet the Schrödinger equation in non-relativistic quantum mechanics:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \ .$$

Show that the Schrödinger equation has wave-like solutions of the form

$$\psi = Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t} \ ,$$

and give a relation between k and ω. Show also that the Schrödinger equation takes the same form after a Galilean boost of velocity v (i.e. $x' = x - vt$ and $t' = t$) when written in terms of

$$\psi' = \psi e^{-imvx'/\hbar} e^{-imv^2t'/(2\hbar)} \ .$$

2. A wrong-way triangle inequality (**)

Suppose U and V are future-pointing time-like 4-vectors. Prove that $U + V$ is also future-pointing and satisfies

$$\sqrt{(U+V)\cdot(U+V)} \geq \sqrt{U\cdot U} + \sqrt{V\cdot V} \ ,$$

and is hence time-like.

Hint: one approach is to first show (e.g. by working in the frame in which $U = (U^0, \mathbf{0})^T$) that V can be written as

$$V = \frac{(U \cdot V)}{(U \cdot U)} U + W \ ,$$

such that $W \cdot U = 0$, where $W^2 < 0$. Then derive a wrong-way Schwarz inequality.

3. The twin paradox strikes back (***)

The resolution of the usual twin paradox is that the twin who goes on the out and back journey changes reference frame, such that the symmetry between the two is broken and they both agree that the one who went to space and back is older.

Suppose instead that the Universe has finite size and periodic boundary conditions, such that if one of the twins heads off into space in some direction they will eventually return from the opposite direction. During the journey this twin will experience no acceleration. When the twin gets back to Earth who is older?

4. The Lorentz transformations from a universal limiting speed (**)

Two inertial frames S and S' are such that S' moves with velocity v in the positive x-direction relative to S. The spatial coordinate axes in S' are parallel to those in S and the two origins coincide at $t = t' = 0$.

(a) Argue that isotropy and homogeneity of spacetime requires that the most general transformation between S and S' is linear and of the form

$$\begin{aligned} t' &= at + bx \\ x' &= ct + dx \\ y' &= y \\ z' &= z \ , \end{aligned}$$

where a, b, c, d depend only on v. *(Hint: consider the length of an object measured before and after a displacement in the x-direction or in time, and similarly the intervals between ticks of a clock.)*

(b) Further argue that

$$x' = f(v^2)x - vf(v^2)t \, , \qquad t' = g(v^2)t - vh(v^2)x \ , \qquad y' = y \ , \qquad z' = z \ .$$

(Hints: Require that all fixed spatial points in S' have velocity v relative to S'. Write $t' - at = bx$ and argue that $a \to -a$ when $v \to -v$.)

Show also that

$$x = f(v^2)x' + vf(v^2)t' \, , \qquad t = g(v^2)t' + vh(v^2)x' \ , \qquad y = y' \ , \qquad z = z' \ .$$

(c) Show that the consistency of the forward transformation and its inverse requires

$$f = g \ , \text{and } f^2 - v^2 fh = 1 \ .$$

(d) If a particle has velocity u' parallel to the x' axis in S' show that its velocity parallel to the x-axis in S is

$$u = \frac{u' + v}{1 + vu'(h/f)} \ .$$

Using the second version of postulate 2 in the lecture notes (universal limiting speed C), show that $h = f/C^2$ and that this leads to the formula for a Lorentz boost.